Writing a python function to calculate the value of $\boldsymbol{J_m(x)}$.

For a general function:

For any general function, the trapezium rule can be used to find an approximate value for the area under a graph. The sum of the area of a series of N trapezia can be calculated to find the total approximate area. Figure 1 below shows this:

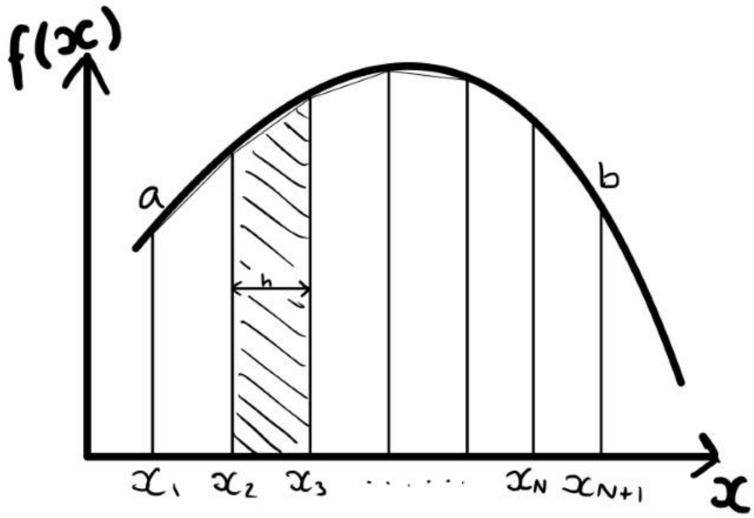

Figure 1- A series of trapezia fit under a curve to approximate the total area under the curve.

Each trapezium has area,

$$A_i = h\left[\frac{f(x_i)+f(x_{i+1})}{2}\right] \tag{1}$$

And so, the total area of the trapezia is,

$$A_{trapzia} = h\left[\frac{f(x_i)+f(x_{i+1})}{2}\right] + h\left[\frac{f(x_{i+1})+f(x_{i+2})}{2}\right] + \cdots + h\left[\frac{f(x_N)+f(x_{N+1})}{2}\right] \tag{2}$$

Starting at $i = 1$, and counting until N, the total area of trapezia is:

$$A_{trapezia} = h\left[\frac{1}{2}f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_N) + \frac{1}{2}f(x_{N+1})\right] \tag{3}$$

Or alternatively, the area under a curve between $x = a$ and $x = b$,

$$A_{curve} = \int_a^b f(x) \approx h\left[\frac{1}{2}f(x_a) + \frac{1}{2}f(x_b) + \sum_{k=1}^{N-1} f(a+kh)\right] \tag{4}$$

For the Bessel function:

A Bessel function, $J_m(x)$ is given by

$$J_m(x) = \frac{1}{\pi}\int_0^{\pi} \cos(m\theta - xsin\theta)\, d\theta \tag{5}$$

And where m is a non-negative integer, and $x \geq 0$.

To find a value for $J_m(x)$, from Equation 5, we need a program that can integrate. The function inside the integral will be $f(\theta)$, seen in Equation 6 below, and this is the function which will need to be inserted into the approximation in Equation 4, for the trapezium rule.

$$f(\theta) = cos\ (m\theta - xsin\theta) \tag{6}$$

$f(\theta)$ will be integrated using the following conditions:

$$N = 10\ 000$$

$$a = 0$$

$$b = \pi$$

Therefore, the approximate integral of the function is given by Equation 7 below:

$$\int_0^{\pi} f(\theta)\, d\theta \approx h\left[\frac{1}{2}f(0) + \frac{1}{2}f(\pi) + \sum_{k=1}^{9\ 999} f(0 + kh)\right] \tag{7}$$

From Figure 1, given that the width of each trapezium is equal, h can be calculated to be:

$$h = x_{i+1} - x_i = \frac{b-a}{N} = \frac{\pi}{10\ 000} \tag{8}$$

By substituting Equation 8 into Equation 6, and simplifying, Equation 9 can be derived:

$$\int_0^{\pi} \cos(m\theta - xsin\theta)\, d\theta \approx \frac{\pi}{10\ 000} \times \left[\frac{1}{2}f(0) + \frac{1}{2}f(\pi) + \sum_{k=1}^{9\ 999} f\left(\frac{k\pi}{10\ 000}\right)\right] \tag{9}$$

