SECOND EDITION

Input–Output Analysis

Foundations and Extensions

Ronald E. Miller and Peter D. Blair

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Input–Output Analysis

The new edition of Ronald Miller and Peter Blair's classic textbook is an essential reference for students and scholars in the input–output research and applications community. The book has been fully revised and updated to reflect important developments in the field since its original publication. New topics covered include social accounting matrices (SAMs) (and extended input–output models) and their connection to input–output data, structural decomposition analysis (SDA), multiplier decompositions, identifying important coefficients, and international input–output models. A new feature of this edition is that it is also supported by an accompanying website with solutions to all problems, a sampling of real-world data sets, and supplemental appendices with further information for more advanced readers.

Input–Output Analysis is an ideal introduction to the subject for advanced undergraduate and graduate students in a wide variety of fields, including economics, regional science, regional economics, city, regional and urban planning, environmental planning, public policy analysis, and public management.

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Second Edition

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and

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Preface

We started working on the first edition of this book (Miller and Blair, 1985) in the late 1970s. At that time, input–output as an academic topic (outside of Wassily Leontief's Harvard research group) was a little more than 25 years old – 1952–1979, give or take a year at either end. We use 1952 because that was when the first author was introduced to input–output analysis in a sophomore-year economics class at Harvard taught by Robert Kuenne, who later claimed that was the first time input–output had been included (anywhere) in an undergraduate economics course.

In 1962, the first author joined the faculty of the Regional Science Department at the University of Pennsylvania. He was asked by then department chair Walter Isard to teach the graduate course in linear models for regional analysis; this was to include a strong input–output component. At that time coverage of the topic in texts was to be found primarily in two chapters of Dorfman, Samuelson and Solow (1958), in Chenery and Clark (1959), in Stone (1961) and in a long chapter on input–output at the regional level in Isard *et al.* (1960); later there were texts by Miernyk (1965), Yan (1969), and Richardson (1972).

The second author of the current text began teaching an applied course covering extensions of the input–output approach to energy, environmental, and other contemporary policy issues of the time in that same regional science program at Penn in the early 1970s, and by the end of that decade the need for a comprehensive and up-to-date textbook became apparent to us. So the first edition of this book very much reflected our shared experiences with students (primarily graduate or undergraduate submatriculants) in mostly regional science and public policy courses at Penn during the 1960s and 1970s. In addition to the basics ("foundations"), many of the additional topics we included ("extensions") reflected our research interests at that time – interregional feedbacks for one of us, energy and environmental applications for the other and spatial aggregation in many-region models as a joint interest.

Over the past decade or so it became increasingly and abundantly clear that the time was ripe for an update/revision. We began to take this notion seriously around 2000–2001 – almost an additional 25 years further into the input–output timeline, so the subject was essentially twice as old as when we wrote the first edition. Activity in the field during that quarter century seems to have exploded. For example:

- The International Input–Output Association (IIOA) was founded (1988).
- The IIOA's journal, *Economic Systems Research*, began publication (1989).
- International conferences were held with increasing frequency and drawing increasing numbers of participants (summarized at the end of Appendix C) and starting recently, "intermediate" input–output meetings are held in nonconference years, co-organized by the IIOA.
- In 1998 Heinz Kurz, Erik Dietzenbacher and Christian Lager published an edited three-volume set of almost 1,500 pages that reproduces some 85 significant input–output papers, along with extensive and detailed introductions to each of the volumes (Kurz, Dietzenbacher and Lager, 1998).

These activities are in part a reflection of enormous increases in computer speed and capacity since the 1950s. The net result is that there is now considerable new material to be examined, digested and considered for inclusion and explanation.

Accordingly, around the end of 2000 we communicated with about 30 of our input–output colleagues throughout the world, asking for help in finding our way through this maze of material. We listed some new topics that we thought should be included (e.g., social accounting matrices or SAMs), some that we might emphasize more (e.g., commodity-by-industry models), some less (e.g., detailed numerical interregional or multiregional examples), and we asked for reactions and suggestions. Additionally, we took into account what we knew of the uses to which the first edition had been put, e.g., as a text for teaching purposes or desk reference for practitioners and researchers.

