# Algebraic Topology

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## Preface

Here is an overview of this part of the book.

1. **General homotopy theory.** This includes category theory; because it started as a part of algebraic topology, we'll speak freely about it here. We'll also cover the general theory of homotopy groups, long exact sequences, and obstruction theory.

2. **Bundles.** One of the major themes of this part of the book is the use of bundles to understand spaces. This will include the theory of classifying spaces; later, we will touch upon connections with cohomology.

3. **Spectral sequences.** It is impossible to describe everything about spectral sequences in the duration of a single course, so we will focus on a special (and important) example: the Serre spectral sequence. As a consequence, we will derive some homotopy-theoretic applications. For instance, we will relate homotopy and homology (via the Hurewicz theorem, Whitehead's theorem, and "local" versions like Serre's mod C theory).

4. **Characteristic classes.** This relates the geometric theory of bundles to algebraic constructions like cohomology described earlier in the book. We will discuss many examples of characteristic classes, including the Thom, Euler, Chern, and Stiefel-Whitney classes. This will allow us to apply a lot of the theory we built up to geometry.

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# Chapter 4

# Basic homotopy theory

## 39 Limits, colimits, and adjunctions

### Limits and colimits

I want to begin by developing a little more category theory.

**Definition 39.1.** Suppose $\mathcal{I}$ is a small category (so that it has a *set* of objects), and let $\mathcal{C}$ be another category. Let $X : \mathcal{I} \to \mathcal{C}$ be a functor. A *cone under* $X$ is a natural transformation $\eta$ from $X$ to a constant functor; to be explicit, this means that for every object $i$ of $\mathcal{I}$ we have a map $\eta_i : X_i \to Y$, and these maps are compatible in the sense that for every $f : i \to j$ in $\mathcal{I}$ the following diagram commutes:

$$\begin{tikzcd} X_i \arrow[r, "\eta_i"] \arrow[d, "f_*"] & Y \arrow[d, "="] \\ X_j \arrow[r, "\eta_j"] & Y \,. \end{tikzcd}$$

A *colimit* of $X$ is an initial cone $(L, \tau_i)$ under $X$; to be explicit, this means that for any cone $(Y, \eta_i)$ under $X$, there exists a unique map $h : L \to Y$ such that $h \circ \tau_i = \eta_i$ for all $i$.

As with any universal property, any two colimits are isomorphic by a unique isomorphism; but existence is another matter. Also, as always for category theoretic concepts, some examples are in order.

**Example 39.2.** If $\mathcal{I}$ is a discrete category (that is, the only maps are identity maps; $\mathcal{I}$ is entirely determined by its set of objects), the colimit of a functor $\mathcal{I} \to \mathcal{C}$ is the coproduct in $\mathcal{C}$ (if this coproduct exists!).

**Example 39.3.** In Lecture 23 we discussed directed posets and the direct limit of a directed system $X : \mathcal{I} \to \mathcal{C}$. The colimit simply generalizes this to arbitrary indexing categories rather than restricting to directed partially ordered sets.

**Example 39.4.** Let $G$ be a group; we can view this as a category with one object, where the morphisms are the elements of the group and composition is given by the group structure. If $\mathcal{C} = \mathbf{Top}$ is the category of topological spaces, a functor $G \to \mathcal{C}$ is simply a group action on a topological space $X$. The colimit of this functor is the orbit space of the $G$-action on $X$.

**Example 39.5.** Let $\mathcal{I}$ be the category whose objects and morphisms are determined by the following directed graph:

$$\begin{tikzcd} & a \arrow[dl] \arrow[dr] & \\ b & & c. \end{tikzcd}$$

The colimit of a diagram $\mathcal{I} \to \mathcal{C}$ is called a *pushout*. With $\mathcal{C} = \mathbf{Top}$, again, a functor $\mathcal{I} \to \mathcal{C}$ is determined by a diagram of spaces:

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ B & & C. \end{tikzcd}$$

The colimit of such a functor is just the pushout $B \cup_A C := B \sqcup C/\sim$, where $f(a) \sim g(a)$ for all $a \in A$. We have already seen this in action before: a special case of this construction appears in the process of attaching cells to build up a CW-complex.

If $\mathcal{C}$ is the category of groups, instead, the colimit of such a functor is the free product quotiented out by a certain relation; this is called the *amalgamated free product*.

**Example 39.6.** Suppose $\mathcal{I}$ is the category defined by the following directed graph:

$$a \rightrightarrows b.$$

The colimit of a diagram $\mathcal{I} \to \mathcal{C}$ is called the *coequalizer* of the diagram. If $\mathcal{C} = \mathbf{Set}$, the coequalizer of $f, g : A \rightrightarrows B$ is the quotient of $B$ by the equivalence relation generated by $f(a) \sim g(a)$ for $a \in A$.

One can also consider cones *over* a diagram $X : \mathcal{I} \to \mathcal{C}$: this is simply a cone in the opposite category.

**Definition 39.7.** The *limit* of a diagram $X : \mathcal{I} \to \mathcal{C}$ is a terminal object in cones over $X$.

**Exercise 39.8.** Revisit the examples provided above: what is the limit of each diagram? For instance, a product is a limit over a discrete category, and the limit of the diagram described in Example 39.4 is just the fixed points.

## Adjoint functors

The notion of a colimit as a special case of the more general concept of an adjoint functor, as long as we are dealing with a cocomplete category as in the following definition.

**Definition 39.9.** A category $\mathcal{C}$ is *cocomplete* if all functors from small categories to $\mathcal{C}$ have colimits. Similarly, $\mathcal{C}$ is *complete* if all functors from small categories to $\mathcal{C}$ have limits.

All examples considered above are both cocomplete and complete.

Let's write $\mathcal{C}^{\mathcal{I}}$ for the category of functors from $\mathcal{I}$ to $\mathcal{C}$, and natural transformations between them.

There is a functor $c : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$, given by sending any object to the constant functor taking on that value. The process of taking the colimit of a diagram supplies us with a functor $\operatorname{colim}_{\mathcal{I}} : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$. We can characterize this functor via the formula

$$\mathcal{C}(\operatorname{colim}_{i\in\mathcal{I}} X_i, Y) = \mathcal{C}^{\mathcal{I}}(X, c_Y)\,,$$

where $X$ is any functor from $\mathcal{I}$ to $\mathcal{C}$, $Y$ is any object of $\mathcal{C}$, and $c_Y$ denotes the constant functor with value $Y$. This formula is reminiscent of the adjunction operator in linear algebra, and is in fact our first example of an adjunction.

**Definition 39.10.** Let $\mathcal{C}, \mathcal{D}$ be categories, and suppose given functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. An *adjunction between* $F$ *and* $G$ is an isomorphism:

$$\mathcal{D}(FX, Y) = \mathcal{C}(X, GY),$$

that is natural in $X$ and $Y$. In this situation, we say that $F$ is a *left adjoint* of $G$ and $G$ is a *right adjoint* of $X$.

This notion was invented by the late MIT Professor Dan Kan.

We've already seen one example of adjoint functors. Here is another one.

**Definition 39.11** (Free groups)**.** There is a forgetful functor $u : \mathbf{Grp} \to \mathbf{Set}$. Any set $X$ gives rise to a group $FX$, the free group on $X$ elements. It is determined by a universal property: set maps $X \to u\Gamma$ are the same as group maps $FX \to \Gamma$, where $\Gamma$ is any group. This is exactly saying that the free group functor the left adjoint to the forgetful functor $u$.

In general, "free objects" come from left adjoints to forgetful functors.

As a general notational practice, try to write the left adjoint as the top arrow:

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G \qquad \text{or} \qquad G : \mathcal{D} \leftrightarrows \mathcal{C} : F\,.$$

### The Yoneda lemma

One of the many important principles in category theory is that an object is determined by the collection of all maps out of it. The Yoneda lemma is a way of making this precise. Observe that for any $X \in \mathcal{C}$ the association $Y \mapsto \mathcal{C}(X,Y)$ gives us a functor $\mathcal{C} \to \mathbf{Set}$. This functor is said to be *corepresentable* by $X$. Suppose that $G : \mathcal{C} \to \mathbf{Set}$ is any functor. An element $x \in G(X)$ determines a natural transformation

$$\theta_x : \mathcal{C}(X, -) \to G$$

in the following way. Let $Y \in \mathrm{ob}\mathcal{C}$. Map $\mathcal{C}(X,Y) \to G(Y)$ by sending $f : X \to Y$ to $f_*(x) \in G(Y)$.

**Lemma 39.12** (Yoneda lemma)**.** *The association $x \mapsto \theta_x$ provides a bijection*

$$\mathrm{nt}(\mathcal{C}(X,-), G) \xrightarrow{\cong} G(X).$$

*Proof.* The inverse is given as follows: Send a natural transformation $\theta : C(X, -) \to G$ to $\theta_X(1_X) \in G(X)$. □

In particular, if $G$ is also corepresentable – $G = \mathcal{C}(Y, -)$, say – then

$$\mathrm{nt}(\mathcal{C}(X,-), \mathcal{C}(Y,-)) \cong \mathcal{C}(Y, X)\,.$$

Simply put, natural transformations $\mathcal{C}(X,-) \to \mathcal{C}(Y,-)$ are in natural bijection with maps $Y \to X$. Consequently natural isomorphisms $\mathcal{C}(X,-) \xrightarrow{\cong} \mathcal{C}(Y,-)$ are in natural bijection with isomorphims $Y \xrightarrow{\cong} X$. So, for example, the object that corepresents a corepresentable functor is unique up to unique isomorphism.

From the Yoneda lemma, we can obtain some pretty miraculous conclusions. For instance, functors with left and/or right adjoints are very well-behaved.

**Theorem 39.13.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. If $F$ admits a right adjoint then it preserves colimits. Dually, if $F$ admits a left adjoint then it preserves limits.*

*Proof.* We'll prove the first statement; the dual proof gives the other. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor that admits a right adjoint $G$, and let $X : \mathcal{I} \to \mathcal{C}$ be a small $\mathcal{I}$-indexed diagram in $\mathcal{C}$. Suppose that the colimit $\mathrm{colim}_{\mathcal{I}} X$ exists. From the definition of colimit, there is for any object $Y$ of $\mathcal{C}$ an isomorphism

$$\mathcal{C}(\mathrm{colim}_{\mathcal{I}} X, Y) \cong \lim{}_{\mathcal{I}} \mathcal{C}(X, Y)\,.$$

Let $Z$ be any object of $\mathcal{D}$, and note the sequence of natural isomorphisms:

$$\begin{aligned} \mathcal{D}(F(\mathrm{colim}_{\mathcal{I}} X), Y) &\cong \mathcal{C}(\mathrm{colim}_{\mathcal{I}} X, G(Y)) \\ &\cong \lim{}_{\mathcal{I}} \mathcal{C}(X, G(Y)) \\ &\cong \lim{}_{\mathcal{I}} \mathcal{D}(F(X), Y) \\ &\cong \mathcal{D}(\mathrm{colim}_{\mathcal{I}} F(X), Y). \end{aligned}$$

The Yoneda lemma now finishes the job. □

# 40 Cartesian closure and compactly generated spaces

A lot of homotopy theory is about loop spaces and mapping spaces. The compact-open topology is available to us – and we'll recall it later. But it suffers from some defects. To clarify how a mapping object should behave in an ideal world, I want to make another category-theoretical digression

**Definition 40.1.** Let $\mathcal{C}$ be a category with finite products. It is *Cartesian closed* if for any object $X$ in $\mathcal{C}$, the functor $X \times -$ has a right adjoint.

**Example 40.2.** If $\mathcal{C} = \mathbf{Set}$,

$$\mathbf{Set}(W \times X, Y) = \mathbf{Set}(X, \mathbf{Set}(W, Y)) \,,$$

so the right adjoint is the co-representable functor $\mathbf{Set}(W, -)$.

So mapping objects are ideally given as the right adjoint to Cartesian product.

Many otherwise well-behaved categories are not Cartesian-closed. The category of abelian groups is an example. There is indeed a functor $Y \mapsto \mathrm{Hom}(W, Y)$, for any abelian group $W$; but it is the right adjoint not of $W \times -$ but rather of $W \otimes -$.

As a general thing, if a functor has a right adjoint then that right adjoint is well-defined up to canonical natural isomorphism; so we will always speak of *the* right adjoint. We'll write the right adjoint to $X \times -$ using exponential notation,

$$Y \mapsto Y^W \,,$$

so that there is a natural bijection

$$\mathcal{C}(W \times X, Y) = \mathcal{C}(X, Y^W) \,.$$

The category of topological spaces is supposed to behave more like **Set** than like **Ab**, but it turns out that **Top** is not Cartesian closed either. Small modifications of it are, however. We describe them below.

To better understand the consequence of Cartesian closure, it's convenient to restate the definition of adjointness. So suppose that

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

is an adjoint pair of functors, so that

$$\mathcal{D}(FX, Y) = \mathcal{C}(X, GY)$$

Take $Y = FX$; the identity $1_{FX}$ corresponds to a map $\eta_X : X \to GFX$, called the *unit* of the adjunction.

Take $X = GY$; the identity $1_{GY}$ corresponds to a map $\epsilon_Y : FGY \to Y$, called the *counit* of the adjunction.

Both these maps are natural transformations of endofunctors of $\mathcal{C}$.

**Example 40.3.** Take for example the free group adjunction

$$F : \mathbf{Set} \rightleftarrows \mathbf{Gp} : u\,.$$

In this case, $\eta : X \to uFX$ sends a set to itself inside the free group it generates; and $\epsilon : FuY \to Y$ sends a word in the free group on the set underlying $Y$ to the product of those elements in $Y$.

**Lemma 40.4.** *The unit and counit of an adjunction make the following diagrams commute.*

$$\begin{tikzcd} GFGY \arrow[r, "G\epsilon_Y"] & GY \\ GY \arrow[u, "\eta_{GY}"] \arrow[ur, "="'] & \end{tikzcd} \qquad \begin{tikzcd} FGFX \arrow[r, "\epsilon_{FX}"] & FX \\ FX \arrow[u, "F\eta_X"] \arrow[ur, "="'] & \end{tikzcd}$$

*Conversely, equipping functors $F$ and $G$ with natural transformations $\eta$ and $\epsilon$ making these diagrams commute establishes $(F, G)$ as an adjoint pair.*

In the Cartesian setting, then, we have natural transformations

$$\eta_X : X \to (W \times X)^W\,, \quad \epsilon_Y : W \times Y^W \to Y\,.$$

In case $\mathcal{C} = \mathbf{Set}$, these are given by

$$x \mapsto (w \mapsto (w, x))\,, \quad (w, f) \mapsto f(w)\,,$$

that is, including a slice and evaluation. You should check that the two triangles commute in this case!

Here are some direct consequences of Cartesian closure. Note: the assumption that finite products exist in $\mathcal{C}$ includes the case of the empty product, which is a terminal object $*$.

**Proposition 40.5.** *Let $\mathcal{C}$ be Cartesian closed.*
*(1) $(X, Z) \mapsto Z^X$ is a functor $\mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$.*
*(2) $\mathcal{C}(X, Y) = \mathcal{C}(*, Y^X)$.*
*(3) $X \times -$ commutes with all colimits.*

## CGHW spaces

The last Proposition shows that **Top** is not Cartesian closed: A standard example from general topology shows that if $Y \to Z$ is a quotient map, the induced map $X \times Y \to X \times Z$ may fail to be a quotient map. But any quotient maps are precisely coequalizers in **Top**.

Henry Whitehead showed that crossing with a compact Hausdorff space *does* preserve quotient maps. This will often suffice, but often not, and the convenience of working in a Cartesian closed category is compelling.

Inspired by Whitehead's theorem, we agree to accept only properties of a space that can be observed by mapping compact Hausdorff spaces into it.

**Definition 40.6.** Let $X$ be a space. A subspace $F \subseteq X$ is said to be *compactly closed* if, for any map $k : K \to X$ from a compact Hausdorff space $K$, the preimage $k^{-1}(F) \subseteq K$ is closed.

It is clear that any closed subset is compactly closed, but there might be compactly closed sets which are not closed in the topology on $X$. This motivates the definition of a $k$-space:

**Definition 40.7.** A topological space $X$ is said to be a *$k$-space* if every compactly closed set is closed.

The $k$ comes either from "kompact" and/or the general topologist John Kelley.

A more categorical characterization of this property is: $X$ is a $k$-space if and only if a map $X \to Y$ is continuous precisely when for every compact Hausdorff space $K$ and map $k : K \to X$ the composite $K \to X \to Y$ is continuous. For instance, compact Hausdorff spaces are $k$-spaces. First countable (so metric spaces) and CW-complexes are also $k$-spaces.

While not all topological spaces are $k$-spaces, any space can be "$k$-ified." The procedure is simple: endow the underlying set of a space $X$ with the topology consisting of all compactly closed sets. The reader should check that this is indeed a topology on $X$. The resulting topological space is denoted $kX$. This construction immediately implies, for instance, that the identity $kX \to X$ is continuous, and is the terminal map to $X$ from a $k$-space.

Let $k\mathbf{Top}$ be the category of $k$-spaces. This is a subcategory of the category of topological spaces, and we will write $i : k\mathbf{Top} \hookrightarrow \mathbf{Top}$ for the inclusion functor. The process of $k$-ification gives a functor $\mathbf{Top} \to k\mathbf{Top}$, which has the property that

$$k\mathbf{Top}(X, kY) = \mathbf{Top}(iX, Y)\,.$$

This is another example of an adjunction! In this case the counit $kiX \to X$ is a homeomorphism. We can conclude from this that the product in $k\mathbf{Top}$ may be computed as

$$X \times^{k\mathbf{Top}} Y = k(iX \times iY)\,.$$

The category $k\mathbf{Top}$ has good categorical properties inherited from $\mathbf{Top}$: it is a complete and cocomplete category. As we will now explain, this category has even better categorical properties than $\mathbf{Top}$ does.

## Mapping spaces

Let $X$ and $Y$ be topological spaces. The set $\mathbf{Top}(X, Y)$ of continuous maps from $X$ to $Y$ admits an interesting topology, the *compact-open topology.* If $X$ and $Y$ are $k$-spaces, we can make a slight modification: Start with topology on $k\mathbf{Top}(X, Y)$ generated by the sets $W(k, U)$, where $k : K \to X$ is a map from a compact Hausdorff space and $U \subseteq Y$ is open, defined by

$$W(k, U) = \{f : X \to Y : f(k(K)) \subseteq U\}\,.$$

This topology is probably not itself determined by maps from compact Hausdorff spaces, so $k$-ify it. We write $Y^X$ for the resulting $k$-space; its underlying set is still the set of continuous maps from $X$ to $Y$. With this definition, we have:

**Proposition 40.8.** *The category $k$**Top** is Cartesian closed.*

*Proof.* See [**?**, Proposition 2.11]. □

## 41 Weak Hausdorff, Basepoints

The ancients came up with a good definition of a topology — but $k$-spaces are better! Sometimes, though, we can be greedy and ask for even more: for instance, we can demand a Hausdorff condition. This leads to a further refinement of $k$-spaces. But "Hausdorff" isn't quite the appropriate condition in the context of $k$-spaces. Rather:

**Definition 41.1.** A space $X$ is *weak Hausdorff* if the image of every continuous map $K \to X$ from a compact Hausdorff space $K$ is closed.

Another way to say this is that the map itself if closed. Clearly Hausdorff implies weak Hausdorff. Every singleton subset of a weak Hausdorff space is closed (since the one-point space is compact Hausdorff).

**Proposition 41.2.** *Let $X$ be a $k$-space.*

1. *$X$ is weak Hausdorff if and only if $\Delta : X \to X \times^{k\mathbf{Top}} X$ is closed. (In algebraic geometry such a condition is termed* separated.*)*
2. *Let $R \subseteq X \times X$ be an equivalence relation. If $R$ is closed, then $X/R$ is weak Hausdorff.*

**Definition 41.3.** A space is *compactly generated* if it is weak Hausdorff and a $k$-space. The category of such spaces is written **CG**.

We have a pair of adjoint pairs:

$$\mathbf{Top} \underset{k}{\overset{i}{\leftrightarrows}} k\mathbf{Top} \underset{j}{\overset{h}{\rightleftarrows}} \mathbf{CG}$$

so

$$ikX \to X \quad , \qquad hjZ \xrightarrow{\cong} Z\,,$$

$$Y \xrightarrow{\cong} kiY \quad , \qquad Y \to jhY\,.$$

The functor $h$ is defined by dividing $Y$ by the intersection of all closed equivalence relations on it. $k$-ification is a right adjoint, but weak Hausdorfification, being formation of a quotient, is a left adjoint. The dangerous and annoying feature of $h$ is that if $Y$ is not weak Hausdorff then the map $Y \to jhY$ is not a bijection; the underlying point-set changes.

**Lemma 41.4.** *If $Y$ is a $k$-space and $Z$ is weak Hausdorff, then $Z^Y$ is weak Hausdorff.*

**Corollary 41.5.** *The category* **CG** *is Cartesian closed.*

We will essentially always be working with either $k$**Top** or **CG**; most of the time either one will do. I will just call the objects "spaces," and (I'm sorry) write **Top** for the corresponding category.

Here's an example of how useful the formation of mapping spaces can be. We already know that a *homotopy* between maps $f, g : X \to Y$ is a map $h : I \times X \to Y$ such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[d, "\mathrm{in}_0"'] \arrow[dr, "f"] & \\
I \times X \arrow[r, "h"] & Y \\
X \arrow[u, "\mathrm{in}_1"] \arrow[ur, "g"'] &
\end{tikzcd}$$

We write $f \sim g$ to indicate that $f$ and $g$ are homotopic. This is an equivalence relation on $\mathbf{Top}(X, Y)$, and we define

$$[X, Y] = \mathbf{Top}(X, Y)/\sim \ .$$

The maps $f$ and $g$ are points in the space $Y^X$, and the homotopy $h$ is the same thing as a path $\hat{f} : I \to Y^X$ from $f$ to $g$. So

$$[X, Y] = \pi_0(Y^X)\,.$$

To talk about the fundamental group, and higher homotopy groups, using this strategy, we have to get basepoints into the picture.

## Basepoints

A *pointed space* is a space $X$ together with a specified "point" in it, with default notation $*$. The point $*$ is called the *basepoint*. This leads some people refer to "based spaces," but to my ear this makes it sound as if we are doing chemistry, or worse, and I prefer "pointed."

This gives a category $\mathbf{Top}_*$ where the morphisms respect the basepoints. This category is complete and cocomplete. For example

$$(X, *) \times (Y, *) = (X \times Y, (*, *))$$

(where of course I mean to be taking the product in $k$**Top** or **CG**). The coproduct is not disjoint union; which basepoint would you pick? So you identify the two basepoints, to get the "wedge" (or in TeX, \vee)

$$X \vee Y = X \sqcup Y/*_X \sim *_Y\,.$$

The one-point space $*$ is the terminal object in $\mathbf{Top}_*$, as in $\mathbf{Top}$, but it is also *initial* in $\mathbf{Top}_*$: there is exactly one map of pointed spaces from it to any $(X, *)$. We have a *pointed category*: A category with initial and terminal objects such that the unique map from the first to the second is an isomorphism.

Pointed categories are (almost) never Cartesian closed: for, as we saw, in a Cartesian closed category $\mathcal{C}$, $\mathcal{C}(X,Y) = \mathcal{C}(*, Y^X)$, and in a pointed category this is a singleton.

But we still know what we would like to take as a "mapping object" in $\mathbf{Top}_*$: Define $Y_*^X$ to be the subspace of $Y^X$ consisting of the pointed maps. As a replacement for Cartesian closure, let's ask: For fixed $X \in \mathbf{Top}_*$, does the functor $Y \mapsto Y_*^X$ have a left adjoint? This would be an analogue in $\mathbf{Top}_*$ of the functor $A \otimes -$ in $\mathbf{Ab}$. Compute:

$$
\begin{tikzcd}
\mathbf{Top}(W, Y^X) \arrow[r, phantom, "="] & \{f : X \times W \to Y\} \\
\mathbf{Top}(W, Y_*^X) \arrow[u, "\subseteq"] \arrow[r, phantom, "="] & \{f(*, w) = * \;\; \forall\, w \in W\} \arrow[u, "\subseteq"] \\
\mathbf{Top}_*(W, Y_*^X) \arrow[u, "\subseteq"] \arrow[r, phantom, "="] & \left\{ \begin{array}{ll} f(*, w) = * & \forall\, w \in W \\ f(x, *) = * & \forall\, x \in X \end{array} \right\} \arrow[u, "\subseteq"]
\end{tikzcd}
$$

So the map $X \times W \to Y$ corresponding to $f : W \to Y_*^X$ sends the wedge $X \vee W \subseteq X \times W$ to the basepoint of $Y$, and hence factors through the *smash product*

$$X \wedge W = X \times W / X \vee W$$

obtained by pinching the "axes" in the product to a point. We have an adjoint pair

$$X \wedge - : \mathbf{Top}_* \rightleftarrows \mathbf{Top}_* : (-)_*^X .$$

You are invited to check the various properties enjoyed by the smash product, analogous to properties of the tensor product. So it's functorial in both variables; the two-point pointed space serves as a unit; and it is associative and commutative. Associativity is a blessing bestowed by assuming compact generation; notice that in forming it we are mixing limits (the product) with colimits (the quotient by the axes), and indeed the smash product turns out *not* to be associative in the full category of spaces.

Based spaces and all spaces are related by another adjoint pair,

$$(-)_+ : \mathbf{Top} \rightleftarrows \mathbf{Top}_* : u$$

where $u$ forgets the basepoint and $(-)_+$ adjoins a disjoint base point. The two-point pointed space is then $*_+$, but everyone writes it as $S^0$. Explain why this is a reasonable symbol for this pointed space.

# 42 Fiber bundles, fibrations, cofibrations

## Fiber bundles

**Definition 42.1.** A *fiber bundle* is a map $p : E \to B$, such that for every $b \in B$, there exists an open subset $U \subseteq B$ containing $b$ and a map $p^{-1}(U) \to p^{-1}(b)$ such that $p^{-1}(U) \to U \times p^{-1}(b)$ is a homeomorphism.

When $p : E \to B$ is a fiber bundle, $E$ is called the *total space*, $B$ the *base space*, and $p$ the *projection*. The point pre-image $p^{-1}(b) \subseteq$ for $b \in B$ is the the *fiber over b*.

Here is an equivalent way of stating Definition 42.1: there is an open cover $\mathcal{U}$ (a "trivializing cover") of $B$ such that for every $U \in \mathcal{U}$ there is a space $F$ and a homeomorphism $p^{-1}(U) \simeq U \times F$ that is compatible with the projections down to $U$.

A trivial example of a fiber bundle is a product projection. Such fiber bundles are in fact called *trivial fiber bundles*, and the definition says that a fiber bundle is a map that is locally (in the base) a trivial fiber bundle.

Fiber bundles are naturally occurring objects. For instance, a covering space $E \to B$ is precisely a fiber bundle with discrete fibers.

**Example 42.2** (The Hopf fibration)**.** The "Hopf fibration" provides a beautiful example of a fiber bundle. Let $S^3 \subset \mathbf{C}^2$ be the 3-sphere. There is a map $p : S^3 \to \mathbf{CP}^1 \cong S^2$ that is given by sending a vector $v$ to the complex line through $v$ and the origin. This is a non-nullhomotopic map, and is a fiber bundle whose fiber is $S^1$.

Here is another way of thinking of the Hopf fibration. Recall that $S^3 = SU(2)$; this contains as a subgroup the collection of matrices $\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$. This subgroup is simply $S^1$, which acts on $S^3$ by translation; the orbit projection is the Hopf fibration is $p$.

The Hopf fibration is a map between smooth manifolds. A theorem of Ehresmann's says that it is not too hard to check the fiber bundle condition for a map between smooth manifolds.

**Theorem 42.3** (Ehresmann)**.** *Suppose $E$ and $B$ are smooth manifolds, and let $p : E \to B$ be a smooth (i.e., $C^\infty$) map. Then $p$ is a fiber bundle if $p$ is a* proper *(preimages of compact sets are compact)* submersion *(that is, $dp : T_e E \to T_{p(e)} B$ is a surjection for all $e \in E$).*

Much of this course will consist of a study of fiber bundles through various essentially algebraic lenses. To bring them into play, we will always require a further condition on our bundles.

**Definition 42.4.** An open cover $\mathcal{U}$ of a space $X$ is *numerable* if there exists a subordinate partition of unity; i.e., there is a family of functions $f_U : X \to [0, 1] = I$, indexed by the elements of $\mathcal{U}$, such that $f^{-1}((0, 1]) = U$ and any $x \in X$ belongs to only finitely many $U \in \mathcal{U}$. The space $X$ is *paracompact* if any open cover admits a numerable refinement. A fiber bundle is *numerable* if it admits a numerable trivializing cover.

So any fiber bundle over a paracompact space is numerable. This isn't too restrictive for us:

**Proposition 42.5.** *CW-complexes are paracompact.*

### Fibrations and path liftings

During the 1940s and '50's, much effort was devoted to extracting homotopy-theoretic features of fiber bundles. It came to be understood that these consequences relied entirely on a "homotopy lifting property." One of the revolutions in topology around 1960 was the realization that it was adantageous to take that property as a *definition*. This extension of the notion of a fiber bundle included wonderful new examples, but still retained the homotopy theoretic consequences. Here is the definition.

**Definition 42.6.** A *(Hurewicz) fibration* is a map $p : E \to B$ that satisfies the *homotopy lifting property* (commonly abbreviated as HLP): Given any $f : W \to E$ and any homotopy $h : I \times W \to B$ with $h(0,w) = pf(w)$, there is a map $\overline{h}$ that lifts $h$ and extends $f$: that is, making the following diagram commute.

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "\mathrm{in}_0"'] & E \arrow[d, "p"] \\ I \times W \arrow[r, "h"'] \arrow[ur, dashed, "\overline{h}"] & B, \end{tikzcd} \tag{4.1}$$

At first sight, this seems like an alarming definition, since the HLP has to be checked for *all* spaces, *all* maps, and *all* homotopies! The HLP is not impossible to check, though.

**Exercise 42.7.** Check that the projection $\mathrm{pr}_1 : B \times F \to B$ is a fibration.

**Exercise 42.8.** Check that the class of fibrations is closed under the following operations.

- Base change: If $p : E \to B$ is a fibration and $X \to B$ is any map, then the induced map $E \times_B X \to X$ is again a fibration.
- Products: If $p_i : E_i \to B_i$ is a family of fibrations then the product map $\prod p_i$ is again a fibration.
- Exponentiation: If $p : E \to B$ is a fibration and $A$ is any space, then $E^A \to B^A$ is again fibration.
- Composition: If $p : E \to B$ and $q : B \to X$ are both fibrations, then the composite $qp : E \to X$ is again a fibration.

**Exercise 42.9.** Let $p : E_0 \to B_0$ be a fibration, and let $f : B \to B_0$ be a homotopy equivalence. Prove that the induced map $B \times_{B_0} E_0 \to E_0$ is a homotopy equivalence. (Warning: this exercise has a lot of technical details!)

There is a simple geometric interpretation of what it means for a map to be a fibration, in terms of "path liftings". We'll use Cartesian closure! The adjoint of the solid arrow part of (4.1) is

$$\begin{tikzcd} E \arrow[r, "p"] & B \\ W \arrow[u, "f"] \arrow[r, "\widehat{h}"'] & B^I \arrow[u, "\mathrm{ev}_0"'] \end{tikzcd} \tag{4.2}$$

By the definition of the pullback, the data of this diagram is equivalent to a map $W \to B^I \times_B E$. Explicitly,

$$B^I \times_B E = \{(\omega, e) \in B^I \times E \text{ such that } \omega(0) = p(e)\}\,.$$

This space comes equipped with a map from $E^I$, given by sending a path $\omega : I \to E$ to

$$\widetilde{p}(\omega) = (p\omega, \omega(0)) \in B^I \times_B E\,.$$

In these terms, giving a lift $\overline{h}$ in (4.1) is equivalent to giving a lift

$$\begin{tikzcd} & E^I \arrow[d, "\widetilde{p}"] \\ W \arrow[r] \arrow[ur, dashed, "\widehat{h}"] & B^I \times_B E \end{tikzcd}$$

This again needs to be checked for every $W$ and every map to $B^I \times_B E$. But at least there is now a universal case to consider: $W = B^I \times_B E$ mapping by the identity map! So $p$ is a fibration if and only if the lift $\lambda$ exists in the following diagram.

$$\begin{tikzcd} & E^I \arrow[d, "\widetilde{p}"] \\ B^I \times_B E \arrow[r, "1"] \arrow[ur, dashed, "\lambda"] & B^I \times_B E \end{tikzcd}$$

The section $\lambda$ is called a *path lifting function*. To understand why, suppose $(\omega, e) \in B^I \times_B E$, so that $\omega$ is a path in $B$ with $\omega(0) = p(e)$. $\lambda(\omega, e)$ is then a path in $E$ lying over $\omega$ and starting at $e$. The path lifting function provides a continuous lift of paths in $B$. The existence (or not) of a *section* of $\widetilde{p}$ provides a single condition that needs to be checked if you want to see that $p$ is a fibration.

There is no mention of local triviality in this definition. You have shown in Exercise 42.7 that a product projection is a fibration. This implies that any (numerable) fiber bundle is a fibration, by the following theorem of Albrecht Dold.

**Theorem 42.10** (Dold)**.** *Let $p : E \to B$ be a map. Assume that there is a numerable cover of $B$, say $\mathcal{U}$, such that for every $U \in \mathcal{U}$ the restriction $p|_{p^{-1}(U)} : p^{-1}U \to U$ is a fibration. (In other words, $p$ is* locally *a fibration over the base). Then $p$ itself is a fibration.*

**Corollary 42.11.** *Any numerable fiber bundle is a fibration.*

# 43 Fibrations and cofibrations

## Comparing fibers over different points

Let $p : E \to B$ be a fibration. Above, we saw that this implies that paths in $B$ "lift" to paths in $E$. Let us consider a path $\omega : I \to B$ with $\omega(0) = a$ and $\omega(1) = b$. Denote by $F_a$ the fiber over $a$. If $p$ is a covering space, then unique path lifting provides us with a homeomorphism $F_a \to F_b$ depending only on the homotopy class of the path $\omega$. Our first goal today is to construct an analogous map for a general fibration.

Consider the solid arrow diagram:

$$\begin{tikzcd}
F_a \arrow[rr] \arrow[d, "\mathrm{in}_0"'] & & E \arrow[d, "p"] \\
I \times F_a \arrow[r, "\mathrm{pr}_1"] \arrow[urr, dashed, "h"] & I \arrow[r, "\omega"] & B.
\end{tikzcd}$$

This commutes since $\omega(0) = a$. Utilizing the homotopy lifting property, there is a dotted arrow that makes the entire diagram commute. If $x \in F_a$, the image $h(1, x)$ is in $F_b$. This supplies us with a map $f : F_a \to F_b$, given by $f(x) = h(1, x)$.

Since we are not working with a covering space, there will in general be many lifts $h$ and so many choices of $f$. We may at least hope that the homotopy class of $f$ is determined by the homotopy class of $\omega$.

So suppose we have two paths $\omega_0, \omega_1$, with $\omega_0(0) = \omega_1(0) = a$, and a homotopy $g : I \times I \to B$ between them (so that $g(0, t) = \omega_0(t)$, $g(1, t) = \omega_1(t)$, $g(s, 0) = a$, $g(s, 1) = b$). Choose lifts $h_0$ and $h_1$ as above. These data are captured by a diagram of the form

$$\begin{tikzcd}
((\partial I \times I) \cup (I \times \{0\})) \times F_a \arrow[rr] \arrow[d, "\mathrm{in}_0"'] & & E \arrow[d, "p"] \\
I \times I \times F_a \arrow[r, "\mathrm{pr}_1"'] \arrow[urr, dashed] & I \times I \arrow[r, "g"'] & B
\end{tikzcd}$$

The map along the top is given by $h_0$ and $h_1$ on $\partial I \times I \times F_a$ and by $pr_2 : I \times F_a \to F_a$ followed by the inclusion on the other summand.

The dotted lift would restrict on $I \times \{1\} \times F_a$ to a homotopy between $f_0$ and $f_1$.

Well, the subspace $(\partial I \times I) \cup (I \times \{0\})$ of $I \times I$ wraps around three edges of the square. It's easy enough to create a homeomorphism with the pair $(I \times I, \{0\} \times I)$, so the HLP (with $W = I \times F_a$) gives us the dotted lift.

We will capture this fact in a statement about the action of paths. Recall that the fundamental group of a pointed space $(X, *)$ acts on the fiber $p^{-1}(*)$ of any covering space $p : E \to X$. We have just shown that it also acts on the homotopy type of the fiber of any fibration over $X$. It's useful to express this in a way that does not specify a basepoint (and hence is meaningful even for spaces that are not path connected). Recall that a *groupoid* is a small category in which every morphism is an isomorphism.

**Definition 43.1.** Let $X$ be a topological space. The *fundamental groupoid* $\Pi_1(X)$ of $X$ is the groupoid whose objects are the points of $X$ and whose morphisms are the homotopy classes of paths in $X$. The composition of compatible paths $\sigma$ and $\omega$ is defined by:

$$(\sigma \cdot \omega)(t) = \begin{cases} \omega(2t) & 0 \le t \le 1/2 \\ \sigma(2t-1) & 1/2 \le t \le 1. \end{cases}$$

The results of the previous sections can be succinctly summarized as follows.

**Proposition 43.2.** *A fibration $p : E \to B$ determines a functor* $\Pi_1(B) \to \mathrm{Ho}\mathbf{Top}$.

Note: Last semester we defined the product of loops as juxtaposition but in the reverse order. That convention would have produced a contravariant functor $\Pi_1(X) \to \mathrm{Ho}\mathbf{Top}$.

## Cofibrations

Let $i : A \to X$ be a map of spaces. If $Y$ is another space, when is the induced map $Y^X \to Y^A$ a fibration? This is asking for the map $i$ to satisfy a condition "dual" to the fibration condition.

By the definition of a fibration, we want a lifting:

$$\begin{tikzcd} W \arrow[r] \arrow[d, "\mathrm{in}_0"'] & Y^X \arrow[d] \\ I \times W \arrow[r] \arrow[ur, dashed] & Y^A . \end{tikzcd}$$

Adjointing over, we get:

$$\begin{tikzcd} A \times W \arrow[r, "i\times 1"] \arrow[d, "1\times\mathrm{in}_0"'] & X \times W \arrow[d] \arrow[ddr] & \\ A \times W \times I \arrow[r] \arrow[drr] & X \times I \times W \arrow[dr, dashed] & \\ & & Y . \end{tikzcd}$$

Adjointing over again, this diagram transforms to:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr] & \\ A \times I \arrow[r] \arrow[drr] & X \times I \arrow[dr, dashed] & \\ & & Y^W . \end{tikzcd}$$

This discussion motivates the following definition of a "cofibration," dual to the notion of fibration.

