18.905: Problem Set III

Due October 19, 2016, in class.

Homework is an important part of this class. I hope you gain from the struggle. Collaboration can be effective, but be sure that you grapple with each problem on your own as well. If you do work with others, you must indicate with whom on your solution sheet. Scores will be posted on the Stellar website.

11. (a) In class I discussed a monoid homomorphism

$$\deg : [S^n, S^n] \to \mathbb{Z}_\times$$

(where $\mathbb{Z}_\times$ is the integers with monoid structure given by multiplication, and $n \geq 1$), given by sending f to its effect on $H_n(S^n)$. I asserted that it was surjective, but this was an induction based on the case $n = 1$. Regard S^1 as the unit circle in the complex plane. In class I claimed that the map $z \mapsto z^d$ has degree d. Please verify this claim.

(b) Regard S^{n-1} as the unit sphere in $\mathbf{R}^n$. Let L be a line through the origin in $\mathbf{R}^n$, and $L^\perp$ its orthogonal complement. Let ρ_L be the linear map given by -1 on L and $+1$ on $L^\perp$. What is $\deg \rho_L$?

(c) What is the degree of the "antipodal map," $\alpha : S^{n-1} \to S^{n-1}$ sending x to $-x$?

(d) The tangent space to a point x on the sphere S^{n-1} can be regarded as the subspace of $\mathbf{R}^n$ of vectors perpendicular to x. A "vector field" on S^{n-1} is thus a continuous function $v : S^{n-1} \to \mathbf{R}^n$ such that $v(x) \perp x$ for all $x \in S^{n-1n}$.

Show that if n is odd then every vector field vanishes at some point on the sphere. (When $n - 1 = 2$, this is the "hairy ball theorem.")

On the other hand, construct a nowhere vanishing vector field on S^{n-1} for any even n.

12. Use the Mayer-Vietoris sequence to compute the homology groups of the projective plane P, the Klein bottle K, and the torus T. (The projective plane is obtained by sewing a disk onto a Möbius band along their boundaries. The Klein bottle is obtained either by sewing two Möbius bands together, or by sewing the two boundary components of a cylinder together in a funny way. A torus is obtained by sewing the boundary components of a cylinder together in a less funny way. In each case, it's a good idea to give yourself a hem: glue open "collars" together.)

(b) Hopefully you computed that $H_2(T)$ is an infinite cyclic group. Say something sensible about whether the "fundamental class" you constructed in Problem **2** is indeed a generator of that abelian group.

13. $\varnothing$.

14. The constructions sketched in Problem **12** are examples of the following general procedure. Take two closed surfaces, Σ_1 and Σ_2, cut a disk out from each one, and glue them together along the hem. This is the *connected sum* $\Sigma_1 \# \Sigma_2$. Write T_1 for the torus, and $T_g = T_1 \# T_{g-1}$. Write P_1 for the projective plane, and $P_g = P_1 \# P_{g-1}$. A theorem of Rado asserts that this is a complete list of compact connected 2-manifolds.

(a) What is the Klein bottle, in this notation?

(b) Complete the work from Problem **12**: compute the homology groups of these closed surfaces.