

# THE TWO-COLUMN SPECHT MODULE OF THE HECKE ALGEBRA OF $S_n$

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## 1. INTRODUCTION

Let  $S_{2n}$  be the symmetric group on  $2n$  indices, let  $\mathcal{H} = \mathcal{H}_{k,q}(S_{2n})$  be the corresponding Hecke algebra with parameter  $q$ , and let  $S = \{T_1, \dots, T_{n-1}\}$  be the simple transpositions generating  $\mathcal{H}$ . Let  $[k]_q = 1 + q + \dots + q^{k-1}$  be the  $q$ -number of  $k$ . Let  $e$  be the smallest positive integer such that  $[e]_q = 0$ , and set  $e = \infty$  if no such integer exists. Either  $q = 1$  and  $e$  is the characteristic of  $k$ , or  $q \neq 1$  and  $q$  is a primitive  $e$ th root of unity.

Let  $V := S^{(2, \dots, 2)}$  be the Specht module corresponding to the young diagram with rows of length 2. The purpose of this writing is to characterize this representation via an isomorphism with another representation of  $\mathcal{H}$ .

**Definition 1.1.** A *crossingless matching* on  $2n$  indices is a partition of  $\{1, \dots, 2n\}$  into parts of size 2 such that no two parts “cross”, i.e. there are no parts  $(a, a')$  and  $(b, b')$  such that  $a < b < a' < b'$ . Then, define  $W_{2n}$  to be the  $\mathbb{C}$ -vector space with basis the set of crossingless matchings on  $2n$  indices.

In order for this to be a  $\mathcal{H}$ -module, endow this with the action given by Figure 1; if this creates a loop, simply scale by  $(1 + q)$ , and otherwise deform into a crossingless matching and scale by  $q^{1/2}$ .

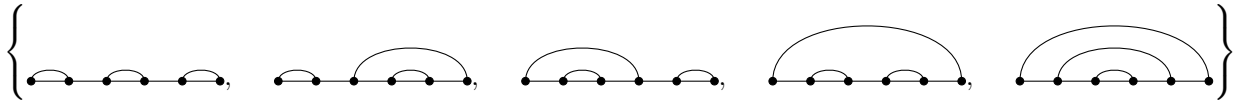
Let the length of an arc  $(i, j)$  be  $l(i, j) := j - i + 1$ . Note that the crossingless matchings can all be identified with a list of  $n$  integers describing the lengths of the arcs from left to right; using this, we may order the crossingless matchings in increasing lexicographical order in order to obtain an order on the basis. Let  $C_n$  be the  $n$ th catalan number, and let the resulting basis for  $W_{2n}$  be  $\{w_1, \dots, w_{C_n}\}$  as illustrated in Figure 2.

We will prove that  $W := W_{2n}$  is isomorphic to  $V$  as representations in the case that  $\mathcal{H}$  is semisimple. To do so, we will prove that  $W$  has an irreducible restriction to  $S_{2n-1} \subset S_{2n}$ ; using the branching theorem, this implies that  $W$  is isomorphic to a Specht module corresponding to a rectangular young diagram.

We will move on to prove that these modules have unique dimension up to transposition of the diagram; then, we will show that  $\dim V = \dim W$  so that  $W$  corresponds to an  $n \times 2$  or  $2 \times n$  diagram. We will then do a short character computation to prove that  $V \cong W$ .



**Figure 1.** Illustration of the actions  $(1 + T_4)w_3$  and  $(1 + T_2)w_3$  in  $W_6$ . In general, we act on basis elements by simple transpositions by deleting loops, deforming into a crossingless matching, and scaling based on whether a loop was deleted.



**Figure 2.** The increasing lexicographical basis for  $W_6$ .

## 2. IRREDUCIBILITY

We can now begin by proving that  $\text{Res}_{\mathcal{H}_{k,p}(S_{2n-1})}^{\mathcal{H}_{k,p}(S_{2n})} W$  is irreducible; in particular, this implies that  $W$  itself is irreducible, as an  $\mathcal{H}$ -subrepresentation is a  $\mathcal{H}_{k,p}(S_{2n-1})$ -subrepresentation.

**Proposition 2.1.** *Set  $\mathcal{H}' := \mathcal{H}_{k,p}(S_{2n-2}) \subset \mathcal{H}$ . Then,  $\text{Res}_{\mathcal{H}'}^{\mathcal{H}} W$  is irreducible if  $e \geq n + 1$ .*

*Proof.* We will prove the equivalent condition that each vector in  $w \in W$  is *cyclic*, i.e.  $\mathcal{H}'w = W$ .

