THERE IS NO FIBONACCI REPRESENTATION OF $\mathscr{H}_{k,q}(S_n)$

MILES JOHNSON & NATALIE STEWART

Let k be a field, and let $q \neq 0 \in k$ be the parameter of the Hecke algebra $\mathscr{H} := \mathscr{H}_{k,q}(S_n)$. We will show that there is no "Fibonacci" representation of $\mathscr{H}$.

First, let F be the vector space with basis given by the strings of length $n+1$ with alphabet $\{p, *\}$ such that no two $*$ symbols appear in a row. This is given an action by the braid group that we will try to emulate. We want an action which is "local", i.e. the simple transposition T_i acts on the string from the ith to the $(i+2)$nd symbol, modifying only the middle character, defined by the following rule:

$$\begin{aligned}
\widehat{(*pp)} &:= a(*pp) \\
\widehat{(*p*)} &:= b(*p*) \\
\widehat{(p*p)} &:= c(p*p) + d(ppp) \\
\widehat{(pp*)} &:= a(pp*) \\
\widehat{(ppp)} &:= d(p*p) + e(ppp).
\end{aligned} \tag{1}$$

For suitable constants $a, b, c, d, e \in k$.

Quadratic Relation. Note that the quadratic relation $T_i^2 = (q-1)T_i + q$ imposes the following restrictions on the constants:

$$\begin{aligned}
a^2 &= (q-1)a + q \\
b^2 &= (q-1)b + q \\
c^2 + d^2 &= (q-1)c + q \\
de &= (q-1)d \\
e^2 + d^2 &= (q-1)e + q \\
dc &= (q-1)d
\end{aligned} \tag{2}$$

Note that we immediately have

$$\begin{aligned}
a, b &\in \left\{ \frac{q-1 \pm \sqrt{(q-1)^2 + 4q}}{2} \right\} \\
&= \left\{ \frac{q-1 \pm (q+1)}{2} \right\} \\
&= \{-1, q\}.
\end{aligned} \tag{3}$$

Further, if $d = 0$ then we have $c, e \in \{-1, q\}$; if $d \neq 0$ then we have that $c = e = (q-1)$, and $d \in \left\{\pm\sqrt{q}\right\}$.

Braid Relations. Here's where we'll run into some issues.

We must verify the relation $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$. Let's begin with the case $d \neq 0$, which we can verify on the following strings:

- $(*ppp)$ requires that $abd = bcd + ade$ and $a^2 e = ae^2 + bd^2$.
  The first of the above equivalently requires

$$ab = b(q-1) + a(q-1) \tag{4}$$

- $(pppp)$ requires that $acd + de^2 = ade$. Equivalently, we require that $e^2 = 0$, i.e. $d = q = 1$. Then, by (4), we require that $ab = 0$, but $a, b \neq 0$; this is a contradiction, so there are no constants a, b, c, d, e which make this a representation of $\mathscr{H}$.

Now suppose that $d = 0$. One thing which is immediately clear is the decomposition into one-dimensional subrepresentations (the spans of each basis vector) if this is a representation. Note that $(*ppp)$ requires that $a^2e = ae^2$, i.e. $a = e$. Similarly, $(*p*p)$ requires that $b = c$. Further, we still satisfy the relations for $(pppp)$, we always satisfy the relations for $(*pp*)$, and the rest of the strings are compatible by symmetry. All that is left are the relations $T_iT_j = T_jT_i$ for $|i-j| > 1$, which are easy to see. Hence, for each q, there are 4 "good" actions on F, each of which decomposes into a direct sum of one-dimensional subrepresentations.

A More General Case. Now, consider a modification of (1):

$$\begin{aligned} \widehat{(*pp)} &:= a(*pp) \\ \widehat{(*p*)} &:= b(*p*) \\ \widehat{(p*p)} &:= c(p*p) + d(ppp) \\ \widehat{(pp*)} &:= g(pp*) \\ \widehat{(ppp)} &:= f(p*p) + e(ppp). \end{aligned} \tag{5}$$

This time, the quadratic relation reads

$$\begin{aligned} a^2 &= (q-1)a + q \\ b^2 &= (q-1)b + q \\ g^2 &= (q-1)g + q \\ c^2 + df &= (q-1)c + q \\ de &= (q-1)d \\ e^2 + df &= (q-1)e + q \\ fc &= (q-1)f \end{aligned} \tag{6}$$

Notably, we have $a, b, g \in \{-1.q\}$ still. If $d = 0$ or $f = 0$, then $c, e \in \{-1, q\}$ still, and if $d = 0$ and $f \neq 0$, then $c = (q-1)$, and hence $c = 1$ and $q = 2$. If $d \neq 0$, we still have that $c = e = (q-1)$, and we have that $df = q$.

Now, suppose that $d, f \neq 0$. Now, $(*ppp)$ requires that $abf = bcf + aef$ and $a^2e = ae^2 + bf^2$, so as before we have that $ab = (a+b)(q-1)$.

$(pppp)$ now requires that $aef = acf + e^2f$. This requires that $e^2 = 0$, so that $q = 1$ and $ab = 0$, a contradiction again.

Now, suppose that $f \neq 0$ and $d = 0$, so that $q = 2$ and $c = 1$. Then, we have that $a(e-1) = e^2$. Then, knowing that $e^2 \neq 0$, we have $e = -1$ so $-2a = 1$, a contradiction. By symmetry, we also arrive at a contradiction if $d \neq 0$ and $f = 0$.

Finally, suppose that $d = f = 0$, and note that we now have $a = e = g$ and $b = c$; hence our case is precisely the case with $d = f$ and $a = g$, and there are exactly four actions of $\mathscr{H}$ on F on which each T_i acts analogously on positions $i, i+1, i+2$ as each other, only modifying position $i+1$. Each of these actions decomposes into a direct sum of 1-dimensional subrepresentations.