

CS350 Homework 1

Russell Miller

January 12, 2011

3.1-1

Let $f(n)$ and $g(n)$ be asymptotically nonnegative functions. Using the basic definition of Θ -notation, prove that $\max(f(n), g(n)) = \Theta(f(n) + g(n))$.

To determine Θ , need to show that there exist positive constants c_1 , c_2 , and N such that $0 \leq c_1(f(n) + g(n)) \leq \max(f(n), g(n)) \leq c_2(f(n) + g(n))$ for all $n \geq N$.

Looking only at the left inequality:

$$c_1(f(n) + g(n)) \leq \max(f(n), g(n))$$

Setting up some premises:

$$f(n) \leq \max(f(n), g(n))$$

$$g(n) \leq \max(f(n), g(n))$$

Algebraically this means:

$$(f(n) + g(n)) \leq 2\max(f(n), g(n))$$

Or:

$$\frac{1}{2}(f(n) + g(n)) \leq \max(f(n), g(n))$$

This gives a value of $c_1 = \frac{1}{2}$. N can be any positive number (e.g., 1).

Next looking at the right inequality:

$$\max(f(n), g(n)) \leq c_2(f(n) + g(n))$$

Even with $c_2 = 1$, this inequality holds for any value of n .

Adding two positive numbers is simply always greater than or equal to either of the numbers individually.

Because these two inequalities hold, the definition of Θ is satisfied.

3-1

Let $p(n) = \sum_{i=0}^d a_i n^i$,

where $a_d > 0$, be a degree- d polynomial in n , and let k be a constant. Use the definitions of the asymptotic notations to prove the following properties.

a. If $k \geq d$, then $p(n) = O(n^k)$.

$p(n)$ can also be written as $a_0 + a_1 n + a_2 n^2 + \dots + a_d n^d$.

This is $d + 1$ terms.

To determine O , need to show that there exist positive constants c and N such that

$0 \leq p(n) \leq cn^k$ for all $n \geq N$.

Now breaking the sum into individual atoms.

$$a_0 \leq cn^k$$

$$a_1 n \leq cn^k$$

$$a_2 n^2 \leq cn^k$$

...

$$a_d n^d \leq cn^k$$

By choosing $c = \max(a_0 \dots a_d)(d + 1)$ and $N = 1$, then since $k \geq d$

it must be that $p(n) \leq cn^k$. Picking the largest coefficient and adding it together the same number of times the coefficients are added together that are less than or equal to that coefficient, then multiplying it by n raised to a power greater or equal to the greatest exponent, means that every aspect of cn^k is greater or equal. Thus the definition of O is satisfied.

b. If $k \leq d$, then $p(n) = \Omega(n^k)$.

Assuming $k \leq d$, rewriting the goal statement.

$$a_0 + a_1n + a_2n^2 + \dots + a_dn^d \geq cn^k$$

This is the same as

$$a_0 + a_1n + a_2n^2 + \dots + a_dn^d - cn^k \geq 0$$

As a side note, since we know $k \leq d$, cn^k is some piece of one of the atoms of $p(n)$. An example of this is that if $k = 2$, part of the left side of this inequality would be $a_2n^2 - cn^2$. This means the coefficient in front of one of the atoms will be subtracted from. Just in case $k = d$, let $c = \frac{a_d}{2}$. This means that in that case, the result of $a_dn^d - cn^d$ would be $\frac{a_d}{2}n^d$. It's important that the atom with a_d in front of it remain positive because in the problem statement, it is the only one guaranteed to be positive.

Knowing that one of the coefficients will change, going to reconstruct $p(n)$ with new coefficients.

$$b_dn^d - b_{d-1}n^{d-1} - \dots - b_1n - b_0 \geq 0$$

The point of this statement is to show that

$$b_dn^d \geq b_{d-1}n^{d-1} + \dots + b_1n + b_0$$

meaning that the highest degree exponent holds the significance of the formula.

Next breaking down the smaller pieces of this. Going to show that $b_dn^d \geq b_in^i$ for each atom $(d-1) \dots 0$.

If you add all of the b_in^i terms on the right, you will get the right side of the inequality. If you try the same thing with b_dn^d you will have an imbalanced left side. So to adjust the break down, the individual pieces need to be $\frac{b_d}{d} \geq b_in^i$.

By dividing the n^i on the right side, you get $\frac{b_dn^{d-i}}{d} \geq b_i$.

By multiplying by the reciprocal of $\frac{b_d}{d}$ you get $n^{d-i} \geq \frac{db_i}{b_d}$.

By taking the $d-i$ root of both sides, you get $n \geq \sqrt[d-i]{\frac{db_i}{b_d}}$.

This suffices as N_i for this piece of the puzzle. Let $N = \max(N_{d-1} \dots N_0)$. This will be a sufficiently large value for N , and with $c = \frac{a_d}{2}$, the definition of Ω is satisfied.

c. If $k = d$, then $p(n) = \Theta(n^k)$.

