Using Planck's radiation law to derive Wien's displacement law:

Equation 1 below shows the formula for Planck's radiation law.

$$\rho(\lambda)d\lambda = \frac{8\pi hc}{\lambda^5\left(e^{hc/\lambda kT} - 1\right)}d\lambda \tag{1}$$

Wien's law can be derived from this by differentiating $\rho(\lambda)$ with respect to x, where x is a dimensionless value:

$$x = \frac{hc}{\lambda kT} \tag{2}$$

By rearranging this, we can also substitute in,

$$\lambda = \frac{hc}{xkT} \tag{3}$$

to eliminate the variable λ from the equation, so that it can be differentiated. By applying these substitutions, we reach Equation 4 below:

$$\rho(\lambda)d\lambda = \frac{8\pi hc}{\left(\frac{hc}{xkT}\right)^5(e^x - 1)}d\lambda \tag{4}$$

Which can be simplified and rearranged to Equation 5:

$$\rho(\lambda)d\lambda = \frac{8\pi k^5T^5}{h^4c^4} * \frac{x^5}{e^x - 1}d\lambda \tag{5}$$

Since *k, h* and *c* are universal constants, and we are assuming that *T* is constant, we can create the substitution in Equation 6 below, where *A* is a constant,

$$A = \frac{8\pi k^5T^5}{h^4c^4} \tag{6}$$

And thus, we can make the substitution of Equation 6 into Equation 5. This creates Equation 7 below, which is in a format much simpler to differentiate than Equation 1,

$$\rho(\lambda)d\lambda = A\frac{x^5}{e^x - 1}d\lambda \tag{7}$$

To differentiate, we can use the quotient rule:

$$\frac{d\rho}{dx} = A * \frac{\frac{d}{dx}(x^5)(e^x-1) - \frac{d}{dx}(e^x-1)(x^5)}{(e^x-1)^2} \tag{8}$$

$$= A * \frac{(5x^4)(e^x-1) - (e^x-0)(x^5)}{(e^x-1)^2} \tag{9}$$

$$= A * \frac{5x^4e^x - 5x^4 - x^5e^x}{(e^x-1)^2} \tag{10}$$

After simplifying, we reach Equation 11:

$$\frac{d\rho}{dx} = Ax^4 * \frac{e^x(5-x)-5}{(e^x-1)^2} \tag{11}$$

Given that $\frac{d\rho}{dx} = 0$, and knowing that $A \neq 0$ from Equation 6, and we're calculating for a point where $x \neq 0$, we can deduce that,

$$e^x(5-x) - 5 = 0 \tag{12}$$

Writing a program to solve Equation 12 using the Newton Raphson method.

The Newton Raphson method can be used to find a relatively accurate value for the solution to the equation. The method involves inputting a value for x, and checking its accuracy, and trying a new value for x, getting closer to the solution until it converges to the correct number. In order to use this method, we must input a value close to the correct value, as an initial estimation. This can be found by plotting a graph of the function and estimating where the intersection is. The curve in Figure 1 is plotted using Python.

