

THE TWO-COLUMN SPECHT MODULE OF THE HECKE ALGEBRA OF S_n

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1. INTRODUCTION

Let S_{2n} be the symmetric group on $2n$ indices, let $\mathcal{H} = \mathcal{H}_{k,q}(S_{2n})$ be the corresponding Hecke algebra with parameter q , and let $S = \{T_1, \dots, T_{n-1}\}$ be the simple transpositions generating \mathcal{H} . Let $[k]_q = 1 + q + \dots + q^{k-1}$ be the q -number of k . Let e be the smallest positive integer such that $[e]_q = 0$, and set $e = \infty$ if no such integer exists. Either $q = 1$ and e is the characteristic of k , or $q \neq 1$ and q is a primitive e th root of unity.

Let $V := S^{(2, \dots, 2)}$ be the Specht module corresponding to the young diagram with rows of length 2. The purpose of this writing is to characterize this representation via an isomorphism with another representation of \mathcal{H} .

Definition 1.1. A *crossingless matching* on $2n$ indices is a partition of $\{1, \dots, 2n\}$ into parts of size 2 such that no two parts “cross”, i.e. there are no parts (a, a') and (b, b') such that $a < b < a' < b'$. Then, define W_{2n} to be the \mathbb{C} -vector space with basis the set of crossingless matchings on $2n$ indices.

In order for this to be a \mathcal{H} -module, endow this with the action given by Figure 1; if this creates a loop, simply scale by $(1 + q)$, and otherwise deform into a crossingless matching and scale by $q^{1/2}$.

Let the length of an arc (i, j) be $l(i, j) := j - i + 1$. Note that the crossingless matchings can all be identified with a list of n integers describing the lengths of the arcs from left to right; using this, we may order the crossingless matchings in increasing lexicographical order in order to obtain an order on the basis. Let C_n be the n th catalan number, and let the resulting basis for W_{2n} be $\{w_1, \dots, w_{C_n}\}$ as illustrated in Figure 2.

We will prove that $W := W_{2n}$ is isomorphic to V as representations in the case that \mathcal{H} is semisimple. To do so, we will prove that W has an irreducible restriction to $S_{2n-1} \subset S_{2n}$; using the branching theorem, this implies that W is isomorphic to a Specht module corresponding to a rectangular young diagram.

We will move on to prove that these modules have unique dimension up to transposition of the diagram; then, we will show that $\dim V = \dim W$ so that W corresponds to an $n \times 2$ or $2 \times n$ diagram. We will then do a short character computation to prove that $V \cong W$.



Figure 1. Illustration of the actions $(1 + T_4)w_3$ and $(1 + T_2)w_3$ in W_6 . In general, we act on basis elements by simple transpositions by deleting loops, deforming into a crossingless matching, and scaling based on whether a loop was deleted.

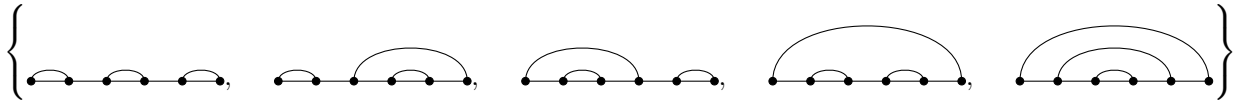


Figure 2. The increasing lexicographical basis for W_6 .

2. IRREDUCIBILITY

We can now begin by proving that $\text{Res}_{\mathcal{H}_{k,p}(S_{2n-1})}^{\mathcal{H}_{k,p}(S_{2n})} W$ is irreducible; in particular, this implies that W itself is irreducible, as an \mathcal{H} -subrepresentation is a $\mathcal{H}_{k,p}(S_{2n-1})$ -subrepresentation.

Proposition 2.1. *Set $\mathcal{H}' := \mathcal{H}_{k,p}(S_{2n-2}) \subset \mathcal{H}$. Then, $\text{Res}_{\mathcal{H}'}^{\mathcal{H}} W$ is irreducible if $e \geq n + 1$.*

Proof. We will prove the equivalent condition that each vector in $w \in W$ is *cyclic*, i.e. $\mathcal{H}'w = W$.

