THERE IS NO FIBONACCI REPRESENTATION OF $\mathscr{H}_{k,q}(S_n)$

MILES JOHNSON & NATALIE STEWART

Let k be a field, and let $q \neq 0 \in k$ be the parameter of the Hecke algebra $\mathscr{H} := \mathscr{H}_{k,q}(S_n)$. We will show that there is no "Fibonacci" representation of $\mathscr{H}$.

First, let F be the vector space with basis given by the strings of length $n+1$ with alphabet $\{p, *\}$ such that no two $*$ symbols appear in a row. This is given an action by the braid group that we will try to emulate. We want an action which is "local", i.e. the simple transposition T_i acts on the string from the ith to the $(i+2)$nd symbol, modifying only the middle character, defined by the following rule:

$$\begin{aligned}
\widehat{(*pp)} &:= a(*pp) \\
\widehat{(*p*)} &:= b(*p*) \\
\widehat{(p*p)} &:= c(p*p) + d(ppp) \\
\widehat{(pp*)} &:= a(pp*) \\
\widehat{(ppp)} &:= d(p*p) + e(ppp).
\end{aligned} \tag{1}$$

For suitable constants $a, b, c, d, e \in k$.

Quadratic Relation. Note that the quadratic relation $T_i^2 = (q-1)T_i + q$ imposes the following restrictions on the constants:

$$\begin{aligned}
a^2 &= (q-1)a + q \\
b^2 &= (q-1)b + q \\
c^2 + d^2 &= (q-1)c + q \\
de &= (q-1)d \\
e^2 + d^2 &= (q-1)e + q \\
dc &= (q-1)d
\end{aligned} \tag{2}$$

Note that we immediately have

$$\begin{aligned}
a, b &\in \left\{ \frac{q-1 \pm \sqrt{(q-1)^2 + 4q}}{2} \right\} \\
&= \left\{ \frac{q-1 \pm (q+1)}{2} \right\} \\
&= \{-1, q\}.
\end{aligned} \tag{3}$$

Further, if $d = 0$ then we have $c, e \in \{-1, q\}$; if $d \neq 0$ then we have that $c = e = (q-1)$, and $d \in \left\{\pm\sqrt{q}\right\}$.

Braid Relations. Here's where we'll run into some issues.

We must verify the relation $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$. We can try to directly confirm this on length-4 strings:

- $(*ppp)$ requires that $adb = bcd + ade$ and $a^2 d = ae^2 + bd^2$. If $d = 0$, then we require that $ae^2 = 0$ with $a, e \neq 0$, a contradiction. Hence we require that $d^2 = q$ and $c = e = (q-1)$.

  With the restrictions given above, the first of the above equivalently requires

$$ab = b(q-1) + a(q-1) \tag{4}$$

- ($pppp$) requires that $acd + de^2 = ade$. Equivalently, we require that $e^2 = 0$, i.e. $d = q = 1$. Then, by (4), we require that $ab = 0$, but $a, b \neq 0$; this is a contradiction, so there are no constants a, b, c, d, e which make this a representation of $\mathscr{H}$.