# MFC Control System Equations
Revised Formulation

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Arduino Code: `https://github.com/millerjd-netizen/Jon_Thesis/tree/main/MFC%20CONTROL%20SYSTEM%20-%20Arduino%20Giga`

## 1 State Space Formulation

The system is modeled as a continuous-time state space system:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{1}$$

where:

- $\mathbf{x} = \begin{bmatrix} \text{pH} \\ \text{EC} \\ \text{TAN} \end{bmatrix}$ is the state vector
- $\mathbf{u} = \begin{bmatrix} Q_{\text{wine}} \\ Q_{\text{urine}} \\ Q_{\text{spirulina}} \end{bmatrix}$ is the input vector (flow rates in mL/min)
- $A$ is the autonomous dynamics matrix (drift without inputs)
- $B$ is the input coupling matrix

## 2 State Variables & Targets

| **State** | **Physical Meaning** | **Target Range** | **Units** |
|---|---|---|---|
| $x_1 = \text{pH}$ | Hydrogen ion activity: $\text{pH} = -\log_{10}[H^+]$ | 6.8 – 7.2 | – |
| $x_2 = \text{EC}$ | Electrical conductivity: $\kappa = \sum_i c_i \lambda_i z_i$ | 5.0 – 10.0 | mS/cm |
| $x_3 = \text{TAN}$ | Total ammonia nitrogen: $[\text{NH}_4^+] + [\text{NH}_3]$ | 50 – 200 | mg-N/L |

Table 1: State variables with physical definitions and operating targets.

| Input | Dilution | pH | EC (mS/cm) | TAN (mg-N/L) |
|---|---|---|---|---|
| $u_1$: Red Wine | 100:1 | 4.0 | 0.02 | 0.1 |
| $u_2$: Fermented Urine | 4:1 | 8.5 | 6.25 | 200 |
| $u_3$: Spirulina (15 g/L) | – | 9.0 | 1.14 | 144 |

Table 2: Substrate properties after dilution. Spirulina EC calculated from ionic conductivity of dissolved minerals ($Na^+$, $K^+$, $PO_4^{3-}$) at 70% dissolution.

# 3 Inputs (Substrate Solutions)

# 4 Autonomous Dynamics (A Matrix)

**Key Question:** *If no inputs are applied ($\mathbf{u} = 0$), how do the states evolve?*

The autonomous dynamics capture natural processes occurring in the anaerobic reactor:

$$\dot{\mathbf{x}} = A\mathbf{x} = \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} \text{pH} \\ \text{EC} \\ \text{TAN} \end{bmatrix} \tag{2}$$

## 4.1 pH Drift ($a_{11}$, $a_{13}$)

In anaerobic digestion, pH is affected by:

- **$CO_2$ production** from fermentation: $CO_2 + H_2O \rightleftharpoons H_2CO_3 \rightarrow$ lowers pH
- **VFA accumulation**: Acetic, propionic acid production $\rightarrow$ lowers pH
- **Ammonia release**: Protein degradation releases $NH_3 \rightarrow$ raises pH

The net effect depends on the balance. For a stable methanogenic reactor at pH 7.0:

$$\frac{d(\text{pH})}{dt} \approx -0.001 \text{ pH units/min} + 0.00002 \cdot \text{TAN} \tag{3}$$

The TAN coupling ($a_{13}$) reflects the ammonia buffering effect: higher TAN provides more alkalinity.

**Source:** Batstone et al., *Water Science & Technology* 45(10):65-73, 2002 (ADM1 model)

## 4.2 EC Drift ($a_{22}$)

EC changes due to:

- **Ion uptake** by bacteria (N, P, K, Mg) $\rightarrow$ decreases EC
- **Cell lysis** releases ions $\rightarrow$ increases EC
- **Precipitation** of struvite ($MgNH_4PO_4$) $\rightarrow$ decreases EC

