Nonlinear Dynamics in the Periodically Forced Bouncing Car: An Experimental and Theoretical Study.

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Abstract

We consider, experimentally and theoretically, a mechanical system consisting of a sliding car on an inclined plane that bounces on an oscillating piston. The main aim of our study is to reproduce the different types of orbits displayed by nonlinear dynamical systems. In particular, we are looking to identify the parameters and initial conditions for which periodic and chaotic (irregular) behavior are exhibited. The data in our experimental model is collected by using infrared lights that are monitored by a Nintendo Wii remote which is linked to a laptop computer via Bluetooth. By varying the pistons frequency and amplitude, it is possible to produce, for relatively small amplitudes, periodic orbits. As the amplitude of the piston is increased (for a fixed piston frequency) we observe bifurcations where the original, stable, periodic orbit is destabilized and replaced by a higher order periodic orbit. For larger amplitudes, periodic orbits are destabilized and replaced by chaotic trajectories. After carefully measuring all experimental parameters, we are able to successfully produce periodic orbits (up to period 3), sticking solutions (the car does not bounce, but gets stuck to the piston), and seemingly chaotic (irregular/unpredictable) behavior that are in very good agreement with the model for the same parameter values.

1. Introduction

Chaos is an area of study in Applied Mathematics that focuses on nonlinear dynamical systems. In mathematics, chaos is defined as seemingly stochastic behavior occurring in deterministic systems. This seemingly stochastic behavior is found to be caused by sensitive dependence to initial conditions; namely, small (infinitesimal) errors in the prescription of the initial conditions are amplified exponentially. This sensitive dependence to initial conditions is the major obstacle when dealing with chaotic systems since

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it precludes our ability to obtain long-term predictions. Henri Poincaré, a mathematician, first pointed out this phenomenology back in the nineteenth century [1]. Chaos Theory, in general, tries to explain naturally occurring phenomena such as the weather, which displays seemingly random behavior although its dynamics is inherently deterministic. Edward Lorentz, a meteorologist, studied a model for the circulation of air in the atmosphere. Using a model with only three degrees of freedom, Lorentz found that for certain parameter values his simplified system exhibited chaotic behavior [2]. Understanding how and when chaos emerges in simple systems will be very useful to understanding other, more complex, phenomena.

The research that we will be conducting is inspired by previous studies performed on a bouncing ball system [3]. In a previous experiment performed by T.M. Mello et al., a ball was dropped onto an oscillating table for the purpose of experimentally constructing an impact map. The impact map records the tables phase ($\theta = \omega t$) and the balls velocity at each impact between the ball and the table. This phase θ, is defined by ω (angular frequency) and t (time). The impact map allows for the observation of the different orbits that a given system might exhibit given the initial condition and parameter values. A period-1 orbit on an impact map would be represented by a single point. On the other hand, a period-2 orbit would be represented on an impact map as two different points, etc. Through the use of impact maps, chaotic motions were explored and the existence of strange attractors was demonstrated in the bouncing ball system [4].

In this paper, we focus on reproducing the different orbits that are seen in nonlinear dynamical systems. What makes our research different from the bouncing ball system is that we will be using an inclined plane and a bouncing car. This allows us to observe the different phenomenology with the car at an angle, instead of strictly vertical as previously seen in the bouncing ball experiments. Furthermore, the car will be periodically forced by a piston up and down this inclined plane. The amplitude, frequency, and initial condition will be manipulated in order to reproduce the different kinds of dynamics.

This paper is organized as follows: In Sec. II we present our methods. In particular, the equations used to model our theoretical system will be discussed and the different phenomenology will be explained. In sec. III the

Singular Value Decomposition is used to analyze data from our theoretical model. Sec. IV describes the experimental bouncing car setup. Next, Sect. V covers the experimental results. This includes the different orbits that were reproduced and the comparison of the experimental and theoretical models. Specifically, we compare qualitatively and quantitatively the experimental and theoretical models. Finally, in Sect. VI we present our conclusions and future challenges.

2. Methods

Let us start with the equation, derived from Newtons law of motion, for a projectile under gravitational pull. In our experiment, this equation is used to model the position of the car with respect to time, where x_0 is the initial position of the car and v_0 is the initial velocity of the car:

$$x(t) = -g_e \left(\frac{t^2}{2}\right) + v_0 \times t + x_0. \tag{1}$$

Determining the effective gravitational pull on an inclined plane ($g_e = g \sin\phi$) is illustrated in figure 1 below.

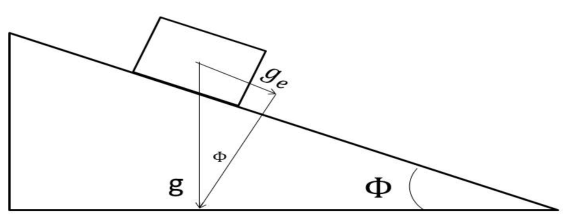

Figure 1: The rectangle in the figure is the car. ϕ is the angle of the plane. Breaking down the gravity vector (g) into component vectors that are perpendicular and parallel to the plane, it is then possible to determine the effective gravitational pull on the car g_e.

