# CS410 HW4

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**2.14 The geometric distribution arises as the distribution of the number of times we flip a coin until it comes up heads. Consider now the distribution of the number of flips $X$ until the $k$th head appears, where each coin flip comes up heads independently with probability $p$. Prove that this distribution is given by**

$$\Pr(X = n) = \binom{n-1}{k-1} p^k (1-p)^{n-k}$$

for $n \geq k$.

The distribution of a binomial random variable is

$$Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

The difference between this and the aforementioned distribution is that this gives the probability that there are $k$ heads. What we want to find is the probability that after finding $k$ heads, we have only flipped $n$ coins.

Assume that on the $n$th flip, we achieve our $k$th heads. Now we know that on the previous $n-1$ flips, we had exactly $k-1$ heads. So the probability that we tossed $k-1$ heads in $n-1$ flips is the same as the probability that the $k$th heads was achieved on the $n$th flip. However, $p$ and $p-1$ are not raised to the $k-1$, and that's because there are in fact $k$ total heads. ■

**3.6 For a coin that comes up heads independently with probability $p$ on each flip, what is the variance in the number of flips until the $k$th head appears?**

The distribution of the number of flips of a coin until the $k$th heads could be viewed as a sum of the distributions of geometric random variables representing the previous $k-1$ heads. Let $X$ be the random variable for the number of coin flips until the $k$th heads, and each $X_i$ be a geometric random variable for the number of flips to get the $i$th heads. For example, $X_1$ is the number of coin flips until the first heads.

$$X = \sum_{i=1}^{k} X_i$$

We know that if each $X_i$ is mutually independent (which we're told each coin flip is)

$$Var[\sum_{i=1}^{k} X_i] = \sum_{i=1}^{k} Var[X_i]$$

We need the variance of $X_i$, which we derived in class.

$$Var[X_i] = \frac{1-p}{p^2}$$

Plugging this back into our distribution of X to find its variance

$$\begin{aligned} Var[X] &= \sum_{i=1}^{k} \frac{1-p}{p^2} \\ &\boxed{= \frac{k(1-p)}{p^2}} \end{aligned}$$

**2.21 Let $a_1, a_2, ..., a_n$ be a random permutation of $\{1, 2, ..., n\}$, equally likely to be any of the $n!$ possible permutations. When sorting the list $a_1, a_2, ..., a_n$, the element $a_i$ must move a distance of $|a - i|$ places from its current position to reach its position in the sorted order. Find**

$$E\left[\sum_{i=1}^{n} |a_i - i|\right],$$

**the expected total distance that elements will have to be moved.**

Let $X_i$ be a random variable for the distance that element $i$ moves, $|a_i - i|$. Let $a_i$ be a random variable for the $i$th element's position in a random permutation.

$$E[a_i] = \sum_{i=1}^{n} i \ \Pr(a_i = i)$$

Because any of the $n$ positions is equally likely,

$$\begin{aligned} \Pr(a_i = i) &= \frac{1}{n} \\ E[a_i] &= \sum_{i=1}^{n} \frac{i}{n} \\ &= \frac{1}{n}(1 + 2 + ... + n) \\ &= \frac{1}{n}\frac{n(n+1)}{2} \\ &= \frac{n+1}{2} \end{aligned}$$

Now we can find $E[X_i]$ by the linearity of expectation.

$$\begin{aligned} E[X_i] &= E[|a_i - i|] \\ &= |E[a_i] - E[i]| \\ &= \left|\frac{n+1}{2} - i\right| \end{aligned}$$

Now we are ready to find $E[X]$, and we'll use linearity of expectation again.

$$\begin{aligned} E[X] &= E\left[\sum_{i=1}^{n} X_i\right] \\ &= \sum_{i=1}^{n} E[X_i] \\ &= \sum_{i=1}^{n} \left|\frac{n+1}{2} - i\right| \\ &= \left|\sum_{i=1}^{n} \frac{n+1}{2} - \sum_{i=1}^{n} i\right| \\ &= \left|\frac{n+1}{2} - \frac{n(n+1)}{2}\right| \\ &\boxed{= \left|\frac{1-n^2}{2}\right|} \end{aligned}$$