18.905: Problem Set V

Due November 16, 2016, in class.

Homework is an important part of this class. I hope you gain from the struggle. Collaboration can be effective, but be sure that you grapple with each problem on your own as well. If you do work with others, you must indicate with whom on your solution sheet. Scores will be posted on the Stellar website. Extra credit for calling attention to mistakes!

19. **(a)** Verify the Lemma stated in lecture on Nov 4: Let I be a directed set, L an abelian group, and $A : I \to \mathbf{Ab}$ an I-directed diagram of abelian groups, with bonding maps $f_{ij} : A_i \to A_j$ for $i \leq j$. A map $A \to c_L$, given by compatible maps $f_i : A_i \to L$, is a direct limit if and only if:
(i) For any $b \in L$ there exists $i \in I$ and $a_i \in A_i$ such that $f_i a_i = b$, and
(ii) For any $a_i \in A_i$ such that $f_i a_i = 0 \in L$, there exists $j \geq i$ such that $f_{ij} a_i = 0 \in A_j$.

20. **(a)** Embed $\mathbb{Z}/p^n\mathbb{Z}$ into $\mathbb{Z}/p^{n+1}\mathbb{Z}$ by sending 1 to p, and write $\mathbb{Z}_{p^\infty}$ for the union. It's called the Prüfer group (at p). Show that $\mathbb{Z}_{p^\infty} \cong \mathbb{Z}[1/p]/\mathbb{Z}$ and that

$$\mathbb{Q}/\mathbb{Z} \cong \bigoplus_p \mathbb{Z}_{p^\infty}$$

where the sum runs over the prime numbers.

(b) Compute $\mathbb{Z}_{p^\infty} \otimes_{\mathbb{Z}} A$ for A each of the following abelian groups: $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}[1/q]$ (for q a prime), and $\mathbb{Z}_{q^\infty}$ (for q a prime).

(c) Compute $\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}[1/p])$ and $\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}_{p^\infty})$, for any abelian group M in terms of the self-map $p : M \to M$.

21. Show that if $f : X \to Y$ induces an isomorphism in homology with coefficients in the prime fields $\mathbb{F}_p$ (for all primes p) and $\mathbb{Q}$, then it induces an isomorphism in homology with coefficients in $\mathbb{Z}$. (Hint: **18 (c)**.)

22. The construction of an isomorphism between the singular homology and the cellular homology of a CW complex carries over *verbatim* with any coefficients. Use this observation to compute the homology of $\mathbb{R}P^n$ with coefficients in $\mathbb{Q}$, $\mathbb{F}_p$, $\mathbb{Z}_{p^\infty}$, and $\mathbb{Z}[1/p]$.

23. $\varnothing$

24. Suppose that $f : X \to Y$ induces an isomorphism in homology with coefficients in $\mathbb{Z}/n\mathbb{Z}$. Show that it induces an isomorphism in homology with coefficients in any abelian group in which every element is killed by some power of n.