The piston follows the harmonic oscillation:

$$s(t) = A\left[\sin(\omega t + \theta_0) + 1\right], \tag{2}$$

where A describes the amplitude of the piston and $\theta = \omega t + \theta_0$ describes the angular phase. 1 is added to the sine function so that the pistons amplitude is always positive. These two equations need to be solved simultaneously to detect the next impact of the car with the piston. Namely, their distance is given by:

$$d(t) = x(t) + s(t), \tag{3}$$

where $d(t) = 0$ implies an impact. Unfortunately, deriving the impacts of this system requires using an implicit equation which potentially has many solutions for each initial condition as depicted in Figure 2.

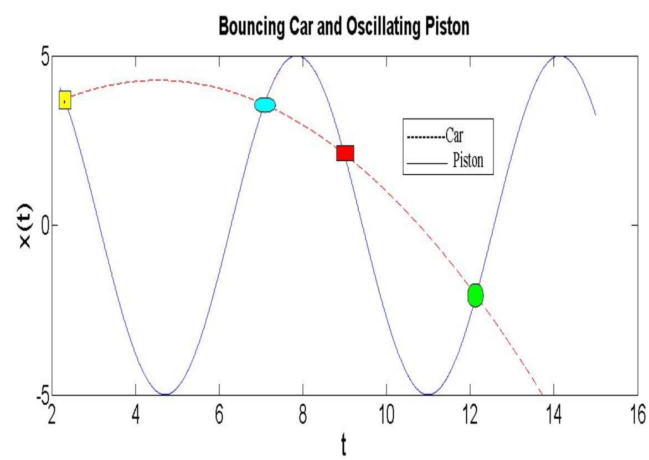

Figure 2: The figure shows that different solutions are possible for a given initial condition and thus solving the implicit function (3) will yield multiple solutions. The yellow rectangle corresponds to the trivial (initial) solution, the blue oval corresponds to the solution of interest (first non-trivial crossing), and the red square and green oval correspond to some other, spurious, solutions.

To circumvent this problem, we used a good initial guess by computing $d(t)$ in a fine grid and detecting its first sign change. Then we refined this initial guess by using a standard nonlinear numerical solver. After each collision, the velocity has to be adjusted by using:

$$v_k = (1 + \alpha)u_k - \alpha v_k^{'}, \tag{4}$$

where u_k is the velocity of the piston ($u(t) = \frac{ds(t)}{dt}$) at the k-th impact, $v_k^{'}$ is the car's velocity just before the k-th impact, v_k is the velocity after the k-th impact, and α describes the loss of energy during the k-th impact ($\alpha = 0$ corresponding to complete energy loss and $\alpha = 1$ corresponding to the energy conserving elastic case). In particular, $(1+\alpha)u_k$ describes the kick to the car's velocity from the piston and -$\alpha v_k^{'}$ describes the restitution coefficiant (α) multiplied by the car's velocity ($v_k^{'}$) right before the k-th impact. Taking the derivative of the piston's position and the car's position, and substituting their values into the impact equation (3), one can then derive an expression for the next impact velocity:

$$v_{k+1} = (1+\alpha)A\omega\cos(\theta_{k+1}) - \alpha[v_k - g[\frac{1}{\omega}(\theta_{k+1} - \theta_k)]] \tag{5}$$

Equation (5) depends on the previous impact velocity (v_k), the next (θ_{k+1}) phase, and previous (θ_k) phase of the piston whose recurrence relation is given by:

$$0 = A[\sin(\theta_k) + 1] + v_k[\frac{1}{\omega}(\theta_{k+1} - \theta_k)] - \frac{1}{2}g[\frac{1}{\omega}(\theta_{k+1} - \theta_k)]^2 - A[\sin(\theta_{k+1}) + 1]. \tag{6}$$

Equation (6) is the implicit equation, produced by substituting the values of $s(t)$ and $x(t)$ into equation (3), that is solved numerically for the solution of interest and was explained in Figure 2.

The different phenomenology that we have observed in this nonlinear dynamical system includes period-1, period-2, period-3, sticking solutions, and chaotic orbits. Figures 3 and 4 below show a period-1 orbit from our theoretical model.