**Definition 43.3.** A *cofibration* is a map $i : A \to X$ that satisfies *homotopy extension property* (sometimes abbreviated as "HEP"): for any space $Y$, there is a dotted map in the following diagram that makes it commute:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \arrow[ddr] & \\ A \times I \arrow[r] \arrow[drr] & X \times I \arrow[dr, dashed] & \\ & & Y . \end{tikzcd}$$

By the universal property of a pushout, this is equivalent to the existence of an extension in

$$\begin{tikzcd} X \cup_A (A \times I) \arrow[r] \arrow[dr] & X \times I \arrow[d, dashed] \\ & Y . \end{tikzcd}$$

Now there is a univeral example, namely $Y = X \cup_A (A \times I)$. So a map $i$ is a cofibration if and only if $X \cup_A (A \times I)$ is a retract of $X \times I$.

pperly draw out
s figure!

**Example 43.4.** $S^{n-1} \hookrightarrow D^n$ is a cofibration.

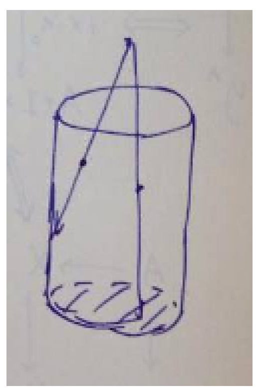

Figure 4.1: Drawing by John Ni.

In particular, setting $n = 1$ in this example, $\{0, 1\} \hookrightarrow I$ is a cofibration.

The class of cofibrations is closed under the following operations.

- Cobase change: if $A \to X$ is a cofibration and $A \to B$ is any map, the pushout $B \to X \cup_A B$ is again a cofibration.
- Coproducts.
- Composition.

Also, it satisfies the condition that we used to motivate the definition: If $A \to X$ is a cofibration and $Y$ is an space, then $Y^X \to Y^A$ is a fibration. In particular, the map

$$ev_{0,1} : Y^I \to Y \times Y$$

given by evaluation at the endpoints of the path is a fibration.

Finally,

**Lemma 43.5.** *Any cofibration is a closed inclusion.*

**Exercise 43.6.** An important (but easy!) fact about fibrations is that the canonical map $X \to *$ from any space $X$ is a fibration. (Afficianados of model categories get excited about this because this says that all objects in the associated model structure on topological spaces is fibrant.) After all, this is a product projection! But the "dual" statement is false. Give an example of a compactly generated space $X$ containing a point $*$ such that $\{*\} \hookrightarrow X$ is not a cofibration.

A basepoint $*$ is *nondegenerate* if $\{*\} \hookrightarrow X$ is a cofibration. If $*$ has a neighborhood in $X$ that contracts to $\{*\}$, the inclusion $* \hookrightarrow X$ is a cofibration. If $*$ is a nondegenerate basepoint of $A$, the evaluation map $\mathrm{ev} : X^A \to X$ is a fibration, The fiber of ev over a basepoint of $X$ is exactly the space of pointed maps $X^A_*$.

# 44 Homotopy fibers

Fix a map $p : E \to Y$. The pullback of $E$ along a map $f : X \to Y$ can vary wildly as $f$ is deformed; it is far from being a homotopy invariant. Just think of the case $X = *$, for example, when the pullback along $f : * \to Y$ is the point preimage $p^{-1}(f(*))$. One of the great features of fibrations is this:

**Proposition 44.1.** *Let $p : E \to Y$ be a fibration and $f_0, f_1 : X \to Y$ two maps. Write $E_0$ and $E_1$ for pullbacks of $E$ along $f_0$ and $f_1$. If $f_0$ and $f_1$ are homotopic then $E_0$ and $E_1$ are homotopy equivalent.*

*Proof.* Let $h : I \times X \to Y$ be a homotopy from $f_0$ to $f_1$. Its adjoint is a path $\widehat{h} : I \to Y^X$ from $f_0$ to $f_1$. The fibration $p : E \to Y$ induces a fibration $E^X \to Y^X$, and the path determines a homotopy equivalence from the fiber over $f_0$ to the fiber over $f_1$. These two fibers are $E_0$ and $E_1$. $\square$

### "Fibrant replacements"

This result makes fibrations extremely valuable in homotopy theory. Not every map is a fibration, but every map can be "replaced" by one, up to homotopy.

**Theorem 44.2.** *For any map $f : X \to Y$, there is a space $T(f)$, along with a fibration $p : T(f) \to Y$ and a homotopy equivalence $X \xrightarrow{\simeq} T(f)$, such that the following diagram* *commutes:*

$$\begin{tikzcd} X \arrow[r, "\simeq"] \arrow[dr, "f"'] & T(f) \arrow[d, "p"] \\ & Y. \end{tikzcd}$$

*Proof.* We already have one example of this in hand: The diagonal map $\Delta : X \to X \times X$ factors as

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[dr, "\Delta"'] & X^I \arrow[d, "\mathrm{ev}_{0,1}"] \\ & X \times X\,. \end{tikzcd}$$

where $p$ evaluates at the end points of the unit interval and $c$ sends $x$ to the constant path at $x$. We've seen that $\mathrm{ev}_{0,1}$ is a fibration. The map $c$ has as homotopy inverse any evaluation map, say $\mathrm{ev}_0$. The composite $\mathrm{ev}_0 \circ c$ is the identity, while $c \circ \mathrm{ev}_0$ is homotopic to the identity via the homotopy $h$ with

$$h(s, \omega)(t) = \omega(st)\,.$$

This "spaghette move" shortens the path from $\omega$ at $s = 1$ to the constant path $c_{\omega(0)}$ at $s = 0$.

Now we can construct the general case by considering the graph of $f$ instead of the diagonal (which is the graph of the identity map). So form the pullback

$$\begin{tikzcd} T(f) \arrow[r] \arrow[d] \arrow[dd, bend right=60, "p"'] & Y^I \arrow[d, "\mathrm{ev}_{0,1}"] \\ X \times Y \arrow[r, "f \times 1"] \arrow[d, "\mathrm{pr}_2"] & Y \times Y \\ Y & \end{tikzcd}$$

so that

$$T(f) = \{(x, \omega) \in X \times Y^I : f(x) = \omega(0)\}\,.$$

The map $T(f) \to X \times Y$ is a base-change of the fibration $\mathrm{ev}_{0,1}$ and so is a fibration; the projection map $\mathrm{pr}_2 : X \times Y \to Y$ is a fibration; so their composite $p$ is a fibration. To get $X$ involved, look at the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Delta"] \arrow[dr, dotted] & Y \arrow[dr, "c"] & \\ X \times X \arrow[dr, "1 \times f"'] & T(f) \arrow[r] \arrow[d] & Y^I \arrow[d, "\mathrm{ev}_{01}"] \\ & X \times Y \arrow[r, "f \times 1"] \arrow[d, "\mathrm{pr}_1"] & Y \times Y \\ & X & \end{tikzcd} \quad .$$

The outside diagram commutes, so we get a map from $X$ to the pullback $T(f)$. In symbols,

$$x \mapsto (x, c_{f(x)})\,.$$

We claim that the composite $T(f) \to X \times Y \to X$ is a homotopy inverse. The composite $X \to T(f) \to X \times Y \to X$ is the identity, so we just need a homotopy joining the other composite to the identity. Such a homotopy $I \times T(f) \to T(f)$ is provided by the same spaghetti move:

$$(s, x, \omega) \mapsto (x, t \mapsto \omega(st))\,.$$

$\square$

**Example 44.3** (Path-loop fibration)**.** If $X = *$, the space $T(f)$ consists of paths $\omega$ in $Y$ such that $\omega(0) = *$. In other words, $T(f) = Y^I_*$; this is called the *path space* of $Y$, and is denoted by $P(Y, *)$, or simply by $PY$. It is contractible, via the spaghetti move. The fiber over $*$ of the fibration $p : T(f) = PY \to Y$ consists of paths that begin and end at $*$, i.e., loops on $Y$ based at $*$. This is denoted $\Omega Y$, and is called the *loop space* of $Y$. The fibration $p : PY \to Y$ is called the *path-loop fibration*.

**Exercise 44.4** ("Cofibrant replacements")**.** In this exercise, you will prove the analogue of Theorem 44.2 for cofibrations. Let $f : X \to Y$ be any map. Show that $f$ factors (functorially) as a composite $X \to M \to Y$, where $X \to M$ is a cofibration and $M \to Y$ is a homotopy equivalence.

## Homotopy fibers

Let $f : E \to Y$ a map. As you move around $Y$, the point preimages generally vary, even up to homotopy type. They may become empty, for example. We saw above that assuming that $f$ is a fibration cures this problem, and we have just seen that any map can be replaced, up to homotopy, by a fibration. So it is sensible to make the following definition:

**Definition 44.5.** Let $(Y, *)$ be a pointed space. The *homotopy fiber* of a map $f : E \to Y$ is the fiber over $*$ of the fibration replacing $f$; that is, it is the pullback in

$$\begin{tikzcd}
F(f,*) \arrow[r] \arrow[d] & T(f) \arrow[r, "\simeq"] \arrow[d, "p"] & X \arrow[dl, "f"] \\
* \arrow[r] & Y &
\end{tikzcd}\,.$$

Homotopy fibers over different points in the same path component are homotopy equivalent.

As a set,

$$F(f, *) = \{(x, \omega) \in E \times Y^I : \omega(0) = f(x), \omega(1) = *\}. \tag{4.3}$$

So an element in the homotopy fiber of $f$ over $*$ is not just a point in $E$ lying over $* \in Y$; it is a point $x \in E$ together with a path in $Y$ joining $f(x)$ to $*$ in $Y$.

The inclusion of the fiber over $* \in Y$, $f^{-1}(*) \hookrightarrow E$, is univeral among maps $g : W \to E$ such that $fg = *$. What is the corresponding universal property of the homotopy fiber? Adjointing a map $W \to T(f)$ gives us a map $g : W \to E$ together with a homotopy from $fg$ to the constant map with value $*$: a *null homotopy* of the composite.

**Remark 44.6.** In forming the homotopy fiber of $f$, we replaced $f$ by a homotopy equivalent fibration and formed the pullback over $*$. We could alternatively have replaced the map $* \to Y$ by a fibration (the path-loop fibration!) and formed the pullback along $f$. The result of this operation is

$$\{(x, \omega) \in X \times Y^I : \omega(0) = *, \omega(1) = f(x)\}.$$

Our description of $F(f, *)$ in (4.3) is almost exactly the same — except that the directions of the paths are reversed. These two ways of thinking of the homotopy fiber are completely equivalent.

# 45 Barratt-Puppe sequence

## Fiber sequences

Recall, from the previous section, that we have a pullback diagram:

$$\begin{tikzcd}
 & F(f,*) \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "\lrcorner"] & PY \arrow[d, "p"] \arrow[dr, "\simeq"] & \\
f^{-1}(*) \arrow[ur] \arrow[r] & X \arrow[r, "f"'] & Y & * \arrow[l]
\end{tikzcd}$$

Consider a pointed map[1] $f : X \to Y$ (so that $f(*) = *$). Then, we will write $Ff$ for the homotopy fiber $F(f, *)$.

Since we're exploring the homotopy fiber $Ff$, we can ask the following, seemingly silly, question: what is the fiber of the canonical map $p : Ff \to X$ (over the basepoint of $X$)? This is precisely the space of loops in $Y$! Since $p$ is a fibration (recall that fibrations are closed under pullbacks), the homotopy fiber of $p$ is also the "strict" fiber! Our expanded diagram is now:

$$\begin{tikzcd}
 & \Omega Y = p^{-1}(*) \arrow[d] & & \\
 & F(f,*) \arrow[r] \arrow[d, "p"] & PY \arrow[d, "p"] \arrow[dr, "\simeq"] & \\
f^{-1}(*) \arrow[ur] \arrow[r] & X \arrow[r, "f"'] & Y & *. \arrow[l]
\end{tikzcd}$$

[1]Some people call such a map "based", but this makes it sound like we're doing chemistry, so we won't use it.

It's easy to see that the composite $Ff \xrightarrow{p} X \xrightarrow{f} Y$ sends $(x,\omega) \mapsto f(x)$; this is a pointed nonconstant map. (Note that the basepoint we're choosing for $Ff$ is the image of the basepoint in $f^{-1}(*)$ under the canonical map $f^{-1}(*) \hookrightarrow Ff$.)

While the composite $fp : Ff \to Y$ is not zero "on the nose", it is nullhomotopic, for instance via the homotopy $h : Ff \times I \to Y$, defined by

$$h(t,(x,\omega)) = \omega(t).$$

**Exercise 45.1.** Let $f : X \to Y$ and $g : W \to X$ be pointed maps. Establish a homeomorphism between the space of pointed maps $W \xrightarrow{p} Ff$ such that $fp = g$ and the space of pointed nullhomotopies of the composite $fg$.

This exercise proves that the homotopy fiber is the "kernel" in the homotopy category of pointed spaces and pointed maps between them.

Define $[W,X]_* = \pi_0(X_*^W)$; this consists of the pointed homotopy classes of maps $W \to X$. We may view this as a pointed set, whose basepoint is the constant map. Fixing $W$, this is a contravariant functor in $X$, so there are maps $[W,Ff]_* \to [W,X]_* \to [W,Y]_*$. This composite is not just nullhomotopic: it is "exact"! Since we are working with pointed sets, we need to describe what exactness means in this context: the preimage of the basepoint in $[W,Y]_*$ is exactly the image of $[W,Ff]_* \to [W,X]_*$. (This is exactly a reformulation of Exercise 45.1.) We say that $Ff \to X \xrightarrow{f} Y$ is a *fiber sequence*.

**Remark 45.2.** Let $f : X \to Y$ be a map of spaces, and suppose we have a homotopy commutative diagram:

$$\begin{tikzcd}
\Omega Y \arrow[r] \arrow[d, "\Omega g"'] & Ff \arrow[r] \arrow[d, dashed] & X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "g"] \\
\Omega Y' \arrow[r] & Ff' \arrow[r] & X' \arrow[r, "f'"'] & Y.
\end{tikzcd}$$

Then the dotted map exists, but it depends on the homotopy $f'h \simeq gf$.

## Iterating fiber sequences

Let $f : X \to Y$ be a pointed map, as before. As observed above, we have a composite map $Ff \xrightarrow{p} X \xrightarrow{f} Y$, and the strict fiber (homotopy equivalent to the homotopy fiber) of $p$ is $\Omega Y$. This begets a map $i(f) : \Omega Y \to Ff$; iterating the procedure of taking fibers gives:

$$\begin{tikzcd}
\cdots \arrow[r] & Fp_3 \arrow[r, "p_4"] & Fp_2 \arrow[r, "p_3"] & Fp_1 \arrow[r, "p_2"] & Ff \arrow[r, "p_1"] & X \arrow[r, "f"] & Y \\
 & \Omega Fp_0 \arrow[u, "\simeq"] \arrow[r, dashed] \arrow[ur, "i(p_2)"] & \Omega X \arrow[u, "\simeq"] \arrow[r, dashed] \arrow[ur, "i(p_1)"] & \Omega Y \arrow[u, "\simeq"] \arrow[ur, "i(f)"] & & &
\end{tikzcd}$$

All the $p_i$ in the above diagram are fibrations. Each of the dotted maps in the above diagram can be filled in up to homotopy. The most obvious guess for what these dotted maps are is simply $\Omega X \xrightarrow{\Omega f} \Omega Y$. But *that is the wrong map*!

The right map turns out to be $\Omega X \xrightarrow{\overline{\Omega f}} \Omega Y$:

**Lemma 45.3.** *The following diagram commutes to homotopy:*

$$\begin{tikzcd} & Fp \\ \Omega X \arrow[ur, "i(p)"] \arrow[r, "\overline{\Omega f}"'] & \Omega Y; \arrow[u] \end{tikzcd}$$

*here, $\overline{\Omega f}$ is the diagonal in the following diagram:*

$$\begin{tikzcd} \Omega X \arrow[r, "-"] \arrow[d, "\Omega f"'] \arrow[dr, "\Omega f"] & \Omega X \arrow[d, "\Omega f"] \\ \Omega Y \arrow[r, "-"'] & \Omega Y, \end{tikzcd}$$

*where the map $- : \Omega X \to \Omega X$ sends $\omega \mapsto \overline{\omega}$.*

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ssible to write this
in symbols.

*Proof.*

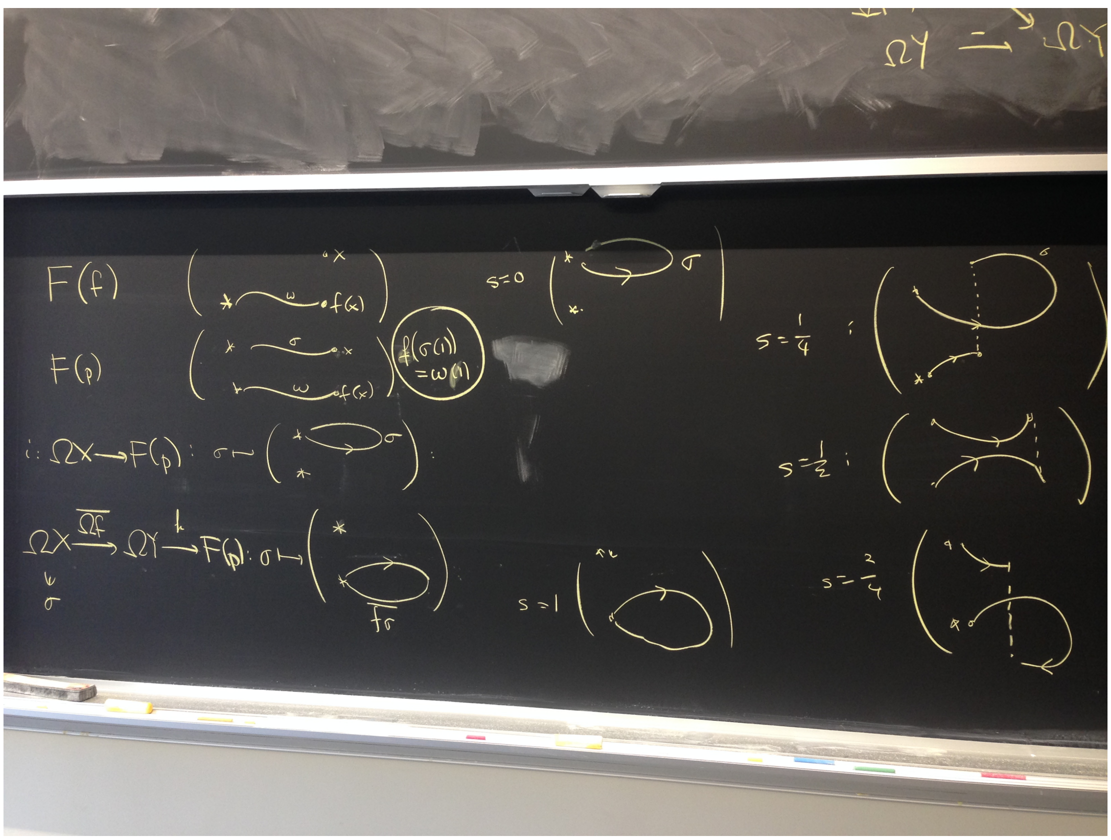


Figure 4.2: A proof of this lemma.

□

By the above lemma, we can extend our diagram to:

$$\begin{tikzcd}
\cdots \arrow[r] & Fp_4 \arrow[r] & Fp_3 \arrow[r] & Fp_2 \arrow[r] & Fp_1 \arrow[r, "p_2"] & Ff \arrow[r, "p_1"] & X \arrow[r, "f"] & Y \\
\cdots \arrow[r] & \Omega Fp_1 \arrow[u, "\simeq"'] \arrow[r, "\overline{\Omega p_2}"'] & \Omega Ff \arrow[u, "\simeq"'] \arrow[ur, "i(p_2)"] \arrow[r, "\overline{\Omega p}"'] & \Omega X \arrow[u, "\simeq"'] \arrow[ur, "i(p_1)"] \arrow[r, "\overline{\Omega f}"'] & \Omega Y \arrow[u, "\simeq"'] \arrow[ur, "i(f)"] & & & \\
\Omega^2 X \arrow[u, "\simeq"'] \arrow[r, "\Omega^2 f"'] & \Omega^2 Y \arrow[u, "\simeq"'] \arrow[ur, "\overline{\Omega i(p_0)}"'] & & & & & &
\end{tikzcd}$$

We have a special name for the sequence of spaces sneaking along the bottom of this diagram:

$$\cdots \to \Omega^2 X \to \Omega^2 Y \to \Omega Ff \to \Omega X \to \Omega Y \to Ff \to X \xrightarrow{f} Y;$$

this is called the *Barratt-Puppe sequence*. Applying $[W, -]_*$ to the Barratt-Puppe sequence of a map $f : X \to Y$ gives a long exact sequence.

The most important case of this long exact sequence comes from setting $W = S^0 = \{\pm 1\}$; in this case, we get terms like $\pi_0(\Omega^n X)$. We can identify $\pi_0(\Omega^n X)$ with $[S^n, X]_*$: to see this for $n = 2$, recall that $\Omega^2 X = (\Omega X)^{S^1}$; because $(S^1)^{\wedge n} = S^n$ (see below for a proof of this fact), we find that

$$(\Omega X)^{S^1} \simeq (X_*^{S^1})_*^{S^1} = X_*^{S^1 \wedge S^1} = X_*^{S^2}, \tag{4.4}$$

as desired.

The space $\Omega X$ is a group in the homotopy category; this implies that $\pi_0 \Omega X = \pi_1 X$ is a group! For $n > 1$, we know that

$$\pi_n(X) = [(D^n, S^{n-1}), (X, *)] = [(I^n, \partial I^n), (X, *)].$$

**Exercise 45.4.** Prove that $\pi_n(X)$ is an abelian group for $n > 2$.

Applying $\pi_0$ to the Barratt-Puppe sequence (see Equation 4.4) therefore gives a long exact sequence (of groups when the homotopy groups are in degrees greater than 0, and of pointed sets in degree 0):

$$\cdots \to \pi_2 X \to \pi_2 Y \to \pi_1 Ff \to \pi_1 X \to \pi_1 Y \to \pi_0 Ff \to \pi_0 X \to \pi_0 X.$$

# 46 Relative homotopy groups

## Spheres and homotopy groups

The functor $\Omega$ (sending a space to its based loop space) admits a left adjoint. To see this, recall that $\Omega X = X_*^{S^1}$, so that

$$\mathbf{Top}_*(W, \Omega X) = \mathbf{Top}_*(S^1 \wedge W, X).$$

**Definition 46.1.** The *reduced suspension* $\Sigma W$ is $S^1 \wedge W$.

If $A \subseteq X$, then

$$X/A \wedge Y/B = (X \times Y)/((A \times Y) \cup_{A\times B} (X \times B)).$$

Since $S^1 = I/\partial I$, this tells us that $\Sigma X = S^1 \wedge X$ can be identified with $I \times X/(\partial I \times X \cup I \times *)$: in other words, we collapse the top and bottom of a cylinder to a point, as well as the line along a basepoint.

The same argument says that $\Sigma^n X$ (defined inductively as $\Sigma(\Sigma^{n-1}X)$) is the left adjoint of the $n$-fold loop space functor $X \mapsto \Omega^n X$. In other words, $\Sigma^n X = (S^1)^{\wedge n} \wedge X$. We claim that $S^1 \wedge S^n \simeq S^{n+1}$. To see this, note that

$$S^1 \wedge S^n = I/\partial I \wedge I^n \wedge \partial I^n = (I \times I^n)/(\partial I \times I^n \cup I \times \partial I^n).$$

The denominator is exactly $\partial I^{n+1}$, so $S^1 \wedge S^n \simeq S^{n+k}$. It's now easy to see that $S^k \wedge S^n \simeq S^{k+n}$.

**Definition 46.2.** The *nth homotopy group* of $X$ is $\pi_n X = \pi_0(\Omega^n X)$.

This is, as we noted in the previous section, $[S^0, \Omega^n X]_* = [S^n, X]_* = [(I^n, \partial I^n), (X, *)]$.

## The homotopy category

Define the *homotopy category of spaces* Ho(**Top**) to be the category whose objects are spaces, and whose hom-sets are given by taking $\pi_0$ of the mapping space. To check that this is indeed a category, we need to check that if $f_0, f_1 : X \to Y$ and $g : Y \to Z$, then $gf_0 \simeq gf_1$ — but this is clear. Similarly, we'd need to check that $f_0 h \simeq f_1 h$ for any $h : W \to X$. We can also think about the homotopy category of pointed spaces (and pointed homotopies) Ho($\mathbf{Top}_*$); this is the category we have been spending most of our time in. Both Ho(**Top**) and Ho($\mathbf{Top}_*$) have products and coproducts, but very few other limits or colimits. From a category-theoretic standpoint, these are absolutely terrible.

Let $W$ be a pointed space. We would like the assignment $X \mapsto X_*^W$ to be a homotopy functor. It clearly defines a functor $\mathbf{Top}_* \to \mathbf{Top}_*$, so this desire is equivalent to providing a dotted arrow in the following diagram:

$$\begin{tikzcd}
\mathbf{Top}_* \arrow[r, "X \mapsto X_*^W"] \arrow[d] & \mathbf{Top}_* \arrow[d] \\
\mathrm{Ho}(\mathbf{Top}_*) \arrow[r, dashed] & \mathrm{Ho}(\mathbf{Top}_*).
\end{tikzcd}$$

Before we can prove this, we will check that a homotopy $f_0 \sim f_1 : X \to Y$ is the same as a map $I_+ \wedge X \to Y$. There is a nullhomotopy if the basepoint of $I$ is one of the endpoints, so a homotopy is the same as a map $I \times X/I \times * \to Y$. The source is just $I_+ \wedge X$, as desired.

A homotopy $f_0 \simeq f_1 : X \to Y$ begets a map $(I_+ \wedge X)^W \to Y_*^W$. For the assignment $X \mapsto X_*^W$ to be a homotopy functor, we need a natural transformation $I_+ \wedge X_*^W \to Y_*^W$, so this map is not quite what's necessary. Instead, we can attempt to construct a map $I_+ \wedge X_*^W \to (I_+ \wedge X)_*^W$.

We can construct a general map $A \wedge X_*^W \to (A \wedge X)_*^W$: there is a map $A \wedge X_*^W \to A_*^W \wedge X_*^W$, given by sending $a \mapsto c_a$; then the exponential law gives a homotopy $A_*^W \wedge X_*^W \to (A \wedge X)_*^W$. This, in turn, gives a map $I_+ \wedge X_*^W \to (I_+ \wedge X)_*^W \to Y_*^W$, thus making $X \mapsto X_*^W$ a homotopy functor.

Motivated by our discussion of homotopy fibers, we can study composites which "behave" like short exact sequences.

**Definition 46.3.** A *fiber sequence* in $\mathrm{Ho}(\mathbf{Top}_*)$ is a composite $X \to Y \to Z$ that is isomorphic, in $\mathrm{Ho}(\mathbf{Top}_*)$, to some composite $Ff \xrightarrow{p} E \xrightarrow{f} B$; in other words, there exist (possibly zig-zags of) maps that are homotopy equivalences, that make the following diagram commute:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
Ff \arrow[r, "p"'] & E \arrow[r, "f"'] & B.
\end{tikzcd}$$

Let us remark here that if $A' \xrightarrow{\sim} A$ is a homotopy equivalence, and $A \to B \to C$ is a fiber sequence, so is the composite $A' \xrightarrow{\sim} A \to B \to C$.

**Exercise 46.4.** Prove the following statements.

- $\Omega$ takes fiber sequences to fiber sequences.
- $\Omega Ff \simeq F\Omega f$. Check this!

We've seen examples of fiber sequences in our elaborate study of the Barratt-Puppe sequence.

**Example 46.5.** Recall our diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & Fp_4 \arrow[r] & Fp_3 \arrow[r] & Fp_2 \arrow[r] & Fp_1 \arrow[r, "p_2"] & Ff \arrow[r, "p_1"] & X \arrow[r, "f"] & Y \\
\cdots \arrow[r] & \Omega Fp_1 \arrow[r, "\overline{\Omega p_2}"] \arrow[u, "\simeq"'] & \Omega Ff \arrow[r, "\overline{\Omega p}"] \arrow[u, "\simeq"'] \arrow[ur, "i(p_2)"] & \Omega X \arrow[r, "\overline{\Omega f}"] \arrow[u, "\simeq"'] \arrow[ur, "i(p_1)"] & \Omega Y \arrow[u, "\simeq"'] \arrow[ur, "i(f)"] & & & \\
\Omega^2 X \arrow[r, "\Omega f"'] \arrow[u, "\simeq"'] & \Omega Y \arrow[u, "\simeq"'] \arrow[ur, "\overline{\Omega i(f)}"'] & & & & & &
\end{tikzcd}$$

The composite $Ff \to X \xrightarrow{f} Y$ is canonically a fiber sequence. The above diagram shows that $\Omega Y \to F \xrightarrow{p} X$ is another fiber sequence: it is isomorphic to $Fp \to F \to X$ in $\mathrm{Ho}(\mathbf{Top}_*)$. Similarly, the composite $\Omega X \xrightarrow{\overline{\Omega f}} \Omega Y \to F$ is another fiber sequence; this implies that $\Omega X \xrightarrow{\Omega f} \Omega Y \to F$ is also an example of a fiber sequence (because these two fiber sequences differ by an automorphism of $\Omega X$)

Applying $\Omega$ again, we get $\Omega F \xrightarrow{\Omega p} \Omega X \xrightarrow{\Omega f} \Omega Y$. Since this is a looping of a fiber sequence, and taking loops takes fiber sequences to fiber sequences (Exercise 46.4), this is another fiber sequence. Looping again gives another fiber sequence $\Omega^2 Y \xrightarrow{\Omega i} \Omega F \xrightarrow{\Omega p} \Omega X$. (For the category-theoretically–minded folks, this is an unstable version of a triangulated category.)

### The long exact sequence of a fiber sequence

As discussed at the end of §45, applying $\pi_0 = [S^0, -]_*$ to the Barratt-Puppe sequence associated to a map $f : X \to Y$ gives a long exact sequence:

$$\begin{tikzcd}
 & \cdots \arrow[r] & \pi_2 Y \arrow[dll] \\
\pi_1 F \arrow[r] & \pi_1 X \arrow[r] & \pi_1 Y \arrow[dll] \\
\pi_0 F \arrow[r] & \pi_0 X. &
\end{tikzcd}$$

of pointed sets. The space $\Omega^2 X$ is an *abelian* group object in Ho(**Top**) (in other words, the multiplication on $\Omega^2 X$ is commutative up to homotopy). This implies $\pi_1(X)$ is a group, and that $\pi_k(X)$ is abelian for $k \geq 2$; hence, in our diagram above, all maps (except on $\pi_0$) are group homomorphisms.

Consider the case when $X \to Y$ is the inclusion $i : A \hookrightarrow X$ of a subspace. In this case,

$$Fi = \{(a, \omega) \in A \times X_*^I | \omega(1) = a\};$$

this is just the collection of all paths that begin at $* \in A$ and end in $A$. This motivates the definition of *relative homotopy groups*:

**Definition 46.6.** Define:

$$\pi_n(X, A, *) = \pi_n(X, A) := \pi_{n-1} Fi = [(I^n, \partial I^n, (\partial I^n \times I) \cup (I^{n-1} \times 0)), (X, A, *)].$$

We have a sequence of inclusions

$$\partial I^n \times I \cup I^{n-1} \times 0 \subset \partial I^n \subset I^n.$$

One can check that

$$\pi_{n-1} Fi = [(I^n, \partial I^n, (\partial I^n \times I) \cup (I^{n-1} \times 0)), (X, A, *)].$$

This gives a long exact sequence on homotopy, analogous to the long exact sequence in relative homology:

$$
\begin{tikzcd}
 & & \cdots \arrow[r] & \pi_2(X,A) \arrow[dll] \\
 \pi_1 A \arrow[r] & \pi_1 X \arrow[r] & \pi_1(X,A) \arrow[dll] & \\
 \pi_0 A \arrow[r] & \pi_0 X & &
\end{tikzcd}
\tag{4.5}
$$

# 47 Action of $\pi_1$, simple spaces, and the Hurewicz theorem

In the previous section, we constructed a long exact sequence of homotopy groups:

$$
\begin{tikzcd}
 & & \cdots \arrow[r] & \pi_2(X,A) \arrow[dll] \\
 \pi_1 A \arrow[r] & \pi_1 X \arrow[r] & \pi_1(X,A) \arrow[dll] & \\
 \pi_0 A \arrow[r] & \pi_0 X, & &
\end{tikzcd}
$$

which looks suspiciously similar to the long exact sequence in homology. The goal of this section is to describe a relationship between homotopy groups and homology groups.

Before we proceed, we will need the following lemma.

**Lemma 47.1** (Excision)**.** *If $A \subseteq X$ is a cofibration, there is an isomorphism*

$$H_*(X,A) \xrightarrow{\simeq} \widetilde{H}_*(X/A).$$

*Under this hypothesis,*

$$X/A \simeq \textit{Mapping cone of } i : A \to X;$$

*here, the mapping cone is the homotopy pushout in the following diagram:*

$$
\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "in_1"] & X \arrow[d] \\
CA \arrow[r] & X \cup_A CA,
\end{tikzcd}
$$

*where $CA$ is the cone on $A$, defined by*

$$CA = A \times I / A \times 0.$$

This lemma is dual to the statement that the homotopy fiber is homotopy equivalent to the strict fiber for fibrations.

Unfortunately, $\pi_*(X, A)$ is definitely not $\pi_*(X/A)$! For instance, there is a cofibration sequence

$$S^1 \to D^2 \to S^2.$$

We know that $\pi_* S^1$ is just $\mathbf{Z}$ in dimension 1, and is zero in other dimensions. On the other hand, we do not, and probably will never, know the homotopy groups of $S^2$. (A theorem of Edgar Brown in [**?**] says that these groups are computable, but this is super-exponential.)

### $\pi_1$-action

There is more structure in the long exact sequence in homotopy groups that we constructed last time, coming from an action of $\pi_1(X)$. There is an action of $\pi_1(X)$ on $\pi_n(X)$: if $x, y$ are points in $X$, and $\omega : I \to X$ is a path with $\omega(0) = x$ and $\omega(1) = y$, we have a map $f_\omega : \pi_n(X, x) \to \pi_n(X, y)$; this, in particular, implies that $\pi_1(X, *)$ acts on $\pi_n(X, *)$. When $n = 1$, the action $\pi_1(X)$ on itself is by conjugation.

In fact, one can also see that $\pi_1(A)$ acts on $\pi_n(X, A, *)$. It follows (by construction) that all maps in the long exact sequence of Equation (4.5) are equivariant for this action of $\pi_1(A)$. Moreover:

**Proposition 47.2** (Peiffer identity). *Let $\alpha, \beta \in \pi_2(X, A)$. Then $(\partial\alpha) \cdot \beta = \alpha\beta\alpha^{-1}$.*

**Definition 47.3.** A topological space $X$ is said to be *simply connected* if it is path connected, and $\pi_1(X, *) = 1$.

Let $p : E \to B$ be a covering space with $E$ and $B$ connected. Then, the fibers are discrete, hence do not have any higher homotopy. Using the long exact sequence in homotopy groups, we learn that $\pi_n(E) \to \pi_n(B)$ is an isomorphism for $n > 1$, and that $\pi_1(E)$ is a subgroup of $\pi_1(B)$ that classifies the covering space. In general, we know from Exercise 47.7 that $\Omega B$ acts on the homotopy fiber $Fp$. Since $Ff$ is discrete, this action factors through $\pi_0(\Omega B) \simeq \pi_1(B)$.

**Definition 47.4.** A space $X$ is said to be *$n$-connected* if $\pi_i(X) = 0$ for $i \leq n$.

Note that this is a well-defined condition, although we did not specify the basepoint: 0-connected implies path connected. Suppose $E \to B$ is a covering space, with the total space $E$ being $n$-connected. Then, Hopf showed that the group $\pi_1(B)$ determines the homology $H_i(B)$ in dimensions $i < n$.

Sometimes, there are interesting spaces which are not simply connected, for which the $\pi_1$-action is nontrivial.

**Example 47.5.** Consider the space $S^1 \vee S^2$. The universal cover is just $\mathbf{R}$, with a 2-sphere $S^2$ stuck on at every integer point. This space is simply connected, so the Hurewicz theorem says that $\pi_2(E) \simeq H_2(E)$. Since the real line is contractible, we can collapse it to a point: this gives a countable bouquet of 2-spheres. As a consequence, $\pi_2(E) \simeq H_2(E) = \bigoplus_{i=0}^{\infty} \mathbf{Z}$.

There is an action of $\pi_1(S^1 \vee S^2)$ on $E$: the action does is shift the 2-spheres on the integer points of $\mathbf{R}$ (on $E$) to the right by 1 (note that $\pi_1(S^1 \vee S^2) = \mathbf{Z}$). This tells us that $\pi_2(E) \simeq \mathbf{Z}[\pi_1(B)]$ as a $\mathbf{Z}[\pi_1(B)]$-module; this is the same action of $\pi_1(E)$ on $\pi_2(E)$. We should be horrified: $S^1 \vee S^2$ is a very simple 3-complex, but its homotopy is huge!

Simply-connectedness can sometimes be a restrictive condition; instead, to simplify the long exact sequence, we define:

**Definition 47.6.** A topological space $X$ is said to be *simple* if it is path-connected, and $\pi_1(X)$ acts trivially on $\pi_n(X)$ for $n \geq 1$.

Note, in particular, that $\pi_1(X)$ is abelian for a simple space.

Being simple is independent of the choice of basepoint. If $\omega : x \mapsto x'$ is a path in $X$, then $\omega_\sharp : \pi_n(X, x) \to \pi_n(X, x')$ is a group isomorphism. There is a (trivial) action of $\pi_1(X, x)$ on $\pi_n(X, x)$, and another (potentially nontrivial) action of $\pi_1(X, x')$ on $\pi_n(X, x')$. Both actions are compatible: hence, if $\pi_1(X, x)$ acts trivially, so does $\pi_1(X, x')$.

If $X$ is path-connected, there is a map $\pi_n(X, *) \to [S^n, X]$. It is clear that this map is surjective, so one might expect a factorization:

$$\begin{tikzcd} \pi_n(X,*) \arrow[rr, two heads] \arrow[dr] & & {[S^n, X]} \\ & \pi_1(X,*)\backslash\pi_n(X,*) \arrow[ur] & \end{tikzcd}$$

**Exercise 47.7.** Prove that $\pi_1(X, *)\backslash\pi_n(X, *) \simeq [S^n, X]$. To do this, work through the following exercises.

Let $f : X \to Y$ be a map of spaces, and let $* \in Y$ be a fixed basepoint of $Y$. Denote by $Ff$ the homotopy fiber of $f$; this admits a natural fibration $p : Ff \to X$, given by $(x, \sigma) \mapsto x$. If $\Omega Y$ denotes the (based) loop space of $Y$, we get an action $\Omega Y \times Ff \to Ff$, given by

$$(\omega, (x, \sigma)) \mapsto (x, \sigma \cdot \omega),$$

where $\sigma \cdot \omega$ is the concatenation of $\sigma$ and $\omega$, defined, as usual, by

$$\sigma \cdot \omega(t) = \begin{cases} \omega(2t) & 0 \leq t \leq 1/2 \\ \sigma(2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

(Note that when $X$ is the point, this defines a "multiplication" $\Omega Y \times \Omega Y \to \Omega Y$; this is associative and unital up to homotopy.) On connected components, we therefore get an action of $\pi_0 \Omega Y \simeq \pi_1 Y$ on $\pi_0 Ff$.

