TIE-DECAY TEMPORAL NETWORKS IN CONTINUOUS TIME AND EIGENVECTOR-BASED CENTRALITIES *

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Abstract. Network theory provides a useful framework for studying interconnected systems of interacting agents. Many networked systems evolve continuously in time, but most existing methods for analyzing time-dependent networks rely on discrete or discretized time. In this paper, we propose a novel approach for studying networks that evolve in continuous time by distinguishing between *interactions*, which we model as discrete contacts, and *ties*, which represent strengths of relationships as functions of time. To illustrate our framework of *tie-decay networks*, we show how to examine — in a mathematically tractable and computationally efficient way — important (i.e., 'central') nodes in networks in which tie strengths decay in time after individuals interact. As a concrete illustration, we introduce a continuous-time generalization of PageRank centrality and apply it to a network of retweets during the 2012 National Health Service controversy in the United Kingdom. Our work also provides guidance for similar generalizations of other tools from network theory to continuous-time networks with tie decay, including for applications to streaming data.

Key words. Networks, Temporal networks, Tie decay, Centrality, PageRank, Continuous time

AMS subject classifications. 05C82, 05C81, 05C50, 91D30, 90B18

1. Introduction.

Networks provide a versatile framework to model and analyze complex systems of interacting entities [42]. In many of these systems, the interaction patterns change in time, and the entities can also leave or enter the system at different times. To accurately model these systems, it is essential to incorporate temporal information about their interactions into network representations [21, 22]. Such time-dependent networks are often called *temporal networks*.

One major challenge in the analysis of temporal networks is that one often has to discretize time by aggregating connections into time windows. Given a discrete or discretized set of interactions, one can then analyze communities, important nodes, and other facets of such networks by examining a multilayer-network representation of these interactions [1,21,27,54]. One challenge with aggregation is there may not be any obvious or even a 'correct' size of a time window (even when such aggregation employs nonuniform time windows [8, 50, 51, 53]). A window that is too small risks missing important structures of networks (e.g., by construing a signal as noise), but using an overly large window may obscure important temporal features of time series. (See [11] for one discussion.) Moreover, in many social systems, interactions are bursty [4, 22, 28], which is a crucial consideration when aggregating interactions [20] and is potentially a major cause of concern when using homogeneous time windows [51]. Bursty interactions not only present a challenge when choosing the width of time windows, but they also challenge *where* to place them; shifting time windows forwards or backwards may significantly alter the statistics of data, even when one does not change the windows' width [28].

From a modeling perspective, aggregation of interactions often may not be an appropriate approach for systems with asynchronous activity or which evolve continuously in time. See [58] for a recent investigation of biological contagions, [61] for a recent study of influential users in social networks, and [62, 63] for a generalization

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of the formalism of 'activity-driven networks' to continuous time. In many cases, contacts in a temporal network can have a noninstantaneous duration, and it can be important to take such information into account [44, 52]. For example, the phone-call data that were studied in Ref. [18] requires contacts to exist for the duration of a phone call. In other cases, interactions can be instantaneous (e.g., a mention in a tweet, or a text message), and their importance decreases over time [7, 33]. For many types of temporal networks (e.g., feeds on social media), there is also a decay in attention span when it comes to reacting to posts [19, 36, 37].

In the present paper, we propose a framework to model temporal networks in which the strength of a connection (i.e., a tie) can evolve continuously in time. For example, perhaps the strength of a tie decays exponentially after the most recent interaction. (See also the decay that occurs in some point-process models, such as Hawkes processes [33].) Our mathematical formulation of such 'tie-decay networks' allows us to examine them using analytical calculations and to implement them efficiently in real-world applications with streaming data. We showcase our formulation by computing continuous-time PageRank centrality scores for a temporal network that we construct from a large collection of Twitter interactions over the course of a year.

The rest of our paper is organized as follows. In Section 2, we formalize our discussion of ties, interactions, and temporal networks. We also introduce the notion of *tie-decay networks*, which is the focus of our study. In Section 3, we formulate how to study eigenvector-based centralities in tie-decay networks. In Section 4, we discuss and compute tie-decay PageRank centralities to examine important agents in a National Health Service (NHS) retweet network. In Section 5, we conclude and discuss the implications of our work. We give proofs of our main theoretical results in Appendix A.

2. Ties, Interactions, and Temporal Networks. Our objective is to construct a continuous-time temporal network that can capture the evolution of relationships between agents. We make an important distinction between 'interactions' and 'ties'. An *interaction* between two agents is an event that takes place at a specific time interval or point in time (e.g., a face-to-face meeting, a text message, or a phone call). In contrast, a *tie* between two agents is a relationship between them, and it can have a weight to represent its strength (such as the strength of a friendship or collaboration). Ties between agents strengthen with repeated interactions, but they can also deteriorate in their absence [7, 41]. In the present paper, we restrict ourselves to modeling instantaneous interactions, but it is possible to generalize our network formulation to incorporate interactions with different durations.

Consider a set of n interacting agents, and let $B(t)$ be the $n \times n$ time-dependent, real, non-negative matrix whose entries $b_{ij}(t)$ represent the connection strengths between agents i and j at time t. To construct a continuous-time temporal network of ties, we make two modeling assumptions about how ties evolve and how interactions strengthen them:

1. In the absence of interactions, we assume that ties decay exponentially, as proposed in [25]. In mathematical terms, $b'_{ij} = -\alpha b_{ij}$ (where the prime represents differentiation with respect to time), so $b_{ij}(t) = b_{ij}(0)e^{-\alpha t}$ for some $\alpha \geq 0$ and initial condition $b_{ij}(0)$.
2. If the agents interact at time $t = \tau$, the strength of the tie grows instantaneously by 1, and it then decays as normal. This choice differs from [25], who reset the strength to 1 after each interaction.

Taken together, these assumptions imply that the temporal evolution of a tie satisfies the ordinary differential equation (ODE)

$$b'_{ij} = -\alpha b_{ij} + \delta(t-\tau)e^{-\alpha(t-\tau)}\,. \tag{1}$$

In equation (1), we represent an instantaneous interaction at $t = \tau$ as a pulse with the Dirac δ-function. If the tie has the resting initial condition $b_{ij}(0) = 0$, the solution to equation (1) is $b_{ij}(t) = H(t-\tau)e^{-\alpha(t-\tau)}$, where $H(t)$ is the Heaviside step function. When there are multiple interactions between agents, we represent them as streams of pulses in the $n \times n$ matrix $\widetilde{A}(t)$. If agent i interacts with agent j at times $\tau_{ij}^{(1)}$, $\tau_{ij}^{(2)}$, $\ldots$, then $\tilde{a}_{ij}(t) = \sum_k \delta(t-\tau_{ij}^{(k)})e^{-\alpha(t-\tau_{ij}^{(k)})}$. We rewrite equation (1) as

$$b'_{ij} = -\alpha b_{ij} + \tilde{a}_{ij}\,, \tag{2}$$

which from a resting[1] initial condition has solution $b_{ij}(t) = \sum_k H(t-\tau_{ij}^{(k)})e^{-\alpha(t-\tau_{ij}^{(k)})}$.

