Final

Please explain all of your answers carefully and clearly. You must submit your solutions electronically by 6pm, December 9, 2011.

(20 pts) **1.** Consider building a network in the following way. Given n machines, insert an edge between any pair of machines with probability $p = c/n$ for some $c \leq n$. For what value of c is the expected number of edges $(n-1)$? For this c, give the best upperbound you can on the probability that there are at least $2(n-1)$ edges in the network.

(15 pts) **2.** Consider a graph $G = (V, E)$ with the following properties: there exists a partitioning of the vertex set V into subsets V_1 and V_2 of sizes r and s (respectively); for all $x \in V_1$ and $y \in V_2$ there is an edge between x and y; there are no edges between vertices in V_1, and no edges between vertices in V_2. Such a graph is denoted $K_{r,s}$ and is called a *complete bipartite graph* (with vertex partitioning into sets of size r and s.) Clearly the number of edges in $K_{r,s}$ is rs.

Prove that there is a two-coloring of edges of $K_{r,s}$ with at most

$$\binom{r}{a}\binom{s}{b}2^{1-ab}$$

monochromatic $K_{a,b}$ as subgraphs.

(20 pts) **3** Let $X_1, X_2, \ldots, X_m$ be independent and identically distributed indicator random variables, and let $\mu = \mathbf{E}[X_i]$ for all i. If we want

$$\Pr\left(\left|\frac{1}{m}\sum_{i=1}^{m} X_i - \mu\right| \geq \epsilon\mu\right) \leq \delta$$

for some $\epsilon > 0$ and $0 \leq \delta \leq 1$, how many indicators m do we require? (Hint: think Chernoff, and derive a bound on m that is a function of ϵ, δ, μ.)

(10 pts) **4.** Mitzenmacher and Upfal, exercise 2.18 (page 41).

(35 pts) **5.** Consider throwing m balls uniformly and independently into bins labeled $0, 1, \ldots, n-1$. We say there is a *k-gap* starting at bin i if bins $i, i+1, \ldots, i+k-1$ are empty.

(a) Determine the expected number (μ) of k-gaps.

(b) Consider the case $n = 10$ and $k = 3$. What is the probability that a 3-gap starts at bin 2, given that a 3-gap starts at bin 1? What is the probability that a 3-gap starts at bin 5, given that a 3-gap starts at bin 1?

(c) Use Chernoff bounds to give an upperbound on the number of k-gaps. If it helps, assume that k divides n. (Hint, use what you learned in part (b).)