And therefore, combining Equations 5 and 9, the value for $J_m(x)$ is given by:

$$J_m(x) = \frac{1}{\pi} \times \frac{\pi}{10\,000} \times \left[\frac{1}{2} f(0) + \frac{1}{2} f(\pi) + \sum_{k=1}^{9\,999} f\left(\frac{k\pi}{10\,000}\right)\right] \tag{10}$$

$$=> \; J_m(x) = \frac{1}{10\,000} \times \left[\frac{1}{2} f(0) + \frac{1}{2} f(\pi) + \sum_{k=1}^{9\,999} f\left(\frac{k\pi}{10\,000}\right)\right] \tag{11}$$

Where the function f is given by Equation 6.

A python program to solve Equation 11, given values for m and x, was written, and used to plot the functions for $m = 0$, $m = 1$, and $m = 2$, as functions of x for the range of x values, $0 \leq x \leq 20$. The plot is shown in Figure 2 below:

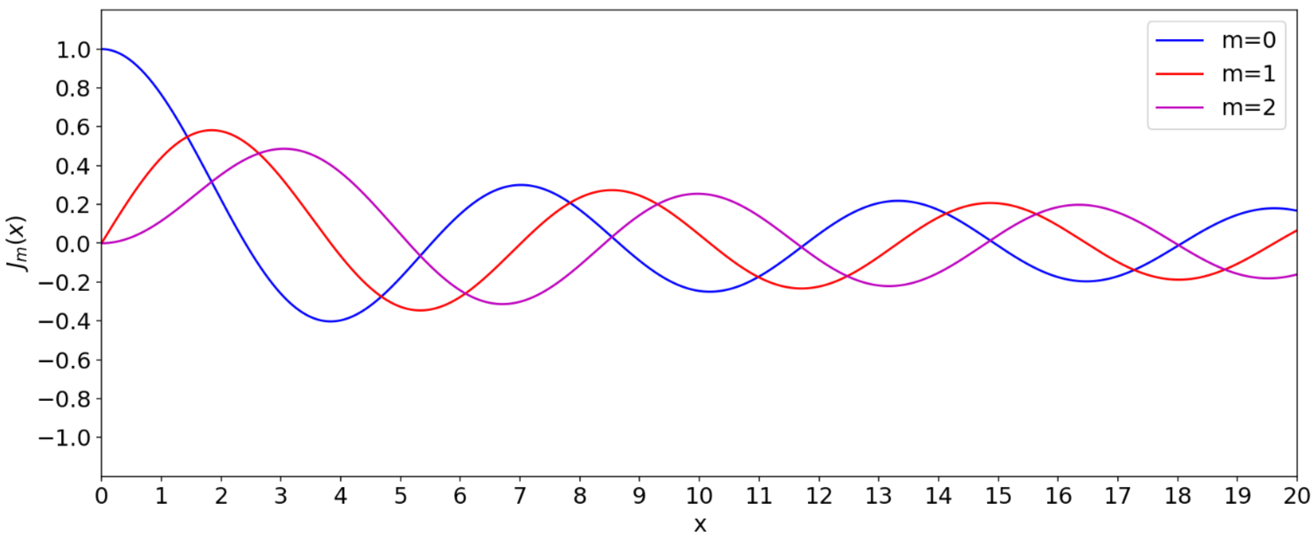

Figure 2- A plot of $J_0(x)$, $J_1(x)$, and $J_2(x)$, as functions of x for the x values, $0 \leq x \leq 20$.

Writing a program that calculates the diffraction pattern of a circular lens

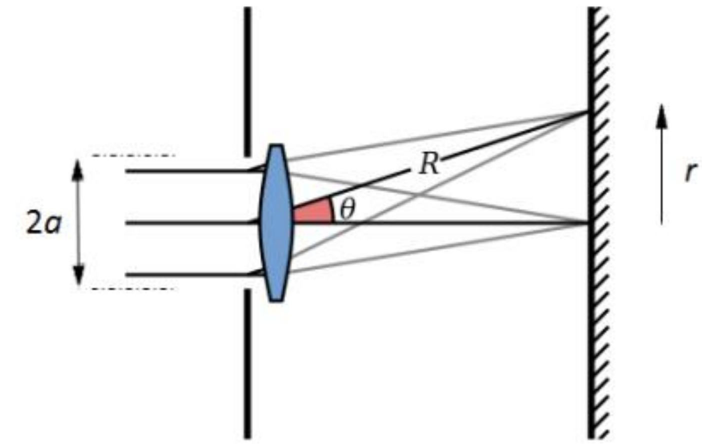

Figure 3- A diagram of the lens of a telescope with a circular aperture, and the path of the light rays passing through it.