In practice — and, specifically, in data-driven applications — we can easily obtain $B(t)$ if we discretize time so that there is at most one interaction during each time step of length Δt (e.g., in a Poisson process). Such time discretization is common in the simulation of stochastic dynamical systems, such as in Gillespie algorithms [10,49,59]. In this case, we let $A(t)$ be the $n \times n$ matrix in which an entry $a_{ij}(t) = 1$ if node i interacts with node j at time t and $a_{ij}(t) = 0$ otherwise. At each time step, $A(t)$ has at most one nonzero entry for a directed network (or two of them, for an undirected network). Therefore,

$$B(t+\Delta t) = e^{-\alpha\Delta t}B(t) + A(t+\Delta t)\,. \tag{3}$$

Equivalently, if interactions between pairs of agents occur at times $\tau^{(\ell)}$ (it can be a different pair at different times) such that $0 \leq \tau^{(0)} < \tau^{(1)} < \ldots < \tau^{(T)}$, then at $t \geq \tau^{(T)}$, we have

$$B(t) = \sum_{k=0}^{T} e^{-\alpha(t-\tau^{(k)})}A(\tau^{(k)})\,. \tag{4}$$

If there are no interactions at time t, then $A(t)$ is a matrix of 0 entries.

Our continuous-time approach avoids having to impose a hard partition of the interactions into bins. However, one still needs to choose a value for the decay parameter α. Another benefit of our approach is that it eliminates the placement of the time windows as a potential source of bias [28]. When choosing a value for α, it may intuitive to think about the *half-life* $\tau_{1/2}$ of a tie, as it gives the amount of time for a tie to lose half of its strength in the absence of new interactions. Given $\alpha > 0$, the half-life of a tie is $\tau_{1/2} = \alpha^{-1}\ln 2$.

In Fig. 1A, we illustrate the evolution of a tie in our modeling framework. If agents i and j have never interacted before time t_0, then $b_{ij}(t_0) = 0$. Suppose that they first interact at time $\tau^{(1)} > t_0$ (i.e., $a_{ij}(\tau^{(1)}) = 1$). Their tie strength then increases by 1, so $b_{ij}(\tau^{(1)}) = 1$; it subsequently decays exponentially: $b_{ij}(t > \tau^{(1)}) = e^{-\alpha(t-\tau^{(1)})}$. If agents i and j next interact at time $\tau^{(2)} > \tau^{(1)}$, so that $a_{ij}(\tau^{(2)}) = 1$, their tie strength becomes $b_{ij}(\tau^{(2)}) = e^{-\alpha(\tau^{(2)}-\tau^{(1)})} + 1$, and so on.

[1] It is not unlike a Norwegian blue parrot. (Norwegian blues stun easily.)

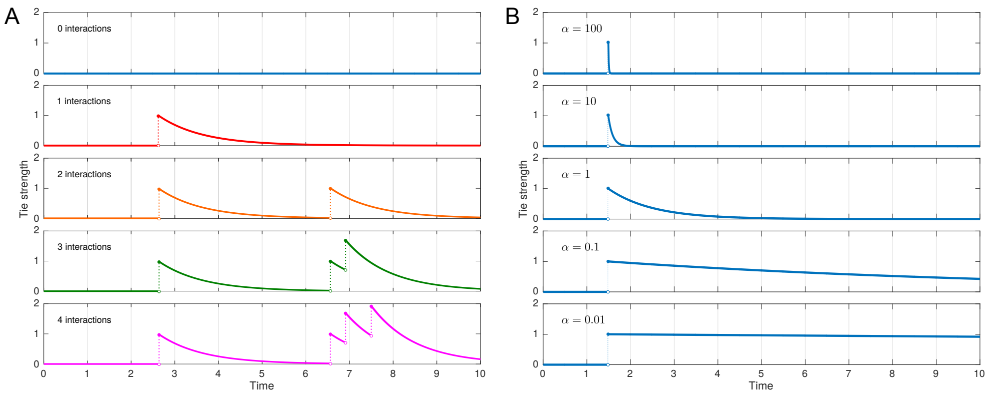

Fig. 1: **A:** Evolution of tie strength with exponential decay, based on equation (2). **B:** The decay rate α determines how fast the tie decays.

3. Eigenvector-Based Centrality Scores in Tie-Decay Networks. One common question that arises when analyzing a network is the following: *What are the most important nodes?* To examine this question, researchers have developed numerous *centrality scores* to quantify the importances of nodes according to different criteria [42]. An important family of centrality scores arise from spectral properties of the adjacency matrix (or other matrices) of a network [6, 39, 48, 54]. Eigenvector-based centrality scores have been very insightful in numerous applications, and one can use efficient numerical algorithms to compute eigenvectors and singular vectors of matrices [16, 56]. Some of the most widely-used spectral centrality scores for directed networks include PageRank [15, 46], which exploits the properties of random walks on a network, and hub and authority scores [29], which exploit both random-walk properties and the asymmetry of connections in directed networks.

In temporal networks, centrality scores must incorporate not only which nodes and edges are present in a network, but also when they are present [26, 47], making the computation of centrality measures a challenging task. Some approaches have exploited numerical methods for dynamical systems to compute specific scores, such as a Katz centrality for temporal networks [18], and others have employed aggregated or multilayer representations of temporal networks to calculate spectral centrality scores [3, 54]. However, these approaches have either been limited to a specific kind of centrality, or they have relied on the judicious aggregation of interactions into time bins. Such an operation is far from straightforward: overly coarse bins obfuscate temporal features, whereas bins that are too small may dilute network structure, resulting in scores that may result more from noise than from signals.

Our tie-decay network formulation in equation (2) allows us to employ efficient numerical techniques to compute a variety of spectral centrality scores. One can tune the decay parameter α (which one can also generalize to be node-specific, tie-specific, or time-dependent) to consider different temporal scales. A key benefit of our framework is that we can easily incorporate both new interactions and new nodes as a network evolves.

