School of Engineering, Computer, and Mathematical Sciences

Engineering Mathematics

ENGE 401

2022 Semester 1

Engineering Mathematics - ENGE401
Course Manual
Auckland University Of Technology

This project is on GitHub, find it
and download the source files at:
https://github.com/millecodex/ENGE401

Version 2.7 by Jeff Nijsse, 2022.
Jeff.Nijsse@aut.ac.nz
Version 1.0 by Peter Watson, 2010.

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1 | Algebra

Engineering Mathematics begins by reviewing foundational algebra. Many of the skills used in this chapter are foundational mathematical tools that you will need to keep using repeatedly both in this course and beyond. Refer to the course website on `canvas.aut.ac.nz` for additional review material covering the basics of algebra.

1.1 Algebra Review

Some of the foundational algebraic properties will be covered here. This section is not comprehensive, and the student should refer to an introductory algebra book if some of these properties are not clear.

Order of Operations

An equation is a mathematical expression separated by two lines of *equal* length (=). There must be symbols (either numbers or algebraic letters) on both sides of the equality. For example, $5x = 25$ is an equation, however, $5(1) + 25$ is just an expression. An equation can be solved; in the previous example, $x = 5$ is a solution, whereas an expression may be simplified: $5(1) + 25 = 30$. Conversely, factoring the quadratic $x^2 - 4x + 4 = (x - 2)^2$ is *not* solving the expression.

When evaluating a mathematical expression the order of operations is important. The acronym **BEDMAS** is used for simplifying expressions starting with **B**rackets, **E**xponents, **D**ivision, **M**ultiplication, **A**ddition, and ending with **S**ubtraction. This means you must resolve any brackets before evaluating exponents, e.g. $(3 - 1)^2 = 2^2 = 4$, and handle multiplication before addition, e.g. $3 * 4 + 2 = 12 + 2 = 14$, and so on.

Solving Equations

The *reverse* is true for solving equations. First you must "undo" any subtraction or addition to isolate the variable.

EXAMPLE Solve the equation for x: $x - 11 = 7$

SOLUTION To isolate x we will add 11 to both sides: $x - 11\mathbf{+11} = 7\mathbf{+11}$

And simplify: $x = 18$

EXAMPLE Solve the equation: $2x + 5 = 10$

SOLUTION To isolate x first we have to subtract 5 from both sides: $2x = 10 - 5$

And then we divide both sides by 2: $\frac{\cancel{2}x}{\cancel{2}} = \frac{10 - 5}{2}$

And simplify: $x = \frac{5}{2}$

> A general rule for solving equations is that you can do any mathematical operation to the equation as long as you do it to both sides. For example, add 5 to both sides, divide both sides by 2, multiply both sides by $\sin(x)$, raise to an exponent, and so on.

EXAMPLE Find x: $x^2 + 1 = 3$

SOLUTION Subtract 1 from both sides: $x^2 = 2$. Recall BEDMAS in reverse order, now we have an exponent. To solve for a power of 2, take the square root of both sides: $\sqrt{x^2} = \sqrt{2}$, and simplify to $x = \sqrt{2}$.

Rearranging Equations

Rearranging equations is a handy tool in analysis to see how variables influence relationship. Sometimes this is called isolating for x or solving for some variable. The equation for a straight line is $y = mx + c$ (see page 7) with two variables x and y. The equation is arranged so that variable y is alone on the left-hand side. This makes y the subject. The equation can also be written where x is the subject. Steps to rearrange are steps where all operations on the new subject are undone in reverse order of operations.

y	$=$	$mx + c$	Operations on the *new* subject x are multiply by m first then add c
$y - x$	$=$	mx	First, we undo 'add c' by subtracting c from both sides
$\dfrac{y - c}{m}$	$=$	$\dfrac{\cancel{m}x}{\cancel{m}}$	Then, we undo 'multiply m' by dividing both sides by m
x	$=$	$\dfrac{y - c}{m}$	By convention, the subject is on the left-hand side of the equals sign

EXAMPLE Make v (velocity) the subject in the formula for kinetic energy: $E_k = \frac{1}{2}mv^2$

SOLUTION There is no addition or subtraction to handle here. First we will decouple (remove) the $\frac{1}{2}$ and the mass, m. Lastly we will take care of the exponent by using a square root.

$$\text{multiply both sides by 2:} \quad 2E_k = mv^2$$

$$\text{divide both sides by } m\text{:} \quad \frac{2E_k}{m} = v^2$$

$$\text{square root both sides to isolate } v\text{:} \quad \sqrt{\frac{2E_k}{m}} = v$$

Exponents

Indices go by a few different names, sometimes they are called powers or exponents. In the expression x^3 the index, exponent, or power is 3. This does not have to be an integer, or even a number: x^n has index n; $x^{\frac{2}{3}}$ has a fractional exponent; x^{-1} has a negative exponent; and $x^{\cos x}$ has another expression for its power.

The Rules of Exponents

- x^n: x is called the base and n the exponent (or power)
- When **multiplying** exponents of the same base, **add** the exponents:

$$x^3 \times x^4 = x^{3+4} = x^7 \qquad \text{or} \qquad a^{-2}a^3 = a^{-2+3} = a^1$$

- When **dividing** exponents of the same base, **subtract** the exponents:

$$\frac{x^6}{x^5} = x^{6-5} = x^1 \qquad \text{or} \qquad \frac{y^7}{y^{-2}} = y^{7-(-2)} = y^9$$

- If an expression is raised to *another* power, multiply the exponents:

$$x^{3^4} = x^{3\times 4} = x^{12}$$

Negative Exponents

One of the most important rules for manipulating mathematics is the exponent of -1. A negative power is equivalent to the inverse of the same expression with a positive power. (Inverse means one divided by the same expression.)

Examples:

$$x^{-1} = \frac{1}{x} \qquad x^{-4} = \frac{1}{x^4} \qquad \frac{2}{7y^3} = \frac{2y^{-3}}{7} \qquad 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

Fractional Exponents

Exponents can be decimal numbers, integers, expressions, variables, and also fractions. Fractional exponents can written using a root sign, this is called surd form. $x^{\frac{1}{2}}$ is also known as the square root of x. Using rules of exponents you can see that $x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x$. Converting between surd and index form is quite handy, especially when we get to differentiation using the power rule.

Index form:	$x^{\frac{1}{2}}$	$x^{\frac{3}{4}}$	$64^{\frac{1}{3}}$	$x^{\frac{a}{b}}$
Surd form:	$\sqrt{x}$	$\sqrt[4]{x^3}$	$\sqrt[3]{64}$	$\sqrt[b]{x^a}$

Laws of Exponents Summary

multiply: $x^a x^b = x^{a+b}$

inverse: $x^{-a} = \dfrac{1}{x^a}$

divide: $\dfrac{x^a}{x^b} = x^{a-b}$

root: $x^{\frac{a}{b}} = \sqrt[b]{x^a}$

power: $(x^a)^b = x^{ab}$

zero: $x^0 = 1$

Expanding and Factorising

Multiplying algebraic expressions is called *expanding* and the reverse process is called *factorising*. We usually use the word factor**is**ing in New Zealand, however, most textbooks use the term factoring. We will use both terms interchangeably in this course. Factorising and expanding can be viewed as opposite operations; one undoes the other.

Example Expand the following algebraic expression (remove the brackets): $x(x-7)$

Solution $x(x-7) = x^2 - 7x$

Example Expand: $(x+3)(x-3)$. Note there is a mnemonic FOIL that may help you remember how to expand here: First, Outside, Inside, Last.

Solution $= x^2 + 3x - 3x - 9 = x^2 - 9$

Example Expand: $x(x+1)(x-2)$

Solution Begin by expanding the first two terms:

$$\begin{aligned} &= (x^2 + x)(x-2) \\ &= x^3 - 2x^2 + x - 2 \end{aligned}$$

Factoring involves removing common terms from expressions and then writing them as products. Recall that product means multiply and can be shown algebraically by writing terms in brackets.

Example Factor the following algebraic expression: $2x - 4x^2 + 6x^3$

Solution Remove a common factor of $2x$: $2x - 4x^2 + 6x^3 = 2x(1 - 2x^2 + 3x^3)$

Example Factorise: $x^2 - 5x - 6$. Note that this is a quadratic equation and will factor into two sets of brackets.

Solution

For these examples you are required to find a pair of numbers that add together to give -5 and multiply together to give -6. In this case the numbers are -6 and $+1$. So the answer is

$$x^2 - 5x - 6 = (x-6)(x+1)$$

This can easily be verified by expanding the brackets.

EXAMPLE Factor: $x^2 - 4x + 4$

SOLUTION $= (x-2)(x-2) = (x-2)^2$

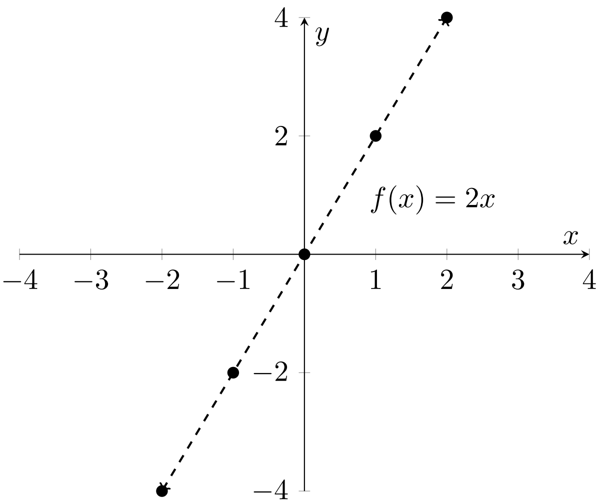
1.2 Functions

A function is a mathematical relationship between groups. Given an element in one group, the function says how to get to the other group. For example the function could be a formula that says if you have x, the output is $2x$. This can be written as $f(x) = 2x$ where $f(x)$ is called function notation and in Cartesian coordinates also means $y = f(x)$. We can depict this function visually using x and y coordinates. Begin by selecting some inputs (x values) and then calculate the outputs ($f(x)$ values) from the formula $f(x) = 2x$.

This method is called making a table of values and in this example any real number for x produces exactly one output for y (also a real number). The figure below plots the points on an (x, y) grid. Connecting the points creates the line $y = 2x$.

inputs x	outputs $f(x) = 2x$	
-2	$f(-2) = 2(-2)$	$= -4$
-1	$f(-1) = 2(-1)$	$= -2$
0	$f(0) = 2(0)$	$= 0$
1	$f(1) = 2(1)$	$= 2$
2	$f(2) = 2(2)$	$= 4$

A **table of values** for the function $f(x) = 2x$



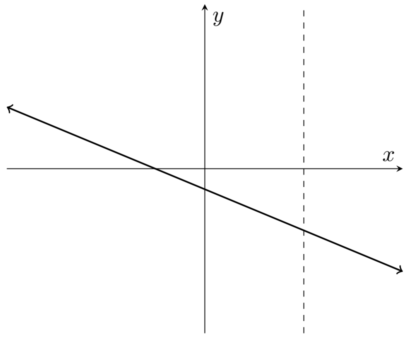
A **plot of the points** showing a linear relationship

Lets write a precise definition of a function:

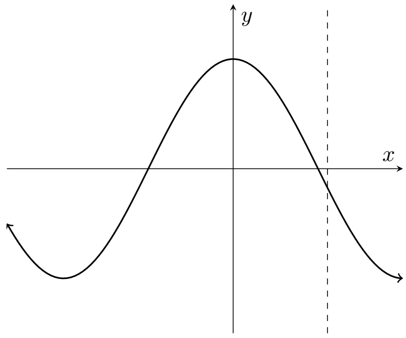
> A function $f(x)$ has exactly one output value, y, for any given input value, x.

In the line plotted above, we see that every x value has only one corresponding y value. This means that $y = 2x$ is a function. Conversely, a relationship that has two or more output values is **not** a function. In the figure below the circle has two output values at $x = 1$; a and b, and this violates the definition of a function.

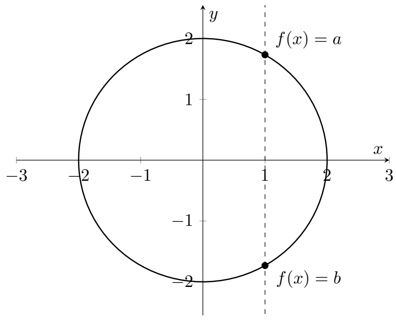
The dashed line in the figures represents what is called the *vertical line test*. If a vertical line passes through more than one point on a curve, then it is not considered a function. The linear and cosine functions both pass the test and are considered *functions*.



(a) linear function



(b) cosine function



(c) circle

Domain & Range

The domain of a function is the set of all inputs that are valid; usually these are the $x-$values. The range of a function is the set of all outputs that are valid; usually these are the $y-$values. For the line we plotted above, $f(x) = 2x$, any value could be substituted into the function, therefore the domain was all the real numbers. This is written as: Domain $x \in \mathbb{R}$. Similarly the range was all the $y-$values, or $y \in \mathbb{R}$.

EXAMPLE What is the domain and range for the circle with equation $x^2 + y^2 = 2^2$?

SOLUTION See the graph for the circle (c) above. The domain is all the x-values that are included; this is everything between -2 and $+2$ inclusive.

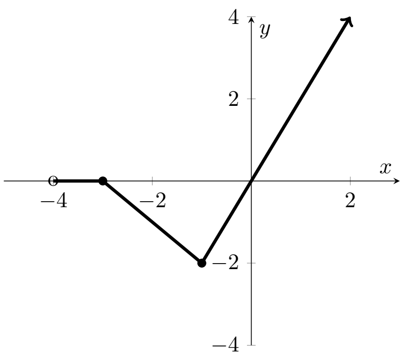
Therefore the domain is:

$$\{-2 \leq x \leq 2,\ x \in \mathbb{R}\}$$

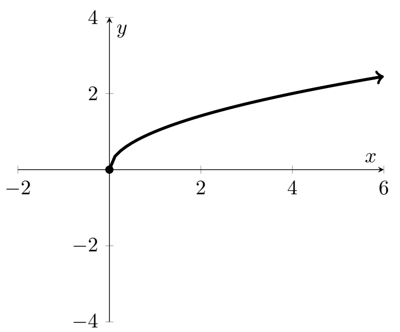
And the range is similar:

$$\{-2 \leq y \leq 2,\ y \in \mathbb{R}\}$$

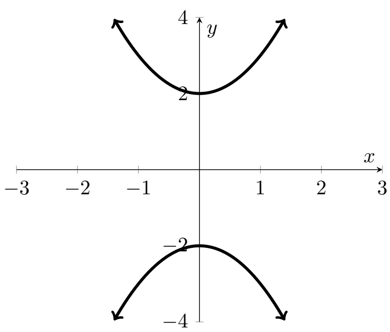
EXAMPLE Find the domain and range for the graphs:



(a) Piece-wise function



(b) Square root function



(c) Parabolas

SOLUTION

(a) $x = -4$ is **not** included in the domain, but everything bigger than -4 is.
Domain:
$\{x > -4,\ x \in \mathbb{R}\}$
Range: $\{y \geq -2,\ y \in \mathbb{R}\}$

(b) The arm of a root function goes to infinity (one-half of a parabola) in both positive x and y.
Domain: $\{x \geq 0,\ x \in \mathbb{R}\}$
Range: $\{y \geq 0,\ y \in \mathbb{R}\}$
or:
D: $\{0 \leq x \leq \infty, x \in \mathbb{R}\}$
R: $\{0 \leq y \leq \infty, y \in \mathbb{R}\}$

(c) Here there is a gap between $y = -2$ and $y = 2$. This must be excluded from the range.
Domain: $\{x \in \mathbb{R}\}$
Range:
$\{y \geq 2,\ \text{and } y \leq -2,\ y \in \mathbb{R}\}$, or:
Range: $(\infty, -2]$ and $[2, \infty)$

We have shown some different notation in the last example. The round brackets are exclusive (∞, ∞) and the square brackets are inclusive $[-2, 2]$. Infinity is not a number and does not behave like one, so it always gets a round bracket; you can never reach infinity to include it.

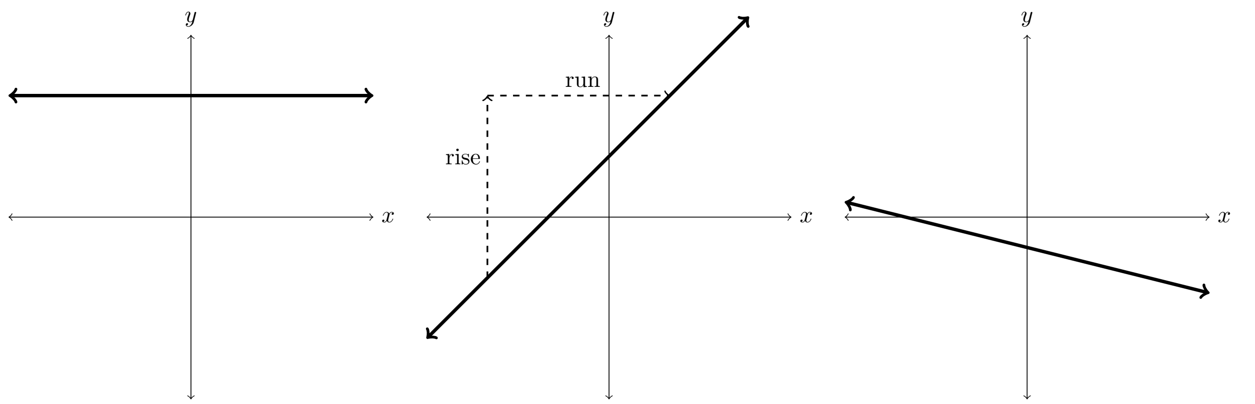
Number Classification

In the preceding examples for domain and range we said the set included "all the real numbers", denoted by the symbol $\mathbb{R}$. This includes all the rational and irrational numbers. Rational numbers are those can be expressed as a *ratio*, or a fraction: $\frac{a}{b}$. The irrational numbers are those that cannot be expressed as a fraction, such as $\pi = 3.14159265\ldots$ because the decimal numbers never repeat. There are many other categories of numbers, for example:

real	$\mathbb{R}$	includes all rational and irrational numbers; the number line
natural	$\mathbb{N}$	the "counting" numbers: starting at 1,2,3,...
integers	$\mathbb{Z}$	the plus/neg natural number and zero: $\ldots, -2, -1, 0, 1, 2, \ldots$
rational	$\mathbb{Q}$	can be written as a fraction: $\frac{a}{b}, b \neq 0$
complex	$\mathbb{C}$	numbers in the complex plane including imaginary numbers: $\sqrt{-1} = i$

Linear Functions

Linear functions can represented nicely as a straight line on a standard Cartesian (x, y) coordinate system. The following are all examples of linear functions, and not surprisingly, can be drawn as a lines.



(a) zero slope (b) positive slope (c) negative slope

Linear functions have a few characteristics that we will get used to manipulating. The slope of a line is often represented by the letter m and can be calculated by taking any two points on the line (x_1, y_1), and (x_2, y_2) and using the formula: $m = \frac{y_2-y_1}{x_2-x_1}$. This is also known as the *gradient*, a term that will be used often in calculus.

$$\text{slope} = \text{m} = \text{gradient} = \frac{rise}{run} = \frac{\Delta y}{\Delta x} = \frac{y_2-y_1}{x_2-x_1}$$

The standard form for an equation of a line is: $y = mx + c$ where m is the slope described above, and c is the $y-$intercept. Alternatively if you know the slope and any given point (x_1, y_1), the equation of a line is $y - y_1 = m(x - x_1)$ where x_1 and y_1 are the coordinates of a point on the line.

Note that a vertical line has an *undefined* slope. Using the formula above, a vertical line has a slope of $m = \frac{\Delta y}{0}$ because it has the same x values everywhere. Dividing by zero is *undefined* (try on your calculator) and therefore a vertical line is not considered a function.

EXERCISE Does a vertical line pass or fail the vertical line test? Why?

Quadratic Functions

A quadratic relationship scales with the square of the input values. The following are all examples of quadratic functions:

(a) $y = x^2 - 5$

(b) $3x^2 + x - 1 = 0$

(c) $s(t) = -4.9t^2 - 15t + 3$

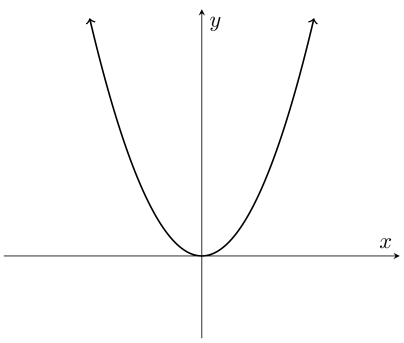
(d) $b^2 + 7b - 1 = 0$

(e) $ax^2 + bx + c = 0$

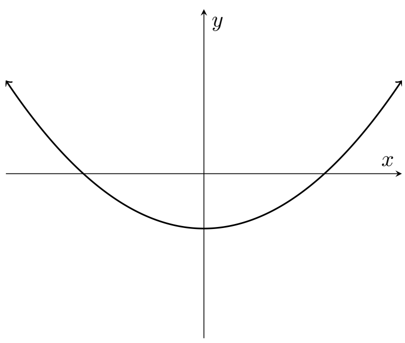
(f) $27 = 3x^2$

Note that they all have a power of 2 in the equation, and that is the highest exponent. This type of relationship is also referred to as parabolic. When a parabola equation is plotted, the solutions represent where the function crosses the $x-$axis. These points are called roots.

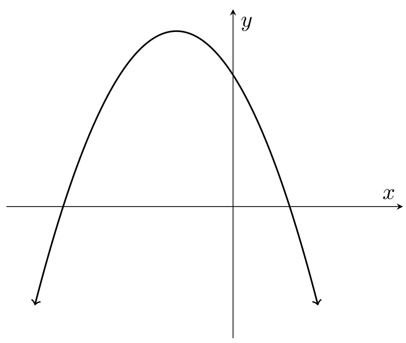
EXAMPLE Find the roots of the following parabolas.



(a) $y = x^2$



(b) $y = 0.3x^2 - 1$



(c) $f(x) = -x^2 - 2 * x + 3$

SOLUTION The roots are where the function intersects the $x-$axis. The $x-$axis is where $y = 0$, so we will substitute $y = 0$ into the functions and solve the equations for x.

(a) Substituting in $y = 0$ gives the equation $0 = x^2$. This solves directly for $x = 0$. Therefore the root to $y = x^2$ is 0.

(b) Solve the equation

$$\begin{aligned} 0 &= 0.3x^2 - 1 \\ 1 &= 0.3x^2 \\ \frac{1}{0.3} &= x^2 \\ \sqrt{\frac{1}{0.3}} &= x \end{aligned}$$

Therefore $x = \pm 1.826$

(c)

$$\begin{aligned} 0 &= -x^2 - 2x + 3 \quad \text{here you can divide by } -1 \\ 0 &= x^2 + 2x - 3 \quad \text{and factor} \\ 0 &= (x+3)(x-1) \end{aligned}$$

Therefore $x = -3$ and $x = 1$

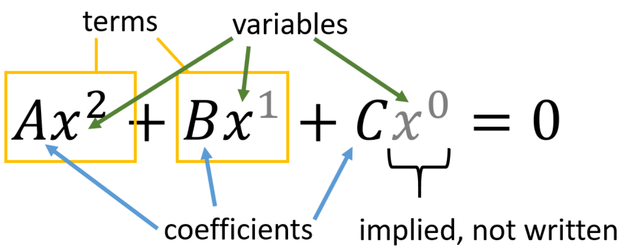
The parabola from part (c) above was solved by factoring. Not all quadratic equations can be solved in this manner. The standard form of a quadratic equation is written $ax^2 + bx + c = 0$. If we solve this equation for x we get the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where a, b, and c are coefficients $(a \neq 0)$. Note here there are two possible solutions because of the plus-minus sign $(\pm)$.

1.3 Polynomials

A polynomial is a type of function that comes up a lot. The quadratic equations above are all examples of polynomials. The standard form of a quadratic equation is shown below with some of the terminology.



The prefix *poly* means many, and polynomials are not limited to three terms. The general form of a polynomial can be written as:

$$A_1x^n + A_2x^{n-1} + A_3x^{n-2} + \cdots + A_nx^1 + C = 0$$

where the terms are written in decreasing powers of x, with

- $n \geq 0, n \in \mathbb{Z}$. This means the exponents must be integers.
- $A_1, \ldots, A_n$ are real numbers.

- C is a constant.
- The order or degree of the polynomial is n (the highest exponent).
- Here, x is the variable. You may have more than one variable in a polynomial, for example $4x^2 + y - xy + 4$ is a valid polynomial.

1.4 Systems of Equations

A system of equations means having more than one relationship represented within a common context. Revenue and costs may have different functions but both relate to the same product. We will study systems composed of two equations and two unknowns. Systems with more equations and more variables are possible and will be covered in future courses. Three approaches to solve systems of equations will be covered here:

- graphing
- elimination
- substitution

It helps if you can visualise the shape of the two functions so that the meaning of the solution is clear in your mind. Later in the course we will find the area between two curves using integration where the intersection of these two curves represents the solution to a system of two equations.

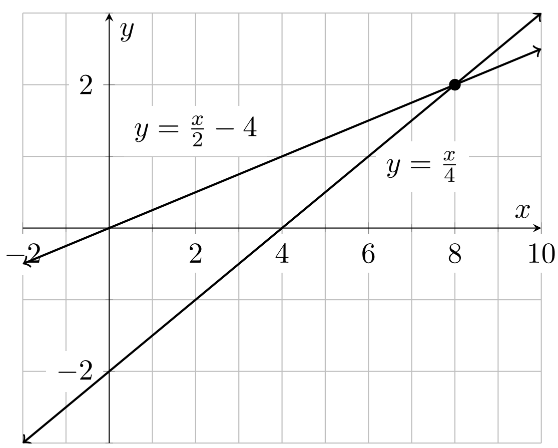
We know that linear equations in two variables are represented by straight lines. Straight lines will always intersect unless they are parallel. The coordinates of the point of intersection of the straight lines is called the solution. you could use a graphical method or one of the two algebraic methods (substitution or elimination) to find the solution. We will start with a graphical method.

Solution by Graphing

Example Find the solution to the set of linear relationships given by:

$$2y = x - 4$$
$$y = \frac{x}{4}$$

Solution Here we have two linear equations (we know they are linear because the highest exponent is 1) and if the lines intersect, that point of intersection represents a solution.



From the plot we can see the point of intersection is $(8, 2)$. Therefore the solution to the system of linear equations is $(8, 2)$. Try plotting the lines yourself with desmos.

Its not always convenient to graph a system of equations to find the solution; you may not have access to a computer, or the solution may not be integers. Solving by direct substitution is the next method.

Solution by Substitution

EXAMPLE Consider the system of equations

$$x^2 + y^2 = 25 \tag{1}$$
$$3y + x = 15 \tag{2}$$

SOLUTION Without knowing what the functions look like or plotting them, we can isolate a variable and substitute it into the other equation. Rearrange equation (2) to isolate x and substitute into equation (1):

$$x = 15 - 3y$$

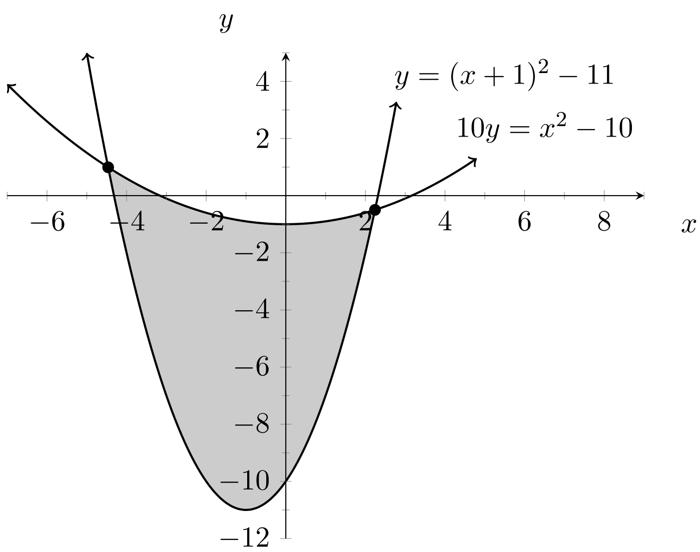
Equation (1) becomes

$$\begin{aligned}
(15-3y)^2 + y^2 &= 25 && \text{(expand and simplify)}\\
225 - 90y + 9y^2 + y^2 &= 25 \\
10y^2 - 90y + 200 &= 0 && \text{(divide by 10)}\\
y^2 - 9y + 20 &= 0 && \text{(factor the quadratic)}\\
(y-5)(y-4) &= 0 \\
\text{Either} \quad y - 5 = 0 \quad &\text{so} \quad y = 5 \\
\text{or} \quad y - 4 = 0 \quad &\text{so} \quad y = 4
\end{aligned}$$

Lastly, back-substitute the $y-$values into equation (2):

When $y = 5$, $3y + x = 15 \rightsquigarrow 15 + x = 15 \iff x = 0$. This means $(0, 5)$ is a solution. Similarly, when $y = 4$, $x = 3$ and so $(3, 4)$ is the second solution. A system with a quadratic equation (degree 2) will have up to 2 solutions.

EXAM QUESTION Find the points of intersection of the two parabolas shown on the plot.



SOLUTION Divide the 2nd equation by 10 to isolate y. Now, set the equations equal and use the quadratic formula to solve:

$$
\begin{aligned}
(x+1)^2 - 11 &= -\frac{x^2}{10} - 1 && \text{(set eq1=eq2)} \\
x^2 + 2x - 10 &= -\frac{1}{10}x^2 - 1 && \text{(expand the brackets)} \\
11x^2 + 20x - 90 &= 0 && \text{(collect like terms)} \\
x &= \frac{-20 \pm \sqrt{400 - 4(11)(-90)}}{22} \\
x &= -4.463 \text{ or } 2.241
\end{aligned}
$$

Substitute both x values back into either equation to find the y values

$$y = 0.992 \text{ or } -0.498$$

Therefore the points of intersection are $(-4.463, 0.992)$, and $(2.241, -0.498)$.

Solution by Elimination

The third method is called elimination and works by eliminating one of the variables from the equation set, then solving for the other variable.

EXAMPLE Solve the system by the method of elimination:

$$
\begin{aligned}
4x - 3y &= 5 && (1) \\
4x + y &= 1 && (2)
\end{aligned}
$$

SOLUTION If we subtract equation (2) from equation (1) columnwise then the x term is eliminated because $4x - 4x = 0$.

$$
\begin{array}{rrl}
 & 4x - 3y & = 5 \\
- & 4x + y & = 1 \\
\hline
 & 0x - 4y & = 4
\end{array}
$$

Now there is one equation with one unknown: $-4y = 4$ so $y = -1$. Back-substitute into either previous equation to solve for $x = \frac{1}{2}$. Therefore the solution is $x = \frac{1}{2}, y = -1$.

EXAMPLE Solve the system by the method of elimination:

$$
\begin{aligned}
3a - 7b &= -3 && (1) \\
b &= \frac{6a - 4}{2} && (2)
\end{aligned}
$$

SOLUTION The first step is to write the equations so the variables line up in columns. Multiply equation (2) by 2 to get $2b = 6a - 4$ and rearrange the terms to match equation (1).

The system now looks like:

$$
\begin{aligned}
3a - 7b &= -3 && (1) \\
6a - 2b &= 4 && (3)
\end{aligned}
$$

We can't eliminate any variables because the coefficients are different.

Multiply equation (1) by 2.

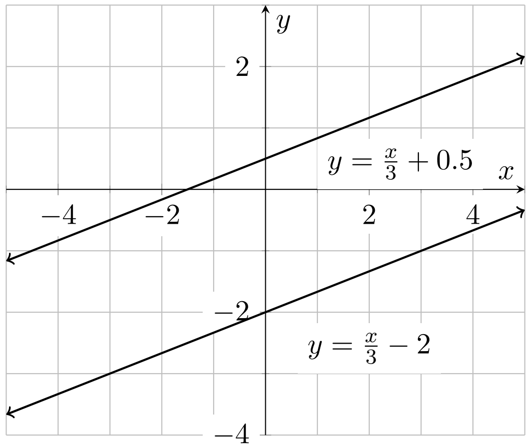
$$
\begin{aligned}
6a - 14b &= -6 && (4) \\
6a - 2b &= 4 && (3)
\end{aligned}
$$

Now calculate (4) $-$ (3).

$$
\begin{array}{rrl}
 & 6a - 14b & = -6 \\
- & 6a - 2b & = 4 \\
\hline
 & 0 + 12b & = 10 \\
\hline
 & b & = \frac{5}{6}
\end{array}
$$

Back-substitute b into equation (1) to solve for a: $3a - 7(\frac{5}{6}) = -3$. Verify that $a = \frac{17}{18}$.

Special Cases



Consider the parallel lines shown. What is the solution to this system? Parallel lines will never meet by definition and so will have no point of intersection. In this case the system has no solution – which is a perfectly valid solution! Notice the slope of both lines is $\frac{1}{3}$ which means they are parallel.

Sometimes the two equations will be two different representations of the same line. Imagine two line plotted on top of one another. In this case the system has an infinite number of solutions because every single point on the first function is also on the second function.

$\boxed{\text{Example}}$ If you start with a linear equation such as $y - 2x = 3$ and multiply each term by a constant you will get an equivalent equation. If you multiply the equation by another number you will get a further equivalent equation.

$$y - 2x = \quad 3$$

Multiply by 2

$$2y - 4x = \quad 6$$

Multiply by -3

$$-3y + 6x = -9$$

$\boxed{\text{Solution}}$ We know these are both just different ways of writing the original equation $y - 2x = 3$ or $y = 2x + 3$. To say the system has an infinite number of solutions we are really saying every point on $y = 2x + 3$ is a solution.

Guidelines for Solving Systems of Equations

These guideline provide a useful way to tackle any problems where equations are involved.

1. Identify the variables. We often call them x and y, but you may chose any name or letter you want.
2. Express all unknown quantities in terms of the variables.
3. Set up a system of equations using the facts provided by the problem.
4. Solve the system of equations and use the solution to check it satisfies the conditions of the problem. Write a sentence describing the answer to the original problem.

EXAMPLE It takes a boat travelling downstream 1 hour to cover the 20 mile distance. On the return trip it take the boat 2.5 hours. What is the speed of the boat and the speed of the current?

SOLUTION This is about a boat travelling with the current and against the current and depends on you knowing that velocities are vectors that can be added and subtracted. Let the speed of the boat be x mi/h and the speed of the current be y mi/h.

$$\begin{aligned}\text{Upstream speed} &= x - y \\ \text{Downstream speed} &= x + y\end{aligned}$$

$$\begin{aligned}\text{Speed} &= \frac{\text{Total distance}}{\text{Total time}} \\ \text{so Total distance} &= \text{Speed} \times \text{Total time} \\ 20 \text{ miles } &= (x+y) \times 1 \text{ hour} \\ 20 &= x + y \qquad (1) \\ \text{Also} \quad 20 \text{ miles } &= (x-y) \times \frac{5}{2} \text{ hours} \\ 8 &= x - y \qquad (2)\end{aligned}$$

Now we can add equations (1) and (2) and y is eliminated

$$\begin{aligned}28 &= 2x \\ x &= 14\end{aligned}$$

Back-substitute into either equation (1) or (2) to solve for $y = 6$.

Check: The boat travels at 14 mi/h and the current travels at 6 mi/h so the effective speed of the boat is 20 mi/h. At 20 mi/h the 20 mi trip took 1 h. Upstream the speed is 8 mi/h.

$$\begin{aligned}\text{Total time} &= \frac{\text{Total distance}}{\text{Speed}} \\ &= \frac{20}{8} = \frac{5}{2} \text{ h}\end{aligned}$$

Therefore the speed of the boat is 14 mi/h and the speed of the current is 6 mi/h.

1.5 Chapter Exercises

§1.1 Algebra Review

1. Remove the brackets and simplify

(a) $-(x+y)$ (b) $-3(5x-2y)$
(c) $\left(x^2+5x-1\right)-(2x-3)$ (d) $(2x-1)(2x+1)$

2. Calculate the value of

(a) $(15.3)^0$ (b) 10^{-2} (c) π^π (d) $\frac{4}{7}\times 0.5$

3. Simplify

(a) $\left(3a^2b\right)^2$ (b) $\left(\frac{x}{3}\right)^3 x^3$ (c) $\dfrac{abc}{a^{-2}b^2 2c}$ (d) $\dfrac{\frac{1}{2}x^2}{\frac{3}{x}}$

4. Evaluate

(a) $\sqrt[4]{2.7}$ accurate to 2 decimal places. (b) $\left(\frac{1}{8}\right)^{-\frac{1}{5}}$ to 3 decimal places.

5. Isolate the variable x

(a) $7x = 4x - t$ (b) $b = \sqrt{x+a}$ (c) $\dfrac{1}{x^2} = g - h$

6. Make t the subject of the equation

(a) $v = u + at$ (b) $l = l_0(1+\alpha t)$ (c) $\dfrac{t+s}{t} = \alpha$

7. Rearrange the equations to

(a) Make R the subject: $V = IR$ (b) Make θ the subject: $P = Fv\cos\theta$
(c) Make r the subject: $F = \frac{q_1q_2}{r^2}$

8. Factorise the expressions by removing the common factors

(a) $7y^2 - 14z^2$ (b) $x(y-2)+x^2$
(c) $(a+c)^2 - 4(a+c)$ (d) $SA = 2\pi rh + 2\pi r^2$

9. Factorise the quadratics

(a) $x^2+11x+28$ (b) $2x^2-5x-12$
(c) b^2-b-20 (d) $3x^2-7x+2$

10. Solve the equations

(a) $7x - 16 = \frac{2}{3}x + 4$

(b) $(x-2)^2 = 15$

(c) $x^2 - 2x - 8 = 0$

(d) $2x^2 + 5x - 4 = 0$

(e) $x^3 - 2x^2 - x - 1 = 0$

(f) $x\,(x-1)\,(x+2) = \frac{1}{6}x$

§1.2 Functions

1. If $f(x) = x^2$ and $g(x) = x^3$ evaluate

(a) $f(2)$

(b) $f(-2)$

(c) $g(3)$

(d) $f(g(2))$

2. The volume of a pipe with length l, inner radius r and outer radius R is $V = \pi\left(R^2 - r^2\right)l$. Find the volume when $R = 3.1$m, $r = 2.2$m and $l = 5.3$m.

3. Sketch the following graphs (without using Desmos).

(a) $y = 4x - 4$

(b) $2y = \frac{x}{3} + 1$

(c) $y = x^2 - 3$

(d) $y = -(x+1)^2$

4. Find the x and y intercepts for the graph of $y = x^2 - 3$

5. Use the grid provided to answer the following questions.

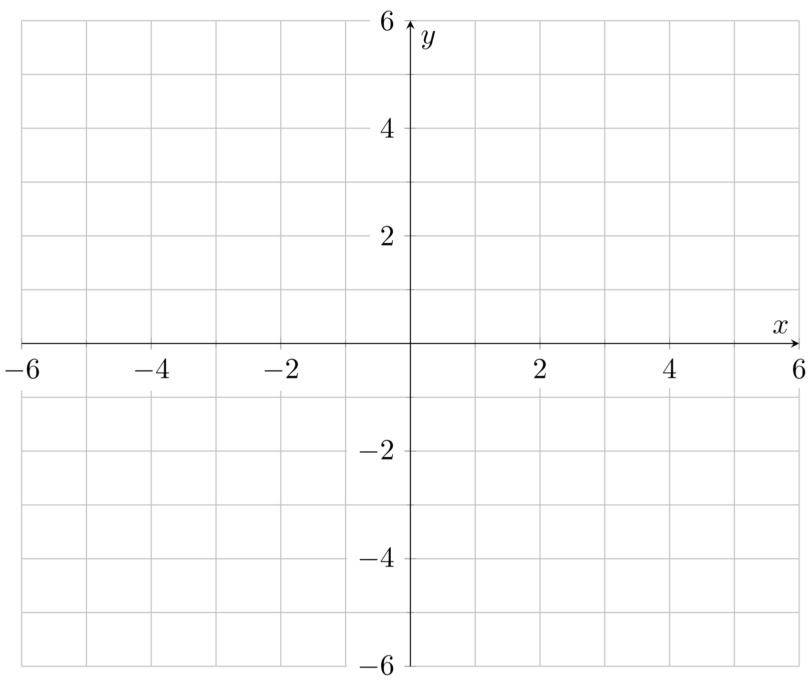
(a) Show the points $A\,(3,5)$ and $B\,(-2,-5)$ on the graph.

(b) Calculate the slope of the line through A and B. i.e. the slope of $\bar{AB}$.

(c) What is the equation of the line AB?

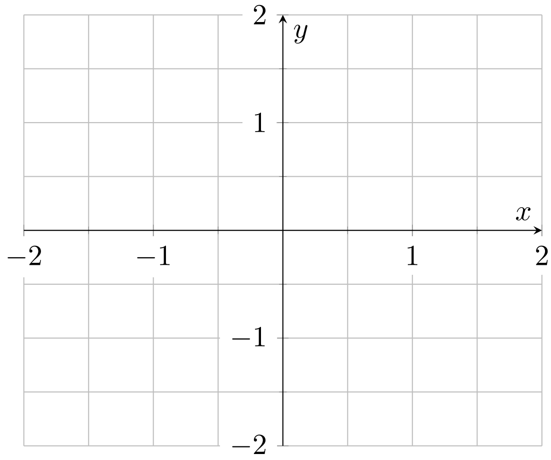
(d) What is the equation of the line parallel to the line AB through the point $(-2,3)$?

(e) Starting from the point B on the graph frame draw a line with a slope of $\frac{4}{5}$.