We will first prove that  $w_1$  is cyclic, for which it is sufficient to prove that every basis vector of  $W$  is in  $Aw_1$ . fix some basis vector  $w_k$ , and suppose that it contains arc  $(1, j)$ , Then, the vector

$$w' := (1 + T_2)(1 + T_4) \dots (1 + T_{j-2})w_1$$

contains an arc  $(1, j)$  and all other arcs are of the form  $(a, a + 1)$  for some  $a$ . We may separately act on the subset of arcs with  $1 < a < j$  and with  $a > j$ ; this process gave our base case of  $W_4$ , and allows us to recurse to  $W_{2m}$  with  $m < n$ , outlining explicit vectors  $h \in \mathcal{H}'$  with  $hw_1 = w_i$ . Hence it is sufficient to prove that  $w_1$  is generated by every  $w \neq 0 \in W$ .

Since the image of  $(1 + T_i)$  has arc  $(i, i + 1)$ , it is isomorphic as a vector space to  $W_{2n}$  with  $2n - 2$  vertices.<sup>1</sup> Further, all actions of  $(1 + T'_j) \in \mathcal{H}_{k,p}(S_{2n-4})$  act identically (through the isomorphism) with simple transpositions other than  $1 + T'_i$ , which acts equivalently to  $q^{-1}(1 + T_i)(1 + T_{i+1})(1 + T_{i-1})$  as illustrated in Figure 3; hence, if the image of  $(1 + T_i)w$  in  $W_{2n-2}$  generates an ideal containing  $w'_1$ .

Then  $(1 + T_i)w$  generates an ideal containing the pre-image of  $w'_1$ , which may either be  $w_1$  if  $i$  is odd or the basis vector containing  $(i - 1, i + 2)$ ,  $(i, i + 1)$ , and all other arcs length 2 if  $i$  is even.

We are now ready to make the central claim in our proof:

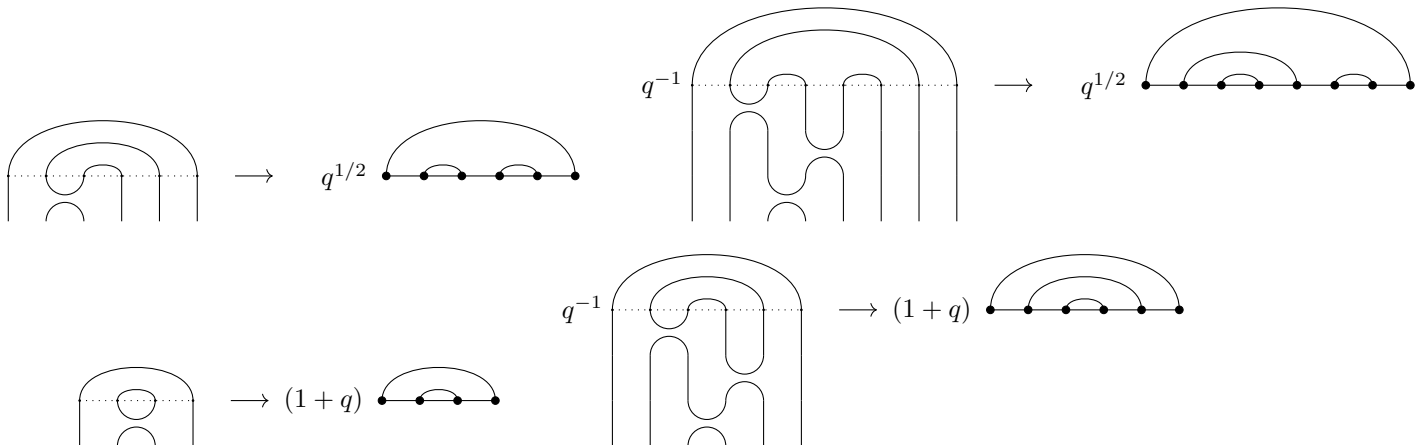
**Claim.** *Suppose  $n > 1$ . Then, the intersection  $K := \bigcap_{i=1}^{2n-2} \ker(1 + T_i)$  is trivial if  $e$  does not divide  $n + 1$ .*

This claim is necessary for irreducibility, as  $K$  is a proper subrepresentation of  $W$ .

Suppose this claim is true. We will use induction on  $n$  to prove irreducibility; the base case  $n = 1$  is clear, so suppose  $W_{2n-2}$  is irreducible for all  $e \geq n$  or, and pick some  $w \in W_{2n}$ . Then, pick some  $1 + T_i$  such that  $(1 + T_i)w \neq 0$ , and pick some action  $h' \in W_{2n-2}$  which takes the image of  $(1 + T_i)w$  in  $W_{2n-2}$  to  $w'_1$ ; this pulls back to an action  $h \in W_{2n}$  such that  $h(1 + T_i)w$  is the preimage of  $w'_1$ .