Net drift in steady-state operation:

$$\frac{d(\text{EC})}{dt} \approx -0.001 \text{ mS/cm/min} \tag{4}$$

**Source:** Tchobanoglous et al., *Wastewater Engineering*, 5th ed., 2014

### 4.3 TAN Consumption ($a_{33}$)

TAN is consumed by:

- **Bacterial assimilation**: Nitrogen incorporation into biomass
- **Ammonia stripping**: $NH_3$ volatilization (minor at pH $< 8$)

From ADM1 kinetics for mesophilic anaerobic digestion:

$$\frac{d(\text{TAN})}{dt} = -k_{\text{assim}} \cdot X \approx -0.3 \text{ mg-N/L/min} \tag{5}$$

where $k_{\text{assim}} \approx 0.05$ d$^{-1}$ and $X$ is biomass concentration.
**Source:** Batstone et al., *Water Science & Technology* 45(10):65-73, 2002

### 4.4 Complete A Matrix

$$A = \begin{bmatrix} -0.001 & 0 & +0.00002 \\ 0 & -0.001 & 0 \\ 0 & 0 & -0.0003 \end{bmatrix} \text{ [units: min}^{-1}\text{]} \tag{6}$$

**Interpretation:**

- Without inputs, pH drifts down slowly ($CO_2$/VFA production)
- EC drifts down slowly (ion uptake)
- TAN decreases (bacterial consumption) at $\sim$0.3 mg-N/L/min
- System is **stable** (all eigenvalues negative) but drifts away from setpoints

**Justification for constant drift rates:** Research on anaerobic digestion kinetics has shown that when operating within optimal ranges (pH 6.8–7.5, TAN 50–200 mg-N/L), microbial reaction rates follow near zero-order kinetics with respect to substrate concentration, meaning consumption rates remain approximately constant rather than concentration-dependent (Batstone et al., *Water Science & Technology* 45(10):65-73, 2002; Siegrist et al., *Water Science & Technology* 45(10):93-100, 2002).

## 5 Input Coupling Matrix (B Matrix)

The B matrix describes how each input affects each state. For a well-mixed reactor with volume $V$:

$$B = \begin{bmatrix} \frac{\partial \dot{\text{pH}}}{\partial Q_1} & \frac{\partial \dot{\text{pH}}}{\partial Q_2} & \frac{\partial \dot{\text{pH}}}{\partial Q_3} \\ \frac{\partial \dot{\text{EC}}}{\partial Q_1} & \frac{\partial \dot{\text{EC}}}{\partial Q_2} & \frac{\partial \dot{\text{EC}}}{\partial Q_3} \\ \frac{\partial \dot{\text{TAN}}}{\partial Q_1} & \frac{\partial \dot{\text{TAN}}}{\partial Q_2} & \frac{\partial \dot{\text{TAN}}}{\partial Q_3} \end{bmatrix} \tag{7}$$

## 5.1 Derivation of Numerical Coefficients

For a CSTR (continuously stirred tank reactor) with volume $V = 1000$ mL, the mixing equation gives:

$$\frac{dx}{dt} = \frac{Q}{V}(x_{\text{in}} - x) \tag{8}$$

At the operating point (pH = 7.0, EC = 5.0, TAN = 100), the coefficients are:

### 5.1.1 Wine ($u_1$): pH 4.0, EC 0.02, TAN 0.1

$$b_{11} = \frac{1}{V}(\text{pH}_{\text{wine}} - \text{pH}_{\text{reactor}}) = \frac{1}{1000}(4.0 - 7.0) = -0.003 \text{ pH/(mL/min)} \tag{9}$$

$$b_{21} = \frac{1}{V}(\text{EC}_{\text{wine}} - \text{EC}_{\text{reactor}}) = \frac{1}{1000}(0.02 - 5.0) = -0.005 \text{ (mS/cm)/(mL/min)} \tag{10}$$

$$b_{31} = \frac{1}{V}(\text{TAN}_{\text{wine}} - \text{TAN}_{\text{reactor}}) = \frac{1}{1000}(0.1 - 100) = -0.10 \text{ (mg-N/L)/(mL/min)} \tag{11}$$

