Bayesian Hierarchical Models

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Background

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  - $D =$ the dataset
- Conditional probability (Bayes Rule) provides the "correct" way to update beliefs, given the data collected (and a probability model).

ALGEBRA

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- If H is a vector, denominator can be difficult to compute.

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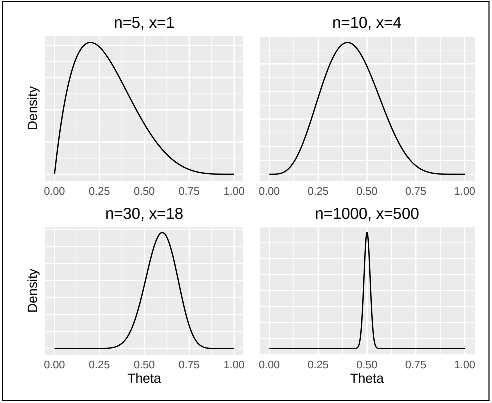
Example: posterior probability distributions for θ



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  - Q. What sort of prior does the decision *Do Not Reject* imply?

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- The conjugate prior for a Bernoulli R.V. is Beta(α, β).

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- $\text{E}[\theta|x] = \frac{x+\alpha}{n+\alpha+\beta}$, $\text{var}[\theta|x] = \frac{(x+\alpha)(n-x+\beta)}{(n+\alpha+\beta)^2(n+1+\alpha+\beta)}$

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&= \Sigma_\theta P(x_{\text{new}}|\theta, x) P(\theta|x) \\
&= \Sigma_\theta P(x_{\text{new}}|\theta) P(\theta|x)
\end{aligned}$$

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Experiment (j)	1	2	3	4	5
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- Q. Define $\theta_j = \mathrm{E}_x[\frac{x_j}{n_j}]$. Suppose $\theta_4 = 0.3$, how does this influence your belief about θ_5?

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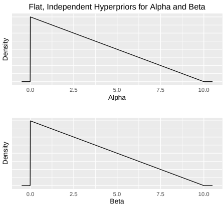
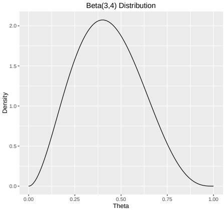
- Q. Can you give an example where it would be useful to insist that whatever the true value of α, $\alpha = \beta$?

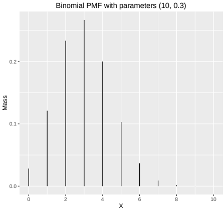
STRUCTURE OF THE MODEL



(a) $P(\phi)$



(b) $P(\theta|\phi = (3,4))$



(c) $P(x|\theta = 0.3)$

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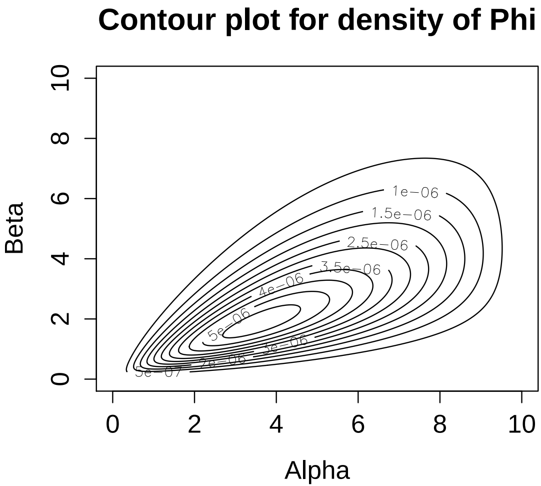
  $P(\theta|x) = \Sigma_\phi P(\phi, \theta|x)$ are the marginal posteriors.
  $P(x_{new}|x) = \Sigma_\phi \Sigma_\theta P(\phi|x)P(\theta|\phi, x)P(x_{new}|\phi, \theta, x)$.