If  $i$  is odd then we have  $h(1 + T_i)w = w_1$  and we are done. Otherwise,  $h(1 + T_i)w$  is a basis vector with the arc  $(i - 1, i + 2)$  and all other arcs having length 2; hence  $q^{-1/2}(1 + T_{i-1})h(1 + T_i)w = w_1$  and we are done.

<sup>1</sup>This isomorphism “ignores” the arc  $(i, i + 1)$ .



**Figure 3.** The correspondence between the action of  $(1 + T_2)$  on  $w'_5 \in W_6$  and the action of  $q^{-1}(1 + T_3)(1 + T_4)(1 + T_2)$  on the corresponding vector in  $W_8$  having arc  $(3, 4)$  first, then on  $w'_2 \in W_4$ . This demonstrates that the action works with and without creating a loop.

*Proof of claim.* Note that  $\bigcap \ker(1 + T_i) = \ker(\bigoplus (1 + T_i))$ . Further, we may postcompose this with the isomorphism  $\text{im}(1 + T_i) \simeq W_{2n-2}$  in order to arrive at a  $(n-2)C_{n-1} \times C_n$  matrix  $A$  whose kernel is  $K$ . Further, we may apply elementary row operations to this matrix, postcomposing with a change of basis, and preserve the kernel of this matrix as well.

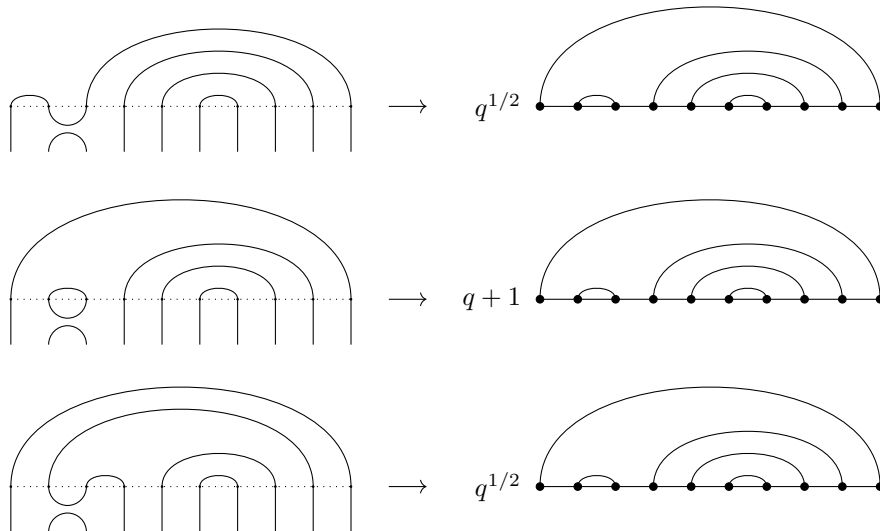
We may characterize these rows as follows: the row corresponding to  $(1 + T_i)$  and mapping into the element  $w_k \in W$  is of the form  $[a_1, \dots, a_{C_n}]$  where  $a_k = 1 + q$ ,  $a_j = q^{1/2}$  whenever  $(1 + T_i)w_j = q^{1/2}w_k$ , and 0 otherwise. In particular, we may characterize some particular rows with few nonzero entries.

First, the row of  $A$  corresponding to  $(1 + T_n)$  and  $w_{C_n}$  will be of the form  $[0, \dots, 0, q^{1/2}, q+1]$ . Similarly, row corresponding to  $(1 + T_{n-1})$  and  $w_{C_{n-1}}$  will be of the form  $[0, \dots, 0, q^{1/2}, 0, q+1, q^{1/2}]$ . An inductive process is illustrated in Figure 4. in order to construct a row corresponding to each  $(1 + T_{n-i})$  with nonzero components appearing in the order  $q^{1/2}, q+1, q^{1/2}$  and having the last two of these appear above the first two in the row corresponding to  $(1 + T_{n-i-1})$ . This process works for  $1 \leq i \leq n-1$ , and two-nonzero-element rows, which are in column-alignment with the appropriate elements in  $(1 + T_2)$  and  $(1 + T_{n-1})$ .