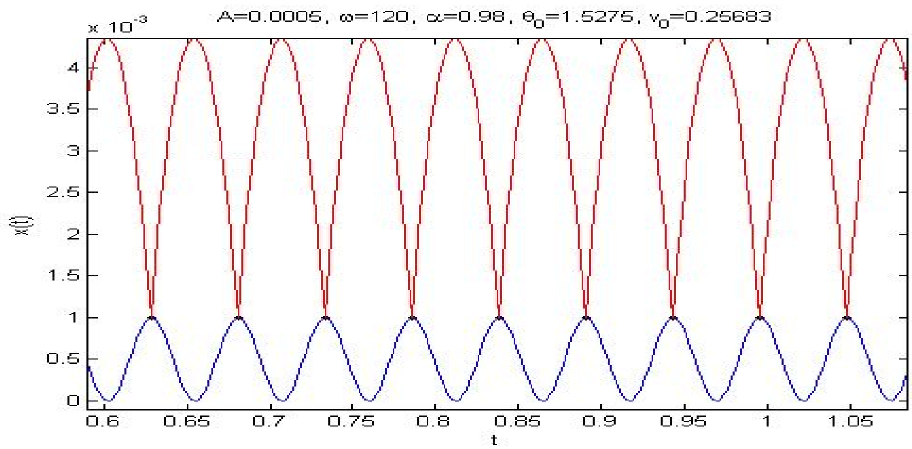

Figure 3: Period-1 orbit obtained from our theoretical model. The red parabolae represent the bouncing car and the blue sine function represents the oscillating piston. The black dots depict the impacts.

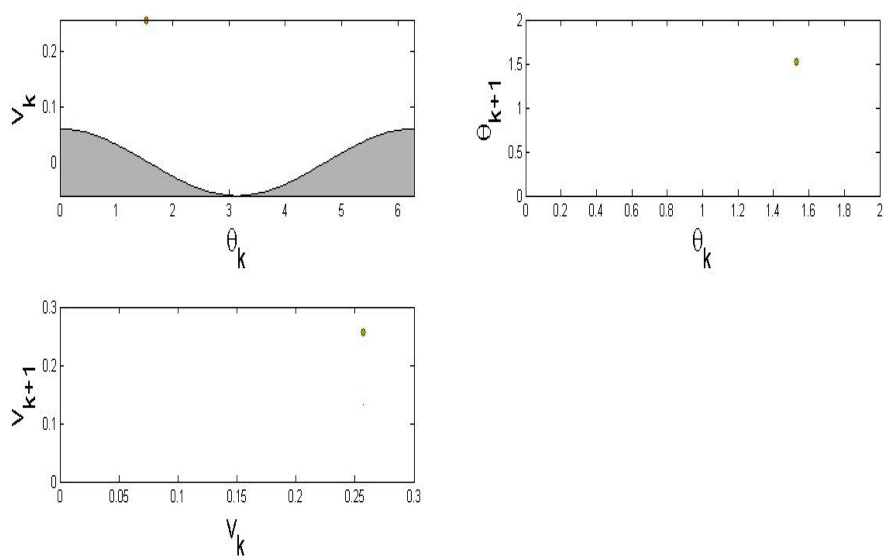

Figure 4: Period-1 orbit represented on an impact map. This impact map corresponds to the orbit depicted in Figure 3. Period-1 corresponds to a single dot on the impact map.

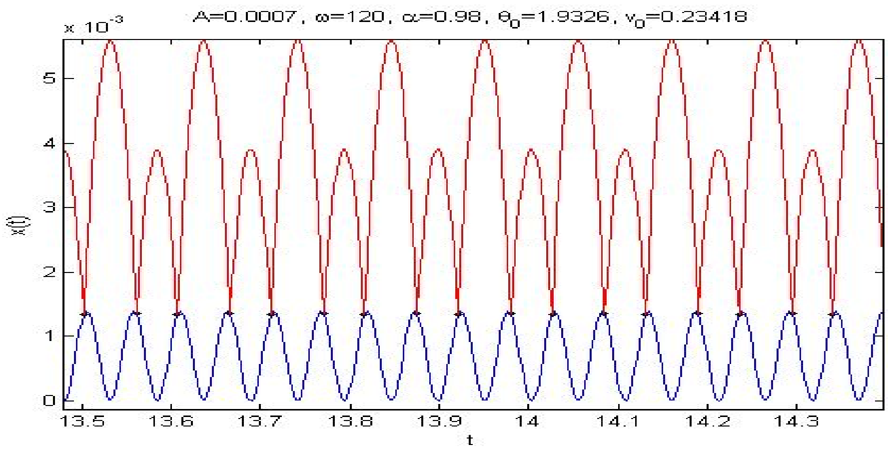

Figure 5: Period-2 orbit from theoretical model.