JOINT AND MARGINAL POSTERIORS

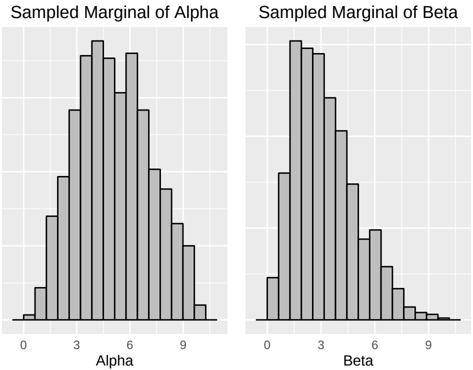
$$
\begin{aligned}
P(\phi, \theta|x) &\propto P(\phi)P(\theta|\phi)P(x|\theta) \\
&\propto P(\phi)\Pi_{j=1}^{4}\theta_j^{\alpha-1}(1-\theta_j)^{\beta-1}\frac{1}{B(\alpha,\beta)}\Pi_{j=1}^{4}\theta_j^{x_j}(1-\theta_j)^{n_j-x_j} \\
&= e^{-\alpha-\beta}\Pi_{j=1}^{4}\theta_j^{\alpha-1+x_j}(1-\theta_j)^{\beta-1+n_j-x_j}\frac{1}{B(\alpha,\beta)} \\
P(\theta|\phi, x) &= \Pi_{j=1}^{4}\theta_j^{\alpha-1+x_j}(1-\theta_j)^{\beta-1+n_j-x_j}\frac{1}{B(\alpha+x_j,\beta+n_j-x_j)} \\
P(\phi|x) &= \frac{P(\phi,\theta|x)}{P(\theta|\phi,x)} \propto e^{-\alpha-\beta}\Pi_{j}^{4}\frac{B(\alpha+x_j,\beta+n_j-x_j)}{B(\alpha,\beta)}
\end{aligned}
$$

Contour plot of $P(\phi|x)$

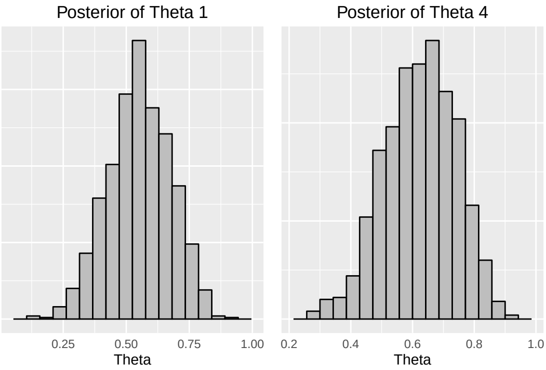


SAMPLED POINTS FROM JOINT PMF OF $\phi|x$

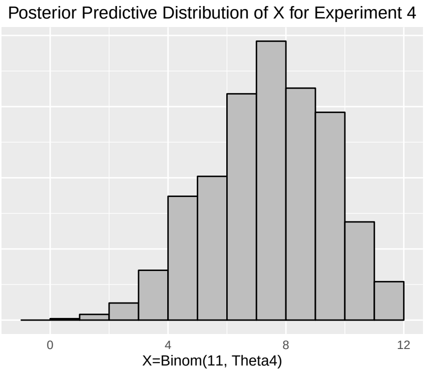
By rejection sampling from the numerically computed joint posterior of ϕ, I find parameter values that my posteriors of interest depend on:



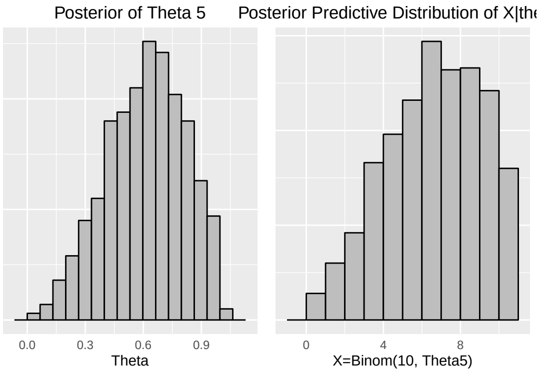
Empirical $P(\theta|x)$ for θ_1 and θ_4



Empirical $P(x_{new}|x)$ for experiment 4 (θ_4)



EMPIRICAL $P(x_{new}|x)$ FOR EXPERIMENT 5 (θ_5)



Given my (too cautious) hyperpriors, not much can be said about experiment five.

MORE CONCEPTS/QUESTIONS

- Posterior predictive checks

More concepts/Questions

- Posterior predictive checks
- Sensitivity analysis

More concepts/Questions

- Posterior predictive checks
- Sensitivity analysis
- Proper/improper priors $\rightarrow$ proper/improper posteriors

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- Posterior predictive checks
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Questions?

Fin.