There is a canonical map

$$Ff \times \Omega Y \to Ff \times_X Ff,$$

given by $((x, \sigma), \omega) \mapsto ((x, \sigma), (x, \sigma) \cdot \omega)$. Prove that this map is a homotopy equivalence (so that the action of $\Omega Y$ on $Ff$ is "free"), and conclude that two elements in $\pi_0 Ff$ map to the same element of $\pi_0 X$ if and only if they are in the same orbit under the action of $\pi_1 Y$.

Let $X$ be path connected, with basepoint $* \in X$. Conclude that $\pi_1(X, *)\backslash\pi_n(X, *) \simeq [S^n, X]$ by proving that the surjection $\pi_n(X, *) \to [S^n, X]$ can be identified with the orbit projection for the action of $\pi_1(X, *)$ on $\pi_n(X, *)$.

If $X$ is simple, then the quotient $\pi_1(X, *)\backslash\pi_n(X, *)$ is simply $\pi_n(X, *)$, so Exercise 47.7 implies that $\pi_n(X, *) \cong [S^n, X]$ — independently of the basepoint; in other words, these groups are canonically the same, i.e., two paths $\omega, \omega' : x \to y$ give the same map $\omega_\sharp = \omega'_\sharp : \pi_n(X, x) \to \pi_n(X, y)$.

**Exercise 47.8.** A *$H$-space* is a pointed space $X$, along with a pointed map $\mu : X \times X \to X$, such that the maps $x \mapsto \mu(x, *)$ and $x \mapsto \mu(*, x)$ are both pointed homotopic to the identity. In this exercise, you will prove that path connected $H$-spaces are simple.

Denote by $\mathcal{C}$ the category of pairs $(G, H)$, where $G$ is a group that acts on the group $H$ (on the left); the morphisms in $\mathcal{C}$ are pairs of homomorphisms which are compatible with the group actions. This category has finite products. Explain what it means for an object of $\mathcal{C}$ to have a "unital multiplication", and prove that any object $(G, H)$ of $\mathcal{C}$ with a unital multiplication has $G$ and $H$ abelian, and that the $G$-action on $H$ is trivial. Conclude from this that path connected $H$-spaces are simple.

## Hurewicz theorem

**Definition 47.9.** Let $X$ be a path-connected space. The Hurewicz map $h : \pi_n(X, *) \to H_n(X)$is defined as follows: an element in $\pi_n(X, *)$ is represented by $\alpha : S^n \to X$; pick a generator $\sigma \in H_n(S^n)$, and send

$$\alpha \mapsto \alpha_*(\sigma) \in H_n(X).$$

We will see below that $h$ is in fact a homomorphism.

This is easy in dimension 0: a point is a 0-cycle! In fact, we have an isomorphism $H_0(X) \simeq \mathbf{Z}[\pi_0(X)]$. (This isomorphism is an example of the Hurewicz theorem.)

When $n = 1$, we have $h : \pi_1(X, *) \to H_1(X)$. Since $H_1(X)$ is abelian, this factors as $\pi_1(X, *) \to \pi_1(X, *)^{ab} \to H_1(X)$. The Hurewicz theorem says that the map $\pi_1(X, *)^{ab} \to H_1(X)$ is an isomorphism. We will not prove this here; see [3, Theorem 2A.1] for a proof.

The Hurewicz theorem generalizes these results to higher dimensions:

**Theorem 47.10** (Hurewicz)**.** *Suppose $X$ is a space for which $\pi_i(X) = 0$ for $i < n$, where $n \geq 2$. Then the Hurewicz map $h : \pi_n(X) \to H_n(X)$ is an isomorphism.*

Before the word "isomorphism" can make sense, we need to prove that $h$ is a homomorphism. Let $\alpha, \beta : S^n \to X$ be pointed maps. The product $\alpha\beta \in \pi_n(X)$ is the composite:

$$\alpha\beta : S^n \xrightarrow{\delta,\text{ pinching along the equator}} S^n \vee S^n \xrightarrow{\beta\vee\alpha} X \vee X \xrightarrow{\nabla} X,$$

where $\nabla : X \vee X \to X$ is the fold map, defined by:

$$\begin{tikzcd}
X \arrow[d] \arrow[dr, "1"] & \\
X \vee X \arrow[r, "\nabla"] & X \\
X \arrow[u] \arrow[ur, "1"'] &
\end{tikzcd}$$

To show that $h$ is a homomorphism, it suffices to prove that for two maps $\alpha, \beta : S^n \to X$, the induced maps on homology satisfy $(\alpha+\beta)_* = \alpha_* + \beta_*$ — then,

$$h(\alpha+\beta) = (\alpha+\beta)_*(\sigma) = \alpha_*(\sigma) + \beta_*(\sigma) = h(\alpha) + h(\beta).$$

To prove this, we will use the pinch map $\delta : S^n \to S^n \vee S^n$, and the quotient maps $q_1, q_2 : S^n \vee S^n \to S^n$; the induced map $H_n(S^n) \to H_n(S^n) \oplus H_n(S^n)$ is given by the diagonal map $a \mapsto (a, a)$. It follows from the equalities

$$(f \vee g)\iota_1 = f,\ (f \vee g)\iota_2 = g,$$

where $\iota_1, \iota_2 : S^n \hookrightarrow S^n \vee S^n$ are the inclusions of the two wedge summands, that the map $(f \vee g)_*((\iota_1)_* + (\iota_2)_*)$ sends $(x, 0)$ to $f_*(x)$, and $(0, x)$ to $g_*(x)$. In particular,

$$(x, x) \mapsto f_*(x) + g_*(x),$$

so the composite $H_n(S^n) \to H_n(X)$ sends $x \mapsto (x, x) \mapsto f_*(x) + g_*(x)$. This composite is just $(f+g)_*(x)$, since the composite $(f \vee g)\delta$ induces the map $(f+g)_*$ on homology.

It is possible to give an elementary proof of the Hurewicz theorem, but we won't do that here: instead, we will prove this as a consequence of the Serre spectral sequence.

**Example 47.11.** Since $\pi_i(S^n) = 0$ for $i < n$, the Hurewicz theorem tells us that $\pi_n(S^n) \simeq H_n(S^n) \simeq \mathbf{Z}$.

**Example 47.12.** Recall the Hopf fibration $S^1 \to S^3 \xrightarrow{\eta} S^2$. The long exact sequence on homotopy groups tells us that $\pi_i(S^3) \xrightarrow{\simeq} \pi_i(S^2)$ for $i > 2$, where the map is given by $\alpha \mapsto \eta\alpha$. As we saw above, $\pi_3(S^3) = \mathbf{Z}$, so $\pi_3(S^2) \simeq \mathbf{Z}$, generated by $\eta$.

One can show that $\pi_{4n-1}(S^{2n}) \otimes \mathbf{Q} \simeq \mathbf{Q}$. A theorem of Serre's says that, other than $\pi_n(S^n)$, these are the only non-torsion homotopy groups of spheres.

# 48 Examples of CW-complexes

## Bringing you up-to-speed on CW-complexes

**Definition 48.1.** A *relative CW-complex* is a pair $(X, A)$, together with a filtration

$$A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots \subseteq X,$$

such that for all $n$, the space $X_n$ sits in a pushout square:

$$\begin{tikzcd}
\coprod_{\alpha\in\Sigma_n} S^{n-1} \arrow[r] \arrow[d, "\text{attaching maps}"'] & \coprod_{\alpha\in\Sigma_n} D^n \arrow[d, "\text{characteristic maps}"] \\
X_{n-1} \arrow[r] & X_n,
\end{tikzcd}$$

and $X = \varinjlim X_n$.

If $A = \emptyset$, this is just the definition of a CW-complex. In this case, $X$ is also compactly generated. (This is one of the reasons for defining compactly generated spaces.) Often, $X$ will be a CW-complex, and $A$ will be a subcomplex. If $A$ is Hausdorff, then so is $X$.

If $X$ and $Y$ are both CW-complexes, define

$$(X \times^k Y)_n = \bigcup_{i+j=n} X_i \times Y_j;$$

this gives a CW-structure on the product $X \times^k Y$. Any closed smooth manifold admits a CW-structure.

**Example 48.2** (Complex projective space)**.** The complex projective $n$-space $\mathbf{CP}^n$ is a CW-complex, with skeleta $\mathbf{CP}^0 \subseteq \mathbf{CP}^1 \subseteq \cdots \subseteq \mathbf{CP}^n$. Indeed, any complex line through the origin meets the hemisphere defined by $\begin{pmatrix} z_0 \\ \vdots \\ z_n \end{pmatrix}$ with $||z|| = 1$, $\Im(z_n) = 0$, and $\Re(z_n) \geq 0$. Such a line meets this hemisphere (which is just $D^{2n}$) at one point — unless it's on the equator; this gives the desired pushout diagram:

$$\begin{tikzcd}
S^{2n-1} \arrow[r] \arrow[d] & D^{2n} \arrow[d] \\
\mathbf{CP}^{n-1} \arrow[r] & \mathbf{CP}^n.
\end{tikzcd}$$

**Example 48.3** (Grassmannians)**.** Let $V = \mathbf{R}^n$ or $\mathbf{C}^n$ or $\mathbf{H}^n$, for some fixed $n$. Define the Grassmannian $\mathrm{Gr}_k(\mathbf{R}^n)$ to be the collection of $k$-dimensional subspaces of $V$. This is equivalent to specifying a $k \times n$ rank $k$ matrix.

For instance, $\mathrm{Gr}_2(\mathbf{R}^4)$ is, as a set, the disjoint union of:

$$\begin{pmatrix} & 1 & & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} & & 1 & * \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & * & * & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} & 1 & * & \\ & 1 & * & \end{pmatrix}, \begin{pmatrix} 1 & * & * & \\ & & 1 & * \end{pmatrix}, \begin{pmatrix} 1 & * & * & \\ 1 & * & * & \end{pmatrix}.$$

Motivated by this, define:

**Definition 48.4.** The $j$-skeleton of $\mathrm{Gr}(V)$ is

$$\mathrm{sk}_j \mathrm{Gr}_k(V) = \{A : \text{row echelon representation with at most } j \text{ free entries}\}.$$

For a proof that this is indeed a CW-structure, see [**?**, §6].

The top-dimensional cell tells us that

$$\dim \mathrm{Gr}_k(\mathbf{R}^n) = k(n-k).$$

The complex Grassmannian has cells in only even dimensions. We know the homology of Grassmannians: Poincaré duality is visible if we count the number of cells. (Consider, for instance, in $\mathrm{Gr}_2(\mathbf{R}^4)$).

# 49 Relative Hurewicz and J. H. C. Whitehead

Here is an "alternative definition" of connectedness:

**Definition 49.1.** Let $n \geq 0$. The space $X$ is said to be $(n-1)$-*connected* if, for all $0 \leq k \leq n$, any map $f : S^{k-1} \to X$ extends:

$$\begin{tikzcd} S^{k-1} \arrow[r] \arrow[d] & X \\ D^k \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

When $n = 0$, we know that $S^{-1} = \emptyset$, and $D^0 = *$. Thus being $(-1)$-connected is equivalent to being nonempty. When $n = 1$, this is equivalent to path connectedness. You can check that this is exactly the same as what we said before, using homotopy groups.

As is usual in homotopy theory, there is a relative version of this definition.

**Definition 49.2.** Let $n \geq 0$. Say that a pair $(X, A)$ is $n$-*connected* if, for all $0 \leq k \leq n$, any map $f : (D^k, S^{k-1}) \to (X, A)$ extends:

$$\begin{tikzcd} (D^k, S^{k-1}) \arrow[r, "f"] \arrow[d, dashed] & (X, A) \\ (A, A) \arrow[ur] & \end{tikzcd}$$

up to homotopy. In other words, there is a homotopy between $f$ and a map with image in $A$, such that $f|_{S^{k-1}}$ remains unchanged.

0-connectedness implies that $A$ meets every path component of $X$. Equivalently:

**Definition 49.3.** $(X, A)$ is $n$-connected if:

- when $n = 0$, the map $\pi_0(A) \to \pi_0(X)$ surjects.
- when $n > 0$, the canonical map $\pi_0(A) \xrightarrow{\simeq} \pi_0(X)$ is an isomorphism, and for all $a \in A$, the group $\pi_k(X, A, a)$ vanishes for $1 \leq k \leq n$. (Equivalently, $\pi_0(A) \xrightarrow{\simeq} \pi_0(X)$ and $\pi_k(A, a) \to \pi_k(X, A)$ is an isomorphism for $1 \leq k < n$ and is onto for $k = n$.)

## The relative Hurewicz theorem

Assume that $\pi_0(A) = * = \pi_0(X)$, and pick $a \in A$. Then, we have a comparison of long exact sequences, arising from the classical (i.e., non-relative) Hurewicz map:

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_1(A) \arrow[r] \arrow[d, "h"] & \pi_1(X) \arrow[r] \arrow[d, "h"] & \pi_1(X,A) \arrow[r] \arrow[d, "h"] & \pi_0(A) \arrow[r] \arrow[d, "h"] & \pi_0(X) \arrow[d, "h"] & \\
\cdots \arrow[r] & H_1(A) \arrow[r] & H_1(X) \arrow[r] & H_1(X,A) \arrow[r] & H_0(A) \arrow[r] & H_0(X) \arrow[r] & H_0(X,A)
\end{tikzcd}$$

To define the relative Hurewicz map, let $\alpha \in \pi_n(X, A)$, so that $\alpha : (D^n, S^{n-1}) \to (X, A)$; pick a generator of $H_n(D^n, S^{n-1})$, and send it to an element of $H_n(X, A)$ via the induced map $\alpha_* : H_n(D^n, S^{n-1}) \to H_n(X, A)$.

Because $H_n(X, A)$ is abelian, the group $\pi_1(A)$ acts trivially on $H_n(X, A)$; in other words, $h(\omega(\alpha)) = h(\alpha)$. Consequently, the relative Hurewicz map factors through the group $\pi_n^\dagger(X, A)$, defined to be the quotient of $\pi_n(X, A)$ by the normal subgroup generated by $(\omega\alpha)\alpha^{-1}$, where $\omega \in \pi_1(A)$ and $\alpha \in \pi_n(X, A)$. This begets a map $\pi_n^\dagger(X, A) \to H_n(X, A)$.

**Theorem 49.4** (Relative Hurewicz)**.** *Let $n \geq 1$, and assume $(X, A)$ is $n$-connected. Then $H_k(X, A) = 0$ for $0 \leq k \leq n$, and the map $\pi_{n+1}^\dagger(X, A) \to H_{n+1}(X, A)$ constructed above is an isomorphism.*

We will prove this later using the Serre spectral sequence.

## The Whitehead theorems

J. H. C. Whitehead was a rather interesting character. He raised pigs.

Whitehead was interested in determining when a continuous map $f : X \to Y$ that is an isomorphism in homology or homotopy is a homotopy equivalence.

**Definition 49.5.** Let $f : X \to Y$ and $n \geq 0$. Say that $f$ is a *$n$-equivalence*[2] if, for every $* \in Y$, the homotopy fiber $F(f, *)$ is $(n-1)$-connected.

For instance, $f$ being a 0-equivalence simply means that $\pi_0(X)$ surjects onto $\pi_0(Y)$ via $f$. For $n > 0$, this says that $f : \pi_0(X) \to \pi_0(Y)$ is a bijection, and that for every $* \in X$:

$$\pi_k(X, *) \to \pi_k(Y, f(*)) \text{ is } \begin{cases} \text{an isomorphism} & 1 \leq k < n \\ \text{onto} & k = n. \end{cases}$$

[2]Some sources sometimes use "$n$-connected".

Using the "mapping cylinder" construction (see Exercise 44.4), we can always assume $f : X \to Y$ is a cofibration; in particular, that $X \hookrightarrow Y$ is a closed inclusion. Then, $f : X \to Y$ is an $n$-equivalence if and only if $(Y, X)$ is $n$-connected.

**Theorem 49.6** (Whitehead)**.** *Suppose $n \geq 0$, and $f : X \to Y$ is $n$-connected. Then:*

$$H_k(X) \xrightarrow{f} H_k(Y) \text{ is } \begin{cases} \textit{an isomorphism} & 1 \leq k < n \\ \textit{onto} & k = n. \end{cases}$$

*Proof.* When $n = 0$, because $\pi_0(X) \to \pi_0(Y)$ is surjective, we learn that $H_0(X) \simeq \mathbf{Z}[\pi_0(X)] \to \mathbf{Z}[\pi_0(Y)] \simeq H_0(Y)$ is surjective. To conclude, use the relative Hurewicz theorem. (Note that the relative Hurewicz dealt with $\pi_n^\dagger(X, A)$, but the map $\pi_n(X, A) \to \pi_n^\dagger(X, A)$ is surjective.) □

The case $n = \infty$ is special.

**Definition 49.7.** $f$ is a *weak equivalence* (or an $\infty$-equivalence, to make it sound more impressive) if it's an $n$-equivalence for all $n$, i.e., it's a $\pi_*$-isomorphism.

Putting everything together, we obtain:

**Corollary 49.8.** *A weak equivalence induces an isomorphism in integral homology.*

How about the converse?

If $H_0(X) \to H_0(Y)$ surjects, then the map $\pi_0(X) \to \pi_0(Y)$ also surjects. Now, assume $X$ and $Y$ path connected, and that $H_1(X)$ surjects onto $H_1(Y)$. We would like to conclude that $\pi_1(X) \to \pi_1(Y)$ surjects. Unfortunately, this is hard, because $H_1(X)$ is the abelianization of $\pi_1(X)$. To forge onward, we will simply give up, and assume that $\pi_1(X) \to \pi_1(Y)$ is surjective.

Suppose $H_2(X) \to H_2(Y)$ surjects, and that $f_* : H_1(X) \xrightarrow{\simeq} H_1(Y)$. We know that $H_2(Y, X) = 0$. On the level of the Hurewicz maps, we are still stuck, because we only obtain information about $\pi_2^\dagger$. Let us assume that $\pi_1(X)$ is trivial[3]. Under this assumption, we find that $\pi_1(Y) = 0$. This implies $\pi_2(Y, X)$ is trivial. Arguing similarly, we can go up the ladder.

**Theorem 49.9** (Whitehead)**.** *Let $n \geq 2$, and assume that $\pi_1(X) = 0 = \pi_1(Y)$. Suppose $f : X \to Y$ such that:*

$$H_k(X) \to H_k(Y) \text{ is } \begin{cases} \textit{an isomorphism} & 1 \leq k < n \\ \textit{onto} & k = n; \end{cases}$$

*then $f$ is an $n$-equivalence.*

Setting $n = \infty$, we obtain:

[3]This is a pretty radical assumption; for the following argument to work, it would technically be enough to ask that $\pi_1(X)$ acts trivially on $\pi_2(Y, X)$: but this is basically impossible to check.

**Corollary 49.10.** *Let $X$ and $Y$ be simply-connected. If $f$ induces an isomorphism in homology, then $f$ is a weak equivalence.*

This is incredibly useful, since homology is actually computable! To wrap up the story, we will state the following result, which we will prove in a later section.

**Theorem 49.11.** *Let $Y$ be a CW-complex. Then a weak equivalence $f : X \to Y$ is in fact a homotopy equivalence.*

# 50 Cellular approximation, cellular homology, obstruction theory

In previous sections, we saw that homotopy groups play well with (maps between) CW-complexes. Here, we will study maps between CW-complexes themselves, and prove that they are, in some sense, "cellular" themselves.

## Cellular approximation

**Definition 50.1.** Let $X$ and $Y$ be CW-complexes, and let $A \subseteq X$ be a subcomplex. Suppose $f : X \to Y$ is a continuous map. We say that $f|_A$ is skeletal[4] if $f(\Sigma_n) \subseteq Y_n$.

Note that a skeletal map might not take cells in $A$ to cells in $Y$, but it takes $n$-skeleta to $n$-skeleta.

**Theorem 50.2** (Cellular approximation)**.** *In the setup of Definition 50.1, the map $f$ is homotopic to some other continuous map $f' : X \to Y$, relative to $A$, such that $f'$ is skeletal on all of $X$.*

To prove this, we need the following lemma.

**Lemma 50.3** (Key lemma)**.** *Any map $(D^n, S^{n-1}) \to (Y, Y_{n-1})$ factors as:*

$$\begin{tikzcd} (D^n, S^{n-1}) \arrow[r] \arrow[dr, dashed] & (Y, Y_{n-1}) \\ & (Y_n, Y_{n-1}) \arrow[u] \end{tikzcd}$$

*"Proof."* Since $D^n$ is compact, we know that $f(D^n)$ must lie in some finite subcomplex $K$ of $Y$. The map $D^n \to K$ might hit some top-dimensional cell $e^m \subseteq K$, which does not have anything attached to it; hence, we can homotope this map to miss a point, so that it contracts onto a lower-dimensional cell. Iterating this process gives the desired result. $\square$

Using this lemma, we can conclude the cellular approximation theorem.

[4] Some would say cellular.

*"Proof" of Theorem 50.2.* We will construct the homotopy $f \simeq f'$ one cell at a time. Note that we can replace the space $A$ by the subspace to which we have extended the homotopy.

Consider a single cell attachment $A \to A \cup D^m$; then, we have

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\text{skeletal}"'] & A \cup D^m \arrow[dl, "\text{may not be skeletal}"] \\ Y & \end{tikzcd}$$

Using the "compression lemma" from above, the rightmost map factors (up to homotopy) as the composite $A \cup D^m \to Y_m \to Y$. Unfortunately, we have not extended this map to the whole of $X$, although we could do this if we knew that the inclusion of a subcomplex is a cofibration. But this is true: there is a cofibration $S^{n-1} \to D^n$, and so any pushout of these maps is a cofibration! This allows us to extend; we now win by iterating this procedure. □

As a corollary, we find:

**Exercise 50.4.** The pair $(X, X_n)$ is $n$-connected.

## Cellular homology

Let $(X, A)$ be a relative CW-complex with $A \subseteq X_{n-1} \subseteq X_n \subseteq \cdots \subseteq X$. In the previous part that $H_*(X_n, X_{n-1}) \simeq \widetilde{H}_*(X_n/X_{n-1})$. More generally, if $B \to Y$ is a cofibration, there is an isomorphism (see [1, p. 433]):

provide a link!

$$H_*(Y, B) \simeq \widetilde{H}_*(Y/B).$$

Since $X_n/X_{n-1} = \bigvee_{\alpha \in \Sigma_n} S^n_\alpha$, we find that

$$H_*(X_n, X_{n-1}) \simeq \mathbf{Z}[\Sigma_n] = C_n.$$

The composite $S^{n-1} \to X_{n-1} \to X_{n-1}/X_{n-2}$ is called a relative attaching map.

There is a boundary map $d : C_n \to C_{n-1}$, defined by

$$d : C_n = H_n(X_n, X_{n-1}) \xrightarrow{\partial} H_{n-1}(X_{n-1}) \to H_{n-1}(X_{n-1}, X_{n-2}) = C_{n-1}.$$

**Exercise 50.5.** Check that $d^2 = 0$.

Using the resulting chain complex, denoted $C_*(X, A)$, one can prove that there is an isomorphism

$$H_n(X, A) \simeq H_n(C_*(X, A)).$$

(In the previous part, we proved this for CW-pairs, but not for relative CW-complexes.) The incredibly useful cellular approximation theorem therefore tells us that the effect of maps on homology can be computed.

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Of course, the same story runs for cohomology: one gets a chain complex which, in dimension $n$, is given by

$$C^n(X, A; \pi) = \mathrm{Hom}(C_n(X, A), \pi) = \mathrm{Map}(\Sigma_n, \pi),$$

where $\pi$ is any abelian group.

## Obstruction theory

Using the tools developed above, we can attempt to answer some concrete, and useful, questions.

**Question 50.6.** Let $f : A \to Y$ be a map from a space $A$ to $Y$. Suppose $(X, A)$ is a relative CW-complex. When can we find an extension in the diagram below?

$$\begin{tikzcd}
X \arrow[dr, dashed] & \\
A \arrow[u, hook] \arrow[r, "f"'] & Y
\end{tikzcd}$$

The lower level obstructions can be worked out easily:

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d] & X_0 \arrow[r, hook] \arrow[dl, dashed] & X_1 \arrow[dll, dashed] \\
\emptyset \neq Y & &
\end{tikzcd}$$

Thus, for instance, if two points in $X_0$ are connected in $X_1$, we only have to check that they are also connected in $Y$.

For $n \geq 2$, we can form the diagram:

$$\begin{tikzcd}
\coprod_{\alpha\in\Sigma_n} S^{n-1}_\alpha \arrow[r, "f"] \arrow[d, hook] & X_{n-1} \arrow[r, "g"] \arrow[d] & Y \\
\coprod_{\Sigma_n} D^n_\alpha \arrow[r] & X_n \arrow[ur, dashed] &
\end{tikzcd}$$

The desired extension exists if the composite $S^{n-1}_\alpha \xrightarrow{f_\alpha} X_{n-1} \to Y$ is nullhomotopic.

Clearly, $g \circ f_\alpha \in [S^{n-1}, Y]$. To simplify the discussion, let us assume that $Y$ is simple; then, Exercise 47.7 says that $[S^{n-1}, Y] = \pi_{n-1}(Y)$. This procedure begets a map $\Sigma_n \xrightarrow{\theta} \pi_{n-1}(Y)$, which is a $n$-cochain, i.e., an element of $C^n(X, A; \pi_{n-1}(Y))$. It is clear that $\theta = 0$ if and only if the map $g$ extends to $X_n \to Y$.

**Proposition 50.7.** *$\theta$ is a cocycle in $C^n(X, A; \pi_{n-1}(Y))$, called the "obstruction cocycle".*

*Proof.* $\theta$ gives a map $H_n(X_n, X_{n-1}) \to \pi_{n-1}(Y)$. We would like to show that the composite

$$H_{n+1}(X_{n+1}, X_n) \xrightarrow{\partial} H_n(X_n) \to H_n(X_n, X_{n-1}) \xrightarrow{\theta} \pi_{n-1}(Y)$$

is trivial. We have the long exact sequence in homotopy of a pair (see Equation (4.5)):

$$\begin{tikzcd}
\pi_{n+1}(X_{n+1},X_n) \arrow[r, two heads] \arrow[d] & H_{n+1}(X_{n+1},X_n) \arrow[d, "\partial"] \\
\pi_n(X_n) \arrow[r] \arrow[d] & H_n(X_n) \arrow[d] \\
\pi_n(X_n,X_{n-1}) \arrow[r, two heads] \arrow[d, "\partial"'] & H_n(X_n,X_{n-1}) \arrow[d, "\theta"] \\
\pi_{n-1}(X_{n-1}) \arrow[r, "g_*"'] & \pi_{n-1}(Y)
\end{tikzcd}$$

This diagram commutes, so $\theta$ is indeed a cocycle. □

Our discussion above allows us to conclude:

**Theorem 50.8.** *Let $(X, A)$ be a relative CW-complex and $Y$ a simple space. Let $g : X_{n-1} \to Y$ be a map from the $(n-1)$-skeleton of $X$. Then $g|_{X_{n-2}}$ extends to $X_n$ if and only if $[\theta(g)] \in H^n(X, A; \pi_{n-1}(Y))$ is zero.*

**Corollary 50.9.** *If $H^n(X, A; \pi_{n-1}(Y)) = 0$ for all $n > 2$, then any map $A \to Y$ extends to a map $X \to Y$ (up to homotopy[5]); in other words, there is a dotted lift in the following diagram:*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & Y \\
X \arrow[ur, dashed] &
\end{tikzcd}$$

For instance, every map $A \to Y$ factors through the cone if $H^n(CA, A; \pi_{n-1}(Y)) \simeq \widetilde{H}^{n-1}(A; \pi_{n-1}(Y)) = 0$.

# 51 Conclusions from obstruction theory

The main result of obstruction theory, as discussed in the previous section, is the following.

**Theorem 51.1** (Obstruction theory)**.** *Let $(X, A)$ be a relative CW-complex, and $Y$ a simple space. The map $[X, Y] \to [A, Y]$ is:*

1. *is onto if $H^n(X, A; \pi_{n-1}(Y)) = 0$ for all $n \geq 2$.*

2. *is one-to-one if $H^n(X, A; \pi_n(Y)) = 0$ for all $n \geq 1$.*

[5] In fact, this condition is unnecessary, since the inclusion of a subcomplex is a cofibration.

**Remark 51.2.** The first statement implies the second. Indeed, suppose we have two maps $g_0, g_1 : X \to Y$ and a homotopy $h : g_0|_A \simeq g_0|_A$. Assume the first statement. Consider the relative CW-complex $(X \times I, A \times I \cup X \times \partial I)$. Because $(X, A)$ is a relative CW-complex, the map $A \hookrightarrow X$ is a cofibration; this implies that the map $A \times I \cup X \times \partial I \to X \times I$ is also a cofibration.

$$\begin{aligned} H^n(X \times I, A \times I \cup X \times \partial I; \pi) &\simeq \widetilde{H}^n(X \times I/(A \times I \cup X \times \partial I); \pi) \\ &= H^n(\Sigma X/A; \pi) \simeq \widetilde{H}^{n-1}(X/A; \pi). \end{aligned}$$

We proved the following statement in the previous section.

**Proposition 51.3.** *Suppose $g : X_{n-1} \to Y$ is a map from the $(n-1)$-skeleton of $X$ to $Y$. Then $g|_{X_{n-2}}$ extends to $X_n \to Y$ iff $[\theta(g)] = 0$ in $H^n(X, A; \pi_{n-1}(Y))$.*

An immediate consequence is the following.

**Theorem 51.4** (CW-approximation)**.** *Any space admits a weak equivalence from a CW-complex.*

This tells us that studying CW-complexes is not very restrictive, if we work up to weak equivalence.

It is easy to see that if $W$ is a CW-complex and $f : X \to Y$ is a weak equivalence, then $[W, X] \xrightarrow{\simeq} [W, Y]$. We can now finally conclude the result of Theorem 49.11:

**Corollary 51.5.** *Let $X$ and $Y$ be CW-complexes. Then a weak equivalence $f : X \to Y$ is a homotopy equivalence.*

## Postnikov and Whitehead towers

Let $X$ be path connected. There is a space $X_{\leq n}$, and a map $X \to X_{\leq n}$ such that $\pi_i(X_{\geq n}) = 0$ for $i > n$, and $\pi_i(X) \xrightarrow{\simeq} \pi_i(X_{\leq n})$ for $i \leq n$. This pair $(X, X_{\leq n})$ is essentially unique up to homotopy; the space $X_{\leq n}$ is called the *nth Postnikov section* of $X$. Since Postnikov sections have "simpler" homotopy groups, we can try to understand $X$ by studying each of its Postnikov sections individually, and then gluing all the data together.

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Suppose $A$ is some abelian group. We saw, in the first partthat there is a space $M(A, n)$ with homology given by:

$$\widetilde{H}_i(M(A, n)) = \begin{cases} A & i = n \\ 0 & i \neq n. \end{cases}$$

This space was constructed from a free resolution $0 \to F_1 \to F_0 \to A \to 0$ of $A$. We can construct a map $\bigvee S^n \to \bigvee S^n$ which realizes the first two maps; coning this off gets $M(A, n)$. By Hurewicz, we have:

$$\pi_i(M(A, n)) = \begin{cases} 0 & i < n \\ A & i = n \\ ?? & i > n \end{cases}$$

It follows that, when we look at the $n$th Postnikov section of $M(A,n)$, we have:

$$\pi_i(M(A,n)_{\le n}) = \begin{cases} A & i = n \\ 0 & i \ne n. \end{cases}$$

In some sense, therefore, this Postnikov section is a "designer homotopy type". It deserves a special name: $M(A,n)_{\le n}$ is called an *Eilenberg-MacLane space*, and is denoted $K(A,n)$. By the fiber sequence $\Omega X \to PX \to X$ with $PX \simeq *$, we find that $\Omega K(\pi,n) \simeq K(\pi,n-1)$. Eilenberg-MacLane spaces are unique up to homotopy.

Note that $n = 1$, $A$ does not have to be abelian, but you can still construct $K(A,1)$. This is called the *classifying space* of $G$; such spaces will be discussed in more detail in the next chapter. Examples are in abundance: if $\Sigma$ is a closed surface that is not $S^2$ or $\mathbf{R}^2$, then $\Sigma \simeq K(\pi_1(\Sigma),1)$. Perhaps simpler is the identification $S^1 \simeq K(\mathbf{Z},1)$.

**Example 51.6.** We can identify $K(\mathbf{Z},2)$ as $\mathbf{CP}^\infty$. To see this, observe that we have a fiber sequence $S^1 \to S^{2n+1} \to \mathbf{CP}^n$. The long exact sequence in homotopy tells us that the homotopy groups of $\mathbf{CP}^n$ are the same as the homotopy groups of $S^1$, until $\pi_* S^{2n+1}$ starts to interfere. As $n$ grows, we obtain a fibration $S^1 \to S^\infty \to \mathbf{CP}^\infty$. Since $S^\infty$ is weakly contractible (it has no nonzero homotopy groups), we get the desired result.

**Example 51.7.** Similarly, we can identify $K(\mathbf{Z}/2\mathbf{Z},1)$ as $\mathbf{RP}^\infty$.

Since $\pi_1(K(A,n)) = 0$ for $n > 1$, it follows that $K(A,n)$ is automatically a simple space. This means that

$$[S^k, K(A,n)] = \pi_k(K(A,n)) = H^n(S^k, A).$$

In fact, a more general result is true:

**Theorem 51.8** (Brown representability)**.** *If $X$ is a CW-complex, then* $[X, K(A,n)] = H^n(X;A)$.

We will not prove this here, but one can show this simply by showing that the functor $[-, K(A,n)]$ satisfies the Eilenberg-Steenrod axioms. Somehow, these Eilenberg-MacLane spaces $K(A,n)$ completely capture cohomology in dimension $n$.

If $X$ is a CW-complex, then we may assume that $X_{\le n}$ is also a CW-complex. (Otherwise, we can use cellular approximation and then kill homotopy groups.) Let us assume that $X$ is path connected; then $X_{\le 1} = K(\pi_1(X),1)$. We may then form a (commuting) tower:

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \cdots \arrow[l] \\
 & X_{\leq 3} \arrow[d] & K(\pi_3(X),3) \arrow[l] \\
 & X_{\leq 2} \arrow[d] & K(\pi_2(X),2) \arrow[l] \\
X \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuur] & X_{\leq 1} \arrow[r, equal] & K(\pi_1(X),1),
\end{tikzcd}
$$

since $K(\pi_n(X), n) \to X_{\leq n} \to X_{\leq n-1}$ is a fiber sequence. This decomposition of $X$ is called the *Postnikov tower* of $X$.

Denote by $X_{>n}$ the fiber of the map $X \to X_{\leq n}$ (for instance, $X_{>1}$ is the universal cover of $X$); then, we have

$$
\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d, equal] & \vdots \arrow[d] & \cdots \arrow[l] \\
X_{>3} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & X_{\leq 3} \arrow[d] & K(\pi_3(X),3) \arrow[l] \\
X_{>2} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & X_{\leq 2} \arrow[d] & K(\pi_2(X),2) \arrow[l] \\
X_{>1} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & X_{\leq 1} \arrow[r, equal] \arrow[d] & K(\pi_1(X),1) \\
X \arrow[r, equal] & X \arrow[r] & *
\end{tikzcd}
$$

The leftmost tower is called the *Whitehead tower* of $X$, named after George Whitehead.

I can take the fiber of $X_{>1} \to X$, and I get $K(\pi_1(X), 0)$; more generally, the fiber of

$X_{>n} \to X_{>n-1}$ is $K(\pi_n(X), n-1)$. This yields the following diagram:

$$
\begin{tikzcd}
\cdots & \vdots \arrow[d] & \vdots \arrow[d, equal] & \vdots \arrow[d] & \cdots \\
K(\pi_3(X),2) \arrow[r] & X_{>3} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & X_{\leq 3} \arrow[d] & K(\pi_3(X),3) \arrow[l] \\
K(\pi_2(X),1) \arrow[r] & X_{>2} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & X_{\leq 2} \arrow[d] & K(\pi_2(X),2) \arrow[l] \\
K(\pi_1(X),0) \arrow[r] & X_{>1} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & X_{\leq 1} \arrow[d] \arrow[r, equal] & K(\pi_1(X),1) \\
 & X \arrow[r, equal] & X \arrow[r] & *
\end{tikzcd}
$$

We can construct Eilenberg-MacLane spaces as cellular complexes by attaching cells to the sphere to kill its higher homotopy groups. The complexity of homotopy groups, though, shows us that attaching cells to compute the cohomology of Eilenberg-MacLane spaces is not feasible.

# Chapter 5

# Vector bundles

## 52 Vector bundles, principal bundles

Each point $x$ in a smooth manifold $M$ admits a "tangent space." This is a real vector space, whose elements are equivalence classes of smooth paths $\sigma : \mathbf{R} \to M$ such that $\sigma(0) = x$. The equivalence relation retains only the velocity vector at $t = 0$. These vector spaces "vary smoothly" over the manifold. The notion of a vector bundle is a topological extrapolation of this idea.

Let $X$ be a topological space. To begin with, we will define a *vector space over* $B$ to be a map $E \to B$ along with the following extra data:

- an "addition" $\mu : E \times_B E \to E$, compatible with the maps down to $B$;
- a "zero section" $s : B \to E$ such that the composite $B \xrightarrow{s} E \to B$ is the identity;
- an inverse $\chi : E \to E$, compatible with the map down to $B$; and
- an action of $\mathbf{R}$:

$$\begin{tikzcd}
\mathbf{R} \times E \arrow[r, equal] \arrow[dr, "p\circ\mathrm{pr}_2"'] & (B \times \mathbf{R}) \times_B E \arrow[r] \arrow[d] & E \arrow[dl, "p"] \\
& B &
\end{tikzcd}$$

These data are required to render each fiber a real vector space of finite dimension.

**Example 52.1.** A "trivial" example of a vector space over $B$ is the projection $B \times V \to B$ where $V$ is a real vector space of finite dimension $n$. This is the *trivial vector space* over $B$ of dimension $n$.

Vector spaces over $X$ form a category in which the morphisms are maps covering the identity map of $X$ that are linear on each fiber.

**Example 52.2.** Let $p : S \to \mathbf{R}$ have $p^{-1}(0) = \mathbf{R}$ and $p^{-1}(s) = 0$ for $s \neq 0$. With the evident structure maps, this is a perfectly good ("skyscraper") vector space over $\mathbf{R}$. This example is peculiar, however; it is not locally constant. Our definition of vector bundles will exlude it and similar oddities. Sheaf theory is the proper home for examples like this.

But this example occurs naturally even if you restrict to trivial bundles and maps between them. The trivial bundle $\mathrm{pr}_1 : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ has as an endomorphism the map

$$(s,t) \mapsto (s, st)\,.$$

This map is an isomorphism on almost all fibers, but is zero over $s = 0$. So if you want to form a kernel or a cokernel, you will get the skyscraper vector space over $\mathbf{R}$. This puts limitations on the operations we can form on vector bundles, if we want them to result in vector bundles.

**Definition 52.3.** A *vector bundle* over $B$ is a vector space $E$ over $B$ that is locally trivial; that is, every point $b \in B$ has a neighborhood over which $E$ is isomorphic to a trivial bundle.