3.1. Tie-Decay PageRank Centrality. PageRank centrality (or simply 'PageRank') is a widely-used eigenvector-based centrality score for time-independent networks [46]. The PageRank score of a node corresponds to its stationary distribution in a teleporting random walk [15,39]. In this type of random walk, a walker stationed at a node continues its walk by following an outgoing edge with probability $\lambda \in [0,1)$ (where, in most versions, one chooses the edge with a probability proportional to its weight), and it 'teleports' to some other node in the network with probability $1-\lambda$. It is common to choose the destination node uniformly at random, but many other choices are possible [15,31]. In the present paper, we employ uniform teleportation with $\lambda = 0.85$ (which is the most common choice). Let B be the adjacency matrix of a weighted network with n nodes, so b_{ij} represents the weight of a directed tie from node i to node j. The $n \times 1$ vector $\boldsymbol{\pi}$ of PageRank scores, with $\boldsymbol{\pi} \geq 0$ and $||\boldsymbol{\pi}||_1 = 1$, is the leading-eigenvector solution of the eigenvalue problem

$$G^T \boldsymbol{\pi} = \boldsymbol{\pi} \,, \tag{5}$$

where G is the $n \times n$ rate matrix of a teleporting random walk:

$$\begin{aligned} G &= \lambda \left(D^\dagger B + \mathbf{c}\mathbf{v}^T \right) + (1-\lambda)\mathbb{1}\mathbf{v}^T \,, \\ &= \lambda P + (1-\lambda)\mathbb{1}\mathbf{v}^T \,, \end{aligned} \tag{6}$$

where $P = D^\dagger B + \mathbf{c}\mathbf{v}^T$; the matrix D is the diagonal matrix of weighted out-degrees, so $d_{ii} = \sum_k b_{ik}$ and $d_{ij} = 0$ when $i \neq j$; and $D^\dagger$ is its Moore–Penrose pseudo-inverse. The $n \times 1$ vector $\mathbf{c}$ is an indicator of 'dangling nodes' (i.e., nodes with 0 out-degree): $c_i = 1 - d_{ii}^\dagger \sum_k b_{ik}$, so $c_i = 1$ if the out-degree of i is 0, and $c_i = 0$ otherwise. Additionally, $\mathbb{1}$ is the $n \times 1$ vector of 1s, and the $n \times 1$ distribution vector $\mathbf{v}$ encodes the probabilities of each node to receive a teleported walker. In our paper, we use $v_i = 1/n$ for all i.

The perturbations to $D^\dagger B$ introduced by $\mathbf{v}$ and $\mathbf{c}$ ensure the ergodicity of the Markov chain, so G^T has a unique right leading eigenvector $\boldsymbol{\pi}$, and all of its entries are strictly positive. To calculate $\boldsymbol{\pi}$, one can perform a power iteration on G^T [56], but in practice we do not need to explicitly construct G^T. The iteration

$$\boldsymbol{\pi}^{(k+1)} = \lambda P^T \boldsymbol{\pi}^{(k)} + (1-\lambda)\mathbf{v} \,, \tag{7}$$

with $\boldsymbol{\pi}^{(0)} = (\mathbf{0})$ or $\boldsymbol{\pi}^{(0)} = \mathbf{v}$, converges to $\boldsymbol{\pi}$ and preserves the sparsity of P. This choice, which ensures that computations are efficient, is equivalent to a power iteration [15].

To compute time-dependent PageRank scores from the tie-strength matrix $B(t)$, we define the temporal transition matrix

$$P(t) = D^\dagger(t) B(t) + \mathbf{c}(t)\mathbf{v}^T \,, \tag{8}$$

where $D(t)$ is the diagonal matrix of weighted out-degrees (i.e., the row sums of $B(t)$) at time t. The rank-1 correction $\mathbf{c}(t)\mathbf{v}^T$ depends on time, because the set of dangling nodes can change in time (though $\mathbf{v}$ remains fixed). The iteration to obtain the time-dependent vector of PageRank scores $\boldsymbol{\pi}(t)$ is now given by

$$\boldsymbol{\pi}^{(k+1)}(t) = \lambda P^T(t) \boldsymbol{\pi}^{(k)}(t) + (1-\lambda)\mathbf{v} \,. \tag{9}$$

To understand the temporal evolution of $\boldsymbol{\pi}(t)$, we begin by establishing some properties of the temporal transition matrix $P(t)$ in the following lemma.

Lemma 1. *When there are no new interactions between times t and $t + \Delta t$, the entries of $A(t+\Delta t)$ are all 0 and $P(t+\Delta t) = P(t)$. If there is a single new interaction* *between nodes i and j, so that $a_{ij}(t+\Delta t) = 1$, then $P(t+\Delta t) = P(t) + \Delta P$, where*

$$\Delta P = \frac{1}{1+e^{-\alpha\Delta t} d_{ii}(t)} \mathbf{e}_i \mathbf{e}_j^T - \frac{1}{d_{ii}(t)\,(1+e^{-\alpha\Delta t} d_{ii}(t))} \mathbf{e}_i \mathbf{e}_i^T B(t) - c_i(t) v_i \mathbf{e}_i \mathbb{1}^T\,, \tag{10}$$

and $\mathbf{e}_i$ and $\mathbf{e}_i$ are, respectively, the i-th and j-th canonical vectors.

The first term in the right-hand side of equation (10) is a matrix whose only nonzero entry is the (i, j)-th term, the second term corresponds to a rescaling of the i-th row of $B(t)$, and the third term is the perturbation due to teleportation. An important implication of Lemma 1 is that when there are no new interactions, the PageRank scores do not change: $\boldsymbol{\pi}(t+\Delta t) = \boldsymbol{\pi}(t)$. If each node or tie has different decay rates (so that now we index α as α_i or α_{ij}), then this is not necessarily the case.

When there are new interactions, the following result sets an upper bound on how much the PageRank scores can change.

THEOREM 2. *Suppose that there is a single interaction between times t and $t+\Delta t$ from node i to j, so that the change ΔP in the transition matrix satisfies equation (10). It follows that*

$$||\boldsymbol{\pi}(t+\Delta t) - \boldsymbol{\pi}(t)||_1 \leq \frac{2\lambda}{1-\lambda} \min\left\{\pi_i(t),\, \frac{1}{1+e^{-\alpha\Delta t} d_{ii}(t)} - \frac{c_i(t)}{2}\right\}. \tag{11}$$

We present two corollaries of Theorem 2.