### 5.1.2 Urine ($u_2$): pH 8.5, EC 6.25, TAN 200

$$b_{12} = \frac{1}{1000}(8.5 - 7.0) = +0.0015 \text{ pH/(mL/min)} \tag{12}$$

$$b_{22} = \frac{1}{1000}(6.25 - 5.0) = +0.00125 \text{ (mS/cm)/(mL/min)} \tag{13}$$

$$b_{32} = \frac{1}{1000}(200 - 100) = +0.10 \text{ (mg-N/L)/(mL/min)} \tag{14}$$

### 5.1.3 Spirulina ($u_3$): pH 9.0, EC 1.14, TAN 144

$$b_{13} = \frac{1}{1000}(9.0 - 7.0) = +0.002 \text{ pH/(mL/min)} \tag{15}$$

$$b_{23} = \frac{1}{1000}(1.14 - 5.0) = -0.00386 \text{ (mS/cm)/(mL/min)} \tag{16}$$

$$b_{33} = \frac{1}{1000}(144 - 100) = +0.044 \text{ (mg-N/L)/(mL/min)} \tag{17}$$

## 5.2 Complete B Matrix

$$B = \begin{bmatrix} -0.003 & +0.0015 & +0.002 \\ -0.005 & +0.00125 & -0.00386 \\ -0.10 & +0.10 & +0.044 \end{bmatrix} \tag{18}$$

**Units:** Row 1: pH/(mL/min), Row 2: (mS/cm)/(mL/min), Row 3: (mg-N/L)/(mL/min)
**Key observations:**

- Wine (column 1): Decreases all states (acidic, low EC, dilutes TAN)
- Urine (column 2): Increases all states (alkaline, high EC, high TAN)
- Spirulina (column 3): Increases pH and TAN, but **decreases EC** (its EC of 1.14 is below reactor EC of 5.0)

## 6 Complete State Space Model

$$\underbrace{\begin{bmatrix} \dot{\text{pH}} \\ \dot{\text{EC}} \\ \dot{\text{TAN}} \end{bmatrix}}_{\dot{\mathbf{x}}} = \underbrace{\begin{bmatrix} -0.001 & 0 & +0.00002 \\ 0 & -0.001 & 0 \\ 0 & 0 & -0.0003 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \text{pH} \\ \text{EC} \\ \text{TAN} \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} -0.003 & +0.0015 & +0.002 \\ -0.005 & +0.00125 & -0.00386 \\ -0.10 & +0.10 & +0.044 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} Q_{\text{wine}} \\ Q_{\text{urine}} \\ Q_{\text{spirulina}} \end{bmatrix}}_{\mathbf{u}} \tag{19}$$

## 7 Physical Basis for Input Properties

### 7.1 Wine pH (4.0 after 100:1 dilution)

Tartaric acid dissociation:

$$\text{H}_2\text{T} \rightleftharpoons \text{H}^+ + \text{HT}^- \quad pK_{a1} = 2.98 \tag{20}$$

At 100:1 dilution, wine pH rises from ∼3.5 to ∼4.0 due to reduced acid concentration.
**Source:** Ribéreau-Gayon et al., *Handbook of Enology Vol. 2*, Wiley, 2006

### 7.2 Urine EC (6.25 mS/cm after 4:1 dilution)

Fermented urine EC ≈ 25 mS/cm (undiluted). Major ionic contributors:

$$\kappa = \sum_i c_i \lambda_i |z_i| \quad \text{(Kohlrausch's Law)} \tag{21}$$

| **Ion** | **Conc. (mM)** | $\lambda^0$ (S·cm²/mol) | **Contribution (%)** |
|---|---|---|---|
| $NH_4^+$ | 350 | 73.5 | 35% |
| $Na^+$ | 150 | 50.1 | 15% |
| $K^+$ | 50 | 73.5 | 8% |
| $Cl^-$ | 200 | 76.3 | 25% |
| $PO_4^{3-}$ | 30 | 69.0 | 10% |

Table 3: Ionic contributions to fermented urine conductivity.