**Remark 52.4.** As in our definition of fiber bundles, we will always assume that a vector bundle admits a numerable trivializing cover. And, to repeat, the fiber dimensions of our vector bundles will always be finite. On the other hand, there is nothing to stop us from replacing $\mathbf{R}$ with $\mathbf{C}$ or even with the quaternions $\mathbf{H}$, and talking about complex or quaternionic vector bundles.

If $p : E \to B$ is a vector bundle, then $E$ is called the *total space*, $p$ is called the *projection map*, and $B$ is called the *base space*. There are various notations in use for vector bundles, and we will switch among them. So we will use a Greek letter like $\xi$ or $\zeta$ to denote the entire structure, and $E(\xi), B(\xi)$ denotes the total space and base space. We may write $\xi \downarrow B$ to indicate a vector bundle over $B$, and, indeed, use $\downarrow$ rather than $\to$ for projection maps in general.

**Example 52.5.** The $n$-dimensional trivial bundle $B \times \mathbf{R}^n \downarrow B$ will be denote by $n\epsilon$.

**Example 52.6.** At the other extreme, Grassmannians support highly nontrivial vector bundles. We can form Grassmannians over any one of the three (skew)fields $\mathbf{R}, \mathbf{C}, \mathbf{H}$. Write $K$ for one of them, and consider the (left) $K$-vector space $K^n$. The *Grassmannian* (or *Grassmann manifold*) $\mathrm{Gr}_k(K^n)$ is the space of $k$-dimensional $K$-subspaces of $K^n$. As we saw last term, this is a topologized as a quotient space of a Stiefel variety of $k$-frames in $K^n$. To each point in $\mathrm{Gr}_k(K)$ is associated a $k$-dimensional subspace of $K^n$. This provides us with a $k$-dimensional $K$-vector space $\xi_{n,k}$ over $\mathrm{Gr}_k(K^n)$, with total space

$$E(\xi_{n,k}) = \{(V, x) \in \mathrm{Gr}_k(K^n) \times K^n : x \in V\}$$

This is the *canonical* or *tautologous* vector bundle over $\mathrm{Gr}_k(K^n)$. It occurs as a subbundle of $n\epsilon$.

**Exercise 52.7.** Prove that $\xi_{n,k}$, as defined above, is locally trivial, so is a vector bundle over $\mathrm{Gr}_k(K^n)$.

For instance, when $k = 1$, we have $\mathrm{Gr}_1(\mathbf{R}^n) = \mathbf{RP}^{n-1}$. The tautologous bundle $\xi_{n,1}$ is 1-dimensional; it is a *line bundle*, the canonical line bundle over $\mathbf{RP}^{n-1}$. We may write $\gamma$ for this line bundle.

**Example 52.8.** Let $M$ be a smooth manifold. Define $\tau_M$ to be the tangent bundle $TM \to M$ over $M$. For example, if $M = S^{n-1}$, then

$$TS^{n-1} = \{(x, v) \in S^{n-1} \times \mathbf{R}^n : v \cdot x = 0\}.$$

## Constructions with vector bundles

One cannot take the kernels or cokernels of a map of vector bundles; but just about anything which can be done for vector spaces can also be done for vector bundles:

1. The pullback of a vector bundle is again a vector bundle: If $p' : E' \to B'$ is a vector bundle then the map $p$ in the diagram below is also a vector bundle.

$$\begin{tikzcd} E \arrow[r, "\overline{f}"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\ B \arrow[r, "f"'] & B' \end{tikzcd}$$

   For instance, if $B = *$, the pullback is just the fiber of $E'$ over the point $* \to B'$. The cover by products of the elements of trivializing covers trivialize the product

   If $\xi$ is the bundle $E' \to B'$, we denote the pullback $E \to B$ by $f^*\xi$.

   There's a convenient way to think of a pullback: the top map $\overline{f}$ in the pullback diagram has two key properties: It covers $f$, and it is a linear isomorphism on fibers. These conditions suffice to present $p$ as the pullback of $p'$ along $f$.

2. If $p : E \to B$ and $p' : E' \to B'$, then we can take the product $E \times E' \xrightarrow{p \times p'} B \times B'$.

3. If $B = B'$, we can form the pullback:

$$\begin{tikzcd} E \oplus E' \arrow[r] \arrow[d] & E \times E' \arrow[d] \\ B \arrow[r, "\Delta"] & B \times B \end{tikzcd}$$

   The bundle $E \oplus E'$ is called the *Whitney sum*. For instance,

$$n\epsilon = \epsilon \oplus \cdots \oplus \epsilon.$$

4. If $E, E' \to B$ are two vector bundles over $B$, we can form another vector bundle $E \otimes E' \to B$ by taking the fiberwise tensor product. Likewise, taking the fiberwise Hom produces a vector bundle $\mathrm{Hom}(E, E') \to B$.

**Example 52.9.** Recall from Example 52.6 the tautological bundle $\gamma$ over $\mathbf{RP}^{n-1}$. The tangent bundle $\tau_{\mathbf{RP}^{n-1}}$ also lives over $\mathbf{RP}^{n-1}$. As this is the first explicit pair of vector bundles over the same space, it is natural to wonder what is the relationship between these two bundles. We claim that

$$\tau_{\mathbf{RP}^{n-1}} = \mathrm{Hom}(\gamma, \gamma^{\perp}).$$

To see this, make use of the 2-fold covering map $S^{n-1} \to \mathbf{RP}^{n-1}$. The projection map is smooth, and covered by a fiberwise isomorphism of tangent bundles. The fibers $T_x S^{n-1}$ and $T_{-x} S^{n-1}$ are both identified with the orthogonal complement of $\mathbf{R}x$ in $\mathbf{R}^n$, and the differential of the antipodal map sends $v$ to $-v$. So the tangent vector to $\pm x \in \mathbf{RP}^{n-1}$ represented by $(x, v)$ is the same as the tangent vector represented by $(-x, -v)$. This describes the vector bundle $\mathrm{Hom}(\gamma, \gamma^{\perp})$.

**Exercise 52.10.** Prove that $\mathrm{Gr}_k(K^n)$, for $K = \mathbf{R}, \mathbf{C}, \mathbf{H}$, then

$$\tau_{\mathrm{Gr}_k(K^n)} = \mathrm{Hom}(\xi_{n,k}, \xi_{n,k}^{\perp}).$$

## Metrics and splitting exact sequences

A map of vector bundles, $\xi \to \eta$, over a fixed base can be identified with a section of $\mathrm{Hom}(\xi, eta)$. We have seen that the kernel and cokernel of a homomorphism will be vector bundles only of the rank is locally constant.

**Exercise 52.11.** Prove the converse.

In particular, we can form kernels of surjections and cokernels of injections; and consider short exact sequences of vector bundles. It is a characteristic of topology, as opposed to analyic or algebraic geometry, that short exact sequences of vector bundles always split. To see this we use a "metric."

**Definition 52.12.** A *metric* on a vector bundle is a continuous choice of inner products on the fibers.

**Lemma 52.13.** *Any (numerable) vector bundle $\xi$ over $X$ admits a metric.*

*Proof.* This will use the fact that if $g, g'$ are both inner products on a vector space then $tg + (1-t)g'$ is another; and more generally that the space of metrics forms a real affine space.

Pick a trivializing open cover $\mathcal{U}$ for $\xi$, and for each $U \in \mathcal{U}$ an isomorphism $\xi|_U \cong U \times V_U$. Pick an inner product $g_U$ on each of the vector spaces $V_U$. Pick a partition of unity subordinate to $\mathcal{U}$; that is, functions $\phi_U : U \to [0,1]$ such that the preimage of the complement of 0 is $U$ and

$$\sum_{x \in U} \phi_U(x) = 1\,.$$

Now the sum

$$g \coloneqq \sum_U \phi_U g_U$$

is a metric on $\xi$. $\square$

**Corollary 52.14.** *Any exact sequence* $0 \to \xi' \to \xi \to \xi'' \to 0$ *of vector bundles (over the same base) splits.*

*Proof.* Pick a metric for $\xi$. Using it, form the orthogonal complement ${\xi'}^{\perp}$. The composite

$$ {\xi'}^{\perp} \hookrightarrow \xi \to \xi'' $$

is an isomorphism: the dimensions of the fibers are the same. This provides a splitting of the surjection $\xi \to \xi''$ and hence of the short exact sequence. $\square$

# 53 Principal bundles, associated bundles

## $I$-invariance

We will denote by $\mathrm{Vect}(B)$ the set of isomorphism classes of vector bundles over $B$. (Justify the use of the word "set"!)

Consider a vector bundle $\xi \downarrow B$. If $f : B' \to B$, taking the pullback gives a vector bundle denoted $f^*\xi$. This operation descends to a map $f^* : \mathrm{Vect}(B) \to \mathrm{Vect}(B')$; we therefore obtain a functor $\mathrm{Vect} : \mathbf{Top}^{op} \to \mathbf{Set}$. One might expect this functor to give some interesting invariants of topological spaces.

**Theorem 53.1.** *Let* $I = \Delta^1$. *Then* Vect *is* $I$*-invariant. In other words, the projection* $X \times I \to X$ *induces an isomorphism* $\mathrm{Vect}(X) \to \mathrm{Vect}(X \times I)$.

One important corollary of this result is:

**Corollary 53.2.** Vect *is a homotopy functor.*

*Proof.* Consider two homotopic maps $f, g : B \to B'$, so there exists a homotopy $H : B' \times I \to B$. If $\xi \downarrow B$, we need to prove that $f_0^*\xi \simeq f_1^*\xi$. This is far from obvious.

Consider the following diagram.

$$
\begin{tikzcd}
B' \times I \arrow[r, "H"] \arrow[d, "\mathrm{pr}"'] & B \\
B' &
\end{tikzcd}
$$

The leftmost map is an isomorphism under Vect, by Theorem 53.1. Let $\eta \downarrow B$ be a vector bundle such that $\mathrm{pr}^*\eta \simeq f^*\xi$. For any $t \in I$, define a map $\in_t \colon B' \to B' \times I$ sends $x \mapsto (x, t)$. We then have isomorphisms:

$$ f_t^*\xi \simeq \in_t^* f^*\xi \simeq \in_t^* \mathrm{pr}^*\eta \simeq (\mathrm{pr} \circ \in_t)^*\eta \simeq \eta, $$

as desired. $\square$

It is easy to see that $\mathrm{Vect}(X) \to \mathrm{Vect}(X \times I)$ is injective. In the next lecture, we will prove surjectivity, allowing us to conclude Theorem 53.1.

### Principal bundles

**Definition 53.3.** Let $G$ be a topological group[1]. A *principal $G$-bundle* is a right action of $G$ on $P$ such that:

- $G$ acts freely.
- The orbit projection $P \to P/G$ is a fiber bundle.

These are not unfamiliar objects, as the next example shows.

**Example 53.4.** Suppose $G$ is discrete. Then the fibers of the orbit projection $P \to P/G$ are all discrete. Therefore, the condition that $P \to P/G$ is a fiber bundle is simply that it's a covering projection (the action is "properly discontinuous").

As a special case, let $X$ be a space with universal cover $\widetilde{X} \downarrow X$. Then $\pi_1(X)$ acts freely on $\widetilde{X}$, and $\widetilde{X} \downarrow X$ is the orbit projection. It follows from our discussion above that this is a principal bundle. Explicit examples include the principal $\mathbf{Z}/2$-bundle $S^{n-1} \downarrow \mathbf{RP}^{n-1}$, and the Hopf fibration $S^{2n-1} \downarrow \mathbf{CP}^{n-1}$, whcih is a principle $S^1$-bundle.

By looking at the universal cover, we can classify covering spaces of $X$. Remember how that goes: if $F$ is a set with left $\pi_1(X)$-action, the dotted map in the diagram below is the desired covering space.

$$\begin{tikzcd}
\widetilde{X} \times F \arrow[r] \arrow[d, "p \circ \mathrm{pr}_1"'] & \widetilde{X} \times F/\sim \arrow[dl, "q"] \\
X &
\end{tikzcd}$$

Here, we say that $(y, gz) \sim (yg, z)$, for elements $y \in \widetilde{X}$, $z \in F$, and $g \in \pi_1(X)$.

Fix $y_0 \in \widetilde{X}$ over $* \in X$. Then it is easy to see that $F \xrightarrow{\simeq} q^{-1}(*)$, via the map $z \mapsto (y_0, z)$. This is all neatly summarized in the following theorem from point-set topology.

**Theorem 53.5** (Covering space theory)**.** *There is an equivalence of categories:*

$$\{\textit{Left } \pi_1(X)\textit{-sets}\} \xrightarrow{\simeq} \{\textit{Covering spaces of } X\},$$

*with inverse functor given by taking the fiber over the basepoint and lifting a loop in $X$ to get a map from the fiber to itself.*

Example 53.4 shows that covering spaces are special examples of principal bundles. The above theorem therefore motivates finding a more general picture.

[1]We will only care about discrete groups and Lie groups.

**Construction 53.6.** Let $P \downarrow B$ is a principal $G$-bundle. If $F$ is a left $G$-space, we can define a new fiber bundle, exactly as above:

$$\begin{tikzcd} P \times F \arrow[r] \arrow[d] & P \times F/\sim \arrow[dl, "q"] \\ B & \end{tikzcd}$$

This is called an *associated bundle*, and is denoted $P \times_G F$.

We must still justify that the resulting space over $B$ is indeed a new fiber bundle with fiber $F$. Let $x \in B$, and let $y \in P$ over $x$. As above, we have a map $F \to q^{-1}(*)$ via the map $z \mapsto [y, z]$. We claim that this is a homeomorphism. Indeed, define a map $q^{-1}(*) \to F$ via

$$[y', z'] = [y, gz'] \mapsto gz',$$

where $y' = yg$ for some $g$ (which is necessarily unique).

**Exercise 53.7.** Check that these two maps are inverse homeomorphisms.

**Definition 53.8.** A vector bundle $\xi \downarrow B$ is said to be an $n$-plane bundle if the dimensions of all the fibers are $n$.

Let $\xi \downarrow B$ be an $n$-plane bundle. Construct a principal $\mathrm{GL}_n(\mathbf{R})$-bundle $P(\xi)$ by defining

$$P(\xi)_b = \{\text{bases for } E(\xi)_b = \mathrm{Iso}(\mathbf{R}^n, E(\xi)_b)\}.$$

To define the topology, note that (topologically) we have

$$P(B \times \mathbf{R}^n) = B \times \mathrm{Iso}(\mathbf{R}^n, \mathbf{R}^n),$$

where $\mathrm{Iso}(\mathbf{R}^n, \mathbf{R}^n) = \mathrm{GL}_n(\mathbf{R})$ is given the usual topology as a subspace of $\mathbf{R}^{n^2}$.

There is a right action of $\mathrm{GL}_n(\mathbf{R})$ on $P(\xi) \downarrow B$, given by precomposition. It is easy to see that this action is free and simply transitive. One therefore has a principal action of $\mathrm{GL}_n(\mathbf{R})$ on $P(\xi)$. The bundle $P(\xi)$ is called the *principalization* of $\xi$.

Given the principalization $P(\xi)$, we can recover the total space $E(\xi)$. Consider the associated bundle $P(\xi) \times_{\mathrm{GL}_n(\mathbf{R})} \mathbf{R}^n$ with fiber $F = \mathbf{R}^n$, with $\mathrm{GL}_n(\mathbf{R})$ acting on $\mathbf{R}^n$ from the left. Because this is a linear action, $P(\xi) \times_{\mathrm{GL}_n(\mathbf{R})} \mathbf{R}^n$ is a vector bundle. One can show that

$$P(\xi) \times_{\mathrm{GL}_n(\mathbf{R})} \mathbf{R}^n \simeq E(\xi).$$

Fix a topological group $G$. Define $\mathrm{Bun}_G(B)$ as the set of isomorphism classes of $G$-bundles over $B$. An isomorphism is a $G$-equivariant homeomorphism over the base. Again, arguing as above, this begets a functor $\mathrm{Bun}_G : \mathbf{Top} \to \mathbf{Set}$. The above discussion gives a natural isomorphism of functors:

$$\mathrm{Bun}_{\mathrm{GL}_n(\mathbf{R})}(B) \simeq \mathrm{Vect}(B).$$

The $I$-invariance theorem will therefore follow immediately from:

**Theorem 53.9.** $\mathrm{Bun}_G$ *is $I$-invariant.*

**Remark 53.10.** Principal bundles allow a description of "geometric structures on $\xi$". Suppose, for instance, that we have a metric on $\xi$. Instead of looking at all ordered bases, we can attempt to understand all ordered orthonormal bases in each fiber. This give the *frame bundle*

$$\mathrm{Fr}(B) = \{\text{ordered orthonormal bases of } E(\xi)_b\};$$

these are isometric isomorphisms $\mathbf{R}^n \to E(\xi)_b$. Again, there is an action of the orthogonal group on $\mathrm{Fr}(B)$: in fact, this begets a principal $O(n)$-bundle. Such examples are in abundance: consistent orientations give an $SO(n)$-bundle. Trivializations of the vector bundle also give principal bundles. This is called "reduction of the structure group".

One useful fact about principal $G$-bundles (which should not be too surprising) is the following statement.

**Theorem 53.11.** *Every morphism of principal $G$-bundles is an isomorphism.*

*Proof.* Let $p : P \to B$ and $p' : P' \to B$ be two principal $G$-bundles over $B$, and let $f : P \to P'$ be a morphism of principal $G$-bundles. For surjectivity of $f$, let $y \in P'$. Consider $x \in P$ such that $p(x) = p'(y)$. Since $p(x) = p'f(x)$ we conclude that $y = f(x)g$ for some $g \in G$. But $f(x)g = f(xg)$, so $xg$ maps to $y$, as desired. To see that $f$ is injective, suppose $f(x) = f(y)$. Now $p(x) = p'f(x) = p(y)$, so there is some $g \in G$ such that $xg = y$. But $f(y) = f(xg) = f(x)g$, so $g = 1$, as desired. We will leave the continuity of $f^{-1}$ as an exercise to the reader. $\square$

Theorem 53.11 says that if we view $\mathrm{Bun}_G(B)$ as a category where the morphisms are given by morphisms of principal $G$-bundles, then it is a groupoid.

## 54 $I$-invariance of $\mathrm{Bun}_G$, and $G$-CW-complexes

Let $G$ be a topological group. We need to show that the functor $\mathrm{Bun}_G : \mathbf{Top}^{op} \to \mathbf{Set}$ is $I$-invariant, i.e., the projection $X \times I \xrightarrow{\mathrm{pr}} X$ induces an isomorphism $\mathrm{Bun}_G(X) \xrightarrow{\simeq} \mathrm{Bun}_G(X \times I)$. Injectivity is easy: the composite $X \xrightarrow{\mathrm{in}_0} X \times I \xrightarrow{\mathrm{pr}} X$ gives you a splitting $\mathrm{Bun}_G(X) \xrightarrow{\mathrm{pr}_*} \mathrm{Bun}_G(X \times I) \xrightarrow{\mathrm{in}_0} \mathrm{Bun}_G(X)$ whose composite is the identity.

The rest of this lecture is devoted to proving surjectivity. We will prove this when $X$ is a CW-complex (Husemoller does the general case; see [**?**, §4.9]). We begin with a small digression.

### $G$-CW-complexes

We would like to define CW-complexes with an action of the group $G$. The naïve definition (of a space with an action of the group $G$) will not be sufficient; rather, we will require that each cell have an action of $G$.

In other words, we will build $G$-CW-complexes out of "$G$-cells". This is supposed to be something of the form $D^n \times H\backslash G$, where $H$ is a closed subgroup of $G$. Here, the space $H\backslash G$ is the orbit space, viewed as a right $G$-space. The boundary of the $G$-cell $D^n \times H\backslash G$ is just $\partial D^n \times H\backslash G$. More precisely:

**Definition 54.1.** A $G$-CW-complex is a (right) $G$-space $X$ with a filtration $0 = X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X$ such that for all $n$, there exists a pushout square:

$$\begin{tikzcd} \coprod \partial D^n_\alpha \times H_\alpha\backslash G \arrow[r] \arrow[d] & \coprod D^n_\alpha \times H_\alpha\backslash G \arrow[d] \\ X_{n-1} \arrow[r] & X_n, \end{tikzcd}$$

and $X$ has the direct limit topology.

Notice that a CW-complex is a $G$-CW-complex for the trivial group $G$.

**Theorem 54.2.** *If $G$ is a compact Lie group and $M$ a compact smooth $G$-manifold, then $M$ admits a $G$-CW-structure.*

This is the analogue of the classical result that a compact smooth manifold is homotopy equivalent to a CW-complex, but it is much harder to prove the equivariant statement.

Note that if $G$ acts principally (Definition 53.3) on $P$, then every $G$-CW-structure on $P$ is "free", i.e., $H_\alpha = 0$.

1. If $X$ is a $G$-CW-complex, then $X/G$ inherits a CW-structure whose $n$-skeleton is given by $(X/G)_n = X_n/G$.

2. If $P \to X$ is a principal $G$-bundle, then a CW-structure on $X$ lifts to a $G$-CW-structure on $P$.

## Proof of $I$-invariance

Recall that our goal is to prove that every $G$-bundle over $X \times I$ is pulled back from some vector bundle over $X$.

As a baby case of Theorem 53.1 we will prove that if $X$ is contractible, then any principal $G$-bundle over $X$ is trivial, i.e., $P \simeq X \times G$ as $G$-bundles.

Let us first prove the following: if $P \downarrow X$ has a section, then it's trivial. Indeed, suppose we have a section $s : X \to P$. Since $P$ has an action of the group on it, we may extend this to a map $X \times G \to P$ by sending $(x, g) \mapsto gs(x)$. As this is a map of $G$-bundles over $X$, it is an isomorphism by Theorem 53.11, as desired.

To prove the statement about triviality of any principal $G$-bundle over a contractible space, it therefore suffices to construct a section for any principal $G$-bundle. Consider the constant map $X \to P$. Then the following diagram commutes up to homotopy, and hence (by Exercise **??**(1)) there is an *actual* section of $P \to X$, as desired.

$$\begin{tikzcd} & P \arrow[d] \\ X \arrow[ur, "\text{const}"] \arrow[r] & X \end{tikzcd}$$

For the general case, we will assume $X$ is a CW-complex. For notational convenience, let us write $Y = X \times I$. We will use descending induction to construct the desired principal $G$-bundle over $X$.

To do this, we will filter $Y$ by subcomplexes. Let $Y_0 = X \times 0$; in general, we define

$$Y_n = X \times 0 \cup X_{n-1} \times I.$$

It follows that we may construct $Y_n$ out of $Y_{n-1}$ via a pushout:

$$\begin{tikzcd} \coprod_{\alpha \in \Sigma_{n-1}} (\partial D^{n-1} \times I \cup D^{n-1}_\alpha \times 0) \arrow[r] \arrow[d, "\coprod_{\alpha \in \Sigma_{n-1}} f_\alpha \times 1_I \cup \phi_\alpha \times 0"'] & \coprod_\alpha (D^{n-1}_\alpha \times I) \arrow[d] \\ Y_{n-1} \arrow[r] & Y_n, \end{tikzcd}$$

where the maps $f_\alpha$ and $\phi_\alpha$ are defined as:

$$\begin{tikzcd} \partial D^{n-1}_\alpha \arrow[r, "f_\alpha"] \arrow[d] & X_{n-2} \arrow[d] \\ D^{n-1}_\alpha \arrow[r, "\phi_\alpha"'] & X_{n-1} \end{tikzcd}$$

In other words, the $f_\alpha$ are the attaching maps and the $\phi_\alpha$ are the characteristic maps.

Consider a principal $G$-bundle $P \xrightarrow{p} Y = X \times I$. Define $P_n = p^{-1}(Y_n)$; then we can build $P_n$ from $P_{n-1}$ in a similar way:

$$\begin{tikzcd} \coprod_\alpha (\partial D^{n-1}_\alpha \times I \cup D^{n-1}_\alpha \times 0) \times G \arrow[r] \arrow[d] & \coprod_\alpha (D^{n-1}_\alpha \times I) \times G \arrow[d] \\ P_{n-1} \arrow[r] & P_n \end{tikzcd}$$

Note that this isn't *quite* a $G$-CW-structure. Recall that we are attempting to fill in a dotted map:

$$\begin{tikzcd} P \arrow[r, dashed] \arrow[d] & P_0 \arrow[d] \\ Y \arrow[r, "\text{pr}"'] & Y_0 = X \end{tikzcd}$$

sh this...

I'm constructing this inductively– we have $P_{n-1} \to P_0$. So I want to define $\coprod_\alpha (D^{n-1}_\alpha \times$

$I) \times G \to P_0$ that's equivariant. That's the same thing as a map $\coprod_\alpha (D^{n-1}_\alpha \times I) \to P_0$ that's compatible with the map from $\coprod(\partial D^{n-1}_\alpha \times I \cup D^{n-1}_\alpha \times 0)$. Namely, I want to fill in:

$$\begin{tikzcd}
\coprod_\alpha(\partial D^{n-1}_\alpha \times I \cup D^{n-1}_\alpha \times 0) \arrow[r] \arrow[d] & \coprod_\alpha(D^{n-1}_\alpha \times I) \arrow[d] \arrow[dddr, dashed] & \\
\coprod_\alpha(\partial D^{n-1}_\alpha \times I \cup D^{n-1}_\alpha \times 0) \times G \arrow[r] \arrow[d] & \coprod_\alpha(D^{n-1}_\alpha \times I) \times G \arrow[d] \arrow[ddr, dashed] & \\
P_{n-1} \arrow[r] \arrow[drr, "\text{induction}"'] & P_n \arrow[dr] & \\
& & P_0 \arrow[d] \\
& & X
\end{tikzcd} \tag{5.1}$$

Now, I know that $(D^{n-1} \times I, \partial D^{n-1} \times I \cup D^{n-1} \times 0) \simeq (D^{n-1} \times I, D^{n-1} \times 0)$. So what I have is:

$$\begin{tikzcd}
D^{n-1} \times 0 \arrow[r, "\text{induction}"] \arrow[d] & P_0 \arrow[d] \\
D^{n-1} \times I \arrow[r, "\phi \circ pr"'] \arrow[ur, dashed] & X
\end{tikzcd}$$

So the dotted map exists, since $P_0 \to X$ is a fibration!

OK, so note that I haven't checked that the outer diagram in Equation 5.1 commutes, because otherwise we wouldn't get $P_n \to P_0$.

**Exercise 54.3.** Check my question above.

Turns out this is easy, because you have a factorization:

$$\begin{tikzcd}
D^{n-1} \times 0 \arrow[r] \arrow[d] & P_{n-1} \arrow[r, "\text{induction}"] & P_0 \arrow[d] \\
D^{n-1} \times I \arrow[rr, "\phi \circ pr"'] \arrow[urr, dashed] & & X
\end{tikzcd}$$

Oh my god, look what time it is! Oh well, at least we got the proof done.

# 55 Classifying spaces: the Grassmann model

We will now shift our focus somewhat and talk about classifying spaces for principal bundles and for vector bundles. We will do this in two ways: the first will be via the Grassmann model and the second via simplicial methods.

**Lemma 55.1.** *Over a compact Hausdorff space, any $n$-plane bundle embeds in a trivial bundle.*

*Proof.* Let $\mathcal{U}$ be a trivializing open cover of the base $B$; since $B$ is compact, we may assume that $\mathcal{U}$ is finite with $k$ elements. There is no issue with numerability, so there is a subordinate partition of unity $\phi_i$. Consider an $n$-plane bundle $E \to B$. By trivialization, there is a fiberwise isomorphism $p^{-1}(U_i) \xrightarrow{f_i} \mathbf{R}^n$ where the $U_i \in \mathcal{U}$. A map to a trivial bundle is the same thing as a bundle map in the following diagram:

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \mathbf{R}^N \arrow[d] \\ B \arrow[r] & * \end{tikzcd}$$

We therefore define $E \to (\mathbf{R}^n)^k$ via

$$e \mapsto (\phi_i(p(e))f_i(e))_{i=1,\cdots,k}.$$

This is a fiberwise linear embedding, generally called a "Gauss map". Indeed, observe that this map has no kernel on every fiber, so it is an embedding. $\square$

The trivial bundle has a metric on it, so choosing the orthogonal complement of the embedding of Lemma 55.1, we obtain:

**Corollary 55.2.** *Over a compact Hausdorff space, any $n$-plane bundle has a complement (i.e. a $\xi^\perp$ such that $\xi \oplus \xi^\perp$ is trivial).*

Another way to say this is that if $B$ is a compact Hausdorff space with an $n$-plane bundle $\xi$, there is a map $f : X \to \mathrm{Gr}_n(\mathbf{R}^{kn})$; this is exactly the Gauss map. It has the property that taking the pullback $f^*\gamma^n$ of the tautologous bundle over $\mathrm{Gr}_n(\mathbf{R}^{kn})$ gives back $\xi$.

In general, we do not have control over the number $k$. There is an easy fix to this problem: consider the tautologous bundle $\gamma^n$ over $\mathrm{Gr}_n(\mathbf{R}^\infty)$, defined as the union of $\mathrm{Gr}_n(\mathbf{R}^m)$ and given the limit topology. This is a CW-complex of finite type (i.e. finitely many cells in each dimension). Note that $\mathrm{Gr}_n(\mathbf{R}^m)$ are not the $m$-skeleta of $\mathrm{Gr}_n(\mathbf{R}^\infty)$!

The space $\mathrm{Gr}_n(\mathbf{R}^\infty)$ is "more universal":

**Lemma 55.3.** *Any (numerable) $n$-plane bundle is pulled back from $\gamma^n \downarrow \mathrm{Gr}_n(\mathbf{R}^\infty)$ via the Gauss map.*

Lemma 55.3 is a little bit tricky, since the covering can be wildly uncountable; but this is remedied by the following bit of point-set topology.

**Lemma 55.4.** *Let $\mathcal{U}$ be a numerable cover of $X$. Then there's another numerable cover $\mathcal{U}'$ such that:*

1. *the number of open sets in $\mathcal{U}'$ is countable, and*

*2. each element of $\mathcal{U}'$ is a disjoint union of elements of $\mathcal{U}$.*

If $\mathcal{U}$ is a trivializing cover, then $\mathcal{U}'$ is also a trivializing cover.

*Proof.* See [**?**, Proposition 3.5.4]. □

It is now an exercise to deduce Lemma 55.3. The main result of this section is the following.

**Theorem 55.5.** *The map $[X, \mathrm{Gr}_n(\mathbf{R}^\infty)] \to \mathrm{Vect}_n(X)$ defined by $[f] \mapsto [f^*\gamma^n]$ is bijective, where $[f]$ is the homotopy class of $f$ and $[f^*\gamma^n]$ is the isomorphism class of the bundle $f^*\gamma^n$.*

This is why $\mathrm{Gr}_n(\mathbf{R}^\infty)$ is also called the *classifying space* for $n$-plane bundles. The Grassmannian provides a very explicit geometric description for the classifying space of $n$-plane bundles. There is a more abstract way to produce a classifying space for principal $G$-bundles, which we will describe in the next section; the Grassmannian is the special case when $G = \mathrm{GL}_n(\mathbf{R})$.

*Proof.* We have already shown surjectivity, so it remains to prove injectivity. Suppose $f_0, f_1 : X \to \mathrm{Gr}_n(\mathbf{R}^\infty)$ such that $f_0^*\gamma^n$ and $f_1^*\gamma^n$ are isomorphic over $X$. We need to construct a homotopy $f_0 \simeq f_1$. For ease of notation, let us identify $f_0^*\gamma^n$ and $f_1^*\gamma_n$ with each other; call it $\xi : E \downarrow X$.

The maps $f_i$ are the same thing as Gauss maps $g_i : E \to \mathbf{R}^\infty$, i.e., maps which are fiberwise linear embeddings. The homotopy $f_0 \simeq f_1$ is created by saying that we have a homotopy from $g_0$ to $g_1$ through Gauss maps, i.e., through other fiberwise linear embeddings.

In fact, we will prove a much stronger statement: *any two* Gauss maps $g_0, g_1 : E \to \mathbf{R}^\infty$ are homotopic through Gauss maps. This is very far from true if I didn't have a $\mathbf{R}^\infty$ on the RHS there.

Let us attempt (and fail!) to construct an affine homotopy between $g_0$ and $g_1$. Consider the map $tg_0 + (1-t)g_1$ for $0 \leq t \leq 1$. In order for these maps to define a homotopy via Gauss maps, we need the following statement to be true: for all $t$, if $tg_0(v) + (1-t)g_1(v) = 0 \in \mathbf{R}^\infty$, then $v = 0$. In other words, we need $tg_0 + (1-t)g_1$ to be injective. Of course, this is not guaranteed from the injectivity of $g_0$ and $g_1$!

Instead, we will construct a composite of affine homotopies between $g_0$ and $g_1$ using the fact that $\mathbf{R}^\infty$ is an infinite-dimensional Euclidean space. Consider the following two linear isometries:

$$\begin{tikzcd} & \mathbf{R}^\infty = \langle e_0, e_1, \cdots \rangle \arrow[dl, "e_i \mapsto e_{2i}"', "\alpha"] \arrow[dr, "e_i \mapsto e_{2i+1}", "\beta"'] & \\ \mathbf{R}^\infty & & \mathbf{R}^\infty \end{tikzcd}$$

Then, we have four Gauss maps: $g_0$, $\alpha \circ g_0$, $\beta \circ g_1$, and $g_1$. There are affine homotopies through Gauss maps:

$$g_0 \simeq \alpha \circ g_0 \simeq \beta \circ g_1 \simeq g_1.$$

We will only show that there is an affine homotopy through Gauss maps $g_0 \simeq \alpha \circ g_0$; the others are left as an exercise. Let $t$ and $v$ be such that $tg_0(v) + (1-t)\alpha g_0(v) = 0$. Since $g_0$ and $\alpha g_0$ are Gauss maps, we may suppose that $0 < t < 1$. Since $\alpha g_0(v)_i$ has only even coordinates, it follows by definition of the map $\alpha$ that $g_0(v)$ only had nonzero coordinates only in dimensions congruent to 0 mod 4. Repeating this argument proves the desired result. $\square$

## 56 Simplicial sets

In order to discuss the simplicial model for classifying spaces of $G$-bundles, we will embark on a long digression on simplicial sets (which will last for three sections). We begin with a brief review of some of the theory of simplicial objects (see also Part **??**).

### Review

We denote by $[n]$ the set $\{0, 1, \cdots, n\}$, viewed as a totally ordered set. Define a category $\mathbf{\Delta}$ whose objects are the sets $[n]$ for $n \geq 0$, with morphisms order preserving maps. There are maps $d^i : [n] \to [n+1]$ given by omitting $i$ (called coface maps) and codegeneracy maps $s^i : [n] \to [n-1]$ that's the surjection which repeats $i$. As discussed in Exercise **??**, any order-preserving map can be written as the composite of these maps, and there are famous relations that these things satisfy. They generate the category $\mathbf{\Delta}$.

There is a functor $\Delta : \mathbf{\Delta} \to \mathbf{Top}$ defined by sending $[n] \mapsto \Delta^n$, the standard $n$-simplex. To see that this is a functor, we need to show that maps $\phi : [n] \to [m]$ induce maps $\Delta^n \to \Delta^m$. The vertices of $\Delta^n$ are indexed by elements of $[n]$, so we may just extend $\phi$ as an affine map to a map $\Delta^n \to \Delta^m$.

Let $X$ be a space. The set of singular $n$-simplices $\mathbf{Top}(\Delta^n, X)$ defines the singular simplicial set $\mathrm{Sin} : \mathbf{\Delta}^{op} \to \mathbf{Set}$.

**Definition 56.1.** Let $\mathcal{C}$ be a category. Denote by $s\mathcal{C}$ the category of simplicial objects in $\mathcal{C}$, i.e., the category $\mathrm{Fun}(\mathbf{\Delta}^{op}, \mathcal{C})$. We write $X_n = X([n])$, called the $n$-simplices.

Explicitly, this gives an object $X_n \in \mathcal{C}$ for every $n \geq 0$, as well as maps $d_i : X_{n+1} \to X_n$ and $s_i : X_{n-1} \to X_n$ given by the face and degeneracy maps.

**Example 56.2.** Suppose $\mathcal{C}$ is a small category, for instance, a group. Notice that $[n]$ is a small category, with:

$$[n](i,j) = \begin{cases} \{\leq\} & \text{if } i \leq j \\ \emptyset & \text{else.} \end{cases}$$

We are therefore entitled to think about $\mathrm{Fun}([n], \mathcal{C})$. This begets a simplicial set $N\mathcal{C}$, called the *nerve of* $\mathcal{C}$, whose $n$-simplices are $(N\mathcal{C})_n = \mathrm{Fun}([n], \mathcal{C})$. Explicitly, an $n$-simplex in the nerve is $(n+1)$-objects in $\mathcal{C}$ (possibly with repetitions) and a chain of $n$ composable morphisms. The face maps are given by composition (or truncation, at the end of the chain of morphisms). The degeneracy maps just compose with the identity at that vertex.

For example, if $G$ is a group regarded as a category, then $(NG)_n = G^n$.

## Realization

The functor Sin transported us from spaces to simplicial sets. Milnor described a way to go the other way.

Let $X$ be a simplicial set. We define the realization $|X|$ as follows:

$$|X| = \left( \coprod_{n \geq 0} \Delta^n \times X_n \right) / \sim,$$

where $\sim$ is the equivalence relation defined as:

$$\Delta^m \times X_m \ni (v, \phi^* x) \sim (\phi_* v, x) \in \Delta^n \times X$$

for all maps $\phi : [m] \to [n]$ where $v \in \Delta^m$ and $x \in X_n$.

**Example 56.3.** The equivalence relation is telling us to glue together simplices as dictated by the simplicial structure on $X$. To see this in action, let us look at $\phi^* = d_i : X_{n+1} \to X_n$ and $\phi_* = d^i : \Delta^n \to \Delta^{n+1}$. In this case, the equivalence relation then says that $(v, d_i x) \in \Delta^n \times X_n$ is equivalent to $(d^i v, x) \in \Delta^{n+1} \times X_{n+1}$. In other words: the $n$-simplex indexed by $d_i x$ is identified with the $i$th face of the $(n+1)$-simplex indexed by $x$.

There's a similar picture for the degeneracies $s^i$, where the equivalence relation dictates that every element of the form $(v, s_i x)$ is already represented by a simplex of lower dimension.

**Example 56.4.** Let $n \geq 0$, and consider the simplicial set $\mathrm{Hom}_{\mathbf{\Delta}}(-, [n])$. This is called the "simplicial $n$-simplex", and is commonly denoted $\mathbf{\Delta}^n$ for good reason: we have a homeomorphism $|\mathbf{\Delta}^n| \simeq \Delta^n$. It is a good exercise to prove this using the explicit definition.

For any simplicial set $X$, the realization $|X|$ is naturally a CW-complex, with

$$\mathrm{sk}_n |X| = \left( \coprod_{k \leq n} \Delta^k \times X_k \right) / \sim .$$

The face maps give the attaching maps; for more details, see [**?**, Proposition I.2.3]. This is a very combinatorial way to produce CW-complexes.