6. Complete the table of values for the function $g(x) = x^3 - x$ and sketch the graph

x	$g(x) = x^3 - x$
-1.5	
-1	
-0.5	
0	
0.5	
1	
1.5	



7. Graph the three functions on a common screen using desmos. How are the graphs related?

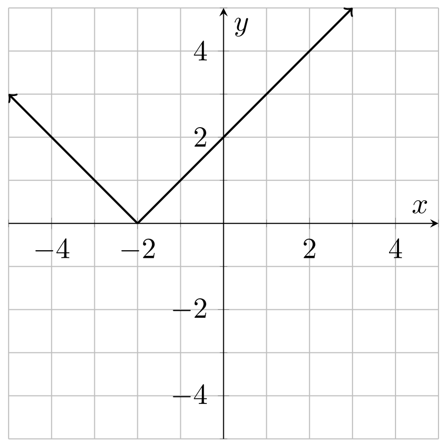
(a) $y = x^2$, $\quad y = -x^2$, $\quad y = x^2 \sin x$

(b) $y = e^x$, $\quad y = -e^x$, $\quad y = e^x \sin 5\pi x$

8. **(a)** Sketch the piece-wise function:

$$f(x) = \begin{cases} 0 & \text{for } x < -2 \\ 2 & \text{for } x > 2 \\ x & \text{for } -2 \le x \le 2 \end{cases}$$

(b) Write the equation of the function:



§1.3 Polynomials

1. What is the degree of the polynomial?

(a) $y = x^2$

(b) $4x^3 - 2x^2 + 6 = 0$

(c) $a^{99} - 99a^2 = -99$

(d) $ey^4 - ey^3 + ey^2 - ey^1 + e$

2. Are the following considered polynomials?

(a) π **(b)** π^x **(c)** $Ax^2 + Bx + C$ **(d)** $4x^3 - 2x^{\frac{3}{2}} + 1 = 0$

3. Make up your own polynomials of degree 1,2,3, and 4.

§1.4 Systems of Equations

1. Solve the system of equations graphically. For the non-linear ones you may want to use desmos .

(a) $\begin{aligned} y &= 2x+6 \\ y &= -x+5 \end{aligned}$

(b) $\begin{aligned} \frac{3y}{4}+1 &= x \\ -x &= y \end{aligned}$

(c) $\begin{aligned} -x+\frac{1}{2}y &= -5 \\ 2x-y &= 10 \end{aligned}$

(d) $\begin{aligned} \frac{x^2}{9}+\frac{y^2}{18} &= 1 \\ y &= -x^2+6x-2 \end{aligned}$

2. Solve the system using substitution.

(a) $\begin{aligned} x-y &= 2 \\ 2x+3y &= 9 \end{aligned}$

(b) $\begin{aligned} 2x-3y &= 12 \\ -x+\frac{3}{2}y &= 4 \end{aligned}$

(c) $\begin{aligned} x+y &= 8 \\ y &= -8-\frac{x}{8} \end{aligned}$

(d) $\begin{aligned} x+y^2 &= 0 \\ 2x+5y^2 &= 75 \end{aligned}$

3. Solve the system by eliminating a variable.

(a) $\begin{aligned} x+2y &= 5 \\ 2x+3y &= 8 \end{aligned}$

(b) $\begin{aligned} x^2-2y &= 1 \\ x^2+5y &= 29 \end{aligned}$

(c) $\begin{aligned} 3x^2-y^2 &= 11 \\ x^2+4y^2 &= 8 \end{aligned}$

(d) $\begin{aligned} 12x+15y &= -18 \\ 2x+5y &= -3 \end{aligned}$

§1.5 Word Problems

1. A rectangle has an area of 180 cm^2 and a perimeter of 54cm. What are its dimensions?

2. The admission fee at an amusement park is \$1.50 for children and \$4.00 for adults. On a certain day, 2200 people entered the park and the admission fees collected totalled \$5050. How many children and how many adults were admitted?

3. A woman keeps fit by bicycling and running every day. One Monday she spends $\frac{1}{2}$h at each activity and covers a total of $12\frac{1}{2}$mi. On Tuesday she runs for 12min and cycles for 45min, covering a total of 16mi. Assuming her running and cycling speeds don't change from day to day, find these speeds.

4. A customer in a coffee shop purchases a blend of two coffees: Kenyan, costing \$3.50 a pound, and Sri Lankan, costing \$5.60 a pound. He buys 3lb of the blend, which costs him \$11.55. How many pounds of each kind went into the mixture?

2 | Trigonometry

In the first half of this chapter the three trigonometric functions (sine, cosine and tangent) will be viewed as functions of angles. The second half of the chapter will approach trigonometry as functions of real numbers.

2.1 The Unit Circle

The convention for measuring angles can be seen in the diagram below, beginning from the positive x-axis and rotating a point from $(1, 0)$ counter-clockwise through (x, y). This traces out an angle, θ. The relationship between the angle and the arc length defines a radian. When the arc length equals the radius, the angle $\theta = 1$ radian. The convention is that the positive direction for measuring angles is anticlockwise and the negative direction for measuring angles is clockwise. This setup also defines a right-angled triangle with hypotenuse r. As the radius rotates, all the different right-triangles are drawn. See the link for the animated version.

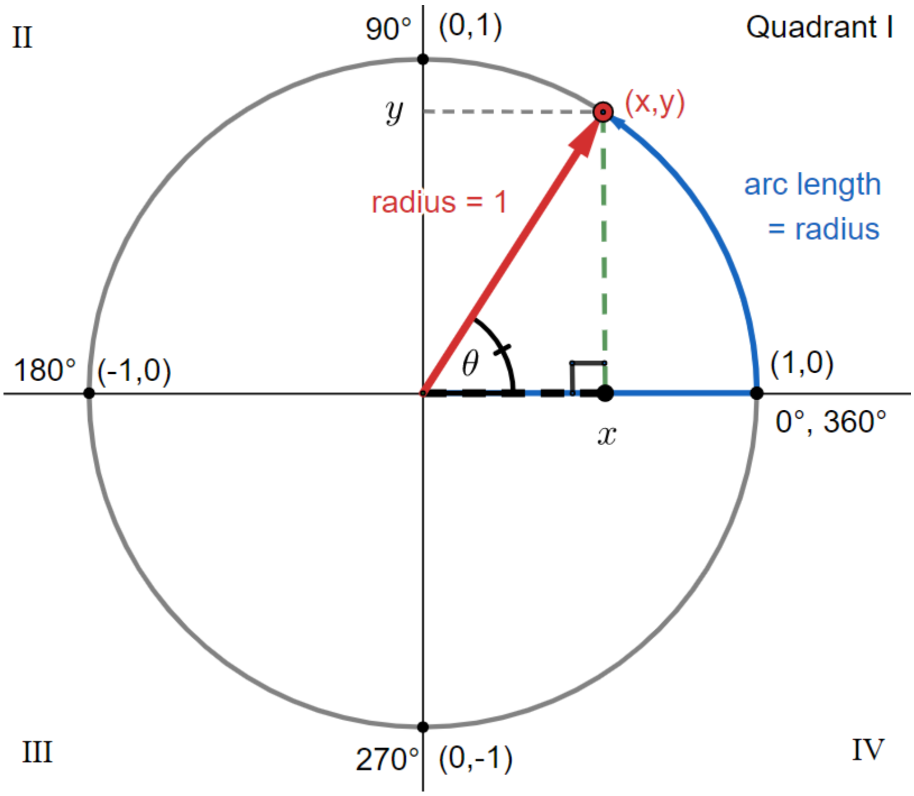


Figure 2.1: The unit circle has a radius of 1. When the arc length from $(1, 0)$ to (x, y) is equal to the radius, the angle θ is 1 radian. There are 2π radians in one complete rotation. Click here for an animated version in Desmos.

Relationship between Degrees and Radians

The abbreviation for radians is *rad.* One complete revolution is 360° which is equivalent to 2π radians. It will help to remember one of these conversion factors:

$$1 \text{ rev} = 2\pi \text{ radians} = 360^\circ \qquad \text{or} \qquad \pi \text{ radians} = 180^\circ$$

EXAMPLE

(a) Convert 36° to radians (b) Convert $\frac{\pi}{3}$ rad to degrees (c) Convert 1 rad to degrees

SOLUTION Begin with the conversion factor and then modify as necessary.

(a)

$$\begin{aligned} 180^\circ &= \pi \text{ rad} \\ 1^\circ &= \frac{\pi}{180} \text{ rad} \\ 1^\circ \times 36 &= \frac{\pi}{180} \times 36 = \frac{\pi}{5} \text{ rad} \end{aligned}$$

(b)

$$\begin{aligned} \pi \text{ rad} &= 180^\circ \\ \frac{\pi}{3} \text{ rad} &= \frac{180^\circ}{3} = 60^\circ \end{aligned}$$

(c)

$$\begin{aligned} \pi \text{ rad} &= 180^\circ \\ 1 \text{ rad} &= \frac{180^\circ}{\pi} \\ &\approx 57.3^\circ \end{aligned}$$

Note the similarity between the answers to (b) and (c). This is because $\frac{\pi}{3} = 1.0472$ so you would expect the values in degrees to be similar. Always include units with your angle whether they be degrees, radians, or even *gradians*. There are 400 gradians in a circle. This unit is not often used, however, your calculator supports all three.

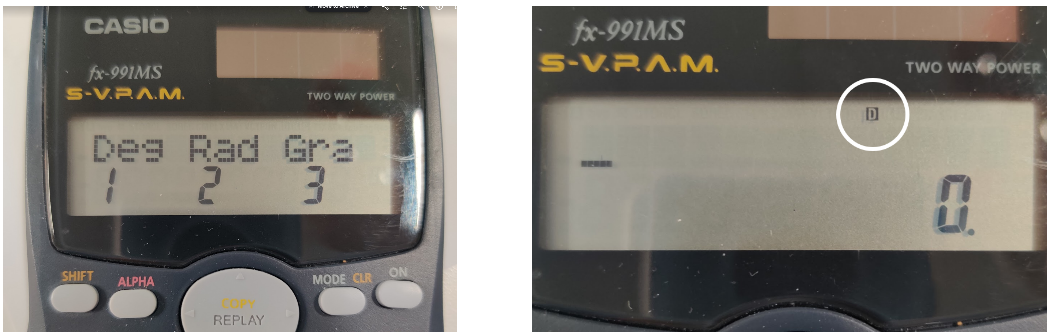


Figure 2.2: Your calculator can switch between degrees, radians, and gradians. On the right is a 'D' indicating degree mode.

Arc Length

Let θ be the angle subtended at the centre for the ends of an arc of any circle then the fraction of the circumference of the circle is $\frac{\theta}{2\pi}$ if θ is measured in radians and $\frac{\theta}{360^\circ}$ if θ is measured in degrees. The length of the circumference of any circle whose radius is r is $2\pi r$. If θ is measured in radians

$$\text{length of arc} = \frac{\theta}{2\pi} \times 2\pi r = \theta r$$

And if θ is measured in degrees

$$\text{length of arc} = \frac{\theta}{360^\circ} \times 2\pi r$$

EXAMPLE Find the length of an arc that subtends an angle of 45° at the centre of a circle whose radius is 9 cm.

SOLUTION

(a) Method 1:

$$\begin{aligned} \text{Length of arc} &= \frac{\theta}{360^\circ} \times 2\pi r \\ &= \frac{45}{360} \times 2\pi \times 9 \\ &= 2.25\pi\text{cm} \\ &\approx 7.07\text{cm} \end{aligned}$$

If an exact answer is required you should leave the answer as 2.25π cm. (Or $\frac{9\pi}{4}$ cm.)

(b) Method 2: Change degrees to radians first

$$\begin{aligned} 180^\circ &= \pi\text{rad} \\ 1^\circ &= \frac{\pi}{180} \\ 45^\circ &= \frac{\pi}{180} \times 45 \\ &= \frac{\pi}{4} \end{aligned}$$

Now, the length of arc $= r\theta = 9 \times \frac{\pi}{4} = \frac{9\pi}{4}$cm.

2.2 Right-Angled Triangles

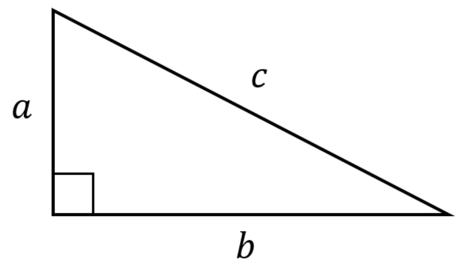
A right angled triangle is a three-sided figure ('tri-gon') with one of the interior angles being 90°. The unit circle in Figure 2.1 defines all the possible right angled triangles by rotating the point (x, y) about the origin. Measuring the side-length of triangle ('metry') is where the name trigonometry comes from. If the ratio of two of the three side lengths is known then the angle must be fixed. Similarly, if the angle is known, the ratio of the side-lengths is fixed.

Pythagoras

Pythagoras is one of the most well known historical figures in mathematics and philosophy, primarily for his eponymous theorem. Given a right-angled triangle, the square of the hypotenuse equals the sum of the squares of the other sides.

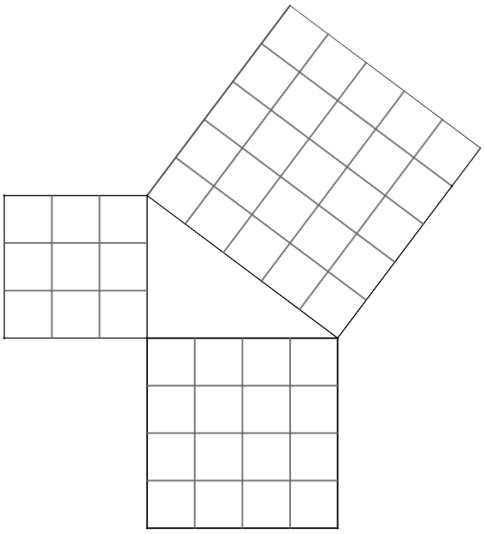
The Pythagorean Theorem:

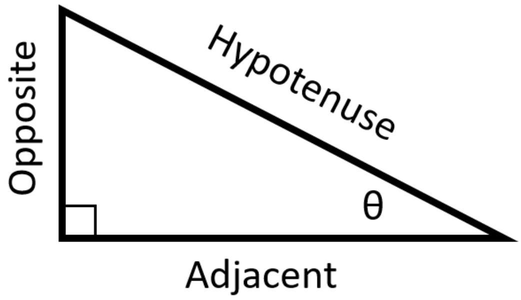
$$a^2 + b^2 = c^2$$



This theorem has many proofs, and one graphical one is shown below. The squares are all the same size. Interestingly this proof does not require any math (symbols) at all!

This is a 3-4-5 triangle which is useful in building to determine if something is square: measure sides to 3 and 4 (any units), then check that the hypotenuse is 5.





Sine, Cosine, & Tangent

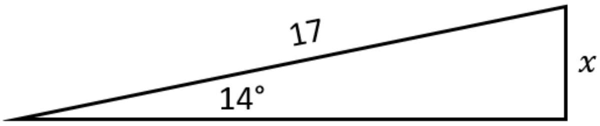
The longest side in a right-triangle is always the hypotenuse, with the other two sides being named with respect to the angle of interest, θ. Notice that if θ moves to the other corner then the sides opposite and adjacent are *reversed*. These names are helpful in defining the trigonometric ratios of sine, cosine and tan.

The phrase **SOH-CAH-TOA** is useful to remember the ratios.

$$\text{S-ine } \theta = \frac{\text{O-pposite}}{\text{H-ypotenuse}}; \qquad \text{C-osine } \theta = \frac{\text{A-djacent}}{\text{H-ypotenuse}}; \qquad \text{T-angent } \theta = \frac{\text{O-pposite}}{\text{A-djacent}}$$

$$\text{S} = \text{O/H} \qquad \text{C} = \text{A/H} \qquad \text{T} = \text{O/A}$$

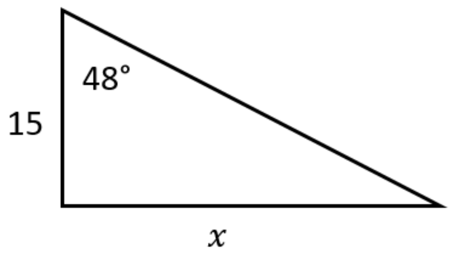
EXAMPLE Find the unknown side x



SOLUTION Write the sine ratio and solve the equation for x.

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$\sin(14) = \frac{x}{17}$$
$$x = 17\sin(14)$$
$$x \approx 4.11$$

EXAMPLE Find the unknown side x



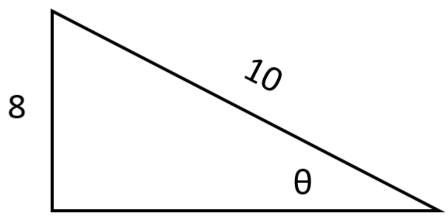
Solution Write the tangent ratio and solve the equation for x.

$$\begin{aligned}\tan\theta &= \frac{\text{opposite}}{\text{adjacent}}\\ \tan(48) &= \frac{x}{15}\\ x &= 15\tan(48)\\ x &\approx 16.7\end{aligned}$$

The calculator gives approximate values of the trigonometric ratios. You must look at your question to check whether angles are in degrees or radians and ensure the calculator is first set in the right mode. Questions where degrees are to be used will give angles marked with a ° symbol.

When solving for an angle on your calculator you select the appropriate trigonometry ratio and use the *shift* button with sin, cos or tan to find $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$.

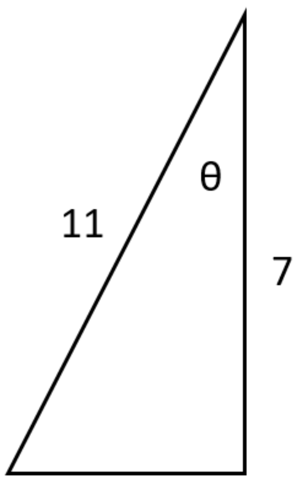
Example Find the unknown angle θ



Solution Write the sine ratio and solve the equation for θ by using the inverse-trig function on your calculator: $\sin^{-1}$.

$$\begin{aligned}\sin\theta &= \frac{\text{opposite}}{\text{hypotenuse}}\\ \sin\theta &= \frac{8}{10}\\ \sin\theta &= 0.8\\ \theta &= \sin^{-1}(0.8)\\ \theta &= 53.1^\circ\end{aligned}$$

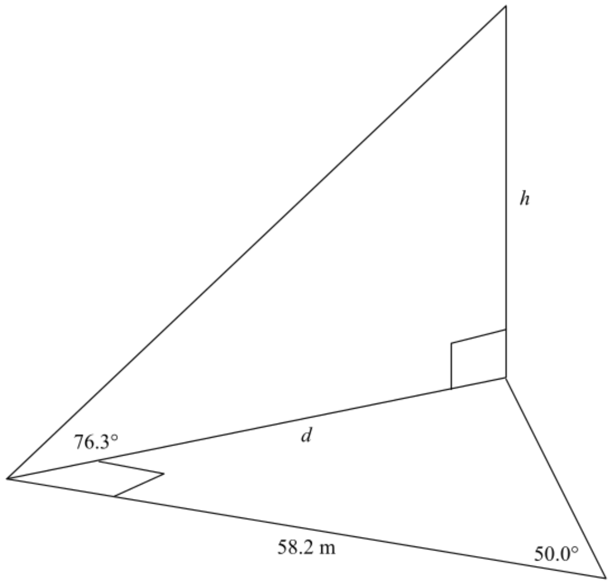
Example Find the unknown angle θ



Solution Write the cosine ratio and solve the equation for θ by using the inverse-trig function on your calculator: $\cos^{-1}$.

$$\begin{aligned}\cos\theta &= \frac{\text{adjacent}}{\text{hypotenuse}}\\ \cos\theta &= \frac{7}{11}\\ \theta &= \cos^{-1}\left(\frac{7}{11}\right)\\ \theta &= 50.5^\circ\end{aligned}$$

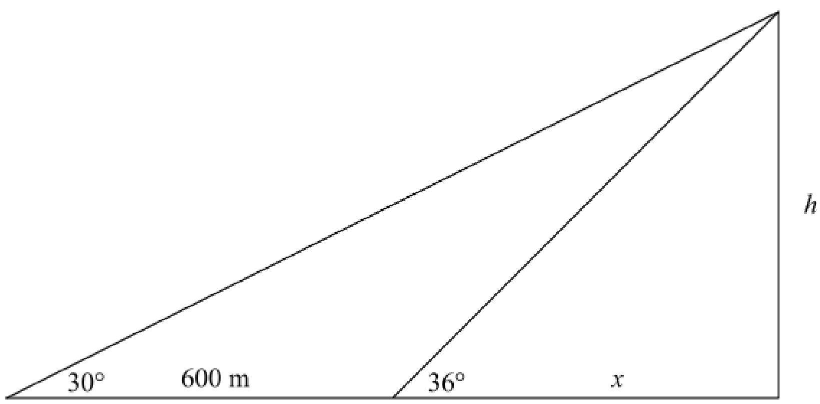
Example The height of a steep cliff is to be measured from a point on the opposite side of the river. The following diagram shows the measurements taken. Estimate the height of the cliff.



Solution

$$\begin{aligned}
\frac{d}{58.2} &= \tan 50.0° \\
d &= 58.2 \times \tan 50.0° \\
\frac{h}{d} &= \tan 76.3° \\
h &= d \times \tan 76.3° \\
&= 58.2 \times \tan 50.0° \times \tan 76.3° \\
&\approx 284.5\text{m}
\end{aligned}$$

Example To estimate the height of a mountain above a level plane the angle of elevation of the top of the mountain is measured to be 30°. 600m closer to the mountain across the plane it is found that the angle of elevation is 36°. Estimate the height of the mountain.



Solution

$$\frac{h}{x} = \tan 36° \text{and} \frac{h}{x+600} = \tan 30°$$

We want h so we eliminate x between these two equations

$$\begin{aligned}
x &= \frac{h}{\tan 36°} \text{and} x + 600 = \frac{h}{\tan 30°} \\
\frac{h}{\tan 30°} &= \frac{h}{\tan 36°} + 600 \\
\frac{h}{\tan 30°} - \frac{h}{\tan 36°} &= 600
\end{aligned}$$

$$\begin{aligned}
h\left(\frac{1}{\tan 30^\circ} - \frac{1}{\tan 36^\circ}\right) &= 600 \\
h\left(\frac{\tan 36^\circ - \tan 30^\circ}{\tan 30^\circ \tan 36^\circ}\right) &= 600 \\
h &= 600 \times \frac{\tan 30^\circ \tan 36^\circ}{\tan 36^\circ - \tan 30^\circ} \\
&\approx 600 \times 2.811603815 \\
&\approx 1687 \text{ m}
\end{aligned}$$

Identities

The unit circle has equation $x^2 + y^2 = 1$ and we define $x = \cos\theta$ and $y = \sin\theta$ so

$$x^2 + y^2 = 1 \rightsquigarrow (\cos\theta)^2 + (\sin\theta)^2 = 1$$

This is always written

$$\boxed{\sin^2\theta + \cos^2\theta = 1}$$

This is an *identity* which means it is true for all values of θ. There are many identities in trigonometry, we will only use the Pythagorean identity (above) and one more. Given $\sin = \frac{\text{opp}}{\text{adj}}$ we can solve for opp = (sin)(hyp). Similarly from cosine: adj = (cos)(hyp). Substituting these into the tangent relationship:

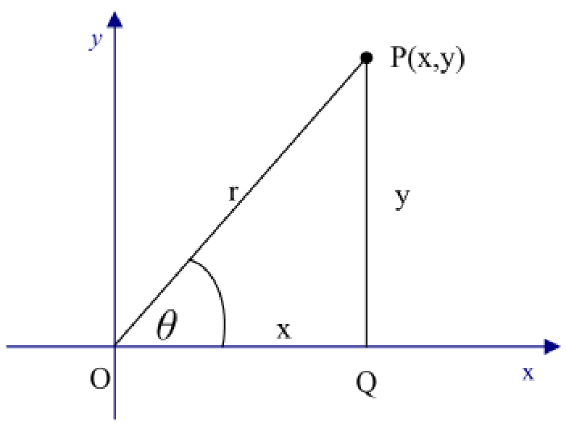
$$\boxed{\tan = \frac{\text{opp}}{\text{adj}} = \frac{(\sin)(\cancel{\text{hyp}})}{(\cos)(\cancel{\text{hyp}})} = \frac{\sin}{\cos}}$$

The sine and cosine law are two more unique relationships that we will cover in section 2.4.

All Students Take Calculus

In the previous section the angles were between 0° and 90°. In this section the angles can take any value. Initially we consider angles between 0° and 360° and relate these to the radian measure between 0 and 2π. We remind you that angles are measured anticlockwise from the positive x-axis.

If the point $P(x, y)$ is in the first quadrant, θ is the angle between OP and the positive x-axis and we complete the right triangle then we have created the following situation.



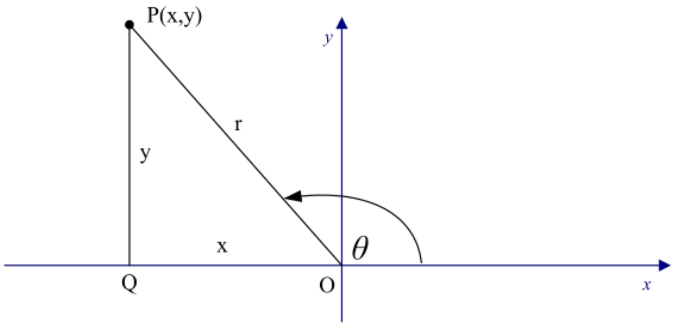
Let the hypotenuse be r then

$$r = \sqrt{x^2 + y^2}$$

Therefore

$$\sin\theta = \frac{y}{r},\ \cos\theta = \frac{x}{r} \text{and } \tan\theta = \frac{y}{x}$$

We now let θ be any angle and define sine, cosine and tangent in the same way. For instance if $P(x, y)$ is in the second quadrant: $\sin\theta = \frac{y}{r}$ because y is positive $\sin\theta$ will be positive. (r is always positive.)



$\cos\theta = \frac{x}{r}$ and since x is negative $\cos\theta$ will be negative. Tangent: $\tan\theta = \frac{y}{x}$ and negative for the same reason.

This pattern can be extended to quadrants 3 and 4. The mnemonic (**A**ll **S**tudents **T**ake **C**alculus) might help you remember which one is positive although you can always work it out if you need to. This means all are positive in the first quadrant, only sine is positive in the second quadrant, only tangent is positive in the third quadrant and only cosine is positive in the fourth quadrant.

The value of a trigonometric function consists of two parts the numerical part and the sign. you must get both parts correct. In the previous section you related the values of a terminal point to another point in the first quadrant. A point on the unit circle could be in any one of the four quadrants.

Quadrant	x-coordinate	y-coordinate	cos	sin	tan
I	$+$	$+$	$+$	$+$	$+$
II	$-$	$+$	$-$	$+$	$-$
III	$-$	$-$	$-$	$-$	$+$
IV	$+$	$-$	$+$	$-$	$-$

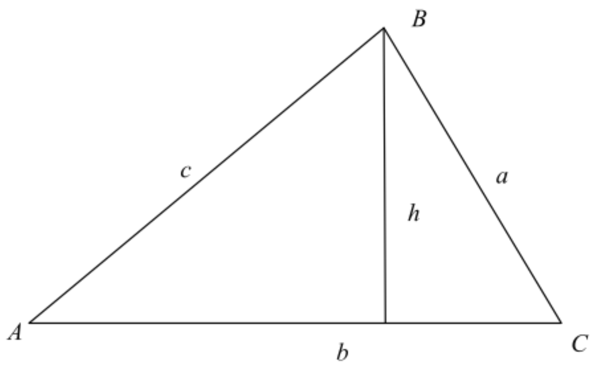
The Area of a Triangle

The fundamental formula for the area of a triangle is Area $= \frac{1}{2} \times$ base $\times$ height Using the trigonometric functions the height can be replaced and the formula becomes:

$$\text{Area} = \frac{1}{2} \times \text{product of two sides} \times \text{sine of the included angle}$$

The formula is particularly easy to remember in symbolic form. Let the triangle have vertices A, B and C, so the the sides opposite these angles are a, b and c respectively. Then the area can be expressed symbolically as

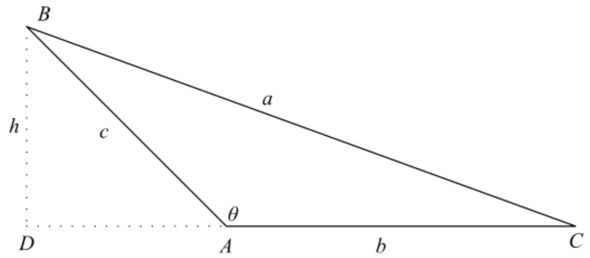
$$\text{Area} = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B$$



$$\sin A = \frac{h}{c} \text{ and } h = c\sin A$$
$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height}$$

$$= \frac{1}{2} \times b \times c \sin A = \frac{1}{2} bc \sin A$$

It depends where you draw h and which angle you choose to use as to which formula you finish up with. The key point to remember is b and c are two sides and A is the angle between them. The triangle above shows A as an acute angle (between 0° and 90°). If the angle is obtuse (between 90° and 180°) the formula still holds.



The angle is in the second quadrant and $\sin(180 - \theta) = \sin\theta$, so $\sin(180 - \theta) = \frac{h}{c}$ can be written as $\sin\theta = \frac{h}{c}$ or $h = c\sin\theta$ or $h = c\sin A$ where A is obtuse. So the area is $\frac{1}{2}bc\sin A$

EXAMPLE A triangle has two sides of 5 cm and 8 cm and the angle between them is 150°. Find its area.

SOLUTION

$$\text{Area} = \frac{1}{2} \times 5 \times 8 \times \sin 150°$$

It helps to remember that $\sin 150 = \sin 30 = \frac{1}{2}$

$$\text{Area} = \frac{1}{2} \times 5 \times 8 \times \frac{1}{2} = 10 \text{ cm}^2$$

2.3 Trig Functions of Real Numbers

In this next part of the chapter the three main trigonometric functions (sine, cosine and tangent) will be studied. They will be viewed as functions of real numbers rather than angles. The trigonometric functions defined in these two ways are identical and there is a simple rule connecting the domains. Why do we show you the two approaches? Trigonometry will be used to solve a variety of problems and these can be divided into two groups, dynamic problems and static problems. When dynamic problems (such as problems involving motion) are being solved real numbers will be used. When static problems (such as finding distances and angles for triangles) are being solved angles will be used.

You should be familiar with the fundamental graphs of $y = \sin x$ and $y = \cos x$. These graphs are the basis of this section. `Desmos` can easily show you the shape of $y = \sin x$ and $y = \cos x$ so if you are asked to draw a rough sketch of these curves you should plot a few key points and draw a smooth curve between them. You will usually be given the required domain however if you are not you would choose to draw these for one complete cycle $(0 - 2\pi)$. To sketch $y = \sin x$ it is enough to select as key points $x = 0$, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, 2π.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = \sin x$	0	1	0	-1	0

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = \cos x$	1	0	-1	0	1

You will be aware that these curves repeat this pattern every 2π where x extends in both the positive and negative directions. 0 to 2π represents one complete cycle. Mathematically we say

$$\sin(x + 2n\pi) = \sin x \text{ for any integer } n$$
$$\cos(x + 2n\pi) = \cos x \text{ for any integer } n$$

Aside: instead of "for any integer n" we can write $\forall n \in \mathbb{Z}$. A function that displays this characteristic is described as *periodic* and for $y = \sin x$ and $y = \cos x$ the *period* is 2π.

Transformations

The following six transformations can be applied to any function including sine, cosine, and tangent.

(a) Vertical shift **(b)** Horizontal shift **(c)** Vertical stretch

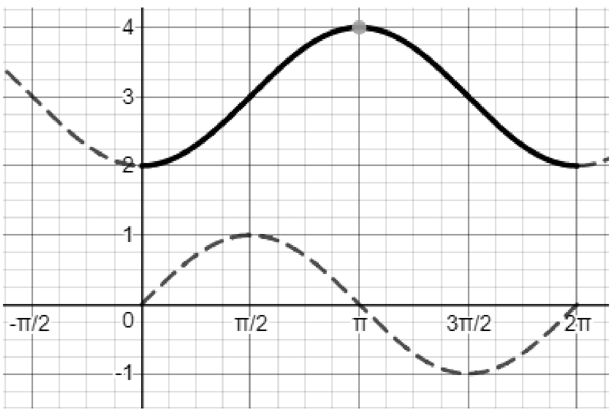
(d) Horizontal stretch **(e)** Reflection in the x-axis **(f)** Reflection in the y-axis

EXAMPLE Sketch $y = \sin\left(x - \frac{\pi}{2}\right) + 3$. This may be considered as a sine curve shifted $\frac{\pi}{2}$ units to the right. Note for periodic functions like a sine wave a horizontal shift is called a *phase shift.* and 3 units upwards.

SOLUTION 1. Beginning with the special points for $\sin x$, a table shows the evolution of the points.

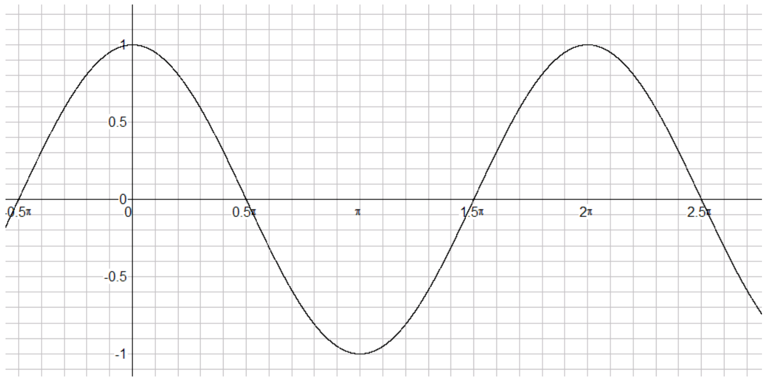
x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\leftarrow$ x-values, plot these
$\sin x$	0	1	0	-1	0	
$\sin\left(x - \frac{\pi}{2}\right)$	-1	0	1	0	-1	
$\sin\left(x - \frac{\pi}{2}\right) + 3$	2	3	4	3	2	$\leftarrow$ y-values, plot these

2. Plot the transformed values with the original special points and connect with a smooth, continuous line. `Desmos` confirms our transformation. The dashed plot shows $y = \sin x$ as reference.



EXAMPLE Sketch $y = \cos(x - \frac{\pi}{6})$.

SOLUTION You could use a table of values however this is $y = \cos x$ with a horizontal shift of $\frac{\pi}{6}$ to the right. Sketch the transformation on top of the graph of $y = \cos x$.

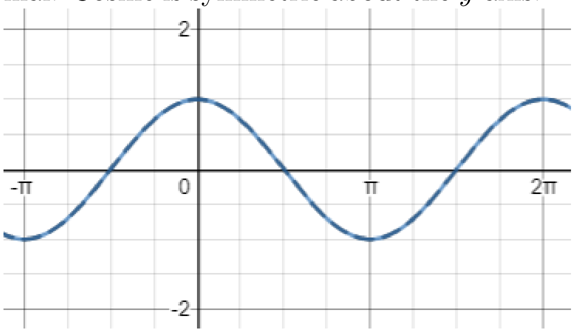


In general $y = a\sin x$ represents a vertical stretch of $y = \sin x$ by a. If a is negative the transformation can either be described as a negative stretch or (preferably) as a *stretch* of $|a|$ followed by a *reflection* in the x-axis. Recall the reflection of $y = f(x)$ in the x-axis is $y = -f(x)$. The number $|a|$ is called the *amplitude* for both $y = \sin x$ and $y = \cos x$ shown below.

If $0 < a < 1$ the fractional stretch causes the curve to shrink vertically. For instance the curve $y = \sin x$ has a maximum value of 1 and a minimum value of -1. the curve $y = \frac{1}{2}\sin x$ has a maximum value of $\frac{1}{2}$ and a minimum value of $-\frac{1}{2}$.

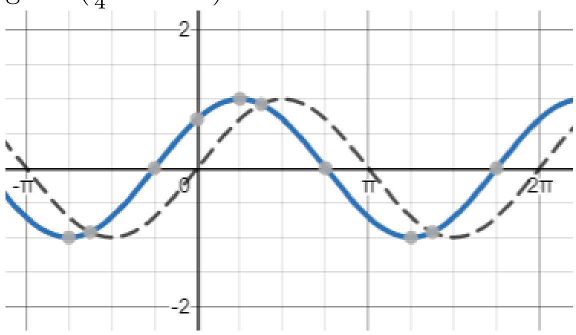
(a) Example Sketch $y = \cos(-t)$

Notice this reflection is on top of the original. Cosine is symmetric about the y-axis.



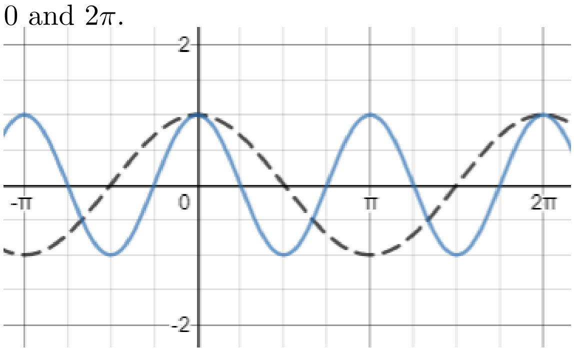
(b) Example Sketch $y = \sin(x + \frac{\pi}{4})$

The angle is shifted to the left by 45 degrees ($\frac{\pi}{4}$ radians).



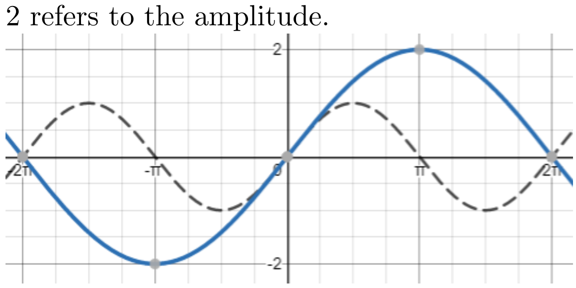
(c) Example Sketch $y = \cos(2x)$

Here there are 2 complete cycles between 0 and 2π.



(d) Example Sketch $y = 2\sin(0.5x)$

Two transformations applied together. The 2 refers to the amplitude.



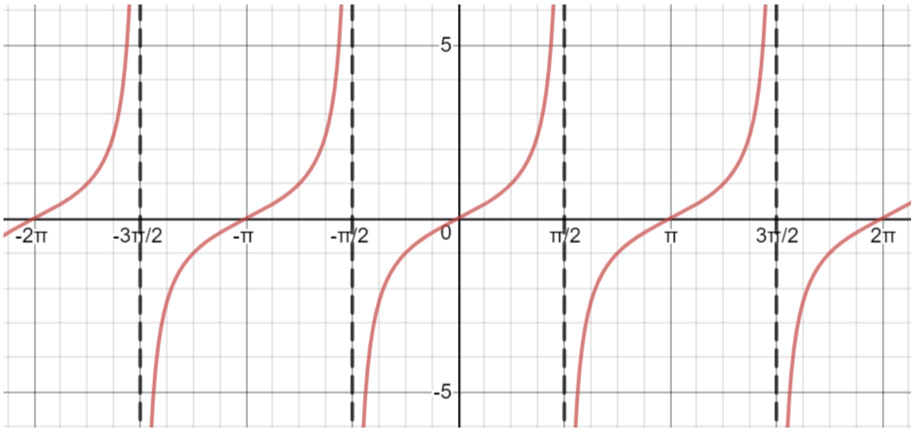


Figure 2.3: The graph of $y = \tan x$. The asymptotes are shown with dashed lines, repeating every π radians.

Tangent

There are other trig functions that we will not be covering here, for example the reciprocal functions are cosecant= $\frac{1}{\sin\theta}$, secant= $\frac{1}{\cos\theta}$, and the inverse tangent function, cotangent= $\frac{1}{\tan\theta}$. By focussing on sine, cosine and tangent the majority of problems we encounter can be solved. Tangent is the odd function out of the trio of common ones.

Previously we learnt that the period for the sine and cosine functions was 2π. Tangent is also a periodic function and it has a period of π (not 2π). This means that it goes through one complete cycle every π. Recall $\tan x = \frac{\sin x}{\cos x}$. To analyse the behaviour of the tangent function it helps if you know what you are looking for. Some key values of tangent will show the pattern.

x	$\sin x$	$\cos x$	$\tan x$
$-\frac{\pi}{2}$	-1	0	$\frac{-1}{0} = -\infty$
$-\frac{\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2} \div \frac{\sqrt{2}}{2} = -1$
0	0	1	$\frac{0}{1} = 0$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2} \div \frac{\sqrt{2}}{2} = 1$
$\frac{\pi}{2}$	1	0	$\frac{1}{0} = \infty$

As x takes values from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, $\tan(x)$ takes values from $-\infty$ to ∞. This pattern is repeated every π. A desmos graph can show this relationship. The dashed lines are the asymptotes.

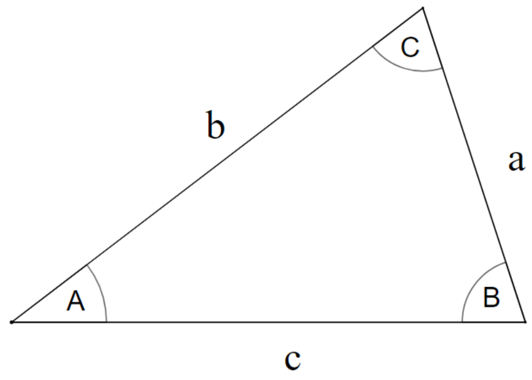
The graph can be seen to have point symmetry. If you rotate the tangent curve through a half turn using the origin as axis the curve will lie on top of itself. This is a pictorial representation of an odd function.

Note: Do not confuse $y = \tan x$ with $y = x^3$. While they may appear to be similar in shape the only similarities are that they both pass through the origin and continue towards ∞ in the first quadrant and $-\infty$ in the third quadrant.

2.4 Applications

The Sine Rule

The Sine Rule is a relationship that allows you to find the sides and angles in triangles *without* a right angle. In the next section we use the Cosine Rule to find sides and angles in triangles also, so as you study these two sections you need to learn which problems require the Sine Rule and which require the Cosine Rule. Previously, we met the formula for the area of a triangle given two sides and the included angle $\left(\text{area} = \frac{1}{2}ab\sin C\right)$. The Sine Rule and Cosine Rule also require specific combinations of sides and angles. Using standard side and angle labelling for a triangle $\rightarrow \Delta ABC$ and let the sides be lowercase a, b and c where a is opposite $\angle A$ etc.



The Sine Rule states that in any triangle

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \qquad \text{or, inversely as:} \qquad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Textbooks may use the term "The Law of Sines" meaning the same relationship.

Proof of the Sine Rule

The Sine Rule is easy to prove beginning with the formula for the area of a triangle, and using the diagram above as reference with the base as c. (Refer to the diagram on page 26.)

$$\text{Area} = \frac{1}{2}(\text{base})(\text{height}) = \frac{1}{2}(c)(b\sin A) = \cdots = \frac{1}{2}ab\sin C = \frac{1}{2}ac\sin B$$

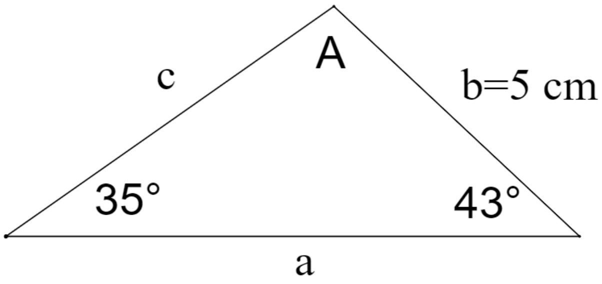
Focussing only on the three terms on the right, multiply by 2:

$$bc\sin A = ab\sin C = ac\sin B$$

And dividing by abc

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

EXAMPLE Find side lengths a and c in the following triangle.