The intensity of light passing through a circular lens is given by,

$$I(r) = I_0 \left(\frac{2J_1(x)}{x}\right)^2 \tag{12}$$

Where,

$$x = ka\,sin\theta = \frac{2\pi}{\lambda} a \frac{r}{R} \tag{13}$$

Given that the focal ratio, $\frac{R}{2a}$, is 10,

$$a = \frac{R}{20} \tag{14}$$

By substituting Equation 14 into Equation 13, Equation 15 can be derived. This shows x as a function of r, since we're calculating for a constant wavelength, λ.

$$x = \frac{\pi}{10\lambda} r \tag{15}$$

To write a program to plot Equation 12, the program from part 2a is used to find the Bessel function $J_1(x)$, and Equation 15 is inserted as the value for x, writing the formula for intensity, I, as a function of r. The value for I_0 in the formula is 1 so that the plot shows the relative intensity, $\frac{I(r)}{I_0}$, and for the wavelength of the light, λ, 600nm was used, as it's in the range of visible light. Thus, using these substitutions, a program is used to plot the function in Equation 16 below. The resulting plot can be seen in Figure 3.

$$\therefore \frac{I(r)}{I_0} = \left[\frac{2\times 600\times 10^{-9}\times r\int_0^{\pi}\cos\left(\theta-\left(\frac{\pi}{10\times 600\times 10^{-9}}\times r\sin\theta\right)\right)d\theta}{0.1\pi^2}\right]^2 \tag{16}$$

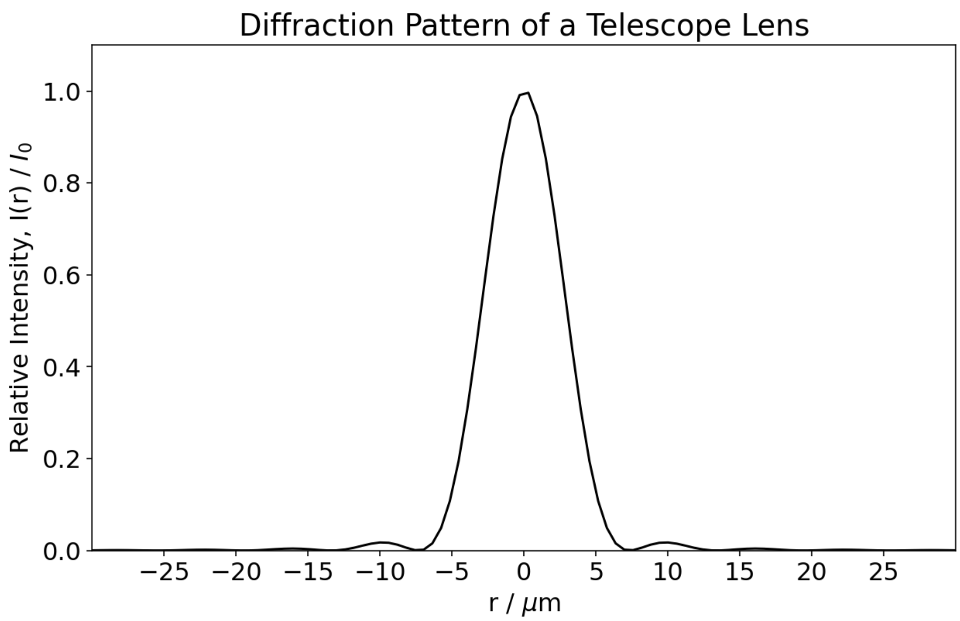

Figure 3- A plot of the relative intensity of light passing through a circular lens, as a function of r to show the diffraction pattern of the lens.

The graph shows a large bright central dot of light, with rings of light surrounding it, getting dimmer as r increases. This aligns with what we know the result should be, from our knowledge of diffraction though circular apertures.