We will first prove that w_1 is cyclic, for which it is sufficient to prove that every basis vector of W is in Aw_1 . fix some basis vector w_k , and suppose that it contains arc $(1, j)$, Then, the vector

$$w' := (1 + T_2)(1 + T_4) \dots (1 + T_{j-2})w_1$$

contains an arc $(1, j)$ and all other arcs are of the form $(a, a + 1)$ for some a . We may separately act on the subset of arcs with $1 < a < j$ and with $a > j$; this process gave our base case of W_4 , and allows us to recurse to W_{2m} with $m < n$, outlining explicit vectors $h \in \mathcal{H}'$ with $hw_1 = w_i$. Hence it is sufficient to prove that w_1 is generated by every $w \neq 0 \in W$.

Since the image of $(1 + T_i)$ has arc $(i, i + 1)$, it is isomorphic as a vector space to W_{2n} with $2n - 2$ vertices.¹ Further, all actions of $(1 + T'_j) \in \mathcal{H}_{k,p}(S_{2n-4})$ act identically (through the isomorphism) with simple transpositions other than $1 + T'_i$, which acts equivalently to $q^{-1}(1 + T_i)(1 + T_{i+1})(1 + T_{i-1})$ as illustrated in Figure 3; hence, if the image of $(1 + T_i)w$ in W_{2n-2} generates an ideal containing w'_1 .

Then $(1 + T_i)w$ generates an ideal containing the pre-image of w'_1 , which may either be w_1 if i is odd or the basis vector containing $(i - 1, i + 2)$, $(i, i + 1)$, and all other arcs length 2 if i is even.

We are now ready to make the central claim in our proof:

Claim. *Suppose $n > 1$. Then, the intersection $K := \bigcap_{i=1}^{2n-2} \ker(1 + T_i)$ is trivial if e does not divide $n + 1$.*

This claim is necessary for irreducibility, as K is a proper subrepresentation of W .

Suppose this claim is true. We will use induction on n to prove irreducibility; the base case $n = 1$ is clear, so suppose W_{2n-2} is irreducible for all $e \geq n$ or, and pick some $w \in W_{2n}$. Then, pick some $1 + T_i$ such that $(1 + T_i)w \neq 0$, and pick some action $h' \in W_{2n-2}$ which takes the image of $(1 + T_i)w$ in W_{2n-2} to w'_1 ; this pulls back to an action $h \in W_{2n}$ such that $h(1 + T_i)w$ is the preimage of w'_1 .

If i is odd then we have $h(1 + T_i)w = w_1$ and we are done. Otherwise, $h(1 + T_i)w$ is a basis vector with the arc $(i - 1, i + 2)$ and all other arcs having length 2; hence $q^{-1/2}(1 + T_{i-1})h(1 + T_i)w = w_1$ and we are done.

¹This isomorphism “ignores” the arc $(i, i + 1)$.