SOLUTION

(1) To find c

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

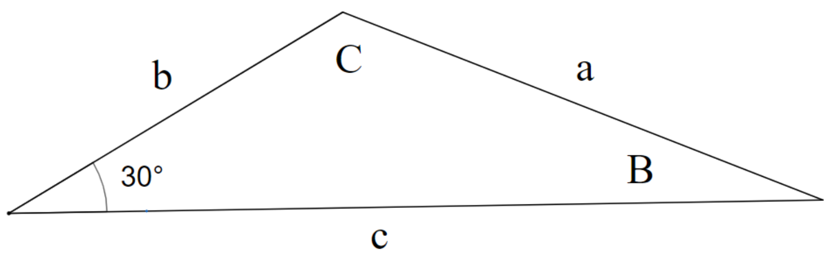
$$\begin{aligned}\frac{c}{\sin 43} &= \frac{5}{\sin 35}\\ c &= \frac{5\sin 43}{\sin 35}\\ &\approx 5.95\text{cm}(2\text{ dp})\end{aligned}$$

(2) To find a (i) $A = 180° - (35° + 43°) = 180° - 78° = 102°$ (ii) It is usually wise to go back to the original data (i.e. use b and B rather than c and C).

$$\begin{aligned}\frac{a}{\sin A} &= \frac{b}{\sin B}\\ \frac{a}{\sin 102} &= \frac{5}{\sin 35}\\ a &= \frac{5\sin 102}{\sin 35}\\ &\approx 8.53\text{cm}(2\text{ dp})\end{aligned}$$

These two calculations illustrate the first two cases in which the Sine Rule is used. You will notice that the triangle has been completely solved in the course of this example. We started with one side and two angles and we found the other two sides and the other angle.

EXAMPLE Given $a = 30°$, $a = 8$ and $b = 7$ solve the triangle (i.e. find B, C and c).



SOLUTION

(a) Find B

$$\begin{aligned}\frac{\sin B}{b} &= \frac{\sin A}{a}\\ \sin B &= \frac{b\sin A}{a}\\ &= \frac{7\sin 30}{8} = 0.4375\\ B &= \sin^{-1} 0.4375 \approx 25.94°\end{aligned}$$

(b) Find C

$$\begin{aligned}C &= 180° - (30° + 25.94°)\\ &= 124.06°\end{aligned}$$

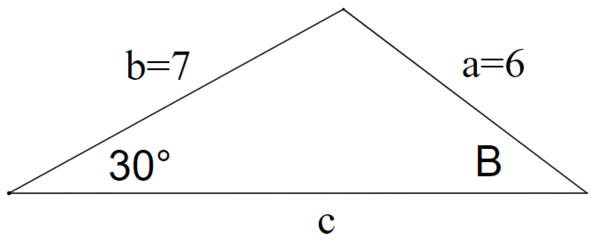
(c) Find c

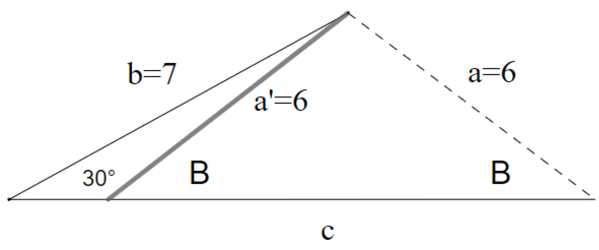
$$\begin{aligned}\frac{c}{\sin C} &= \frac{a}{\sin A}\\ c &= \frac{a\sin C}{\sin a}\\ &= \frac{8\sin 124.06}{\sin 30}\\ &\approx 13.3\end{aligned}$$

The third case is not as straight forward and referred to as the ambiguous case. Given two sides and an angle there could be no triangle formed, one triangle formed or two triangles formed depending on the length of the side opposite the given angle.

EXAMPLE The Ambiguous Case: Given $A = 30°$, $a = 6$ and $b = 7$ solve the triangle.

SOLUTION In this case $b \sin A = 3.5$ and $b = 7$ so as a lies between 3.5 and 7. This is the ambiguous case, therefore there are two solutions as the side lengths and angle can produce two different triangles.





First solution (Proceed as before)

(a) Find B

$$\begin{aligned}\frac{\sin B}{b} &= \frac{\sin A}{a}\\ \sin B &= \frac{b \sin A}{a}\\ &= \frac{7 \sin 30}{6} = 0.58\dot{3}\\ B &= \sin^{-1} 0.58\dot{3} \approx 35.7°\end{aligned}$$

(b) Find C
First recognize this angle is obtuse $> 90°$

$$\begin{aligned}C &= 180° - (30° + 35.69°)\\ &= 114.31°\end{aligned}$$

(c) Find c

$$\begin{aligned}\frac{c}{\sin C} &= \frac{a}{\sin A}\\ c &= \frac{a \sin C}{\sin A}\\ &= \frac{6 \sin 114.31}{\sin 30}\\ &\approx 10.9\end{aligned}$$

Second solution

(a) Find the second value of B
$B = 180° - 35.69° = 144.31°$

(b) Find C
$C = 180° - (30° + 144.31°) = 180° - 174.31° = 5.69°$

(Or if you remember the rule that the exterior angle of a triangle is the sum of the two interior opposite angles $C + 30° = 35.69°$ so $C = 35.69° - 30° = 5.69°$)

(3) Find c

$$\begin{aligned}\frac{c}{\sin C} &= \frac{a}{\sin A}\\ c &= \frac{a \sin C}{\sin A}\\ &= \frac{6 \sin 5.69}{\sin 30}\\ &\approx 1.2\end{aligned}$$

Both answers here are correct mathematically, however, depending on the situation one answer refers to a specific triangle which should be checked with a diagram.

The Cosine Rule

The Cosine Rule, sometimes called 'The Law of Cosines', is useful for a non-right triangle when we know all three side lengths without an angle, or two sides and the angle between them.

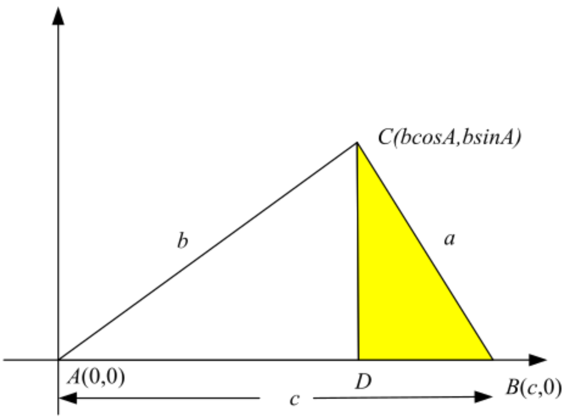
For any triangle side sides a, b, and c and angle A opposite side a:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Note the similarity to the Pythagorean theorem (see page 21).

Proof of The Cosine Rule

To prove: For any triangle ΔABC, $a^2 = b^2 + c^2 - 2bc\cos A$



By Pythagoras' Theorem

$$\begin{aligned}
BC^2 &= CD^2 + DB^2 \\
a^2 &= (c - b\cos A)^2 + (b\sin A)^2 \\
&= c^2 - 2bc\cos A + b^2\cos^2 A + b^2\sin^2 A \\
&= c^2 - 2bc\cos A + b^2\left(\cos^2 A + \sin^2 A\right) \\
&= c^2 - 2bc\cos A + b^2 \text{as} \cos^2 A + \sin^2 A = 1
\end{aligned}$$

This is usually written

$$a^2 = b^2 + c^2 - 2bc\cos A$$

The diagram has been drawn to simplify the way the proof unfolds. You will see that by placing the vertex A at the origin the side a is found in terms of b, c, and A. The proof would have been the same had A and B been as shown and C placed in the second quadrant. (Thus producing a triangle with an obtuse angle at A.) This rule is symmetrical. You need to be given two sides and the included angle (b, c and A) and the formula allows you to calculate a. Most textbooks will therefore show you three equivalent formulae:

$$\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos A \\
b^2 &= c^2 + a^2 - 2ca\cos B \\
c^2 &= a^2 + b^2 - 2ab\cos C
\end{aligned}$$

EXAMPLE Given $a = 5$, $b = 6$ and $C = 130°$, find c.

SOLUTION

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ &= 5^2 + 6^2 - 2 \times 5 \times 6 \times \cos 130 \\ c^2 &\approx 99.567 \\ c &\approx \sqrt{99.567} \approx 9.978 \end{aligned}$$

EXAMPLE Find the angle given three side lengths; given $a = 5$, $b = 6$ and $c = 9$, find A.

SOLUTION Because a is the smallest side A will most certainly be an acute angle.

$$\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \\ &= \frac{6^2 + 9^2 - 5^2}{2 \times 6 \times 9} \\ &\approx 0.852 \\ A &\approx \cos^{-1}(0.852) \\ &\approx 31.6° \end{aligned}$$

2.5 Chapter Exercises

§2.1 Unit Circle

1. Find the radian measure of the angle with the given degree measurements.

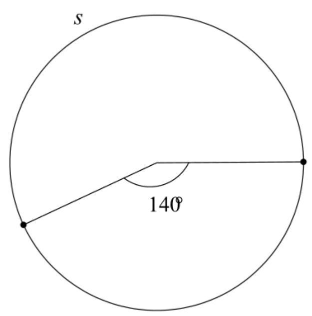
(a) $36°$ (b) $-480°$
(c) $60°$ (d) $-135°$

2. Find the degree measure of the angle with the given radian measure.

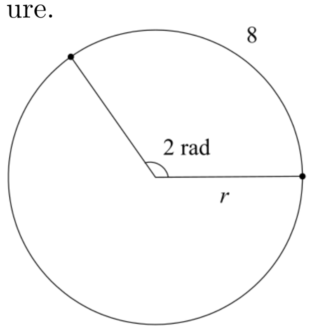
(a) $\frac{3\pi}{4}$ (b) $\frac{5\pi}{6}$
(c) -1.5 (d) $-\frac{\pi}{12}$

3. Arc length

(a) Find the length of the arc s in the figure. The radius is 5.



(b) Find the radius r of the circle in the figure.



4. Find the length of an arc that subtends a central angle of 2rad in a circle of radius 2mi.

5. Find the radius of the circle if an arc of length 6m on the circle subtends a central angle of $\pi/6$ rad.

6. Pittsburgh, Pennsylvania and Miami, Florida lie approximately on the same meridian. Pittsburgh has a latitude of 40.5° N and Miami is 25.5° N. Find the distance between these two cities. (The radius of the earth is 3960mi.)

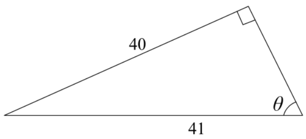
7. Find the distance the earth travels in one day in its path around the sun. Assume the year has 365 days and that the path of the earth around the sun is a circle of radius 93 million miles.

§2.2 Right Angled Triangles

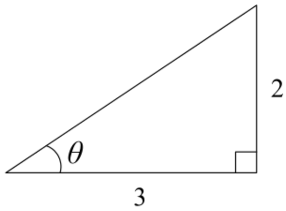
1. Find the exact value of $\sin\theta$, $\cos\theta$ and $\tan\theta$ of the angle θ in the triangle.

2. Find $\sin\theta$, $\cos\theta$ and $\tan\theta$ in the following triangles.

(a)

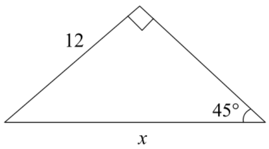


(b)

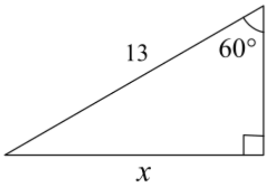


3. Find the side length labelled x for the following triangles.

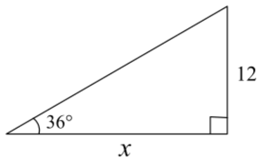
(a)



(b)

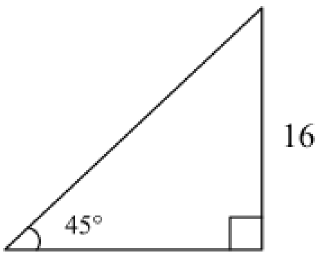


(c)

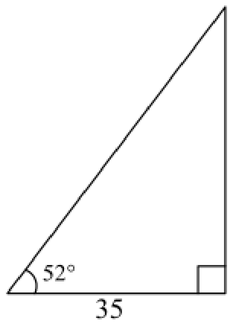


4. Solve the triangles.

(a)



(b)



5. The angle of elevation to the top of the Empire State Building in New York is found to be 11° from the ground at a distance of 1mi from the base of the building. Using this information, find the height of the Empire State Building.

6. A laser beam is to be directed towards the centre of the moon but the beam strays 0.5° from its intended path.

(a) How far has the beam diverged from its assigned target when it reaches the moon? (The distance of the earth to the moon is 240 000mi.)

(b) The radius of the moon is about 1000mi. Will the beam strike the moon?

7. A water tower is located 325ft from a building. From a window in the building it is observed that the angle of elevation to the top of the tower is 39° and the angle of depression to the bottom of the tower is 25°. How tall is the tower? How high is the window?

8. Find the area of a triangle with sides of length 7 and 9 and included angle 72°.

9. A triangle has an area of $16in^2$, and two of the sides of the triangle have lengths 5in and 7in. Find the angle included by these two sides.

§2.3 Trigonometric Functions

1. Sketch the functions by hand.

(a) $y = 1 + \sin x$ **(b)** $y = 1 - \cos x$

(c) $y = -2\sin x$ **(d)** $y = 4 - 2\cos x$

(e) $y = |\cos x|$

2. Find the amplitude and period of the function and sketch its graph.

(a) $y = \cos 4x$ **(b)** $y = 3\sin 3x$

(c) $y = 10\sin \frac{1}{2}x$ **(d)** $y = -\cos \frac{1}{3}x$

(e) $y = 3\cos 3\pi x$

3. Find the amplitude, period and phase shift of the function, and plot using `desmos`.

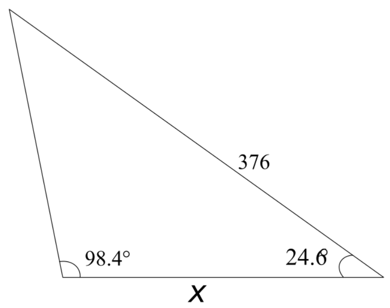
(a) $y = \cos\left(x - \frac{\pi}{2}\right)$ **(b)** $y = -\sin\left(x - \frac{\pi}{6}\right)$

(c) $y = 2\sin\left(\frac{2}{3}x - \frac{\pi}{6}\right)$ **(d)** $y = 3\cos\pi\left(x + \frac{1}{2}\right)$

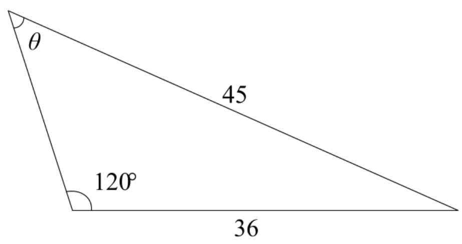
(e) $y = -\frac{1}{2}\cos\left(2x - \frac{\pi}{3}\right)$ **(f)** $y = \sin(3x + \pi)$

§2.4 Applications

1. Use the Sine Rule to find side x or angle θ

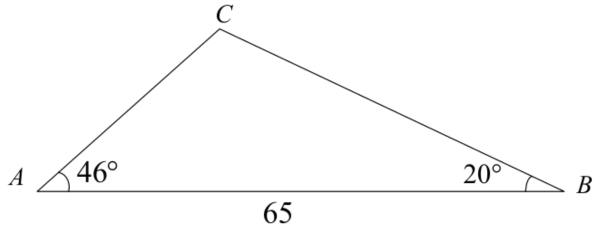


(a)



(b)

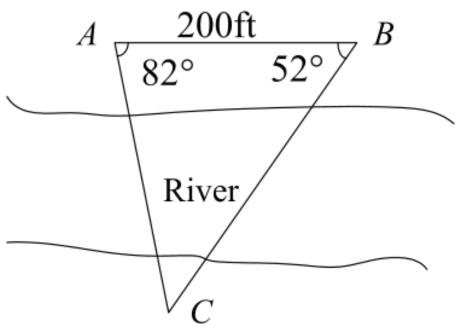
2. Solve the triangle using the Sine Rule.



3. Sketch each triangle and then solve using the Sine Rule.

(**a**) $\angle A = 50°$, $\angle B = 68°$, $c = 230$ (**b**) $\angle B = 29°$, $\angle C = 51°$, $b = 44$

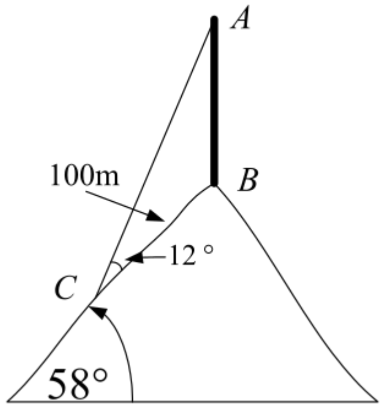
4. To find the distance across a river, a surveyor chooses points A and B, which are 200ft apart on one side of the river. She then chooses a reference point C on the opposite side of the river and finds that $\angle BAC \approx 82°$ and $\angle ABC \approx 52°$. Find the approximate distance from A to C.



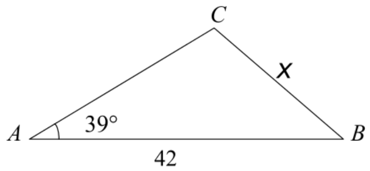
5. The path of a satellite circling the earth causes it to pass directly over two tracking stations A and B, which are 50mi apart. When the satellite is on one side of thetwo stations, the angle of elevation at A and B are measured to be 87.0° and 84.2°, respectively.

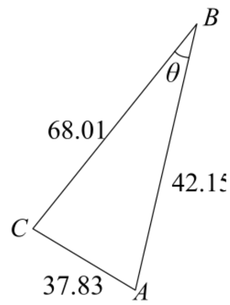
(**a**) How far is the satellite from station A? (**b**) How high is the satellite above the ground?

6. A communication tower is located at the top of a steep hill. The angle of inclination of the hill is 58°. A guy wire is attached to the top of the tower and to the ground, 100m downhill from the base of the tower. The angle between the slope of the hill and the guy wire is measured as 12°. Find AC, the length of cable required for the guy wire.

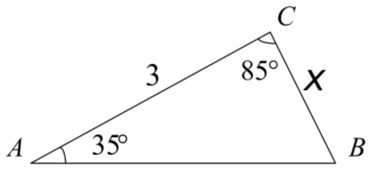


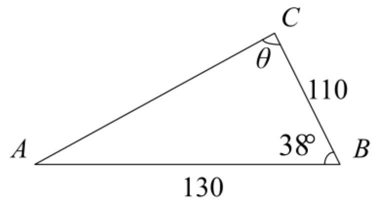
7. Use the Cosine Rule to find side x given $AC = 44.3$ and θ.

(a)


(b)


8. Use either the Sine Rule or Cosine Rule as appropriate to find x and θ.

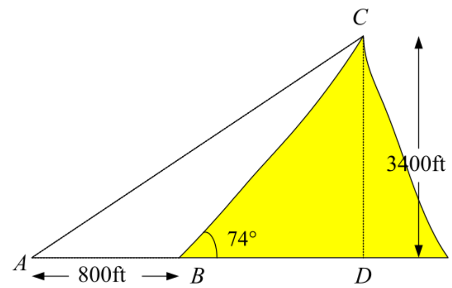
(a)


(b)


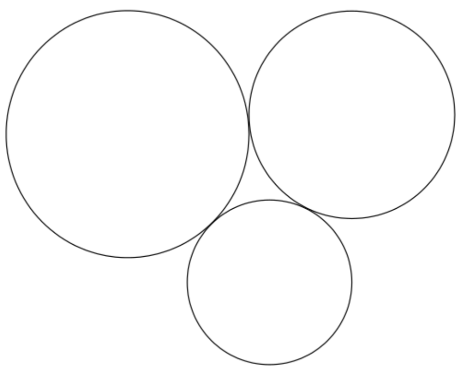
9. Two straight roads diverge at an angle of 65°. Two cars leave the intersection at 2.00 P.M., one travelling at 50 mi/hr and the other at 30 mi/hr. How far apart are the cars at 2.30 P.M.?

10. A pilot flies in a straight path for 1 hr 30 min. She then makes a course correction, heading 10° to the right of her original course, and flies for 2 hr in the new direction. If she maintains a constant speed of 625 mi/h how far is she from her starting point?

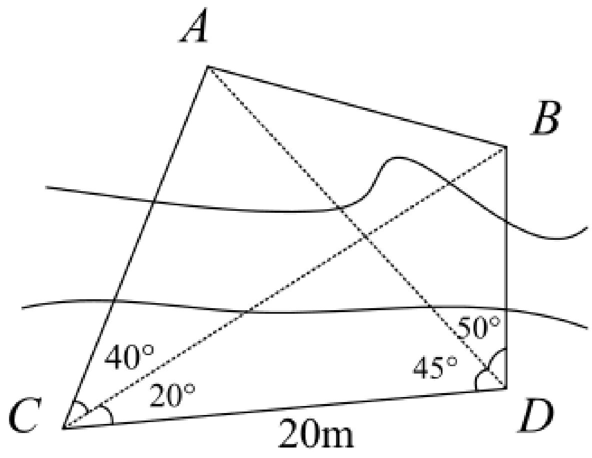
11. A steep mountain is inclined 74° to the horizontal and rises 3400 ft above the surrounding plain. A cable car is to be installed from a point 800 ft from the base to the top of the mountain, as shown. Find the shortest length of cable needed.



12. Three circles of radii 4, 5, and 6 cm respectively are mutually tangent. Find the area enclosed between the circles.



13. A surveyor wishes to find the distance between two points A and B on the opposite side of a river. on her side of the river she chooses two points C and D that are 20 m apart and measures the angles shown. Find the distance between A and B.

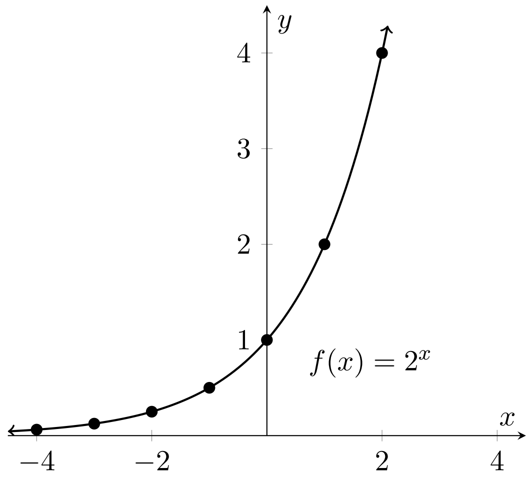


3 | Exponential & Logarithmic Functions

We defined the polynomial a^x in Chapter 1 (page 9), however x was restricted to rational numbers. (Recall rational numbers can be expressed as a fraction: $\frac{a}{b}$.) We now want to explore a^x where and x is any real number. Consider $y = 2^x$. This is an *exponential* function because the variable, x, is in the exponent. The graph is shown below. Notice it will never go below $y = 0$.

variable x	function $f(x) = 2^x$	
-4	$f(-4) = 2^{-4}$	$= 0.0625$
-3	$f(-3) = 2^{-3}$	$= 0.125$
-2	$f(-2) = 2^{-2}$	$= 0.25$
-1	$f(-1) = 2^{-1}$	$= 0.5$
0	$f(0) = 2^0$	$= 1$
1	$f(1) = 2^1$	$= 2$
2	$f(2) = 2^2$	$= 4$

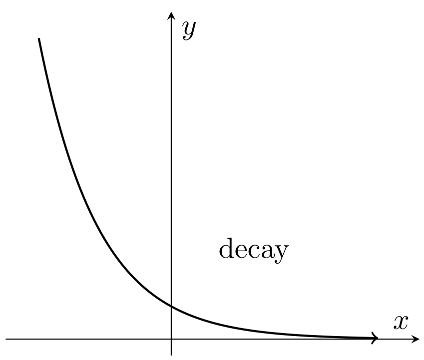
A **table of values** for the function $f(x) = 2^x$



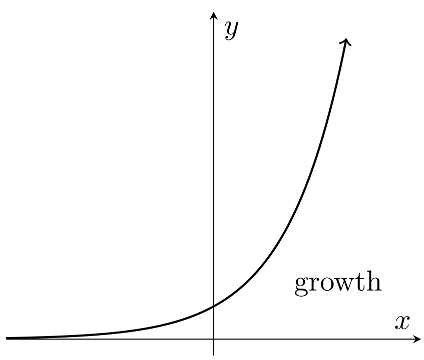
A **plot of the points** showing an exponential relationship

The Exponential Function $f(x) = a^x$

Let $y = a^x$ where $a > 0$. Here, $f(x) = a^x$ is called an *exponential* function where a is called the *base*. The domain is the set of real numbers, $\mathbb{R}$. The range is $(0, \infty)$. The x-axis forms an asymptote.



When $0 < a < 1$, $y = a^x$ 'decays' as x gets big



When $a > 1$, $y = a^x$ 'grows' as x gets big

EXAMPLE Find the equation of the exponential function that passes through $(0,1)$ and $(3,125)$.

SOLUTION An exponential function that passes through $(0,1)$ is of the form $f(x) = a^x$. As $f(3) = 125$ we substitute $x = 3$ and get

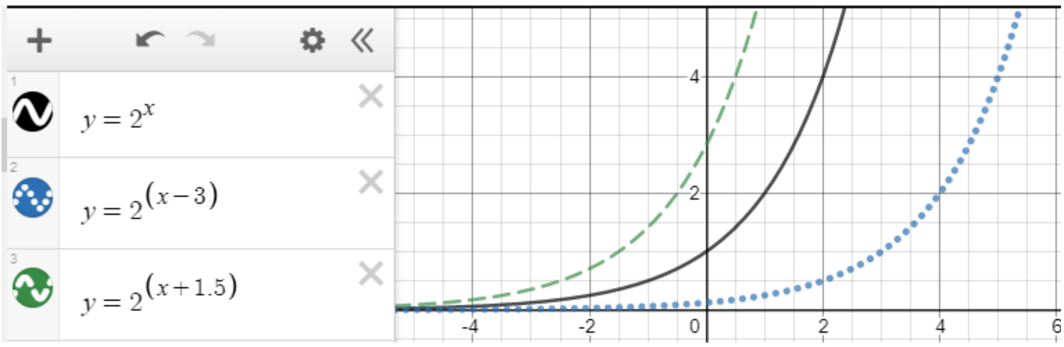
$$a^3 = 125$$
$$a = \sqrt[3]{125} = 5$$
$$\therefore f(x) = 5^x \text{ satisfies the conditions.}$$

Transformations of Exponential Functions

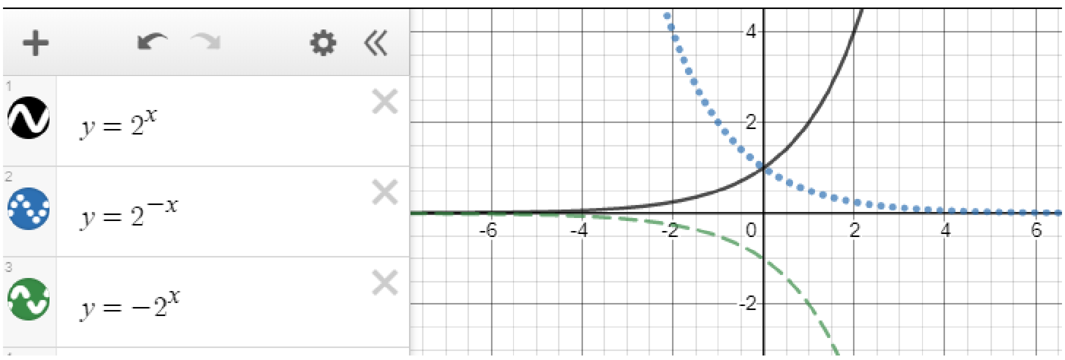
Recall the transformations we have met so far:

Vertical stretch of a	$y = f(x) \rightsquigarrow y = af(x)$
Horizontal stretch of $\frac{1}{b}$	$y = f(x) \rightsquigarrow y = f(bx)$
Vertical shift of $c \uparrow$	$y = f(x) \rightsquigarrow y = f(x) + c$
Horizontal shift of $d \longrightarrow$	$y = f(x) \rightsquigarrow y = f(x - d)$
Reflection in x-axis	$y = f(x) \rightsquigarrow y = -f(x)$
Reflection in y-axis	$y = f(x) \rightsquigarrow y = f(-x)$

Each of these transformations can be applied to an exponential function. Horizontal shifts are 'attached' to the x with brackets and opposite to convention. A shift to the right is represented by negative, e.g., $y = 2^(x - 3)$. Vertical shifts are positive to shift up and negative to shift down.



Reflections are accomplished by multiplying by negative one. To reflect in the x-axis, multiply the whole function by -1, e.g., $y = -2^x$. And for a reflection in the y-axis multiply the x variable by -1, e.g., $y = 2^{-x}$.



See the lecture notes for more examples.

EXERCISE Use Desmos to verify that $y = 2^{-x}$ and $y = \left(\frac{1}{2}\right)^x$ are equivalent.

3.1 The Natural Exponential Function

The *natural exponential function* has wide application in mathematics engineering. It arises naturally and crops up in applications such as finance, population, radioactivity, charge on a capacitor, and more. We have defined $f(x) = a^x$ and there is a particular value of a that we denote by the letter e. It is an irrational number (like π, $\sqrt{2}$ etc.) and has a button on your calculator. To 10 decimal places it is

$$e \approx 2.7182818285\ldots$$

The natural exponential function $f(x) = e^x$ is often simply referred to as *the* exponential function.

Compound interest can demonstrate an example of how the value above is found. Imagine a bank that pays 100% interest on your money. Given an initial deposit of \$1, at the end of year you will receive \$1 in interest payment and have a total of \$2. Compounded interest allows this to happen at intervals smaller than 1 year. If the interest is compounded twice per year, then after 6 months, you will receive 50% interest and have \$1.50. In the second half of the year you now have an additional \$0.50 available to earn the second half interest. Now, $\$1.50 \times 50\% = \$0.75$, so at the end of the year you have \$1.50+\$0.75=\$2.25.

Lets say the interest is compounded monthly, then after 1 month you will receive $\frac{1}{12} \times 100\% = 8.33\%$ interest for \$1+\$0.0833=\$1.0833. The second month will earn the same rate (8.33%) on \$1.0833, for a total of \$1.1736. After 12 months, your dollar will now be worth $\$1.00 \times \left(1 + \frac{1}{12}\right)^{12} = \$2.6130$.

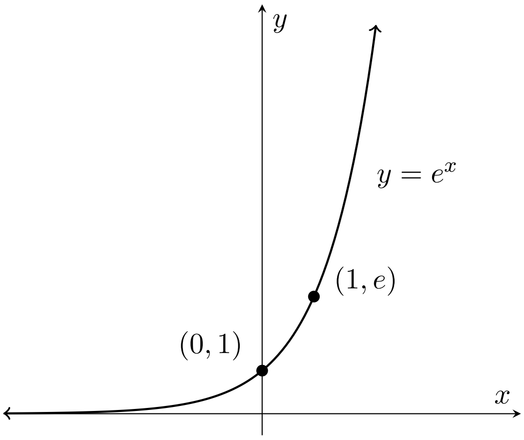
Repeat this process for shorter and shorter time intervals. What is the most money you can have? Compounding every second would get close to the maximum but there would still be gaps *between* the seconds. Closing these gaps yields a *continuous* function.

compounding periods	interest (\$)	total (\$)
1 (yearly)	1.00	2.00
2	1.25	2.25
3	1.3704	2.3704
4 (quarterly)	1.4414	2.4414
5	1.48832	2.48832
6	1.521626	2.521626
12 (monthly)	1.613035	2.613035
52 (weekly)	1.692597	2.692597
365 (daily)	1.714567	2.714567
continuous	1.718282	**2.7182818**... $=\boldsymbol{e}$

As a function, $f(x) = e^x$, is plotted. Note the similarity to $y = 2^x$ (page 41).

The exponential function, e, is naturally occurring, and useful mathematically as we will see in the next chapter on differentiation.

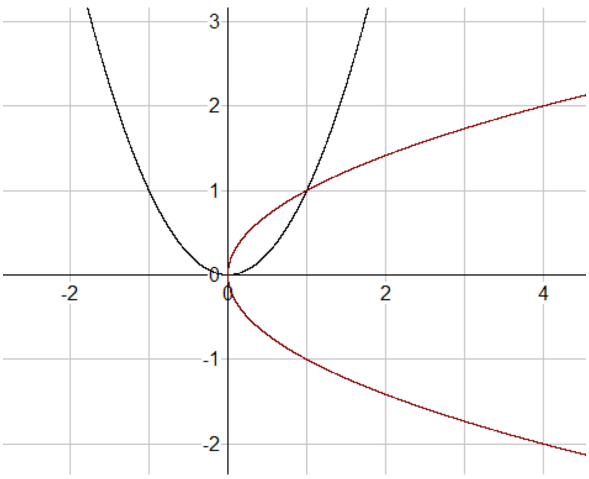
Replacing the x axis with time we can imagine the compounding benefits (or detriments) of the exponential function as time increases.

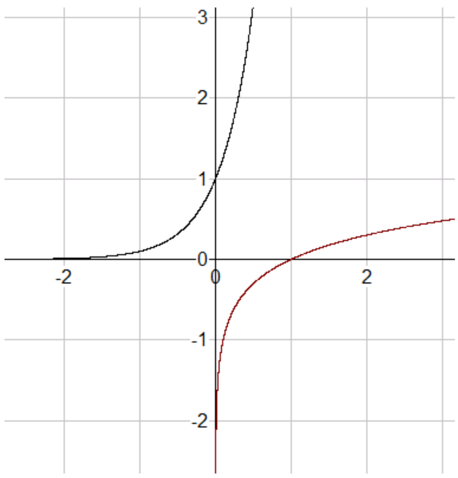


3.2 Logarithmic Functions

The *logarithmic function* is the inverse of the function $f(x) = a^x$. Recall the inverse of a function is the reflection of the function in the line $y = x$. Mathematically this is equivalent to swapping the x and the y in $y = a^x$. So $x = a^y$ is the inverse of $y = a^x$. We have another notation for the inverse of a function, which is a little more complicated. Let $y = f(x)$ be a function of x then $y = f^{-1}(x)$ is the inverse of this function.

Sometimes the inverse of the function is also a function. For example the inverse of $y = x^2$ is $x = y^2$. $y = x^2$ is a function (vertical line test always applies), whereas $x = y^2$ is not a function (vertical line test is broken).





The inverse of $y = 10^x$ is $x = 10^y$. $y = 10^x$ is a function (vertical line test always applies) and so is $x = 10^y$.

Another useful fact to remember about inverses concerns the domain and range. The *domain* of f is the *range* of f^{-1} and the *range* of f is the *domain* of f^{-1}.

We have a notation for $x = a^y$ it is $y = \log_a x$:

$$y = \log_a x \Leftrightarrow x = a^y$$

In $x = a^y$ substitute $y = \log_a x$ and we get $x = a^{\log_a x}$. This means that given a base of a the power (or exponent) to which a must be raised to get x is $\log_a x$.

Problems involving logarithms will often require us to switch back and forth between $y = \log_a x$ and $x = a^y$, however it is also helpful if you can remember to substitute for y and write $x = a^{\log_a x}$ so that you can say “the logarithm is the power”.

EXAMPLE

(a) $\log_{10} 100 = 2$ because $10^2 = 100$

(b) $\log_3 81 = 4$ because $3^4 = 81$

(c) $\log_{10} 0.01 = -2$ because $10^{-2} = 0.01$

The graph of $y = \log_a x$

The exponential function $y = a^x$ where $a > 0$ is now known and its domain is $\mathbb{R}$ and its range is the positive real numbers. We can write $\mathbb{R}^+$, meaning the positive real numbers, instead of $(0, \infty)$.

The graph of $f(x) = a^x$ can be reflected in the line $y = x$ and the result is $f^{-1}(x) = \log_a x$.

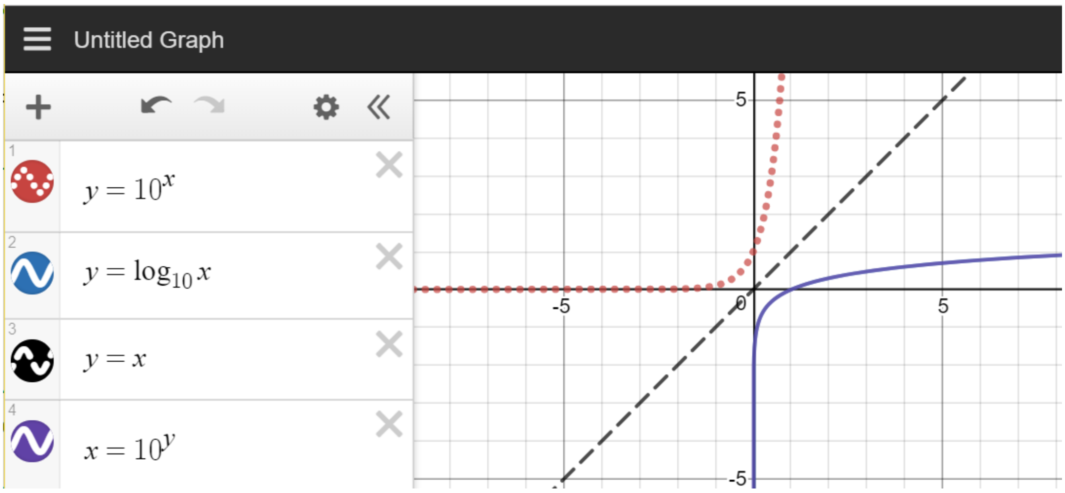
Graphing Exercise On the same set of axes draw:

(a) $y = 10^x$ **(b)** $y = \log_{10} x$ **(c)** $y = x$ **(d)** $x = 10^y$

Make a comment about each statement below.

1. Check the graphs in (a), (b) and (c) are you confident that $y = \log_{10} x$ is the reflection of $y = 10^x$ in the line $y = x$?
2. When you enter $x = 10^y$ describe what takes place.

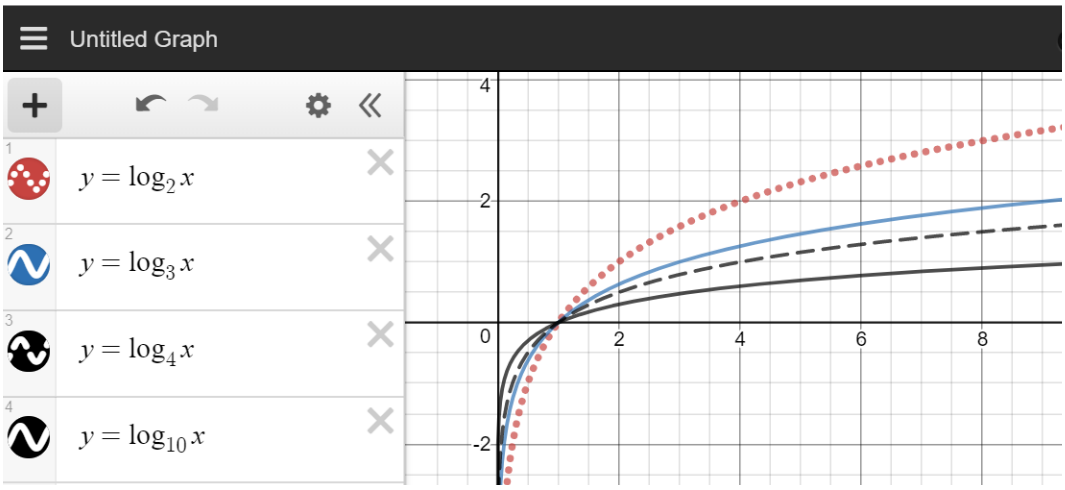
Graphing Exercise Solution Using desmos we can see the different plots. Plots (b) and (d) are equivalent, so are on top of each other.



Graphing Exercise On the same set of axes draw:

(a) $y = \log_2 x$ **(b)** $y = \log_3 x$ **(c)** $y = \log_4 x$ **(d)** $y = \log_{10} x$

Graphing Exercise Solution Using desmos we can see the relationship between the different bases in the log equation:

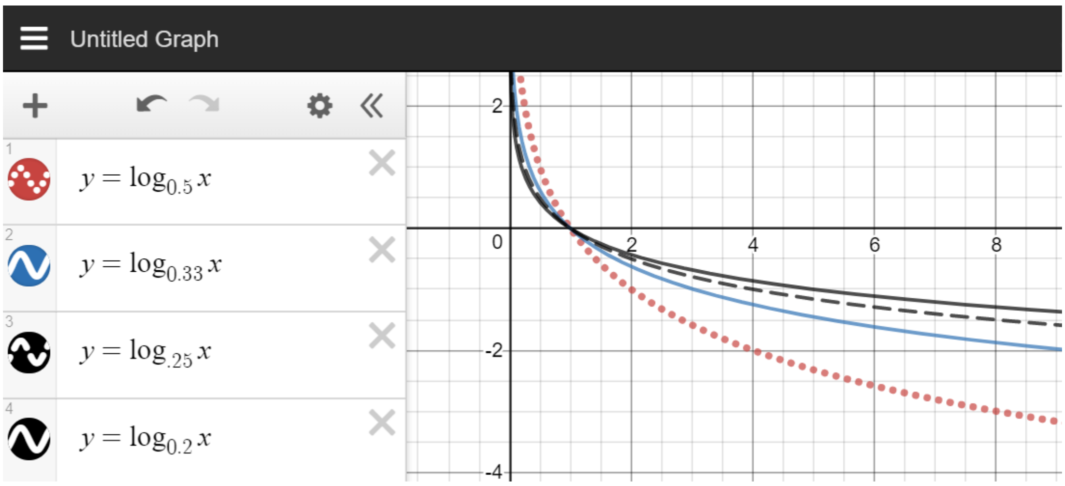


Notice the point that is common to all curves and the behaviour of the family of curves for $x > 1$ and for $0 < x < 1$.

Property 1: A property of logarithms is $\log_a 1 = 0$ and this can be seen on the graphs where every graph goes through $(1, 0)$.