COROLLARY 3. *If i is a dangling node at time t, then*

$$||\boldsymbol{\pi}(t+\Delta t) - \boldsymbol{\pi}(t)||_1 \leq \frac{2\lambda}{1-\lambda} \min\left\{\pi_i(t), \frac{1}{2}\right\}. \tag{12}$$

COROLLARY 4. *If i has one or more outgoing edges at time t, then*

$$||\boldsymbol{\pi}(t+\Delta t) - \boldsymbol{\pi}(t)||_1 \leq \frac{2\lambda}{1-\lambda} \min\left\{\pi_i(t),\, \frac{1}{1+e^{-\alpha\Delta t} d_{ii}(t)}\right\}. \tag{13}$$

We give proofs of Lemma 1, Theorem 2, and Corollaries 3 and 4 in Appendix A.

3.1.1. Temporal Power Iteration. To calculate the PageRank scores at time $t+\Delta t$, we use the power iteration in equation (9) to update the PageRank vector using $\boldsymbol{\pi}(t)$ as the initial value. That is,

$$\boldsymbol{\pi}^{(0)}(t+\Delta t) = \boldsymbol{\pi}(t)\,. \tag{14}$$

The relative error of the computed PageRank vector at iteration k is

$$\left|\left|e_{\text{rel}}^{(k)}\right|\right|_1 = \left|\left|\boldsymbol{\pi}(t+\Delta t) - \boldsymbol{\pi}^{(k)}(t+\Delta t)\right|\right|_1\,. \tag{15}$$

A result from [5] (Theorem 6.1) shows that $\left|\left|e_{\text{rel}}^{(k)}\right|\right|_1 \leq \lambda^k \left|\left|e_{\text{rel}}^{(0)}\right|\right|_1$. From $\boldsymbol{\pi}^{(0)}(t+\Delta t) = \boldsymbol{\pi}(t)$ and Theorem 2, it then follows that

$$\begin{aligned} \left|\left|e_{\text{rel}}^{(k)}\right|\right|_1 &\leq \lambda^k\, ||\boldsymbol{\pi}(t+\Delta t) - \boldsymbol{\pi}(t)||_1 \\ &\leq \frac{2\lambda^{k+1}}{1-\lambda} \min\left\{\pi_i(t),\, \frac{1}{1+e^{-\alpha\Delta t} d_{ii}(t)} - \frac{c_i(t)}{2}\right\}. \end{aligned} \tag{16}$$

Thus, we may select a level of error tolerance ϵ such that $\left|\left|e_{\text{rel}}^{(k^*)}\right|\right|_1 \leq \epsilon$ for some number of iterations k^*. This k^* represents the maximum number of iterations required to reach a relative error of at most ϵ, and may be computed as

$$(17) \quad k^* = \frac{\ln(\epsilon) - \ln(2) + \ln(1-\lambda) - \ln\left(\min\left\{\pi_i(t)\,,\ \frac{1}{1+e^{-\alpha\Delta t}d_{ii}(t)} - \frac{c_i(t)}{2}\right\}\right)}{\ln(\lambda)} - 1\,.$$

In practice [15], we can instead track the residual after k iterations:

$$\begin{aligned}\mathbf{r}^{(k)}(t+\Delta t) &= \boldsymbol{\pi}^{(k+1)}(t+\Delta t) - \boldsymbol{\pi}^{(k)}(t+\Delta t)\\ &= (1-\lambda)\mathbf{v} - \left(I_n - \lambda P^T(t+\Delta t)\right)\boldsymbol{\pi}^{(k)}(t+\Delta t)\,.\end{aligned} \tag{18}$$

We can use this residual to bound the relative error,

$$\begin{aligned}\left|\left|\boldsymbol{\pi}(t+\Delta t) - \boldsymbol{\pi}^{(k)}(t+\Delta t)\right|\right|_1 &= \left|\left|\left(I_n - \lambda P^T(t+\Delta t)\right)^{-1}\mathbf{r}^{(k)}(t+\Delta t)\right|\right|_1\\ &\leq \frac{1}{1-\lambda}\left|\left|\mathbf{r}^{(k)}(t+\Delta t)\right|\right|_1\,,\end{aligned} \tag{19}$$

and thereby monitor the convergence of the power iteration. In our experiments, we always obtain $\|\boldsymbol{\pi}^{(k+1)}(t+\Delta t) - \boldsymbol{\pi}^{(k)}(t+\Delta t)\|_{l_1} < 10^{-6}$ in two iterations or fewer (see Sec. 4.2).

4. The National Health Service (NHS) Retweet Network. We now compute tie-decay continuous-time PageRank scores to track the evolution of node importances over time in a large data set of time-annotated interactions on Twitter.

Twitter is a social-media platform that has become a prominent channel for organizations, individuals, and 'bots' to broadcast events, share ideas, report events, and socialize by posting messages (i.e., 'tweets') of at most 140 characters in length [30]. (Recently, Twitter expanded the maximum tweet length to 280 characters, but the maximum was 140 characters at the time that our data were collected.) Data from Twitter has allowed researchers to study patterns and trends associated with a plethora of large-scale political and social events and processes, such as protests and civil unrest, public health, and information propagation [2,3,9,13,17,40,45,55].

Twitter accounts (which can represent an individual, an organization, a bot, etc.) can interact in several ways, and there are various ways to represent such interactions in the form of a network. For example, accounts can subscribe to receive other accounts' tweets ('follow'), can mention each other in a tweet ('mention'), can pass to their followers a tweet that was posted by someone else ('retweet'), and so on. These interactions represent an explicit declaration of interest from a source account about a target, and one can thus encode them using directed networks [2]. These interactions are often time-resolved, so it is sensible to analyze them as time-dependent networks [3].

We study a retweet network, which we construct from a data set of tweets about the UK's National Health Service (NHS) that were posted after the controversial Health and Social Care Act of 2012 [35]. Our data set covers over five months of time and includes tweets in English that include the term 'NHS'. Specifically, we consider retweets that involve — either as authors or retweeters — the $10,000$ most-active Twitter accounts (according to the number of posted tweets in our data) from 5 March 2012 to the 21 August 2012 (see Fig. 2). All data were collected by Sinnia[2].

[2]http://www.sinnia.com/.

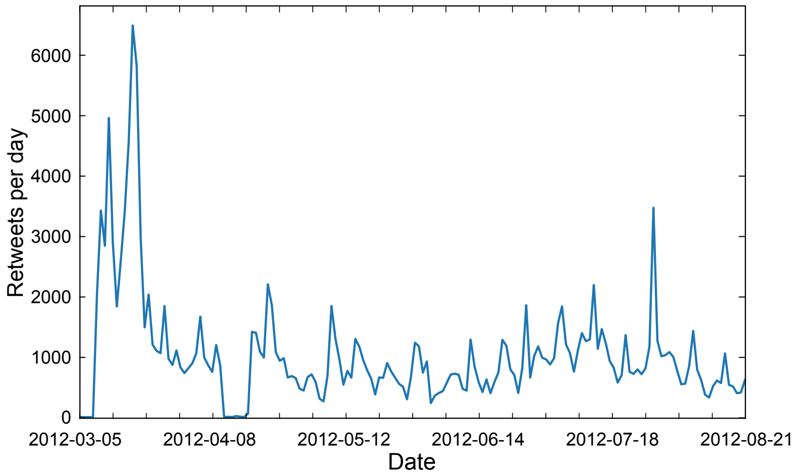

Fig. 2: Number of daily retweets among the 10,000 most-active Twitter accounts in the NHS data set.