This construction yields a  $n \times C_n$  submatrix of  $A$  which has nonzero entries on the rightmost column, and has (by removing zero columns) the same column space as the following:

$$\begin{bmatrix} q+1 & q^{1/2} & & & & & \\ q^{1/2} & q+1 & q^{1/2} & & & & \\ & q^{1/2} & q+1 & q^{1/2} & & & \\ & & \ddots & \ddots & & & \\ & 0 & & & q^{1/2} & q+1 & q^{1/2} \\ & & & & q^{1/2} & q+1 & \end{bmatrix}$$

We may recursively compute that this matrix has determinant  $[n+1]_q$ , which vanishes exactly when  $e \mid n+1$ ; henceforth assume that this does not vanish. Then, this matrix is invertible, so we may use elementary row operations to transform it into the identity; these correspond to elementary row operations on  $A$  which yield a row  $[0, \dots, 0, 1]$ . Hence any vector  $w \in K$  has a zero coefficient corresponding to  $w_{C_n}$ . We will now prove that every nonzero vector  $w \in K$  must have a nonzero coefficient corresponding to  $w_{C_n}$ , giving that  $K = 0$ .  $\square$



**Figure 4.** Illustrated is the row constructed for transposition  $(1 + T_2)$ ; clearly these are the only basis elements mapping to multiples of the desired element, and they relate to each other. In general, the basis elements considered are the preimages of  $w_{C_{n-1}}$  in the isomorphism  $\text{im}(1 + T_i) \simeq W_{2n-2}$ .

## 3. CORRESPONDENCE WITH THE SPECHT MODULE

Suppose  $\mathcal{H}$  is semisimple; then  $W$  is a Specht module corresponding to a rectangular region. We may now use dimension, for which one should note that there is an easy bijection between standard tableaux on  $n = 2 + \cdots + 2$  and the Dyck paths on  $2n$  points by specifying that the number in the  $(i, 1)$  box of the tableau is the index of the  $i$ th “up” path. Hence  $V_{(2, \dots, 2)}$  has dimension the  $n$ th catalan number  $C_n$ .

Similarly, we may biject the Dyck paths on  $2n$  points with the crossingless matchings, by making the value at point  $i$  the number of “open crossings” at that point, i.e. the number of arcs  $(a, b)$  with  $a \leq i < b$ . Hence  $V$  and  $W$  have the same dimension. This pins the shape of the diagram corresponding to  $W$  as follows.

**Proposition 3.1.** *Let  $V_1$  and  $V_2$  be two specht modules corresponding to  $a_1 \times b_1$  and  $a_2 \times b_2$  rectangular young diagrams. Then,  $\dim V_1 = \dim V_2$  if and only if the diagram of  $V_1$  is the same as or a transposition of the diagram of  $V_2$ .*

*Proof.* Recall that, if they have the same diagram then  $V_1 \cong V_2$ , and if they’re transposed from each other,  $V_1 \cong V_2 \otimes U$  where  $U$  is the alternating representation; hence  $\dim V_1 = \dim V_2 \cdot \dim U = \dim V_2$ .

Suppose WLOG that  $a_1 < a_2 < b_2 < b_1$ . By the hook-length formula, it is sufficient to give a bijective correspondence between the boxes in  $V_2$  and  $V_1$  such that the hook-length in  $V_2$  is larger than the hook-length in  $V_1$ , and at least once strictly larger. We can give this correspondence by listing the boxes beginning at the bottom right corner, then increasing up left-to-right diagonals, and note that this satisfies our conditions.  $\square$

Let  $V' := V_{(n, n)}$ . Then,  $W$  is isomorphic to exactly one of  $V$  and  $V'$ . It is hence sufficient to prove that  $W$  is not isomorphic to  $V'$ , and we may do this via a character computation.

## 4. SUMMARY OF RESULTS

What follows is a color-coded list of propositions on crossingless matchings. Those which are blue are **confirmed**, those which are olive are **in progress/soon to be proven**, and those which are red are **conjectural**.

**Proposition 4.1.** *Suppose  $n > 1$ . Then, the intersection  $K := \bigcap_{i=1}^{2n-2} \ker(1 + T_i)$  is trivial if  $e$  does not divide  $n + 1$ .*

**Proposition 4.2.** *Define the subalgebra  $\mathcal{H}' := \mathcal{H}_{k,p}(S_{2n-2}) \subset \mathcal{H}$ . Then,  $\text{Res}_{\mathcal{H}'}^{\mathcal{H}} W$  is irreducible if  $e \geq n + 1$ .*

**Proposition 4.3.** *Suppose  $\mathcal{H}$  is semisimple, i.e.  $e > 2n$ . Then,  $V \simeq W$ .*

**Proposition 4.4.** *The intersection  $K := \bigcap_{i=1}^{2n-2} \ker(1 + T_i)$  is nontrivial when  $e = n + 1$ .*

**Proposition 4.5.**  *$W$  is reducible when  $e \leq n + 1$ .*

**Proposition 4.6.** *Even if  $e \leq 2n$ , we have  $V \simeq W$ .*