CS410 Final Exam

Russell Miller

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1. Consider building a network in the following way. Given n machines, insert an edge between any pair of machines with probability $p = c/n$ for some $c \leq n$. For what value of c is the expected number of edges $(n-1)$? For this c, give the best upper bound you can on the probability that there are at least $2(n-1)$ edges in the network.

Let us define a random variable X to represent the total number of edges. Let us also consider an individual edge between two machines. Let X_i be an indicator random variable that is 1 if and only if there is an edge between those two machines. Then X will be a binomial random variable that is equal to the sum of these indicators.

$$X = \sum_{i=1}^{m} X_i$$

where m is the total number of combinations of 2 machines. We can see that $m = \binom{n}{2}$.
Now we will find the expected number of edges $E[X]$.

$$E[X] = E[\sum_{i=1}^{\binom{n}{2}} X_i] = \sum_{i=1}^{\binom{n}{2}} E[X_i] = \binom{n}{2} p$$

We're told that $p = c/n$.

$$E[X] = \binom{n}{2}\left(\frac{c}{n}\right) = \frac{c(n-1)}{2}$$

In order to find c, let us assume that $E[X] = n - 1$ and solve for c.

$$\begin{aligned} \frac{c(n-1)}{2} &= n-1 \\ c(n-1) &= 2(n-1) \\ c &= 2 \end{aligned}$$

Let's get the variance of X in attempt to find a good bound. The variance of a Binomial random variable with parameters n and p is $np(1-p)$, but $X \sim B(\binom{n}{2}, 2/n)$.

$$Var[X] = \binom{n}{2}\left(\frac{2}{n}\right)(1 - \left(\frac{2}{n}\right)) = (n-1)(1 - \left(\frac{2}{n}\right)) = \frac{(n-1)(n-2)}{n}$$

We now have a Chebyshev bound.

$$Pr(|X - E[X]| \geq a) \leq \frac{Var[X]}{a^2}$$

We need to find a, and we're bounding the condition of $X \geq 2E[X]$. If $X = 2E[X]$, then $X - E[X] = E[X] = a$.

$$Pr(|X - E[X]| \geq E[X]) \leq \frac{Var[X]}{(E[X])^2} = \frac{\frac{(n-1)(n-2)}{n}}{(n-1)^2} = \frac{n-2}{n(n-1)}$$

2. Consider a graph $G = (V, E)$ with the following properties: there exists a partitioning of the vertex set V into subsets V_1 and V_2 of sizes r and s (respectively); for all $x \in V_1$ and $y \in V_2$ there is an edge between x and y; there are no edges between vertices in V_1, and no edges between vertices in V_2. Such a graph is denoted $K_{r,s}$ and is called a complete bipartite graph (with vertex partitioning into sets of size r and s.) Clearly the number of edges in $K_{r,s}$ is rs.
Prove that there is a two-coloring of edges of $K_{r,s}$ with at most

$$\binom{r}{a}\binom{s}{b}2^{1-ab}$$

monochromatic $K_{a,b}$ as subgraphs.