**Source:** Udert et al., *Water Research* 40(9):1803-1812, 2006

### 7.3 Spirulina EC (1.14 mS/cm at 15 g/L)

Spirulina mineral content (mg/100g dry weight):

- Na: 1048, K: 1363, Ca: 120, Mg: 195, P: 1180, Cl: 400

EC calculated using ionic conductivity with 70% dissolution factor:

$$\text{EC} = 0.5 + \sum_{\text{ions}} \frac{c_i \cdot \lambda_i}{1000} \cdot 0.70 \cdot 0.80 = 1.14 \text{ mS/cm} \tag{22}$$

**Source:** Vonshak, *Spirulina platensis: Physiology*, Taylor & Francis, 1997

### 7.4 Spirulina TAN (144 mg-N/L at 15 g/L)

Protein content provides nitrogen:

$$\text{TAN} = (\text{conc.}) \times (\text{protein } \%) \times (\text{N in protein}) \times (\text{availability}) \tag{23}$$

$$\text{TAN} = 15\frac{\text{g}}{\text{L}} \times 0.60 \times 0.16 \times 0.10 \times 1000 = 144 \text{ mg-N/L} \tag{24}$$

where 10% availability represents immediately soluble nitrogen (full release requires proteolysis over days).

**Source:** Becker, E.W., *Biotechnology Advances* 25:207-210, 2007

## 8 Trace Metal Dynamics

While pH, EC, and TAN are the primary control variables, trace metals are tracked as secondary states. Spirulina (15 g/L) provides:

| Metal | In Spirulina (mg/L) | Target Range | Status |
|---|---|---|---|
| Fe | 4.28 | 1.0 – 10.0 | OK |
| Zn | 0.30 | 0.1 – 1.0 | OK |
| Cu | 0.92 | 0.01 – 0.5 | High |
| Mn | 0.29 | 0.1 – 1.0 | OK |
| Se | 0.001 | 0.01 – 0.1 | Low |
| Ni | 0 | 0.05 – 1.0 | None |
| Co | 0 | 0.01 – 0.3 | None |

Table 4: Trace metal content from spirulina. Note: Spirulina contains no Ni or Co.

Metal consumption by bacteria follows first-order kinetics:

$$\frac{d[\text{Metal}]}{dt} = -k_m \cdot [\text{Metal}] \quad \text{where } k_m \approx 0.0005 \text{ min}^{-1} \tag{25}$$

**Source:** Fermoso et al., *Bioresource Technology* 100:1983-1991, 2009

## 9 Control Law

Priority-based bang-bang control with the following logic:

$$\mathbf{u} = \begin{cases} [Q_{\max}, 0, 0]^T & \text{if pH} > 7.2 \quad \text{(dose wine)} \\ [0, Q_{\max}, 0]^T & \text{if pH} < 6.8 \quad \text{(dose urine)} \\ [0, Q_{\max}, 0]^T & \text{if TAN} < 50 \text{ and EC} < 5.0 \quad \text{(dose urine)} \\ [0, 0, Q_{\max}]^T & \text{if TAN} < 50 \text{ and EC} \geq 5.0 \quad \text{(dose spirulina)} \\ [0, Q_{\max}, 0]^T & \text{if EC} < 5.0 \quad \text{(dose urine)} \\ [0, 0, 0]^T & \text{otherwise} \end{cases} \tag{26}$$

where $Q_{\max} = 5$ mL/min (peristaltic pump flow rate).

### 9.1 Priority Rationale

1. **pH (highest priority)**: Bacterial survival requires pH 6.5–7.5. Outside this range, methanogenesis stops.
2. **TAN**: Nitrogen is essential for growth. At pH 7.0, toxic free ammonia is only 0.56% of TAN.
3. **EC**: Affects power output but bacteria tolerate 1–20 mS/cm.
4. **Trace metals (lowest priority)**: Required for enzyme function but needed only in $\mu$g/L to mg/L quantities; deficiencies develop slowly over days to weeks, allowing periodic rather than continuous correction.

## 10 References


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