The geometric realization functor and the singular simplicial set give two functors going back and forth between spaces and simplicial sets. It is natural to ask: do they form an adjoint pair? The answer is yes:

$$\begin{tikzcd} s\mathbf{Set} \arrow[r, bend left, "|-|"] \arrow[r, phantom, "\perp"] & \mathbf{Top} \arrow[l, bend left, "\mathrm{Sin}"] \end{tikzcd}$$

For instance, let $X$ be a space. There is a continuous map $\Delta^n \times \mathrm{Sin}_n(X) \to X$ given by $(v, \sigma) \mapsto \sigma(v)$. The equivalence relation defining $|\mathrm{Sin}(X)|$ says that the map factors through the dotted map in the following diagram:

$$\begin{tikzcd} |\mathrm{Sin}(X)| \arrow[r, dashed] & X \\ \coprod \Delta^n \times \mathrm{Sin}_n(X) \arrow[u, two heads] \arrow[ur] & \end{tikzcd}$$

The resulting map is the counit of the adjunction.

Likewise, we can write down the unit of the adjunction: if $K \in s\mathbf{Set}$, the map $K \to \mathrm{Sin}|K|$ sends $x \in K_n$ to the map $\Delta^n \to |K|$ defined via $v \mapsto [(v, x)]$.

This is the beginning of a long philosophy in semi-classical homotopy theory, of taking any homotopy-theoretic question and reformulating it in simplicial sets. For instance, one can define homotopy groups in simplicial sets. For more details, see [**?**].

We will close this section with a definition that we will discuss in the next section. Let $\mathcal{C}$ be a category. From our discussion above, we conclude that the realization $|N\mathcal{C}|$ of its nerve is a CW-complex, called the *classifying space* $B\mathcal{C}$ of $\mathcal{C}$; the relation to the notion of classifying space introduced in §55 will be elucidated upon in a later section.

## 57 Properties of the classifying space

One important result in the story of geometric realization introduced in the last section is the following theorem of Milnor's.

**Theorem 57.1** (Milnor)**.** *Let $X$ be a space. The map $|\mathrm{Sin}(X)| \to X$ is a weak equivalence.*

As a consequence, this begets a functorial CW-approximation to $X$. Unforunately, it's rather large.

In the last section, we saw that $|-|$ was a left adjoint. Therefore, it preserves colimits (Theorem 39.13). Surprisingly, it also preserves products:

**Exercise 57.2** (Hard)**.** Let $X$ and $Y$ be simplicial sets. Their product $X \times Y$ is defined to be the simplicial set such that $(X \times Y)_n = X_n \times Y_n$. Under this notion of product, there is a homeomorphism

$$|X \times Y| \xrightarrow{\simeq} |X| \times |Y|.$$

It is important that this product is taken in the category of $k$-spaces.

Last time, we defined the classifying space $B\mathcal{C}$ of $\mathcal{C}$ to be $|N\mathcal{C}|$.

**Theorem 57.3.** *The natural map $B(\mathcal{C} \times \mathcal{D}) \xrightarrow{\simeq} B\mathcal{C} \times B\mathcal{D}$ is a homeomorphism*[2]*.*

[2] Recall that if $\mathcal{C}$ and $\mathcal{D}$ are categories, the product $\mathcal{C} \times \mathcal{D}$ is the category whose objects are pairs of objects of $\mathcal{C}$ and $\mathcal{D}$, and whose morphisms are pairs of morphisms in $\mathcal{C}$ and $\mathcal{D}$.

*Proof.* It is clear that $N(\mathcal{C} \times \mathcal{D}) \simeq N\mathcal{C} \times N\mathcal{D}$. Since $B\mathcal{C} = |N\mathcal{C}|$, the desired result follows from Exercise 57.2. $\square$

In light of Theorem 57.3, it is natural to ask how natural transformations behave under the classifying space functor. To discuss this, we need some categorical preliminaries.

The category **Cat** is Cartesian closed (Definition **??**). Indeed, the right adjoint to the product is given by the functor $\mathcal{D} \mapsto \mathrm{Fun}(\mathcal{C}, \mathcal{D})$, as can be directly verified.

Consider the category [1]. This is particularly simple: a functor $[1] \to \mathcal{C}$ is just an arrow in $\mathcal{C}$. It follows that a functor $[1] \to \mathcal{D}^{\mathcal{C}}$ is a natural transformation between two functors $f_0$ and $f_1$ from $\mathcal{C}$ to $\mathcal{D}$. By our discussion above, this is the same as a functor $\mathcal{C} \times [1] \to \mathcal{D}$.

By Theorem 57.3, we have a homeomorphism $B([1] \times \mathcal{C}) \simeq B[1] \times B\mathcal{C}$. One can show that $B[1] = \Delta^1$, so a natural transformation between $f_0$ and $f_1$ begets a map $\Delta^1 \times B\mathcal{C} \to B\mathcal{D}$ between $Bf_0$ and $Bf_1$. Concretely:

**Lemma 57.4.** *A natural transformation $\theta : f_0 \to f_1$ where $f_0, f_1 : \mathcal{C} \to \mathcal{D}$ induces a homotopy $Bf_0 \sim Bf_1 : B\mathcal{C} \to B\mathcal{D}$.*

An interesting comment is in order. The notion of a homotopy is "reversible", but that is definitely not true for natural transformations! The functor $B$ therefore "forgets the polarity in **Cat**".

Lemma 57.4 is quite powerful: consider an adjunction $L \dashv R$ where $L : \mathcal{C} \to \mathcal{D}$; then we have natural transformations given by the unit $1_{\mathcal{C}} \to RL$ and the counit $LR \to 1_{\mathcal{D}}$. By Lemma 57.4 we get a homotopy equivalence between $B\mathcal{C}$ and $B\mathcal{D}$. In other words, two categories that are related by any adjoint pair are homotopy equivalent.

A special case of the above discussion yields a rather surprising result. Consider the category [0]. Let $\mathcal{D}$ be another category such that there is an adjoint pair $L \dashv R$ where $L : [0] \to \mathcal{D}$. Then $L$ determines an object $*$ of $\mathcal{D}$. Let $d$ be any object of $\mathcal{D}$. We have the counit $LR(d) \to d$; but $LR(d) = *$, so there is a unique morphism $* \to X$. (To see uniqueness, note that the adjunction $L \dashv R$ gives an identification $\mathcal{D}(*, X) = \mathcal{C}(0, 0) = 0$.) In other words, such a category $\mathcal{D}$ is simply a category with an initial object.

Arguing similarly, any category $\mathcal{D}$ with adjunction $L \dashv R$ where $L : \mathcal{D} \to [0]$ is simply a category with a terminal object. From our discussion above, we conclude that if $\mathcal{D}$ is any category with a terminal (or initial) object, then $B\mathcal{D}$ is contractible.

## 58 Classifying spaces of groups

The constructions of the previous sections can be summarized in a single diagram:

$$\begin{tikzcd} \mathbf{Cat} \arrow[r, "\text{nerve}"] & s\mathbf{Set} \arrow[d, "|-|"] \\ \mathbf{Gp} \arrow[u, hook] \arrow[r, "B"'] & \mathbf{Top} \end{tikzcd}$$

The bottom functor is defined as the composite along the outer edge of the diagram. The space $BG$ for a group $G$ is called the *classifying space of* $G$. At this point, it is far from clear what $BG$ is classifying. The goal of the next few sections is to demystify this definition.

**Lemma 58.1.** *Let $G$ be a group, and $g \in G$. Let $c_g : G \to G$ via $x \mapsto gxg^{-1}$. Then the map $Bc_g : BG \to BG$ is homotopic to the identity.*

*Proof.* The homomorphism $c_g$ is a functor from $G$ to itself. It suffices to prove that there is a natural transformation $\theta$ from the identity to $c_g$. This is rather easy to define: it sends the only object to the only object: we define $\theta_* : * \to *$ to be the map given by $* \xrightarrow{g} *$ specified by $g \in \mathrm{Hom}_G(*, *) = G$. In order for $\theta$ to be a natural transformation, we need the following diagram to commute, which it obviously does:

$$\begin{tikzcd} * \arrow[r, "g"] \arrow[d, "1"'] & * \arrow[d, "gxg^{-1}"] \\ * \arrow[r, "g"'] & *. \end{tikzcd}$$

□

Groups are famous for acting on objects. Viewing groups as categories allows for an abstract definition a group action on a set: it is a functor $G \to \mathbf{Set}$. More generally, if $\mathcal{C}$ is a category, an action of $\mathcal{C}$ is a functor $\mathcal{C} \xrightarrow{X} \mathbf{Set}$. We write $X_c = X(c)$ for an object $c$ of $\mathcal{C}$.

**Definition 58.2.** The "translation" category $X\mathcal{C}$ has objects given by

$$\mathrm{ob}(X\mathcal{C}) = \coprod_{c \in \mathcal{C}} X_c,$$

and morphisms defined via $\mathrm{Hom}_{X\mathcal{C}}(x \in X_c, y \in X_d) = \{f : c \to d : f_*(x) = y\}$.

There is a projection $X\mathcal{C} \to \mathcal{C}$. (For those in the know: this is a special case of the Grothendieck construction.)

**Example 58.3.** The group $G$ acts on itself by left translation. We will write $\widetilde{G}$ for this $G$-set. The translation category $\widetilde{G}G$ has objects as $G$, and maps $x \to y$ are elements $yx^{-1}$. This category is "unicursal", in the sense that there is exactly one map from one object to another object. Every object is therefore initial and terminal, so the classifying space of this category is trivial by the discussion at the end of §57. We will denote by $EG$ the classifying space $B(\widetilde{G}G)$. The map $\widetilde{G}G \to G$ begets a canonical map $EG \to BG$.

The $G$ also acts on itself by right translation. Because of associativity, the right and left actions commute with each other. It follows that the right action is equivariant with respect to the left action, so we get a right action of $G$ on $EG$.

**Claim 58.4.** This action of $G$ on $EG$ is a principal action, and the orbit projection is $EG \to BG$.

To prove this, let us contemplate the set $N(\widetilde{G}G)_n$. An element is a chain of composable morphisms. In this case, it is actually just a sequence of $n+1$ elements in $G$, i.e., $N(\widetilde{G}G)_n = G^{n+1}$. The right action of $G$ is simply the diagonal action. We claim that this is a free action. More precisely:

**Lemma 58.5** (Shearing). *If $G$ is a group and $X$ is a $G$-set, and if $X \times^\Delta G$ has the diagonal $G$-action and $X \times G$ has $G$ acting on the second factor by right translation, then $X \times^\Delta G \simeq X \times G$ as $G$-sets.*

*Proof.* Define a bijection $X \times^\Delta G \mapsto X \times G$ via $(x, g) \mapsto (xg^{-1}, g)$. This map is equivariant since $(x, g) \cdot h = (xh, gh)$, while $(xg^{-1}, g) \cdot h = (xg^{-1}, gh)$. The element $(xh, gh)$ is sent to $(xh(gh)^{-1}, gh)$, as desired. The inverse map $X \times G \to X \times^\Delta G$ is given by $(x, g) \mapsto (xg, g)$. $\square$

We know that $G$ acts freely on $N(\widetilde{G}G)_n$, soo a nonidentity group element is always going to send a simplex to another simplex. It follows that $G$ acts freely on $EG$.

To prove the claim, we need to understand the orbit space. The shearing lemma shows that quotienting out by the action of $G$ simply cancels out one copy of $G$ from the product $N(\widetilde{G}G) = G^n$. In symbols:

$$N(\widetilde{G}G)/G \simeq G^n \simeq (NG)_n.$$

Of course, it remains to check the compatibility with the face and degeneracy maps. We will not do this here; but one can verify that everything works out: the realization is just $BG$!

We need to be careful: the arguments above establish that $EG/G \simeq BG$ when $G$ is a finite group. The case when $G$ is a topological group is more complicated. To describe this generalization, we need a preliminary categorical definition.

Let $\mathcal{C}$ be a category, with objects $\mathcal{C}_0$ and morphisms $\mathcal{C}_1$. Then we have maps $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \xrightarrow{\text{compose}} \mathcal{C}_1$ and two maps (source and target) $\mathcal{C}_1 \to \mathcal{C}_0$, and the identity $\mathcal{C}_0 \to \mathcal{C}_1$. One can specify the same data in any category $\mathcal{D}$ with pullbacks. Our interest will be in the case $\mathcal{D} = \mathbf{Top}$; in this case, we call $\mathcal{C}$ a "category in **Top**".

Let $G$ be a topological group acting on a space $X$. We can again define $XG$, although it is now a category in **Top**. Explicitly, $(XG)_0 = X$ and $(XG)_1 = G \times X$ as spaces. The nerve of a topological category begets a simplicial space. In general, we will have

$$(N\mathcal{C})_n = \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \times \cdots \times_{\mathcal{C}_0} \mathcal{C}_1.$$

The geometric realization functor works in exactly the same way, so the realization of a simplicial space gets a topological space. The above discussion passes through with some mild topological conditions on $G$ (namely, if $G$ is an absolute neighborhood retract of a Lie group); we conclude:

**Theorem 58.6.** *Let $G$ be an absolute neighborhood retract of a Lie group. Then $EG$ is contractible, and $G$ acts from the right principally. Moreover, the map $EG \to BG$ is the orbit projection.*

A generalization of this result is:

**Exercise 58.7.** Let $X$ be a $G$-set. Show that

$$EG \times_G X \simeq B(XG).$$

## 59 Classifying spaces and bundles

Let $\pi : Y \to X$ be a map of spaces. This defines a "descent category" $\check{C}(\pi)$ whose objects are the points of $Y$, whose morphisms are points of $Y \times_X Y$, and whose structure morphisms are the obvious maps. Let $cX$ denote the category whose objects and morphisms are both given by points of $X$, so that the nerve $NcX$ is the constant simplicial object with value $X$. There is a functor $\check{C}(\pi) \to cX$ specified by the map $\pi$.

Let $\mathcal{U}$ be a cover of $X$. Let $\check{C}(\mathcal{U})$ denote the descent category associated to the obvious map $\epsilon : \coprod_{U \in \mathcal{U}} U \to X$. It is easy to see that $\epsilon : B\check{C}(\mathcal{U}) \simeq X$ if $\mathcal{U}$ is numerable. The morphism determined by $x \in U \cap V$ is denoted $x_{U,V}$. Suppose $p : P \to X$ is a principal $G$-bundle. Then $\mathcal{U}$ trivializes $p$ if there are homeomorphisms $t_U : p^{-1}(U) \xrightarrow{\simeq} U \times G$ over $U$. Specifying such homeomorphisms is the same as a trivialization of the pullback bundle $\epsilon^* P$.

This, in turn, is the same as a functor $\theta_P : \check{C}(\mathcal{U}) \to G$. To see this, we note that the $G$-equivariant composite $t_V \circ t_U^{-1} : (U \cap V) \times G \to (U \cap V) \times G$ is determined by the value of $(x, 1) \in (U \cap V) \times G$. The map $U \cap V \to G$ is denoted $f_{U,V}$. Then, the functor $\theta_P : \check{C}(\mathcal{U}) \to G$ sends every object of $\check{C}(\mathcal{U})$ to the point, and $x_{U,V}$ to $f_{U,V}(x)$.

On classifying spaces, we therefore get a map $X \xleftarrow{\simeq} B\check{C}(\mathcal{U}) \xrightarrow{\theta_P} BG$, where the map on the left is given by $\epsilon$.

**Exercise 59.1.** Prove that $\theta_P^* EG \simeq \epsilon^* P$.

This suggests that $BG$ is a classifying space for principal $G$-bundles (in the sense of §55). To make this precise, we need to prove that two principal $G$-bundles are isomorphic if and only if the associated maps $X \to BG$ are homotopic.

To prove this, we will need to vary the open cover. Say that $\mathcal{V}$ *refines* $\mathcal{U}$ if for any $V \in \mathcal{U}$, there exists $U \in \mathcal{U}$ such that $V \subseteq U$. A *refinement* is a function $p : \mathcal{V} \to \mathcal{U}$ such that $V \subseteq p(V)$. A refinement $p$ defines a map $\coprod_{V \in \mathcal{V}} V \to \coprod_{U \in \mathcal{U}} U$, denoted $\rho$.

As both $\coprod_{V \in \mathcal{V}} V$ and $\coprod_{U \in \mathcal{U}} U$ cover $X$, we get a map $\check{C}(\mathcal{V}) \to \check{C}(\mathcal{U})$ over $cX$. Taking classifying spaces begets a diagram:

$$\begin{tikzcd} B\check{C}(\mathcal{V}) \arrow[r] \arrow[dr] & B\check{C}(\mathcal{U}) \arrow[d] \\ & X \end{tikzcd}$$

Let $t$ be trivialization of $P$ for the open cover $\mathcal{U}$. The construction described above begets a functor $B\check{C}(\mathcal{U}) \to BG$, so we get a trivialization $s$ for $\mathcal{V}$. This is a homeomorphism $s_V : p^{-1}(V) \to V \times G$ which fits into the following diagram:

$$\begin{tikzcd}
p^{-1}(V) \arrow[r, "s_V", "\sim"'] \arrow[d] & V \times G \arrow[d] \\
p^{-1}(\rho(V)) \arrow[r, "\sim", "t_{\rho(V)}"'] & \rho(V) \times G
\end{tikzcd}$$

By construction, there is a large commutative diagram:

$$\begin{tikzcd}
B\check{C}(\mathcal{V}) \arrow[r] \arrow[rr, bend left] \arrow[dr, "\sim"] & B\check{C}(\mathcal{U}) \arrow[r] \arrow[d, "\sim"] & BG \\
& X. &
\end{tikzcd}$$

This justifies dropping the symbol $\mathcal{U}$ in the notation for the map $\theta_P$.

Consider two principal $G$-bundles over $X$:

$$\begin{tikzcd}
P \arrow[r, "\simeq"] \arrow[dr] & Q \arrow[d] \\
& X,
\end{tikzcd}$$

and suppose I have trivializations $(\mathcal{U}, t)$ of $P$ and $(\mathcal{W}, s)$ of $Q$. Let $\mathcal{V}$ be a common refinement, so that there is a diagram:

$$\begin{tikzcd}
& \check{C}(\mathcal{U}) \arrow[dr, "\theta_P^{\mathcal{U}}"] & \\
\check{C}(\mathcal{V}) \arrow[ur] \arrow[dr] \arrow[rr, bend left=10, "\theta_P^{\mathcal{V}}"] \arrow[rr, bend right=10, "\theta_Q^{\mathcal{V}}"'] & & G \\
& \check{C}(\mathcal{W}) \arrow[ur, "\theta_Q^{\mathcal{W}}"'] &
\end{tikzcd}$$

Included in the diagram is a mysterious natural transformation $\beta : \theta_P^{\mathcal{V}} \to \theta_Q^{\mathcal{V}}$, whose construction is left as an exercise to the reader. Its existence combined with Lemma 57.4 implies that the two maps $\theta_P, \theta_Q : B\check{C}(\mathcal{V}) \simeq X \to BG$ are homotopic, as desired. Should we describe this? It's rather technical...

### Topological properties of $BG$

Before proceeding, let us summarize the constructions discussed so far. Let $G$ be some topological group (assumed to be an absolute neighborhood retract of a Lie group). We constructed $EG$, which is a contractible space with $G$ acting freely on the right (this works for any topological group). There is an orbit projection $EG \to BG$, which is a principal $G$-bundle under our assumption on $G$. The space $BG$ is universal, in the sense that there is a bijection

$$\mathrm{Bun}_G(X) \xleftarrow{\simeq} [X, BG]$$

given by $f \mapsto [f^*EG]$.

Let $E$ be a space such that $G$ acts on $E$ from the left. If $P \to B$ is any principal $G$-bundle, then $P \times E \to P \times_G E$ is another principal $G$-bundle. In the case $P = EG$, it follows that if $E$ is a contractible space on which $G$ acts, then the quotient $EG \times_G E$ is a model for $BG$. Recall that $EG$ is contractible. Therefore, if $E$ is a contractible space on which $G$ acts freely, then the quotient $G \backslash E$ is a model for $BG$. Of course, one can run the same argument in the case that $G$ acts on $E$ from the right. Although the construction with simplicial sets provided us with a very concrete description of the classifying space of a group $G$, we could have chosen any principal action on a contractible space in order to obtain a model for $BG$.

Suppose $X$ is a pointed path connected space. Remember that $X$ has a contractible path space $PX = X^I_*$. The canonical map $PX \to X$ is a fibration, with fiber $\Omega X$.

Consider the case when $X = BG$. Then, we can compare the above fibration with the fiber bundle $EG \to BG$:

$$\begin{tikzcd}
G \arrow[r] \arrow[d] & \Omega BG \arrow[d] \\
* \simeq EG \arrow[r, dashed] \arrow[d] & PBG \simeq * \arrow[d] \\
BG \arrow[r, equal] & BG
\end{tikzcd}$$

The map $EG \to BG$ is nullhomotopic; a choice of a nullhomotopy is exactly a lift into the path space. Therefore, the dotted map $EG \to PBG$ exists in the above diagram. As $EG$ and $PBG$ are both contractible, we conclude that $\Omega BG$ is weakly equivalent to $G$. In fact, this weak equivalence is a $H$-map, i.e., it commutes up to homotopy with the multiplication on both sides.

**Remark 59.2** (Milnor)**.** If $X$ is a countable CW-complex, then $\Omega X$ is not a CW-complex, but it is *homotopy* equivalent (not just weakly equivalent) to one. Moreover, $\Omega X$ is weakly equivalent to a topological group $GX$ such that $BGX \simeq X$.

## Examples

We claim that $BU(n) \simeq \mathrm{Gr}_n(\mathbf{C}^\infty)$. To see this, let $V_n(\mathbf{C}^\infty)$ is the contractible space of complex $n$-frames in $\mathbf{C}^\infty$, i.e., isometric embeddings of $\mathcal{C}^n$ into $\mathcal{C}^\infty$. The Lie group $U(n)$ acts principally on $V_n(\mathbf{C}^\infty)$ by precomposition, and the quotient $V_n(\mathbf{C}^\infty)/U(n)$ is exactly the Grassmannian $\mathrm{Gr}_n(\mathbf{C}^\infty)$. As $\mathrm{Gr}_n(\mathbf{C}^\infty)$ is the quotient of a principal action of $U(n)$ on a contractible space, our discussion in the previous section implies the desired claim.

Let $G$ be a compact Lie group (eg finite).

**Theorem 59.3** (Peter-Weyl)**.** *There exists an embedding $G \hookrightarrow U(n)$ for some $n$.*

Since $U(n)$ acts principally on $V_n(\mathbf{C}^\infty)$, it follows $G$ also acts principally on $V_n(\mathcal{C}^\infty)$. Therefore $V_n(\mathcal{C}^\infty)/G$ is a model for $BG$. It is not necessarily that this the most economic description of $BG$.

For instance, in the case of the symmetric group $\Sigma_n$, we have a much nicer geometric description of the classifying space. Let $\mathrm{Conf}_n(\mathbf{R}^k)$ denote embeddings of $\{1, \cdots, n\} \to \mathbf{R}^k$ (ordered distinct $n$-tuples). This space is definitely *not* contractible! However, the classifying space $\mathrm{Conf}_n(\mathbf{R}^\infty)$ is contractible. The symmetric group obviously acts freely on this (for finite groups, a principal action is the same as a free action). It follows that $B\Sigma_n$ is the space of *un*ordered configurations of $n$ distinct points in $\mathbf{R}^\infty$. Using Cayley's theorem from classical group theory, we find that if $G$ is finite, a model for $BG$ is the quotient $\mathrm{Conf}_n(\mathbf{R}^\infty)/G$.

We conclude this chapter with a construction of Eilenberg-Maclane spaces via classifying spaces. If $A$ is a topological abelian group, then the multiplication $\mu : A \times A \to A$ is a homomorphism. Applying the classifying space functor begets a map $m : BA \times BA \to BA$. If $G$ is a finite group, then $BA = K(A, 1)$. The map $m$ above gives a topological abelian group model for $K(A, 1)$. There is nothing preventing us from iterating this construction: the space $B^2A$ sits in a fibration

$$BA \to EBA \simeq * \to B^2A.$$

It follows from the long exact sequence in homotopy that the homotopy groups of $B^2A$ are the same as that of $BA$, but shifted up by one. Repeating this procedure multiple times gives us an explicit model for $K(A, n)$:

$$B^nA = K(A, n).$$

# Chapter 6

# Spectral sequences

> Spectral sequences are one of those things for which anybody who is anybody must suffer through. Once you've done that, it's like linear algebra. You stop thinking so much about the 'inner workings' later.
>
> – Haynes Miller

## 60 The spectral sequence of a filtered complex

Our goal will be to describe a method for computing the homology of a chain complex. We will approach this problem by assuming that our chain complex is equipped with a filtration; then we will discuss how to compute the associated graded of an induced filtration on the homology, given the homology of the associated graded of the filtration on our chain complex.

We will start off with a definition.

**Definition 60.1.** A *filtered chain complex* is a chain complex $C_*$ along with a sequence of subcomplexes $F_sC_*$ such that the group $C_n$ has a filtration by

$$F_0C_n \subset F_1C_n \subseteq \cdots ,$$

such that $\bigcup F_sC_n = C_n$.

The differential on $C_*$ begets the structure of a chain complex on the associated graded $\mathrm{gr}_sC_n = F_sC_n/F_{s-1}C_n$; in other words, the differential on $C_*$ respects the filtration, hence begets a differential $d : \mathrm{gr}_sC_n \to \mathrm{gr}_sC_{n-1}$.

The canonical example of a filtered chain complex to keep in mind is the homology of a filtered space (such as a CW-complex). Let $X$ be a filtered space, i.e., a space equipped with a filtration $X_0 \subseteq X_1 \subseteq \cdots$ such that $\bigcup X_n = X$. We then have a filtration of the chain complex $C_*(X)$ by the subcomplexes $C_*(X_n)$.

For ease of notation, let us write

$$E^0_{s,t} = \mathrm{gr}_sC_{s+t} = F_sC_{s+t}/F_{s-1}C_{s+t},$$

so the differential on $C_*$ gives a differential $d^0 : E^0_{s,t} \to E^0_{s,t-1}$. A first approximation to the homology of $C_*$ might therefore be the homology $H_{s+t}(\mathrm{gr}_s C_*)$. We will denote this group by $E^1_{s,t}$. This is the homology of the associated graded of the filtration $F_* C_*$.

We can get an even better approximation to $H_* C_*$ by noticing that there is a differential even on $E^1_{s,t}$. By construction, there is a short exact sequence of chain complexes

$$0 \to F_{s-1}C_* \to F_s C_* \to \mathrm{gr}_s C_* \to 0,$$

so we get a long exact sequence in homology. The differential on $E^1_{s,t}$ is the composite of the boundary map in this long exact sequence with the natural map $H_*(F_{s-1}C_*) \to H_*(\mathrm{gr}_{s-1} C_*)$; more precisely, it is the composite

$$d^1 : E^1_{s,t} = H_{s+t}(\mathrm{gr}_s C_*) \xrightarrow{\partial} H_{s+t-1}(F_{s-1}C_*) \to H_{s+t-1}(\mathrm{gr}_{s-1} C_*) = E^1_{s-1,t}.$$

It is easy to check that $(d^1)^2 = 0$.

This construction is already familiar from cellular chains: in this case, $E^1_{s,t}$ is exactly $H_{s+t}(X_s, X_{s-1})$, which is exactly the cellular $s$-chains when $t = 0$ (and is 0 if $t \neq 0$). The $d^1$ differential is constructed in exactly the same way as the differential on cellular chains.

In light of this, we define $E^2_{s,t}$ to be the homology of the chain complex $(E^1_{*,*}, d^1)$; explicitly, we let

$$E^2_{s,t} = \ker(d^1 : E^1_{s,t} \to E^1_{s-1,t}) / \operatorname{im}(d^1 : E^1_{s+1,t} \to E^1_{s,t}).$$

Does this also have a differential $d^2$? The answer is yes. We will inductively define $E^r_{s,t}$ via a similar formula: if $E^{r-1}_{*,*}$ and the differential $d^{r-1} : E^{r-1}_{s,t} \to E^{r-1}_{s-r+1,t+r-2}$ are both defined, we set

$$E^r_{s,t} = \ker(d^{r-1} : E^{r-1}_{s,t} \to E^{r-1}_{s-r+1,t+r-2}) / \operatorname{im}(d^{r-1} : E^{r-1}_{s+r-1,t-r+2} \to E^{r-1}_{s,t}).$$

The differential $d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$ is defined as follows. Let $[x] \in E^r_{s,t}$ be represented by an element of $x \in E^1_{s,t}$, i.e., an element of $H_{s+t}(\mathrm{gr}_s C_*)$. As above, the boundary map induces natural maps $\partial : H_{s+t}(\mathrm{gr}_s C_*) \to H_{s+t-1}(F_{s-1}C_*)$ and $\partial : H_{s+t-1}(F_{s-r}C_*) \to H_{s+t-1}(\mathrm{gr}_{s-r} C_*)$. The element $\partial x \in H_{s+t-1}(F_{s-1}C_*)$ in fact lifts to an element of $H_{s+t-1}(F_{s-r}C_*)$. The image of this element under $\partial$ inside $H_{s+t-1}(\mathrm{gr}_{s-r} C_*) = E^1_{s-r,t+r-1}$ begets a class in $E^r_{s-r,t+r-1}$; this is the desired differential.

**Exercise 60.2.** Fill in the missing details in this construction of $d^r$, and show that $(d^r)^2 = 0$.

We have proven most of the statements in the following theorem.

**Theorem-Definition 60.3.** *Let $F_* C$ be a filtered complex. Then there exist natural*

1. *bigraded groups $(E^r_{s,t})_{s \geq 0, t \in \mathbf{Z}}$ for any $r \geq 0$, and*

2. *differentials $d^r : E^r_{s,t} \to E^r_{s-r,t+r-1}$ for any $r \geq 0$.*

*such that $E^{r+1}_{s,t}$ is the homology of $(E^r_{*,*}, d^r)$, and $(E^0, d^0)$ and $(E^1, d^1)$ are as above. If $F_*C$ is bounded below, then this spectral sequence* converges to $\mathrm{gr}_* H_*(C)$*, in the sense that there is an isomorphism:*

$$E^\infty_{s,t} \simeq \mathrm{gr}_s H_{s+t}(C). \tag{6.1}$$

This is called a *homology spectral sequence*. One should think of each $E^r_{*,*}$ as a "page", with lattice points $E^r_{s,t}$. We still need to describe the symbols used in the formula (6.1).

There is a filtration $F_s H_n(C) := \mathrm{im}(H_n(F_s C) \to H_n(C))$, and $\mathrm{gr}_s H_*(C)$ is the associated graded of this filtration. Taking formula (6.1) literally, we only obtain information about the associated graded of the homology of $C_*$. Over vector spaces, this is sufficient to determine the homology of $C_*$, but in general, one needs to solve an extension problem.

To define the notation $E^\infty$ used above, let us assume that the filtration $F_*C$ is bounded below (so $F_{-1}C = 0$). It follows that $E^0_{s,t} = F_s C_{s+t}/F_{s-1}C_{s+t} = 0$ for $s < 0$, so the spectral sequence of Theorem-Definition 60.3 is a "right half plane" spectral sequence. It follows that in our example, the differentials from the group in position $(s,t)$ must have vanishing $d^{s+1}$ differential.

In turn, this implies that there is a surjection $E^{s+1}_{s,t} \to E^{s+2}_{s,t}$. This continues: we get surjections

$$E^{s+1}_{s,t} \to E^{s+2}_{s,t} \to E^{s+3}_{s,t} \to \cdots,$$

and the direct limit of this directed system is defined to be $E^\infty_{s,t}$.

For instance, in the case of cellular chains, we argued above that $E^1_{s,t} = H_{s+t}(X_s, X_{s-1})$, so that $E^1_{s,t} = 0$ if $t \neq 0$, and the $d^1$ differential is just the differential in the cellular chain complex. It follows that $E^2_{s,t} = H^{cell}_s(X)$ if $t = 0$, and is 0 if $t \neq 0$. All higher differentials are therefore zero (because either the target or the source is zero!), so $E^r_{s,t} = E^2_{s,t}$ for every $r \geq 2$. In particular $E^\infty_{s,t} = H^{cell}_s(X)$ when $t = 0$, and is 0 if $t \neq 0$. There are no extension problems either: the filtration on $X$ is bounded below, so Theorem-Definition 60.3 implies that $\mathrm{gr}_s H_{s+0}(X) = H_s(X) \simeq H^{cell}_s(X) = E^\infty_{s,t}$.

In a very precise sense, the datum of the spectral sequence of a filtered complex $F_*C_*$ determines the homology of $C_*$:

**Corollary 60.4.** *Let $C \xrightarrow{f} D$ be a map of filtered complexes. Assume that the filtration on $C$ and $D$ are bounded below and exhaustive. Assume also that $E^r(f)$ is an isomorphism for some $r$. Then $f_* : H_*(C) \to H_*(D)$ is an isomorphism.*

*Proof.* The map $E^r(f)$ is an isomorphism which is also also a chain map, i.e., it is compatible with the differential $d^r$. It follows that $E^{r+1}(f)$ is an isomorphism. By induction, we conclude that $E^\infty_{s,t}(f)$ is an isomorphism for all $s, t$. Theorem-Definitino 60.3 implies that the map $\mathrm{gr}_s(f_*) : \mathrm{gr}_s H_*(C) \to \mathrm{gr}_s H(D)$ is an isomorphism.

We argue by induction using the short exact sequence:

$$0 \to F_s H_*(C) \to F_{s+1} H_*(C) \to \mathrm{gr}_{s+1} H_*(C) \to 0.$$

We have $\mathrm{gr}_0 H_n(C) = F_0 H_n(C) = \mathrm{im}(H_n(F_0 C) \to H_n(C))$, so the base case follows from the five lemma. In general, $f$ induces an isomorphism an isomorphism on the groups on the left (by the inductive hypothesis) and right (by the above discussion), so it follows that $F_s f_*$ is an isomorphism by the five lemma. Since the filtration $F_* C_*$ was exhaustive, it follows that $f_*$ is an isomorphism. $\square$

## Serre spectral sequence

In this book, we will give two constructions of the Serre spectral sequence. The second will appear later. Fix a fibration $E \xrightarrow{p} B$, with $B$ a CW-complex. We obtain a filtration on $E$ by taking the preimage of the $s$-skeleton of $B$, i.e., $E_s = p^{-1}\mathrm{sk}_s B$. It follows that there is a filtration on $S_*(E)$ given by

$$F_s S_*(E) = \mathrm{im}(S_*(p^{-1}\mathrm{sk}_s(B)) \to S_* E).$$

This filtration is bounded below and exhaustive. The resulting spectral sequence of Theorem-Definition 60.3 is the Serre spectral sequence.

Let us be more explicit. We have a pushout square:

$$\begin{tikzcd}
E_{s-1} \arrow[r] \arrow[d] & E_s \arrow[d] \\
B_{s-1} \arrow[r] & B_s \\
\coprod_{\alpha\in\Sigma_s} S^{s-1}_\alpha \arrow[r] \arrow[u] & \coprod_{\alpha\in\Sigma_s} D^s_\alpha \arrow[u]
\end{tikzcd}$$

Let $F_\alpha$ be the preimage of the center of $\alpha$ cell. In particular, we have a pushout:

$$\begin{tikzcd}
E_{s-1} \arrow[r] & E_s \\
\coprod_{\alpha\in\Sigma_s} S^{s-1}_\alpha \times F_\alpha \arrow[r] \arrow[u] & \coprod_{\alpha\in\Sigma_s} D^s_\alpha \times F_\alpha \arrow[u]
\end{tikzcd}$$

We know that

$$E^1_{s,t} = H_{s+t}(E_s, E_{s-1}) = \bigoplus_{\alpha\in\Sigma_s} H_{s+t}(D^s_\alpha \times F_\alpha, S^{s-1}_\alpha \times F_\alpha).$$

We can suggestively view this as $\bigoplus_{\alpha\in\Sigma_s} H_{s+1}((D^s_\alpha, S^{s-1}_\alpha)\times F_\alpha)$. By the Künneth formula (at least, if our coefficients are in a field), this is exactly $\bigoplus_{\alpha\in\Sigma_s} H_t(F_\alpha)$. In analogy with our discussion above regarding the spectral sequence coming from the cellular chain complex, one would like to think of this as "$C_s(B; H_t(F_\alpha))$". Sadly, there are many things wrong with writing this.

For instance, suppose $B$ isn't connected. The fibers $F_\alpha$ could have completely different homotopy types, so the symbol $C_s(B; H_t(F_\alpha))$ does not make any sense. Even if

$B$ was path-connected, there would still be no canonical way to identify the fibers over different points. Instead, we obtain a functor $H_t(p^{-1}(-)) : \Pi_1(B) \to \mathbf{Ab}$, i.e., a "local coefficient system" on $B$. So, the right thing to say is "$E^2_{s,t} = H_s(B; \underline{H_t(\text{fiber})})$".

To define precisely what $H_s(B; \underline{H_t(\text{fiber})})$ means, let us pick a basepoint in $B$, and build the universal cover $\widetilde{B} \to B$. This has an action of $\pi_1(B, *)$, so we obtain an action of $\pi_1(B, *)$ on the chain complex $S_*(\widetilde{B})$. Said differently, $S_*(\widetilde{B})$ is a chain complex of right modules over $\mathbf{Z}[\pi_1(B)]$. If $B$ is connected, a local coefficient system on $B$ is the same thing as a (left) action of $\pi_1(B)$ on $H_t(p^{-1}(*))$. Then, we define a chain complex:

$$S_*(B; \underline{H_t(p^{-1}(*))}) = S_*(\widetilde{B}) \otimes_{\mathbf{Z}[\pi_1(B)]} H_t(p^{-1}(*));$$

the differential is induced by the $\mathbf{Z}[\pi_1(B)]$-equivariant differential on $S_*(\widetilde{B})$. Our discussion above implies that the homology of this chain complex is the $E^2$-page.

We will always be in the case where that local system is trivial, so that $H_*(B; \underline{H_*(p^{-1}(*))})$ is just $H_*(B; H_*(p^{-1}(*)))$. For instance, this is the case if $\pi_1(B)$ acts trivially on the fiber. In particular, this is the case if $B$ is simply connected.

# 61 Exact couples

Let us begin with a conceptual discussion of exact couples. As a special case, we will recover the construction of the spectral sequence associated to a filtered chain complex (Theorem-Definition 60.3).

**Definition 61.1.** An *exact couple* is a diagram of (possilby (bi)graded) abelian groups

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

which is exact at each joint.

As $jkjk = 0$, the map $E \xrightarrow{jk} E$ is a differential, denoted $d$. An exact couple determines a "derived couple":

$$\begin{tikzcd} A' \arrow[rr, "i' = i|_{\mathrm{im}\, i}"] & & A' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd} \tag{6.2}$$

where $A' = \mathrm{im}(i)$ and $E' = H_*(E, d)$. Iterating this procedure, we get exact sequences

$$\begin{tikzcd} A^r \arrow[rr, "i_r"] & & A^r \arrow[dl, "j_r"] \\ & E^r \arrow[ul, "k_r"] & \end{tikzcd}$$

where the next exact couple is the derived couple of the preceding exact couple.

It remains to define the maps in the above diagram. Define $j'(ia) = ja$. *A priori*, it is not clear that this well-defined. For one, we need $[ja] \in E'$; for this, we must check that $dja = 0$, but $d = jk$, and $jkja = 0$ so this follows. We also need to check that $j'$ is well-defined modulo boundaries. To see this, suppose $ia = 0$. We then need to know that $ja$ is a boundary. But if $ia = 0$, then $a = ke$ for some $e$, so $ja = jke = de$, as desired.