The pattern you observe as the base gets bigger might not be evident for values of the base between 0 and 1. On the same set of axes draw the following:

(a) $y = \log_{\frac{1}{2}} x$ **(b)** $y = \log_{\frac{1}{3}} x$ **(c)** $y = \log_{\frac{1}{4}} x$ **(d)** $y = \log_{\frac{1}{5}} x$



To say the pattern is the same you have to be careful to describe the base. Explain how changing the base gives the same pattern as for (a) to (d).

Property 2: A second property of logarithms is $\log_a a = 1$. That is $a^1 = a$ or the power to which you have to raise a to get a is 1. On your curves above locate a point on each curve that shows this. You should in each case be looking for the point $(a, 1)$.

Property 3: A third property of logarithms is $\log_a a^x = x$. This useful property must be understood if logarithm problems are to be mastered. You should understand what $\log_a a^x$ is saying. a is the base so $\log_a a^x = x$ says "The power to which a must be raised to get a^x is x."

Base-10

When the base is 10 we write $y = \log x$ which *implies* that $y = \log_{10} x$. If you see no base you assume it is base 10. Desmos and other mathematics programs recognise "log" as being "logarithm of base 10." The log key on the calculator gives the base-10 logarithm of any positive number.

$$y = \log x \Leftrightarrow 10^y = x$$

Natural Logarithms

When the base is e we write $y = \log_e x = \ln x$. If you see $\ln x$ you can assume it is $\log_e x$. This is sometimes called the *natural* logarithm of x. Software should recognise "ln" as meaning "logarithm to the base e." The ln key on the calculator gives the natural logarithm of any positive number.

$$y = \ln x \Leftrightarrow e^y = x$$

Graphing Exercise Use Desmos to sketch the following. Describe in words how each curve is related to $y = \ln x$.

(a) $y = \ln x$ **(b)** $y = \ln(-x)$ **(c)** $y = -\ln x$

(d) $y = \ln(x - 1)$ **(e)** $y = \ln(x) - 1$ **(f)** $y = \ln(-1 - x)$

3.3 The Laws of Logarithms

The logarithmic laws are listed beside their equivalent exponent laws (see Section 1.1 Exponents) to show the symmetry between the two. The terms *logarithm* and *exponent* are very similar.

	Logarithms					Exponents		
Law 1:	$\log_x(ab)$	$=$	$\log_x(a) + \log_x(b)$			$x^a x^b$	$=$	x^{a+b}
Law 2:	$\log_x\left(\frac{a}{b}\right)$	$=$	$\log_x(a) - \log_x(b)$			$\frac{x^a}{x^b}$	$=$	x^{a-b}
Law 3:	$\log_x(a^b)$	$=$	$b \cdot \log_x(a)$			$(x^a)^b$	$=$	x^{ab}
	$\log_x\left(\frac{1}{x^a}\right)$	$=$	$-a$			x^{-a}	$=$	$\frac{1}{x^a}$
	$\log_x 1$	$=$	0			x^0	$=$	1
	$\log_x(x)$	$=$	1			x^1	$=$	x
recall:	$\log_a(x)$	$=$	y	converts to	$\Leftrightarrow$	x	$=$	a^y

For all of the above logarithms, $\log_x$ can be replaced with the natural logarithm, $\log_e$, or ln. So for example:

Law 1:	$\ln(ab)$	$=$	$\ln(a) + \ln(b)$
Law 2:	$\ln\left(\frac{a}{b}\right)$	$=$	$\ln(a) - \ln(b)$
Law 3:	$\ln(a^b)$	$=$	$b \cdot \ln(a)$

Example Expand using the logarithm laws

(a) $\log\sqrt{3} = \log 3^{\frac{1}{2}} = \frac{1}{2}\log 3$

(b) $\ln\left(\frac{a\sqrt{b}}{\sqrt[3]{c}}\right) = \ln\left(ab^{\frac{1}{2}}c^{-\frac{1}{3}}\right) = \ln a + \ln b^{\frac{1}{2}} + \ln c^{-\frac{1}{3}} = \ln a + \frac{1}{2}\ln b - \frac{1}{3}\ln c$

Example Evaluate

(a) $\log_2 112 - \log_2 7 = \log_2 \frac{112}{7} = \log_2 16 = \log_2 2^4 = 4\log_2 2 = 4$

(b) $\log_2 8^{23} = \log_2\left(2^3\right)^{23} = \log_2\left(2^{69}\right) = 69\log_2 2 = 69$

(c) $\log\sqrt{0.001} = \log(0.001)^{\frac{1}{2}} = \frac{1}{2}\log 0.001 = \frac{1}{2}\log 10^{-3} = \frac{1}{2}\times -3 = -\frac{3}{2} = -1\frac{1}{2}$

(d) $e^{2\ln 4} = \left(e^{\ln 4}\right)^2 = 4^2 = 16$ (Let $\ln 4 = x$ then $e^x = 4$ so $e^{\ln x} = 4$)

Example Rewrite as a single logarithm term using the logarithm laws

(a) $\log 12 + \frac{1}{2}\log 5 - \log 3 = \log\frac{12\sqrt{5}}{3} = \log 4\sqrt{5}$

(b) $\log_3\left(x^2-1\right) - \log_3(x-1) = \log_3\frac{x^2-1}{x-1} = \log_3\frac{(x+1)(x-1)}{x-1} = \log_3(x+1)$

3.4 Exponential and Logarithmic Equations

The types of problems we meet in this section will be able to be rearranged so that they look like

$$a^{f(x)} = b$$

Where a and b are real numbers, x is the unknown variable we are trying to find and $f(x)$ is an expression in x. The technique we will use will be the same for every problem we solve.

Step 1 Our first step is to inspect the problem to see if the unknown variable is in the exponent.

Step 2 Now that we have established that we are solving an exponential equation we rearrange it until it is in the form $a^{f(x)} = b$

Step 3 Take the logarithm of both sides. In most practical situations we either take logarithms to the base 10 or logarithms to the base e. there are three situations

Case 1 The problem has reduced to $10^{f(x)} = b$. Take logarithms to the base 10.

Case 2 The problem has reduced to $e^{f(x)} = b$. Take logarithms to the base e.

Case 3 The problem has reduced to $a^{f(x)} = b$ where a is neither 10 nor e. You can take logarithms to the base 10 or e as you wish, either is correct.

Step 4 Solve the equation you obtain.

Example 1 Solve $3^x = 5$. This is an example of case 3 above.

SOLUTION Two methods will be explored below:

Method 1: Take logarithms to the base 10

$$\begin{aligned} \log 3^x &= \log 5 \\ x \log 3 &= \log 5 \\ x &= \frac{\log 5}{\log 3} \\ &\approx 1.46497 \end{aligned}$$

Method 2: Take logarithms to the base e

$$\begin{aligned} \ln 3^x &= \ln 5 \\ x \ln 3 &= \ln 5 \\ x &= \frac{\ln 5}{\ln 3} \\ &\approx 1.46497 \end{aligned}$$

This shows the same result whether you take logarithms to base 10 or logarithms to base e.

Example 2 Solve $3^{2x+1} = 5$

SOLUTION Take logarithms to base 10:

$$\begin{aligned} \log 3^{2x+1} &= \log 5 \\ (2x+1) \log 3 &= \log 5 \\ 2x + 1 &= \frac{\log 5}{\log 3} \\ 2x &= \frac{\log 5}{\log 3} - 1 \\ x &= \frac{1}{2}\left(\frac{\log 5}{\log 3} - 1\right) \\ &\approx 0.2325 \end{aligned}$$

Example 3 Solve $4\left(1 + 10^{5x}\right) = 9$

SOLUTION This must first be rearranged:

$$\begin{aligned} 1 + 10^{5x} &= \frac{9}{4} = 2.25 \\ 10^{5x} &= 2.25 - 1 = 1.25 \\ \log 10^{5x} &= \log 1.25 \\ 5x &= \log 1.25 \\ x &= \frac{1}{5} \log 1.25 \\ &\approx 0.01938 \end{aligned}$$

Example 4 Solve

$$\frac{10}{1+e^{-x}} = 3$$

SOLUTION

$$\begin{aligned} 1 + e^{-x} &= \frac{10}{3} \\ e^{-x} &= \frac{10}{3} - 1 \\ &= \frac{10-3}{3} = \frac{7}{3} \\ -x &= \ln 7 - \ln 3 \\ x &\approx -0.85 \text{ (2 dp)} \end{aligned}$$

Solving Logarithmic Equations

Whereas exponential equations have the unknown variable in an exponent, logarithmic equations are equations containing the logarithm of an unknown variable.

Step 1 Recognise you are dealing with a logarithmic equation by inspecting the problem to see if you have a logarithm of a term containing the unknown variable.

Step 2 Rearrange the equation until it is in the form $\log_a (\text{unknown}) = b$.

Step 3 Take 'antilogarithms.' That is, use the conversion: $\log_a x = b$ then $a^b = x$.

Step 4 Solve the equation for the unknown variable.

Example 1 Solve $\ln x = 5$

SOLUTION Take antilogarithms with base e:

$$\begin{aligned} e^5 &= x \\ x &\approx 148.4131 \end{aligned}$$

Example 2 Solve $5 + 4\log(5x) = 17$

SOLUTION Isolate the log and solve:

$$\begin{aligned} 4\log(5x) &= 17 - 5 = 12 \\ \log 5x &= 3 \\ 10^3 &= 5x \\ x &= \frac{1000}{5} = 200 \end{aligned}$$

Check: Substitute $x = 200$

$$\begin{aligned} \text{LHS} &= 5 + 4\log(5 \times 200) \\ &= 5 + 4\log 1000 = 5 + 4\log 10^3 \\ &= 5 + 4 \times 3 = 5 + 12 = 17 = \text{RHS} \end{aligned}$$

Example 3 Solve $\log x + \log(x-1) = \log 4x$
Solution

$$\begin{aligned}\log x + \log(x-1) - \log 4x &= 0\\ \log\frac{x(x-1)}{4x} &= 0\\ \log\frac{1}{4}(x-1) &= 0\\ 10^0 = \frac{1}{4}(x-1) &= 1\\ x-1 = 4;\ x &= 5\end{aligned}$$

Check:

$$\begin{aligned}\text{LHS} &= \log 5 + \log 4 = \log(5\times 4) = \log 20\\ \text{RHS} &= \log 4\times 5 = \log 20 = \text{LHS}\end{aligned}$$

Example 4 The velocity of a sky diver t seconds after jumping is given by

$$v(t) = 80\left(1 - e^{-0.2t}\right)$$

After how many seconds is the velocity 70 ft/s?
Solution

$$\begin{aligned}70 &= 80\left(1 - e^{-0.2t}\right)\\ \frac{70}{80} &= 1 - e^{-0.2t}\\ e^{-0.2t} &= 1 - \frac{70}{80}\\ &= 1 - \frac{7}{8} = \frac{1}{8} = 0.125\\ -0.2t &= \ln 0.125\\ t &= \frac{\ln 0.125}{-0.2} \approx 10.39\text{s}\end{aligned}$$

3.5 Exponential Modelling

The natural exponential function, e^x, is used in a variety of situations where there is exponential growth or decay.

Example The exponential function can be used to model the way populations grow and diseases spread. The following example is about the spread of an infectious disease in a small city whose population is 10,000. After t days the number of people who have caught the disease is modelled by the function

$$f(t) = \frac{10000}{5 + 2495e^{-0.84t}}$$

(a) How many people had the disease initially? *Initially* in this sense means right at the beginning, or when the clock is still at zero. Substitute $t = 0$ into the function:

$$f(0) = \frac{10000}{5 + 2495e^0} = \frac{10000}{2500} = 4 \text{ people}$$

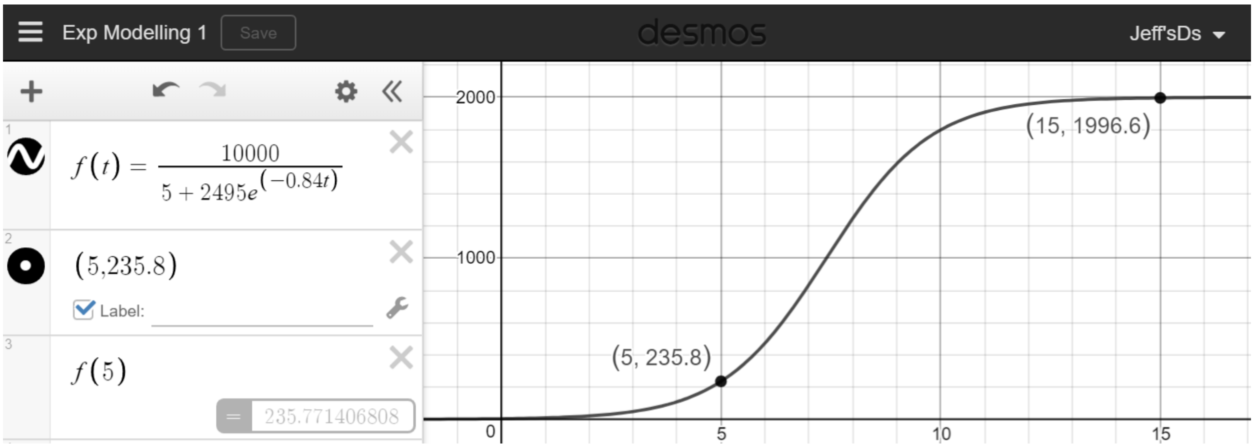
(b) How many people have the disease after 1 day? Substitute $t = 1$ into the function:

$$f(1) = \frac{10000}{5 + 2495e^{-0.84}} = \frac{10000}{5 + 2495(0.4317)} = \frac{10000}{1082.1} \approx 9.24;\ 9 \text{ people}$$

(c) How many people have the disease after 5 days? Following from part (b), find $f(5)$:

$$f(5) = \frac{10000}{5 + 2495e^{-0.84(5)}} = \frac{10000}{42.41} \approx 235.77;\ 236 \text{ people}$$

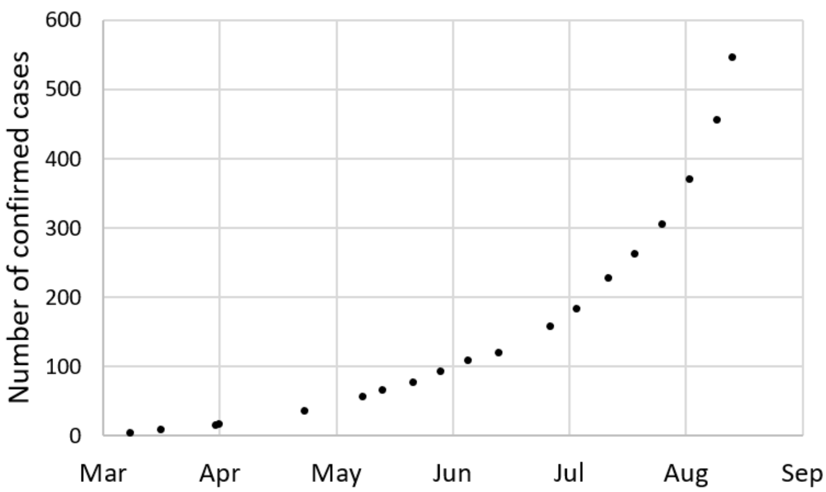
(d) Use desmos to graph the function and describe its behaviour.



The graph has distinctive characteristics. It starts at a particular non-zero value (when $t = 0$) and increases slowly at first then more rapidly. It slows down after a time and levels off because the exponential function in the denominator $\rightsquigarrow 0$ when $t \rightsquigarrow \infty$. Graphs with these characteristics are called logistic curves. The particular model is called a logistic growth model.

EXAM QUESTION

In the winter of 2019 the number of people contracting the measles virus grew substantially in the Auckland region. The plot shows the number of confirmed cases with data indicating 546 cases as of August 20, 2019.



(a) What type of mathematical model (function) would you expect to fit the data?

This data most closely resembles an exponential function.

(b) Using Excel, a mathematical formula has been fit to the data: $f(t) = 8.06e^{0.0272t}$ where t is measured in days. How many cases were originally reported?

Sub in $t = 0$ to find the initial number of cases in the data.

$$f(0) = 8.06e^0 = 8.06 \text{ cases}$$

Remember the model is an approximation of the data so it is likely that there were 8 cases to start with.

(c) How many days have passed between the initial data point and August 20?

Solve for t knowing that $f(t) = 546$

$$\begin{aligned} 546 &= 8.06e^{0.0272t} \\ \frac{546}{8.06} &= e^{0.0272t} \\ \ln\left(\frac{546}{8.06}\right) &= 0.0272t \\ t &= \frac{4.2157}{0.0272} = 154.989 \approx 155 \text{ days} \end{aligned}$$

3.6 Chapter Exercises

§3.1 e^x functions

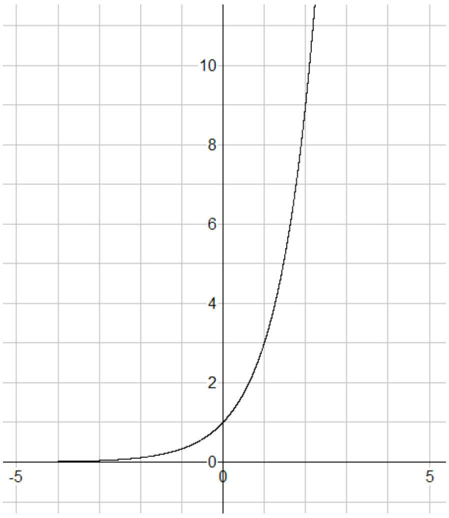
1. Use your calculator to evaluate to 5dp

(a) e^4 **(b)** $2e^{-0.7}$ **(c)** $e^{3.1}$ **(d)** e^e

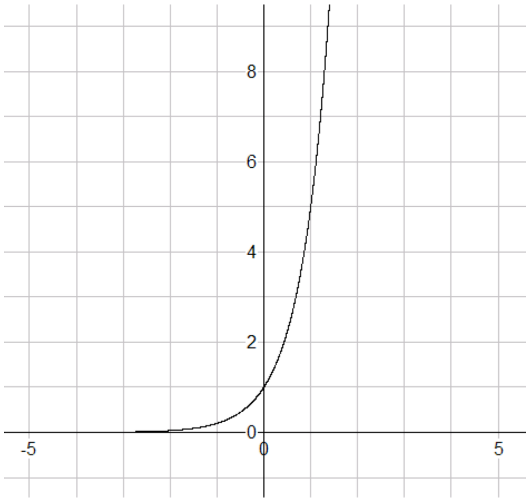
2. Use Desmos to sketch:

(a) $f(x) = e^{-x}$ **(b)** $g(x) = 2e^{0.1x}$ **(c)** $h(x) = -2.1e^{-0.12x}$

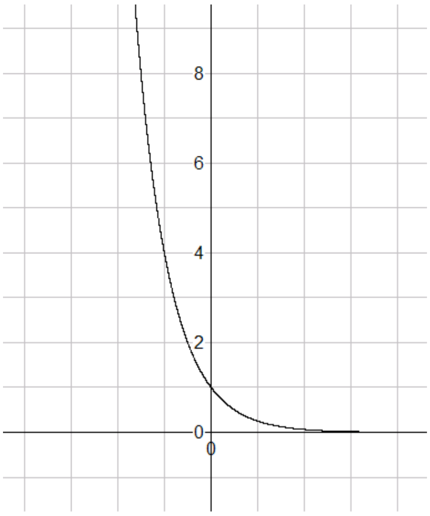
3. Find the exponential function $f(x) = a^x$ whose graph is given.



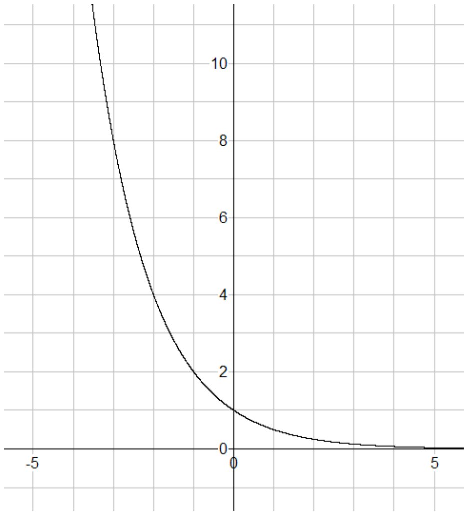
(a)



(b)



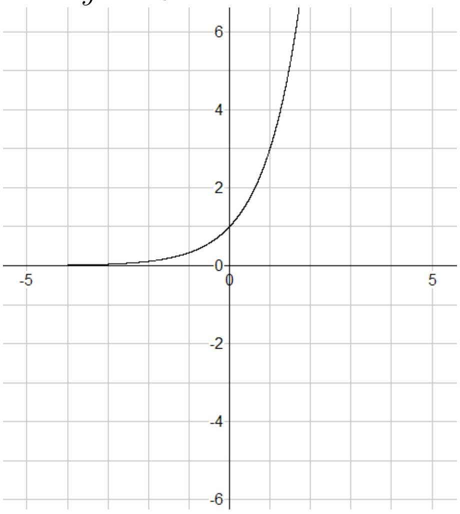
(c)



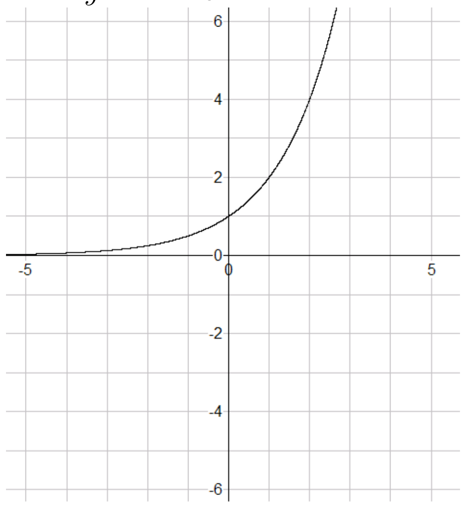
(d)

4. Sketch the transformed graph

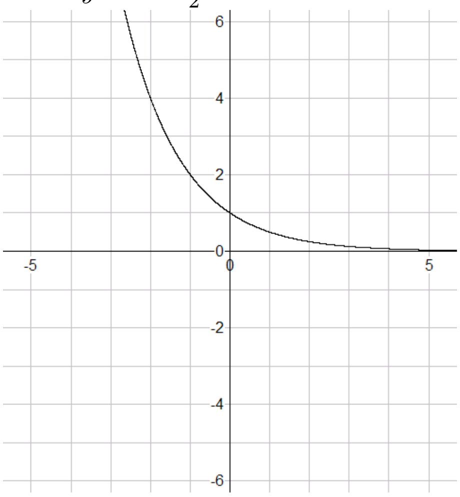
(a) The graph is $y = 3^x$.
Draw $y = -3^x$.



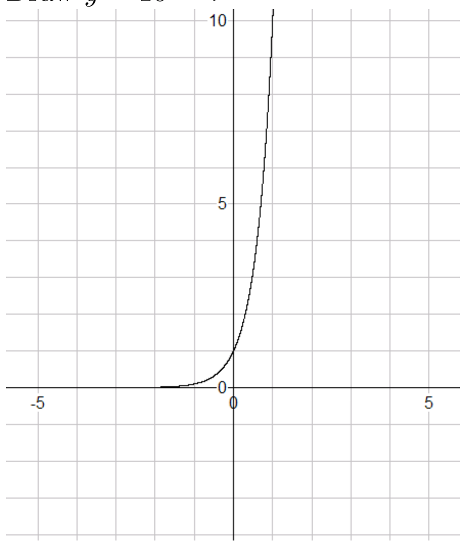
(b) The graph is $y = 2^x$.
Draw $y = 2^x - 3$.



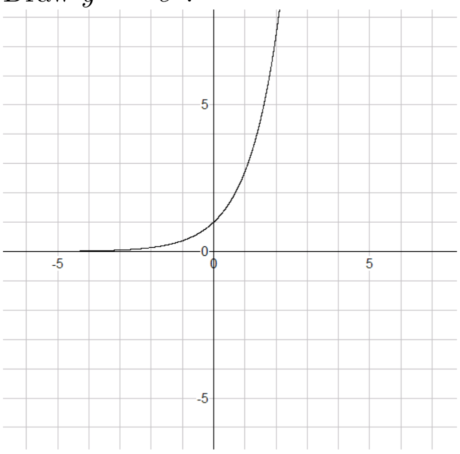
(c) The graph is $y = \frac{1}{2}^x$.
Draw $y = 4 + \frac{1}{2}^x$.



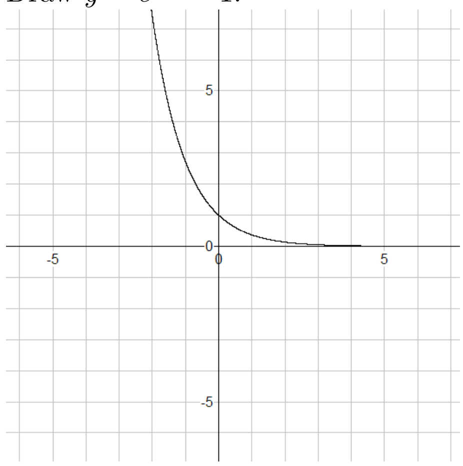
(d) The graph is $y = 10^x$.
Draw $y = 10^{x+3}$.



(e) The graph is $y = e^x$.
Draw $y = -e^x$.



(f) The graph is $y = e^{-x}$.
Draw $y = e^{-x} - 1$.



§3.2 Logarithmic Functions

1. Express the equation in exponential form.

(**a**) $\log_5 25 = 2$ (**b**) $\log_5 1 = 0$ (**c**) $\log_8 2 = \frac{1}{3}$

(**d**) $\log_2\left(\frac{1}{8}\right) = -3$ (**e**) $\ln 5 = x$ (**f**) $\ln y = 5$

2. Express the equation in logarithmic form

(**a**) $5^3 = 125$ (**b**) $10^{-4} = 0.0001$ (**c**) $8^{-1} = \frac{1}{8}$

(**d**) $2^{-3} = \frac{1}{8}$ (**e**) $e^x = 2$ (**f**) $e^3 = y$

3. Evaluate the expression

(**a**) $\log_3 3$ (**b**) $\log_3 1$ (**c**) $\log_3 3^2$

(**d**) $\log_6 36$ (**e**) $\log_9 81$ (**f**) $\log_7 7^{10}$

(**g**) $\log_3\left(\frac{1}{27}\right)$ (**h**) $\log_{10}\sqrt{10}$ (**i**) $\log_5 0.2$

(**j**) $2^{\log_2 37}$ (**k**) $3^{\log_3 8}$ (**l**) $e^{\ln\sqrt{5}}$

(**m**) $\log_8 0.25$ (**n**) $\ln e^4$ (**o**) $\ln(1/e)$

4. Use the definition of the logarithmic function to find x.

(**a**) $\log_2 x = 5$ (**b**) $\log_2 16 = x$ (**c**) $\log_3 243 = x$

(**d**) $\log_{10} x = 2$ (**e**) $\log_x 16 = 4$ (**f**) $\log_x 8 = \frac{3}{2}$

5. Use the Laws of Logarithms to rewrite the expression in a form with no logarithms of products, quotients roots or powers.

(**a**) $\log_2(2x)$ (**b**) $\log_2(x(x-1))$ (**c**) $\log 6^{10}$

(**d**) $\log_2\left(AB^2\right)$ (**e**) $\log_3\left(x\sqrt{y}\right)$ (**f**) $\log_5\sqrt[3]{x^2+1}$

(**g**) $\ln\sqrt{ab}$ (**h**) $\ln\left(x\sqrt{\frac{y}{z}}\right)$ (**i**) $\log\sqrt[4]{x^2+y^2}$

6. Evaluate the expressions

(**a**) $\log_5\sqrt{125}$ (**b**) $\log 2 + \log 5$ (**c**) $\log_4 192 - \log_4 3$

(**d**) $\ln 6 - \ln 15 + \ln 20$ (**e**) $10^{2\log 4}$ (**f**) $\log\left(\log 1000^{10,000}\right)$

7. Rewrite the expression as a single logarithm

(**a**) $\log_3 5 + 5\log_3 2$

(**b**) $\log_2 A + \log_2 B - 2\log_2 C$

(**c**) $\ln 5 + 2\ln x + 3\ln\left(x^2+5\right)$

§3.3 Logarithmic Equations

1. Find the solution of the exponential equation, correct to four decimal places.

(a) $e^x = 16$ **(b)** $10^{2x} = 5$ **(c)** $3e^x = 10$

(d) $e^{1-4x} = 2$ **(e)** $8^{0.4x} = 5$ **(f)** $5^{-x/100} = 2$

(g) $5^x = 4^{x+1}$ **(h)** $2^{3x+1} = 3^{x-2}$ **(i)** $100\,(1.04)^{2t} = 300$

2. Solve the equations for x

(a) $x^2 2^x - 2^x = 0$ **(b)** $4x^3e^{-3x} - 3x^4e^{-3x} = 0$

(c) $e^{4x} + 4e^{2x} - 21 = 0$ **(d)** $\ln x = 10$

(e) $\log(3x+5) = 2$ **(f)** $2 - \ln(3-x) = 0$

(g) $\log x + \log(x-1) = \log(4x)$ **(h)** $\log_5(x+1) - \log_5(x-1) = 2$

§3.4 Modelling

1. A radioactive substance decays in such a way that the amount of mass remaining after t days is given by the function

$$m(t) = 13e^{-0.015t}$$

Where $m(t)$ is measured in kilograms.

(a) Find the mass at time $t = 0$.

(b) How much of the mass remains after 45 days?

2. A sky diver jumps from a reasonable height above the ground. The air resistance she experiences is proportional to her velocity, and the constant of proportionality is 0.2. It can be shown that the downward velocity of the sky diver at time t is given by

$$v(t) = 80\left(1 - e^{-0.2t}\right)$$

where t is measured in seconds and $v(t)$ is measured in feet per second (ft/s).

(a) Find the initial velocity of the sky diver.

(b) Find the velocity after 5s and after 10s.

(c) Draw a graph of the velocity function $v(t)$.

(d) The maximum velocity of a falling object with wind resistance is called the *terminal velocity*. From the graph in part (c) find the terminal velocity of the sky diver.

3. The population of a certain species of bird is limited by the type of habitat required for nesting. The population behaves according to the *logistic growth model*

$$n(t) = \frac{5600}{0.5 + 27.5e^{-0.044t}}$$

where t is measured in years.

(a) Find the initial bird population.

(b) Draw a graph of the function $n(t)$.

(c) What size does the population approach as time goes on?

4. A 15g sample of radioactive iodine decays in such a way that the mass remaining after t days is given by $m(t) = 15e^{-0.087t}$ where $m(t)$ is measured in grams. After how many days is there only 5g remaining?

5. A small lake is stocked with a certain species of fish. The fish population is modelled by the function

$$P = \frac{10}{1 + 4e^{-0.8t}}$$

where P is the number of fish in thousands and t is measured in years since the lake was stocked.

(a) Find the fish population after 3 years.

(b) After how many years will the fish population reach 5000 fish?

4 | Differentiation

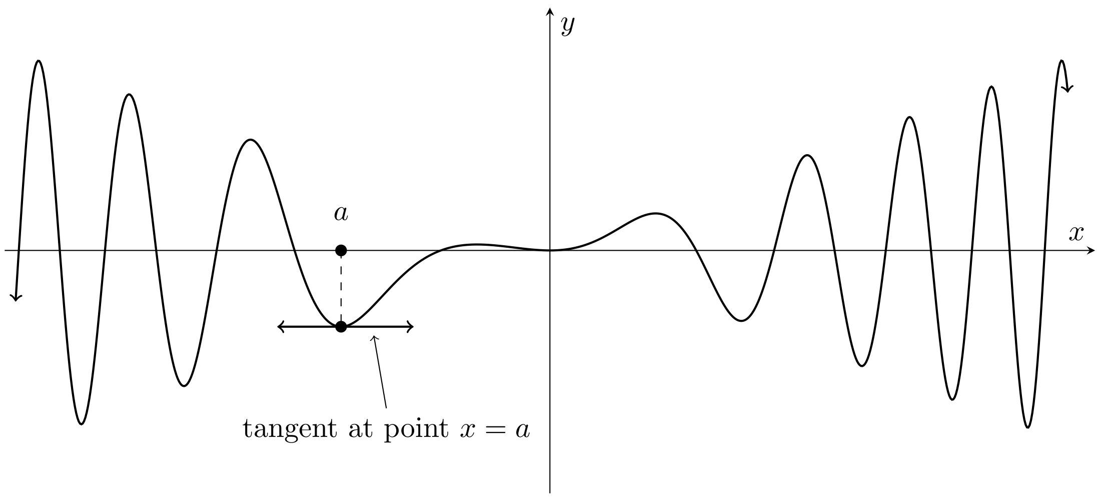


Figure 4.1: The function is smooth everywhere; there are no gaps or sharp transitions, and therefore it has a well defined tangent line at any point. One of these tangents is shown at $x = a$. The function that defines *all* possible tangents is called the derivative.

Before sharpening our pencil consider the Prologue to a book titled "*Calculus Made Easy*[1]" by Silvanus P. Thompson published in 1910:

> Considering how many fools can calculate, it is surprising that it should be thought either a difficult or a tedious task for any other fool to learn how to master the same tricks.
>
> Some calculus-tricks are quite easy. Some are enormously difficult. The fools who write the textbooks of advanced mathematics — and they are mostly clever fools — seldom take the trouble to show you how easy the easy calculations are. On the contrary, they seem to desire to impress you with their tremendous cleverness by going about it in the most difficult way.
>
> Being myself a remarkably stupid fellow, I have had to unteach myself the difficulties, and now beg to present to my fellow fools the parts that are not hard. Master these thoroughly, and the rest will follow. What one fool can do, another can.

[1] The full and verbose title: Calculus Made Easy: Being a Very-Simplest Introduction to Those Beautiful Methods of Reckoning Which are Generally Called by the Terrifying Names of the Differential Calculus and the Integral Calculus.

Thompson goes on to describe the differential operator, ‘d’:

> The preliminary terror, which chokes off most fifth-form boys from even attempting to learn how to calculate, can be abolished once for all by simply stating what is the meaning–in common-sense terms–of the two principal symbols that are used in calculating.
>
> These dreadful symbols are:
>
> (1) **d** which merely means “a little bit of.”
>
> Thus dx means a little bit of x; or du means a little bit of u. Ordinary mathematicians think it more polite to say “an element of,” instead of “a little bit of.” Just as you please. But you will find that these little bits (or elements) may be considered to be indefinitely small.

For the second symbol, the integrand, $\int$, you will have to jump to the next Chapter.

Rate of Change

We will begin with the concept of average speed. If you travel a distance of 120 km in 2 hours then your average speed is 60 kph.

$$\text{Average speed} = \frac{\text{distance travelled}}{\text{time elapsed}}$$

A distance/time graph can be drawn. The average speed can be expressed using function notation

$$\text{Average speed} = \frac{s(b) - s(a)}{b - a}$$

Finding the average rate of change is important in many contexts and in fact the average rate of change can be defined for any function.

$$\text{The average rate of change of the function } y = f(x) \text{ is } \frac{\text{change in } y}{\text{change in } x} \text{ or } \frac{f(b) - f(a)}{b - a} \tag{1}$$

The average rate of change is the slope of the **secant line** between $x = a$ and $x = b$ on the graph of f, that is the slope of the line that passes through $(a, f(a))$ and $(b, f(b))$.

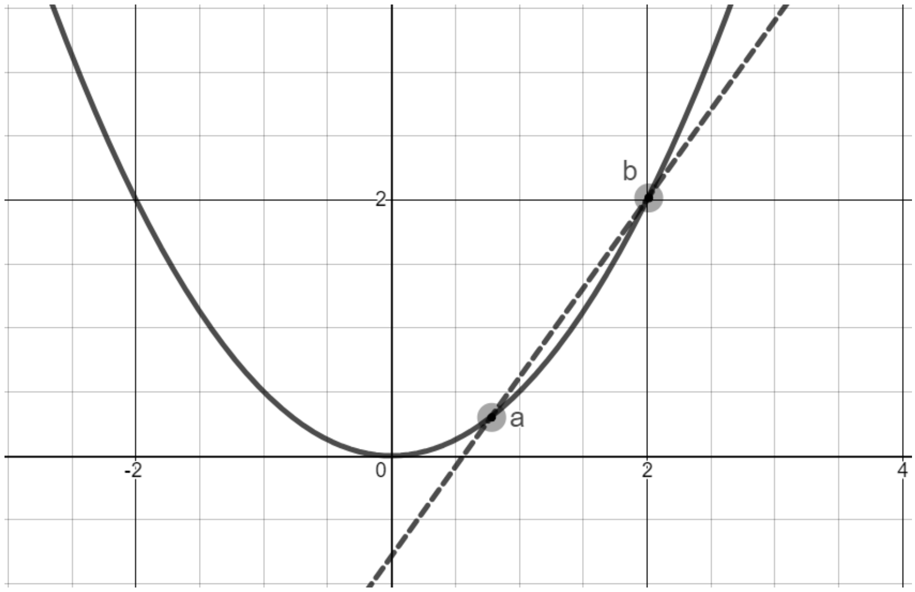


Figure 4.2: As the two points come closer together the secant line ab gets shorter. When b is at a the secant line becomes a tangent line. See this animated with Desmos (check the link on blackboard).

EXAMPLE Calculate the average rate of change for the function $f(x) = x^2 + 4$ between the following points:

(a) $x = 2$ and $x = 6$

SOLUTION

Using the function notation in (1) above,

$$\frac{f(2) - f(6)}{2 - 6} = \frac{8 - 40}{-4} = 8$$

(b) $x = 5$ and $x = 10$

SOLUTION

$$\frac{f(5) - f(10)}{5 - 10} = \frac{29 - 104}{-5}$$
$$= 15$$

(c) $x = a$ and $x = a + h$ $(h \neq 0)$

SOLUTION

$$\frac{f(a) - f(a+h)}{a - (a+h)}$$
$$= \frac{a^2 + 4 - ([a+h]^2 + 4)}{-h}$$

Can this be simplified further?

Tangents

We now investigate the process of changing the value of $(b - a)$ in formula (1) the slope of the secant approaches the slope of the tangent at $x = a$. However, as $b - a$ is made smaller and smaller, eventually it will get to zero. This is necessary to find the exact tangent, except now the denominator of equation (1) is zero! A new definition can help us avoid this *math error*.

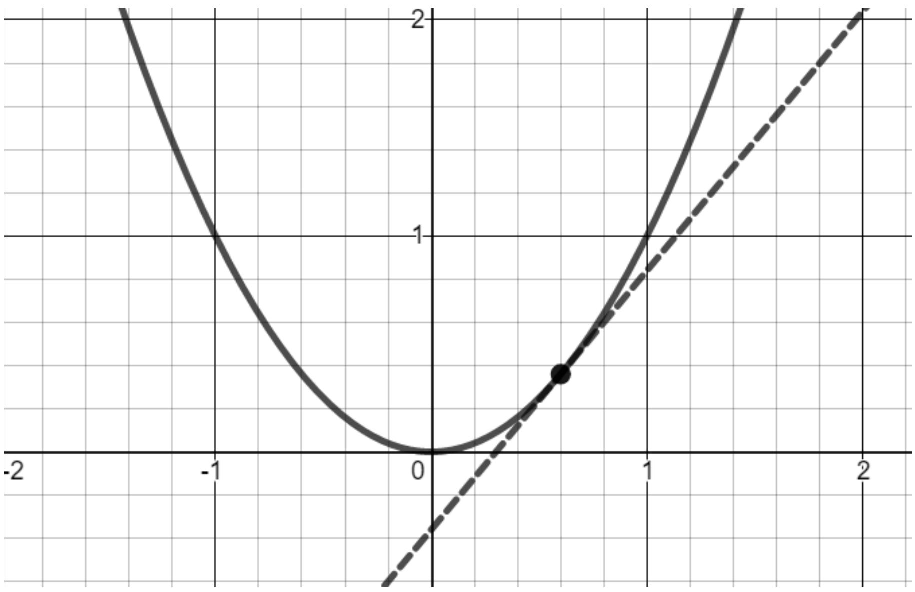


Figure 4.3: A tangent can only touch the curve once. The slope of the tangent is perpendicular to the radius at the tangent point. This slope is called the rate of change or *derivative* of the function at the point. See this animated with Desmos (check the link on blackboard).

Definition: The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that the *limit* exists. $\lim\limits_{x \to a}$ is the "limit as x approaches a."

This means that as the value of x gets close to a the function remains smooth. Imagine zooming in on a function, from far away it may appear smooth, but up close it could have some gaps or discontinuities. Limits do not exist at sharp transitions in a graph, or where the function does not exist (think of piecewise functions).

We sometimes refer to the slope of the tangent line to a curve at a point as the slope of the curve at that point. The idea is that if we zoom in far enough towards the point then the curve looks almost like a straight line. The more we zoom in the more the parabola looks like a straight line.

Using function notation for the tangent line is usually easier to use and is often preferred. The slope of the secant line between $x = a$ and $x = a + h$ is $\dfrac{f(a+h) - f(a)}{h}$. This looks familiar from EXAMPLE 3 above. We can now use this as a definition:

$$\text{slope } = m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This is limit notation, $\lim\limits_{h \to 0}$, and we would say 'the limit as h approaches 0'. Note that if $h = 0$ the function is now undefined (math error). So h is allowed to get close to zero, but not actually equal zero.

4.1 Derivatives from 1st Principles

Because the expression $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ occurs so widely it is given a special name and notation.

The derivative of a function at a number a, denoted by $f'(a)$ is

Definition of the derivative: $$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

The process of finding the derivative using the above definition is called finding the derivative *from first principles*. So far we have found a derivative of a function f at a fixed number a. If we replace a in this equation with a variable x we obtain

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$ *Note the difference from above

given any number x for which this limit exists. We assign to x the number $f'(x)$. So we can regard f' as a new function which we call the derived function or the derivative of x.

Derivatives of Polynomial Functions

The constant function $f(x) = c$ is considered a polynomial of degree zero. Using the method of first principles we can find the derivative as follows:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0$$

Higher Power Polynomials

When $f(x) = x$ it can be shown from first principles that $f'(x) = 1$. Similarly when $f(x) = x^2$it can be shown that $f'(x) = 2x$ and when $f(x) = x^3$ it can be shown that $f'(x) = 3x^2$.

EXAMPLE Find $f'(x)$ for $f(x) = x^4$ from first principles.