, a data analytics company, using Twitter Gnip PowerTrack API[3]. From these data, we construct a tie-decay temporal network in which the interactions are retweets.

4.1. Tie-Decay PageRank Centrality in the NHS Retweet Network. The temporal network of retweets from the NHS data has the tie-strength matrix $\mathbf{B}(t)$ (see equation (3)) and starts from the initial condition $\mathbf{B}(0) = 0$. We construct three tie-decay networks: ones whose values of α correspond to tie half-lives of 1 hour, 1 day, and 1 week. We compute temporal PageRank scores of all Twitter accounts for each of these networks.

In Fig. 3, we show an example of the effect of the value of α on PageRank scores. We compute temporal PageRank scores for networks in which the tie half-life is 1 hour, 1 day, and 1 week. In each panel in Fig. 3, we plot the Twitter account with the largest PageRank score at every time point; the transitions between white and gray shading indicate when some other account takes over as having the top score. When the half-life is short (i.e., α is large), interactions produce feeble ties that die off quickly unless there are frequent and sustained interactions between the nodes. Consequently, the PageRank scores of the Twitter accounts change wildly in time, and (as the top panel in Fig. 3 illustrates) such a short half-life implies that the Twitter account with the top PageRank score changes frequently. When the half-life is longer (e.g., 1 day), ties are better able to 'build momentum' and strengthen from interactions that otherwise would be too distant in time. The ability to build and maintain ties results in fewer transitions between which Twitter accounts hold the top spot in the ranking. The middle panel in Fig. 3 shows that we indeed observe more stability in the transitions when the half-life is 1 day. Finally, when the half-life of a tie is 1 week, two specific accounts (@DrEoinClarke and @marcuschown) dominate the ranking; they alternate between the top and second spots.

In this case study, two accounts (@DrEoinClarke and @marcuschown) come out as dominant as we tune the half-life of ties to larger values. The first of these, Eoin

[3]See https://gnip.com/realtime/powertrack/.

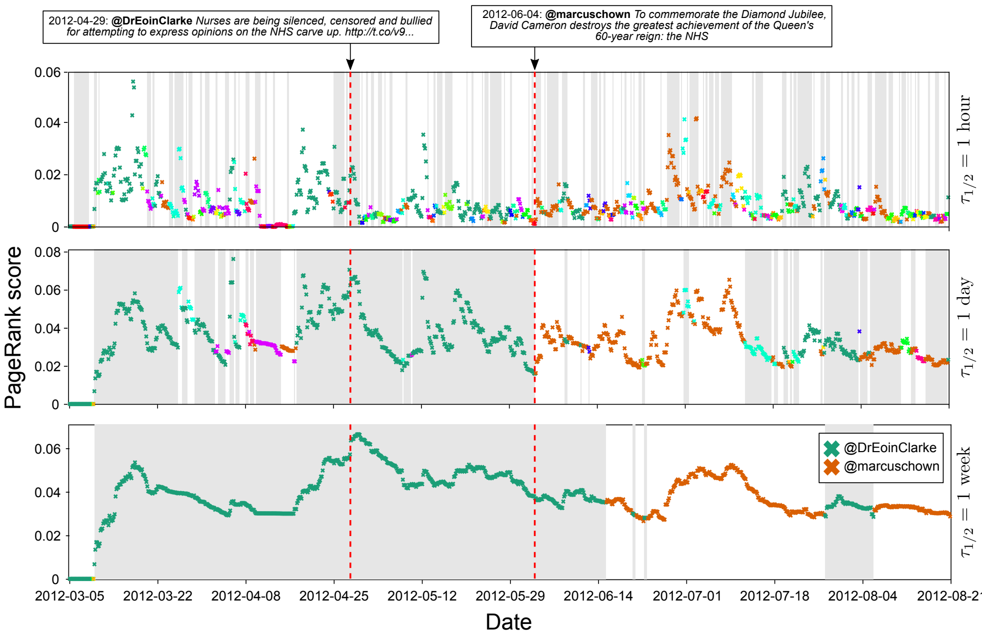

Fig. 3: (Color online) Top Twitter accounts, according to tie-decay PageRank, in the temporal NHS retweet network with three values of tie half-life. Each mark color is associated with a unique Twitter account, and the alternating gray and white background color indicates intervals in which the same account holds the top tie-decay PageRank score. Transitions in color (from white to gray, and vice versa) indicate when another account achieves the top score. The first dashed red line indicates the moment that @DrEoinClarke posted the tweet in the associated box; the second line corresponds to a tweet by @marcuschown.

Clarke (@DrEoinClarke), is a Labour-party activist and an outspoken critic of the UK's coalition government's (of 2012) stance on the NHS. Marcus Chown (@marcuschown) is a science writer, journalist, and broadcaster who was also an outspoken critic of the UK Government's NHS policy during 2012. Their dominance becomes apparent as we increase the half-life of the ties. On 29 April, @DrEoinClarke posted a tweet that gathered significant attention. This tweet yields a short-lived boost in his PageRank score when $\tau_{1/2}$ is 1 hour and 1 day, and it yields a more sustained increase when $\tau_{1/2}$ is 1 week. On 4 June, @marcuschown posted a tweet that results in a boost in his PageRank score. When $\tau_{1/2}$ is 1 day, the number of retweets of this tweet are enough to carry him to the top spot. However, when $\tau_{1/2}$ is 1 week, the retweets are not enough to overtake @DrEoinClarke, whose ties remain strong.