Define $k' : H(E, d) \to \operatorname{im} i$ via $k'([e]) \mapsto ke$. As before, we need to check that this is well-defined. For instance, we have to check that $ke \in \operatorname{im} i$. Since $de = 0$ and $d = jk$, we learn that $jke = 0$. Thus $ke$ is killed by $j$, and therefore, by exactness, is in the image of $i$. We also need to check that $k'$ is independent of the choice of representative of the homology class. Say $e = de'$. Then $kd = kde' = kjke' = 0$.

**Exercise 61.2.** Check that these maps indeed make diagram (6.2) into an exact couple.

It follows that we obtain a spectral sequence, in the sense of Theorem-Definition 60.3.

**Exercise 61.3.** By construction,

$$A^r = \operatorname{im}(i^r|_A) = i^r A.$$

Show, by induction, that

$$E^r = \frac{k^{-1}(i^r A)}{j(\ker i^r)}$$

and that

$$i_r(a) = ia,\ j_r(i^r a) = [ja],\ k_r(e) = ke.$$

Intuitively: an element of $E^1$ will survive to $E^r$ if its image in $A^1$ can be pulled back under $i^{r-1}$. The differential $d^r$ is obtained by the homology class of the pushforward of this preimage via $j$ to $E^1$.

**Remark 61.4.** In general, the groups in consideration will be bigraded. It is clear by construction that $\deg(i') = \deg(i)$, $\deg(k') = \deg(k)$, and $\deg(j') = \deg(j) - \deg(i)$. It follows by an easy inductive argument that

$$\deg(d^r) = \deg(j) + \deg(k) - (r-1)\deg(i).$$

The canonical example of an exact couple is that of a filtered complex; the resulting spectral sequence is precisely the spectral sequence of Theorem-Definition 60.3. If $C_*$ is a filtered chain complex, we let $A_{s,t} = H_{s+t}(F_s C_*)$, and $E^1_{s,t} = E_{s,t} = H_{s+t}(\mathrm{gr}_s C_*)$. The exact couple is precisely that which arises from the long exact sequence in homology associated to the short exact sequence of chain complexes

$$0 \to F_{s-1}C_* \to F_s C_* \to \mathrm{gr}_s C_* \to 0.$$

Note that in this case, the exact couple is one of bigraded groups, so Remark 61.4 dictates the bidegrees of the differentials.

We will conclude this section with a brief discussion of the convergence of the spectral sequence constructed above. Assume that $i : A \to A$ satisfies the property that

$$\ker(i) \cap \bigcap i^r A = 0.$$

Let $\widetilde{A}$ be the colimit of the directed system

$$A \xrightarrow{i} A \xrightarrow{i} A \to \cdots$$

There is a natural filtration on $\widetilde{A}$. Let $I$ denote the image of the map $A \to \widetilde{A}$; the kernel of this map is $\bigcup \ker(i^r)$. The groups $i^r I$ give an exhaustive filtration of $\widetilde{A}$, and the quotients $i^r I/i^{r+1} I$ are all isomorphic to $I/iI$ (since $i$ is an isomorphism on $\widetilde{A}$). Then we have an isomorphism

$$E^\infty \simeq I/iI. \tag{6.3}$$

Indeed, we know from Exercise 61.3 that

$$E^\infty \simeq \frac{k^{-1}\left(\bigcap i^r A\right)}{j\left(\bigcup \ker i^r\right)};$$

by our assumption on $i$, this is

$$\frac{\ker(k)}{j\left(\bigcup \ker i^r\right)} \simeq \frac{j(A)}{j\left(\bigcup \ker i^r\right)}.$$

But there is an isomorphism $A/iA \to j(A)$ which clearly sends $iA+\bigcup \ker i^r$ to $j\left(\bigcup \ker i^r\right)$. By our discussion above, $A/\bigcup \ker i^r \simeq I$, and $iA/\bigcup \ker i^r \simeq iI$. Modding out by $iI$ on both sides, we get (6.3).

## 62 The homology of $\Omega S^n$, and the Serre exact sequence

The goal of this section is to describe a computation of the homology of $\Omega S^n$ via the Serre spectral sequence, as well as describe a "degenerate" case of the Serre spectral sequence.

### The homology of $\Omega S^n$

Let us first consider the case $n = 1$. The space $\Omega S^1$ is the base of a fibration $\Omega S^1 \to PS^1 \to S^1$. Comparing this to the fibration $\mathbf{Z} \to \mathbf{R} \to S^1$, we find that $\Omega S^1 \simeq \mathbf{Z}$. Equivalently, this follows from the discussion in §59 and the observation that $S^1 \simeq K(\mathbf{Z}, 1)$.

Having settled that case, let us now consider the case $n > 1$. Again, there is a fibration $\Omega S^n \to PS^n \to S^n$. In general, if $F \to E \to B$ is a fibration and the space

$F$ has torsion-free homology, we can (via the universal coefficients theorem) rewrite the $E^2$-page:

$$E^2_{s,t} = H_s(B; H_t(F)) \simeq H_s(B) \otimes H_t(F).$$

Since $S^n$ has torsion-free homology, the Serre spectral sequence (see §60) runs:

$$E^2_{s,t} = H_s(S^n) \otimes H_t(\Omega S^n) \Rightarrow H_*(PS^n) = \mathbf{Z}.$$

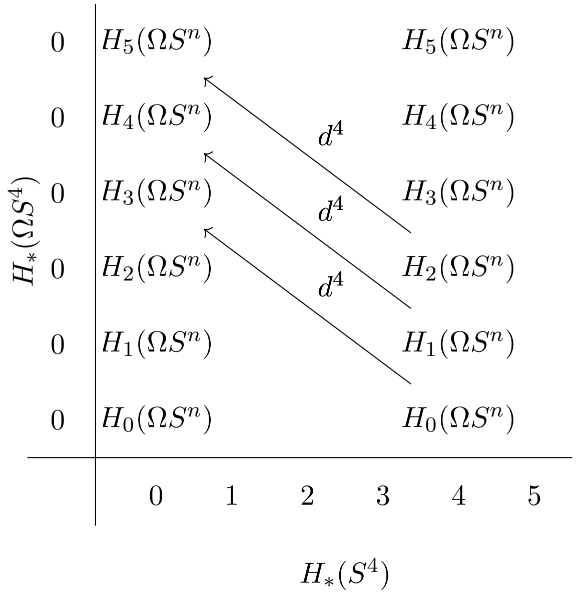
Since $H_s(S^n)$ is concentrated in degrees 0 and $n$, we learn that $E^2$-page is concentrated in columns $s = 0, n$. For instance, if $n = 4$, then the $E^2$-page (without the differentials drawn in) looks like:

$$
H_*(\Omega S^4)\quad
\begin{array}{c|cccccc}
0 & H_5(\Omega S^n) & & & & H_5(\Omega S^n) & \\
0 & H_4(\Omega S^n) & & & & H_4(\Omega S^n) & \\
0 & H_3(\Omega S^n) & & & & H_3(\Omega S^n) & \\
0 & H_2(\Omega S^n) & & & & H_2(\Omega S^n) & \\
0 & H_1(\Omega S^n) & & & & H_1(\Omega S^n) & \\
0 & H_0(\Omega S^n) & & & & H_0(\Omega S^n) & \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5
\end{array}
$$

$$H_*(S^4)$$

We know that $H_0(\Omega S^n) = \mathbf{Z}$. Since the target has homology concentrated in degree 0, we know that $E^2_{n,0}$ has to be killed. The only possibility is that it is hit by a differential, or that it supports a nonzero differential.

There are not very many possibilities for differentials in this spectral sequence. In fact, up until the $E^n$-page, there are no differentials (either the target or source of the differential is zero), so $E^2 \simeq E^3 \simeq \cdots \simeq E^n$. On the $E^n$-page, there is only one possibility for a differential: $d^n : E^2_{n,0} \to E^n_{0,n-1}$. This differential has to be a monomorphism because if it had anything in its kernel, that will be left over in the position. In our example above (with $n = 4$), we have




However, we still do not know the group $E^n_{0,n-1}$. If it is bigger than $\mathbf{Z}$, then $d^n$ is not surjective. There can be no other differentials on the $E^r$-page for $r \geq n+1$ (because of sparsity), so the $d^n$ differential is our last hope in killing everything in degree $(0, n-1)$. This means that $d^n$ is an epimorphism. We find that $E^n_{0,n-1} = H_{n-1}(\Omega S^n) \simeq \mathbf{Z}$, and that $d^n$ is an isomorphism.

We have now discovered that $H_{n-1}(\Omega S^n) \simeq \mathbf{Z}$ — but there is a lot more left in the $E^2$-page! For instance, we still have a $\mathbf{Z}$ in $E^n_{n,n-1}$. Because $H^*(PS^n)$ is concentrated in degree 0, this, too, must die! We are in exactly the same situation as before, so the same arguments show that the differential $d^n : E^n_{n,n-1} \to E^n_{0,2(n-1)}$ has to be an isomorphism. Iterating this argument, we find:

$$H_q(\Omega S^n) \simeq \begin{cases} \mathbf{Z} & \text{if } (n-1)|q \geq 0 \\ 0 & \text{else} \end{cases}$$

This is a great example of how useful spectral sequences can be.

**Remark 62.1.** The loops $\Omega X$ is an associative $H$-space. Thus, as is the case for any $H$-space, the homology $H_*(\Omega X; R)$ is a graded associative algebra. Recall that the suspension functor $\Sigma$ is the left adjoint to the loops functor $\Omega$, so there is a unit map $A \to \Omega\Sigma A$. This in turn begets a map $\widetilde{H}_*(A) \to H_*(\Omega\Sigma A)$.

Recall that the universal tensor algebra $\mathrm{Tens}(\widetilde{H}_*(A))$ is the free associative algebra on $\widetilde{H}_*(A)$. Explicitly:

$$\mathrm{Tens}(\widetilde{H}_*(A)) = \bigoplus_{n \geq 0} \widetilde{H}_*(A)^{\otimes n}.$$

In particular, by the universal property of $\mathrm{Tens}(\widetilde{H}_*(A))$, we get a map $\alpha : \mathrm{Tens}(\widetilde{H}_*(A)) \to H_*(\Omega\Sigma A)$.

**Theorem 62.2** (Bott-Samelson). *The map $\alpha$ is an isomorphism if $R$ is a PID and $H_*(A)$ is torsion-free.*

For instance, if $A = S^{n-1}$ then $\Omega S^n = \Omega\Sigma A$. Theorem 62.2 then shows that

$$H_*(\Omega S^n) = \mathrm{Tens}(\widetilde{H}_*(S^{n-1})) = \langle 1, x, x^2, x^3, \cdots \rangle,$$

where $|x| = n - 1$. It is a mistake to call this "polynomial", since if $n$ is even, $x$ is an odd class (in particular, $x$ squares to zero by the Koszul sign rule).

Theorem 62.2 suggests thinking of $\Omega\Sigma A$ as the "free associative algebra" on $A$. Let us make this idea more precise.

**Remark 62.3.** The space $\Omega A$ is homotopy equivalent to a topological monoid $\Omega_M A$, called the *Moore loops* on $A$. This means that $\Omega_M A$ has a *strict* unit and is *strictly* associative (i.e., not just up to homotopy). Concretely,

$$\Omega_M A := \{(\ell, \omega) : \ell \in \mathbf{R}_{\geq 0}, \omega : [0, \ell] \to A, \omega(0) = * = \omega(\ell)\},$$

topologized as a subspace of the product. There is an identity class $1 \in \Omega_M A$, given by $1 = (0, c_*)$ where $c_*$ is the constant loop at the basepoint $*$. The addition on this space is just given by concatenatation. In particular, the lengths get added; this overcomes the obstruction to $\Omega A$ not being strictly associative, so the Moore loops $\Omega_M A$ are indeed strictly associative. If the basepoint is nondegenerate, it is not hard to see that the inclusion $\Omega A \hookrightarrow \Omega_M A$ is a homotopy equivalence.

Given the space $A$, we can form the free monoid $\mathrm{FreeMon}(A)$. The elements of this space are just formal sequences of elements of $A$ (with topology coming from the product topology), and the multiplication is given by juxtaposition. Let us adjoin the element $1 = *$. As with all free constructions, there is a map $A \to \mathrm{FreeMon}(A)$ which is universal in the sense that any map $A \to M$ to a monoid factors through $\mathrm{FreeMon}(A)$.

The unit $A \to \Omega\Sigma A$ is a map from $A$ to a monoid, so we get a monoid map $\beta : \mathrm{FreeMon}(A) \to \Omega\Sigma A$.

**Theorem 62.4** (James). *The map $\beta : \mathrm{FreeMon}(A) \to \Omega\Sigma A$ is a weak equivalence if $A$ is path-connected.*

The free monoid looks very much like the tensor product, as the following theorem of James shows.

**Theorem 62.5** (James). *Let $J(A) = \mathrm{FreeMon}(A)$. There is a splitting:*

$$\Sigma J(A) \simeq_w \Sigma \left( \bigvee_{n \geq 0} A^{\wedge n} \right).$$

Applying homology to the splitting of Theorem 62.5 shows that:

$$\widetilde{H}_*(J(A)) \simeq \bigoplus_{n \geq 0} \widetilde{H}_*(A^{\wedge n}).$$

Assume that our coefficients are in a PID, and that $\widetilde{H}_*(A)$ is torsion-free; then this is just $\bigoplus_{n\geq 0}\widetilde{H}_*(A)^{\otimes n}$. In particular, we recover our computation of $H_*(\Omega S^n)$ from these general facts.

### The Serre exact sequence

Suppose $\pi : E \to B$ is a fibration over a path-connected base. Assume that $\widetilde{H}_s(B) = 0$ for $s < p$ where $p \geq 1$. Let $* \in B$ be a chosen basepoint. Denote by $F$ the fiber $\pi^{-1}(*)$. Assume $\widetilde{H}_t(F) = 0$ for $t < q$, where $q \geq 1$. We would like to use the Serre spectral sequence to understand $H_*(E)$. As always, we will assume that $\pi_1(B)$ acts trivially on $H_*(F)$.

Recall that the Serre spectral sequence runs

$$E^2_{s,t} = H_s(B; H_t(F)) \Rightarrow H_{s+t}(E).$$

Our assumptions imply that $E^2_{0,0} = \mathbf{Z}$, and $E^2_{0,t} = 0$ for $t < q$. Moreover, $E^2_{s,0} = 0$ for $s < p$. In particular, $E^2_{0,q+t} = H_{q+t}(F)$ and $E^2_{p+k,0} = H_{p+k}(B)$ — the rest of the spectral sequence is mysterious.

By sparsity, the first possible differential is $d^p : H_p(B) \to H_{p-1}(F)$, and $d^{p+q}$ : $H_{p+1}(B) \to H_p(F)$. In the mysterious zone, there are differentials that hit $E^2_{p,q}$.

Again by sparsity, the only differential is $d^s : E^s_{s,0} \to E^s_{0,s-1}$ for $s < p + q - 1$. This is called a *transgression*. It is the last possible differential which has a chance at being nonzero. This means that the cokernel of $d^s$ is $E^\infty_{0,s-1}$. There is also a map $E^\infty_{s,0} \to E^s_{s,0}$. We obtain a mysterious composite

$$0 \to E^\infty_{s,0} \to E^s_{s,0} \simeq H_s(B) \xrightarrow{d^s} E^s_{0,s-1} \simeq H_{s-1}(F) \to E^\infty_{0,s-1} \to 0. \tag{6.4}$$

Let $n < p + q - 1$. Recall that $F_s H_n(E) = \operatorname{im}(H_*(\pi^{-1}(\mathrm{sk}_s(B))) \to H_*(E))$, so $F_0 H_n(E) = E^\infty_{0,n}$. Here, we are using the fact that $F_{-1}H_*(E) = 0$. In particular, there is a map $E^\infty_{0,n} \to H_n(E)$. By our hypotheses, there is only one other potentially nonzero filtration in this range of dimensions, so we have a short exact sequence:

$$0 \to F_0 H_n(E) = E^\infty_{0,n} \to H_n(E) \to E^\infty_{n,0} \to 0 \tag{6.5}$$

Splicing the short exact sequences (6.4) and (6.5), we obtain a long exact sequence:

$$H_{p+q-1}(F) \to \cdots \to H_n(F) \to H_n(E) \to H_n(B) \xrightarrow{\text{transgression}} H_{n-1}(F) \to H_{n-1}(E) \to \cdots$$

This is called the *Serre exact sequence*. In this range of dimensions, homology behaves like homotopy.

## 63 Edge homomorphisms, transgression

Recall the Serre spectral sequence for a fibration $F \to E \to B$ has $E^2$-page given by

$$E^2_{s,t} = H_s(B; H_t(F)) \Rightarrow H_{s+t}(E).$$

If $B$ is path-connected, $\widetilde{H}_t(F) = 0$ for $t < q$, $\widetilde{H}_s(B) = 0$ for $s < p$, and $\pi_1(B)$ acts trivially on $H_*(F)$, we showed that there is a long exact sequence (the Serre exact sequence)

$$H_{p+q-1}(F) \xrightarrow{\bullet} H_{p+q-1}(E) \to H_{p+q-1}(B) \to H_{p+q-2}(F) \to \cdots \tag{6.6}$$

Let us attempt to describe the arrow marked by $\bullet$.

Let $(E^r_{p,q}, d^r)$ be any spectral sequence such that $E^r_{p,q} = 0$ if $p < 0$ or $q < 0$; such a spectral sequence is called a *first quadrant* spectral sequence. The Serre spectral sequence is a first quadrant spectral sequence. In a first quadrant spectral sequence, the $d^2$-differential $d^2 : E^2_{0,t} \to E^2_{-2,t+1}$ is zero, since $E^2_{s,t}$ vanishes for $s < 0$. This means that $H_t(F) = H_0(B; H_t(F)) = E^2_{0,t}$ surjects onto $E^3_{0,t}$. Arguing similarly, this surjects onto $E^4_{0,t}$. Eventually, we find that $E^r_{0,t} \simeq E^{t+2}_{0,t}$ for $r \geq t+2$. In particular,

$$E^{t+2}_{0,t} \simeq E^\infty_{0,t} \simeq \mathrm{gr}_0 H_t(E) \simeq F_0 H_t(E),$$

which sits inside $H_t(E)$. The composite

$$E^2_{0,t} = H_t(F) \to E^3_{0,t} \to \cdots \to E^{t+2}_{0,t} \subseteq F_0 H_t(E) \to H_t(E)$$

is precisely the map $\bullet$! Such a map is known as an *edge homomorphism.*

The map $F \to E$ is the inclusion of the fiber; it induces a map $H_t(F) \to H_t(E)$ on homology. We claim that this agrees with $\bullet$. Recall that $F_0 H_t(E)$ is defined to be $\mathrm{im}(H_t(F_0 E) \to H_t(E))$. In the construction of the Serre spectral sequence, we declared that $F_0 E$ is exactly the preimage of the zero skeleton. Since $B$ is simply connected, we find that $F_0 E$ is exactly the fiber $F$.

To conclude the proof of the claim, consider the following diagram:

$$\begin{tikzcd} F \arrow[r] \arrow[d] & F \arrow[d] \\ F \arrow[r] \arrow[d] & E \arrow[d] \\ * \arrow[r, hook] & B \end{tikzcd}$$

The naturality of the Serre spectral sequence implies that there is an induced map of spectral sequences. Tracing through the symbols, we find that this observation proves our claim.

The long exact sequence (6.6) also contains a map $H_s(E) \to H_s(B)$. The group $F_s H_s(E) = H_s(E)$ maps onto $\mathrm{gr}_s H_s(E) \simeq E^\infty_{s,0}$. If $F$ is connected, then $H_s(B) = H_s(B; H_0(F)) = E^2_{s,0}$. Again, the $d^2$-differential $d^2 : E^2_{s+2,-1} \to E^2_{s,0}$ is trivial (since the source is zero). Since $E^3 = \ker d^2$, we have an injection $E^3_{s,0} \to E^2_{s,0}$. Repeating the same argument, we get injections

$$E^\infty_{s,0} = E^{s+1}_{s,0} \to \cdots \to E^2_{s,0} \to E^2_{s,0} = H_s(B).$$

Composing with the map $H_s(E) \to E^\infty_{s,0}$ gives the desired map $H_s(E) \to H_s(B)$ in the Serre exact sequence. This composite is also known as an edge homomorphism.

As above, this edge homomorphism is the map induced by $E \to B$. This can be proved by looking at the induced map of spectral sequences coming from the following map of fiber sequences:

$$\begin{tikzcd} F \arrow[r] \arrow[d] & * \arrow[d] \\ E \arrow[r] \arrow[d] & B \arrow[d] \\ B \arrow[r] & B \end{tikzcd}$$

The topologically mysterious map is the boundary map $\partial : H_{p+q-1}(B) \to H_{p+q-2}(F)$. Such a map is called a *transgression*. Again, let $(E^r_{s,t}, d^r)$ be a first quadrant spectral sequence. In our case, $E^2_{n,0} = H_n(B)$, at least $F$ is connected. As above, we have injections

$$i : E^n_{n,0} \to \cdots \to E^3_{n,0} \to E^2_{n,0} = H_n(B).$$

Similarly, we have surjections

$$s : E^2_{0,n-1} \to E^3_{0,n-1} \to \cdots \to E^n_{0,n-1}.$$

There is a differential $d^n : E^n_{n,0} \to E^n_{0,n-1}$. The transgression is defined as the *linear relation* (not a function!) $E^2_{n,0} \to E^2_{0,n-1}$ given by

$$x \mapsto i^{-1} d^n s^{-1}(x).$$

However, the reader should check that in our case, the transgression is indeed a well-defined function.

Topologically, what is the origin of the transgression? There is a map $H_n(E, F) \xrightarrow{\pi_*} H_n(B, *)$, as well as a boundary map $\partial : H_n(E, F) \to H_{n-1}(F)$. We claim that:

$$\operatorname{im} \pi_* = \operatorname{im}(E^n_{n,0} \to H_n(B) = E^2_{n,0}), \quad \partial \ker \pi_* = \ker(H_{n-1}(F) = E^2_{0,n-1} \to E^n_{0,n-1}).$$

*Proof sketch.* Let $x \in H_n(B)$. Represent it by a cycle $c \in Z_n(B)$. Lift it to a chain in the total space $E$. In general, this chain will not be a cycle (consider the Hopf fibration). The differentials record this boundary; let us recall the geometric construction of the differential. Saying that the class $x$ survives to the $E^n$-page is the same as saying that we can find a lift to a chain $\sigma$ in $E$, with $d\sigma \in S_{n-1}(F)$. Then $d^n(x)$ is represented by the class $[dc] \in H_{n-1}(F)$. This is precisely the trangression.

Informally, we lift something from $H_n(B)$ to $S_n(E)$; this is well-defined up to something in $F$. In particular, we get an element in $H_n(E, F)$. We send it, via $\partial$, to an element of $H_{n-1}(F)$ — and this is precisely the transgression. $\square$

### An example

We would like to compare the Serre exact sequence (6.6) with the homotopy exact sequence:

$$* \to \pi_{p+q-1}(F) \to \pi_{p+q-1}(E) \to \pi_{p+q-1}(B) \xrightarrow{\partial} \pi_{p+q-2}(F) \to \cdots$$

There are Hurewicz maps $\pi_{p+q-1}(X) \to H_{p+q-1}(X)$. We claim that there is a map of exact sequences between these two long exact sequences.

$$\begin{tikzcd}
H_{p+q-1}(E) \arrow[r, "\pi_*"] & H_{p+q-1}(B) \arrow[r, "\partial"'] & H_{p+q-2}(F) \arrow[r] & \cdots \\
\pi_{p+q-1}(E) \arrow[r, "\pi_*"'] \arrow[u, "h"'] & \pi_{p+q-1}(B) \arrow[r] \arrow[u, "h"] & \pi_{p+q-2}(F) \arrow[r] \arrow[u, "h"] & \cdots
\end{tikzcd}$$

The leftmost square commutes by naturality of Hurewicz. The commutativity of the righmost square is not immediately obvious. For this, let us draw in the explicit maps in the above diagram:

$$\begin{tikzcd}
 & & H_{p+q-1}(E,F) \arrow[dl] \arrow[dr] & & \\
H_{p+q-1}(E) \arrow[r, "\pi_*"] & H_{p+q-1}(B) \arrow[rr, "\partial"'] & & H_{p+q-2}(F) \arrow[r] & \cdots \\
\pi_{p+q-1}(E) \arrow[r, "\pi_*"'] \arrow[u, "h"'] \arrow[dr] & \pi_{p+q-1}(B) \arrow[rr] \arrow[u, "h"] & & \pi_{p+q-2}(F) \arrow[r] \arrow[u, "h"] & \cdots \\
 & \pi_{p+q-1}(E,F) \arrow[u, "s"', "\cong"] \arrow[uuur] \arrow[urr] & & &
\end{tikzcd}$$

The map marked $s$ is an isomorphism (and provides the long arrow in the above diagram, which makes the square commute), since

$$\pi_n(E,F) = \pi_{n-1}(\mathrm{hofib}(F \to E)) = \pi_{n-1}(\Omega B) = \pi_n(B).$$

Let us now specialize to the case of the fibration

$$\Omega X \to PX \to X.$$

Assume that $X$ is connected, and $* \in X$ is a chosen basepoint. Let $p \geq 2$, and suppose that $\widetilde{H}_s(X) = 0$ for $s < p$. Arguing as in §62, we learn that the Serre spectral sequence we know that the homology of $\Omega X$ begins in dimension $p-1$ since $PX \simeq *$, so $q = p-1$. Likewise, if we knew $\widetilde{H}_n(\Omega X) = 0$ for $n < p-1$, then the same argument shows that $\widetilde{H}_n(X) = 0$ for $n < p$.

### A surprise gust: the Hurewicz theorem

The discussion above gives a proof of the Hurewicz theorem; this argument is due to Serre.

**Theorem 63.1** (Hurewicz, Serre's proof). *Let $p \geq 1$. Suppose $X$ is a pointed space with $\pi_i(X) = 0$ for $i < p$. Then $\widetilde{H}_i(X) = 0$ for $i < p$ and $\pi_p(X)^{ab} \to H_p(X)$ is an isomorphism.*

*Proof.* Let us assume the case $p = 1$. This is classical: it is Poincaré's theorem. We will only use this result when $X$ is a loop space, in which case the fundamental group is already abelian.

Let us prove this by induction, using the loop space fibration. By assumption, $\pi_i(\Omega X) = 0$ for $i < p-1$. By our inductive hypothesis, $\widetilde{H}_i(\Omega X) = 0$ for $i < p-1$, and $\pi_{p-1}(\Omega X) \xrightarrow{\simeq} H_{p-1}(\Omega X)$. By our discussion above, we learn that $\widetilde{H}_i(X) = 0$ for $i < p$. The Hurewicz map $\pi_p(X) \xrightarrow{h} H_p(X)$ fits into a commutative diagram:

$$\begin{tikzcd}
\pi_{p-1}(\Omega X) \arrow[r, "\simeq"] & H_{p-1}(\Omega X) \\
\pi_p(X) \arrow[u, "\simeq"] \arrow[r, "h"'] & H_p(X) \arrow[u, "\simeq"', "\text{transgression}"]
\end{tikzcd}$$

It follows from the Serre exact sequence that the transgression is an isomorphism. $\square$

## 64 Serre classes

**Definition 64.1.** A class $\mathbf{C}$ of abelian groups is a *Serre class* if:

1. $0 \in \mathbf{C}$.
2. if I have a short exact sequence $0 \to A \to B \to C \to 0$, then $A \& C \in \mathbf{C}$ if and only if $B \in \mathbf{C}$.

Some consequences of this definition: a Serre class is closed under isomorphisms (easy). A Serre class is closed under subobjects and quotients, because there is a short exact equence

$$0 \to A \hookrightarrow B \to B/A \to 0.$$

Consider an exact sequence $A \to B \to C$ (not necessarily a *short* exact sequence). If $A, C \in \mathbf{C}$, then $B \in \mathbf{C}$ because we have a short exact sequence:

$$\begin{tikzcd}
 & & & \operatorname{coker} i \arrow[r] \arrow[d] & 0 \\
 & A \arrow[r, "i"] \arrow[d] & B \arrow[r, "p"] \arrow[ur] & C & 0 \\
0 \arrow[r] & \ker p \arrow[ur] & & &
\end{tikzcd}$$

Some examples are in order.

**Example 64.2.** 1. $\mathbf{C} = \{0\}$, and $\mathbf{C}$ the class of all abelian groups.

2. Let $\mathbf{C}$ be the class of all torsion abelian groups. We need to check that $\mathbf{C}$ satisfies the second condition of Definition **??**. Consider a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0.$$

We need to show that $B$ is torsion if $A$ and $C$ are torsion. To see this, let $b \in B$. Then $p(b)$ is killed by some integer $n$, so there exists $a \in A$ such that $i(a) = nb$. SInce $A$ is torsion, it follows that $b$ is torsion, too.

3. Let $\mathcal{P}$ be a set of primes. Define:

$$\mathbf{C}_{\mathcal{P}} = \{A : \text{if } p \notin \mathcal{P}, \text{ then } p : A \xrightarrow{\simeq} A, \text{ i.e., } A \text{ is a } \mathbf{Z}[1/p]\text{-module}\}$$

Let $\mathbf{Z}_{(\mathcal{P})} = \mathbf{Z}[1/p : p \notin \mathcal{P}] \subseteq \mathbf{Q}$.

For instance, if $\mathcal{P}$ is the set of all primes, then $\mathbf{C}_{\mathcal{P}}$ is the Serre class of all abelian groups. If $\mathcal{P}$ is the set of all primes other than $\ell$, then $\mathbf{C}_{\mathcal{P}}$ is the Serre class consisting of all $\mathbf{Z}[1/\ell]$-modules. If $\mathcal{P} = \{\ell\}$, then $\mathbf{C}_{\{\ell\}} =: \mathbf{C}_\ell$ is the Serre class of all $\mathbf{Z}_{(\ell)}$-modules. If $\mathcal{P} = \emptyset$, then $\mathbf{C}_\emptyset$ is all rational vector spaces.

4. If $\mathbf{C}$ and $\mathbf{C}'$ are Serre classes, then so is $\mathbf{C} \cap \mathbf{C}'$. For instance, $\mathbf{C}_{\text{tors}} \cap \mathbf{C}_{\text{fg}}$ is the Serre class $\mathbf{C}_{\text{finite}}$. Likewise, $\mathbf{C}_p \cap \mathbf{C}_{\text{tors}}$ is the Serre class of all $p$-torsion abelian groups.

Here are some straightforward consequences of the definition:

1. If $C_\bullet$ is a chain complex, and $C_n \in \mathbf{C}$, then $H_n(C_\bullet) \in \mathbf{C}$.

2. Suppose $F_*A$ is a filtration on an abelian group. If $A \in \mathbf{C}$, then $\text{gr}_n A \in \mathbf{C}$ for all $n$. If $F_*A$ is finite and $\text{gr}_n A \in \mathbf{C}$ for all $n$, then $A \in \mathbf{C}$.

3. Suppose we have a spectral sequence $\{E_r\}$. If $E^2_{s,t} \in \mathbf{C}$, then $E^r_{s,t} \in \mathbf{C}$ for $r \geq 2$. It follows that if $\{E^r\}$ is a right half-plane spectral sequence, then $E^{s+1}_{s,t} \twoheadrightarrow E^{s+2}_{s,t} \twoheadrightarrow \cdots \twoheadrightarrow E^\infty_{s,t} \in \mathbf{C}$.

   Thus, if the spectral sequence comes from a filtered complex (which is bounded below, such that for all $n$ there exists an $s$ such that $F_s H_n(C) = H_n(C)$, i.e., the homology of the filtration stabilizes), then $E^\infty_{s,t} = \text{gr}_s H_{s+t}(C)$. This means that if the $E^2_{s,t} \in \mathbf{C}$ for all $s + t = n$, then $H_n(C) \in \mathbf{C}$.

To apply this to the Serre spectral sequence, we need an additional axiom for Definition 64.1:

2. if $A, B \in \mathbf{C}$, then so are $A \otimes B$ and $\text{Tor}_1(A, B)$.

All of the examples given above satisfy this additional axiom.

**Terminology 64.3.** $f : A \to B$ is said to be a $\mathbf{C}$-epimorphism if $\operatorname{coker} f \in \mathbf{C}$, a $\mathbf{C}$-monomorphism if $\ker f \in \mathbf{C}$, and a $\mathbf{C}$-isomorphism if it is a $\mathbf{C}$-epimorphism and a $\mathbf{C}$-monomorphism.

**Proposition 64.4.** *Let $\pi : E \to B$ be a fibration and $B$ path connected, such that the fiber $F = \pi^{-1}(*)$ is path connected. Suppose $\pi_1(B)$ acts trivially on $H_*(F)$.*

*Let $\mathbf{C}$ be a Serre class satisfiying Axiom 2. Let $s \geq 3$, and assume that $H_n(E) \in \mathbf{C}$ where $1 \leq n < s-1$ and $H_t(B) \in \mathbf{C}$ for $1 \leq t < s$. Then $H_t(F) \in \mathbf{C}$ for $1 \leq t < s-1$.*

*Proof.* We will do the case $s = 3$, for starters. We're gonna want to relate the low-dimension homology of these groups. What can I say? We know that $H_0(E) = \mathbf{Z}$ since it's connected. I have $H_1(E) \to H_1(B)$, via $\pi$. This is one of the edge homomorphisms, and thus it surjects (no possibility for a differential coming in). I now have a map $H_1(F) \to H_1(E)$. But I have a possible $d^2 : H_2(B) \to H_1(F)$, which is a transgression that gives:

$$H_2(B) \xrightarrow{\partial} H_1(F) \to H_1(E) \to H_1(B) \to 0$$

Let me take a step back and say something general. You might be interested in knowing when something in $H_n(F)$ maps to zero in $H_n(E)$. I.e., what's the kernel of $H_n(F) \to H_n(E)$. The sseq gives an obstruction to being an isomorphism. The only way that something can be killed by $H_n(F) \to H_n(E)$ is described by:

$$\ker(H_n(F) \to H_n(E)) = \bigcup \left(\text{im of } d^r \text{ hitting } E^r_{0,n}\right)$$

You can also say what the cokernel is: it's whatever's left in $E^\infty_{s,t}$ with $s + t = n$. These obstruct $H_n(F) \to H_n(E)$ from being surjective.

In the same way, I can do this for the base. If I have a class in $H_n(E)$, that maps to $H_n(B)$, the question is: what's the image? Well, the only obstruction is the possibility is that the element in $H_n(B)$ supports a nonzero differential. Thus:

$$\operatorname{im}(H_n(E) \xrightarrow{\pi_*} H_n(B)) = \bigcap \left(\ker(d^r : E^r_{r,0} \to \cdots)\right)$$

Again, you can think of the sseq as giving obstructions. And also, the obstruction to that map being a monomorphism that might occur in lower filtration along the same total degree line.

Back to our argument. We had the low-dimensional exact sequence:

$$H_2(B) \xrightarrow{\partial} H_1(F) \to H_1(E) \to H_1(B) \to 0$$

Here $p = 3$, so we have $H_2(B) \in \mathbf{C}$ and $H_1(E) \in \mathbf{C}$. Thus $H_1(F) \in \mathbf{C}$. That's the only thing to check when $p = 3$.

Let's do one more case of this induction. What does this say? Now I'll do $p = 4$. We're interested in knowing if $E^2_{0,3} \in \mathbf{C}$. There are now two possible differentials! I have $H_2(F) = E^2_{0,2} \twoheadrightarrow E^3_{0,2}$. This quotient comes from $d^2 : E^2_{2,1} \to E^2_{0,2}$. Now, $d^3 : E^3_{3,0} \to E^3_{0,2}$ which gives a surjection $E^3_{0,2} \twoheadrightarrow E^4_{0,2} \simeq E^\infty_{0,2} \hookrightarrow H_2(E)$. Now, our assumptions were that $E^2_{2,1}, E^3_{3,0}, H_2(E) \in \mathbf{C}$. Thus $E^3_{0,2} \in \mathbf{C}$ and so $E^2_{0,2} = H_2(F) \in \mathbf{C}$. Ta-da! ∎

We're close to doing actual calculations, but I have to talk about the multiplicative structure on the Serre sseq first.

## 65 Mod C Hurewicz, Whitehead, cohomology spectral sequence

We had $\mathbf{C}_{fg}$ and $\mathbf{C}_{tors}$, and

$$\mathbf{C}_{\mathcal{P}} = \{A|\ell : A \xrightarrow{\simeq} A, \ell \notin \mathcal{P}\}, \quad \mathbf{C}_p = \mathbf{C}_{\{p\}}, \quad \mathbf{C}_{p'} = \mathbf{C}_{\text{not } p}$$

Another one is $\mathbf{C}_{p'} \cap \mathbf{C}_{tors}$, which consists of torsion groups such that $p$ is an isomorphism on $A$. There is therefore no $p$-torsion, and it has only prime-to-$p$ torsion. This is the same thing as saying that $A \otimes \mathbf{Z}_{(p)} = 0$.

**Theorem 65.1** (Mod $\mathbf{C}$ Hurewicz)**.** *Let $X$ be simply connected and $\mathbf{C}$ a Serre class such that $A, B \in \mathbf{C}$ implies that $A \otimes B, \mathrm{Tor}_1(A, B) \in \mathbf{C}$ (this is axiom 2). Assume also that if $A \in \mathbf{C}$, then $H_j(K(A, 1)) = H_j(BA) \in \mathbf{C}$ for all $j > 0$. (This is valid for all our examples, and is what is called Axiom 3.)*

*Let $n \geq 1$. Then $\pi_i(X) \in \mathbf{C}$ for any $1 < i < n$ if and only if $H_i(X) \in \mathbf{C}$ for any $1 < i < n$, and $\pi_n(X) \to H_n(X)$ is a mod $\mathbf{C}$ isomorphism.*

**Example 65.2.** For $1 < i < n$, the group $\widetilde{H}_i(X)$ is:

1. torsion;
2. finitely generated;
3. finite;
4. $- \otimes \mathbf{Z}_{(p)} = 0$

if and only if $\pi_i(X)$ for $1 < i < n$.

*Proof.* Look at $\Omega X \to PX \to X$. Then $\pi_1 \Omega X \in \mathbf{C}$. Look at Davis+Kirk. □

There's a Whitehead theorem that comes out of this, that I want to state for you.

**Theorem 65.3** (Mod $\mathbf{C}$ Whitehead theorem)**.** *Let $\mathbf{C}$ be a Serre class satisfying axioms 1, 2, 3, and:*

*(2′) $A \in \mathbf{C}$ implies that $A \otimes B \in \mathbf{C}$ for any $B$.*

*This is satisfied for all our examples except $\mathbf{C}_{fg}$.*

*Suppose I have $f : X \to Y$ where $X, Y$ are simply connected. Suppose $\pi_2(X) \to \pi_2(Y)$ is onto. Let $n \geq 2$. Then $\pi_i(X) \to \pi_i(Y)$ is a $\mathbf{C}$-isomorphism for $2 \leq i \leq n$ and is a $\mathbf{C}$-epimorphism for $i = n$, with the same statement for $H_i$.*

These kind of theorems help us work locally at a prime, and that's super. You'll see this in the next assignment, which is mostly up on the web. You'll also see this in calculations which we'll start doing in a day or two.