SOLUTION

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h} \\
&= \lim_{h\to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} \\
&= \lim_{h\to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} \\
&= \lim_{h\to 0} (4x^3 + 6x^2h + 4xh^2 + h^3) \text{ here, substitute } h = 0 \\
f'(x) &= 4x^3
\end{aligned}$$

This pattern will follow for any similar polynomial: If $f(x) = x^n$ then $f'(x) = nx^{n-1}$. Or alternatively

The Power Rule for Differentiation: $\dfrac{d}{dx}(x^n) = nx^{n-1}$

This pattern implies that n must be a positive integer. It can be shown that from the definition of a derivative $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$ or $y = x^{-1}$ then $\frac{dy}{dx} = -1 \times x^{-2}$, which proves the power rule for $n = -1$.

Similarly if the exponent is a fraction it can be shown that the power rule holds e.g. if $f(x) = \sqrt{x}$ then $f'(x) = \frac{1}{2\sqrt{x}}$ or $f(x) = x^{\frac{1}{2}}$ then $f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$. It can be shown that the power rule holds for any real number n.

The Natural Exponential Function

Recall the natural exponential function from section 3.1. Here we can see why precisely it is so special. Using limit notation, we can say that e is the number such that $\lim\limits_{h\to 0} \frac{e^h - 1}{h} = 1$. The derivative is:

The Derivative of the Natural Exponential Function

$$\frac{d}{dx}\left(e^x\right) = e^x$$

The natural exponential function is unique because **it has its own derivative!** Geometrically this means that the slope of the tangent at any point is the same as the y-coordinate, $f(x)$, of that point.

4.2 Standard Derivatives

The basic functions have easily repeatable patterns to find their derivatives. The common ones are summarized in the table below:

Function $f(x)$	Derivative $f'(x)$	Notes
A	0	A is constant
x	1	power rule for x^1
Ax	A	A is a constant multiple
x^n	nx^{n-1}	power rule - general form
e^x	e^x	exponential
$\ln(x)$	$\frac{1}{x}$	logarithmic
$\sin(x)$	$\cos(x)$	trigonometric
$\cos(x)$	$-\sin(x)$	
$\tan(x)$	$\sec^2(x)$	

EXAMPLE Find the following derivatives:

(a) $f(x) = 7$
7 is constant (no variables: 'x'), so $f'(x) = 0$

(b) $f(x) = 7x$
7 is a constant multiple: $f'(x) = 7$

(c) $f = 7x^2$
using the power rule: $f' = 7(2)x^1 = 14x$

(d) $f(x) = \pi x^{-3}$
pi is a constant: $f'(x) = -3\pi x^{-4}$

(e) $g(x) = x^{\frac{3}{4}}$
use the power rule for fractional exponents: $g'(x) = \dfrac{3}{4}x^{-\frac{1}{4}}$

(f) $f(x) = \dfrac{e^x}{3}$
the exponential always has its own derivative, $\frac{1}{3}$ is a constant: $f'(x) = \frac{1}{3}e^x$

(g) $s(t) = 2\sin t$
2 is a constant: $s'(t) = 2\cos t$

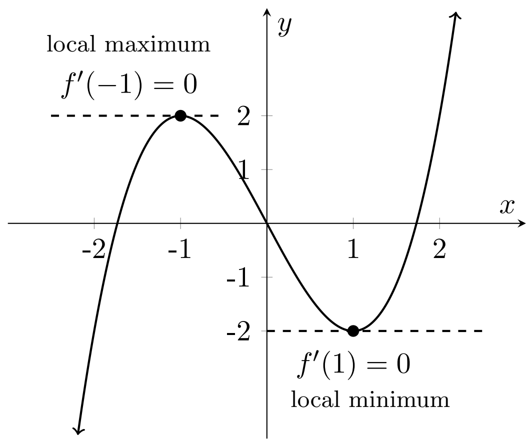
(h) $f = \dfrac{\cos r}{3}$
$\frac{1}{3}$ is a constant: $f' = -\frac{1}{3}\sin r$

(i) $f(x) = \tan(\pi x)$
this requires the Chain Rule (4.4): $f'(x) = \pi\sec^2(\pi x)$

4.3 Maximums, Minimums, and Tangents

Turning points occur at the boundary between regions in a function. A function that transitions from increasing to decreasing must have a turning point. In the function below, $f(-1) = 2$ means the point $(-1, 2)$ is a turning point. More specifically this is a local maximum of our function. It is the highest point in its *local* neighbourhood. Notice that the other boundary has different properties: it is the lowest point. So $f(1) = -2$ means the point $(1, -2)$ is a local minimum.

The boundary means that the slope of the *tangent* has gone from increasing to decreasing. Or from positive to negative. During this transition it had to go through zero, and so a turning point can be found anywhere the slope is equal to zero: $f'(x) = 0$. Remember that the first derivative represents the slope of the function.



EXAMPLE Find the turning points for

$$f(x) = x^3 - 3x$$

SOLUTION Set the derivative equal to zero and solve for values of x.

$$\begin{aligned} \frac{dy}{dx} &= 3x^2 - 3 \\ 0 &= 3x^2 - 3 \\ 3 &= 3x^2 \\ 1 &= x^2 \\ x &= 1, \text{and } x = -1 \end{aligned}$$

Therefore the $x-$values of the turning points are 1 and -1. Note there are two turning points in our function so there should be two corresponding solutions.

$$\begin{aligned} f(1) &= (1)^3 - 3(1) \\ &= 1 - 3 = -2 \end{aligned}$$

Therefore $(1, -2)$ is the first turning point. Similarly, $(-1, 2)$ is found as the other turning point. These are both shown in the figure above.

EXAM QUESTION Given the cubic function $f(x) = 2x^3 - 3x^2 - 12x + 1$, find **(a)** the equation of the tangent line to $f(x)$ at the point $x = -2$, and **(b)** all turning points.

SOLUTION

(a) Find the slope of the function at $x = -2$ by finding the derivative:

$$f'(x) = 6x^2 - 6x - 12$$
$$f'(-2) = 6(4) - 6(-2) - 12 = 24$$

find the corresponding y-value:

$$f(-2) = 2(-8) - 3(4) + 24 + 1 = -3$$

Use the standard point-formula for an equation of a line:

$$y - y_1 = m(x - x_1)$$
$$y - (-3) = 24(x - -2)$$
$$y = 24x + 45$$

(b) Set the derivative equal to zero and solve for the x-values:

$$0 = 6x^2 - 6x - 12$$
$$0 = x^2 - x - 2$$
$$0 = (x - 2)(x + 1)$$
$$x = 2, \text{ or } x = -1$$

find the y-values:

$$f(2) = -19, \text{ and } f(-1) = 8$$

Therefore the two turning points are $(2, -19)$ and $(-1, 8)$. Without a graph, we don't know the nature of these points – are they concave-up or concave-down?

Second Derivative Test

Given a graph it is easy to determine if points are maxima or minima. However, the graph is not always available, so it helps to have a test to determine if a point is a maximum or minimum.

We know first derivative represents the slope of the function. Looking at the parabola $y = x^2$ and following the function from left to right we can plot some values of the first derivative. The graph tells us that the turning point is at $(0, 0)$ and is a minimum. The value of the 1st derivative is -2,-1,0,1,2 : these numbers are increasing. This trend is always true for a local minimum.

How do we test for this trend of increasing slopes? The rate of change of the *slopes* is the second derivative, and if this number (at the point $(0, 0)$) is positive, then the point is a minimum.

This is called the second derivative test.

Second Derivative Test

- If $f'(x) = 0$, and $f''(x) > 0$, then the point $(x, f(x))$ is a local minimum. The graph in this neighbourhood is concave up.

- If $f'(x) = 0$, and $f''(x) < 0$, then the point $(x, f(x))$ is a local maximum. The graph in this neighbourhood is concave down.

Example Use the second derivative test to determine if the turning point $(-1, 2)$ is a maximum or minimum on the function $f(x) = x^3 - 3x$

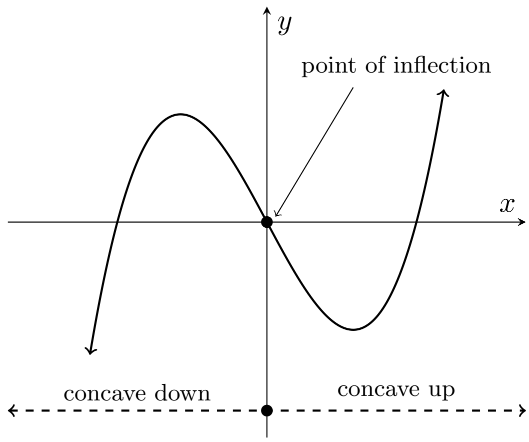
Solution Find the second derivative and substitute the turning point.

$$\begin{aligned} f'(x) &= 3x^2 - 3 \\ f''(x) &= 6x \\ f''(-1) &= 6(-1) = -6 \\ -6 &< 0 \end{aligned}$$

therefore the turning point $(-1, 2)$ is a local maximum. Confirm by looking at the graph on page 65.

Points of Inflection

There is one last feature of the initial function that can be found. There is a point in between the maximum and minimum points where the change in slopes of the function change from decreasing to increasing. This is called a point of inflection. It is where the concavity of the function changes between down and up.



The second derivative test can be used to determine if a function is concave-up (> 0) or concave down (< 0). Note there is no $=$ in these inequalities. In between is where the point of inflection can be found:

> Second Derivative Test for Concavity
>
> - If $f''(x) = 0$, then the point $(x, f(x))$ is a point of inflection.

Example Find where the function changes from concave-up to concave-down.

$$f(x) = 3x^3 - 12x^2 + 7$$

Solution Find the second derivative and set equal to zero to find the point of inflection.

$$\begin{aligned} f'(x) &= 9x^2 - 24x \\ f''(x) &= 18x - 24 \\ 0 &= 18x - 24 \end{aligned}$$

$$\frac{4}{3} = x$$

Sub back into $f(x)$ to find the $y-$value:

$$f\left(\frac{4}{3}\right) = 3\left(\frac{4}{3}\right)^3 - 12\left(\frac{4}{3}\right)^2 + 7$$

$= -7\frac{2}{9}$

Therefore, the point $\left(\frac{4}{3}, -7\frac{2}{9}\right)$ is a point of inflection.

4.4 The Product, Quotient, and Chain Rules

The Product Rule

Let $f(x) = x$ and $g(x) = x^2$. What is the derivative of $f(x) \times g(x)$? The question helps to show that the answer is NOT $f'(x) \times g'(x)$

$f(x) \times g(x) = x \times x^2 = x^3$ and we know the derivative of x^3 is $3x^2$. Also we know that $f'(x) = 1$ and $g'(x) = 2x$ so $f'(x) \times g'(x) = 1 \times 2x = 2x$ not $3x^2$.

So the derivative of the product of two functions is not the product of the derivatives of each function. In symbols this can be written

$$(fg)' \neq f'g'$$

Theorem If f and g are both differentiable then

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)\frac{d}{dx}\left[g(x)\right] + g(x)\frac{d}{dx}\left[f(x)\right]$$

The Product rule is often seen in an abbreviated form as $(uv)' = uv' + vu'$.

EXAMPLE Find the derivative of $f(x) = x^2 e^x$.

SOLUTION Use the product rule:

$$f'(x) = (2x)(e^x) + (x^2)(e^x)$$

This could be simplified by factoring: $f'(x) = e^x(2x + x^2)$ but is not mandatory.

EXAMPLE Differentiate $f = 4\pi x \sin x$.

SOLUTION Use the product rule. 4π is a constant.

$$f' = (4\pi)(\sin x) + (4\pi x)(\cos x)$$

EXAMPLE If $g(x) = \frac{e^x}{3}\sqrt{x+2}$, find $g'(x)$.

SOLUTION Convert the root to power form and use the product rule.

$$\begin{aligned}
g(x) &= \frac{1}{3}e^x(x+2)^{\frac{1}{2}} \\
g'(x) &= \left[\frac{1}{3}e^x\right]\left[(x+2)^{\frac{1}{2}}\right] + \left[\frac{1}{3}e^x\right]\left[\frac{1}{2}(x+2)^{-\frac{1}{2}}\right] \\
&= \frac{e^x}{3}\sqrt{x+2} + \frac{e^x}{6\sqrt{x+2}}
\end{aligned}$$

The Quotient Rule

Let $u = f(x)$ and $v = g(x)$ be differentiable functions of x then we can show that

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}\left[f(x)\right] - f(x)\frac{d}{dx}\left[g(x)\right]}{\left[g(x)\right]^2}$$

or in abbreviated form as

$$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$$

EXAMPLE Differentiate with the quotient rule: $y = \frac{(s-1)(s+3)}{e^{2s}}$

SOLUTION Expand the numerator first and then differentiate

$$\begin{aligned} y &= \frac{s^2 + 2s - 3}{e^{2s}} \\ y' &= \frac{(2s+2)(e^{2s}) - (s^2 + 2s - 3)(e^{2s}) \cdot 2}{(e^{2s})^2} \\ y' &= \frac{-2(s^2 + s - 4)}{e^{2s}} \end{aligned}$$

EXAMPLE Find f', given $f = \frac{\sin x}{x}$

SOLUTION

$$f' = \frac{[\cos(x) \cdot x] - \sin(x)}{x^2} = \frac{x\cos(x) - \sin(x)}{x^2}$$

EXAMPLE Find f', given $f = \dfrac{\ln x}{e^x}$

SOLUTION

$$\begin{aligned} f' &= \frac{(\frac{1}{x})(e^x) - (\ln x)(e^x)}{e^{x^2}} \\ &= \frac{1 - x\ln x}{xe^x} \end{aligned}$$

The Chain Rule

When functions are combined with other functions, they are often called composite functions. These require special treatment when differentiating.

Let $f(x) = x^2$ and $g(x) = 2x+1$ then $(f \circ g)$ This means f 'composed of' g is $f\left(g\left(x\right)\right) = f(2x+1) = (2x+1)^2$.

Also, g 'composed of' f would be: $(g \circ f)(x) = g\left(x^2\right) = 2\left(x^2\right) + 1 = 2x^2 + 1$.

The differentiation rules we have met so far allow us to differentiate pairs of functions that have been added, subtracted, multiplied or divided. They do not allow us to differentiate an expression that is made from a function that is within another function.

The following are all examples of composite functions.

1. We can differentiate x^2 but we can't use the same procedure to differentiate $(1-x)^2$. Here we can imagine if $f(x) = x^2$ and $g(x) = 1 - x$ then $(f \circ g)(x) = f(1-x) = (1-x)^2$.
2. We can differentiate $\frac{1}{x^2}$ but we can't use the same procedure to differentiate $\frac{1}{x^2+1}$.
3. We can differentiate e^x but we can't use the same procedure to differentiate e^{x^2}.

A name often used for functions of this type is *function of a function.*

Once we recognise we are dealing with a composite function we need a procedure to differentiate it. You will find that you are far more likely to be required to differentiate a composite function in a practical situation than a simple one. It can be proved that the derivative of the composite function $f \circ g$ is the product of the derivatives of f and g. This important rule is given the name the *Chain Rule.* A substitution method is often used to add clarity to the differentiation process.

Let $y = u^2$ and let $u = 1 - x$. Then $\frac{dy}{du} = 2u$ and $\frac{du}{dx} = -1$. Now $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2u \times (-1) = -2u = -2(1-x) = 2(x-1)$.

The Leibniz form of the Chain Rule $\dfrac{dy}{dx} = \dfrac{dy}{\cancel{du}}\dfrac{\cancel{du}}{dx}$ is what gives the rule its name. Because of the apparent cancelling it is particularly easy to learn in this form.

As an aside let us verify the rule for this example. Given $y = (1-x)^2$. We will expand the right hand side of the equation. It becomes $y = x^2 - 2x + 1$. So $y' = 2x - 2 = 2(x-1)$ as before.

Using function notation the Chain Rule states: If f and g are both differentiable and $F = f \circ g$ is the composite function $F(x) = f(g(x))$, then F is differentiable and $F' = f'(g(x))g'(x)$.

A Comment on the Leibniz form of the Chain Rule

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ gives the impression that the du could cancel but remember we have not defined du. We have defined $\frac{dy}{du}$ as the rate of change of y with respect to u and $\frac{du}{dx}$ as the rate of change of u with respect to x. However the apparent cancelling helps us to remember the way the differentials are arranged. it also helps us to accept the extension of the Chain Rule to cover a function of a function of a function etc. e.g.

$$\text{Let } y = f(u),\ u = g(v) \text{ and } v = h(x)$$
$$\text{Then } \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

EXAMPLE Find $F'(x)$ when $F(x) = \frac{1}{x^2+1}$.

SOLUTION Using function notation $F(x) = (f \circ g)(x) = f(g(x))$

where

$$f(u) = u^{-1} \text{ and } g(x) = x^2 + 1$$

$$f'(u) = -u^{-2} \text{and} g'(x) = 2x$$

and

$$\begin{aligned} F'(x) &= f'(g(x))g'(x) \\ &= \frac{-1}{(x^2+1)^2} \cdot 2x \\ &= \frac{-2x}{(x^2+1)^2} \end{aligned}$$

Using the Leibniz notation let $u = x^2 + 1$ and $y = u^{-1}$ then

$$\begin{aligned} F'(x) &= \frac{dy}{du}\frac{du}{dx} = -u^{-2}(2x) \\ &= \frac{-1}{(x^2+1)^2}(2x) = \frac{-2x}{(x^2+1)^2} \end{aligned}$$

To use the method we need to bring a new variable into the problem we are trying to solve. It is recommended that you use the variable u wherever possible so that you follow through using a pattern you are familiar with.

In summary: if g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'[g(x)] \cdot g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

The Chain Rule will be found in many situations where functions are added, subtracted, multiplied or divided. As an example we will focus on combining the Chain Rule with the Product Rule, however any combination of these rules could be found in a problem.

EXAMPLE Differentiate xe^{-x^2}.

SOLUTION We can see that there is a product of two functions present in this example, i.e. $f(x) = x$ and $g(x) = e^{-x^2}$. Also $g(x)$ is a composite function.

We have from the Product Rule

$$(fg)' = fg' + gf'$$

We can see that f, g and $f' = 1$ can be substituted immediately and only g' requires some effort to be worked out. $g(x)$ is a composite function so $g'(x)$ is computed using the Chain Rule.

Let $u = -x^2$ then $\frac{du}{dx} = -2x$. Also $g(u) = e^u$ so $\frac{dg}{du} = e^u$.

$$\begin{aligned}\frac{dg}{dx} &= \frac{dg}{\cancel{du}} \cdot \frac{\cancel{du}}{dx}\\ &= e^u \cdot -2x\\ &= e^{-x^2} \cdot -2x\\ &= -2x\ e^{-x^2}\end{aligned}$$

EXAMPLE Find $a'(x)$ given $a(x) = 4\pi x \tan(x - \frac{\pi}{4})$

SOLUTION Use the product and chain rules

$$\begin{aligned}a' &= 4\pi \tan\left(x - \frac{\pi}{4}\right) + 4\pi x \sec^2\left(x - \frac{\pi}{4}\right)(1)\\ a' &= 4\pi\left[xsec^2\left(x - \frac{\pi}{4}\right) + tan\left(x - \frac{\pi}{4}\right)\right]\end{aligned}$$

So $g'(x) = -2x\ e^{-x^2}$

Putting this all together

$$\begin{aligned}(fg)' &= fg' + gf'\\ &= x \cdot -2x\ e^{-x^2} + e^{-x^2} \cdot 1\\ &= e^{-x^2}\left[1 - 2x^2\right]\end{aligned}$$

EXAM QUESTION Find $\frac{dy}{dx}$ given $y = x^3 e^{2x}$

SOLUTION First use the product rule; there is a chain rule on e^{2x}:

$$\begin{aligned}\frac{dy}{dx} &= 3x^2(e^{2x}) + x^3(e^{2x})(2)\\ \frac{dy}{dx} &= x^2 e^{2x}(3 + 2x)\end{aligned}$$

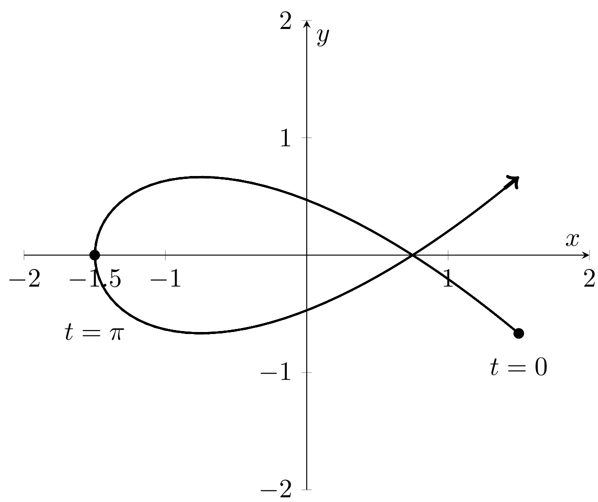
4.5 Parametric Differentiation

Parametric Curves x and y are both given as functions of a third variable t (called the *parameter*). Let the equations be

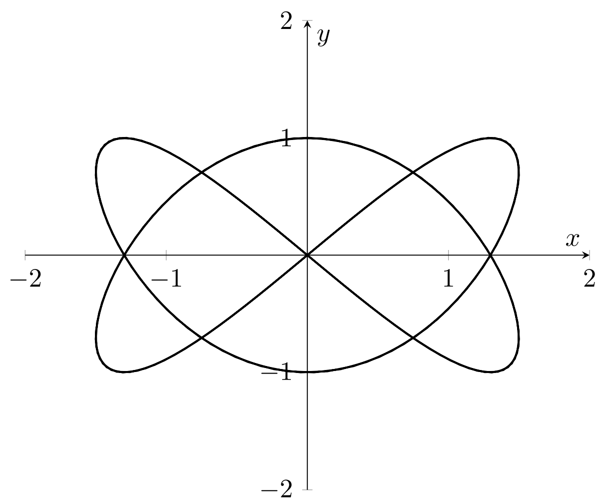
$$x = f(t) \text{ and } y = g(t)$$

Each value of t gives a point (x, y). As t varies the point $(x, y) = (f(t), g(t))$ traces out a curve in the coordinate plane called a *parametric curve*. This is useful for functions that violate the vertical line test (see Section 1.2) such as a circle, $x^2 + y^2 = r^2$, because only proper functions can be differentiated.

Both the fish and the Pokemon-looking-thingy below can be plotted using parametric equations.



(a) fish: $x = \dfrac{3 \cdot \cos t}{2}$, $y = -\dfrac{2}{3}\cos\left(\dfrac{3t}{2}\right)$



(b) Pokemon: $x = -\dfrac{3}{2}\sin(t)$, $y = \sin\left(\dfrac{3t}{2}\right)$

To find the point at the mouth of the fish in terms of t, we must solve the equations for $x = -1.5$, or $y = 0$. As t increases, the plot is drawn.

$$\begin{aligned} x &= \frac{3 \cdot \cos t}{2} \\ -1.5 &= \frac{3 \cdot \cos t}{2} \\ -1 &= \cos t \\ t &= \pi \end{aligned}$$

check that $y = 0$

$$\begin{aligned} y &= -\frac{2}{2} \cos\left(\frac{3\pi}{2}\right) \\ y &= -\frac{2}{3} \cdot 0 \\ y &= 0 \end{aligned}$$

The point $t = 0$ corresponds to $x = 1.5$, and $y = -\frac{2}{3}$ = which is shown as a point on the tail.

EXERCISE In which direction is the Pokemon drawn? Where does it start?

Using the Chain Rule to find the Derivative $\frac{dy}{dx}$

Given the parametric equations $x = f(t)$ and $y = g(t)$ define a parametric curve. If f and g are both differentiable the Chain Rule gives

$$\frac{dy}{dt} = \frac{dy}{\cancel{dx}} \cdot \frac{\cancel{dx}}{dt}$$

provided y is also a differentiable function of x. So provided $\frac{dx}{dt} \neq 0$

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

When dividing fractions remember to 'invert and multiply'

$$\frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dx}$$

EXAMPLE Find the derivative, $\frac{dy}{dx}$, of the fish in part (a).

SOLUTION Differentiate each equation and combine with the chain rule to find $\frac{dy}{dx}$.

$$\begin{aligned} x &= \frac{3 \cos t}{2} \\ \frac{dx}{dt} &= -\frac{3}{2} \sin t \end{aligned}$$

$$\begin{aligned} y &= -\frac{2}{3} \cos\left(\frac{3t}{2}\right) \\ \frac{dy}{dt} &= -\frac{2}{3} \cdot -\sin\left(\frac{3t}{2}\right) \cdot \frac{3}{2} \\ \frac{dy}{dt} &= \sin\left(\frac{3t}{2}\right) \end{aligned}$$

Now, using the chain rule, note $\frac{dx}{dt}$ must be inverted:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= \sin\left(\frac{3t}{2}\right) \cdot \frac{1}{-\frac{3}{2}\sin t}$$
$$= \frac{-2\sin\frac{3t}{2}}{3\sin t}$$

Compare your derivatives, y' and x', with the superhero logo plot from part (b).

4.6 Related Rates

The concept of related rates is best understood by exploring some examples.

EXAM QUESTION Air is being pumped into a spherical balloon so that its volume is increasing at a rate of 50 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?
SOLUTION The volume of a sphere is $\frac{4}{3}\pi r^3$. Find the derivative with respect to radius.

$$\frac{dV}{dr} = 4\pi r^2$$

We are looking for "how fast" (time) and radius (r), or $\frac{dr}{dt}$. From the Chain Rule we can write

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

Substituting $\frac{dV}{dt} = 50$ and $\frac{dV}{dr} = 4\pi r^2$ we get

$$50 = 4\pi r^2 \cdot \frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{50}{4\pi r^2}$$

Now we substitute $r = 25$. (Diameter = 50 so radius = 25)

$$\frac{dr}{dt}_{r=25} = \frac{50}{4\pi(25)^2}$$
$$= \frac{1}{50\pi} \approx 0.00637$$

Therefore the radius is increasing at the rate of $\dfrac{1}{50\pi}$ cm/s.

EXAMPLE A ladder 5 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at the rate of 0.5 m/s how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3 m from the wall?

SOLUTION Let the origin be placed at the corner where the wall meets the floor, let x be the distance of the foot of the ladder from the wall and let y be the distance of the top of the ladder from the corner. The ladder forms a right angled triangle whose sides are x, y and with hypotenuse 5. We are given that $\frac{dx}{dt} = 0.5$ and are asked to find $\frac{dy}{dt}$ when $x = 3$.

Pythagoras' theorem gives

$$x^2 + y^2 = 5^2 \tag{1}$$

Differentiate equation (1) with respect to t. Note that this derivative uses a technique called implicit differentiation.

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Solve for $\frac{dy}{dt}$

$$\frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$$

Using the Pythagorean theorem with $x = 3$ and the hypotenuse $= 5$, $y = 4$

Substitute $\frac{dx}{dt} = 0.5$, $x = 3$ and $y = 4$

$$\begin{aligned} \frac{dy}{dt} &= -\frac{3}{4} \cdot 0.5 \\ &= -0.375 \end{aligned}$$

The top of the ladder is moving vertically downwards at the rate of 0.375 m/s.

EXAMPLE A water tank has the shape of an inverted circular cone with a base radius of 2 m and height of 4 m. If water is being pumped into the tank at a rate of 2 m^3/min find the rate at which the water level is rising when the water is 3 m deep.

SOLUTION Let V, r and h be the volume of water the radius of the surface and the height at time t. We are given

$$\frac{dV}{dt} = 2\mathrm{m}^3/\mathrm{min}$$

We are asked to find $\frac{dh}{dt}$ when $h = 3$.

Draw a diagram to show that the relationship between r and h can be found by similar triangles.

$$\begin{aligned} \frac{r}{h} &= \frac{2}{4} \\ r &= \frac{h}{2} \qquad (1) \end{aligned}$$

The formula for the volume is

$$V = \frac{1}{3}\pi r^2 h \qquad (2)$$

Substituting equation (1) in equation (2)

$$\begin{aligned} V &= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h \\ &= \frac{\pi}{12}h^3 \end{aligned}$$

Differentiate with respect to t

$$\begin{aligned} \frac{dV}{dt} &= \frac{\pi}{12} \cdot 3h^2 \cdot \frac{dh}{dt} \\ &= \frac{\pi}{4}h^2 \cdot \frac{dh}{dt} \end{aligned}$$

So

$$\frac{dh}{dt} = \frac{4}{\pi h^2} \cdot \frac{dV}{dt}$$

Substitute $h = 3$ and $\frac{dV}{dt} = 2$

$$\begin{aligned} \frac{dh}{dt} &= \frac{4}{\pi\,(3)^3} \cdot 2 \\ &= \frac{8}{9\pi} \end{aligned}$$

The water level is rising at the rate of $\frac{8}{9\pi}$ m/min.

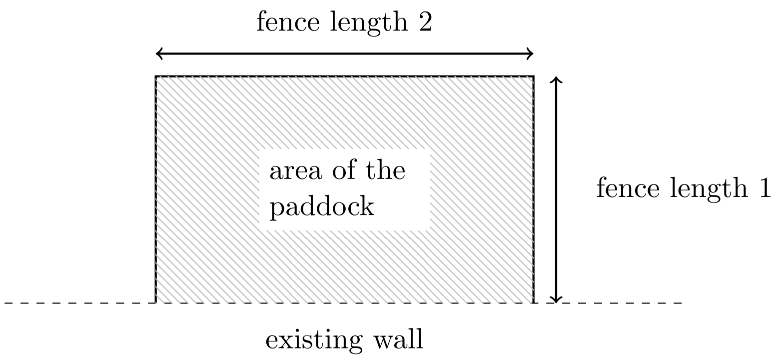
4.7 Optimisation

Optimisation is the process of using calculus to find the best result for a situation involving a changing quantity (variable). Examples include maximizing profit, minimizing cost, maximizing volume, minimizing amount materials used, and so on. As long as the quantities in question can be represented by a function, calculus can used to find the special points of the function and their nature.

A general method to approach an optimisation problem

1. Write down the known variables and draw a diagram
2. Form an equation of the situation by placing the unknown variable in terms of the known variables
3. Use the facts of the problem to reduce the expression until it becomes a relationship between the unknown quantity and one of the known quantities
4. Optimize by finding a maximum or minimum value

EXAMPLE A farmer wishes to fence a paddock using an existing wall as one side of the paddock. She has 100 meters of fencing and wants to know the dimensions of the paddock to enclose the maximum area.



SOLUTION Following the steps outlined above, draw a diagram and label the variables. Let T represent the total fence length which cannot be more than 100m. Write this as an equation: $T = 100$. There are 3 individual lengths that make up the total, so let l represent the long side of the paddock, and w represent the short side.

$$T = l + 2w$$
$$100 = l + 2w$$

The quantity to be optimised is area (A) of the paddock (remember it is a rectangle):

$$A = lw$$

The next step is to represent the quantity to be optimised (area) as a function of one other variable. We can rearrange our earlier equation to solve for length: $l = 100 - 2w$. This can now be substituted into the area equation:

$$A = lw$$

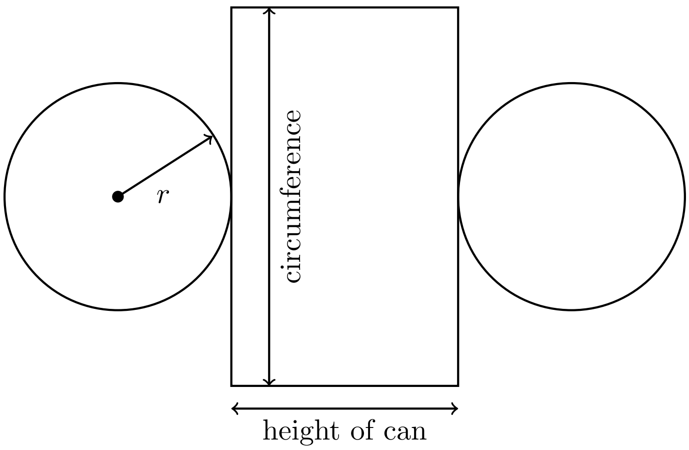
$$\begin{aligned} &= (100 - 2w)\,w \\ &= 100w - 2w^2 \end{aligned}$$

This is now an equation that can optimised using calculus.

$$\begin{aligned} \frac{dA}{dw} &= 100 - 4w \\ 0 &= 100 - 4w \\ w &= 25 \end{aligned}$$

Therefore the paddock width is 25 meters to maximize the total area. Going back to the original constraint of 100m total length means that the paddock length is $100 - 2(25) = 50$ meters.

Example A cylindrical can is to be made to hold 1 litre of oil. Find the dimensions that will minimise the cost of the aluminium to manufacture the can. Note that 1 L = 1000 cm^3.



Solution The minimum cost of the aluminium will be the minimum surface area of the cylinder. The can can be deconstructed into two circles and a rectangle. We will label the variables required to calculate area.

Let SA represent surface area of the can. The total surface area is:

$$\begin{aligned} SA &= \pi r^2 + \pi r^2 + (\text{height} * \text{circumference}) \\ &= 2\pi r^2 + 2\pi r h \end{aligned}$$

Note that we have 2 variables that are unknown, r, and h. We need to express one variable in terms of the other in order to proceed. Use the additional information in the problem. The volume of the can must be 1000 cm^3.

$$\begin{aligned} V &= \pi r^2 h \\ 1000 &= \pi r^2 h \\ h &= \frac{1000}{\pi r^2} \end{aligned}$$

Substitute this form for h into the surface area function:

$$\begin{aligned} \text{SA} &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + 2\pi r \left(\frac{1000}{\pi r^2}\right) \end{aligned}$$

$$= 2\pi r^2 + \frac{2000}{r}$$

This function is now the surface area in terms of a single variable, r, and can be optimised:

$$\frac{d(\text{SA})}{dr} = 4\pi r - \frac{2000}{r^2}$$
$$0 = 4\pi r - \frac{2000}{r^2}$$
$$= 4\pi r^3 - 2000$$
$$r^3 = \frac{2000}{4\pi}$$
$$r = 5.419\,\text{cm}$$

Therefore the final dimensions of the can optimised for minimum cost are radius= 5.42 cm, and height= 10.8 cm.

4.8 Chapter Exercises

§4.1 Differentiation from First Principles

1. Find the derivative of the function at the given value using the method of first principles.

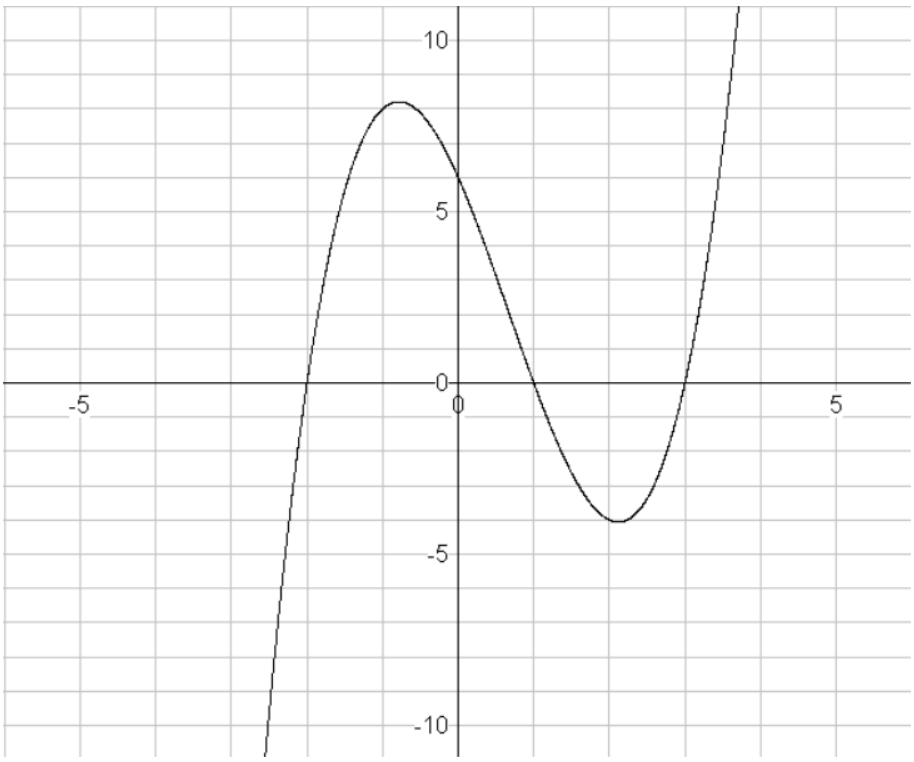
(**a**) $f(x) = 5x^2 + 3x - 1$ at the number 2

(**b**) $f(x) = 1 - 3x^2$ at the number 2

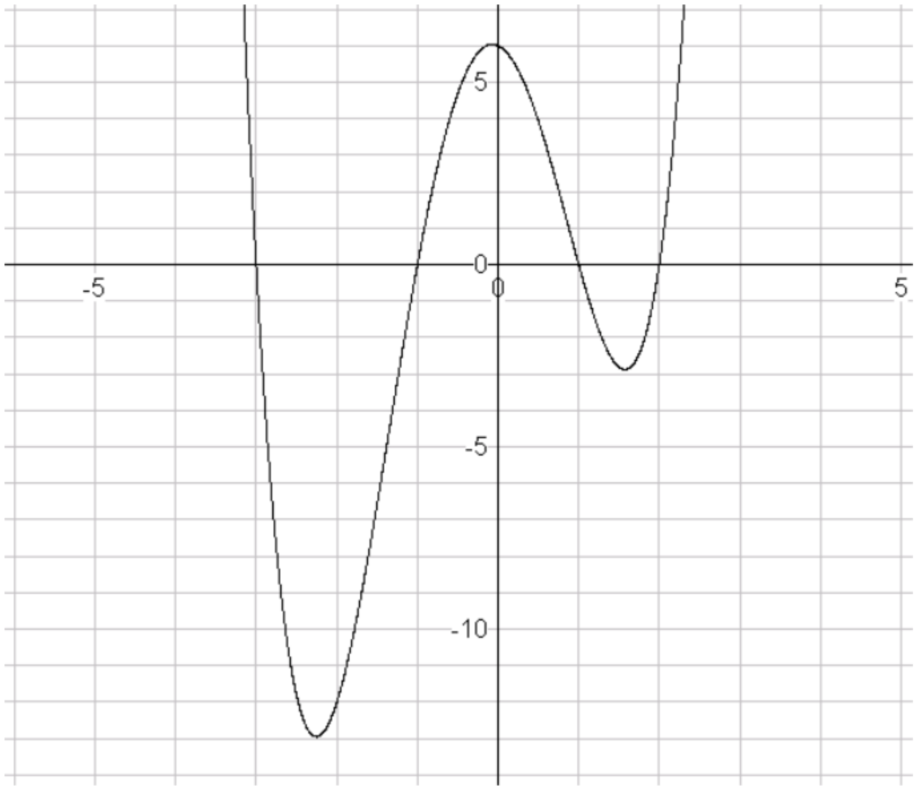
(**c**) $f(x) = x^4$ at the number 1, given $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$

Graphing Exercises

2. The graph of $f(x) = (x-1)(x+2)(x-3)$ is drawn below. On the same set of axes, sketch the graph of the derived function, $f'(x)$. What is the shape of the graph of the derived function?

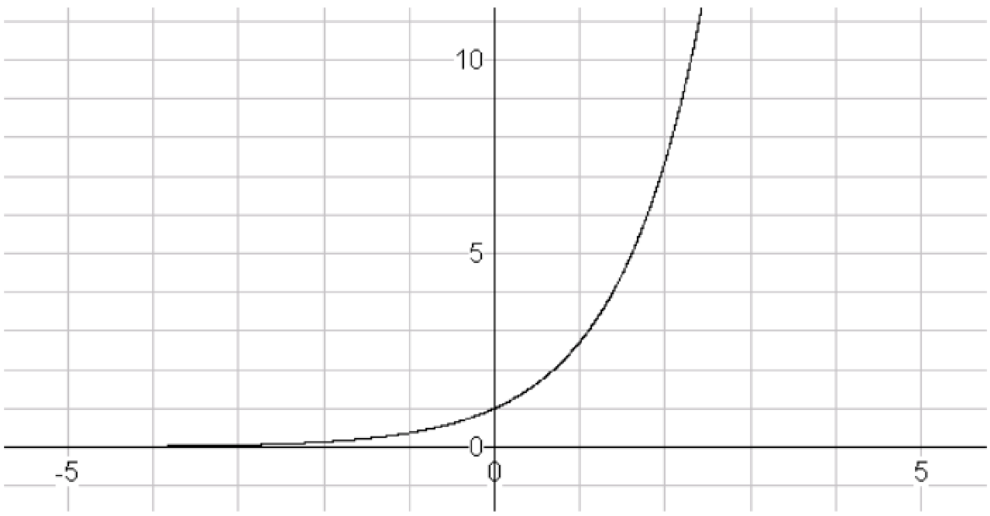


3. The graph of $f(x) = (x+1)(x-1)(x-2)(x+3)$ is drawn below. On the same set of axes, sketch the graph of the derived function, $f'(x)$



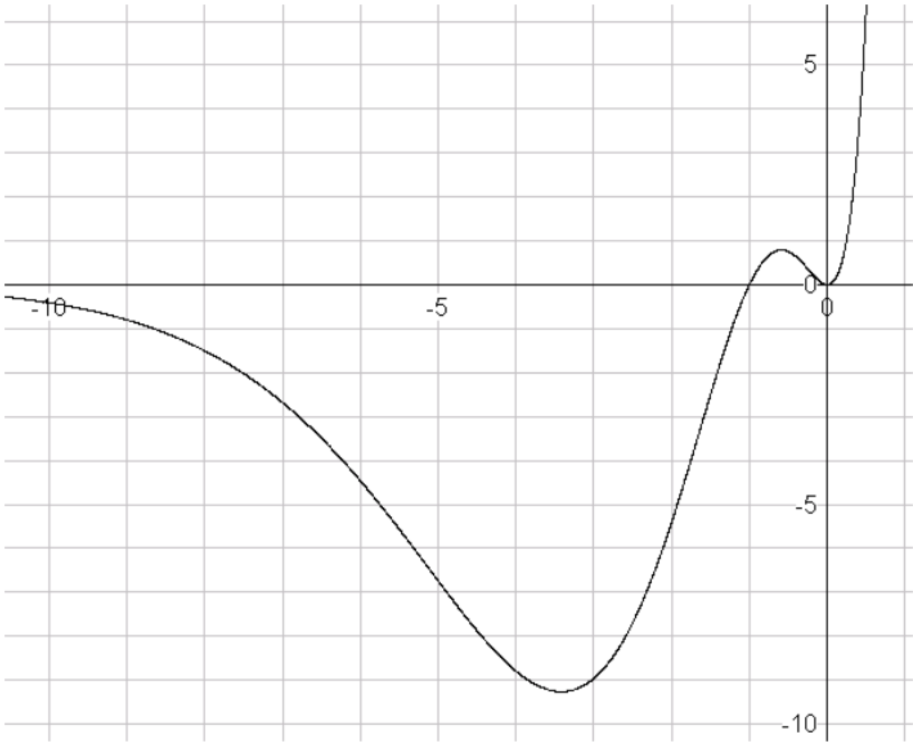
What is the order of $f(x)$? What is the order of $f'(x)$?