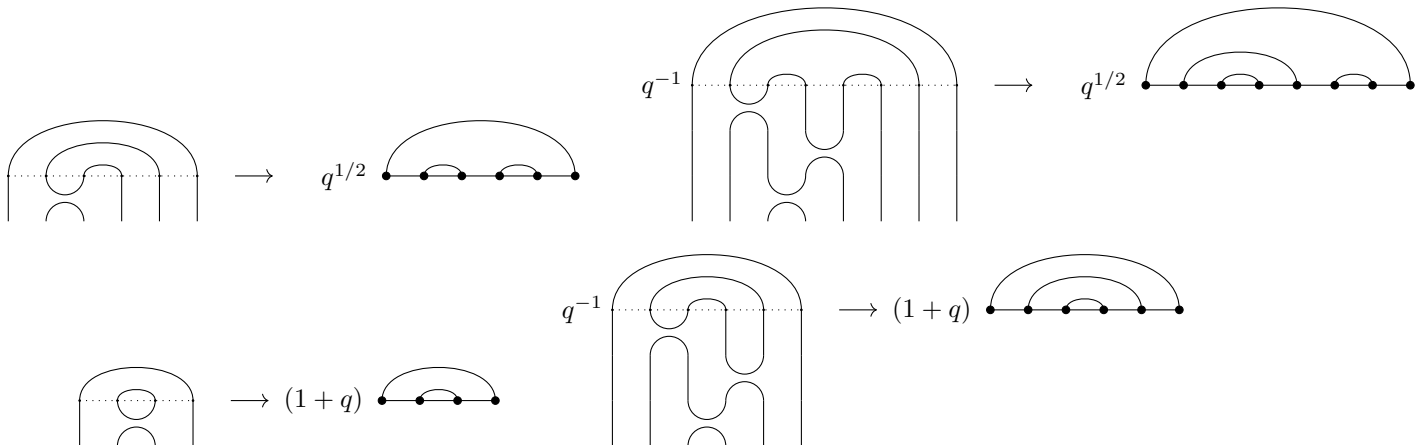


Figure 3. The correspondence between the action of $(1 + T_2)$ on $w'_5 \in W_6$ and the action of $q^{-1}(1 + T_3)(1 + T_4)(1 + T_2)$ on the corresponding vector in W_8 having arc $(3, 4)$ first, then on $w'_2 \in W_4$. This demonstrates that the action works with and without creating a loop.

Proof of claim. Note that $\bigcap \ker(1 + T_i) = \ker(\bigoplus (1 + T_i))$. Further, we may postcompose this with the isomorphism $\text{im}(1 + T_i) \simeq W_{2n-2}$ in order to arrive at a $(n-2)C_{n-1} \times C_n$ matrix A whose kernel is K . Further, we may apply elementary row operations to this matrix, postcomposing with a change of basis, and preserve the kernel of this matrix as well.

We may characterize these rows as follows: the row corresponding to $(1 + T_i)$ and mapping into the element $w_k \in W$ is of the form $[a_1, \dots, a_{C_n}]$ where $a_k = 1 + q$, $a_j = q^{1/2}$ whenever $(1 + T_i)w_j = q^{1/2}w_k$, and 0 otherwise. In particular, we may characterize some particular rows with few nonzero entries.

First, the row of A corresponding to $(1 + T_n)$ and w_{C_n} will be of the form $[0, \dots, 0, q^{1/2}, q+1]$. Similarly, row corresponding to $(1 + T_{n-1})$ and $w_{C_{n-1}}$ will be of the form $[0, \dots, 0, q^{1/2}, 0, q+1, q^{1/2}]$. An inductive process is illustrated in Figure 4. in order to construct a row corresponding to each $(1 + T_{n-i})$ with nonzero components appearing in the order $q^{1/2}, q+1, q^{1/2}$ and having the last two of these appear above the first two in the row corresponding to $(1 + T_{n-i-1})$. This process works for $1 \leq i \leq n-1$, and two-nonzero-element rows, which are in column-alignment with the appropriate elements in $(1 + T_2)$ and $(1 + T_{n-1})$.

This construction yields a $n \times C_n$ submatrix of A which has nonzero entries on the rightmost column, and has (by removing zero columns) the same column space as the following:

$$\begin{bmatrix} q+1 & q^{1/2} & & & & & \\ q^{1/2} & q+1 & q^{1/2} & & & & 0 \\ & q^{1/2} & q+1 & q^{1/2} & & & \\ & & \ddots & \ddots & & & \\ & 0 & & & q^{1/2} & q+1 & q^{1/2} \\ & & & & q^{1/2} & q+1 & \end{bmatrix}$$

We may recursively compute that this matrix has determinant $[n+1]_q$, which vanishes exactly when $e \mid n+1$; henceforth assume that this does not vanish. Then, this matrix is invertible, so we may use elementary row operations to transform it into the identity; these correspond to elementary row operations on A which yield a row $[0, \dots, 0, 1]$. Hence any vector $w \in K$ has a zero coefficient corresponding to w_{C_n} . We will now prove that every nonzero vector $w \in K$ must have a nonzero coefficient corresponding to w_{C_n} , giving that $K = 0$. \square