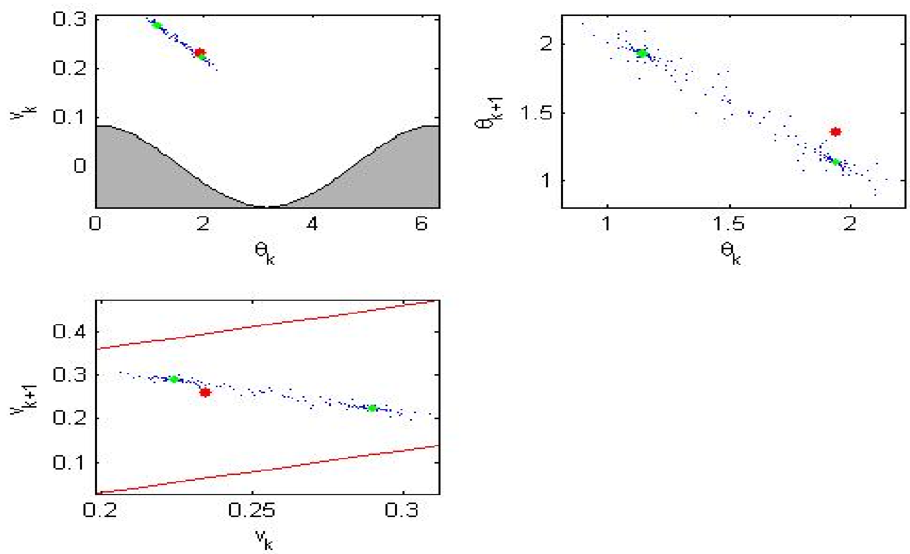

Figure 6: Period-2 orbit represented on an impact map. This impact map corresponds to the orbit depicted in figure 5. Period-2 corresponds to two points on an impact map (green points). The red dot is the initial impact and the little blue dots are all the phases this system explored before settling down into a period-2.

Periodic orbits correspond to patterns that repeat themselves and the period is the time it takes for the repetition to occur. Period-1 orbits have a period exactly equal to the piston, hence the name period-1. Similarly, period-2 orbits require twice the period of the piston to complete a full cycle (See figures 5 and 6 for an example of a period-2 orbit from our theoretical model). Chaotic orbits, however, exhibit behavior that does not repeat itself or follow any particular pattern [1]. Figures 7 and 8 depict an example of a chaotic orbit from our theoretical model with no apparent pattern.

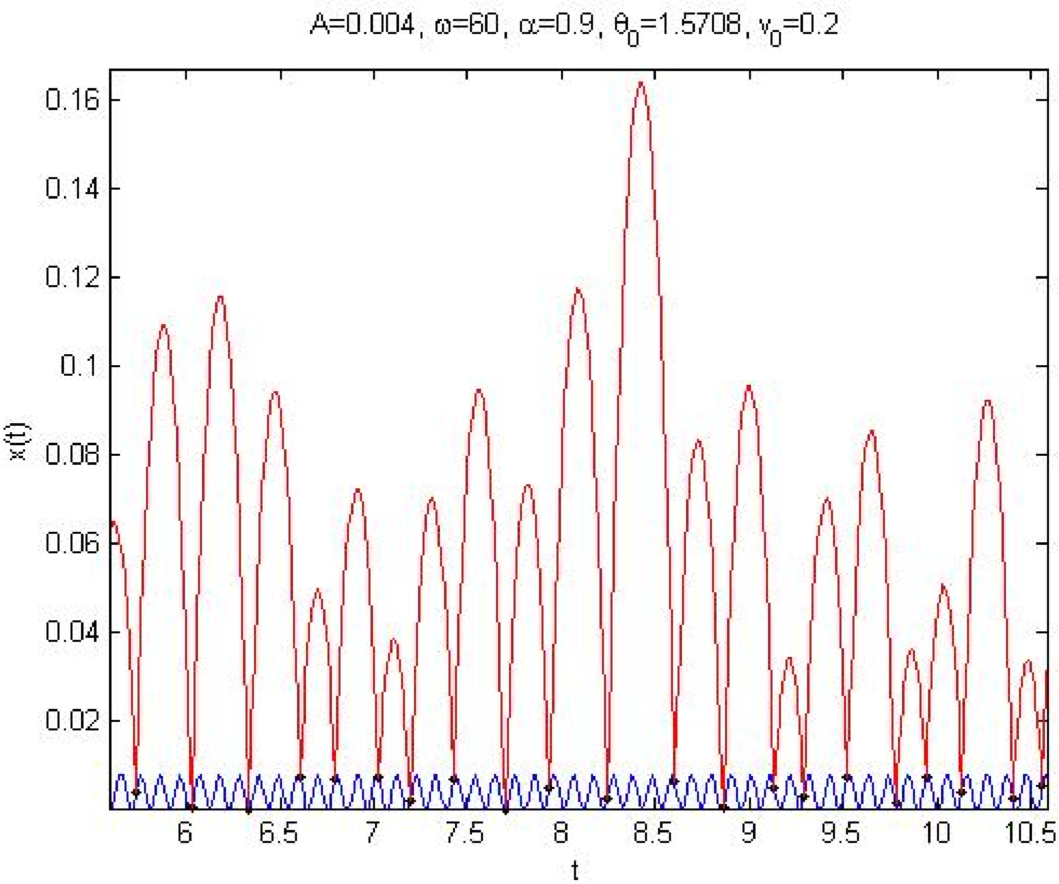

Figure 7: Chaotic orbit from theoretical model.