$p(n) = \Theta(n^k)$ can be rewritten as

$$c_1n^k \leq p(n) \leq c_2n^k \text{ for some nonnegative constants } c \text{ and } N, \text{ and where } n \geq N.$$

The left inequality here has already been proven in part b.

The right inequality was proven in part a.

Thus the definition of Θ has already been satisfied.

3-4

Let $f(n)$ and $g(n)$ be asymptotically positive functions. Prove or disprove each of the following conjectures.

- a. $f(n) = O(g(n))$ implies $g(n) = O(f(n))$.

Assume $f(n) = O(g(n))$.

This means, for some positive constant c , $f(n) \leq cg(n)$, for large enough n .

Let $f(n) = n$ and $g(n) = n^2$. By choosing $c = 1$ and $N = 1$, it is clear that $n \leq n^2$ for $n \geq N$.

However there is no value of c and N where $n^2 \leq cn$.

Therefore, this conjecture cannot be satisfied.

- b. $f(n) + g(n) = \Theta(\min(f(n), g(n)))$.

This can be rewritten as $c_1(\min(f(n), g(n))) \leq f(n) + g(n) \leq c_2(\min(f(n), g(n)))$.

Let $f(n) = n$ and $g(n) = n^2$. For any positive value of c_1 it is easy to see that the left inequality holds.

It is always true that $n \leq n^2 + n$.

However, with the right inequality there does not exist a value for c_2 where $n^2 + n \leq cn$.

Quadratic growth versus linear growth is fundamentally always greater.

Therefore, this conjecture cannot be satisfied.

- c. $f(n) = O(g(n))$ implies $\lg(f(n)) = O(\lg(g(n)))$, where $\lg(g(n)) \geq 1$ and $f(n) \geq 1$ for all sufficiently large n .

Starting with the definition of O , $f(n) \leq c(g(n))$.

Then take the logarithm of both sides to get $\lg(f(n)) \leq \lg(cg(n))$.

Using the product logarithm rule that is the same as $\lg(f(n)) \leq \lg c + \lg(g(n))$.

By transitivity, if $\lg(f(n)) \leq \lg c + \lg(g(n))$ and $\lg c + \lg(g(n)) \leq c(\lg(g(n)))$, then $\lg(f(n)) \leq c(\lg(g(n)))$. So first going to prove that $\lg c + \lg(g(n)) \leq c(\lg(g(n)))$.

To make this more clear, specifying a base for the logarithms in the inequality. Choosing 10.

$\log_{10} c + \log_{10} g(n) \leq c(\log_{10} g(n))$

Using this base, and choosing $c = 10$, $1 + \log_{10} g(n) \leq 10 \log_{10} g(n)$.

The $\log_{10} g(n)$ will have equal values on both sides, and obviously multiplying by 10 will produce a larger value than adding 1 for any positive value of n .

Therefore, this conjecture is satisfied.

- d. $f(n) = O(g(n))$ implies $2^{f(n)} = O(2^{g(n)})$.

By the definition of O , $f(n) \leq cg(n)$.

If we let $f(n) = 5n$ and $g(n) = 3n$, it is clear that this holds by choosing $c = 2$.

For $2^{f(n)} = O(2^{g(n)})$ to also hold, it must hold that $2^{f(n)} \leq c2^{g(n)}$.

Distributing the functions, $2^{5n} \leq c2^{3n}$.

For some simple examples, using $c = 2$ again, and checking some values of n , this evaluates to $32 \leq 8$ for $n = 1$.

$1024 \leq 32$ for $n = 2$.

$32728 \leq 128$ for $n = 3$.

It is very clear to see that as n gets larger, the difference gets much larger. This is another way of saying that the difference of the exponents far outweighs any value of c . Therefore, this conjecture cannot be satisfied.

e. $f(n) = O((f(n))^2)$.

By the definition of O , $f(n) \leq c((f(n))^2)$.

Let $f(n) = n$. There is no value of c that can cause n^2 to be less than n . While this is a rather simple example, this statement defines O .

f. $f(n) = O(g(n))$ **implies** $g(n) = \Omega(f(n))$.

By the definition of O , $f(n) \leq cg(n)$ for some constant c .

Let $f(n) = n$ and $g(n) = n^2$. O is satisfied for these functions with any value of c .

Using any value of c , Ω is also satisfied for $g(n) \geq cf(n)$.

If a function is an upper bound for another function, it is bounded below by that function!

g. $f(n) = \Theta(f(n/2))$.

By the definition of Θ , $c_1 f(\frac{n}{2}) \leq f(n) \leq c_2 f(\frac{n}{2})$.

Choose $c_1 = 1$ and $c_2 = 4$. The n value for each of these forms of f will be less for the lower bound, and more for the upper bound. Therefore, Θ is satisfied.