4. The graph of $f(x) = e^x$ is drawn below. On the same set of axes, sketch the graph of the derived function, $f'(x)$



What is the shape of the graph of the derived function?

5. The graph of $f(x) = 10x^2e^x(x+1)$ is drawn below. On the same set of axes, sketch the graph of the derived function, $f'(x)$



6. Differentiate

(a) $f(x) = \frac{1}{x^2}$ **(b)** $y = \sqrt[3]{x^2}$ **(c)** $y = x^5$

(d) $y = \frac{1}{x^3}$ **(e)** $y = x^{-4}$ **(f)** $y = x^{3/4}$

(g) $y = \frac{1}{\sqrt{x}}$

§4.2 Standard Derivatives

1. Find $\frac{\mathrm{dy}}{\mathrm{dx}}$

(a) $y = \frac{2}{5}x^5$ **(b)** $y = -10$

(c) $y = -3x^4$ **(d)** $y = \frac{2}{x^4}$

(e) $y = \frac{1}{3x^3}$

2. Differentiate

(a) $f(t) = \sqrt{t}$ **(b)** $f(t) = \sqrt{t^3}$

(c) $f(z) = \sqrt[3]{z^5}$ **(d)** $f(x) = 2x^{3.2}$

3. Find the derivative

(a) $f(x) = 2x^3 - 3x^2 + 4x - 1$ **(b)** $f(x) = x^2 + x + 1 + \frac{1}{x}$

(c) $x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5$

4. (a) Given $s = 4t^2 - 7t + 5$ find $\frac{ds}{dt}$

(b) Find $\frac{d(3x)}{dx}$

(c) Find $\frac{d(3u^4)}{du}$

(d) Given $f(x) = 2x - 3$ find $Df(x)$

(e) Given $f(x) = e^x - x$ find $f'(x)$. Sketch the graph.

(f) Differentiate $f(x) = (3x)^3$

(g) Differentiate $g(x) = \left(x^3\right)^5$

(h) Find the derivative of $f(x) = e^x - x^e$

5. Find the derived function

(a) $f(x) = \frac{1}{x^3} - \frac{1}{\sqrt[4]{x^3}}$

(b) $f(x) = \frac{\sqrt{x}+\sqrt[3]{x}}{\sqrt[4]{x}}$

(c) $g(x) = ex^2 + 2e^x + xe^2 + x^{e^2}$

(d) $f(x) = \sqrt[3]{x} + \sqrt[5]{2}$

6. Sketch the graphs and label your axes in radians

(a) $f'(x)$ where $f(x) = \sin x$

(b) $g'(x)$ where $g(x) = \cos x$

(c) $h'(x)$ where $h(x) = \tan x$

7. Differentiate the trigonometric functions

(a) $y = x^2 \sin x$

(b) $f(x) = \sqrt{x} \sin x$

(c) $h(x) = \tan(5x)$

(d) $y = \frac{x}{\cos x}$

(e) $g(t) = \cos(\omega t + \delta)$

8. Find the tangent line to the curve $y = e^x \cos x$ at the point $(0, 1)$.

9. A ladder 10m long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall and let x be the distance between the bottom of the ladder and the wall. If the bottom of the ladder slides away from the wall, how fast is x changing with respect to θ when $\theta = \frac{\pi}{3}$?

10. Find the derivative

(a) $\sin 4x$

(b) $\frac{2}{\pi} \sin \pi x$

(c) $5 \cos 3x$

(d) $\tan 3x$

(e) $3 \tan(x + 2)$

(f) $\sin^3 x$

(g) $\sin^2 3x$

(h) $\sin^3 (x - 1)^2$

(i) $\tan^2 2x$

§4.3 Maximums, Minimums, and Tangents

1. The function $y = 7x - 3$ has no turning point. Why?

2. Turning points are sometimes referred to as stationary points. Find the stationary points of the following functions:

(a) $y = \frac{x^3}{3} - 4x$

(b) $y = x^2 + \frac{16}{x}$

(c) $f(x) = -x^4 + 4x^3 - 4x^2 + 1$

3. Find the turning points and use the second derivative test to show if they are local maxima or local minima:

(a) $y = 4 - (3 - x)^2$

(b) $y = \frac{x^3}{3} - x$

(c) $f(x) = (3 - x)(x + 2)(x + 5)$

4. Consider $f(x) = xe^x$. Do you think this function has a local max or min? Think of the graph of $y = e^x$. Plot the function using Desmos and find the turning point.

5. Find the equations of the tangent line and the normal line to the curve $y = x\sqrt{x}$ at the point $(1, 1)$.

6. Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal.

7. At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?

8. Find a point a on the curve $f(x) = x^3 + 2x^2 + 3x + 4$ where $f'(a) = 2$

9. Find where the tangent line to the function $f(x) = x^3 - x + 1$ is parallel to the line $y = x$.

Graphing Exercise

10. If $f(x) = 2x^2 - x^3$ find f', f'', f''', $f^{(4)}$. Use `Desmos` to graph f, f', f'', and f''' on a common screen. Describe whether these graphs are consistent with a geometric interpretation of these derivatives.

11. If $f(x) = \frac{1}{x}$ find $f'(x)$ and $f''(x)$ then graph f, $f'$and f'' on a common screen. Are your answers reasonable?

§4.4 Product, Quotient, & Chain Rules

1. Use the product rule to find the derivative

(a) $f(x) = xe^x$

(b) $g(x) = x^2e^x$

2. Differentiate with respect to x

(a) x^3e^x

(b) $x^{-3}e^x$

(c) $(x + 1)\,e^x$

(d) $(x + 2)\,(x - 2)\,e^x$

3. Use the quotient rule to find the derivative

(a) $y = \frac{3x+1}{2x-1}$

(b) $y = \frac{e^x}{x+1}$

(c) $f(t) = \frac{2t}{1+t^2}$ find $\frac{df}{dt}$

(d) $f(x) = \frac{A}{B+Ce^x}$

4. Differentiate

(a) $\frac{3x^2}{1-x}$

(b) $\frac{x}{x+2}$

(c) $\frac{\sqrt{x}}{x+2}$

(d) $\frac{2x^2+3x+2}{e^x}$

Exercises involving Tangent and Normals

5. Find the equation of the tangent line and normal line to the curve $y = xe^x$ at the point $(0, 0)$.

6. The curve $y = 1/(1+x^2)$ has the name witch of Maria Agnesi.

(a) Find the equation of the tangent line to this curve at the point $\left(1, \frac{1}{2}\right)$.

(b) Use Desmos to draw the graph of the curve and the tangent line on the same grid.

7. The curve $y = x/(1+x^2)$ is called a serpentine.

(a) Find the equation of the tangent line at the point $\left(1, \frac{1}{2}\right)$.

(b) Use Desmos to draw the graph of the curve and the tangent line on the same grid.

Composite Functions

8. Let $f(x) = \sqrt{x}$ and $g(x) = x^3$ find $(f \circ g)(x)$ and $(g \circ f)(x)$.

9. Given $h(x) = e^x$ and $j(x) = \frac{x^2}{2}$ find $(h \circ j)(x)$ and $(j \circ h)(x)$.

10. The chain rule

(a) Find $F'(x)$ when $F(x) = \sqrt{1+x^2}$.

(b) Given $y = \left(1-x^2\right)^5$ find $\frac{dy}{dx}$.

(c) Find the derivative of e^{x^2}.

(d) Find the derived function for e^{e^x}.

11. Differentiate with respect to x

(a) $(3x+2)^3$ **(b)** $(5x+3)^{\frac{3}{5}}$

(c) $\frac{1}{2x+1}$ **(d)** $\frac{3}{(4-x)^3}$

(e) $\sqrt{2x-5}$ **(f)** $\sqrt[3]{5-x^2}$

(g) $\frac{1}{\sqrt{x+2}}$ **(h)** e^{2x^3}

12. You may have to combine rules to differentiate the following

(a) $y = x^2 e^{-2x}$ **(b)** $(1-2x)^2 e^{-x}$

(c) $\frac{p+1}{\sqrt{p^2+1}}$ **(d)** $\left(x^2+3\right)^2 (x-4)$

(e) $(x-3)^3 (x+2)$ **(f)** $\sqrt{x+1}\,(x-1)^2$

(g) $e^{x^2}\sqrt{x+1}$ **(h)** $\frac{x^2+x+2}{(x+1)^2}$

§4.5 Parametric Differentiation

1. Use desmos to sketch the parametric curves

(a) $x = t^2 - 2t$ and $y = t+1$ where $0 \leqslant t \leqslant 4$ **(b)** $x = \cos t$ and $y = \sin t$ where $0 \leqslant t \leqslant 2\pi$

(c) $x = 2\cos t$ and $y = \sin t$ for $0 \leqslant t \leqslant 2\pi$ **(d)** $x = \cos t$ and $y = \sin 2t$ where $0 \leqslant t \leqslant 2\pi$

(e) $x = 1.5\cos t - \cos 40t$ and $y = 1.5\sin t - \sin 40t$ where $0 \leqslant t \leqslant 2\pi$

2. Given $y = 2t$ and $x = t^2$

  (a) Find $\frac{dy}{dx}$.

  (b) By finding the appropriate value of t show that $(1, 2)$ lies on the parametric curve.

  (c) Find the equation of the tangent line to the parametric curve at $(1, 2)$.

3. Given $x = \cos t$ and $y = \sin t$ find the equation of the tangent line at the point $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$. Where does this curve have horizontal or vertical tangent lines?

4. Given $x = \cos t$ and $y = \sin 2t$ find the equation of the tangent line at the point $\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$. Where does this curve have horizontal or vertical tangent lines?

5. Find $\frac{dy}{dx}$ in terms of the parameter

  (a) $x = t^2 + t \qquad y = t^3 - t^2$

  (b) $x = e^t \cos t \qquad y = e^t \sin t$

  (c) $x = at^2 \qquad y = 2at$ (a is constant)

  (d) $x = a \sin\theta \qquad y = b\cos\theta$

  (e) $x = 2\sin\theta \qquad y = \cos 2\theta$

  (f) $x = a\,(\theta - \sin\theta) \qquad y = a\,(1 - \cos\theta)$

§4.6 Related Rates

1. If V is the volume of a cube with edge length x and the cube is expanding as time passes, find $\frac{dV}{dt}$ in terms of $\frac{dx}{dt}$.

2. Each side of a square is increasing at a rate of 6cm/s. At what rate is the area of the square increasing when the area of the square is $16cm^2$?

3. **(a)** If A is the area of a circle with radius r and the circle expands as time passes find $\frac{dA}{dt}$ in terms of $\frac{dr}{dt}$.

  (b) Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 1m/s how fast is the area of the spill increasing when the radius is 30m?

4. If a snowball melts so that its surface area decreases at a rate of $1cm^2$/min, find the rate at which the diameter decreases when the diameter is 10 cm. The following steps may help you to solve this problem.

  (a) What quantities are given in the problem?

  (b) What is the unknown?

  (c) Draw a picture of the situation for any time t.

  (d) Write an equation that relates the quantities.

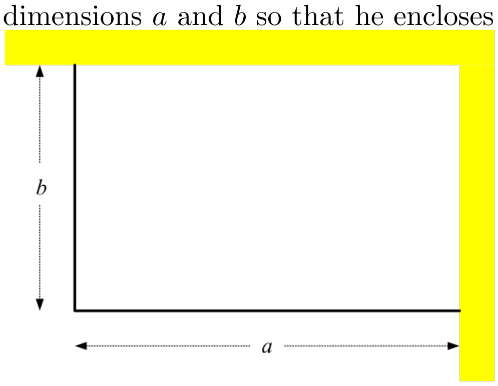
  (e) Solve for the unknown quantities.

5. At noon ship A is 150km west of a ship B. Ship A is sailing east at 35km/h and ship B is sailing north at 25km/h. How fast is the distance between the ships changing at 4:00 P.M.?

6. A plane flying horizontally at an altitude of 2km and a speed of $800km$/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 3km away from the station.

§4.7 Optimisation

1. Divide 50 into two parts such that the product of the two parts is a maximum.
2. Find the number that exceeds its square by the greatest amount.
3. A farmer has 2400m of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the greatest area?
4. A farmer wishes to fence off a corner of a field where there is an existing hedge on two sides. The hedge is to be used to fence the two sides. If he has 300m of fencing available, find the dimensions a and b so that he encloses the maximum area.



5. The volume of the cone is given by the formula $V(r) = \frac{10\pi r^2}{3} - \frac{\pi r^3}{3}$. Find the radius of the cone at its maximum volume.
6. The total cost of holding a large event is composed of the venue hire, V, times the venue tax, T, plus the entertainment, E. These values can all be estimated based on the number of people attending the event, p, such that: $V(p) = 50p+80$, $T(p) = 0.002p-0.4$, and $E(p) = 36000-60p$. Find the minimum cost.
7. Find the area of the greatest rectangle that can be inscribed in a semicircle of radius r.
8. If $1200cm^2$ of material is available to make a box with a square base and an open top. Find the largest possible volume of the box. (Hint: There is no wasted material.)

5 | Integration $\int$

In mathematics we are often given a function f and asked to find a function F whose derivative is f. F in this situation is called the integral of f. It is usual to develop a list of integrals by differentiating a range of functions then using those to work backwards. The terms integration and anti-differentiation are synonymous. Generally we will use the term integration, however, both are acceptable.

The process of 'reversing' or 'undoing' a derivative has its own symbol, the integrand: $\int$

$$\int f'(x)dx = f(x) + C$$

5.1 Standard Integrals

It is not the intention here to list all of the rules that are required however at this stage let us explore the Power Rule to establish a rule for integrating an expression of the form $y = x^n$

function $f(x)$	derivative $f'(x)$
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$

or

f	f'
x	1
$\frac{1}{2}x^2$	x
$\frac{1}{3}x^3$	x^2
$\frac{1}{4}x^4$	x^3

now, in reverse

derivative $f'(x)$	integral $\int f'(x) = f(x)$
1	$x + C$
x	$\frac{1}{2}x^2 + C$
x^2	$\frac{1}{3}x^3 + C$
x^3	$\frac{1}{4}x^4 + C$

This establishes the pattern and if you think about the rule for differentiating $y = x^n$ you can soon establish the rule for integrating x^n.

The **Power Rule** for integrating polynomials

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \text{ where } n \neq -1$$

If $f(x) = x^{-1}$ then the integral of f is $\ln|x| + c$ or $\ln|kx|$

All of the differentiation rules we have met so far lead to integration rules. For instance we can establish standards for $\sin x$, $\cos x$, and $\sec^2 x$. The standard integrals are summarized in the table below.

Function $f(x)$	Integral $\int f(x)dx$	Notes
1	$x + C$	constant
A	$Ax + C$	A is constant
$x^n, n \neq -1$	$\frac{x^{n+1}}{n+1} + C$	power rule general form
e^x	$e^x + C$	exponential
$\frac{1}{x}$	$\ln \lvert x \rvert + C$	special case: x^{-1}
$\ln x$	$x \ln \lvert x \rvert - x + C$	
$\sin(x)$	$-\cos(x) + C$	trigonometric
$\cos(x)$	$\sin(x) + C$	
$\tan(x)$	$\ln \lvert \sec x \rvert + C$	
$\sec^2(x)$	$\tan(x) + C$	

To allow us to combine these integrals and thus extend the range of questions we can tackle we use two important rules for integrals

Sum Rule The integral of the sum of two functions is the sum of the integrals of the functions.

$$\int [f(x) + g(x)]\ dx = \int f(x)\ dx + \int g(x)\ dx$$

This is easily extended to the sum or difference of a number of functions.

Constant Multiple Rule The integral of a constant times a function is the constant times the integral of the function.

$$\int cf(x)\ dx = c \int f(x)\ dx$$

Indefinite Integrals

Although $\int f(x)dx$ looks very similar to $\int_a^b f(x)\ dx$ they are quite different and must not be confused or used in place of each other. $\int f(x)\ dx$ is a function of x or a family of functions of x and $\int_a^b f(x)\ dx$ is a number. They are connected of course, provided $f(x)$ is a continuous function of x on $[a, b]$. In this case the Evaluation Theorem gives the connection between them.

$$\int_a^b f(x)\ dx = \int f(x)\ dx \Big]_a^b$$

The indefinite integral represents either a particular integral or a family of integrals. These will use a constant C where C takes a different value for each member of the family. C is called the *constant of integration*.

Example Integrate the following functions; find $\int f(x)$.

(a) $f(x) = 3x^2$

Solution Applying the power rule:

$$\int 3x^2\,dx = \frac{3x^{2+1}}{2+1} + C$$
$$= x^3 + C$$

(b) $f(x) = 7$

Solution Here we are integrating a constant:

$$\int 7\,dx = 7x + C$$

(c) $f(x) = x^{\frac{2}{3}}$

Solution The power rule still applies to fractional indices:

$$\int x^{\frac{2}{3}}\,dx = \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + C$$
$$= \frac{3x^{5/3}}{5} + C$$

(d) $f(x) = \frac{1}{2\sqrt{x}} + \frac{1}{\sqrt{2}}$

Solution Here we need to combine the sum rule and the power rule:

$$\int f(x)\,dx = \int (\frac{1}{2\sqrt{x}})\,dx + \int (\frac{1}{\sqrt{2}})\,dx$$
$$= \frac{1}{2}\int x^{-\frac{1}{2}} + \frac{1}{\sqrt{2}}x + C$$
$$= \sqrt{x} + \frac{x}{\sqrt{2}} + C$$

These four examples are all *indefinite* integrals and have an unknown constant in the answer. The following section will introduce definite integrals.

Example Find $f(x)$ given $f''(x) = 6$

Solution Here we have a *second* derivative, indicated by the double-prime symbol, f''. Knowing that $\int f'(x) = f(x)$ we can safely assume that

$$\boxed{\int f''(x)\,dx = f'(x).}$$

So $f'(x) = \int f''(x) = \int 6\,dx = 6x + C$. Now we need to integrate a second time to get f.

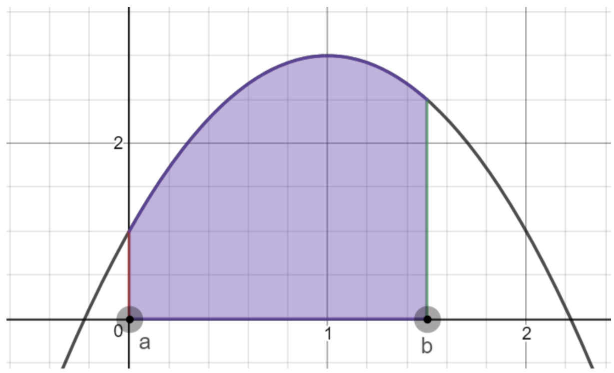
$$\int (6x + C)\,dx = 6x^2 + Cx + D$$

We end up with two unknown values, C and D as opposed to just a single value.

The previous two examples of equations involving derivatives. Any equation involving derivatives of a function is called a *differential equation*. We will look into this subject in the next chapter.

5.2 Area

In this section we attempt to find the area under a curve. That is the area that lies between a curve and the x-axis from $x = a$ to $x = b$. The area is bounded by the x-axis, a continuous curve $y = f(x)$ and the two vertical lines $x = a$ and $x = b$. This is shown in the figure with the area shaded in. Note the area stops at the axis. Area as calculated by integration is always in reference to the axis.

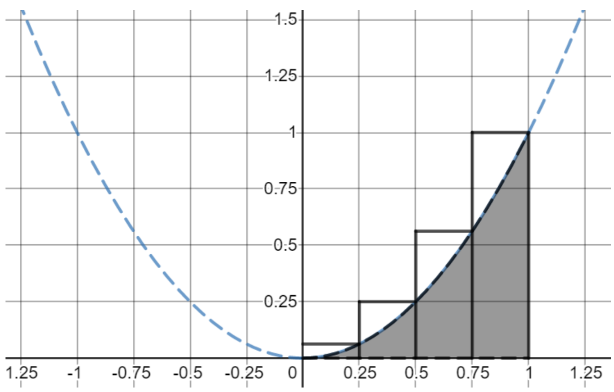


Previously, when we wanted to find the slope of a tangent line we found the slope of a secant line and applied the limiting process: $\lim_{h\to 0}$. A similar procedure will be used to find the area. We first approximate the area with rectangular strips then we take the limit of the areas of these rectangular strips by making the strips narrower and narrower and thus the number of strips between $x = a$ and $x = b$ greater and greater.

Area with Riemann Sums

EXAMPLE Given the parabola $y = x^2$, use rectangles to find the area under this curve between 0 and 1.

SOLUTION



Consider 4 strips by constructing vertical lines at $x = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$, and 1 as shown in the diagram. Rectangles are constructed using the right-hand boundary and we know from inspection this will be larger than the actual area.

$$\begin{aligned}\text{Right sum} &= \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2 + \frac{1}{4}(1)^2 \\ &= \frac{1}{64} + \frac{1}{16} + \frac{9}{64} + \frac{1}{4}\frac{15}{32} = 0.46875\end{aligned}$$

Repeating this process with a larger number of rectangles will improve the accuracy of the method. Using a spreadsheet like Excel shows a convergence on a value of $\frac{1}{3}$ as n rectangles increase. The sum using the left-hand boundary is included for comparison.

n	Left sum	Right sum
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
100	0.3283500	0.3383500
1000	0.3328335	0.3338335
∞	$\frac{1}{3}$	$\frac{1}{3}$

It can be seen that a very accurate approximation to the area can be obtained as the number of rectangles increases. It should be clear that as $n \to \infty$ both the left sum and the right sum approach the area under the curve we write

$$A = \lim_{n\to\infty} \text{ Left Sum } = \lim_{n\to\infty} \text{ Right Sum}$$

This process can be generalised by selecting any height within each rectangular strip and finding the area of each strip using this height. Let there be n strips and consider the i^{th} strip. Select a value of x in the i^{th} strip call it x_i. The height of this rectangle will be $f(x_i)$. Consider the situation described above where the area is bounded by the x-axis, a continuous curve $y = f(x)$ and the two vertical lines $x = a$ and $x = b$.

With n rectangles the length of the base of each rectangle is $\Delta x = \frac{b-a}{n}$. The area of the i^{th} rectangle is $f(x_i)\Delta x$.

The sum of all the rectangles is

$$\sum_{i=1}^{n} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x$$

And

$$A = \lim_{n\to\infty} \left[\sum_{i=1}^{n} f(x_i)\Delta x\right]$$

If f is a continuous function defined on the interval $[a, b]$ then as $n \to \infty$ the number represented by $\sum_{i=1}^{n} f(x_i)\Delta x \to A$ the area under the curve $y = f(x)$. This number is called the definite integral of f from a to b and is denoted by $\int_a^b f$ or $\int_a^b f(x)dx$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \left[\sum_{i=1}^{n} f(x_i)\Delta x\right]$$

This process is called a *Riemann sum* after the German mathematician Bernard Riemann (1826-1866) who defined the integral in this way. The symbol $\int$ was introduced by Leibniz and is called the *integral sign.*

Definite Integrals

The method of computing Riemann sums is often long and to achieve a result that is accurate enough requires a computer. Both Isaac Newton and Gottfried Leibniz discovered a much simpler way based on the integral. This discovery is called *The Evaluation Theorem*.

Evaluating Definite Integrals
Given F is an integral of f i.e. $F' = f$, provided f is continuous on the interval $[a, b]$ then

$$\int_a^b f(x)dx = F(b) - F(a)$$

This is an amazing result in view of the fact that it replaces such a complex procedure as finding Riemann sums over greater and greater numbers of elementary rectangles.

EXAMPLE Evaluate $\int_0^1 x^2 dx$.

SOLUTION Because we know a particular integral of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$ We have from the Evaluation Theorem

$$\int_0^1 x^2 dx = F(1) - F(0) = \frac{1}{3} \cdot 1^3 - \frac{1}{3} \cdot 0^3 = \frac{1}{3}$$

Looking back at the calculation of left sum and right sum above we can now see that the actual area that we were endeavouring to calculate was in fact $1/3$ or $0.\dot{3}$.

These are some of the different notations for using the Evaluation Theorem

$$F(b) - F(a) = F(x)|_a^b = [F(x)]_a^b = F(x)]_a^b$$

Area with Definite Integrals

Areas above the x-axis have *positive* definite integrals and areas below the x-axis have *negative* definite integrals. If the context of the question is to evaluate a definite integral then use the definition. If the question is about **area** you must find the parts of the question that have areas above the x-axis and those parts that have areas below the x-axis and evaluate them separately. The definite integral calculates the result as the net sum of the positive and negative areas. To find the total area take the absolute value of the individual parts.

EXAMPLE Find the area under the curve $y = x^3 - x$ between $x = -1$ and $x = 1$

SOLUTION This can be factored to give $y = x(x + 1)(x - 1)$. This cubic curve crosses the x-axis at -1, 0, and 1. Here, this area must be found in two parts.

Figure: A cubic showing how area 'under' the curve is evaluated. The area for $-1 \leq x \leq 0$ is positive (above the axis), and the area for $0 \leq x \leq 1$ is negative.

This is expected to be negative because it is below the axis:

$$\int_0^1 (x^3 - x)dx$$
$$= \frac{x^4}{4} - \frac{x^2}{2}\Big|_0^1$$
$$= [\frac{1}{4} - \frac{1}{2}] - 0$$
$$= -\frac{1}{4}$$

This is expected to be positive:

$$\int_{-1}^0 (x^3 - x)dx$$
$$= \frac{x^4}{4} - \frac{x^2}{2}\Big|_{-1}^0$$
$$= 0 - [\frac{1}{4} - \frac{1}{2}]$$
$$= +\frac{1}{4}$$

Therefore the total area is:

$$\Big| -\frac{1}{4}\Big| + \frac{1}{4} = \frac{1}{2} \text{ units}^2$$

We will compare with a single integral from -1 to 1.

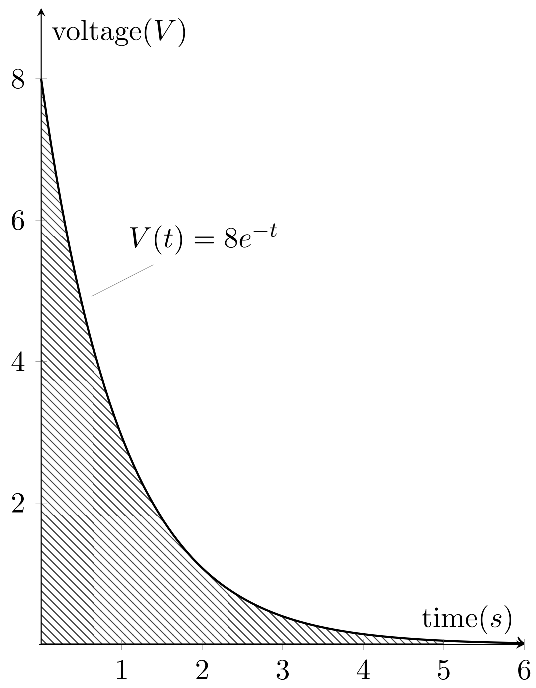
$$\int_{-1}^1 (x^3 - x)dx = \frac{x^4}{4} - \frac{x^2}{2}|_{-1}^1 = -\frac{1}{4} - (-\frac{1}{4}) = 0$$

Area must be non-negative, and so this result is nonsensical given the context.

EXAMPLE The energy, or electrical charge, that a capacitor can discharge is found by taking the integral of the voltage-time function. This can neatly be represented as the area under the voltage-time curve. Find the total discharge from the capacitor after 5 seconds. The units for charge are coulombs.

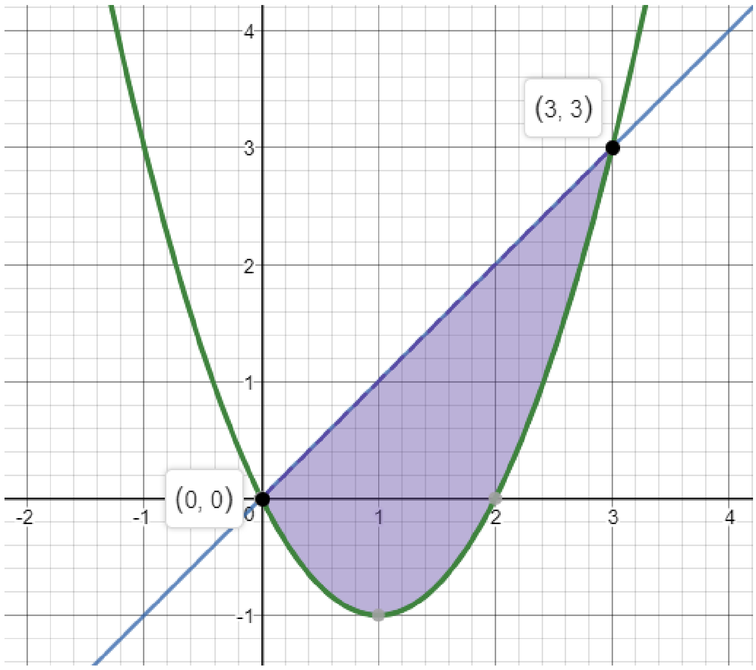
SOLUTION Integrate the $V(t)$ function to find the area:

$$\begin{aligned} \text{area} &= \int_a^b V(t)dt \\ &= 8\int_0^5 \left(e^{-t}\right) dt \\ &= 8\left[\left(-1e^{-t}\right)\Big|_0^5\right] \\ &= 8\left[-e^{-5} - (-e^0)\right] = 8.054 \text{ coulombs} \end{aligned}$$



The standard integral to calculate area is bounded by the axis as seen in the previous examples. In the following example we see that when finding area between intersecting curves, a new strategy can be applied.

EXAMPLE Find the area between the two curves: $f(x) = x$ and $g(x) = x^2 - 2x$. They intersect as shown at $(0,0)$, and $(3,3)$.



SOLUTION When inspecting the shaded region, the straight line function, $y = x$ is above the parabola. We could handle this with two separate integrals: find $\int f(x)$ and then subtract $\int g(x)$, or we can simplify and find $\int f(x) - g(x)$:

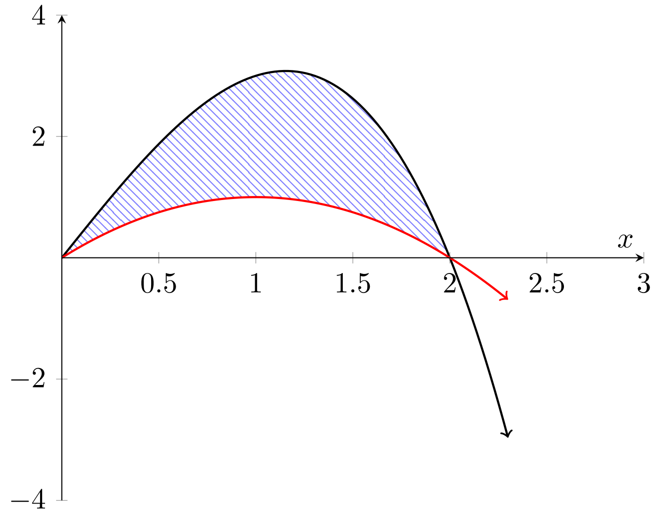
$$\begin{aligned}
\text{Area} &= \int_a^b [f(x) - g(x)]dx \\
&= \int_0^3 [x - (x^2 - 2x)]dx \\
&= \int_0^3 [-x^2 + 3x]dx
\end{aligned}$$

$$\begin{aligned}
&= -\frac{x^3}{3} + \frac{3x^2}{2}\Big|_0^3 \\
&= \left(-9 + \frac{27}{2}\right) - 0 = \frac{9}{2} \text{ units}^2
\end{aligned}$$

Area between functions
Let $f(x)$ be the upper function and $g(x)$ be the lower function, then

$$\text{Area } = \int_a^b [f(x) - g(x)]dx$$

EXAMPLE A logo is formed by the shaded area between the cubic function $f(x) = 4x - x^3$ and a parabola $g(x) = 2x - x^2$. The two curves intersects at $x = 0$ and $x = 2$. Find the shaded area.



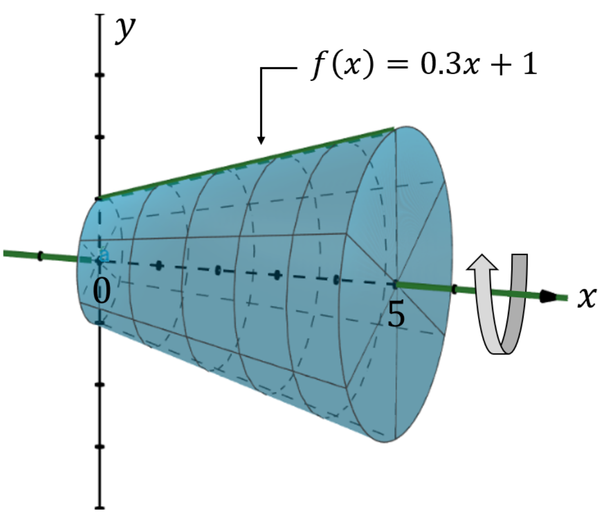
SOLUTION Identify the top curve $f(x)$ and subtract the bottom $g(x)$:

$$\begin{aligned}
\text{Shaded area} &= \int_a^b (f(x) - g(x))dx \\
&= \int_0^2 \left((4x - x^3) - (2x - x^2)\right) dx \\
&= \int_0^2 \left((2x - x^3 + x^2)\right) dx \\
&= \left[\left(2\frac{x^2}{2} - \frac{x^4}{4} + \frac{x^3}{3}\right)\Big|_0^2\right] \\
&= \left[\left(4 - \frac{16}{4} + \frac{8}{3}\right) - (0)\right] \\
&= \frac{8}{3}\ \text{units}^2
\end{aligned}$$

5.3 Volume

If a function is revolved around an axis it creates a volume between the axis and the function. Similar to how if we integrate a function, it results in an area — if we integrate an *area* it results in a volume.

EXAMPLE A connector was obtained by revolving the function $f(x) = 0.3x + 1$ around the x-axis for $0 \le x \le 5$. Calculate the volume of the connector.



SOLUTION
Given the function of the outer boundary of the shape, we must square the function and integrate this result.

$$\begin{aligned}
\text{volume} &= \int_a^b \pi[f(x)]^2 dx \\
&= \pi \int_0^5 (0.3x + 1)^2\, dx
\end{aligned}$$

Use the power rule and chain rule to integrate:

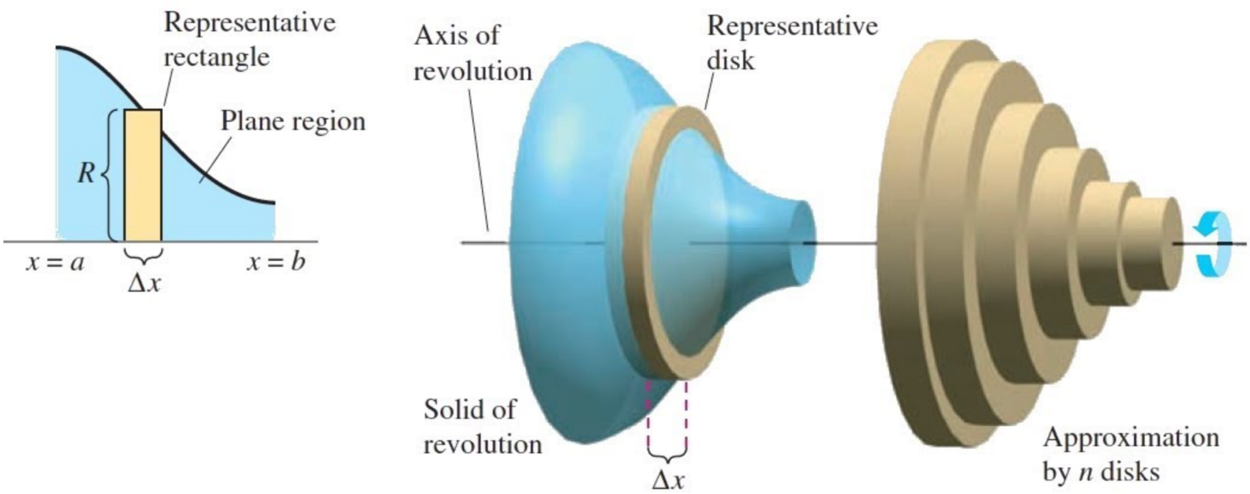
$$= \pi \frac{(0.3x + 1)^3}{0.3(3)}\Big|_0^5$$

Evaluate at the boundary points:

$$= \frac{10\pi}{9}\left[\left(\frac{5}{2}\right)^3 - 1\right] = \frac{65\pi}{4}$$

$$\approx 51.05\ \text{units}^3$$

The previous example used the disc-method to calculate a volume. Beginning with a rectangular section of width Δx, and length $f(x)$, just like the Riemann Sum for calculating area on page 89. This height becomes a radius when rotated around an axis of revolution, labelled R below. The *volume* of a single disc is $\pi R^2 \Delta x$.



Finding the volume of all the discs involves integrating over the length of the axis from a to b.

$$\text{Volume} = \int \pi R^2 \Delta x$$

The radius is the function evaluation, $f(x)$, and lowercase delta, δ, means infinitesimal thin discs as the number of them goes to infinity, $n \to \infty$, we can generalize the formula for volume:

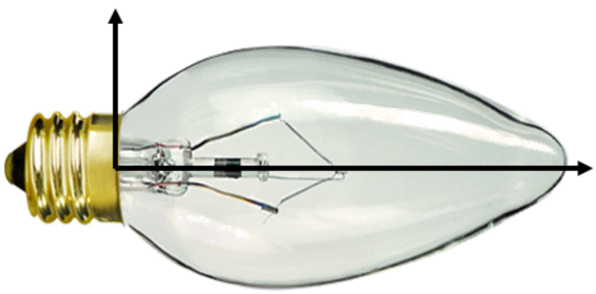
Volume rotated around the x-axis

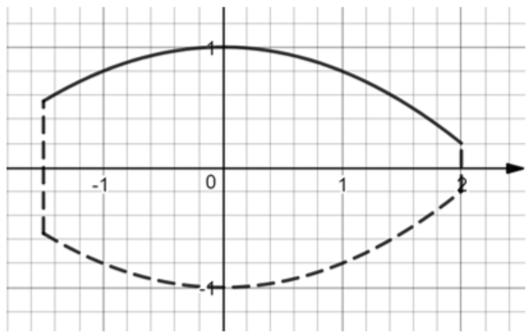
$$\text{Volume} = \int_a^b \pi y^2 dx = \int_a^b \pi (f(x))^2 dx$$

Volume rotated around the y-axis

$$\text{Volume} = \int_c^d \pi x^2 dy = \int_c^d \pi (f(y))^2 dy$$

Example Fluorescent and incandescent light bulbs are often filled with the inert gas krypton. Find the volume of krypton gas required to fill the bulb shown.





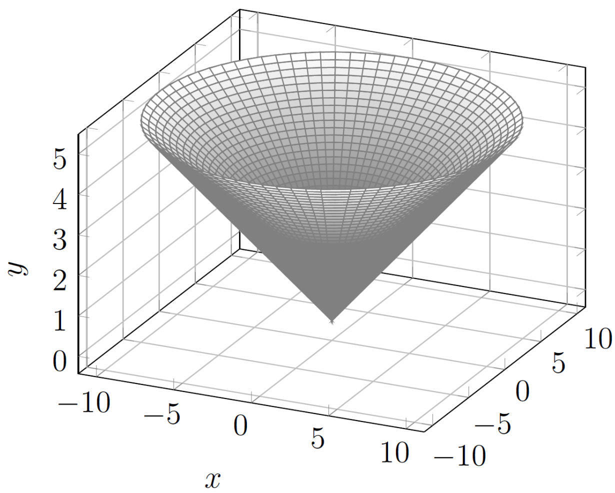
The function has been estimated to be:

$$f(x) = 1 - \frac{x^2}{5} \qquad \text{for } -1.5 \le x \le 2 \text{ cm}$$

SOLUTION You will have to calculate $\left(1 - \frac{x^2}{5}\right)^2$ before integrating

$$\begin{aligned}
\text{volume} &= \int_a^b \pi[f(x)]^2 dx \\
&= \pi \int_{-1.5}^{2} \left(1 - \frac{x^2}{5}\right)^2 dx \\
&= \pi \int_{-1.5}^{2} \left(\frac{x^4}{25} - \frac{2x^2}{5} + 1\right) dx \\
&= \pi \left(\frac{x^5}{125} - \frac{2x^3}{15} + x\right)\Bigg|_{-1.5}^{2} \\
&= \pi \left[\left(\frac{32}{125} - \frac{16}{15} + 2\right) - (-0.06075 - -0.45 - 1.5)\right] = 2.3\pi \approx 7.23 \ \text{cm}^3
\end{aligned}$$

EXAM QUESTION Calculate the volume of the container found by rotating the curve $y = \sqrt{x^3}$ around the y–axis for $0 \le y \le 5$.



Here the volume is created by rotation about the y-axis, and therefore we need to adjust our formula. First we will solve the equation $y = \sqrt{x^3}$ for x, and then integrate.

SOLUTION Volume $= \int_a^b \pi f(y)^2 dy$

Rearrange the function to isolate x:

$$\begin{aligned}
y &= \sqrt{x^3} \\
y^2 &= x^3 \\
\sqrt[3]{y^2} &= x
\end{aligned}$$

Integrate to find volume:

$$\begin{aligned}
V &= \pi \int_0^5 (y^{\frac{2}{3}})^2 \, dy \\
&= \pi \int_0^5 y^{\frac{4}{3}} \, dy \\
&= \pi \frac{y^{\frac{7}{3}}}{\frac{7}{3}}\Bigg|_0^5
\end{aligned}$$

$$= \frac{3\pi}{7}5^{\frac{7}{3}} - 0 \qquad\qquad \approx 57.6 \text{ units}^3$$

5.4 Integration by Substitution

Earlier we stated that every differentiation rule leads to an integration rule. The chain rule for differentiation leads to the substitution rule for integration, i.e. the *substitution rule for integration.*

A series of examples will illustrate how integration by substitution works.

Example Find the integral: $\int 2x\sqrt{1+x^2}\ dx$.

Solution This cannot be integrated using the techniques discussed so far. This is an example of a function that can be integrated using the substitution rule. They are often recognised by noting the presence of a composite function; the $\sqrt{1+x^2}$ can be seen as $\sqrt{u}$ where the $1+x^2$ under the root is replaced by a new variable.

