HW #1

ECE 472: Robotics and Computer Vision
Instructor: Kristen Dana, Fall 2019
Rutgers University

Seth Karten
Due Date: Sep 12th

Problem 1

Are the following vectors basis vectors for $\mathbb{R}^3$? Why or why not?

$$\mathbf{w_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{w_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

To show that $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$ are basis vectors for $\mathbb{R}^3$, we must show that $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$ are orthonormal vectors and span $\mathbb{R}^3$. Note that for all $1 \leq i, j \leq 3$, by taking the dot product of each combination of vectors, we have,

$$\mathbf{w_i} \cdot \mathbf{w_j} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

By the definition of orthonormal vectors in $\mathbb{R}^3$, $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$ are orthonormal. These three vectors must span all of $\mathbb{R}^3$ since there are three vectors and each is respectively orthogonal to each other. Thus they form an basis for $\mathbb{R}^3$.

Problem 2

Do you think the following can also be basis vectors for $\mathbb{R}^3$? Why or why not?

$$\mathbf{t_1} = \begin{bmatrix} 0.707 \\ -0.707 \\ 0 \end{bmatrix}, \mathbf{t_2} = \begin{bmatrix} -0.707 \\ 0.707 \\ 0 \end{bmatrix}, \mathbf{t_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

These vectors can not be basis vectors for $\mathbb{R}^3$ since $\mathbf{t_1} \cdot \mathbf{t_2} \neq 0$. This implies that $\mathbf{t_1}$ is not orthonormal of $\mathbf{t_2}$. In fact, $\mathbf{t_1}$ is a multiple of $\mathbf{t_2}$ since $\mathbf{t_2} = -\mathbf{t_2}$. It follows that they are not independent. Thus, the vectors are not a basis for $\mathbb{R}^3$.

Problem 3

Do the following vectors span $\mathbb{R}^3$? Why or why not?

$$\mathbf{w_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{w_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{w_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w_4} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

By taking a linear combination of vectors $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$, one can span $\mathbb{R}^3$ since $\mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3}$ forms a basis in $\mathbb{R}^3$. Let $s, t, u \in \mathbb{R}$. Then

$$s\mathbf{w_1} + t\mathbf{w_2} + u\mathbf{w_3} + 0\mathbf{w_4}$$

spans all of $\mathbb{R}^3$.

Problem 4

Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1.9 \\ 1 & 3 \\ 1 & 3.9 \\ 1 & 5 \end{bmatrix}$ and let $b = \begin{bmatrix} 1 \\ 3.2 \\ 5 \\ 7.2 \\ 9.3 \\ 11.1 \end{bmatrix}$. Then we can fit the line using the least squares estimation in the form of $y = mx + b_1$, where b_1 and m are found by the following equation:

$$\begin{bmatrix} b_1 \\ m \end{bmatrix} = (A^\intercal A)^{-1} A^\intercal b.$$

Substituting, we have,

$$\begin{bmatrix} b_1 \\ m \end{bmatrix} = \left(\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1.9 & 3 & 3.9 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1.9 \\ 1 & 3 \\ 1 & 3.9 \\ 1 & 5 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1.9 & 3 & 3.9 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 3.2 \\ 5 \\ 7.2 \\ 9.3 \\ 11.1 \end{bmatrix}.$$

This becomes,

$$\begin{bmatrix} b_1 \\ m \end{bmatrix} = \begin{bmatrix} 6 & 14.8 \\ 14.8 & 53.82 \end{bmatrix}^{-1} \begin{bmatrix} 36.8 \\ 126.07 \end{bmatrix}$$

. Note that the determinant of $A^\intercal A$ is $6 * 53.82 - 14.8 * 14.8 = 103.88$. By inverting the matrix, we get,

$$\begin{bmatrix} b_1 \\ m \end{bmatrix} = \frac{1}{103.88} \begin{bmatrix} 53.82 & -14.8 \\ -14.8 & 6 \end{bmatrix} \begin{bmatrix} 36.8 \\ 126.07 \end{bmatrix}.$$

Finally, this simplifies to

$$\begin{bmatrix} b_1 \\ m \end{bmatrix} = \begin{bmatrix} 1.104543704 \\ 2.038698498 \end{bmatrix}.$$

Verifying this in Python, we get the same result.

Problem 5

The conjecture that there are only 2 independent columns in the matrix

$$A = \begin{bmatrix} 4.29 & 2.2 & 5.51 \\ 5.20 & 10.1 & -8.24 \\ 1.33 & 4.8 & -6.62 \end{bmatrix}$$

can be tested through SVD. If the singular values of the SVD are nonzero, then the rank is 3. The singular values are 16.14479065, 7.6505272, and 0.01638304. Since each singular value is non-zero, the rank of the matrix is 3. However, effective rank can be considered 2 since our third singular value is very close to 0.