CS410 Take-home Midterm

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Textbook 2.22

Let $a_1, a_2, ..., a_n$ be a list of n distinct numbers. We say that a_i and a_j are *inverted* if $i < j$ but $a_i > a_j$. The *Bubblesort* sorting algorithm swaps pairwise adjacent inverted numbers in the list until there are no more inversions, so the list is in sorted order. Suppose that the input to Bubblesort is a random permutation, equally likely to be any of the $n!$ permutations of n distinct numbers. Determine the expected number of inversions that need to be corrected by Bubblesort.

First we define the random variable X to represent the number of inversions required for the list to be sorted. X is made up of a set of indicator random variables X_{ij} where each $X_{ij} = 1$ when a_i and a_j are inverted. That is, $i < j$ and $a_i > a_j$. Since we know that

$$X = \sum_{i<j} X_{ij}$$

by the Linearity of Expectation, we know that

$$E[X] = \sum_{i<j} E[X_{ij}]$$

and since j is defined as being adjacent to i

$$= \sum_{i=1}^{n} \sum_{j=i+1}^{n} Pr(X_{ij} = 1)$$

The probability of a pair of numbers being inverted is simply $\frac{1}{2}$. a_i and a_j are two random digits, and it's equally likely that either could be greater than the other.

$$Pr(X_{ij}) = \frac{1}{2}$$

Plugging this back in

$$\begin{aligned} E[X] &= \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{1}{2} \\ &= \binom{n}{2} \left(\frac{n-1}{2}\right) \end{aligned}$$

$$\boxed{E[X] = \frac{n(n-1)}{4}}$$