Let $u = 1+x^2$. The integral now becomes $\int 2x\sqrt{u}\ dx$. This is not yet ready because there are two different variables: u **and** x. Lets differentiate the new equation:

$$u = 1 + x^2$$
$$\frac{du}{dx} = 2x$$

And we can isolate the dx with the intention of replacing it in the original integral, so $dx = \dfrac{du}{2x}$.

If we now look back at the original question we are ready to substitute expressions containing u for expressions containing x.

$$\begin{aligned}\int 2x\sqrt{1+x^2}\ dx &= \int \sqrt{u}\cdot 2x\ dx \\ &= \int \sqrt{u}\cdot \cancel{2x}\ \frac{du}{\cancel{2x}} \\ &= \int u^{\frac{1}{2}}\ du \\ &= \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} = \frac{2}{3}\sqrt[2]{u^3} + C\end{aligned}$$

One last step is to switch back to the original variable x. Make the final substitution $u = 1+x^2$.

$$= \frac{2\sqrt{(1+x^2)^3}}{3} + C$$

The Substitution Rule can be stated formally as follows. If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then $\int f(g(x))\, g'(x)\ dx = \int f(u)\ du$. dx and du are known as differentials. The Substitution Rule permits us to replace $g'(x)\ dx$ with du.

Example Find $\int x\sin\left(x^2\right)\ dx$

Solution Using the procedure from the previous example try $u = x^2$. This gives $du = 2x\ dx$.

$$\begin{aligned}\int x\sin\left(x^2\right)\,dx &= \int \sin\left(x^2\right)\cdot x\,dx\\ &= \int \sin(u)\cdot \cancel{x}\cdot \frac{du}{2\cancel{x}}\\ &= \frac{1}{2}\int \sin(u)\,du\\ &= -\frac{1}{2}\cos u + C\\ &= -\frac{1}{2}\cos\left(x^2\right) + C\end{aligned}$$

It is clear the challenge of the Substitution Rule is to come up with a suitable substitution. Earlier examples often have fairly obvious substitutions but they can quickly become quite complicated and result in many false starts.

EXAMPLE Find $\int \frac{x}{\sqrt{1-x^2}}\,dx$

SOLUTION Let $u = 1 - x^2$. Then $du = -2x\,dx$. So $x\,dx = -\frac{1}{2}du$. These are now substituted into the original expression

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du = -\frac{1}{2}\int u^{-1/2}\,du = -\frac{1}{2}\left(2u^{1/2}\right) + C = -\sqrt{1-x^2} + C$$

In each of the above 3 examples you could use desmos to compare the original function and the integral to see if the result is reasonable. For example 3 try graphing the original function $y = \frac{x}{\sqrt{1-x^2}}$ and the integral $y = -\sqrt{1-x^2}$ on the same axes and check the original function represents the slope of the tangent lines to the curve $y = -\sqrt{1-x^2}$.

Evaluating Definite Integrals by Substitution

We will look at two ways to evaluate a definite integral.

Method 1 Find the indefinite integral then use the Evaluation Theorem.

Method 2 Make the substitution to the integrand and differential (remembering to change the limits to the new variable).

EXAMPLE Evaluate $\int_0^8 \sqrt{x+1}\,dx$. Graphing provides us with the following:

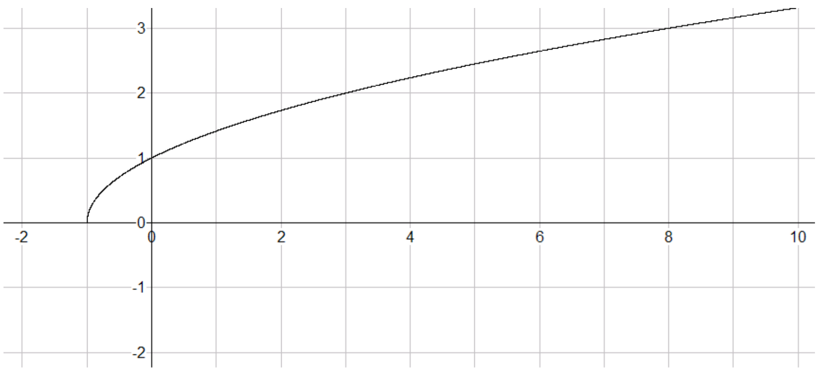


Figure: The curve is clearly continuous. If we let $u = x + 1$ then $u' = 1$, this is also continuous.

SOLUTION **Method 1** Substitute $u = x + 1$ then $du = dx$. Substituting these values in the indefinite integral we get

$$\int \sqrt{x+1}\, dx = \int \sqrt{u}\, du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(x+1)^{3/2} + C$$

So

$$\begin{aligned}
\int_0^8 \sqrt{x+1}\, dx &= \int \sqrt{x+1}\, dx \Big]_0^8 \\
&= \frac{2}{3}(x+1)^{3/2}\Big]_0^8 \\
&= \frac{2}{3}(9)^{3/2} - \frac{2}{3}(1)^{3/2} \\
&= \frac{2}{3}27 - \frac{2}{3}1 \\
&= \frac{2}{3}(27-1) = \frac{2}{3}26 = \frac{52}{3} = 17\frac{1}{3}
\end{aligned}$$

Method 2 Again we let $u = x + 1$ so $du = dx$. Also we calculate the new limits for u using $u = x + 1$.

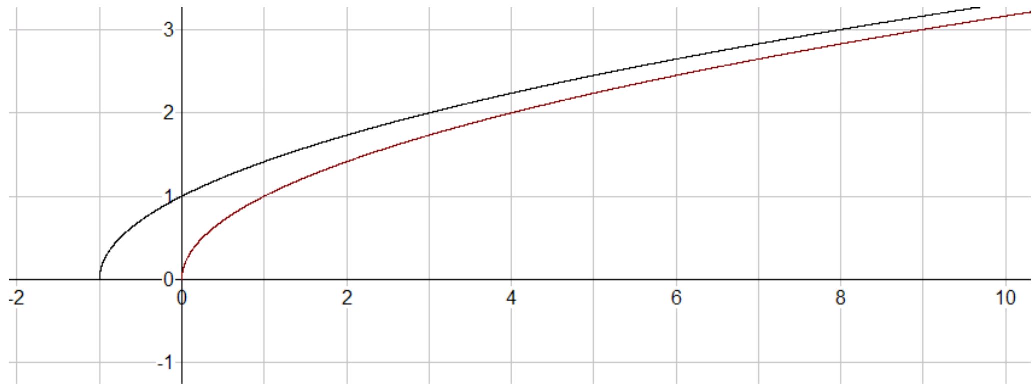
$$\begin{aligned}
x = 0 &\text{ gives } u = 0 + 1 = 1 \\
x = 8 &\text{ gives } u = 8 + 1 = 9
\end{aligned}$$

Now the definite integral is transformed into a definite integral in u.

$$\begin{aligned}
\int_0^8 \sqrt{x+1}\, dx &= \int_1^9 \sqrt{u}\, du \\
&= \frac{2}{3}u^{3/2}\Big]_1^9 \\
&= \frac{2}{3}9^{3/2} - \frac{2}{3}1^{3/2} \\
&= \frac{2}{3}(27-1) = 17\frac{1}{3}
\end{aligned}$$

A check on the graph will show that an area of $17\frac{1}{3}$ appears to be reasonable.

Method 2 is usually preferred as the step where the indefinite integral is first calculated has been neatly incorporated into the method. The difficulty with method 2 is that once the values for u have been calculated the integral is completely transformed and we never return to the original question. We answer a different question that, because of the transformation, has the same answer. This concept might cause confusion. However a graph of the situation shows what has happened.



It is clear that the area under the first graph between 0 and 8 is the same as the area under the second graph between 1 and 9.

5.5 Integration by Parts

Recall the rule for the differentiation of a product

$$\frac{d}{dx}\left(f(x) \cdot g(x)\right) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

We can integrate each side and write the process as follows

$$\begin{aligned} f(x) \cdot g(x) &= \int \left[f(x) \cdot g'(x) + g(x) \cdot f'(x)\right]\, dx \\ &= \int f(x) \cdot g'(x)\, dx + \int g(x) \cdot f'(x)\, dx \end{aligned}$$

This is rewritten in a particular way to become the formula for *integration by parts*.

$$\int f(x) \cdot g'(x)\, dx = f(x) \cdot g(x) - \int g(x) \cdot f'(x)\, dx \qquad (1)$$

$$\text{or} \int f(x) \cdot g'(x)\, dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)\, dx \qquad (2)$$

This formula is written in a number of different ways in textbooks. Here are two ways

Let $u = f(x)$ and $v = g(x)$ then $du = f'(x)\, dx$ and $dv = g'(x)\, dx$ so using the substitution rule equation (1) becomes

$$\int u\, dv = uv - \int v\, du \qquad (3)$$

Let $\int f(x) \cdot g'(x)\, dx$ be regarded as the the integral of the product of two functions then $g = \int g'$. We can write equation (2) as

$$\boxed{\int fg' = fg - \int f'g} \qquad (4)$$

Equations (3) and (4) are the common forms that are best to remember.

The success of this method depends on the discovery that a simpler integral results from this process. Sometimes the process results in a worse situation than you started with so should be abandoned. Sometimes you produce a pattern which leads to a solution after 2 or more applications of the integration by parts rule. The patterns that produce solvable problems can be discovered as different questions are tried.

EXAMPLE Find $\int x \cos x \, dx$. This can be seen as the integral of a product. The two functions are $f(x) = x$ and $g(x) = \cos x$. So $f'(x) = 1$ and $\int g(x) = \sin x$ are easy to find. Notice though that $f'(x) = 1$ gives us a clue that *integration by parts* is going to be a fruitful method.
SOLUTION Using equation (2)

$$\begin{aligned}
\int x \cos x \, dx &= f(x) \cdot g(x) - \int f'(x) \cdot g(x) \, dx \\
&= x \cdot \sin x - \int 1 \cdot \sin x \, dx \\
&= x \cdot \sin x - \int \sin x \, dx \\
&= x \cdot \sin x - (-\cos x) + C \\
&= x \ \sin x + \cos x + C
\end{aligned}$$

EXAMPLE Use integration by parts to find $\int \ln x \, dx$. This is a function that has arisen in the course as the integral of $\frac{1}{x}$, however we are now able to use this fact to help with this question.
SOLUTION Notice there is only one function here so we have to create two functions by stating $\ln x = 1 \cdot \ln x$. The two functions are therefore $f(x) = \ln x$ and $g(x) = 1$. Can we find f' and $\int g$? Yes!

$$\begin{aligned}
\int \ln x \, dx &= \int \ln x \cdot 1 \, dx \\
&= \ln x \cdot x - \int \frac{1}{x} \cdot x \, dx \\
&= x \ \ln x - \int 1 \, dx \\
&= x \ \ln x - x + C
\end{aligned}$$

EXAMPLE Find $\int e^x \cos x \, dx$ This is an example where a pattern is established and perseverance leads to the solution.
SOLUTION Let $f(x) = \cos x$ and $g(x) = e^x$. Can we find f' and $\int g$? Yes!

$$\begin{aligned}
\int e^x \cos x \, dx &= \cos x \cdot e^x - \int -\sin x \cdot e^x \, dx \\
&= e^x \cos x + \int e^x \sin x \, dx
\end{aligned}$$

This is an integral that is very similar in appearance to the original question with the $\cos x$ transformed into $\sin x$. We continue by repeating the integration by parts process with the new integral $\int e^x \sin x \, dx$

$$\int e^x \cos x \, dx = e^x \cos x + \overbrace{\sin x \cdot e^x - \int \cos x \cdot e^x dx}^{\int e^x \sin x \, dx}$$

$$= e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

Notice the original integral has now appeared on the right hand side of the equation. We now simply solve the equation for $\int e^x \cos x \, dx$ and add the constant of integration.

$$\begin{aligned} 2\int e^x \cos x \, dx &= e^x \cos x + e^x \sin x \\ \int e^x \cos x \, dx &= \frac{e^x}{2}(\cos x + \sin x) + C \end{aligned}$$

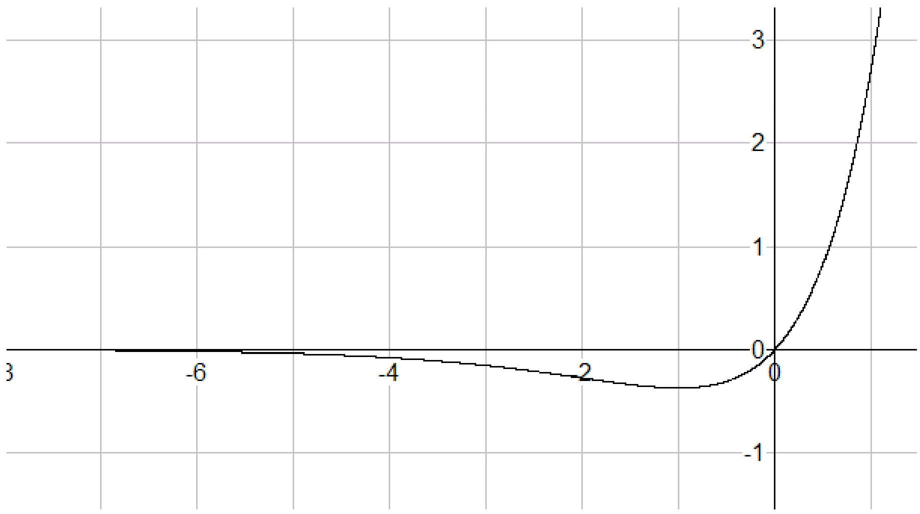
This can easily be verified by differentiating.

Definite Integrals

Integration by parts can be combined with the Evaluation Theorem to evaluate definite integrals. If we assume f' and g' are continuous then we can use the Evaluation Theorem and write equation (1) as follows

$$\int_a^b f(x) \cdot g'(x) \, dx = f(x) \cdot g(x)]_a^b - \int_a^b g(x) \cdot f'(x) \, dx \tag{5}$$

EXAMPLE Evaluate $\int_0^1 xe^x \, dx$. A graph of $y = xe^x$ shows the area required and gives us an idea of the answer to expect.



SOLUTION Notice x becomes simpler when differentiated and e^x is unchanged when it is integrated

$$\begin{aligned} \int_0^1 xe^x \, dx &= xe^x]_0^1 - \int_0^1 1 \cdot e^x \, dx \\ &= xe^x]_0^1 - e^x]_0^1 \\ &= \left(\left(1e^1\right) - \left(0e^0\right)\right) - \left(e^1 - e^0\right) \\ &= e^1 - e^1 + e^0 \\ &= e^0 = 1 \end{aligned}$$

5.6 Applications of Integration

Rectilinear Motion

We will use integration to analyse the motion of an object moving in a straight line. Let the position function for the object be $s = f(t)$ where t is the time. The velocity function is $v(t) = s'(t)$. Therefore the position function is the integral of the velocity function. Also the acceleration function is $a(t) = v'(t)$ so the velocity function is the integral of the acceleration function. We can obtain the position function from the acceleration function by integrating twice. This process will generate two constants of integration so we need two additional pieces of information to find the particular solution. Usually $s(0)$ and $v(0)$ are given.

EXAMPLE A particle moves in a straight line with an acceleration of $a(t) = 4t + 2$. If the initial velocity is -4cm/s and the initial displacement is 5cm, find the position function.

SOLUTION

As $v'(t) = a(t) = 4t + 2$ we can integrate $a(t)$ to obtain $v(t)$

$$\begin{aligned} v(t) &= \int a(t)dt \\ &= \int (4t+2)dt \\ &= 4\frac{t^2}{2} + 2t + C_1 \\ &= 2t^2 + 2t + C_1 \end{aligned}$$

Substitute $t = 0$ since we know the initial velocity $v(0) = -4$:

$$\begin{aligned} v(0) &= 2 \cdot 0^2 + 2 \cdot 0 + C_1 \\ C_1 &= -4 \text{ and therefore} \\ v(t) &= 2t^2 + 2t - 4 \end{aligned}$$

Integrate $v(t)$ to obtain $s(t)$:

$$\begin{aligned} s(t) &= \int v(t)dt \\ &= 2\frac{t^3}{3} + 2\frac{t^2}{2} - 4t + C_2 \\ &= \frac{2}{3}t^3 + t^2 - 4t + C_2 \end{aligned}$$

Substitute $t = 0$ because we are given the initial displacement i.e. $s(0) = 5$

$$\begin{aligned} s(0) &= \frac{2}{3} \cdot 0^3 + 0^2 - 4 \cdot 0 + C_2 \\ C_2 &= 5 \text{ and therefore} \\ s(t) &= \frac{2}{3}t^3 + t^2 - 4t + 5 \end{aligned}$$

The gravitational force near the Earth's surface that produces a downwards acceleration is about -9.8 m/s^2, represented by the vector quantity, $\vec{g}$.

EXAMPLE A ball is thrown vertically upwards with a speed of 24.5m/s from the edge of a cliff that is 147m above the ground.

(a) Find the position function.

(b) Find the time when the ball reaches its maximum height.

(c) Find the maximum height above the ground.

(d) When does the ball hit the ground?

Solution

(a) It is usual to choose the positive direction to be upwards this means that $\vec{g} = -9.8$ m/s^2. Integrating the acceleration function yields the velocity function.

$$\begin{aligned} v(t) &= \int a(t)dt \\ &= \int -9.8dt \\ &= -9.8t + C_1 \end{aligned}$$

Substituting $v(0) = 24.5$ we get that $C_1 = 24.5$

$$v(t) = -9.8t + 24.5$$

Because $s'(t) = v(t)$ we integrate $v(t)$

$$\int v(t) = s(t) = -9.8\frac{t^2}{2} + 24.5t + C_2$$

Substitute $s(0) = 147$. We get $C_2 = 147$.

$$\therefore s(t) = -4.9t^2 + 24.5t + 147$$

This expression will hold true until the ball is acted on by an external force (hits the ground).

(b) The maximum height is reached when $v(t) = 0$; for a moment the vertical velocity will be zero before it begins to fall.

$$\begin{aligned} -9.8t + 24.5 &= 0 \\ 24.5 &= 9.8t \end{aligned}$$

$$t = \frac{24.5}{9.8} = 2.5 \text{ s}$$

(c) The maximum height is found by substituting $t = 2.5$ into $s(t)$

$$\begin{aligned} s(2.5) &= -4.9 \cdot 2.5^2 + 24.5 \cdot 2.5 + 147 \\ &= 177.6 \text{ m} \end{aligned}$$

(d) The ball hits the ground when the displacement, $s(t) = 0$.

$$\begin{aligned} -4.9t^2 + 24.5t + 147 &= 0 \\ t^2 - 5t - 30 &= 0 \end{aligned}$$

Solve for t using the quadratic formula:

$$\begin{aligned} t &= \frac{5 \pm \sqrt{25 - 4 \times 1 \times -30}}{2} \\ &= \frac{5 \pm \sqrt{145}}{2} \\ &= 8.5 \text{ s, or } -3.5\text{s} \end{aligned}$$

The negative result will be discarded, however, represents a solution to the parabola in an algebraic sense. We did not test whether the value of t when $v(t) = 0$ gives a maximum or minimum. A maximum can be assumed knowing that -9.8 m/s^2 has a negative coefficient.

Work

The strategy we use to allow us to apply calculus to a problem in engineering is the same as we used to evaluate areas. The physical quantity is divided up into a large number of small parts, each one approximately equal to the theoretical quantity it represents. These are then summed and a limit taken as the number of small parts, $n \to \infty$. This process allows us to evaluate the resulting integral.

You will recall that from Newton's Second Law of Motion

$$F = ma = m\frac{d^2s}{dt^2} \tag{1}$$

Where $s(t)$ is the position function, m is the mass of an object and F is the force required to produce an acceleration of a. (Where $a = \frac{d^2s}{dt^2}$).

We usually measure mass in kilograms (kg), distance in metres (m) and force in newtons (N)

($N = kg \cdot m/s^2$). If the acceleration is constant then the force to produce that acceleration will also be constant.

Work = force × distance or

$$W = Fd \tag{2}$$

If F is in newtons and d is in metres then W is in newton-metres. One newton-metre is called a joule (J).

EXAMPLE A mass of 3.5 kg lifted 0.5 m requires a force of $F = ma = 3.5 \times 9.8 = 34.3$N. This is the force required to counter the force exerted by gravity. Calculate the work done.

SOLUTION The work done is calculated using equation 2

$$W = Fd = 34.3 \times 0.5 = 17.15 \text{ J}$$

If the force is variable this formula can no longer be applied. Let the force acting on an object as it moves along the x-axis in a positive direction from a to b be $f(x)$, where f is a continuous function of x. Divide the interval from a to b into n subintervals of width Δx where $\Delta x = \frac{b-a}{n}$. For simplicity we let the end points of these subintervals be x_0, x_1, x_2, ..., x_n We select the i^{th} subinterval and select a representative x-value in this interval x_i^*. The work done when we move the object from x_{i-1} to x_i is

$$W_i \approx f(x_i^*)\Delta x$$

The total work done is

$$W = \sum_{i=1}^{n} f(x_i^*)\Delta x \tag{3}$$

As we did with area we find the limit as $n \to \infty$ of this sum. As this is a Riemann sum the limit is a definite integral

$$W = \sum_{i=1}^{n} f(x_i^*)\Delta x = \int_a^b f(x)\, dx \tag{4}$$

EXAMPLE If the force on a particle is given by the equation $f(x) = 3x^2 - 2x$ N, how much work is done moving the particle from $x = 2$ to $x = 3$?

SOLUTION The graph of $f(x) = 3x^2 - 2x$ shows that for the interval $[2, 3]$ the area is above the x-axis.

$$\begin{aligned}
W &= \int_a^b f(x)\, dx \\
&= \int_2^3 \left(3x^2 - 2x\right)\, dx \\
&= 3\frac{x^3}{3} - 2\frac{x^2}{2}\Big]_2^3 \\
&= x^3 - x^2\Big]_2^3 \\
&= \left(3^3 - 3^2\right) - \left(2^3 - 2^2\right) \\
&= 18 - 4 = 14 \text{ J}
\end{aligned}$$

EXAM QUESTION Hooke's Law states that the force required to maintain a spring stretched x units beyond its natural length is proportional to x, the spring displacement in meters, such that $f(x) = kx$ where k is the spring constant and f the force. This law holds provided x does not get too large.

The work required to stretch the spring can be calculated by integrating the force function:

$$W = \int_a^b f(x)\,dx\,.$$

Find the work done in stretching a spring from 12 cm to 17 cm given that it takes 4 newtons of force to stretch the spring from 10 cm to 12 cm.
SOLUTION

First you will need to find the spring constant k:

$$\begin{aligned} f(x) &= kx \\ 4\,\mathrm{N} &= k(0.12 - 0.10) \\ k &= \frac{4}{0.02} = 200\ \frac{\text{newtons}}{\text{meter}} \end{aligned}$$

Integrate the force function for find the work done in stretching the spring:

$$\begin{aligned} W &= \int_{0.12}^{0.17} 200x\,dx \\ &= 100x^2\Big|_{0.12}^{0.17} \\ &= 100(0.17^2 - 0.12^2) \\ &= 1.45\ \text{J} \end{aligned}$$

5.7 Chapter Exercises

§5.1 Standard Integrals

1. Find the general integral in each case.

(a) $4x^7$ **(b)** 10 **(c)** $\frac{8x^3}{3}$

(d) $3.2e^x$ **(e)** $\frac{1}{2x}$ **(f)** $\sqrt{x}$

(g) $x^{1.2}$ **(h)** x^π **(i)** $\frac{4}{\sqrt[3]{x^4}}$

(j) $x^{-\frac{2}{3}}$ **(k)** $(2x+1)^2$ **(l)** $e^x + x^e$

2. Find the indefinite integrals.

(a) $\int \left[x - \frac{1}{x^2}\right]\, dx$ **(b)** $\int x\sqrt{x}\, dx$ **(c)** $\int (\sin x - 2\cos x)\, dx$

(d) $\int (1-x)(2-x)\, dx$ **(e)** $\int \left(\sin^2 x + \cos^2 x\right)\, dx$ **(f)** $\int \left(\frac{2}{x} + \frac{3}{\sqrt{x}}\right)\, dx$

(g) $\int \frac{\cos x}{2}\, dx$ **(h)** $\int \frac{1}{2}\sec^2 x\, dx$ **(i)** $\int \left(x + 1 + \frac{1}{x}\right)\, dx$

(j) $\int \pi\left(r^2 - x^2\right)\, dx$ **(k)** $\int \frac{1}{2x}\, dx$ **(l)** $\int 2\pi r\, dr$

§5.2 Area

1. Evaluate the integrals.

(a) $\int_{-1}^{2} x^5\, dx$ **(b)** $\int_1^3 \left(1 + 2x - 4x^3\right)\, dx$ **(c)** $\int_0^1 x^{2/3}\, dx$

(d) $\int_1^3 e^x\, dx$ **(e)** $\int_1^3 \frac{1}{x}\, dx$

2. Find the area under the curve.

(a) $y = x^3$ between $x = -1$ and $x = 3$

(b) $y = (x+1)(x-1)$ between $x = -2$ and $x = 3$

3. Find the integral and evaluate according to the limits.

(a) $\int_0^2 (x-1)(x+5)\, dx$ **(b)** $\int_1^2 \frac{x^3-6}{x^2}\, dx$ **(c)** $\int_2^3 \frac{dx}{3x}$

(d) $\int_0^1 3e^x\, dx$ **(e)** $\int_{-2}^{2} x^4\, dx$ **(f)** $\int_{-1}^{1} x^3\, dx$

§5.3 Volume

1. Determine the volumes of the solids of revolution which are generated by rotating the following areas for one complete revolution about the x-axis.

(a) $y = 2x + 1$ between $x = 0$ and $x = 3$
(b) $y = 6x^2 + 1$ between $x = 1$ and $x = 4$
(c) $y = -3x$ between $x = -2$ and $x = 0$
(d) $5\sqrt{x}$ for $1 \le x \le 2$

2. Determine the volume found when the function is rotated one revolution around the y-axis.

(a) $y = x - 1$ between $y = -1$ and $y = 3$
(b) $y = x^2 - 3$ between $y = -3$ and $y = 0$
(c) $y = \sqrt{x}$ for $0 \le y \le 2$
(d) $y = \sqrt{x + 4}$ from $y = 0$ to $y = 4$

3. The volume for a cylinder with radius r and height h is given by $V = \pi r^2 h$ Use integration to show that this formula is correct. Hint: Which curve $y = f(x)$ is rotated around the x-axis to form a cylindrical solid of rotation?

§5.4 Integration by Substitution

1. Use the given substitution to find the integral.

(a) $\int e^{4x}\, dx, \quad u = 4x$
(b) $\int \tan x\, dx, \quad u = \cos x$
(c) $\int e^{\sin\theta} \cos\theta\, d\theta, \quad u = \sin\theta$
(d) $\int (4x - 3)^{15}\, dx, \quad u = 4x - 3$
(e) $\int x e^{x^2}\, dx, \quad u = x^2$
(f) $\int \frac{x}{x^2+1}\, dx, \quad u = x^2 + 1$
(g) $\int 2x \left(x^2 + 2\right)^3\, dx \quad u = x^2 + 2$
(h) $\int \sin^2 x \cos x\, dx, \quad u = \sin x$
(i) $\int \frac{x}{\sqrt{1+x}}\, dx, \quad u = 1 + x$
(j) $\int \frac{dx}{\sqrt{x}+x}, \quad u = \sqrt{x}$

2. Integrate the following. The substitution has not been given.

(a) $\int \sqrt{2x + 1}\, dx$
(b) $\int \frac{dx}{\sqrt{x+1}}$
(c) $\int \frac{e^x}{e^x+1}\, dx$
(d) $\int \frac{\ln x}{x}\, dx$
(e) $\int \frac{\cos x}{\sin x+1}\, dx$
(f) $\int \cos \frac{x}{2}\, dx$
(g) $\int \cos 2x\, dx$
(h) $\int \frac{e^x+e^{-x}}{2}\, dx$
(i) $\int_0^2 (x - 1)^{11}\, dx$

3. Use an appropriate substitution and then evaluate the integral.

(a) $\int_0^2 x\sqrt{2x^2 + 1}\, dx$
(b) $\int_0^{\frac{\pi}{4}} \sin\left(x + \frac{\pi}{2}\right)\, dx$
(c) $\int_0^1 (3 - 2x)^4\, dx$
(d) $\int_{-2}^{-1} \frac{dx}{(2x+1)^2}$
(e) $\int_4^9 \frac{dx}{2x+1}$
(f) $\int_0^{\frac{\pi}{4}} \tan x\, dx$

§5.5 Integration by Parts

1. Find the integral using integration by parts.

(a) $\int x \ln x\, dx$
(b) $\int x \sin x\, dx$
(c) $\int x \cos 3x\, dx$
(d) $\int e^x \sin x\, dx$
(e) $\int \ln(2x)\, dx$

2. By writing $\sin^2 x$ as $\sin x \cdot \sin x$ and using integration by parts show that

$$\int \sin^2 x\, dx = \frac{1}{2}(x - \sin x \cdot \cos x) + C$$

3. Integrate.

(a) $\int xe^{2x}\, dx$

(b) $\int xe^{-x}\, dx$

(c) $\int x^2 \sin x\, dx$

(d) $\int (x-1)\, e^x\, dx$

(e) $\int \frac{\ln x}{x}\, dx$

4. Integrate and evaluate.

(a) $\int_0^5 xe^{-x}\, dx$

(b) $\int_0^1 x^2 e^x\, dx$

(c) $\int_0^\pi e^x \sin x\, dx$

(d) $\int_1^3 x \ln x\, dx$

(e) $\int_0^\pi \sin^2 x\, dx$

(f) $\int_0^{\frac{\pi}{2}} x \sin x\, dx$

(g) $\int_0^1 xe^{2x}\, dx$

(h) $\int_5^7 x \cos x\, dx$

(i) $\int_1^2 (x-1)\, e^x\, dx$

(j) $\int_2^4 \frac{\ln x}{x}\, dx$

(k) $\int_0^3 x^2 \sin x\, dx$

§5.6 Applications

1. A particle is moving in a straight line with an acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6cm/\text{s}$ and its initial displacement is $s(0) = 9\text{cm}$. Find the position function $s(t)$.

2. A ball is thrown upwards with an initial velocity of 15m/s from the edge of a cliff 140m above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground?

Hits the ground after $t \approx 7.1$s

3. A particle is moving along a straight line with an acceleration of $a(t) = 3 + 4t - 12t^2$. Its initial velocity, $v(0)$ is 4m/s and its initial displacement, $s(0)$ is 5m. Find the position function, $s(t)$.

4. A stone is dropped off a cliff and hits the ground with a speed of 40m/s. What is the height of the cliff?

5. A car is travelling at 50km/h when the brakes are firmly applied producing a constant deceleration of 5 m/s^2. How far will the car travel before coming to rest?

6. A car is travelling at 100km/h when the driver sees a railway crossing 80m ahead and slams on the brakes. What constant deceleration is required so that the car will stop in 80m?

7. Two balls are thrown upwards from the edge of the cliff in exercise 2 above. The first is thrown with a speed of 15m/s and the second is thrown one second later with a speed of 8m/s. Do the balls ever pass each other?

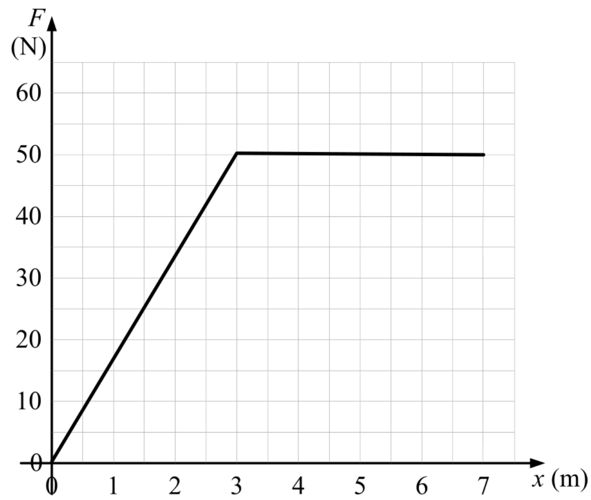
8. A particle moves along a straight line with a velocity function $v(t) = \sin t - \cos t$ and its initial displacement is $s(0) = 0\text{m}$. Find its position function.

9. A car braked with a constant deceleration 5m/s^2, producing skid marks measuring 60m before coming to rest. How fast was the car travelling when the brakes were first applied?

10. A particle is moved along the x-axis by a force given by the equation $f(x) = 6/\left(1+x\right)^2$ N at a point x metres from the origin. Find the work done in moving the particle from the origin to a distance of 5 metres.

11. When a particle is located a distance of x metres from the origin a force of $\cos(\pi x/3)$ Newtons acts on it. How much work is done in moving the particle from $x = 1$ to $x = 2$? Interpret your answer by considering the work done from $x = 1$ to $x = 1.5$ and from $x = 1.5$ to $x = 2$. By drawing the graph of $y = \cos(\pi x/3)$ the reason should become apparent.

12. The graph shows a force function . (Force in Newtons against distance in metres.) The force increases at a constant rate until it reaches its maximum value then remains constant. How much work is done by the force in moving an object a distance of 7m?



13. A spring has a natural length of 20cm. If a 25N force is required to keep it stretched to a length of 30 cm how much work is done stretching it from a length of 20cm to 25cm?

14. Suppose that 2J of work is needed to stretch a spring from its natural length of 20cm to a length of 30cm. How much work is needed to stretch it from a length of 22cm to 28cm?

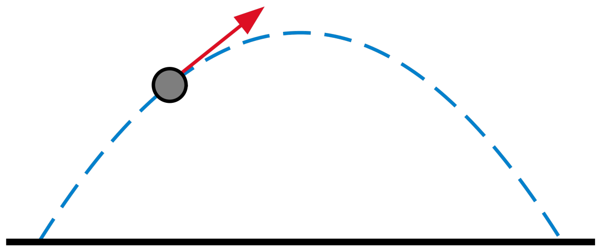
15. If 6J of work is needed to stretch a spring from 10cm to 12cm and 10J is needed to stretch it from 12cm to 14cm, what is the natural length of the spring?

6 | Differential Equations

Any equation involving derivatives of a function can be called a differential equation. For example, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$ says "the derivative equals $2x$." As the derivative of x^2, this simple equation also represents a rate-of-change as a function of x, or, a differential equation. In this chapter we explore this idea further and look at some mathematical models that take the form of differential equations.

The trajectory of a projectile launched from a cannon follows a curve determined by an ordinary differential equation that is derived from Newton's second law. The relationship between the displacement x and the time t of an object under the force F, is given by the differential equation:

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = F(x(t))$$



Change is a familiar concept from the slope of the tangent line that we are familiar with. If we have the function, then we can calculate the slope of the tangent at any point on the axis. Now, if the x-axis represents time, t, this calculation has the possibility of predicting the future! Differential equations can arise when we formulate mathematical models. We can develop our understanding of this process by considering the mathematical models of some physical phenomena.

6.1 Basic Differential Equations

One model of population growth arises from the assumption that the rate at which the population grows is proportional to the size of the population. Let N be the size of the population at time t then the rate of change of population with respect to time is $\frac{\mathrm{d}N}{\mathrm{d}t}$. So the model can be expressed as a differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = kN$$

where k is the constant of proportionality.

To make any sense of this model we need to explore the range of values of N and k. N cannot be zero (otherwise there would be nothing to change). We assume that N is a function of t and that $N(t) > 0$. Similarly for us to have population "growth" $k > 0$

$$\therefore \frac{\mathrm{d}N(t)}{\mathrm{d}t} > 0$$

We have already shown the properties of the exponential function. The general exponential function is

$$N(t) = N_0e^{kt}$$

Let $N(t)$ be N_0 when $t = 0$. i.e. $N(0) = N_0$. N_0 is called the *initial value* of $N(t)$.

Now

$$\begin{aligned}\frac{\mathrm{d}N(t)}{\mathrm{d}t} &= N_0 e^{kt} \times k \\ &= kN(t)\end{aligned}$$

Thus we have shown that $N(t) = N_0 e^{kt}$ is a solution of the differential equation $\frac{\mathrm{d}N(t)}{\mathrm{d}t} = kN(t)$

This solution arises because we are familiar with the behaviour of the exponential function. Our ability to "guess" the answer is limited so the subject of differential equations involves developing techniques to handle physical situations that are more and more realistic and more and more complex. The answer to this problem came so simply to us we have to wonder if there are other equations for $N(t)$ that give the same answer.

The Order of a Differential Equation

The differential equation $\frac{\mathrm{d}N(t)}{\mathrm{d}t} = kN(t)$ is referred to as a first order differential equation because the order of the highest derivative is *one*.

Here is an example of a second order differential equation

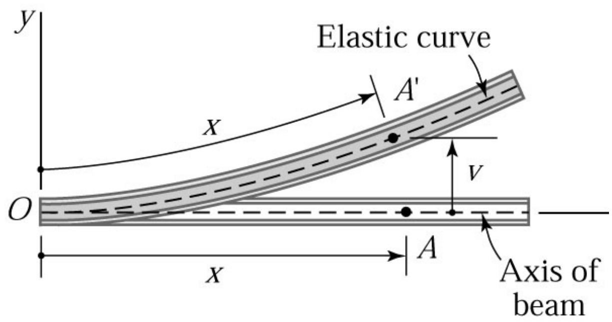
$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -kx$$

This is the differential equation that arises from Hooke's Law. The second derivative is written $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}$. Another application from mechanics is found in construction.

The elastic curve of a beam under a uniform distributed load will be deflected according to:

$$M = EI\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

Where M is the bending moment acting at O. E is Young's modulus of elasticity of the material of the beam, and I is the moment of inertia of the beam section.



The Solution of a Differential Equation

When you are asked to "solve" a differential equation you are expected to find all possible solutions of the differential equation. Thus, the solution is another function.

Example Find all possible solutions of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$. This differential equation could also be expressed as $y' = 2x$.

Solution By integrating we obtain

$$y = x^2 + C$$

Where C is the constant of integration. This is an arbitrary constant and gives us a family of functions all of which are solutions of the differential equation $y' = 2x$. This family of solutions is often referred to as the *general solution*.

In a physical situation we are often provided with additional information and this will allow us to find a *particular solution*. This is easy to visualise when considering $y' = 2x$. The general solution is $y = x^2 + C$ and if you are also told that the curve passes through $(2, 6)$ you can use this information to evaluate C.

$$\begin{aligned} 6 &= 2^2 + C \\ C &= 2 \end{aligned}$$

So the particular solution is $y = x^2 + 2$. To appreciate the concept of this particular solution you only need to use desmos to draw $y = x^2 + C$ for various values of C.

Initial-Value Problems

In a physical problem when you are given the conditions for the particular solution in the form $y(t_0) = y_0$ where t_0 is the initial value of t and y_0 is the initial value of $y(t)$, the point (t_0, y_0) is called an *initial condition* and the problem of finding the particular solution given the differential equation is referred to as an *initial-value problem*.

EXAMPLE For the differential equation $y' = -y^2$

(a) Verify that $y = \frac{1}{t+C}$ is the general solution
(b) Find the solution of the initial-value problem $y' = -y^2$ and $y\,(0) = 0.5$

SOLUTION

(a) Given $y = \frac{1}{t+C} = (t+C)^{-1}$

$$\begin{aligned} y' &= -\,(t+C)^{-2} \\ &= \frac{-1}{(t+C)^2} \\ &= -\left(\frac{1}{t+C}\right)^2 = -y^2 \end{aligned}$$

So $y = \frac{1}{t+C}$ is the general solution of $y' = -y^2$

(b) Substitute $(0, 0.5)$ in $y = \frac{1}{t+C}$

$$\begin{aligned} 0.5 &= \frac{1}{0+C} \\ \frac{1}{2} &= \frac{1}{C} \\ C &= 2 \end{aligned}$$

The particular solution is $y = \frac{1}{t+2}$

EXAM QUESTION

Find the particular solution to the differential equation $1 + x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = x^3$ given that $y(1) = \frac{5}{2}$.

SOLUTION Isolate the derivative, $\frac{\mathrm{d}y}{\mathrm{d}x}$, and integrate to find the function y:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x^3 - 1}{x^2} \\ \int \frac{\mathrm{d}y}{\mathrm{d}x} &= \int x dx - \int \frac{1}{x^2} dx \end{aligned}$$

$$y = \frac{x^2}{2} + \frac{1}{x} + C$$

Use the initial condition $y(1) = \frac{5}{2}$ to solve for the unknown constant:

$$\begin{aligned} \frac{5}{2} &= \frac{1^2}{2} + \frac{1}{1} + C \\ C &= 1 \end{aligned}$$

Therefore the particular solution is $\qquad y = \dfrac{x^2}{2} + \dfrac{1}{x} + 1$

6.2 Separable Equations

In some special cases we can find explicit solutions of differential equations. One type of equation can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{f(x)}{g(y)}$$

Expressed in this form we simply have to recognise that $f(x)$ is a function without any y's in it and $g(y)$ is a function without any x's in it. The variables can be *separated* by cross-multiplying such that:

$$g(y)\,dy = f(x)\,dx$$

Then we integrate both sides

$$\int g(y)\mathrm{d}y = \int f(x)\mathrm{d}x$$

This procedure can be verified by differentiating both sides with respect to x

$$\frac{\mathrm{d}}{\mathrm{d}x}\int g(y)\mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}x}\int f(x)\mathrm{d}x$$

By the chain rule the left hand side becomes

$$\frac{\mathrm{d}}{\mathrm{d}y}\int g(y)\,\mathrm{d}y \times \frac{\mathrm{d}y}{\mathrm{d}x}$$

So

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}y}\int g(y)\mathrm{d}y \times \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\int f(x)\mathrm{d}x \\ g(y) \times \frac{\mathrm{d}y}{\mathrm{d}x} &= f(x) \\ \text{or} \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{f(x)}{g(y)} \end{aligned}$$

EXAMPLE Solve the differential equation by the method of separating variables

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y^2}$$

SOLUTION The solution to a differential equation is another function. Separate the variables and integrate both sides.

$$\begin{aligned} y^2 \mathrm{d}y &= x^2 \mathrm{d}x \\ \int y^2 \mathrm{d}y &= \int x^2 \mathrm{d}x \\ \frac{y^3}{3} &= \frac{x^3}{3} + C \\ y^3 &= x^3 + 3C \\ &= x^3 + C_1 \end{aligned}$$

Where C_1 is a new arbitrary constant. Isolate y to find a function of the form $y = f(x)$:

$$y = \sqrt[3]{x^3 + C_1}$$

Find C_1 given $y(0) = 2$

$$\begin{aligned} 2 &= \sqrt[3]{0^3 + C_1} \\ C_1 &= 8 \\ \text{therefore } y &= \sqrt[3]{x^3 + 8} \end{aligned}$$

EXAMPLE Solve the differential equation:

$$y' = 3x^2 y$$

SOLUTION First write $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 y$$

Separate the variables

$$\frac{1}{y}\mathrm{d}y = 3x^2 \mathrm{d}x$$

Integrate

$$\begin{aligned} \int \frac{1}{y}\mathrm{d}y &= \int 3x^2 \mathrm{d}x \\ \ln|y| &= x^3 + C \qquad (1) \end{aligned}$$

We usually write this by writing in exponential form

$$\begin{aligned} |y| &= e^{x^3 + C} \\ &= e^{x^3} e^C \\ y &= \pm e^C e^{x^3} \end{aligned}$$

And write $A = \pm e^C$ where the value of A is used that satisfies the particular problem

$$y = Ae^{x^3} \qquad (2)$$

In practice we usually jump from line (1) to line (2) and leave out the intermediate steps.