In Fig. 4, we show a complementary illustration of the effect of half-life value on the time-resolved rankings of Twitter accounts. We construct a time-independent network in which we aggregate all of the interactions in our data set — specifically, we consider $B(t)$ for $\alpha = 0$ with the time t set to be 21 August 2012 — and we determine the top-5 accounts by calculating standard PageRank on this network. We then track the time-dependent ranks (where rank 1 is the Twitter account with the largest PageRank score, and so on) of these five accounts for different values of α

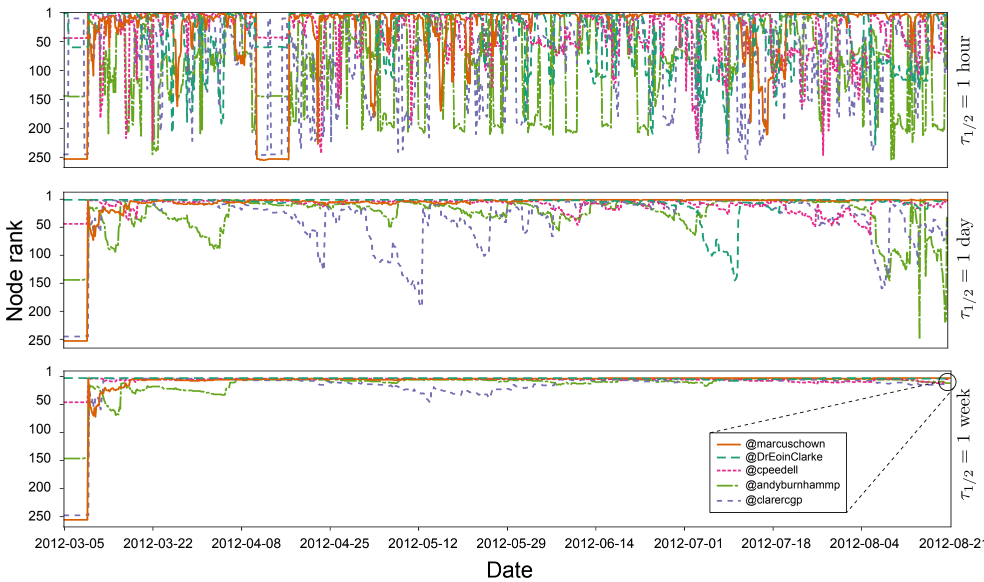

Fig. 4: (Color online) Time series of time-resolved ranks of five prominent Twitter accounts for aggregations of tie-decay networks with three different values of half-life. The most important accounts are higher on the vertical axis.

(or, equivalently, of $\tau_{1/2}$). When $\tau_{1/2} = 1$ hour, these accounts often overtake each other in the rankings, and the changes in rankings can be rather drastic, as some Twitter accounts drop or rise by almost 250 spots. As we consider progressively longer half-lives, we observe less volatility in the rankings.

The experiments in Figs. 3 and 4 demonstrate how one can use α as a tuning parameter to reflect the longevity of relationship values in a network. They also demonstrate the value of our approach for illustrating fluctuations in network structure. When analyzing networks in discrete time, there is a risk that aggregating interactions may conceal intermediate dynamics and nuances of network structure [11]. In contrast, our continuous-time network formulation avoids arbitrary cutoff choices (and potential ensuing biases [28]) when choosing the borders of time bins. It also allows a smoother exploration of network structure at a level of temporal granularity that is encoded in the value of α.

4.2. Computational Efficiency. In applications (e.g., for streaming data), it is often desirable to update the values of time-dependent centrality measures, such as tie-decay PageRank, each time that there is a new interaction. We know from Theorem 2 that there is a bound on the magnitude of the difference between the PageRank vectors at times t and $t + \Delta t$ when there is a new interaction. We thus expect to obtain faster convergence of the power iteration for $t + \Delta t$ when we use the PageRank vector from t as our initial vector. To demonstrate this, we select a period of time with high activity in our NHS data — 08:00 am to 12:00 pm on 18 March 2012 (see Fig. 2) — and calculate the decay PageRank vector at time $t + \Delta t$ with two different starting vectors: the uniform vector $\boldsymbol{\pi}^{(0)}(t + \Delta t) = \frac{1}{n}\mathbb{1}$ and the previous PageRank vector $\boldsymbol{\pi}^{(0)}(t + \Delta t) = \boldsymbol{\pi}(t)$. In time-independent networks, it has been observed that the uniform vector has better convergence properties than any

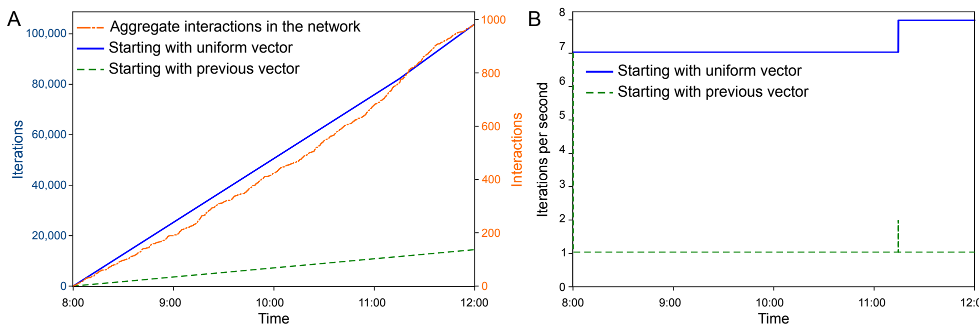

Fig. 5: **A:** Cumulative number of iterations to convergence (where we define convergence as $\|\boldsymbol{\pi}^{(k+1)}(t+\Delta t) - \boldsymbol{\pi}^{(k)}(t+\Delta t)\|_{l_1} < 10^{-6}$) when the starting vector is the uniform vector (continuous blue curve) and the previous PageRank vector (dashed green curve). For context, we also include the aggregate number of interactions in the network (dash-dotted orange curve; right vertical axis). **B:** When analyzing our data, we observe that tie-decay PageRank requires at most 2 iterations to converge when starting from the previous time step's vector, whereas using the uniform vector requires 7 or more iterations. In this example, the half-life of a tie is 1 day.

other starting vector, in the absence of prior knowledge about the final PageRank vector [15]. In our continuous-time network formulation, given the bound between the magnitudes of the vectors at t and $t+\Delta t$, it is intuitive that using the vector from the previous time has computational advantages over other choices. We demonstrate this fact in Fig. 5.

5. Conclusions and Discussion. We have introduced a continuous-time framework, which takes into account tie decays, for studying temporal networks; and we used our approach to generalize PageRank to continuous time. We applied our new tie-decay PageRank to study a network of communication on Twitter, and we found that adjusting the half-life of the decay allows us to examine the temporal dynamics of rankings at different temporal scales. In our tie-decay formalism, a tie between two nodes strengthens through repeated interactions, and it decays in their absence. Such tie-decay networks allow one to tractably analyze time-dependent interactions without having to aggregate interactions into time bins, as is typically done in existing formulations for studying temporal networks [21]. We purposely avoided aggregating interactions using time bins, whose sizes and placement are difficult to determine, by modeling the weakening of ties in time as exponentially decaying with a rate α. In addition to representing the decay of human relationships [7], as we have done in this paper, our framework can also be used more generally to model the decreasing value of old information, decay in other types of interactions, and so on.

