18.905: Problem Set II

Due October 5, 2016, in class.

Homework is an important part of this class. I hope you gain from the struggle. Collaboration can be effective, but be sure that you grapple with each problem on your own as well. If you do work with others, you must indicate with whom on your solution sheet. Scores will be posted on the Stellar website.

5. (a) Let A be a chain complex (of abelian groups). It is *acyclic* if $H(A) = 0$, and *contractible* if it is chain-homotopy-equivalent to the trivial chain complex. Prove that a chain complex is contractible if and only if it is acyclic and for every n the inclusion $Z_nA \hookrightarrow A_n$ is a split monomorphism of abelian groups.

(b) Give an example of an acyclic chain complex that is not contractible.

6. Propose a construction of the product and the coproduct of two spaces in the homotopy category, and check that your proposal serves the purpose.

7.(a) Let S and T be sets and A an abelian group. Establish a bijection between the set of maps of sets from $S \times T$ to A and the set of bilinear maps $\mathbb{Z}S \times \mathbb{Z}T \to A$.

(b) For positive integers m, n, let $\mathbb{Z}/m$, $\mathbb{Z}/n$ denote the cyclic groups of order m, n. Construct a surjective bilinear map $\mu : \mathbb{Z}/m \times \mathbb{Z}/n \to \mathbb{Z}/\gcd\{m, n\}$. Show that any bilinear map $\mathbb{Z}/m \times \mathbb{Z}/n \to A$ factors uniquely as $f \circ \mu$ where $f : \mathbb{Z}/\gcd\{m, n\} \to A$ is a homomorphism.

8. (a) Let $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$ be a short exact sequence. Show that the following three sets are in bijection with one another.

(i) The set of homomorphisms $\sigma : C \to B$ such that $p\sigma = 1_C$.

(ii) The set of homomorphisms $\pi : B \to A$ such that $\pi i = 1_A$.

(iii) The set of homomorphisms $\alpha : A \oplus C \to B$ such that $\alpha(a, 0) = ia$ for all $a \in A$ and $p\alpha(a, c) = c$ for all $(a, c) \in A \oplus C$.

Moreover, show that any homomorphism as in (iii) is an isomorphism.

Any one of these structures is a *splitting* of the short exact sequence, and the sequence is then said to be *split*.

(b) Suppose that

$$\begin{tikzcd} \cdots \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d, "\cong"] & A_{n-1} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & A'_n \arrow[r] & B'_n \arrow[r] & C'_n \arrow[r] & A'_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

is a "ladder": a map of long exact sequences. So both rows are exact and each square commutes. Suppose also that every third vertical map is an isomorphism, as indicated. Prove that these data naturally determine a long exact sequence

$$\cdots \longrightarrow A_n \longrightarrow A'_n \oplus B_n \longrightarrow B'_n \longrightarrow A_{n-1} \longrightarrow \cdots$$

9. (a) ("3×3 lemma.") Let

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A'' & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

be a commutative diagram of abelian groups. Assume that all three columns are exact, that all but one of the rows is exact, and that the compositions in the remaining row are trivial. Prove that the remaining row is also exact. (Hint: view each row as a chain complex ....)

(b) ("Long exact homology sequence of a triple.") Let (C, B, A) be a "triple," so C is a space, B is a subspace of C, and A is a subspace of B. Show that there are natural transformations $\partial : H_n(C, B) \to H_{n-1}(B, A)$ such that

$$\cdots \longrightarrow H_n(B, A) \xrightarrow{i_*} H_n(C, A) \xrightarrow{j_*} H_n(C, B) \xrightarrow{\partial} H_{n-1}(B, A) \longrightarrow \cdots$$

is exact, where $i : (B, A) \to (C, A)$ and $j : (C, A) \to (C, B)$ are the inclusions of pairs. (**9 (a)** might be useful.)

10. This exercise generalizes our computation of the homology of spheres, and introduces several important constructions.

The *cone* on a space X is the quotient space $CX = X \times I / X \times \{0\}$, where I is the unit interval $[0, 1]$. The cone is a pointed space, with basepoint $*$ given by the "cone point," i.e. the image of $X \times \{0\}$. (By convention, the cone on the empty space $\varnothing$ is a single point, the cone point.) Regard X as the subspace of CX of all points of the form $(x, 1)$.

Define the *suspension* of a space X to be $SX = CX/X$. Make SX a pointed space by declaring the image of $X \subseteq CX$ to be the basepoint in SX. (By convention, the quotient $W/\varnothing$ is the disjoint union of W with a single point, which is declared to be the basepoint. So $S\varnothing = */\varnothing$ is the discrete two-point space, with the new point as basepoint.)

The quotient map induces a map of pairs $f : (CX, X) \to (SX, *)$.

(a) Show that CX is contractible.

(b) Show that there is a natural isomorphism $\widetilde{H}_{n-1}(X) \to H_n(SX, *)$, for any n.