Change of subject here. Today I'm going to say a lot of things for which I won't give a proof. I want to talk about cohomology sseq.

### Cohomology sseq

We're building up this powerful tool using spectral sequences. We saw how powerful the cup product was, and that is what cohomology is good for. In cohomology, things get turned upside down:

**Definition 65.4.** A *decreasing filtration* of an object $A$ is

$$A \supseteq \cdots \supseteq F^{-1}A \supseteq F^0A \supseteq F^1A \supseteq F^2A \supseteq \cdots \supseteq 0$$

This is called "bounded above" if $F^0A = A$. Write $\mathrm{gr}^s A = F^sA/F^{s+1}A$.

**Example 65.5.** Suppose $X$ is a filtered space. So there's an increasing filtration $\emptyset = F_{-1}X \subseteq F_0X \subseteq \cdots$. Let $R$ be a commutative ring of coefficients. Then I have $S^*(X)$, where the differential goes up one degree. Define

$$F^sS^*(X) = \ker(S^*(X) \to S^*(F_{s-1}X))$$

For instance, $F^0S^*(X) = S^*(X)$. Thus this is a bounded above decreasing filtration .

My computer will run out of juice so TEXthis up later!

**Example 65.6.** Let $X = E \xrightarrow{\pi} B =$ CW-complex with $\pi_1(B)$ acting trivially on $H_t(F)$. Then $F_sE = \pi^{-1}(\mathrm{sk}_sB)$. Thus I get a filtration on $S^*(E)$, and

$$F^sH^*(X) = \ker(H^*(X) \to H^*(F_{s-1}X))$$

Doing everything the same as before, we get a *cohomology spectral sequence*. Here are some facts.

1. First, you have $E_r^{s,t}$ (note that indices got reversed). There's a differential $d_r : E_r^{s,t} \to E_r^{s+r,t-r+1}$, so that the total degree of the differential is 1.
2. You discover that
$$E_2^{s,t} \simeq H^s(B; H^t(F))$$
3. and $E_\infty^{s,t} \simeq \mathrm{gr}^s H^{s+t}(E)$.
4.

## 66 A few examples, double complexes, Dress sseq

Way back in 905 I remember computing the cohomology ring of $\mathbf{CP}^n$ using Poincaré duality. Let's do it fresh using the fiber sequence

$$S^1 \to S^{2n+1} \to \mathbf{CP}^n$$

where $S^1$ acts on $S^{2n+1}$. Here we know the cohomology of the fiber and the total space, but not the cohomology of the base. Let's look at the cohomology sseq for this. Then

$$E_2^{s,t} = H^s(\mathbf{CP}^n; H^t(S^1)) \simeq H^s(\mathbf{CP}^n) \otimes H^t(S^1) Rightarrow H^{s+t}(S^{2n+1})$$

The isomorphism $H^s(\mathbf{CP}^n; H^t(S^1)) \simeq H^s(\mathbf{CP}^n) \otimes H^t(S^1)$ follows from the UCT.

We know at least that $\mathbf{CP}^n$ is simply connected by the lexseq of homotopy groups. I don't have to worry about local coefficients. Let's work with the case $S^5$. We know that $\mathbf{CP}^n$ is simply connected, so the one-dimensional cohomology is 0. The only way to kill $E_2^{0,1}$ is by sending it via $d_2$ to $E_2^{2,0}$. Is this map surjective? Yes, it's an isomorphism.

Now I'm going to give names to the generators of these things; see the below diagram. $E_2^{2,1}$ is in total degree 3 and so we have to get rid of it. I will compute $d_2$ on this via Leibniz:

$$d_2(xy) = (d_2x)y - xd_2y = (d_2x)y = y^2$$

which gives (iterating the same computation):

$$\begin{tikzcd}
0 & \mathbf{Z}x \arrow[drr, "d_2"'] & 0 & \mathbf{Z}xy \arrow[drr, "d_2"'] & 0 & \mathbf{Z}xy^2 \\
0 & \mathbf{Z} & 0 & \mathbf{Z}y & 0 & \mathbf{Z}y^2 \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}$$

This continues until the end where you reach $\mathbf{Z}xy^{??}$ which is a permanent cycle since it lasts until the $E_\infty$-page.

Another example: let $C_m$ be the cyclic group of order $m$ sitting inside $S^1$. How can we analyse $S^{2n+1}/C_m =: L$? This is the lens space. We have a map $S^{2n+1}/C_m \to S^{2n+1}/S^1 = \mathbf{CP}^n$. This is a fiber bundle whose fiber is $S^1/C_m$. The spectral sequence now runs:

$$E^2_{s,t} = H_s(\mathbf{CP}^n) \otimes H_t(S^1/C_m) \Rightarrow H_{s+t}(L)$$

We know the whole $E^2$ term now:

$$\begin{tikzcd}
0 & \mathbf{Z} & 0 & \mathbf{Z} & 0 & \mathbf{Z} \\
0 & \mathbf{Z} & 0 & \mathbf{Z} \arrow[ull, "m"'] & 0 & \mathbf{Z} \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}$$

In cohomology, we have something dual:

$$\begin{tikzcd}
0 & u \arrow[drr, "m"'] & 0 & uy \arrow[drr, "m"'] & 0 & uy^2 \\
0 & 1 & 0 & y & 0 & y^2 \\
 & 0 & 1 & 2 & 3 & 4
\end{tikzcd}$$

What's the ring structure? We get that $H^*(L) = \mathbf{Z}[y,v]/(my, y^{n+1}, yv, v^2)$ where $|v| = 2n+1$ and $|y| = 2$. By the way, when $m = 1$, this is $\mathbf{RP}^{2n+1}$. This is a computation of the cohomology of odd real projective spaces. Remember that odd projective spaces are orientable and you're seeing that here because you're picking up a free abelian group in the top dimension.

## Double complexes

$A_{s,t}$ is a bigraded abelian group with $d_h : A_{s,t} \to A_{s-1,t}$ and $d_v : A_{s,t} \to A_{s,t-1}$ such that $d_v d_h = d_h d_v$. Assume that $\{(smt) : s+t = n, A_{s,t} \neq 0\}$ is finite for any $n$. Then

$$(tA)_n = \bigoplus_{s+t=n} A_{s,t}$$

Under this assumption, there's only finitely many nonzero terms. I like this personally because otherwise I'd have to decide between the direct sum and the direct product, so we're avoiding that here. It's supposed to be a chain complex. Here's the differential:

$$d(a_{s,t}) = d_h a_{s,t} + (-1)^s d_v a_{s,t}$$

Then $d^2 = 0$, as you can check.

**Question 66.1.** What is $H_*(tA_*)$?

Define a filtration as follows:

$$F_p(tA)_n = \bigoplus_{s+t=n, s\leq p} A_{s,t} \subseteq (tA)_n$$

This kinda obviously gives a filtered complex. Let's compute the low pages of the sseq. What is $\mathrm{gr}_s(tA)$? Well

$$\mathrm{gr}_s(tA)_{s+t} = (F_s/F_{s-1})_{s+t} = A_{s,t}$$

This associated graded object has its own differential $\mathrm{gr}_s(tA)_{s+t} = A_{s,t} \xrightarrow{d_v} A_{s,t-1} = \mathrm{gr}_s(tA)_{s+t-1}$. Let $E^0_{s,t} = \mathrm{gr}_s(tA)_{s+t} = A_{s+t}$, so that $d^0 = d_v$. Then $E^1 = H(E^0_{s,t}, d^0) = H(A_{s,t}; d_v) =: H^v_{s,t}(A)$. So computing $E^1$ is ez. Well, what's $d^1$ then?

To compute $d^1$ I take a vertical cycle that and the differential decreases the ... by 1, so that $d^1$ is induced by $d_h$. This means that I can write $E^2_{s,t} = H^h_{s,t}(H^v(A))$.

**Question 66.2.** You can also do ${}'E^2_{s,t} = H^v_{s,t}(H^h(A))$, right?

Rather than do that, you can define the transposed double complex $A^{\mathsf{T}}_{t,s} = A_{s,t}$, and $d^{\mathsf{T}}_h(a_{s,t}) = (-1)^s d_v(a_{s,t})$ and $d^{\mathsf{T}}_v(a_{s,t}) = (-1)^t d_h a_{s,t}$. When I set the signs up like that, then

$$tA^{\mathsf{T}} \simeq tA$$

*as complexes* and not just as groups (because of those signs). Thus, you get a spectral sequence
$$^{\mathsf{T}}E^2_{s,t} = H^v_{s,t}(H^h(A))$$
converging to the same thing. I'll reserve telling you about Dress' construction until Monday because I want to give a double complex example. It's not ... it's just a very clear piece of homological algebra.

**Example 66.3** (UCT)**.** For this, suppose I have a (not necessarily commutative) ring $R$. Let $C_*$ be a chain complex, bounded below of right $R$-modules, and let $M$ be a left $R$-module. Then I get a new chain complex of abelian groups via $C_* \otimes_R M$. What is $H(C_* \otimes_R M)$? I'm thinking of $M$ as some kind of coefficient. Let's assume that each $C_n$ is projective, or at least flat, for all $n$.

Shall we do this?

Let $M \leftarrow P_0 \leftarrow P_1 \leftarrow \cdots$ be a projective resolution of $M$ as a left $R$-module. Then $H_*(P_*) \xrightarrow{\simeq} M$. Form $C_* \otimes_R P_*$: you know how to do this! I'll define $A_{s,t}$ to be $C_s \otimes_R P_t$. It's got two differentials, and it's a double complex. Let's work out the two sseqs.

Firstly, let's take it like it stands and take homology wrt $P$ first. I'm organizing it so that $C$ is along the base and $P$ is along the fiber. What is the vertical homology $H^v(A_{*,*})$? If the $C$ are projective then tensoring with them is exact, so that $H^v(A_{s,*}) = C_s \otimes_R H_*(P_*)$, so that $E^1_{s,t} = H^v_{s,t}(A_{*,*}) = C_s \otimes M$ if $t = 0$ and 0 otherwise. The spectral sequence is concentrated in one row. Thus,

$$E^2_{s,t} = \begin{cases} H_s(C_* \otimes_R M) & \text{if } t = 0 \\ 0 & \text{else} \end{cases}$$

This is canonically the same thing as $E^\infty_{s,0} \simeq H_s(tA)$.

Let me go just one step further here. The game is to look at the *other* spectral sequence, where I do horizontal homology first. Then $H^h(A_{*,*}) = H_t(C_*) \otimes P_s$ again because the $P_*$ are projective. Thus,

$$E^2_{s,t} = H^v(H^h(A_{*,*})) = \mathrm{Tor}^R_s(H_t(C), M) \Rightarrow H_{s+t}(C_* \otimes_R M)$$

That's the *universal coefficients spectral sequence.*

What happens if $R$ is a PID? Only two columns are nonzero, and $E^2_{0,n} = H_n(C) \otimes_R M$ and $E^2_{1,n-1} = \mathrm{Tor}_1(H_{n-1}(C), M)$. This exactly gives the universal coefficient exact sequence.

Later we'll use this stuff to talk about cohomology of classifying spaces and Grassmannians and Thom isomorphisms and so on.

## 67 Dress spectral sequence, Leray-Hirsch

I think I have to be doing something tomorrow, so no office hours then. The new pset is up, and there'll be one more problem up. There are two more things about spectral sequences, and specifically the multiplicative structure, that I have to tell you about. The construction of the Serre sseq isn't the one that we gave. He did stuff with simplicial homology, but as you painfully figured out, $\Delta^s \times \Delta^t$ isn't another simplex. Serre's solution was to not use simplices, but to use cubes. He defined a new kind of homology using the $n$-cube. It's more complicated and unpleasant, but he worked it out.

## Dress' sseq

Dress made the following variation on this idea, which I think is rather beautiful. We have a trivial fiber bundle $\Delta^t \to \Delta^s \times \Delta^t \to \Delta^s$. Let's do with this what we did with homology in the first place. Dress started with some map $\pi : E \to B$ (not necessarily a fibration), and he thought about the set of maps from $\Delta^s \times \Delta^t \to \Delta^s$ to $\pi : E \to B$. This set is denoted $\mathrm{Sin}_{s,t}(\pi)$. This forgets down to $S_s(B)$. Altogether, this $\mathrm{Sin}_{*,*}(\pi)$ is a functor $\mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op} \to \mathrm{Set}$, forming a "bisimplicial set".

The next thing we did was to take the free $R$-module, to get a bisimplicial $R$-module $R\mathrm{Sin}_{*,*}(\pi)$. We then passed to chain complexes by forming the alternating sum. We can do this in two directions here! (The $s$ is horizontal and $t$ is vertical.) This gives us a double complex. We now get a spectral sequence! I hope it doesn't come as a surprise that you can compute the horizontal – you can compute the vertical differential first, and then taking the horizontal differential gives the homology of $B$ with coefficients in something. Oh actually, the totalization $tR\mathrm{Sin}_{*,*}(\pi) \simeq R\mathrm{Sin}_*(E) = S_*(E)$. We'll have

$$E^2_{s,t} = H_s(B; \text{crazy generalized coefficients}) \Rightarrow H_{s+t}(E)$$

These coefficients may not even be local since I didn't put any assumptions on $\pi$! This is like the "Leray" sseq, set up without sheaf theory. If $\pi$ is a fibration, then those crazy generalized coefficients is the local system given by the homology of the fibers. This gives the Serre sseq.

This has the virtue of being completely natural. Another virtue is that I can form $\mathrm{Hom}(-, R)$, and this gives rise to a multiplicative double complex. Remember that the cochains on a space form a DGA, and that's where the cup product comes from. The same story puts a bigraded multiplication on this double complex, and that's true *on the nose*. That gives rise a multiplicative cohomology sseq.

This is very nice, but the only drawback is that the paper is in German. That was item one in my agenda.

## Leray-Hirsch

This tells you condition under which you can compute the cohomology of a total space. Anyway. We'll see.

Let's suppose I have a fibration $\pi : E \to B$. For simplicity suppose that $B$ is path connected, so that gives meaning to the fiber $F$ which we'll also assume to be path-connected. All cohomology is with coefficients in a ring $R$. I have a sseq

$$E_2^{s,t} = H^s(B; \underline{H^t F}) \Rightarrow H^{s+t}(E)$$

If you want assume that $\pi_1(B)$ acts trivially so that that cohomology in local coefficients is just cohomology with coefficients in $H^*F$. I have an algebra map $\pi^* : H^*(B) \to H^*(E)$, making $H^*(E)$ into a module over $H^*(B)$. We have $E_2^{*,t} = H^*(B; H^t(F))$, and this is a $H^*(B)$-module. That's part of the multiplicative structure, since $E_2^{*,0} = H^*B$. This row acts on every other row by that module structure.

Everything in the bottom row is a permanent cycle, i.e., survives to the $E_\infty$-page. In other words

$$H^*(B) = E_2^{*,0} \twoheadrightarrow E_3^{*,0} \twoheadrightarrow \cdots \twoheadrightarrow E_\infty^{*,0}$$

Each one of these surjections is an algebra map.

What the multiplicative structure is telling us is that $E_r^{*,0}$ is a graded algebra acting on $E_r^{*,t}$. Thus, $E_\infty^{*,t}$ is a module for $H^*(B)$.

Really I should be saying that it's a module for $H^*(B; \underline{H^0(F)})$. Can I guarantee that the $\pi_1(B)$-action on $F$ is trivial. We know that $F \to *$ induces an iso on $H^0$ (that's part of being path-connected). So if you have a fibration whose fiber is a point, there's no possibility for an action. This fibration looks the same as far as $H^0$ of the fiber is concerned. Thus the $\pi_1(B)$-action is trivial on $H^0(F)$, so saying that it's a $H^*(B)$-module is fine.

Where were we? We have module structures all over the place. In particular, we know that $H^*(E)$ is a module over $H^*(B)$ as we saw, and also $E_\infty^{*,t}$ is a $H^*(B)$-module. These better be compatible!

Define an increasing filtration on $H^*(E)$ via $F_tH^n(E) = F^{n-t}H^n(E)$. For instance, $F_0H^n(E) = F^nH^n(E)$. What is that? In our picture, we have the associated quotients along the diagonal on $E_\infty^{s,t}$ given by $s + t = n$. In the end, since we know that $F^{n+1}H^n(E) = 0$, it follows that

$$F_0H^n(E) = F^nH^n(E) = E_\infty^{n,0} = \operatorname{im}(\pi^* : H^n(B) \to H^n(E))$$

With respect to this filtration, we have

$$\mathrm{gr}_t H^*(E) = E_\infty^{*,t}$$

I learnt this idea from Dan Quillen. It's a great idea. This increasing filtration $F_*H^*(E)$ is a filtration by $H^*(B)$-modules, and $\mathrm{gr}_t H^*(E) = E_\infty^{*,t}$ is true as $H^*B$-modules. It's exhaustive and bounded below.

This is a great perspective. Let's use it for something. Let me give you the Leray-Hirsch theorem.

**Theorem 67.1** (Leray-Hirsch)**.** *Let $\pi : E \to B$.*

1. *Suppose $B$ and $F$ are path-connected.*

2. *Suppose that $H^t(F)$ is free*[1] *of finite rank as a $R$-module.*

[1] Everything is coefficients in $R$

3. *Also suppose that $H^*(E) \twoheadrightarrow H^*(F)$. That's a big assumption; it's dual is saying that the homology of the fiber injects into the homology of $E$. This is called "totally non-homologous to zero" – this is a great phrase, I don't know who invented it.*

*Pick an $R$-linear surjection $\sigma : H^*(F) \to H^*(E)$; this defines a map $\overline{\sigma} : H^*(B) \otimes_R H^*(F) \to H^*(E)$ via $\overline{\sigma}(x \otimes y) = \pi^*(x) \cup \sigma(y)$. This is the $H^*(B)$-linear extension. Then $\overline{\sigma}$ is an isomorphism.*

**Remark 67.2.** It's not natural since it depends on the choice of $\sigma$. It tells you that $H^*(E)$ is free as a $H^*(B)$-module. That's a good thing.

*Proof.* I'm going to use our Serre sseq

$$E_2^{s,t} = H^s(B; \underline{H^t F}) \Rightarrow H^{s+t}(E)$$

Our map $H^*(E) \to H^*(F)$ is an edge homomorphism in the sseq, which means that it factors as $H^*(E) \to E_2^{0,*} = H^0(B; \underline{H^*(F)}) \subseteq H^*(F)$. Since $H^*(E) \to H^*(F)$, we have $H^0(B; \underline{H^*(F)}) \simeq H^*(F)$. Thus the $\pi_1(B)$-action on $F$ is trivial.

**Question 67.3.** What's this arrow $H^*(E) \to E_2^{0,*}$? We have a map $H^*(E) \to H^*(E)/F^1 = E_\infty^{0,*}$. This includes into $E_2^{0,*}$.

Now you know that the $E_2$-term is $H^s(B; H^t(F))$. By our assumption on $H^*(F)$, this is $H^s(B) \otimes_R H^t(F)$, as algebras. What do the differentials look like? I can't have differentials coming off of the fiber, because if I did then the restriction map to the fiber wouldn't be surjective, i.e., that $d_r|_{E_r^{0,\infty}} = 0$. The differentials on the base are of course zero. This proves that $d_r$ is zero on every page by the algebra structure! This means that $E_\infty = E_2$, i.e., $E_\infty^{*,t} = H^*(B) \otimes H^t(F)$.

Now I can appeal to the filtration stuff that I was talking about, so that $E_\infty^{*,t} = \mathrm{gr}_t H^*(E)$. Let's filter $H^*(B) \otimes H^*(F)$ by the degree in $H^*(F)$, i.e., $F_q = \bigoplus_{t \leq q} H^*(B) \otimes H^t(F)$. The map $\overline{\sigma} : H^*(B) \otimes H^*(F) \to H^*(E)$ is filtration preserving, and it's an isomorphism on the associated graded. This is the identification $H^*(B) \otimes H^t(F) = E_\infty^{*,t} = \mathrm{gr}_t H^*(E)$. Since the filtrations are exhaustive and bounded below, we conclude that $\overline{\sigma}$ itself is an isomorphism. □

## 68 Integration, Gysin, Euler, Thom

Today there's a talk by

the one

the only

JEAN-PIERRE SERRE

OK let's begin.

## Umkehr

Let $\pi : E \to B$ be a fibration and suppose $B$ is path-connected. Suppose the fiber has no cohomology above some dimension $d$. The Serre sseq has nothing above row $d$.

Let's look at $H^n(E)$. This happens along total degree $n$. We have this neat increasing filtration that I was talking about on Monday whose associated quotients are the rows in this thing. So I can divide out by it (i.e I divide out by $F_{d-1}H^n(E)$). Then I get

$$H^n(E) \twoheadrightarrow H^n(E)/F_{d-1}H^n(E) = E_\infty^{n-d,d} \rightarrowtail E_2^{n-d,d} = H^{n-d}(B; \underline{H^d(F)})$$

That's because on the $E_2$ page, at that spot, there's nothing hitting it, but there might be a differential hitting it. There it is; here's another edge homomorphism.

**Remark 68.1.** This is a *wrong-way map*, also known as an "umkehr" map. It's also called a *pushforward map*, or the *Gysin map*.

We know from the incomprehensible discussion that I was giving on Monday that this was a filtration of modules over $H^*(B)$, so that this map $H^n(E) \to H^{n-d}(B; \underline{H^d(F)})$ is a $H^*(B)$-module map.

**Example 68.2.** $F$ is a compact connected $d$-manifold with a given $R$-orientation. Thus $H^d(F) \simeq R$, given by $x \mapsto \langle x, [F] \rangle$. There might some local cohomology there, but I do get a map $H^n(E; R) \to H^{n-d}(B; \overline{R})$. This is such a map, and it has a name: it's written $\pi_!$ or $\pi_*$. I'll write $\pi_*$.

Of course, if $\pi_1(B)$ fixes $[F] \in H_d(F; R)$, then $\underline{R}$-cohomology is $R$-cohomology. Thus our map is now $H^n(E; R) \to H^{n-d}(B; R)$. Sometimes it's also called a pushforward map. Note that we also get a projection formula

$$\pi_*(\pi^*(b) \cup e) = b \cup \pi_*(e)$$

where $\pi^*$ is the pushforward, $e \in H^n(E)$ and $b \in H^s(B)$. Others call this Frobenius reciprocity.

## Gysin

Suppose $H^*(F) = H^*(S^{n-1})$. In practice, $F \cong S^{n-1}$, or even $F \simeq S^{n-1}$. In that case, $\pi : E \to B$ is called a *spherical fibration* Then the spectral sequence is *even simpler*! It has only two nonzero rows!

Let's pick an orientation for $S^{n-1}$, to get an isomorphism $H^{n-1}(S^{n-1})$. Well the spectral sequence degenerates, and you get a long exact sequence

$$\cdots \to H^s(B) \xrightarrow{\pi^*} H^s(E) \xrightarrow{\pi_*} H^{s-n+1}(B; \underline{R}) \xrightarrow{d_n} H^{s+1}(B) \xrightarrow{\pi^*} H^{s+1}(E) \to \cdots$$

That's called the *Gysin sequence*[2]. Because everything is a module over $H^*(B)$, this is a lexseq of $H^*(B)$-modules.

[2] pronounced Gee-sin

Let me be a little more explicit. Suppose we have an orientation. We now have a differential $H^0(B) \to H^n(B)$. We have the constant function $1 \in H^0(B)$, and this maps to something in $B$. This is called the *Euler class*, and is denoted $e$.

Since $d_n$ is a module homomorphism, we have $d_n(x) = d_n(1 \cdot x) = d_n(1) \cdot x = e \cdot x$ where $x$ is in the cohomology of $B$. Thus our lexseq is of the form

$$\cdots \to H^s(B) \xrightarrow{\pi^*} H^s(E) \xrightarrow{\pi_*} H^{s-n+1}(B; \underline{R}) \xrightarrow{e\cdot -} H^{s+1}(B) \xrightarrow{\pi^*} H^{s+1}(E) \to \cdots$$

### Some facts about the Euler class

Suppose $E \to B$ has a section $\sigma : B \to E$ (so that $\pi\sigma = 1_B$). So, if it came from a vector bundle, I'm asking that there's a nowhere vanishing cross-section of that vector bundle. Let's apply cohomology, so that you get $\sigma^*\pi^* = 1_{H^*(B)}$. Thus $\pi^*$ is monomorphic. In terms of the Gysin sequence, this means that $H^{s-n}(B) \xrightarrow{e\cdot -} H^s(B)$ is zero. But this implies that

$$\boxed{e = 0}$$

Thus, if you don't have a nonzero Euler class then you cannot have a section! If your Euler class is zero sometimes you can conclude that your bundle has a section, but that's a different story.

The Euler class of the tangent bundle of a manifold when paired with the fundamental class is the Euler characteristic. More precisely, if $M$ is oriented connected compact $n$-manifold, then

$$\langle e(\tau_M), [M] \rangle = \chi(M)$$

That's why it's called the Euler class. (He didn't know about spectral sequences or cohomology.)

### Time for Thom

This was done by Rene Thom. Let $\xi$ be a $n$-plane bundle over $X$. I can look at $H^*(E(\xi), E(\xi) - \text{section})$. If I pick a metric, this is $H^*(D(\xi), S(\xi))$, where $D(\xi)$ is the disk bundle[3] and $S(\xi)$ is the sphere bundle. If there's no point-set annoyance, this is $\widetilde{H}^*(D(\xi)/S(\xi))$.

If $X$ is a compact Hausdorff space, then ... The open disk bundle $D^0(\xi) \simeq E(\xi)$. This quotient $D(\xi)/S(\xi) = E(\xi)^+$ since you get the one-point compactification by embedding into a compact Hausdorff space ($D(\xi)$ here) and then quotienting by the complement (which is $S(\xi)$ here). This is called the *Thom space* of $\xi$. There are two notations: some people write $\mathrm{Th}(\xi)$, and some people (Atiyah started this) write $X^\xi$.

**Example 68.3** (Dumb)**.** Suppose $\xi$ is the zero vector bundle. Then your fibration is $\pi : X \to X$. What's the Thom space? The disk bundle is $X$, and the boundary of a disk is empty, so $\mathrm{Th}(0) = X^0 = X \sqcup *$.

[3] $D(\xi) = \{v \in E(\xi) : ||v|| \leq 1\}$.

The Thom space is a pointed space (corresponding to $\infty$ or the point which $S(\xi)$ is collapsed to).

I'd like to study its cohomology, because it's interesting. There's no other justification. Maybe I'll think of it as the relative cohomology.

So, guess what? We've developed sseqs and done cohomology. Anything else we'd like to do to groups and functors and things?

Let's make the spectral sequence relative!

I have a path connected $B$, and I'll study:

$$\begin{tikzcd} F_0 \arrow[r, hook] \arrow[d] & F \arrow[d] \\ E_0 \arrow[r, hook] \arrow[d] & E \arrow[d] \\ B \arrow[r, equal] & B \end{tikzcd}$$

Then if you sit patiently and work through things, we get

$$E_2^{s,t} = H^s(B; H^t(F, F_0)) \Rightarrow_s H^{s+t}(E, E_0)$$

Note that $\Rightarrow_s$ means that $s$ determines the filtration.

Let's do this with the Thom space. We have $D(\xi) \xrightarrow{\simeq} X$. That isn't very interesting. In our case, we have an incredibly simple spectral sequence, where everything on the $E_2$-page is concentrated in row $n$. Thus the $E_2$ page *is* the cohomology of

$$\widetilde{H}^{s+n}(\mathrm{Th}(\xi)) = H^{s+n}(D(\xi), S(\xi)) \simeq H^s(B; \underline{R})$$

where $\underline{R} = \underline{H^n(D^n, S^{n-1})}$. This is a canonical isomorphism of $H^*(B)$-modules.

Suppose your vector bundle $\xi$ is oriented, so that $\underline{R} = R$. Now, if $s = 0$, then I have $1 \in H^0(B)$. This gives $u \in H^n(\mathrm{Th}(\xi))$, which is called the *Thom class*.

The cohomology of $B$ is a free module of rank one over $H^*(B)$, so that $H^*(\mathrm{Th}(\xi))$ is also a $H^*(B)$-module that is free of rank 1, generated by $u$.

Let me finish by saying one more thing. This is why the Thom space is interesting. Notice one more thing: there's a lexseq of a pair

$$\cdots \to \widetilde{H}^s(\mathrm{Th}(\xi)) \to H^s(D(\xi)) \to H^s(S(\xi)) \to \widetilde{H}^{s+1}(\mathrm{Th}(\xi)) \to \cdots$$

We have synonyms for these things:

$$\cdots \to H^{s-n}(X) \to H^s(X) \to H^s(S(\xi)) \to H^{s-n+1}(X) \to \cdots$$

And aha, this is is exactly the same form as the Gysin sequence. Except, oh my god, what have I done here?

Yeah, right! In the Gysin sequence, the map $H^{s-n}(X) \to H^s(X)$ was multiplication by the Euler class. The Thom class $u$ maps to some $e' \in H^n(X)$ via $\widetilde{H}^n(D(\xi), S(\xi)) \to H^n(D(\xi)) \simeq H^n(X)$. And the map $H^{s-n}(X) \to H^s(X)$ is multiplication by $e'$. Guess what? This is the Gysin sequence.

You'll explore more in homework.

I'll talk about characteristic classes on Friday.

# Chapter 7

# Characteristic classes

## 69 Grothendieck's construction of Chern classes

### Generalities on characteristic classes

We would like to apply algebraic techniques to study $G$-bundles on a space. Let $A$ be an abelian group, and $n \geq 0$ an integer.

**Definition 69.1.** A *characteristic class* for principal $G$-bundles (with values in $H^n(-;A)$) is a natural transformation of functors $\mathbf{Top} \to \mathbf{Ab}$:

$$\mathrm{Bun}_G(X) \xrightarrow{c} H^n(X;A)$$

Concretely: if $P \to Y$ is a principal $G$-bundle over a space $X$, and $f : X \to Y$ is a continuous map of spaces, then

$$c(f^*P) = f^*c(P).$$

The motivation behind this definition is that $\mathrm{Bun}_G(X)$ is still rather mysterious, but we have techniques (developed in the last section) to compute the cohomology groups $H^n(X;A)$. It follows by construction that if two bundles over $X$ have two different characteristic classes, then they cannot be isomorphic. Often, we can use characteristic classes to distinguish a given bundle from the trivial bundle.

**Example 69.2.** The Euler class takes an oriented real $n$-plane vector bundle (with a chosen orientation) and produces an $n$-dimensional cohomology class $e : \mathrm{Vect}_n^{or}(X) = \mathrm{Bun}_{SO(n)}(X) \to H^n(X;\mathbf{Z})$. This is a characteristic class. To see this, we need to argue that if $\xi \downarrow X$ is a principal $G$-bundle, we can pull the Euler class back via $f : X \to Y$. The bundle $f^*\xi \downarrow Y$ has a orientation if $\xi$ does, so it makes sense to even talk about the Euler class of $f^*\xi$. Since all of our constructions were natural, it follows that $e(f^*\xi) = f^*e(\xi)$.

Similarly, the mod 2 Euler class is $e_2 : \mathrm{Vect}_n(X) = \mathrm{Bun}_{O(n)}(X) \to H^n(X;\mathbf{Z}/2\mathbf{Z})$ is another Euler class. Since everything has an orientation with respect to $\mathbf{Z}/2\mathbf{Z}$, the mod 2 Euler class is well-defined.

By our discussion in §58, we know that $\mathrm{Bun}_G(X) = [X, BG]$. Moreover, as we stated in Theorem 51.8, we know that $H^n(X; A) = [X, K(A, n)]$ (at least if $X$ is a CW-complex). One moral reason for cohomology to be easier to compute is that the spaces $K(A, n)$ are infinite loop spaces (i.e., they can be delooped infinitely many times). It follows from the Yoneda lemma that characteristic classes are simply maps $BG \to K(A, n)$, i.e., elements of $H^n(BG; A)$.

**Example 69.3.** The Euler class $e$ lives in $H^n(BSO(n); \mathbf{Z})$; in fact, it is $e(\xi)$, the Euler class of the universal oriented $n$-plane bundle over $BSO(n)$. A similar statement holds for $e_2 \in H^n(BO(n); \mathbf{Z}/2\mathbf{Z})$. For instance, if $n = 2$, then $SO(2) = S^1$. It follows that

$$BSO(2) = BS^1 = \mathbf{CP}^\infty.$$

We know that $H^*(\mathbf{CP}^\infty; \mathbf{Z}) = \mathbf{Z}[e]$ — it's the polynomial algebra on the "universal" Euler class! Similarly, $O(1) = \mathbf{Z}/2\mathbf{Z}$, so

$$BO(1) = B\mathbf{Z}/2 = \mathbf{RP}^\infty.$$

We know that $H^*(\mathbf{RP}^\infty; \mathbf{F}_2) = \mathbf{F}_2[e_2]$ — as above, it is the polynomial algebra over $\mathbf{Z}/2\mathbf{Z}$ on the "universal" mod 2 Euler class.

## Chern classes

These are one of the most fundamental example of characteristic classes.

**Theorem 69.4** (Chern classes)**.** *There is a unique family of characteristic classes for complex vector bundles that assigns to a complex $n$-plane bundle $\xi$ over $X$ the $n$th* Chern class *$c_k^{(n)}(\xi) \in H^{2k}(X; \mathbf{Z})$, such that:*

1. *$c_0^{(n)}(\xi) = 1$.*
2. *If $\xi$ is a line bundle, then $c_1^{(1)}(\xi) = -e(\xi)$.*
3. *The* Whitney sum formula *holds: if $\xi$ is a $p$-plane bundle and $\eta$ is a $q$-plane bundle (and if $\xi \oplus \eta$ denotes the fiberwise direct sum), then*

$$c_k^{(p+q)}(\xi \oplus \eta) = \sum_{i+j=k} c_i^{(p)}(\xi) \cup c_j^{(q)}(\eta) \in H^{2k}(X; \mathbf{Z}).$$

*Moreover, if $\xi_n$ is the universal $n$-plane bundle, then*

$$H^*(BU(n); \mathbf{Z}) \simeq \mathbf{Z}[c_1^{(n)}, \cdots, c_n^{(n)}],$$

*where $c_k^{(n)} = c_k^{(n)}(\xi_n)$.*

This result says that all characteristic classes for complex vector bundles are given by polynomials in the Chern classes because the cohomology of $BU(n)$ gives all the characteristic classes. It also says that there are no universal algebraic relations among the Chern classes: you can specify them independently.

**Remark 69.5.** The $(p+q)$-plane bundle $\xi_p \times \xi_q = \mathrm{pr}_1^*\xi_p \oplus \mathrm{pr}_2^*\xi_q$ over $BU(p) \times BU(q)$ is classified by a map $BU(p) \times BU(q) \xrightarrow{\mu} BU(p+q)$. The Whitney sum formula computes the effect of $\mu$ on cohomology:

$$\mu^*(c_k^{(n)}) = \sum_{i+j=k} c_i^{(p)} \times c_j^{(q)} \in H^{2k}(BU(p) \times BU(q)),$$

where, you'll recall,

$$x \times y := \mathrm{pr}_1^* x \cup \mathrm{pr}_2^* y.$$

The Chern classes are "stable", in the following sense. Let $\epsilon$ be the trivial one-dimensional complex vector bundle, and let $\xi$ be an $n$-dimensional vector bundle. What is $c_k^{(n+q)}(\xi \oplus \epsilon^q)$? For this, the Whitney sum formula is valuable.

The trivial bundle is characterized by the pullback:

$$\begin{tikzcd}
X \times \mathbf{C}^n = n\epsilon \arrow[r] \arrow[d] & \mathbf{C}^n \arrow[d] \\
X \arrow[r] & *
\end{tikzcd}$$

By naturality, we find that if $k > 0$, then $c_k^{(n)}(n\epsilon) = 0$. The Whitney sum formula therefore implies that

$$c_k^{(n+q)}(\xi \oplus \epsilon^q) = c_k^{(n)}(\xi).$$

This phenomenon is called stability: the Chern class only depends on the "stable equivalence class" of the vector bundle (really, they are only defined on "K-theory", for those in the know). For this reason, we will drop the superscript on $c_k^{(n)}(\xi)$, and simply write $c_k(\xi)$.

## Grothendieck's construction

Let $\xi$ be an $n$-plane bundle. We can consider the vector bundle $\pi : \mathbf{P}(\xi) \to X$, the projectivization of $\xi$: an element of the fiber of $\mathbf{P}(\xi)$ over $x \in X$ is a line inside $\xi_x$, so the fibers are therefore all isomorphic to $\mathbf{CP}^{n-1}$.

Let us compute the cohomology of $\mathbf{P}(\xi)$. For this, the Serre spectral sequence will come in handy:

$$E_2^{s,t} = H^s(X; H^t(\mathbf{CP}^{n-1})) \Rightarrow H^{s+t}(\mathbf{P}(\xi)).$$

**Remark 69.6.** Why is the local coefficient system constant? The space $X$ need not be simply connected, but $BU(n)$ is simply connected since $U(n)$ is simply connected. Consider the projectivization of the universal bundle $\xi_n \downarrow BU(n)$; pulling back via $f : X \to BU(n)$ gives the bundle $\pi : \mathbf{P}(\xi) \to X$. The map on fibers $H^*(\mathbf{P}(\xi_n)_{f(x)}) \to H^*(\mathbf{P}(\xi_n)_x)$ is an isomorphism which is equivariant with respect to the action of the fundamental group of $\pi_1(X)$ via the map $\pi_1(X) \to \pi_1(BU(n)) = 0$.

Because $H^*(\mathbf{CP}^{n-1})$ is torsion-free and finitely generated in each dimension, we know that

$$E_2^{s,t} \simeq H^s(X) \otimes H^t(\mathbf{CP}^{n-1}).$$

The spectral sequence collapses at $E_2$, i.e., that $E_2 \simeq E_\infty$, i.e., there are no differentials. We know that the $E_2$-page is generated as an algebra by elements in the cohomology of the fiber and elements in the cohomology of the base. Thus, it suffices to check that elements in the cohomology of the fiber survive to $E_\infty$. We know that

$$E_2^{0,2t} = \mathbf{Z}\langle x^t\rangle, \text{ and } E_2^{0,2t+1} = 0,$$

where $x = e(\lambda)$ is the Euler class of the canonical line bundle $\lambda \downarrow \mathbf{CP}^{n-1}$.

In order for the Euler class to survive the spectral sequence, it suffices to come up with a two dimensional cohomology class in $\mathbf{P}(\xi)$ that restricts to the Euler class over $\mathbf{CP}^{n-1}$. We know that $\lambda$ itself is the restriction of the tautologous line bundle over $\mathbf{CP}^\infty$. There is a tautologous line bundle $\lambda_\xi \downarrow \mathbf{P}(\xi)$, given by the tautologous line bundle on each fiber. Explicitly:

$$E(\lambda_\xi) = \{(\ell, y) \in \mathbf{P}(\xi) \times_X E(\xi) | y \in \ell \subseteq \xi_x\}.$$

Thus, $x$ is the restriction $e(\lambda_\xi)|_{\text{fiber}}$ of the Euler class to the fiber. It follows that the class $x$ survives to the $E_\infty$-page.