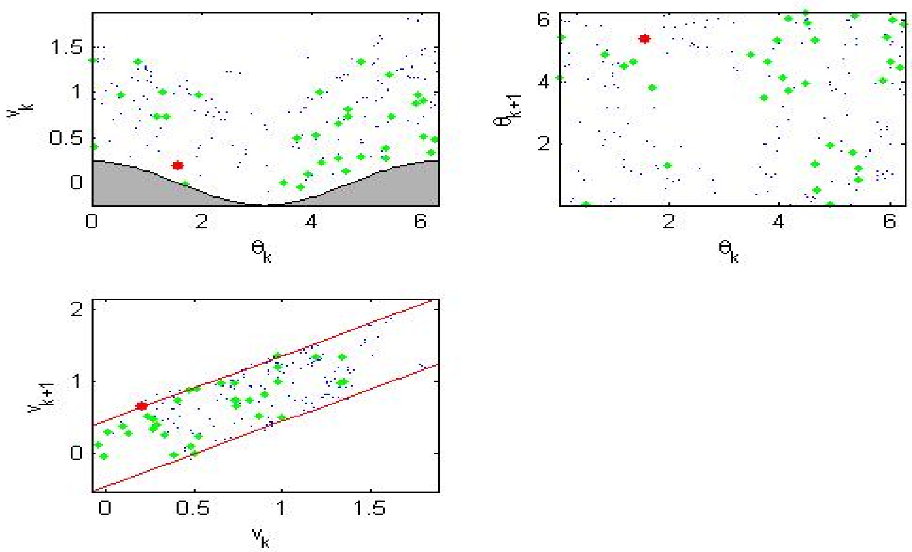

Figure 8: Chaotic orbit impact map corresponding to the orbit depicted in Figure 7.

Although it is possible for chaotic orbits to co-exist with other orbits, chaotic orbits do not appear to have a pattern of repetition. Chaotic orbits explore a wide range of phases and velocities.

3. Experimental Setup

In order to extract the data from our experimental model, infrared lights were built into the car and piston. The next step was to write a computer program that would allow us to use a Nintendo Wii remote to capture the data from the infrared LED lights. In writing the program, we used the Wii library to link the Wii remote to a laptop computer through a bluetooth port [6]. With the programming complete, the Wii remote was placed on a tripod facing the infrared lights. After adjusting the position of the Wii remote to have both infrared lights in range, we began our experimental runs. It is important to note that the first run of every set of data was dedicated to measuring the distance between the car and piston at a fixed, measured, distance. This allowed us to determine the conversion factor between physical units (cm) and Wii remote units. The angle of inclination on our experimental apparatus was fixed at 17.3 degrees. The frequency on our apparatus is in cycles per minute and all data obtained was converted to angular frequency using $\omega = 2\pi f$. All distances on our apparatus are measured in centimeters.

4. Experimental Results

We have been able to produce experimentally period-1, period-2, period-3, sticking solutions, and chaotic orbits with our experimental apparatus. Furthermore, using a curve fitting procedure to obtain more precisely the parameters from our experimental model, we then input those parameters into the theoretical model and observe good quantitative and qualitative agreement as seen in Figures 9 through 16.

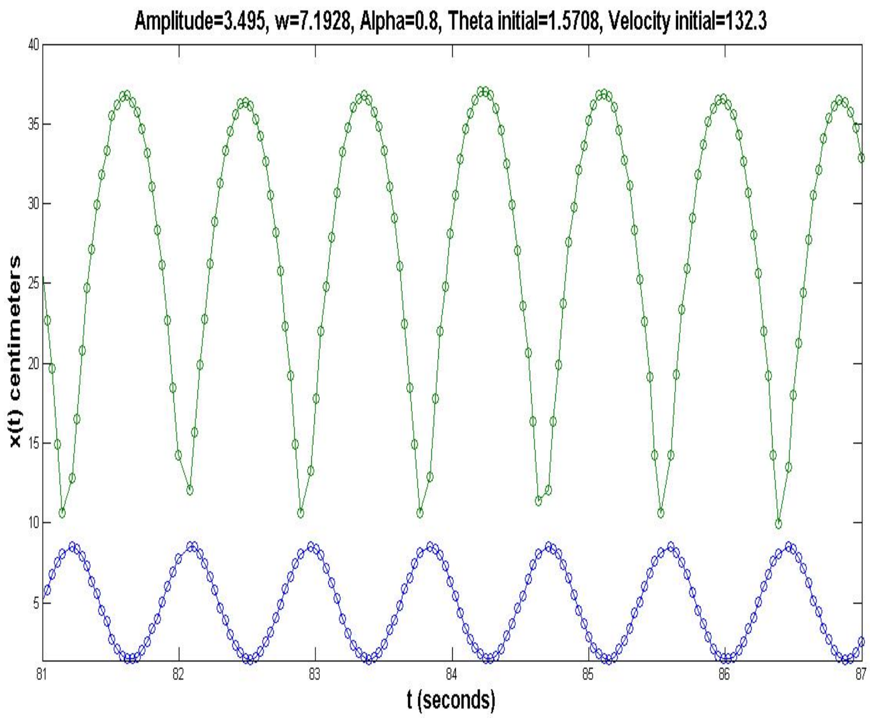

Figure 9: Period-1 orbit from experimental model.