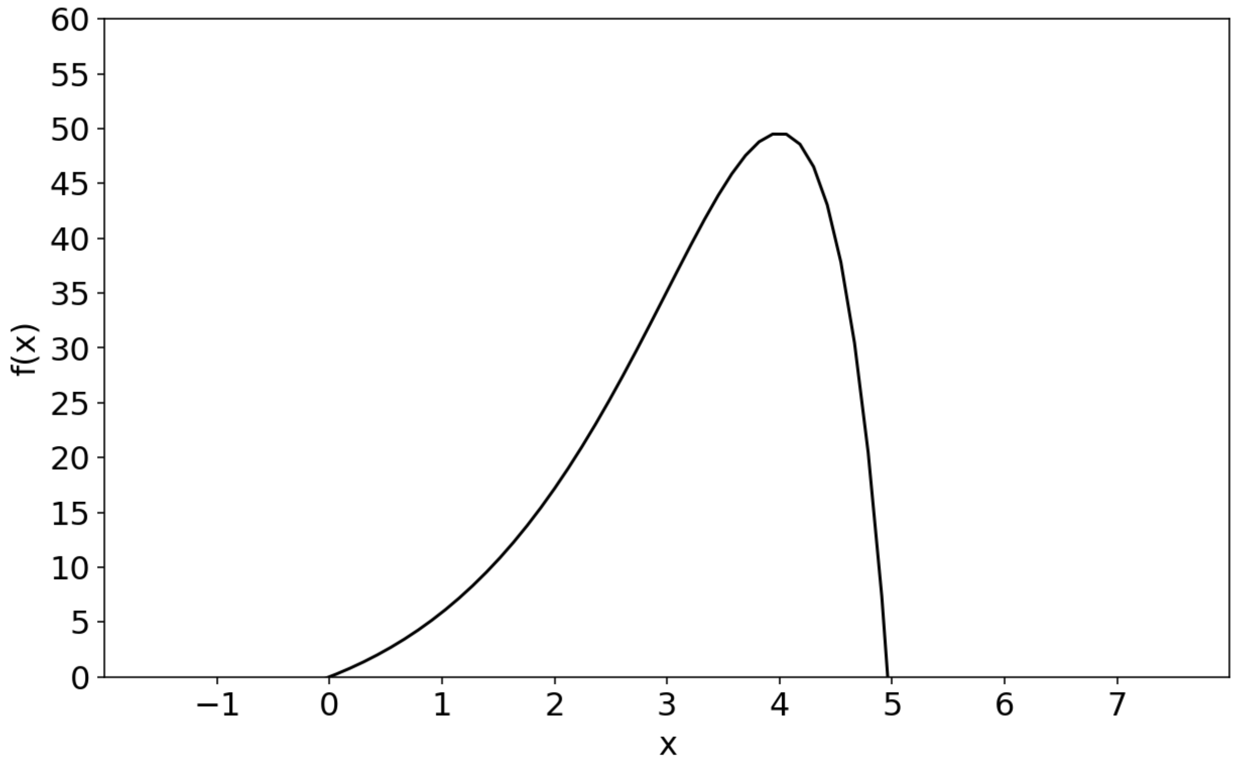

Figure 1- A graph of the function $e^x(5-x)-5=0$ *created in Jupyter Notebook.*

The line intersects the x-axis at $x=0$ and at $x\approx 5$. Using the Newton Raphson method, we can find the exact intercept. The attached file *Newton-Raphson.py* shows the Python code used to find a more accurate value for x to 6 decimal places. Figure 2 shows the output from the program, from which we can deduce that the closest value for x is 4.965114.

```
In [1]: runfile('C:/Users/mp2521/.spyder-py3/temp.py', wdir='C:/Users/mp2521/.spyder-py3')
Counter, i: 2,    xi: 4.966310,    x(i-1)-x(i): 0.033690,    Precision: 0.006738
Counter, i: 3,    xi: 4.965116,    x(i-1)-x(i): 0.001195,    Precision: 0.000241
Counter, i: 4,    xi: 4.965114,    x(i-1)-x(i): 0.000001,    Precision: 0.000000
```

Figure 2- The output from the program used to calculate x.

This calculation of x could then be used to find Wien's constant. Wien's law states that,

$$\lambda_{max} = \frac{B}{T} \tag{13}$$

Where B is Wien's constant. By looking at a standard blackbody curve, the point where the wavelength is at its maximum, is at the point where $\frac{d\rho}{dx} = 0$, and so from Equation 2,

$$\lambda_{max} = \frac{hc}{xkT} \tag{14}$$

Equating Equations 13, and 14, we see that

$$\frac{B}{T} = \frac{hc}{xkT} \quad => \quad B = \frac{hc}{xk} \tag{15}$$

By substituting the following values in for h, c and k, we can calculate Wien's constant, B.

$$h = 6.626071 \times 10^{-34}\ \mathrm{m^2 kg/s}$$

$$c = 2.997925\ \times 10^{8}\ \mathrm{m/s}$$

$$k = 1.380649 \times 10^{-23}\ \mathrm{m^2 kg/s^2 K}$$

$$=> \ B = \frac{6.626071 \times 10^{-34} \times 2.997925 \times 10^{8}}{4.965114 \times 1.380649 \times 10^{-23}} = 2.897772 \times 10^{-3}\ \mathrm{mK}$$

This value aligns almost exactly with the known value of Wien's constant, which is 2.89777×10^{-3} mK [1].

References

[1] Sheldon R., Wien's Constant, TechTarget.com, Updated February 2023, accessed 01/05/2023 https://www.techtarget.com/whatis/definition/Wiensconstant#:~:text=Wien's%20constant%20is%20typically%20represented,%E2%8B%85%2010%2 D3.