We showcased tie-decay temporal networks by investigating the temporal evolution of important accounts in a large collection of tweets about the UK's National Health Service. Specifically, we used tie-decay networks to compute continuous-time PageRank centralities of the nodes, and we provided a numerical scheme and bounds on the PageRank change of the scores upon the arrival of each new interaction. Such bounds are important for studying data streams, in which new data arrives at a potentially alarming rate. By tuning the decay rate of the interactions, we illustrated that PageRank scores change much more drastically when the half-life is short and that they are much more stable for long half-lives.

Our tie-decay approach to continuous-time network dynamics admits a natural generalization to streams of interactions, providing a valuable tool to analyze temporal networks in real time. Streaming data is ubiquitous — it arises in social-media data, sensor streams, communication networks, and more [12,32] — and analyzing tie-decay networks offers a promising approach for studying it. To provide a step towards studying streaming networked systems, we illustrated how to perform an update of tie-decay PageRank from one time step to another, and it will be important to develop such ideas further for other types of computations (such as clustering). In the short term, one can also implement efficient schemes for numerical computations of tie-decay generalizations of other centralities scores (such as hubs and authorities) that are defined from eigenvectors. Our framework permits a tie-decay generalization of personalized PageRank [15,24] (by varying $\mathbf{v}$ in eq. (9)), which can in turn be used to develop new, principled methods for studying community structure in networks that evolve in continuous-time. An extension of our tie-decay framework to noninstantaneous interactions (i.e., taking durations into account) is possible by replacing the term with the Dirac δ in (1) with a function that is nonzero only when a tie exists. For example, if an interaction lasts from τ_{begin} until τ_{end}, then $H(t-\tau_{\text{begin}})-H(t-\tau_{\text{end}})$, where H is the Heaviside function. Alternatively, window [60] or test functions [23] may also be used.

A wealth of other avenues are worth pursuing, including systematically investigating different interaction decay rates (e.g., individual rates for nodes or ties), tie-decay rates other than exponential ones [14,61] (it may be particularly interesting to explore the effects of heavy tails in such rates), developing clustering methods for tie-decay networks, analysis of localization phenomena (and their impact on centralities and clustering) [38, 54], developing and studying random-network null models for such networks, incorporating noninstantaneous interactions, change-point detection, and examining continuous-time networks with multiplex interactions.

Many networked data sets are time-dependent, and it is important to be able to model such systems in continuous time. Tie-decay networks offer a promising approach for further developing these types of analyses.

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Appendix A. Proofs.

A.1. Lemma 1.

Proof. When there is a new interaction from node i to node j, the rows of $P(t+\Delta t)$ that correspond to nodes $k \neq i$ (i.e., nodes from which the new connection did not originate) are unchanged: $p_{kh}(t+\Delta t) = p_{kh}(t)$ for all $h \in \{1, \ldots, n\}$.

To determine the change in the i-th row of $P(t+\Delta t)$, we first examine the i-th row of $B(t+\Delta t)$:

$$b_{ih}(t+\Delta t) = \begin{cases} e^{-\alpha \Delta t} b_{ih}(t)\,, & h \neq j\,, \\ e^{-\alpha \Delta t} b_{ij}(t) + 1\,, & h = j\,. \end{cases} \tag{20}$$

We then consider the change to the rank-1 correction $\mathbf{c}(t)\mathbf{v}^T$. If i is not a dangling node at time t, then $\mathbf{c}(t+\Delta t) = \mathbf{c}(t)$. However, if i is a dangling node at time t, then

$c_i(t) = 1$ and $c_i(t+\Delta t) = 0$. Therefore,

$$c_i(t+\Delta t) - c_i(t) = \begin{cases} 0\,, & c_i(t) = 0\,, \\ -1\,, & c_i(t) = 1\,. \end{cases} \tag{21}$$

Observe that $c_i(t+\Delta t)$ necessarily equals 0 and that $c_i(t) \in \{0, 1\}$. Therefore,

$$c_i(t+\Delta t) - c_i(t) = -c_i(t)\,, \tag{22}$$

so the change to the correction term is

$$c_i(t+\Delta t)v_i - c_i(t)v_i = -c_i(t)v_i\,. \tag{23}$$

The i-th row of $P(t+\Delta t)$ is

$$p_{ih}(t+\Delta t) = \begin{cases} \dfrac{e^{-\alpha\Delta t} b_{ih}(t)}{1 + e^{-\alpha\Delta t}\sum_k b_{ik}(t)} - c_i(t)v_i\,, & h \neq j\,, \\[2ex] \dfrac{e^{-\alpha\Delta t} b_{ij}(t) + 1}{1 + e^{-\alpha\Delta t}\sum_k b_{ik}(t)} - c_i(t)v_i\,, & h = j\,. \end{cases} \tag{24}$$

For $h \neq j$, the difference between $p_{ih}(t+\Delta t)$ and $p_{ih}(t)$ is

$$p_{ih}(t+\Delta t) - p_{ih}(t) = \frac{e^{-\alpha\Delta t} b_{ih}(t)}{1 + e^{-\alpha\Delta t}\sum_k b_{ik}(t)} - \frac{b_{ih}(t)}{\sum_k b_{ik}(t)} - c_i(t)v_i \tag{25}$$

$$= \frac{-b_{ih}(t)}{\sum_k b_{ik}(t)\,[1 + e^{-\alpha\Delta t}\sum_k b_{ik}(t)]} - c_i(t)v_i \tag{26}$$

$$= \frac{-b_{ih}(t)}{d_{ii}(t)\,[1 + e^{-\alpha\Delta t} d_{ii}(t)]} - c_i(t)v_i\,. \tag{27}$$

For the entry $p_{ij}(t+\Delta t)$, we have

$$p_{ij}(t+\Delta t) - p_{ij}(t) = \frac{1 + e^{-\alpha\Delta t} b_{ij}(t)}{1 + e^{-\alpha\Delta t}\sum_k b_{ik}(t)} - \frac{b_{ij}(t)}{\sum_k b_{ik}(t)} - c_i(t)v_i \tag{28}$$

$$= \frac{\sum_k b_{ik}(t) - b_{ij}(t)}{\sum_k b_{ik}(t)\,[1 + e^{-\alpha\Delta t}\sum_k b_{ik}(t)]} - c_i(t)v_i \tag{29}$$

$$= \frac{d_{ii}(t) - b_{ij}(t)}{d_{ii}(t)\,[1 + e^{-\alpha\Delta t} d_{ii}(t)]} - c_i(t)v_i \tag{30}$$

$$= \frac{1}{1 + e^{-\alpha\Delta t} d_{ii}(t)} - \frac{b_{ij}(t)}{d_{ii}(t)\,[1 + e^{-\alpha\Delta t} d_{ii}(t)]} - c_i(t)v_i\,. \tag{31}$$

In matrix terms, the change in $P(t+\Delta t)$ is thus

$$\Delta P = \frac{1}{1 + e^{-\alpha\Delta t} d_{ii}(t)}\mathbf{e}_i\mathbf{e}_j^T - \frac{1}{d_{ii}(t)\,(1 + e^{-\alpha\Delta t} d_{ii}(t))}\mathbf{e}_i\mathbf{e}_i^T B(t) - c_i(t)v_i\mathbf{e}_i\mathbb{1}^T\,, \tag{32}$$

which concludes the proof. ∎

A.2. Theorem 2.