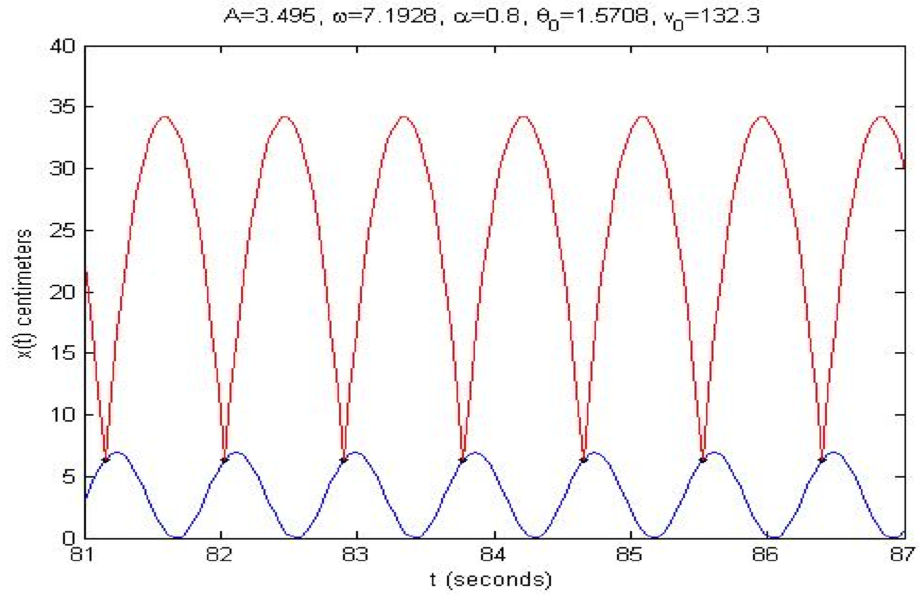

Figure 10: Period-1 orbit reproduced on theoretical model from data in figure 9.

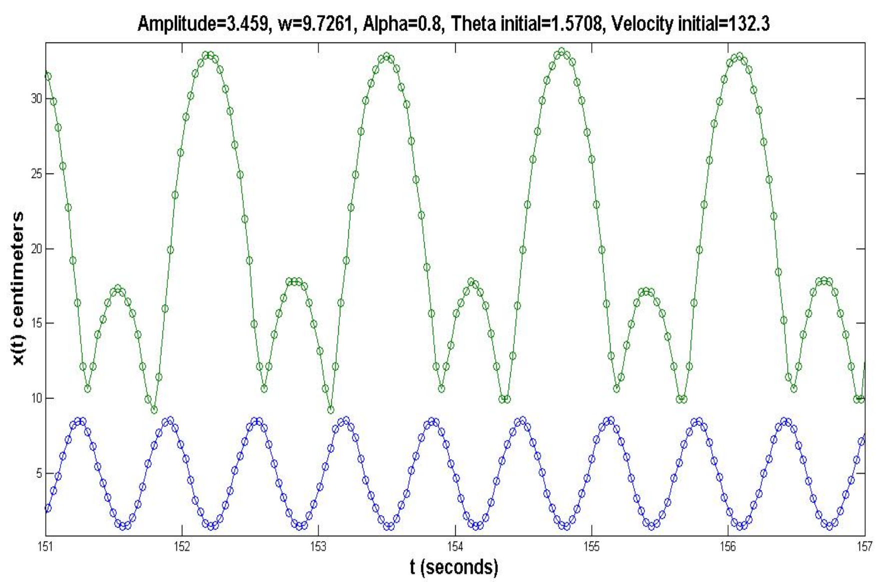

Figure 11: Period-2 orbit from experimental model.