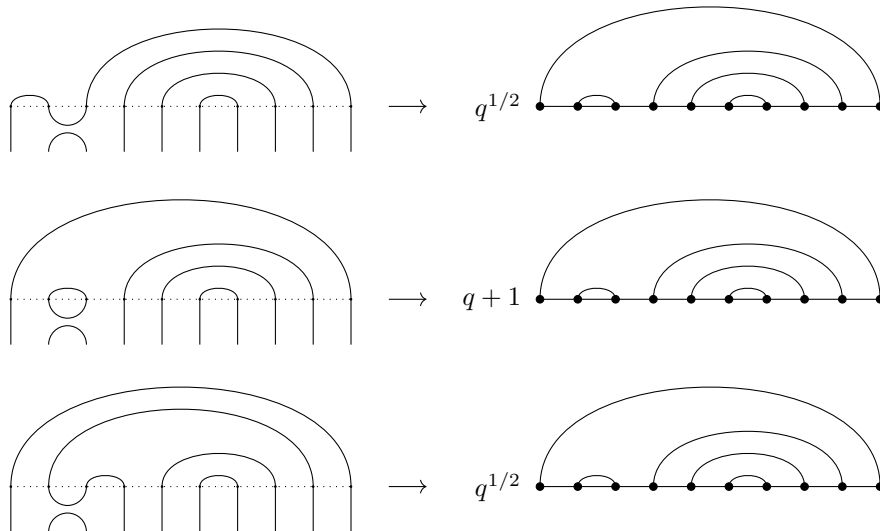


Figure 4. Illustrated is the row constructed for transposition $(1 + T_2)$; clearly these are the only basis elements mapping to multiples of the desired element, and they relate to each other. In general, the basis elements considered are the preimages of $w_{C_{n-1}}$ in the isomorphism $\text{im}(1 + T_i) \simeq W_{2n-2}$.

3. CORRESPONDENCE WITH THE SPECHT MODULE

Suppose \mathcal{H} is semisimple; then W is a Specht module corresponding to a rectangular region. We may now use dimension, for which one should note that there is an easy bijection between standard tableaux on $n = 2 + \cdots + 2$ and the Dyck paths on $2n$ points by specifying that the number in the $(i, 1)$ box of the tableau is the index of the i th “up” path. Hence $V_{(2, \dots, 2)}$ has dimension the n th catalan number C_n .

Similarly, we may biject the Dyck paths on $2n$ points with the crossingless matchings, by making the value at point i the number of “open crossings” at that point, i.e. the number of arcs (a, b) with $a \leq i < b$. Hence V and W have the same dimension. This pins the shape of the diagram corresponding to W as follows.

Proposition 3.1. *Let V_1 and V_2 be two specht modules corresponding to $a_1 \times b_1$ and $a_2 \times b_2$ rectangular young diagrams. Then, $\dim V_1 = \dim V_2$ if and only if the diagram of V_1 is the same as or a transposition of the diagram of V_2 .*

Proof. Recall that, if they have the same diagram then $V_1 \cong V_2$, and if they’re transposed from each other, $V_1 \cong V_2 \otimes U$ where U is the alternating representation; hence $\dim V_1 = \dim V_2 \cdot \dim U = \dim V_2$.

Suppose WLOG that $a_1 < a_2 < b_2 < b_1$. By the hook-length formula, it is sufficient to give a bijective correspondence between the boxes in V_2 and V_1 such that the hook-length in V_2 is larger than the hook-length in V_1 , and at least once strictly larger. We can give this correspondence by listing the boxes beginning at the bottom right corner, then increasing up left-to-right diagonals, and note that this satisfies our conditions. \square

Let $V' := V_{(n, n)}$. Then, W is isomorphic to exactly one of V and V' . It is hence sufficient to prove that W is not isomorphic to V' , and we may do this via a character computation.

4. SUMMARY OF RESULTS

What follows is a color-coded list of propositions on crossingless matchings. Those which are blue are **confirmed**, those which are olive are **in progress/soon to be proven**, and those which are red are **conjectural**.

Proposition 4.1. *Suppose $n > 1$. Then, the intersection $K := \bigcap_{i=1}^{2n-2} \ker(1 + T_i)$ is trivial if e does not divide $n + 1$.*

Proposition 4.2. *Define the subalgebra $\mathcal{H}' := \mathcal{H}_{k,p}(S_{2n-2}) \subset \mathcal{H}$. Then, $\text{Res}_{\mathcal{H}'}^{\mathcal{H}} W$ is irreducible if $e \geq n + 1$.*

Proposition 4.3. *Suppose \mathcal{H} is semisimple, i.e. $e > 2n$. Then, $V \simeq W$.*

Proposition 4.4. *The intersection $K := \bigcap_{i=1}^{2n-2} \ker(1 + T_i)$ is nontrivial when $e = n + 1$.*

Proposition 4.5. *W is reducible when $e \leq n + 1$.*

Proposition 4.6. *Even if $e \leq 2n$, we have $V \simeq W$.*