Proof. The change in PageRank scores with one new interaction is

$$\begin{aligned}\boldsymbol{\pi}(t+\Delta t)-\boldsymbol{\pi}(t) &= \left[\lambda(P(t)^T+\Delta P^T)+(1-\lambda)\mathbf{v}\mathbb{1}^T\right]\boldsymbol{\pi}(t+\Delta t)\\ &\quad-\left[\lambda P(t)^T+(1-\lambda)\mathbf{v}\mathbb{1}^T\right]\boldsymbol{\pi}(t)\\ &=\lambda P(t)^T(\boldsymbol{\pi}(t+\Delta t)-\boldsymbol{\pi}(t))+\lambda\Delta P^T\boldsymbol{\pi}(t+\Delta t)\,.\end{aligned} \tag{33}$$

Rearranging terms gives

$$\left(I_n-\lambda P(t)^T\right)(\boldsymbol{\pi}(t+\Delta t)-\boldsymbol{\pi}(t))=\lambda\Delta P^T\boldsymbol{\pi}(t+\Delta t)\,, \tag{34}$$

which implies that

$$\boldsymbol{\pi}(t+\Delta t)-\boldsymbol{\pi}(t)=\lambda\left(I_n-\lambda P(t)^T\right)^{-1}\Delta P^T\boldsymbol{\pi}(t+\Delta t)\,, \tag{35}$$

where I_n is the $n\times n$ identity matrix. From a Neumann-series expansion [57], we see that $\left|\left|(I_n-\lambda P(t)^T)^{-1}\right|\right|_1$ is bounded above by $1/(1-\lambda)$.

Taking norms on both sides of (35) yields

$$||\boldsymbol{\pi}(t+\Delta t)-\boldsymbol{\pi}(t)||_1\le\frac{\lambda}{1-\lambda}\left|\left|\Delta P^T\right|\right|_1\,. \tag{36}$$

Noting that $\left|\left|\Delta P^T\right|\right|_1=||\Delta P||_\infty$, we use the definition of ΔP from equation (10) to obtain

$$\begin{aligned}||\boldsymbol{\pi}(t+\Delta t)-\boldsymbol{\pi}(t)||_1\le&\frac{\lambda}{(1-\lambda)(1+e^{-\alpha\Delta t}d_{ii}(t))}\left|\left|\mathbf{e}_i\mathbf{e}_j^T-\frac{1}{d_{ii}(t)}\mathbf{e}_i\mathbf{e}_i^TB(t)\right|\right|_\infty\\ &-\frac{\lambda c_i(t)v_i}{1-\lambda}\left|\left|\mathbf{e}_i\mathbb{1}^T\right|\right|_\infty\,.\end{aligned} \tag{37}$$

Recall that $B(t)$ is the tie-strength matrix and that $\mathbf{e}_i$ and $\mathbf{e}_j$, respectively, are the i-th and j-th canonical vectors. Let $Q=\mathbf{e}_i\mathbf{e}_j^T-\frac{1}{d_{ii}(t)}\mathbf{e}_i\mathbf{e}_i^TB(t)$ be the matrix with elements

$$q_{hk}=\begin{cases}1-b_{hk}(t)/d_{ii}(t)\,, & h=i\,,k=j\,,\\ -b_{hk}(t)/d_{ii}(t)\,, & h=i\,,k\neq j\,,\\ 0\,, & \text{otherwise}\,,\end{cases} \tag{38}$$

so Q has nonzero elements only in row i. Noting that $d_{ii}(t)=\sum_k b_{ik}(t)$ and using

$$||Q||_\infty=\max_{1\le h\le n}\sum_{k=1}^n|q_{hk}|=\sum_{k=1}^n|q_{ik}|\,, \tag{39}$$

we see that

$$||Q||_\infty\le 2\,. \tag{40}$$

We also observe that $\mathbf{e}_i\mathbb{1}^T$ is the $n\times n$ matrix whose elements are equal to 1 in row i and are equal to 0 elsewhere. Therefore,

$$\left|\left|\mathbf{e}_i\mathbb{1}^T\right|\right|_\infty=n\,. \tag{41}$$

With $nv_i = 1$ (i.e., uniform teleportation), it follows from equations (37), (40), and (41) that

$$(42) \qquad ||\boldsymbol{\pi}(t+\Delta t) - \boldsymbol{\pi}(t)||_1 \leq \frac{2\lambda}{(1-\lambda)(1+e^{-\alpha\Delta t}d_{ii}(t))} - \frac{\lambda c_i(t)}{1-\lambda}\,.$$

The change in the PageRank vector is also subject to the bound [34, 43]

$$(43) \qquad ||\boldsymbol{\pi}(t+\Delta t) - \boldsymbol{\pi}(t)||_1 \leq \frac{2\lambda}{1-\lambda}\sum_{s\in\mathcal{S}(t+\Delta t)} \pi_s(t) = \frac{2\lambda}{1-\lambda}\pi_i(t)\,,$$

where $\mathcal{S}(t+\Delta t)$ is the set of nodes (in this case, just node i) that experience a change in transition probabilities (i.e., a change in out-edges). Combining the bounds in (42) and (43) yields

$$(44) \qquad ||\boldsymbol{\pi}(t+\Delta t) - \boldsymbol{\pi}(t)||_1 \leq \frac{2\lambda}{1-\lambda}\min\left\{\pi_i(t),\, \frac{1}{1+e^{-\alpha\Delta t}d_{ii}(t)} - \frac{c_i(t)}{2}\right\}\,,$$

which completes our proof. ∎

Note that $d_{ii}(t) > 0 \iff c_i(t) = 0$ and $d_{ii}(t) = 0 \iff c_i(t) = 1$, which guarantees that the quantity on the right-hand side of (44) is always positive. This gives the results in Corollaries 3 and 4.

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