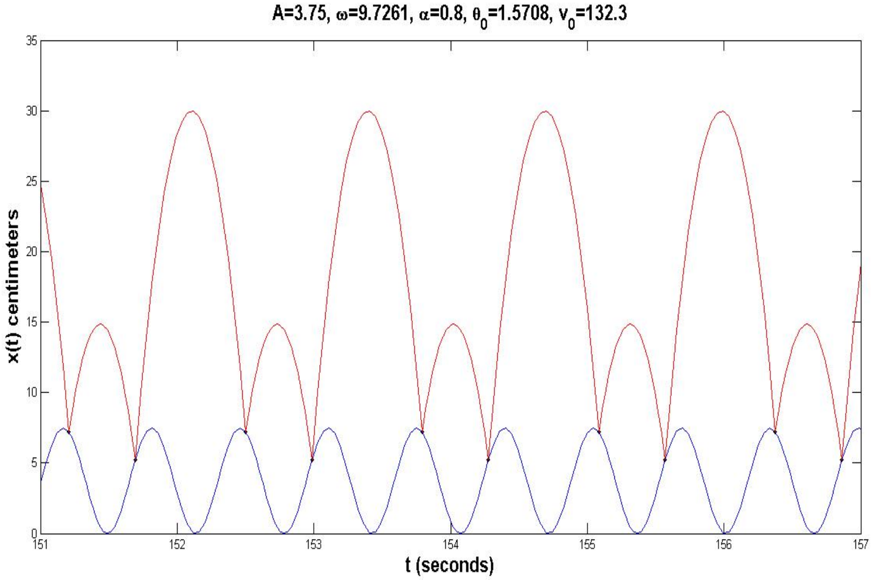

Figure 12: Period-2 orbit reproduced on theoretical model from data in figure 11.

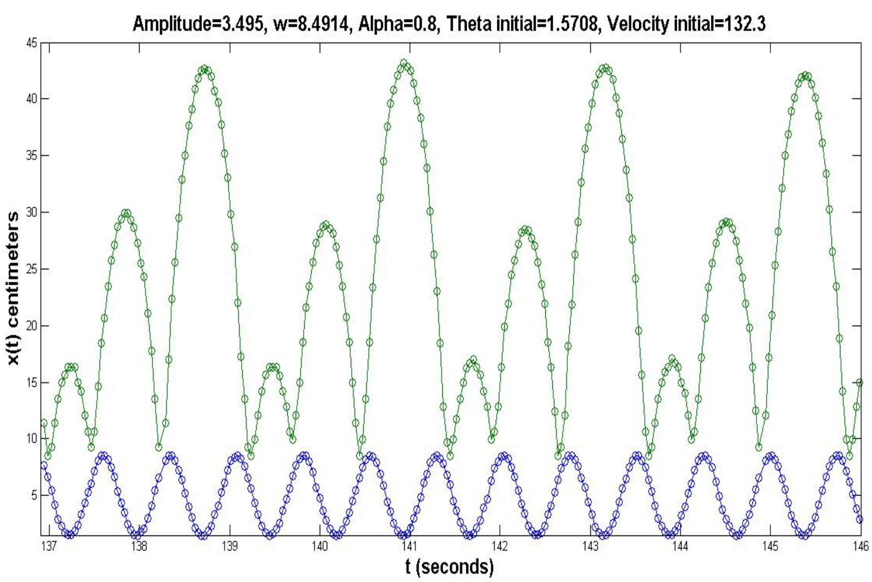

Figure 13: Period-3 orbit from experimental model.

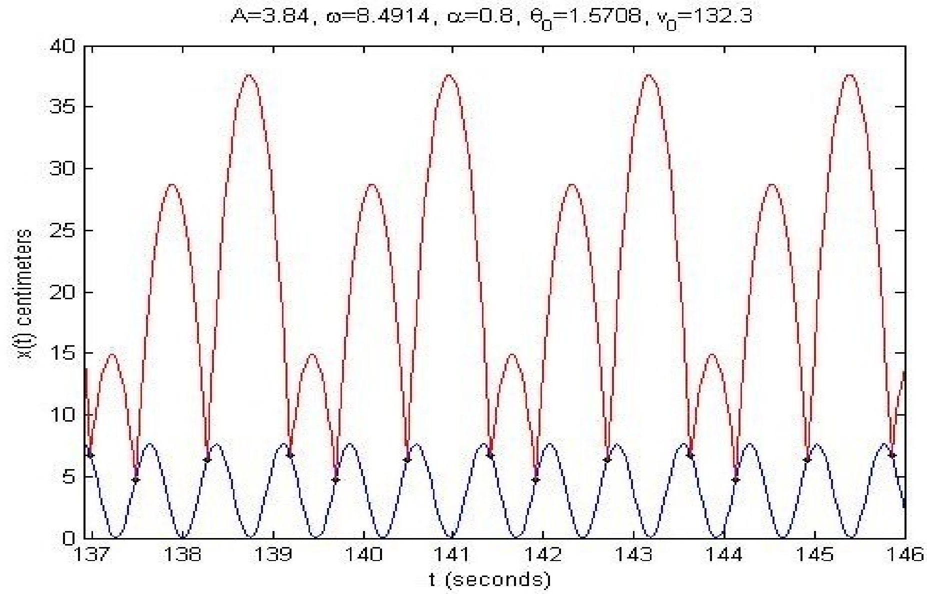

Figure 14: Period-3 orbit reproduced on theoretical model from data in figure 13. The discrepancy between amplitude in both figures is explained in text.