6.3 Chapter Exercises

§6.1 Differential Equations

1. Solve the following differential equations and use the given conditions to find the particular solution.

(a) $\frac{\mathrm{d}y}{\mathrm{d}t} = e^t + 2t \qquad y(0) = 2$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = \sin 2x + \cos 2x \qquad y(0) = 0$

(c) $t\frac{\mathrm{d}y}{\mathrm{d}t} = 1 - t^2 \qquad y\,(1) = 2$

(d) $x^2\frac{\mathrm{d}P}{\mathrm{d}x} = 2 + x \qquad P(1) = 2$

2. Show that $y = x + \frac{1}{x}$ is a solution of the differential equation $xy' + y = 2x$

3. Verify that $y = \sin x \cos x - \cos x$ is a solution of the initial-value problem

$$y' + (\tan x)\,y = \cos^2 x \qquad y(0) = -1$$

on the interval $\frac{-\pi}{2} < x < \frac{\pi}{2}$.

4. Which of the following functions are solutions of the differential equation $y'' + y = \sin x$
(a) $y = \sin x$ (b) $y = \cos x$ (c) $y = \frac{1}{2}x\sin x$ (d) $y = -\frac{1}{2}x\cos x$

5. A mass falling to earth with a constant acceleration of $9.8\mathrm{m/s^2}$ satisfies the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 9.8$$

where v is the velocity at time t. Find an expression for v if

(a) The mass is dropped from a stationary position.

(b) The mass is fired towards earth with an initial velocity of 100m/s.

6. The angular velocity ω of a flywheel under constant braking torque of N is given by the differential equation

$$I\frac{\mathrm{d}\omega}{\mathrm{d}t} + N = 0.$$

where I is the moment of inertia (I is a constant)

(a) Find ω in terms of t given that $\omega = \omega_0$ when $t = 0$.

(b) Calculate the time taken to bring the flywheel to rest from an initial speed of 60π rad/s given that the moment of inertia is 100 kgm^2 under a braking torque of 40 Nm.

§6.2 Separable Equations

1. Solve the following differential equations

(a) $y\frac{\mathrm{d}y}{\mathrm{d}x} = x$ given $y = 4$ when $x = 0$

(b) $\frac{1}{t}\frac{\mathrm{d}v}{\mathrm{d}t} = 2$ given $v = 2$ when $t = 1$

(c) $e^x\frac{\mathrm{d}y}{\mathrm{d}x} + 2 = 0$ given $y = 5$ when $x = 0$

(d) $(x + 2)\frac{\mathrm{d}y}{\mathrm{d}x} = y + 3$ given $y = 0$ when $x = 0$

(e) $\frac{\mathrm{d}i}{\mathrm{d}t} + i = 1$ given $i = 10$ when $t = 0$

(f) $\frac{\mathrm{d}y}{\mathrm{d}x} - 2x = 0$ $y(0) = 2$

2. The rate at which the atoms of a radioactive substance split up is given by $\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N$, where N is the number of atoms present after t seconds and λ is a constant.

(a) Show that $N = N_0e^{-\lambda t}$ where N_0 is the number of atoms present initially (i.e. when $t = 0$).

(b) Find the time in years for half of the atoms of a given mass of radium to disintegrate if $\lambda = 1.37 \times 10^{-11}$ for radium.

3. A body falling in a medium where the resistance is proportional to the velocity v at time t obeys the differential equation $\frac{\mathrm{d}v}{\mathrm{d}t} = 10 - 0.2v$

(a) If the body falls from rest find v in terms of t and show that as $t \to \infty$ the velocity approaches 50m/s.

(b) How long would it take for the body to reach a velocity of 25m/s?

4. On a hot summer's day a bottle of beer is placed in a fridge set at 4°C. The rate of change of the bottle's temperature θ°C after t minutes is given by Newton's Law of Cooling, $\frac{\mathrm{d}\theta}{\mathrm{d}t} = -k(\theta - 4)$ where k is a constant that is specific to the beer.

(a) Find θ in terms of t. If the initial temperature $(t = 0)$ of the bottle of beer was 22°C and it took 10min to cool to 20°C, find the value of k, and the particular solution.

(b) How long would it take the bottle to reach the ideal drinking temperature of 8°C?

5. Brine containing 2 grams of salt per litre flows into a tank initially filled with 50 litres of water containing 10 grams of salt. The brine enters the tank at 5 litres/min, the concentration is kept constant by stirring and the mixture flows out through a tap at the same rate so that the tank at all times continues to contain 50 L of mixture. Let there be Q grams of salt in the tank after t minutes.

(a) Show that $\frac{\mathrm{d}Q}{\mathrm{d}t} = 10 - \frac{Q}{10}$

(b) Solve this differential equation to obtain an expression for Q in terms of t.

(c) How much salt is in the tank after 10 minutes?

(d) Sketch the graph of Q against t.

(e) What happens to Q as $t \to \infty$?

6. A rectangular tank is divided into two equal compartments by a vertical porous membrane. Liquid in one compartment, initially at a depth of 40cm, passes into the other compartment which is initially empty, at a rate proportional to the difference in levels

(a) If the depth of the liquid in the second compartment is $x$cm after t minutes show that $\frac{\mathrm{d}x}{\mathrm{d}t} = k\,(40 - 2x)$.

(b) Show that the solution of this differential equation is $x = 20\left(1 - e^{-2kt}\right)$.

(c) If the level in the second compartment rises 2cm in the first 5 minutes, after how much time will the difference in levels be 2cm?

(d) Interpret what is happening.

7. Under certain conditions the relative density ρ of a gas and its temperature T°C satisfies the differential equation $\frac{\mathrm{d}\rho}{\mathrm{d}T} = -n^2\rho^2$ (where n is a constant). Express ρ in terms of T given that $\rho = 0.002$ when $T = 30$°C and $\rho = 0.0016$ when $T = 45$°C.

7 | Answers

Answers to selected exercises.

CHAPTER 1

§1.1, p.15, Algebra Review

1. **(a)** $-x-y$
(b) $-15x+6y$
(c) x^2+3x+2
(d) $4x^2-1$

2. **(a)** 1
(b) 0.01
(c) 36.462
(d) $\frac{2}{7}$

3. **(a)** $9a^4b^2$
(b) $\frac{x^6}{27}$
(c) $\frac{a^3}{2b}$
(d) $\frac{x^3}{12}$

4. **(a)** 1.28
(b) 1.516

5. **(a)** $x=-\frac{t}{3}$
(b) $x=b^2-a$
(c) $x=\sqrt{\frac{1}{g-h}}$

6. **(a)** $t=\frac{v-u}{a}$
(b) $t=\frac{1}{\alpha}\left(\frac{l}{l_0}-1\right)$
(c) $t=\frac{-s}{1-\alpha}$

7. **(a)** $R=\frac{V}{I}$
(b) $\theta=\cos^{-1}\left(\frac{P}{Fv}\right)$
(c) $r=\sqrt{\frac{q_1q_2}{F}}$

8.
(a) $7(y^2-2z^2)$
(b) $x[(y-2)+x]$
(c) $(a+c)(-3a-3c)$
(d) $2\pi r(h+r)$

9. **(a)** $(x+7)\,(x+4)$
(b) $(2x+3)\,(x-4)$
(c) $(b-5)(b-4)$
(d) $(3x-1)(x-2)$

10. **(a)** 3.16
(b) -1.87 or 5.87
(c) 4 or -2
(d) 0.637 or -3.137
(e) $x=2.547$
(f) $x=\{-2.055,0,1.055\}$

§1.2, p.16, Functions

1. **(a)** 4
(b) 4
(c) 27
(d) 64

2. $79.4\mathrm{m}^3$

3. **(a)** $y=4x-4$
(b) $2y=\frac{x}{3}+1$
(c) $y=x^2-3$
(d) $y=-(x+1)^2$

4. $x-$intercepts 1.73 and -1.73 and $y-$intercept -3

5.

(a) Show the points $A\,(3,5)$ and $B\,(-2,-5)$ on the graph.

(b) 2

(c) $y = 2x - 1$

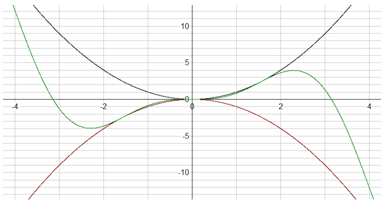
(d) $y = 2x + 7$

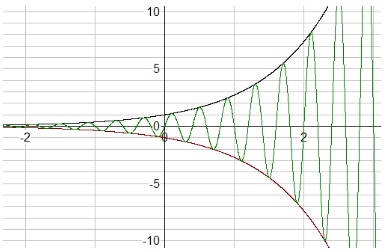
(e) draw a line with a slope of $\frac{4}{5}$

6. Table of Values

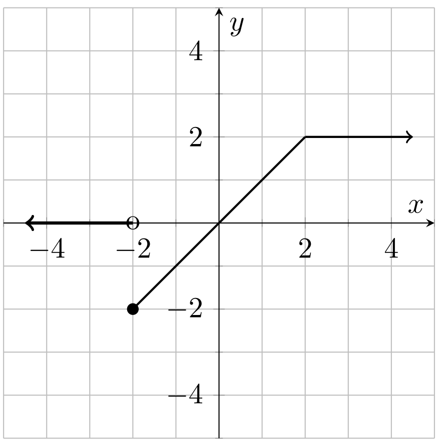
x	$g(x) = x^3 - x$
-1.5	-1.875
-1	0
-0.5	0.375
0	0
0.5	-0.375
1	0
1.5	1.875

7.

(a) 

(b) 

8.

(a) 

(b) $y = |x + 2|$

§1.3, p.17, Polynomials

1. (a) 2

(b) 3

(c) 99

(d) 4

2. (a) No, there is no variable term

(b) No, a variable in the exponent is an *exponential* equation

(c) Yes

(d) No, the powers must be integers

§1.4, p.18, Systems of Equations

1. (a) $(-\frac{1}{3}, 5\frac{1}{3})$

(b) $(0.571, -0.571)$

(c) infinite solutions

(d) $(1.23, 3.87)\,, (-0.35, -4.21)$

2. (a) $(3,1)$

(b) no solution; parallel lines

(c) $(18.29, -10.29)$

(d) $(-25, 5)$, and $(-25, -5)$

3. (a) $(1,2)$

(b) $(-3,4)$ and $(3,4)$

(c) $(-2,-1)$, $(-2,1)$, $(2,-1)$, $(2,1)$

(d) $(-1.5, 0)$

§1.5, p.18, Word Problems

1. 12cm by 15cm

2. The number of children admitted was 1500 and the number of adults was 700.

3. Run $5mi/$h, cycle $20mi/$h

4. 2.5 pounds of Kenyan coffee and 0.5 pounds of Sri Lankan coffee should be mixed.

CHAPTER 2

§2.1, p.35, Unit Circle

1.

(a) $\frac{\pi}{5} \approx 0.628\text{rad}$

(b) $\frac{-8\pi}{3} \approx -8.378\text{rad}$

(c) $\frac{\pi}{3} \approx 1.047\text{rad}$

(d) $\frac{-3\pi}{4} \approx -2.356\text{rad}$

2. (a) $135°$

(b) $150°$

(c) $\frac{-270}{\pi} \approx -85.9°$

(d) $-15°$

3. (a) $\frac{55\pi}{9} \approx 19.2$

(b) 4

4. 4mi

5. $\frac{36}{\pi} \approx 11.459\text{m}$

6. $330\pi \approx 1037\text{mi}$

7. 1.6 million mi

§2.2, p.36, Right Angled Triangles

1. $\sin\theta = \frac{4}{5}, \cos\theta = \frac{3}{5}, \tan\theta = \frac{4}{3}$

2. (a) $\sin\theta = \frac{40}{41}, \cos\theta = \frac{9}{41}, \tan\theta = \frac{40}{9}$

(b) $\sin\theta = \frac{2\sqrt{13}}{13}, \cos\theta = \frac{3\sqrt{13}}{13}, \tan\theta = \frac{2}{3}$

3. (a) $12\sqrt{2}$

(b) $\frac{13\sqrt{3}}{2}$

(c) 16.51658

4. (a) $45°, 16, 16\sqrt{2} \approx 22.63$

(b) $38°, 44.79, 56.85$

5. 1026ft

6. (a) 2100mi

(b) No

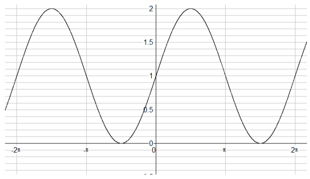
7. 415ft, 152ft

8. 30.0

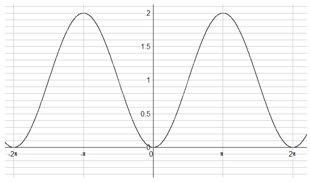
9. $66.1°$

§2.3, p.37, Trigonometric Functions

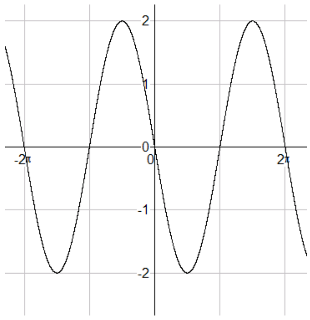
1.



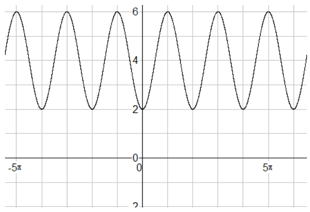
(a)



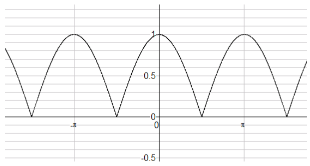
(b)



(c)

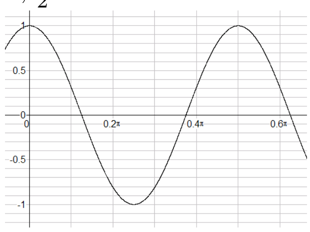


(d)

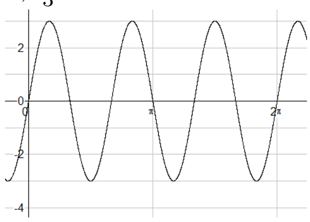


(e)

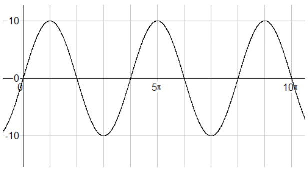
2. (a) $1, \frac{\pi}{2}$



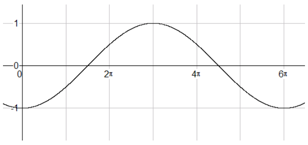
(b) $3, \frac{2\pi}{3}$



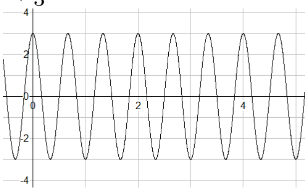
(c) $10, 4\pi$



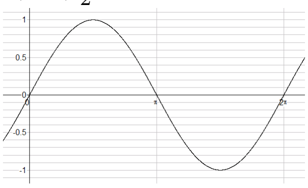
(d) $1, 6\pi$



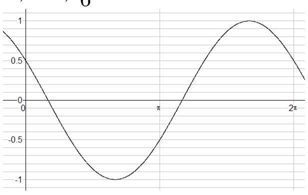
(e) $3, \frac{2}{3}$



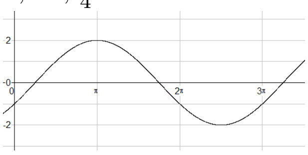
3. (a) $1, 2\pi, \frac{\pi}{2}$



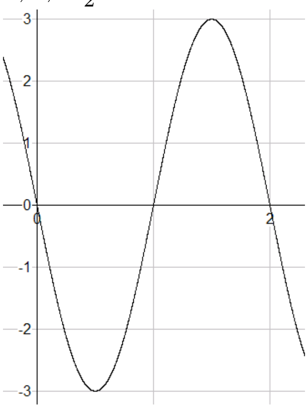
(b) $2, 2\pi, \frac{\pi}{6}$



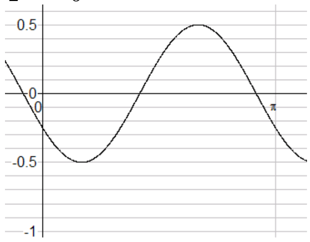
(c) $2, 3\pi, \frac{\pi}{4}$



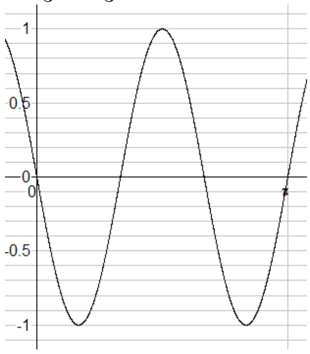
(d) $3, 2, -\frac{1}{2}$



(e) $\frac{1}{2}, \pi, \frac{\pi}{6}$



(f) $1, \frac{2\pi}{3}, -\frac{\pi}{3}$



§2.4, p.37, Applications

1. (a) 318.8
   (b) 44°

2. $\angle C = 114°, a \approx 51, b \approx 24$

3. (a) $\angle C = 62°, a \approx 200 b \approx 242$
   (b) $\angle A = 100°, a \approx 89, c \approx 71$

4. 219ft

5. (a) 1018mi,
   (b) 1017mi

6. 155m

7. (a) 28.9
   (b) 28.89°

8. (a) 2
   (b) 84.6°

9. 23.1mi

10. 2179mi

11. 3835ft

12. $3.85cm^2$

13. 14.3m

CHAPTER 3

§3.1, p.53, e^x functions

1.

(a) 54.59815
(b) 0.99317
(c) 22.19795
(d) 15.15426

2. Use desmos to verify

3. (a) $f(x) = 3^x$
(b) $f(x) = 5^x$
(c) $f(x) = \left(\frac{1}{4}\right)^x$
(d) $f(x) = \left(\frac{1}{2}\right)^x$

4. Use desmos to verify

§3.2, p.56, Logarithmic Functions

1. (a) $5^2 = 25$
(b) $5^0 = 1$
(c) $8^{1/3} = 2$
(d) $2^{-3} = \frac{1}{8}$
(e) $e^x = 5$
(f) $e^5 = y$

2. (a) $\log_5 125 = 3$
(b) $\log_{10} 0.0001 = -4$
(c) $\log_8 \frac{1}{8} = -1$
(d) $\log_2 \frac{1}{8} = -3$
(e) $\ln 2 = x$
(f) $\ln y = 3$

3. (a) 1
(b) 0
(c) 2
(d) 2
(e) 2
(f) 10
(g) -3
(h) $\frac{1}{2}$
(i) -1
(j) 37
(k) 8
(l) $\sqrt{5}$
(m) $-\frac{2}{3}$
(n) 4
(o) -1

4. (a) 32
(b) 4
(c) 5
(d) 100
(e) 2
(f) 4

5. (a) $1 + \log_2 x$
(b) $\log_2 x + \log_2 (x-1)$
(c) $10 \log 6$
(d) $\log_2 A + 2\log_2 B$
(e) $\log_3 x + \frac{1}{2}\log_3 y$
(f) $\frac{1}{3}\log_5 \left(x^2+1\right)$
(g) $\frac{1}{2}(\ln a + \ln b)$
(h) $\ln x + \frac{1}{2}(\ln y - \ln z)$
(i) $\frac{1}{4}\log \left(x^2+y^2\right)$

6. (a) $\frac{3}{2}$
(b) 1
(c) 3
(d) $\ln 8$
(e) 16
(f) $4 + \log 3$

7. (a) $\log_3 160$
(b) $\log_2 \left(AB/C^2\right)$
(c) $\ln \left[5x^2 \left(x^2+5\right)^3\right]$

§3.3, p.57, Logarithmic Equations

1.

(a) 2.7726

(b) 0.3495

(c) 1.2040

(d) 0.0767

(e) 1.9349

(f) -43.0677

(g) 6.2126

(h) -2.9469

(i) 14.0055

2. (a) ± 1

(b) $0, \frac{4}{3}$

(c) $\frac{1}{2}\ln 3 \approx 0.5493$

(d) $e^{10} \approx 22026$

(e) $\frac{95}{3}$

(f) $3 - e^2 \approx -4.3891$

(g) 5

(h) $\frac{13}{12}$

§3.4, p.57, Modelling

1. (a) 13kg

(b) 6.6kg

2. (a) $0 \ \frac{\text{ft}}{\text{s}}$

(b) $50.6 \ \frac{\text{ft}}{\text{s}}$, $69.2 \ \frac{\text{ft}}{\text{s}}$

(c) Use desmos to plot the graph

(d) $80 \ \frac{\text{ft}}{\text{s}}$

3. (a) 200

(b) Use desmos to plot the graph

(c) $11,200$

4. 13 days

5. (a) 7337

(b) 1.73 years

CHAPTER 4

§4.1, p.78, Differentiation from First Principles

1.

(a) 23

(b) -12

(c) 4

2. Use Desmos to plot the graph and compare to yours

3. Use Desmos to check

4. Use Desmos to check

5. Use Desmos to check

6. (a) $f'(x) = -2x^{-3} = \frac{-2}{x^3}$

(b) $f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$

(c) $y' = 5x^4$

(d) $y' = -3x^{-4}$

(e) $y' = -4x^{-5}$

(f) $y' = \frac{3}{4}x^{-\frac{1}{4}}$

(g) $y' = -\frac{1}{2}x^{-\frac{3}{2}}$

§4.2, p.80, Standard Derivatives

1. (a) $2x^4$

(b) 0

(c) $-12x^3$

(d) $-8x^{-5} = \frac{-8}{x^5}$

(e) $-x^{-4} = \frac{-1}{x^4}$

2. (a) $\frac{1}{2}t^{-\frac{1}{2}} = \frac{1}{2\sqrt{t}}$

(b) $\frac{3}{2}t^{\frac{1}{2}} = \frac{3\sqrt{t}}{2}$

(c) $\frac{5}{3}z^{\frac{2}{3}} = \frac{5\sqrt[3]{z^2}}{3}$

(d) $6.4x^{2.2}$

3. (a) $f'(x) = 6x^2 - 6x + 4$

(b) $f'(x) = 2x + 1 - \frac{1}{x^2}$

(c) $8x^7 + 60x^4 - 16x^3 + 30x^2 - 6$

4. (a) $\frac{ds}{dt} = 8t - 7$

(b) $\frac{d(3x)}{dx} = 3$

(c) $\frac{d(3u^4)}{du} = 12u^3$

(d) $Df(x) = 2$

(e) $f'(x) = e^x - 1$

(f) $f'(x) = 81x^2$

(g) $g'(x) = 15x^{14}$

(h) $f'(x) = e^x - ex^{e-1}$

5.

(a) $f'(x) = -3x^{-4} + \frac{3}{4}x^{-1\frac{3}{4}}$
$= -\frac{3}{x^4} + \frac{3}{4\sqrt[4]{x^7}}$

(b) $f'(x) = \frac{1}{4}x^{-\frac{3}{4}} + \frac{1}{12}x^{-\frac{11}{12}}$
$= \frac{1}{4\sqrt[4]{x^3}} + \frac{1}{12\sqrt[12]{x^{11}}}$

(c) $g'(x) = 2ex + 2e^x + e^2 + e^2x^{e^2-1}$

(d) $f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$

6. Check with Desmos

7. (a) $x^2 \cos x + 2x \sin x$

(b) $\sqrt{x} \cos x + \frac{1}{2\sqrt{x}} \sin x$

(c) $5 \sec^2 5x$

(d) $\frac{\cos x + x \sin x}{\cos^2 x}$

(e) $-\omega \sin(\omega t + \delta)$

8. $y - x - 1 = 0$

9. $\frac{dx}{d\theta} = 5\text{m/rad}$

10. (a) $4 \cos 4x$

(b) $2 \cos \pi x$

(c) $-15 \sin 3x$

(d) $3 \sec^2 3x$

(e) $3 \sec^2 (x + 2)$

(f) $3 \sin^2 x \cos x$

(g) $6 \sin 3x \cos 3x$

(h) $6(x-1) \sin^2 (x-1)^2 \cos (x-1)^2$

(i) $4 \tan 2x \sec^2 2x$

§4.3, p.81, Maximums, Minimums, and Tangents

1. $y = 7x - 3$ is a linear function and so has no maximum or minimum value. Its derivative is *constant.*

2. (a) $(2, 0)$ and $(-2, 0)$

(b) $(2, 12)$

(c) $(0, 1)$ and $(1, 0)$ and $(2, 1)$

3. (a) $(3, 4)$ is a local maximum

(b) $(-1, 0.667)$ is a local maximum and $(1, -0.667)$ is a local minimum

(c) $(-3.67, -14.82)$ min and $(1, 36)$ max

4. There is a local minimum at $(-1, -0.368)$ If you know how to use the product rule for differentiation you can find out without using desmos.

5. Tangent line is $2y - 3x + 1 = 0$
Normal line is $3y + 2x - 5 = 0$

6. $(0, 4)$, $(\sqrt{3}, -5)$, $(-\sqrt{3}, -5)$

7. $(\ln 2, 2) \approx (0.69, 2)$

8. $a = -\frac{1}{3}$ or -1

9. $x = \pm\sqrt{\frac{2}{3}}$

10. See Desmos

11. See Desmos

§4.4, p.82, Product, Quotient, & Chain Rules

1. (a) $f'(x) = xe^x + e^x = e^x(x + 1)$

(b) $g'(x) = x^2e^x + 2xe^x = xe^x(x + 2)$

2. (a) $(x^3e^x)' = x^3e^x + 3x^2e^x = x^2e^x(x + 3)$

(b) $(x^{-3}e^x)' = x^{-3}e^x - 3x^{-4}e^x$
$= \frac{e^x}{x^4}(x - 3)$

(c) $((x + 1)e^x)' = e^x(x + 2)$

(d) $((x + 2)(x - 2)e^x)' = e^x(x^2 + 2x - 4)$

3. (a) $y' = \frac{-5}{(2x-1)^2}$

(b) $y' = \frac{xe^x}{(x+1)^2}$

(c) $\frac{df}{dt} = \frac{2-2t^2}{(1+t^2)^2}$

(d) $f'(x) = \frac{-ACe^x}{(B+Ce^x)^2}$

4. (a) $\frac{-3x(x-2)}{(x-1)^2}$

(b) $\frac{2}{(x+1)^2}$

(c) $\frac{2-x}{2\sqrt{x}(x+2)^2}$

(d) $(1 + x - 2x^2)e^{-x}$

5. Tangent $y = x$ Normal $y = -x$

6. (a) $2y + x - 2 = 0$

(b)

7. (a) Horizontal line $y = \frac{1}{2}$

(b)

8. $(f \circ g)(x) = \sqrt{x^3}$
$(g \circ f)(x) = (\sqrt{x})^3$

9. $(h \circ j)(x) = e^{\frac{x^2}{2}} = e^{x^2/2}$
$(j \circ h)(x) = \frac{e^{2x}}{2}$

10. (a) $F'(x) = \frac{x}{\sqrt{1+x^2}}$
(b) $\frac{dy}{dx} = -10x\left(1+x^2\right)^4$
(c) $\left(e^{x^2}\right)' = 2xe^{x^2}$
(d) $\left(e^{e^x}\right)' = e^{e^x} \cdot e^x = e^{e^x+x}$

11. (a) $9(3x+2)^2$
(b) $\frac{3}{(5x+3)^{2/5}}$
(c) $\frac{-2}{(2x+1)^2}$
(d) $\frac{9}{(4-x)^4}$
(e) $\frac{1}{\sqrt{2x-5}}$
(f) $\frac{-2}{3\sqrt[3]{(5-x^2)^2}}$
(g) $\frac{-1}{2\sqrt{(x+2)^3}}$
(h) $6x^2e^{2x^3}$

12. (a) $2xe^{-2x}(1-x)$
(b) $-e^{-x}(1-2x)(5-2x)$
(c) $\frac{1-p}{(p^2+1)\sqrt{p^2+1}}$
(d) $\left(x^2+3\right)\left(5x^2-16x+3\right)$
(e) $(x-3)^2(2x-1)$
(f) $\frac{(x-1)(5x+3)}{2\sqrt{x+1}}$
(g) $\frac{e^{x^2}(2x+1)^2}{2\sqrt{x+1}}$
(h) $\frac{x-3}{(x+1)^3}$

§4.5, p.83, Parametric Differentiation

1. Check with desmos
2. (a) $\frac{dy}{dx} = \frac{1}{t}$
(b) Let $t = 1$
(c) $y - x - 1 = 0$
3. $y + x = \sqrt{2}$, horizontal when $t = \frac{\pi}{2}$ or $\frac{3\pi}{2}$,
Vertical when $t = 0$ or π
4. $2y + 4\sqrt{3}x = 6 + \sqrt{3}$, horizontal when $t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ vertical when $t = 0, \pi$
5. (a) $\frac{t(3t^2+1)}{2t+1}$
(b) $\frac{\cos t+\sin t}{\cos t-\sin t}$
(c) $\frac{1}{t}$
(d) $-\frac{b}{a}\tan\theta$
(e) $-2\sin\theta$ because $\sin 2\theta = 2\sin\theta\cos\theta$
(f) $\frac{\sin\theta}{1-\cos\theta}$

§4.6, p.84, Related Rates

1. $3x^2\frac{dx}{dt}$
2. $\frac{dA}{dt} = 48$ cm^2/s
3. (a) $\frac{dA}{dt} = \frac{dr}{dt} \cdot \frac{dA}{dr}$
(b) $\frac{dA}{dt} = 60\pi$ m^{2}s^{-1}
4. $\frac{-1}{20\pi}$ decreasing at a rate of $\frac{1}{20\pi}$ cm/min
5. $\frac{215}{\sqrt{101}} \approx 21.4$ km/h
6. 666 km/h

§4.7, p.85, Optimisation

1. 25 and 25
2. $\frac{1}{2}$
3. 1200m by 600m
4. 150m by 150m
5. $\frac{20}{3}$ cm
6. $20,032.00 for 399 people
7. Area $= r^2$
8. 20 by 20 by 10, Volume $= 4000cm^3$

CHAPTER 5

§5.1, p.108, Standard Integrals

1. (a) $\frac{x^8}{2}+C$
 (b) $10x+C$
 (c) $\frac{2}{3}x^4+C$
 (d) $3.2e^x$
 (e) $\frac{1}{2}\ln|x|+C$
 (f) $\frac{2x^{3/2}}{3}+C$
 (g) $\frac{x^{2.2}}{2.2}+C$
 (h) $\frac{x^{\pi+1}}{\pi+1}+C$
 (i) $\frac{-12}{\sqrt[3]{x}}+C$
 (j) $3\sqrt[3]{x}+C$
 (k) $\frac{4}{3}x^3+2x^2+x+C$
 (l) $e^x+\frac{x^{e+1}}{e+1}+C$

2. (a) $\frac{x^2}{2}+x^{-1}+C$
 (b) $\frac{2\sqrt{x^5}}{5}+C$
 (c) $-\cos x-2\sin x+C$
 (d) $2x-\frac{3}{2}x^2+\frac{x^3}{3}+C$
 (e) $x+C$
 (f) $2\ln|x|+6\sqrt{x}+C$
 (g) $\frac{1}{2}\sin x+C$
 (h) $\frac{1}{2}\tan x+C$
 (i) $\frac{x^2}{2}+x+\ln|x|+C$
 (j) $\pi r^2x+\frac{\pi x^3}{3}+C$
 (k) $\frac{1}{2}\ln|x|+C$
 (l) πr^2+C

§5.2, p.108, Area

1. (a) $\frac{21}{2}$
 (b) -70
 (c) $\frac{3}{5}$
 (d) e^3-e
 (e) $\ln 3$

2.
 (a) $20\frac{1}{2}$ units^2
 (b) $9\frac{1}{3}$ units^2

3. (a) $\frac{2}{3}$
 (b) $-1\frac{1}{2}$
 (c) $\frac{1}{3}\ln\frac{3}{2}\approx 0.135$
 (d) $3\,(e-1)\approx 5.155$
 (e) $12\frac{4}{5}$
 (f) $\frac{1}{4}-\frac{1}{4}=0$, ($Area=\frac{1}{2}$)

§5.3, p.108, Volume

1. (a) 57π units^3
 (b) $\frac{36828}{5}\pi$ units^3
 (c) 24π units^3
 (d) $\frac{75\pi}{2}$ units^3

2. (a) $\frac{64}{3}\pi$ units^3
 (b) 4.5π units^3
 (c) $\frac{32\pi}{5}$ units^3
 (d) ≈ 308 units^3

3. $V=\pi\int_0^h(r)^2dy$, here r^2 is *constant*
 $V=\pi[r^2y]_0^h=\pi r^2h$

§5.4, p.109, Integration by Substitution

1. (a) $\frac{1}{4}e^{4x}+C$
 (b) $\ln|\cos x|+C$
 (c) $e^{\sin\theta}+C$
 (d) $\frac{1}{64}\left(4x-3\right)^{16}+C$
 (e) $\frac{1}{2}e^{x^2}+C$
 (f) $\frac{1}{2}\ln\left(x^2+1\right)+C$ as x^2+1 is always positive
 (g) $\frac{1}{4}\left(x^2+2\right)^4+C$
 (h) $\frac{1}{3}\sin^3 x+C$
 (i) $\frac{2\sqrt{(1+x)^3}}{3}-2\sqrt{1+x}+C$
 (j) $2\ln(1+\sqrt{x})+C$ as $1+\sqrt{x}$ is always positive

2.

(a) $\frac{1}{3}(2x+1)^{3/2}+C$

(b) $2\sqrt{x+1}+C$

(c) $\ln(e^x+1)+C$ as e^x+1 is always positive

(d) $\frac{1}{2}(\ln x)^2+C, x>0$ because $\frac{\ln x}{x}$ does not exist when $x\le 0$

(e) $\ln|\sin x+1|+C$

(f) $2\sin\frac{x}{2}+C$

(g) $\frac{1}{2}\sin 2x+C$

(h) $\frac{e^x-e^{-x}}{2}+C$

(i) 0, Area $=\frac{1}{12}\times 2=\frac{1}{6}$

3. (a) $4\frac{1}{3}$

(b) $\frac{\sqrt{2}}{2}$

(c) $\frac{121}{5}=24.2$

(d) $-\frac{5}{12}$

(e) $\frac{1}{2}\ln\frac{19}{9}\approx 0.374$

(f) $\ln\frac{\sqrt{2}}{2}=-0.347$, Area $=0.347$ units^2

§5.5, p.109, Integration by Parts

1. (a) $\frac{x^2}{2}\ln x-\frac{x^2}{4}+C$

(b) $\sin x-x\cos x+C$

(c) $\frac{x}{3}\sin 3x+\frac{1}{9}\cos 3x+C$

(d) $\frac{1}{2}e^x(\sin x-\cos x)+C$

(e) $x\ln|2x|-x+C$

2.

3. (a) $\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+C$

(b) $-e^{-x}(x+1)+C$

(c) $-x^2\cos x+2x\sin x+2\cos x+C$

(d) $e^x(x-2)+C$

(e) $\frac{1}{2}(\ln x)^2+C, x>0$ because $\frac{\ln x}{x}$ does not exist when $x\le 0$

4. (a) $-6e^{-5}+1\approx 0.9596$

(b) $e-2\approx 0.718$

(c) $\frac{1}{2}(e^\pi+1)\approx 12.07$

(d) $\frac{9}{2}\ln 3-\frac{1}{2}\ln 1-2\approx 2.9$

(e) $\frac{\pi}{2}\approx 1.57$

(f) 1

(g) $\frac{1}{4}\left(e^2-1\right)\approx 1.597$

(h) $7\sin 7+\cos 7-(5\sin 5+\cos 5)$ ≈ 9.864

(i) $e\approx 2.718$

(j) $\frac{1}{2}\left[(\ln 4)^2-(\ln 2)^2\right]\approx 0.721$

(k) 5.777

§5.6, p.110, Applications

1. $s(t)=t^3+2t^2-6t+9$
2. $s(t)=-4.9t^2+15t+140$, hits ground after $t\approx 7.1$s
3. $s(t)=5+4t+\frac{3}{2}t^2+\frac{2}{3}t^3-t^4$
4. ≈ 81.6m
5. 19.3m
6. ≈ -4.8m/s^2 or $-62500km/\text{h}^2$
7. Pass after approximately 4.6s
8. $s(t)=-\cos t-\sin t+1$
9. $\sqrt{600}\approx 24.5$m/s
10. 5J
11. $\frac{3}{\pi}\left(2-\sqrt{3}\right)\approx 0.256$
12. 275J
13. 0.3125J
14. 1.2J
15. 8cm

CHAPTER 6

§6.1, p.117, Differential Equations

1. (a) $y=e^t+t^2+1$

(b) $y=\frac{1}{2}(\sin 2x-\cos 2x+1)$

(c) $y=\ln|t|-\frac{t^2}{2}+2.5$

(d) $P=-\frac{2}{x}+\ln|x|+4$

2. $y'=1-\frac{1}{x^2}$, so the left hand side of the D.E. is $x(1-\frac{1}{x^2})+x+\frac{1}{x}=2x$
Therefore the L.S.=R.S.

3. The initial value:
$y = \sin(0)\cos(0) - \cos(0) = -1$
The derivative:
$y' = \cos^2 x - \sin^2 x + \sin x$
L.S. of D.E.:
$= \cos^2 x - \sin^2 x + \sin x + \frac{\sin x}{\cos x}(y)$
with $y = \sin x \cos x - \cos x$
L.S $= \cos^2(x) =$ R.S.

4. $y = -\frac{1}{2}x\cos x$

5. (a) $v = 9.8t$ (b) $v = 9.8t + 100$

6. (a) $\omega = \omega_0 - \frac{N}{I}t$
(b) 47.1s

§6.2, p.117, Separable Equations

1. **(a)** $y^2 = x^2 + 16$

(b) $v = t^2 + 1$

(c) $y = 2e^{-x} + 3$

(d) $y = \frac{3}{2}x$

(e) $i = -(9e)^{-t} + 1$

(f) $y = x^2 + 2$

2. 5×10^{10} years

3. (a) $v = 50 - 50e^{-0.2t}$
(b) 3.47s

4. (a) $\theta = 4 + 18e^{-kt}$ $k = 0.011778303$
(b) $127.6989838 \approx 128\,\text{min}$ (3sf)

5. (b) $Q = 100 - 90e^{-0.1t}$
(c) 66.9g
(e) $Q \to 100$g

6. (c) $142.165794 \approx 142$min

7. $\rho = \left(8\frac{1}{3}t + 250\right)^{-1}$

Useful Formula

- Equations of a Line:

$$\begin{aligned} \text{point } (x_1, y_1), \text{ slope } m \text{ form:} \quad & y - y_1 = m(x - x_1) \\ \text{slope } m, \text{ intercept } b \text{ form:} \quad & y = mx + b \end{aligned}$$

- Quadratic Formula: If $ax^2 + bx + c = 0, a \neq 0,$ then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Sine Law: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$; Cosine Law: $a^2 = b^2 + c^2 - 2bc\cos(A)$

• Logarithms					• Exponents		
Law 1:	$\log_x(ab)$	$=$	$\log_x(a) + \log_x(b)$		$x^a x^b$	$=$	x^{a+b}
Law 2:	$\log_x\left(\frac{a}{b}\right)$	$=$	$\log_x(a) - \log_x(b)$		$\dfrac{x^a}{x^b}$	$=$	x^{a-b}
Law 3:	$\log_x(a^b)$	$=$	$b \cdot \log_x(a)$		$(x^a)^b$	$=$	x^{ab}
	$\log_x\left(\frac{1}{x^a}\right)$	$=$	$-a$		x^{-a}	$=$	$\frac{1}{x^a}$
	$\log_x 1$	$=$	0		x^0	$=$	1
	$\log_x(x)$	$=$	1		x^1	$=$	x
	$\log_a(x)$	$=$	y converts to	$\Leftrightarrow$	x	$=$	a^y

• Standard Derivatives

$f(x)$	$f'(x)$	Notes
A	0	A is constant
x	1	power rule for x^1
Ax	A	A is a constant multiple
x^n	nx^{n-1}	power rule general form
e^x	e^x	exponential
$\ln(x)$	$\frac{1}{x}$	logarithmic
$\sin(x)$	$\cos(x)$	trigonometric
$\cos(x)$	$-\sin(x)$	
$\tan(x)$	$\sec^2(x)$	

- Product Rule: $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$
- Quotient Rule: $\left[\dfrac{f(x)}{g(x)}\right]' = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$
- Chain Rule: $f[g(x)]' = f'[g(x)] \cdot g'(x)$

• Standard Integrals

$f(x)$	$\int f(x)dx$	Notes
1	$x + C$	constant
A	$Ax + C$	A is constant
$x^n, n \neq -1$	$\dfrac{x^{n+1}}{n+1} + C$	power rule general form
e^x	$e^x + C$	exponential
$\frac{1}{x}$	$\ln\lvert x\rvert + C$	special case: x^{-1}
$\ln x$	$x\ln\lvert x\rvert - x + C$	natural logarithm: $\log_e$
$\sin(x)$	$-\cos(x) + C$	trigonometric
$\cos(x)$	$\sin(x) + C$	
$\tan(x)$	$\ln\lvert\sec x\rvert + C$	
$\sec^2(x)$	$\tan(x) + C$	

- Integration By Parts: $\displaystyle\int fg' = fg - \int f'g$ or $\displaystyle\int udv = uv - \int vdu$

- Area between the curve $f(x)$ and the x-axis: $A = \displaystyle\int_a^b f(x)dx$

- Area between two curves $f(x)$ and $g(x)$: $A = \displaystyle\int_a^b [f(x) - g(x)]\, dx$

- Volume of revolution about the x-axis: $V = \displaystyle\int_a^b \pi y^2 dx = \pi \int_a^b [f(x)]^2 dx$

- Volume of revolution about the y-axis: $V = \displaystyle\int_a^b \pi x^2 dy = \pi \int_a^b [f(y)]^2 dy$