Using the Leray-Hirsch theorem (Theorem 67.1), we conclude that

$$H^*(\mathbf{P}(\xi)) = H^*(X)\langle 1, e(\lambda_\xi), e(\lambda_\xi)^2, \cdots, e(\lambda_\xi)^{n-1}\rangle.$$

For simplicity, let us write $e = e(\lambda_\xi)$. Unforunately, we don't know what $e^n$ is, although we do know that it is a linear combination of the $e^k$ for $k < n$. In other words, we have a relation

$$e^n + c_1 e^{n-1} + \cdots + c_{n-1}e + c_n = 0,$$

where the $c_k$ are elements of $H^{2k}(X)$. These are the Chern classes of $\xi$. By construction, they are unique!

To prove Theorem 69.4(2), note that when $n = 1$ the above equation reads

$$e + c_1 = 0,$$

as desired.

## 70 $H^*(BU(n))$, splitting principle

Theorem 69.4 claimed that the Chern classes, which we constructed in the previous section, generate the cohomology of $BU$ as a polynomial algebra. Our goal in this section is to prove this result.

### The cohomology of $BU(n)$

Recall that $BU(n)$ supports the universal principal $U(n)$-bundle $EU(n) \to BU(n)$. Given any left action of $U(n)$ on some space, we can form the associated fiber bundle. For instance, the associated bundle of the $U(n)$-action on $\mathbf{C}^n$ yields the universal line bundle $\xi_n$.

Likewise, the associated bundle of the action of $U(n)$ on $S^{2n-1} \subseteq \mathcal{C}^n$ is the unit sphere bundle $S(\xi_n)$, the unit sphere bundle. By construction, the fiber of the map $EU(n) \times_{U(n)} S^{2n-1} \to BU(n)$ is $S^{2n-1}$. Since

$$S^{2n-1} = U(n)/(1 \times U(n-1)),$$

we can write

$$EU(n) \times_{U(n)} S^{2n-1} \simeq EU(n) \times_{U(n)} (U(n)/U(n-1)) \simeq EU(n)/U(n-1) = BU(n-1).$$

In other words, $BU(n-1)$ is the unit sphere bundle of the tautologous line bundle over $BU(n)$. This begets a fiber bundle:

$$S^{2n-1} \to BU(n-1) \to BU(n),$$

which provides an inductive tool (via the Serre spectral sequence) for computing the homology of $BU(n)$. In §68, we observed that the Serre spectral sequence for a spherical fibration was completely described bythe Gysin sequence.

Recall that if $B$ is oriented and $S^{2n-1} \to E \xrightarrow{\pi} B$ is a spherical bundle over $B$, then the Gysin sequence was a long exact sequence

$$\cdots \to H^{q-1}(E) \xrightarrow{\pi_*} H^{q-2n}(B) \xrightarrow{e\cdot} H^q(B) \xrightarrow{\pi^*} H^q(E) \xrightarrow{\pi_*} \cdots$$

Let us assume that the cohomology ring of $E$ is polynomial and concentrated in even dimensions. For the base case of the induction, both these assumptions are satisfied (since $BU(0) = *$ and $BU(1) = \mathbf{CP}^\infty$).

These assumptions imply that if $q$ is even, then the map $\pi_*$ is zero. In particular, multiplication by $e|_{H^{\text{even}}(B)}$ (which we will also denote by $e$) is injective, i.e., $e$ is a nonzero divisor. Similarly, if $q$ is odd, then $e \cdot H^{q-2n}(B) = H^q(B)$. But if $q = 1$, then $H^{q-2n}(B) = 0$; by induction on $q$, we find that $H^{\text{odd}}(B) = 0$. Therefore, if $q$ is even, then $H^{q-2n+1}(B) = 0$. This implies that there is a short exact sequence

$$0 \to H^*(B) \xrightarrow{e\cdot} H^*(B) \to H^*(E) \to 0. \tag{7.1}$$

In particular, the cohomology of $E$ is the cohomology of $B$ quotiented by the ideal generated by the nonzero divisor $e$.

For instance, when $n = 1$, then $B = \mathbf{CP}^\infty$ and $E \simeq *$. We have the canonical generator $e \in H^2(\mathbf{CP}^\infty)$; these deductions tell us the well-known fact that $H^*(\mathbf{CP}^\infty) \simeq \mathbf{Z}[e]$.

Consider the surjection $H^*(B) \xrightarrow{\pi^*} H^*(E)$. Since $H^*(E)$ is polynomial, we can lift the generators of $H^*(E)$ to elements of $H^*(B)$. This begets a splitting $s : H^*(E) \to$

$H^*(B)$. The existence of the Euler class $e \in H^*(B)$ therefore gives a map $H^*(E)[e] \xrightarrow{\overline{s}} H^*(B)$. We claim that this map is an isomorphism.

This is a standard algebraic argument. Filter both sides by powers of $e$, i.e., take the $e$-adic filtration on $H^*(E)[e]$ and $H^*(B)$. Clearly, the associated graded of $H^*(E)[e]$ just consists of an infinite direct sum of the cohomology of $E$. The associated graded of $H^*(B)$ is the same, thanks to the short exact sequence (7.1). Thus the induced map on the associated graded $\mathrm{gr}^*(\overline{s})$ is an isomorphism. In this particular case (but not in general), we can conclude that $\overline{s}$ is an isomorphism: in any single dimension, the filtration is finite. Thus, using the five lemma over and over again, we see that the map $\overline{s}$ an isomorphism on each filtered piece. This implies that $\overline{s}$ itself is an isomorphism, as desired.

This argument proves that

$$H^*(BU(n-1)) = \mathbf{Z}[c_1, \cdots, c_{n-1}].$$

In particular, there is a map $\pi^* : H^*(BU(n)) \to H^*(BU(n-1))$ which an isomorphism in dimensions at most $2n$. Thus, the generators $c_i$ have *unique* lifts to $H^*(BU(n))$. We therefore get:

**Theorem 70.1.** *There exist classes $c_i \in H^{2i}(BU(n))$ for $1 \leq i \leq n$ such that:*

- *the canonical map $H^*(BU(n)) \xrightarrow{\pi_*} H^*(BU(n-1))$ sends*

$$c_i \mapsto \begin{cases} c_i & i < n \\ 0 & i = n, \text{ and} \end{cases}$$

- $c_n := (-1)^n e \in H^{2n}(BU(n))$.

*Moreover,*

$$\boxed{H^*(BU(n)) \simeq \mathbf{Z}[c_1, \cdots, c_n]}.$$

## The splitting principle

**Theorem 70.2.** *Let $\xi \downarrow X$ be an $n$-plane bundle. Then there exists a space $\mathrm{Fl}(\xi) \xrightarrow{\pi} X$ such that:*

1. *$\pi^*\xi = \lambda_1 \oplus \cdots \lambda_n$, where the $\lambda_i$ are line bundles on $Y$, and*

2. *the map $\pi^* : H^*(X) \to H^*(\mathrm{Fl}(\xi))$ is monic.*

*Proof.* We have already (somewhat) studied this space. Recall that there is a vector bundle $\pi : \mathbf{P}(\xi) \to X$ such that

$$H^*(\mathbf{P}(\xi)) = H^*(X)\langle 1, e, \cdots, e^{n-1}\rangle.$$

Moreover, in §69, we proved that there is a complex line bundle over $\mathbf{P}(\xi)$ which is a subbundle of $\pi^*\xi$. In other words, $\pi^*\xi$ splits as a sum of a line bundle and some other bundle (by Corollary 52.14). Iterating this construction proves the existence of $\mathrm{Fl}(\xi)$. ∎

This proof does not give much insight into the structure of $\mathrm{Fl}(\xi)$. Remember that the *frame bundle* $\mathrm{Fr}(\xi)$ of $\xi$: an element of $\mathrm{Fr}(\xi)$ is a linear, inner-product preserving map $\mathbf{C}^n \to E(\xi)$. This satisfies various properties; for instance:

$$E(\xi) = \mathrm{Fr}(\xi) \times_{U(n)} \mathbf{C}^n.$$

Moreover,

$$\mathbf{P}(\xi) = \mathrm{Fr}(\xi) \times_{U(n)} U(n)/(1 \times U(n-1)).$$

The *flag bundle* $\mathrm{Fl}(\xi)$ is defined to be

$$\mathrm{Fl}(\xi) = \mathrm{Fr}(\xi) \times_{U(n)} U(n)/(U(1) \times \cdots \times U(1)).$$

The product $U(1)\times\cdots\times U(1)$ is usually denoted $T^n$, since it is the maximal torus in $U(n)$. For the universal bundle $\xi_n \downarrow BU(n)$, the frame bundle is exactly $EU(n)$; therefore, $\mathrm{Fl}(\xi_n)$ is just the bundle given by $BT^n \to BU(n)$. By construction, the fiber of this bundle is $U(n)/T^n$. In particular, there is a monomorphism $H^*(BU(n)) \hookrightarrow H^*(BT^n)$. The cohomology of $BT^n$ is extremely simple — it is the cohomology of a product of $\mathbf{CP}^\infty$'s, so

$$H^*(BT^n) \simeq \mathbf{Z}[t_1, \cdots, t_n],$$

where $|t_k| = 2$. The $t_i$ are the Euler classes of $\pi_i^*\lambda_i$, under the projection map $\pi_i : BT^n \to \mathbf{CP}^\infty$.

## 71 The Whitney sum formula

As we saw in the previous section, there is an injection $H^*(BU(n)) \hookrightarrow H^*(BT^n)$. What is the image of this map?

The symmetric group sits inside of $U(n)$, so it acts by conjugation on $U(n)$. This action stabilizes this subgroup $T^n$. By naturality, $\Sigma_n$ acts on the classifying space $BT^n$. Since $\Sigma_n$ acts by conjugation on $U(n)$, it acts on $BU(n)$ in a way that is homotopic to the identity (Lemma 58.1). However, each element $\sigma \in \Sigma_n$ simply permutes the factors in $BT^n = (\mathbf{CP}^\infty)^n$; we conclude that $H^*(BU(n); R)$ actually sits inside the invariants $H^*(BT^n; R)^{\Sigma_n}$.

Recall the following theorem from algebra:

**Theorem 71.1.** *Let $\Sigma_n$ act on the polynomial algebra $R[t_1, \cdots, t_n]$ by permuting the generators. Then*

$$R[t_1, \cdots, t_n]^{\Sigma_n} = R[\sigma_1^{(n)}, \cdots, \sigma_n^{(n)}],$$

*where the $\sigma_i$ are the elementary symmetric polynomials, defined via*

$$\prod_{i=1}^n (x - t_i) = \sum_{j=0}^n \sigma_i^{(n)} x^{n-i}.$$

For instance,

$$\sigma_1^{(n)} = -\sum t_i, \ \sigma_n^{(n)} = (-1)^n \prod t_i.$$

If we impose a grading on $R[t_1, \cdots, t_n]$ such that $|t_i| = 2$, then $|\sigma_i^{(n)}| = 2i$. It follows from our discussion in §70 that the ring $H^*(BT^n)^{\Sigma_n}$ has the same size as $H^*(BU(n))$.

Consider an injection of finitely generated abelian groups $M \hookrightarrow N$, with quotient $Q$. Suppose that, after tensoring with any field, the map $M \to N$ an isomorphism. If $Q \otimes k = 0$, then $Q = 0$. Indeed, if $Q \otimes \mathbf{Q} = 0$ then $Q$ is torsion. Similarly, if $Q \otimes \mathbf{F}_p = 0$, then $Q$ has no $p$-component. In particular, $M \simeq N$. Applying this to the map $H^*(BU(n) \to H^*(BT^n)^{\Sigma_n}$, we find that

$$H^*(BU(n); R) \xrightarrow{\simeq} H^*(BT^n; R)^{\Sigma_n} = R[\sigma_1^{(n)}, \cdots, \sigma_n^{(n)}].$$

What happens as $n$ varies? There is a map $R[t_1, \cdots, t_n] \to R[t_1, \cdots, t_{n-1}]$ given by sending $t_n \mapsto 0$ and $t_i \mapsto t_i$ for $i \neq n$. Of course, we cannot say that this map is equivariant with respect to the action of $\Sigma_n$. However, it *is* equivariant with respect to the action of $\Sigma_{n-1}$ on $R[t_1, \cdots, t_n]$ via the inclusion of $\Sigma_{n-1} \hookrightarrow \Sigma_n$ as the stabilizer of $n \in \{1, \cdots, n\}$. Therefore, the $\Sigma_n$-invariants sit inside the $\Sigma_{n-1}$-invariants, giving a map

$$R[t_1, \cdots, t_n]^{\Sigma_n} \to R[t_1, \cdots, t_n]^{\Sigma_{n-1}} \to R[t_1, \cdots, t_{n-1}]^{\Sigma_{n-1}}.$$

We also find that for $i < n$, we have $\sigma_i^{(n)} \mapsto \sigma_i^{(n-1)}$ and $\sigma_n^{(n)} \mapsto 0$.

### Where do the Chern classes go?

To answer this question, we will need to understand the multiplicativity of the Chern class. We begin with a discussion about the Euler class. Suppose $\xi^p \downarrow X, \eta^q \downarrow Y$ are oriented real vector bundles; then, we can consider the bundle $\xi \times \eta \downarrow X \times Y$, which is another oriented real vector bundle. The orientation is given by picking oriented bases for $\xi$ and $\eta$. We claim that

$$e(\xi \times \eta) = e(\xi) \times e(\eta) \in H^{p+q}(X \times Y).$$

Since $D(\xi \times \eta)$ is homeomorphic to $D(\xi) \times D(\eta)$, and $S(\xi \times \eta) = D(\xi) \times S(\eta) \cup S(\xi) \times D(\eta)$, we learn from the relative Künneth formula that

$$H^*(D(\xi \times \eta), S(\xi \times \eta)) \leftarrow H^*(D(\xi), S(\xi)) \otimes H^*(D(\eta), S(\eta)).$$

It follows that

$$u_{\xi \times \eta} = u_\xi \times u_\eta \in H^{p+q}(\mathrm{Th}(\xi) \times \mathrm{Th}(\eta));$$

this proves the desired result since the Euler class is the image of the Thom class under the map $H^n(\mathrm{Th}(\xi)) \to H^n(D(\xi)) \simeq H^n(B)$.

Consider the diagonal map $\Delta : X \to X \times X$. The cross product in cohomology then pulls back to the cup product, and the direct product of fiber bundles pulls back to the Whitney sum. It follows that

$$e(\xi \oplus \eta) = e(\xi) \cup e(\eta).$$

If $\xi^n \downarrow X$ is an $n$-dimensional complex vector bundle, then we defined[1]

$$c_n(\xi) = (-1)^n e(\xi_{\mathbf{R}}).$$

We need to describe the image of $c_n(\xi_n)$ under the map $H^{2n}(BU(n)) \to H^{2n}(BT^n)^{\Sigma_n}$.

Let $f : BT^n \to BU(n)$ denote the map induced by the inclusion of the maximal torus. Then, by construction, we have a splitting

$$f^*\xi_n = \lambda_1 \oplus \cdots \oplus \lambda_n.$$

Thus,

$$(-1)^n e(\xi) \mapsto (-1)^n e(\lambda_1 \oplus \cdots \oplus \lambda_n) = (-1)^n e(\lambda_1) \cup \cdots \cup e(\lambda_n).$$

The discussion above implies that $f^*$ sends the right hand side to $(-1)^n t_1 \cdots t_n = \sigma_n^{(n)}$. In other words, the top Chern class maps to $\sigma_n^{(n)}$ under the map $f^*$.

Our discussion in the previous sections gives a commuting diagram:

$$\begin{tikzcd}
H^*(BU(n)) \arrow[r] \arrow[d] & H^*(BT^n)^{\Sigma_n} \arrow[d] \\
H^*(BU(n-1)) \arrow[r] & H^*(BT^{n-1})^{\Sigma_{n-1}}
\end{tikzcd}$$

Arguing inductively, we find that going from the top left corner to the bottom left corner to the bottom right corner sends

$$c_i \mapsto c_i \mapsto \sigma_i^{(n-1)} \text{ for } i < n.$$

Likewise, going from the top left corner to the top right corner to the bottom right corner sends

$$c_i \mapsto \sigma_i^{(n)} \mapsto \sigma_i^{(n-1)} \text{ for } i < n.$$

We conclude that the map $f^*$ sends $c_i^{(i)} \mapsto \sigma_i^{(i)}$.

## Proving the Whitney sum formula

By our discussion above, the Whitney sum formula of Theorem 69.4 reduces to proving the following identity:

$$\sigma_k^{(p+q)} = \sum_{i+j=k} \sigma_i^{(p)} \cdot \sigma_j^{(q)} \tag{7.2}$$

inside $\mathbf{Z}[t_1, \cdots, t_p, t_{p+1}, \cdots, t_{p+q}]$. Here, $\sigma_i^{(p)}$ is thought of as a polynomial in $t_1, \cdots, t_p$, while $\sigma_i^{(q)}$ is thought of as a polynomial in $t_{p+1}, \cdots, t_{p+q}$. To derive Equation (7.2),

[1]There's a slight technical snag here: a complex bundle doesn't have an orientation. However, its underlying oriented real vector bundle does.

simply compare coefficients in the following:

$$\begin{aligned}
\sum_{k=0}^{p+q} \sigma_k^{(p+q)} x^{p+q-k} &= \prod_{i=1}^{p+q} (x - t_i) \\
&= \prod_{i=1}^{p} (x - t_i) \cdot \prod_{j=p+1}^{p+q} (x - t_j) \\
&= \left( \sum_{i=0}^{p} \sigma_i^{(p)} x^{p-i} \right) \left( \sum_{j=0}^{q} \sigma_j^{(p)} x^{q-j} \right) \\
&= \sum_{k=0}^{p+q} \left( \sum_{i+j=k} \sigma_i^{(p)} \sigma_j^{(q)} \right) x^{p+q-k}.
\end{aligned}$$

## 72 Stiefel-Whitney classes, immersions, cobordisms

There is a result analogous to Theorem 69.4 for all vector bundles (not necessarily oriented):

**Theorem 72.1.** *There exist a unique family of characteristic classes* $w_i : \mathrm{Vect}_n(X) \to H^n(X; \mathbf{F}_2)$ *such that for* $0 \leq i$ *and* $i > n$*, we have* $w_i = 0$*, and:*

1. $w_0 = 1$*;*
2. $w_1(\lambda) = e(\lambda)$*; and*
3. *the Whitney sum formula holds:*

$$w_k(\xi \oplus \eta) = \sum_{i+j=k} w_i(\xi) \cup w_j(\eta)$$

*Moreover:*

$$H^*(BO(n); \mathbf{F}_2) = \mathbf{F}_2[w_1, \cdots, w_n],$$

*where* $w_n = e_2$*.*

**Remark 72.2.** We can express the Whitney sum formula simply by defining the *total Steifel-Whitney class*

$$1 + w_1 + w_2 + \cdots =: w.$$

Then the Whitney sum formula is just

$$w(\xi \oplus \eta) = w(\xi) \cdot w(\eta).$$

Likewise, the Whitney sum formula can be stated by defining the total Chern class.

**Remark 72.3.** Again, the Steifel-Whitney classes are stable:

$$w(\xi \oplus k\epsilon) = w(\xi).$$

Again, Grothendieck's definition works since the splitting principle holds. There is an injection $H^*(BO(n)) \hookrightarrow H^*(B(\mathbf{Z}/2\mathbf{Z})^n)$. To compute $H^*(BO(n))$, our argument for computing $H^*(BU(n))$ does not immediately go through, although there is a fiber sequence

$$S^{n-1} \to EO(n) \times_{O(n)} O(n)/O(n-1) \to BO(n);$$

the problem is that $n-1$ can be even or odd. We still have a Gysin sequence, though:

$$\cdots \to H^{q-n}(BO(n)) \xrightarrow{e\cdot} H^q(BO(n)) \xrightarrow{\pi^*} H^q(BO(n-1)) \to H^{q-n+1}(BO(n)) \to \cdots$$

In order to apply our argument for computing $H^*(BU(n))$ to this case, we only need to know that $e$ is a nonzero divisor. The splitting principle gave a monomorphism $H^*(BO(n)) \hookrightarrow H^*((\mathbf{RP}^\infty)^n)$. The fact that $e$ is a nonzero divisor follows from the observation that under this map,

$$e_2 = w_n \mapsto e_2(\lambda_1 \oplus \cdots \oplus \lambda_n) = t_1 \cdots t_n,$$

using the same argument as in §71; however, $t_1 \cdots t_n$ is a nonzero divisor, since $H^*((\mathbf{RP}^\infty)^n)$ is an integral domain.

## Immersions of manifolds

The theory developed above has some interesting applications to differential geometry.

**Definition 72.4.** Let $M^n$ be a smooth closed manifold. An *immersion* is a smooth map from $M^n$ to $\mathbf{R}^{n+k}$, denoted $f : M^n \looparrowright \mathbf{R}^{n+k}$, such that $(\tau_{M^n})_x \hookrightarrow (\tau_{\mathbf{R}^{n+k}})_{f(x)}$ for $x \in M$.

Informally: crossings are allows, but not cusps.

**Example 72.5.** There is an immersion $\mathbf{RP}^2 \looparrowright \mathbf{R}^3$, known as *Boy's surface*.

**Question 72.6.** When can a manifold admit an immersion into an Euclidean space?

Assume we had an immersion $i : M^n \looparrowright \mathbf{R}^{n+k}$. Then we have an embedding $f : \tau_M \to i^*\tau_{\mathbf{R}^{n+k}}$ into a trivial bundle over $M$, so $\tau_M$ has a $k$-dimensional complement, called $\xi$ such that

$$\tau_M \oplus \xi = (n+k)\epsilon.$$

Apply the total Steifel-Whitney class, we have

$$w(\tau)w(\xi) = 1,$$

since there's no higher Steifel-Whitney class of a trivial bundle. In particular,

$$w(\xi) = w(\tau)^{-1}.$$

**Example 72.7.** Let $M = \mathbf{RP}^n \looparrowright \mathbf{R}^{n+k}$. Then, we know that

$$\tau_{\mathbf{RP}^n} \oplus \epsilon \simeq (n+1)\lambda^* \simeq (n+1)\lambda,$$

where $\lambda \downarrow \mathbf{RP}^n$ is the canonical line bundle. By Remark 72.3, we have

$$w(\tau_{\mathbf{RP}^n}) = w(\tau_{\mathbf{RP}^n} \oplus \eta) = w((n+1)\lambda) = w(\lambda)^{n+1}.$$

It remains to compute $w(\lambda)$. Only the first Steifel-Whitney class is nonzero. Writing $H^*(\mathbf{RP}^n) = \mathbf{F}_2[x]/x^{n+1}$, we therefore have $w(\lambda) = x$. In particular,

$$w(\tau_{\mathbf{RP}^n}) = (1+x)^{n+1} = \sum_{i=0}^{n} \binom{n+1}{i} x^i.$$

It follows that

$$w_i(\tau_{\mathbf{RP}^n}) = \binom{n+1}{i} x^i.$$

The total Steifel-Whitney class of the complement of the tangent bundle is:

$$w(\xi) = (1+x)^{-n-1}.$$

The most interesting case is when $n$ is a power of 2, i.e., $n = 2^s$ for some integer $s$. In this case, since taking powers of 2 is linear in characteristic 2, we have

$$w(\xi) = (1+x)^{-1-2^s} = (1+x)^{-1}(1+x)^{-2^s} = (1+x)^{-1}(1+x^{2^s})^{-1}.$$

As all terms of degree greater than $2^s$ are zero, we conclude that So

$$w(\xi) = 1 + x + x^2 + \cdots + x^{2^s-1} + 2x^s = 1 + x + x^2 + \cdots + x^{2^s-1}.$$

As $x^{2^s-1} \neq 0$, this means that $k = \dim \xi \geq 2^s - 1$. We conclude:

**Theorem 72.8.** *There is no immersion* $\mathbf{RP}^{2^s} \looparrowright \mathbf{R}^{2\cdot 2^s - 2}$.

The following result applied to $\mathbf{RP}^{2^s}$ shows that the above result is sharp:

**Theorem 72.9** (Whitney)**.** *Any smooth compact closed manifold* $M^n \looparrowright \mathbf{R}^{2n-1}$.

However, Whitney's result is *not* sharp for a general smooth compact closed manifold. Rather, we have:

**Theorem 72.10** (Brown–Peterson, Cohen)**.** *A closed compact smooth n-manifold* $M^n \looparrowright \mathbf{R}^{2n-\alpha(n)}$, *where* $\alpha(n)$ *is the number of* 1*s in the dyadic expansion of* $n$.

This result is sharp, since if $n = \sum 2^{d_i}$ for the dyadic expansion, then $M = \prod_i \mathbf{RP}^{2^{d_i}} \not\looparrowright \mathbf{R}^{2n-\alpha(n)-1}$.

## Cobordism, characteristic numbers

If we have a smooth closed compact $n$-manifold, then it embeds in $\mathbf{R}^{n+k}$ for some $k \gg 0$. The normal bundle then satisfies

$$\tau_M \oplus \nu_M = (n+k)\epsilon.$$

A piece of differential topology tells us that if $k$ is large, then $\nu_M \oplus N\epsilon$ is independent of the bundle for some $N$.

This example, combined with Remark **??**, shows that $w(\nu_M)$ is independent of $k$. We are therefore motivated to think of Stiefel-Whitney classes as coming from $H^*(BO; \mathbf{F}_2) = \mathbf{F}_2[w_1, w_2, \cdots]$, where $BO = \varinjlim BO(n)$. Similarly, Chern classes should be thought of as coming from $H^*(BU; \mathbf{Z}) = \mathbf{Z}[c_1, c_2, \cdots]$. This exa

**Definition 72.11.** The characteristic number of a smooth closed compact $n$-manifold $M$ is defined to be $\langle w(\nu_M), [M] \rangle$.

Note that the fundamental class $[M]$ exists, since our coefficients are in $\mathbf{F}_2$, where everything is orientable.

This definition is very useful when thinking about cobordisms.

**Definition 72.12.** Two (smooth closed compact) $n$-manifolds $M, N$ are *(co)bordant* if there is an $(n+1)$-dimensional manifold $W^{n+1}$ with boundary such that

$$\partial W \simeq M \sqcup N.$$

For instance, when $n = 0$, the manifold $* \sqcup *$ is *not* cobordant to $*$, but it is cobordant to the empty set. However, $* \sqcup * \sqcup *$ is cobordant to $*$. Any manifold is cobordant to itself, since $\partial(M \times I) = M \sqcup M$. In fact, cobordism forms an equivalence relation on manifolds.

**Example 72.13.** A classic example of a cobordism is the "pair of pants"; this is the following cobordism between $S^1$ and $S^1 \sqcup S^1$: add image

Let us define

$$\Omega_n^O = \{\text{cobordism classes of } n\text{-manifolds}\}.$$

This forms a group: the addition is given by disjoint union. Note that every element is its own inverse. Moreover, $\bigoplus_n \Omega_n^O = \Omega_*^O$ forms a graded ring, where the product is given by the Cartesian product of manifolds. Our discussion following Definition 72.12 shows that $\Omega_0^O = \mathbf{F}_2$.

**Exercise 72.14.** Every 1-manifold is nullbordant, i.e., cobordant to the point.

Thom made the following observation. Suppose an $n$-manifold $M$ is embedded into Euclidean space, and that $M$ is nullbordant via some $(n+1)$-manifold $W$, so that $\nu_W|_M = \nu_M$. In particular,

$$\langle w(\nu_M), [M] \rangle = \langle w(\nu_W)|_M, [M] \rangle.$$

On the other hand, the boundary map $H_{n+1}(W,M) \xrightarrow{\partial} H_n(M)$ sends the relative fundamental class $[W,M]$ to $[M]$. Thus

$$\langle w(\nu_M), [M]\rangle = \langle w(\nu_M), \partial[W,M]\rangle = \langle \delta w(\nu_M), [W,M]\rangle.$$

However, we have an exact sequence

$$H^n(W) \xrightarrow{i^*} H^n(M) \xrightarrow{\delta} H^{n+1}(W,M).$$

Since $w(\nu_M)$ is in the image of $i^*$, it follows that $\delta w(\nu_M) = 0$. In particular, the characteristic number of a nullbordant manifold is zero. Thus, we find that "Stiefel-Whitney numbers tell all":

**Proposition 72.15.** *Characteristic numbers are cobordism invariants. In other words, characteristic numbers give a map*

$$\Omega_n^O \to \mathrm{Hom}(H^n(BO), \mathbf{F}_2) \simeq H_n(BO).$$

More is true:

**Theorem 72.16** (Thom, 1954)**.** *The map of graded rings $\Omega_*^O \to H_*(BO)$ defined by the characteristic number is an inclusion. Concretely, if $w(M^n) = w(N^n)$ for all $w \in H^n(BO)$, then $M^n$ and $N^n$ are cobordant.*

The way that Thom proved this was by expressing $\Omega_*^O$ is the graded homotopy ring of some space, which he showed is the product of mod 2 Eilenberg-MacLane spaces. Along the way, he also showed that:

$$\Omega_*^O = \mathbf{F}_2[x_i : i \neq 2^s - 1] = \mathbf{F}_2[x_2, x_4, x_5, x_6, x_8, \cdots]$$

This recovers the result of Exercise 72.14 (and so much more!).

## 73 Oriented bundles, Pontryagin classes, Signature theorem

We have a pullback diagram

$$\begin{tikzcd}
BSO(n) \arrow[r] \arrow[d, "\text{double cover}"] & S^\infty \arrow[d] \\
BO(n) \arrow[r, "w_1"'] & B\mathbf{Z}/2\mathbf{Z}
\end{tikzcd}$$

The bottom map is exactly the element $w_1 \in H^1(BO(n); \mathbf{F}_2)$. It follows that a vector bundle $\xi \downarrow X$ represented by a map $f : X \to BO(n)$ is orientable iff $w_1(\xi) = f^*(w_1) = 0$, since this is equivalent to the existence of a factorization:

$$\begin{tikzcd}
 & BSO(n) \arrow[r] \arrow[d] & S^\infty \arrow[d] \\
X \arrow[r, "\xi"'] \arrow[ur, dashed] & BO(n) \arrow[r, "w_1"'] & B\mathbf{Z}/2\mathbf{Z}
\end{tikzcd}$$

The fiber sequence $BSO(n) \to BO(n) \to \mathbf{RP}^\infty$ comes from a fiber sequence $SO(n) \to O(n) \to \mathbf{Z}/2\mathbf{Z}$ of groups. For $n \geq 3$, we can kill $\pi_1(SO(n)) = \mathbf{Z}/2\mathbf{Z}$, to get a double cover $\mathrm{Spin}(n) \to SO(n)$. The group $\mathrm{Spin}(n)$ is called the *spin group*. We have a cofiber sequence

$$B\mathrm{Spin}(n) \to BSO(n) \xrightarrow{w_2} K(\mathbf{Z}/2\mathbf{Z}, 2).$$

If $w_2(\xi) = 0$, we get a further lift in the above diagram, begetting a *spin structure* on $\xi$.

Bott computed that $\pi_2(\mathrm{Spin}(n)) = 0$. However, $\pi_3(\mathrm{Spin}(n)) = \mathbf{Z}$; killing this gives the *string group* $\mathrm{String}(n)$. Unlike $\mathrm{Spin}(n)$, $SO(n)$, and $O(n)$, this is not a finite-dimensional Lie group (since we have an infinite dimensional summand $K(\mathbf{Z}, 2)$). However, it can be realized as a topological group. The resulting maps

$$\mathrm{String}(n) \to \mathrm{Spin}(n) \to SO(n) \to O(n)$$

are just the maps in the Whitehead tower for $O(n)$. Taking classifying spaces, we get

$$\begin{tikzcd}
 & B\mathrm{String}(n) \arrow[d] & \\
 & B\mathrm{Spin}(n) \arrow[r, "p_1/2"] \arrow[d] & K(\mathbf{Z}, 4) \\
 & BSO(n) \arrow[r] \arrow[d] & K(\mathbf{Z}/2\mathbf{Z}, 2) \\
X \arrow[r, "\xi"'] \arrow[ur, dashed] \arrow[uur, dashed] \arrow[uuur, dashed] & BO(n) \arrow[r, "w_1"'] & B\mathbf{Z}/2\mathbf{Z}
\end{tikzcd}$$

Computing the (mod 2) cohomology of $BSO(n)$ is easy. We have a double cover $BSO(n) \to BO(n)$ with fiber $S^0$. Consequently, there is a Gysin sequence:

$$0 \to H^q(BO(n)) \xrightarrow{w_1} H^{q+1}(BO(n)) \xrightarrow{\pi^*} H^{q+1}(BSO(n)) \to 0$$

since $w_1$ is a nonzero divisor. The standard argument shows that

$$H^*(BSO(n)) = \mathbf{F}_2[w_2, \cdots, w_n].$$

However, it is *not* easy to compute $H^*(B\mathrm{Spin}(n))$ and $H^*(B\mathrm{String}(n))$; these are extremely complicated (and only become more complicated for higher connective covers of $BO(n)$). However, we will remark that they are concentrated in even degrees.

To define integral characteristic classes for oriented bundles, we will need to study Chern classes a little more. Let $\xi$ be a complex $n$-plane bundle, and let $\overline{\xi}$ denote the conjugate bundle. What is the total Chern class $c(\overline{\xi})$? Recall that the Chern classes $c_k(\overline{\xi})$ occur as coefficients in the identity

$$\sum c_i(\overline{\xi}) e(\lambda_{\overline{\xi}})^{n-i} = 0,$$

where $\lambda_{\overline{\xi}} \downarrow \mathbf{P}(\overline{\xi})$. Note that $\mathbf{P}(\overline{\xi}) = \mathbf{P}(\xi)$. By construction, $\lambda_{\overline{\xi}} = \overline{\lambda_\xi}$. In particular, we find that

$$e(\lambda_{\overline{\xi}}) = -e(\lambda_\xi).$$

It follows that

$$0 = \sum_{i=0}^{n} c_i(\overline{\xi}) e(\overline{\lambda_\xi})^{n-i} = \sum_{i=0}^{n} c_i(\overline{\xi})(-1)^{n-i} e(\lambda_\xi)^{n-i} = (-1)^n e(\lambda_\xi)^n + \cdots$$

This is *not* monic, and hence doesn't define the Chern classes of $\overline{\xi}$. We do, however, get a monic polynomial by multiplying this identity by $(-1)^n$:

$$\sum_{i=0}^{n} (-1)^i c_i(\overline{\xi}) e(\lambda_\xi)^{n-i} = 0.$$

It follows that

$$\boxed{c_i(\overline{\xi}) = (-1)^i c_i(\xi).}$$

If $\xi$ is a real vector bundle, then

$$c_i(\xi \otimes \mathbf{C}) = c_i(\overline{\xi \otimes \mathbf{C}}) = (-1)^i c_i(\xi \otimes \mathbf{C}).$$

If $i$ is odd, then $2c_i(\xi \otimes \mathbf{C}) = 0$. If $R$ is a $\mathbf{Z}[1/2]$-algebra, we therefore define:

**Definition 73.1.** Let $\xi$ be a real $n$-plane vector bundle. Then the $k$th Pontryagin class of $\xi$ is defined to be

$$p_k(\xi) = (-1)^k c_{2k}(\xi \otimes \mathcal{C}) \in H^{4k}(X; R).$$

Notice that this is 0 if $2k > n$, since $\xi \otimes \mathcal{C}$ is of complex dimension $n$. The Whitney sum formula now says that:

$$(-1)^k p_k(\xi \oplus \eta) = \sum_{i+j=k} (-1)^i p_i(\xi)(-1)^j p_j(\eta) = (-1)^k \sum_{i+j=k} p_i(\xi) p_j(\eta).$$

If $\xi$ is an oriented real $2k$-plane bundle, one can calculate that

$$p_k(\xi) = e(\xi)^2 \in H^{4k}(X; R).$$

We can therefore write down the cohomology of $BSO(n)$ with coefficients in a $\mathbf{Z}[1/2]$-algebra:

| $* =$ | 2 | 4 | 6 | 8 | 10 | 12 |
|---|---|---|---|---|---|---|
| $H^*(BSO(2))$ | $e_2$ | $(e_2^2)$ | | | | |
| $H^*(BSO(3))$ | | $p_1$ | | | | |
| $H^*(BSO(4))$ | | $p_1, e_4$ | | $(e_4^2)$ | | |
| $H^*(BSO(5))$ | | $p_1$ | | $p_2$ | | |
| $H^*(BSO(6))$ | | $p_1$ | $e_6$ | $p_2$ | | $(e_6^2)$ |
| $H^*(BSO(7))$ | | $p_1$ | | $p_2$ | | $p_3$ |

Here, $p_k \mapsto e_{2k}^2$. In the limiting case (i.e., for $BSO = BSO(\infty)$), we get a polynomial algebra on the $p_i$.

## Applications

We will not prove any of the statements in this section; it only serves as an outlook. The first application is the following analogue of Theorem 72.16:

**Theorem 73.2** (Wall). *Let $M^n, N^n$ be oriented manifolds. If all Stiefel-Whitney numbers and Pontryagin numbers coincide, then $M$ is oriented cobordant to $N$, i.e., there is an $(n+1)$-manifold $W^{n+1}$ such that*

$$\partial W^{n+1} = M \sqcup -N.$$

The most exciting application of Pontryagin classes is to Hirzebruch's "signature theorem". Let $M^{4k}$ be an oriented $4k$-manifold. Then, the formula

$$x \otimes y \mapsto \langle x \cup y, [M] \rangle$$

defines a pairing

$$H^{2k}(M)/\text{torsion} \otimes H^{2k}(M)/\text{tors} \to \mathbf{Z}.$$

Poincaré duality implies that this is a perfect pairing, i.e., there is a nonsingular symmetric bilinear form on $H^{2k}(M)/\text{torsion} \otimes \mathbf{R}$. Every symmetric bilinear form on a real vector space can be diagonalized, so that the associated matrix is diagonal, and the only nonzero entries are $\pm 1$. The number of 1s minus the number of $-1$s is called the *signature* of the bilinear form. When the bilinear form comes from a $4k$-manifold as above, this is called the signature of the manifold.

**Lemma 73.3** (Thom). *The signature is an oriented bordism invariant.*

This is an easy thing to prove using Lefschetz duality, which is a deep theorem. Hirzebruch's signature theorem says:

**Theorem 73.4** (Hirzebruch signature theorem). *There exists an explicit rational polynomial $L_k(p_1, \cdots, p_k)$ of degree $4k$ such that*

$$\langle L(p_1(\tau_M), \cdots, p_1(\tau_M)), [M] \rangle = \text{signature}(M).$$

The reason the signature theorem is so interesting is that the polynomial $L(p_1(\tau_M), \cdots, p_1(\tau_M))$ is defined only in terms of the tangent bundle of the manifold, while the signature is defined only in terms of the topology of the manifold. This result was vastly generalized by Atiyah and Singer to the Atiyah-Singer index theorem.

**Example 73.5.** One can show that

$$L_1(p_1) = p_1/3.$$

The Hirzebruch signature theorem implies that $\langle p_1(\tau), [M^4] \rangle$ is divisible by 3.

**Example 73.6.** From Hirzebruch's characterization of the $L$-polynomial, we have

$$L_2(p_1, p_2) = (7p_2 - p_1^2)/45.$$

This imposes very interesting divisibility constraints on the characteristic classes of a tangent bundle of an 8-manifold. This particular polynomial was used by Milnor to produce "exotic spheres", i.e., manifolds which are homeomorphic to $S^7$ but not diffeomorphic to it.

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