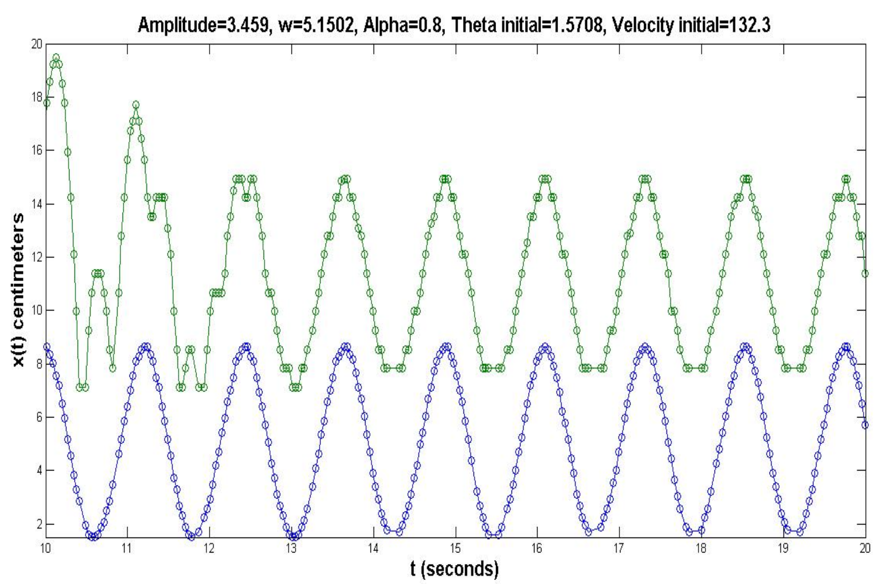

Figure 15: Sticking solution from experimental model.

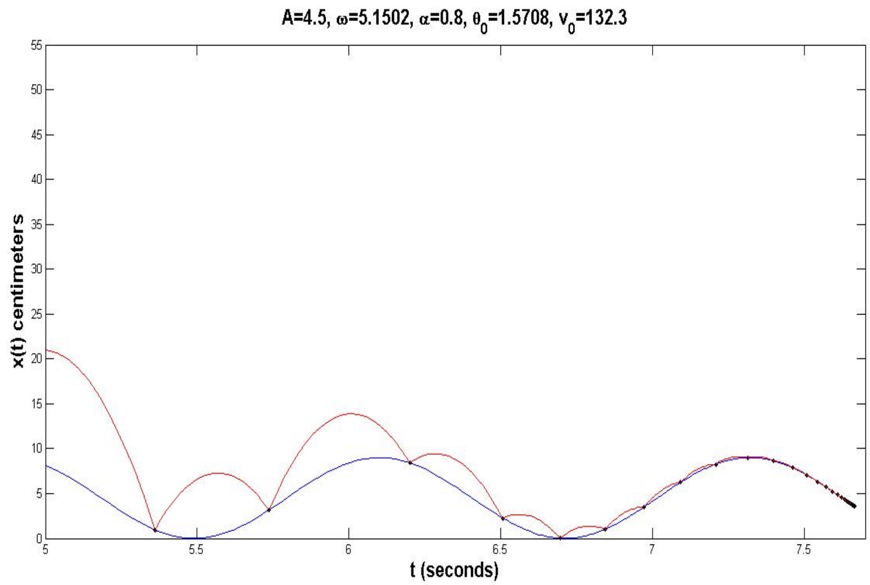

Figure 16: Sticking solution reproduced on theoretical model from data in Figure 15.

Some of our experimental results, when evaluated in the theoretical model, show discrepancies in amplitude due to the fact that there is a spring attached to the back of our experimental car. The spring is adding up to two centimeters to our calculated amplitude, depending on the initial conditions of a particular run; for larger velocities and frequencies, the spring will add less amplitude to our systems. This spring is also causing it to appear that there is no impact between the car and piston (see figure 11 and 13). This can be seen by noticing the gap between the cars parabolic trajectory and the piston (oscillating sine function).

5. Conclusion

In this paper we have formulated a theoretical model that reproduces the different types of orbits that are seen in nonlinear dynamical systems. We then turned to the experimental model where a bouncing car on an inclined plane, a periodically oscillating piston, and a Wii remote were used to reproduce the different nonlinear dynamics and to extract data. Comparison between our experimental and theoretical models shows good qualitative and quantitative agreement. We used data extracted from our experimental apparatus to compare the two models. We found that for period-1, period-2, period-3, and sticking solutions, good quantitative and qualitative results can be observed.

There is much work that can still be done with the bouncing car system. To start, one could better model the behavior of the spring attached to the car so that the system displays better quantitative results. Next, the threshold of the different orbits can be studied for the purpose of observing bifurcation points. Finally, there are still many system parameters to manipulate. One parameter that was not manipulated in this system was the angle of the inclined plane (it was fixed at 17.3 degrees). It would be interesting to see what, if any, new phenomenology can be reproduced through further manipulation of this parameter and others.

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