As a result, we have added some discussion of:

- SAMs (and extended input–output models) and their connection to input–output data;
- Structural decomposition analysis (SDA);
- Multiplier decompositions [Miyazawa, additive (Stone), multiplicative (Pyatt and Round)];
- Identifying important coefficients;
- International input–output models.

We have expanded discussions of:

- The historical background and context for Leontief's work;
- The connection of input–output accounts and national income and product accounts (NIPAs);
- Commodity-by-industry accounting and models;
- Multipliers, including Miyazawa multipliers, net multipliers, elasticity measures, and output-to-output multipliers;
- Location quotients and related techniques for estimating regional technology, including numerical examples and real-world illustrations;
- Energy input–output analysis to include references to econometric extensions;

- Environmental applications to include linear programming and multiobjective programming extensions;
- The hypothetical extraction approach to linkage analysis;
- The Ghosh (supply-side) model;
- The Leontief price model;
- Estimating interregional flows;
- Hybrid methods;
- Mixed exogenous/endogenous models.

In order to keep the new edition to manageable length, there are topics that had to be excluded or treated only very briefly; these include:

- Econometric/input–output model connections;
- Qualitative input–output analysis;
- Recent developments in dynamic input–output modeling;
- Discussions and comparisons of alternative working models (e.g., REMI and IMPLAN in the USA and others elsewhere);
- The role and interpretation of eigenvectors and eigenvalues in input–output models.

The historical material on US input–output data has been reworked and updated, especially to reflect the international movement toward commodity-by-industry formulations. With the ready accessibility of computing capabilities, we have greatly expanded the end-of-chapter problems to include many more realistic examples as well as some real-world examples and applications. Because of the higher level of mathematical competence that we see in our potential readers as compared with 20+ years ago, we have tried to use more compact matrix representations more extensively and whenever possible.

With appreciation we acknowledge many helpful conversations, face-to-face and electronic, with Takahiro Akita, William Beyers, Faye Duchin, Geoffrey Hewings, Takeo Ihara, Andrew Isserman, Randall Jackson, Louis de Mesnard, Jan Oosterhaven, Karen Polenske, Jeffery Round and Guy West. Anne Carter and Joseph Richter helped us fill in the historical record of IIOA meetings. Sandra Svaljek and Ivan Rusan at The Institute of Economics, Zagreb, Croatia, kindly supplied us with a copy of Mijo Sekulić's important 1968 article in *Ekonomska Analiza*, a publication of their Institute. We are grateful to IDE/JETRO (Tokyo), especially Satoshi Inomata, who provided us with many of their important input–output tables and studies using those tables and to leadership and staff of the US Department of Commerce, Bureau of Economic Analysis, especially Mark Planting, who helped us navigate the US input–output tables. Finally, we single out two colleagues with whom we have had almost continuous interaction for years: Erik Dietzenbacher, with whom we have had literally hundreds of discussions and from whom we have had as many suggestions, and Michael Lahr who has been a constant source of critical observations and has recommended and helped us track down countless important references.

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1 Introduction and Overview

1.1 Introduction

Input–output analysis is the name given to an analytical framework developed by Professor Wassily Leontief in the late 1930s, in recognition of which he received the Nobel Prize in Economic Science in 1973 (Leontief, 1936, 1941). One often speaks of a Leontief model when referring to input–output. The term *interindustry analysis* is also used, since the fundamental purpose of the input–output framework is to analyze the interdependence of industries in an economy. Today the basic concepts set forth by Leontief are key components of many types of economic analysis and, indeed, input–output analysis is one of the most widely applied methods in economics (Baumol, 2000). This book develops the framework set forth by Leontief and explores the many extensions that have been developed over the last nearly three quarters of a century.

In its most basic form, an input–output model consists of a system of linear equations, each one of which describes the distribution of an industry's product throughout the economy. Most of the extensions to the basic input–output framework are introduced to incorporate additional detail of economic activity, such as over time or space, to accommodate limitations of available data or to connect input–output models to other kinds of economic analysis tools. This book is an updated and considerably expanded edition of our 1985 textbook (Miller and Blair, 1985).

In this chapter we introduce the basic input–output analysis framework and outline the topics to be covered in the balance of the text. Appendix C provides a historical account of the work leading up to Leontief's formulation and its subsequent development and refinement. More detailed historical accounts of the early development of input–output analysis and input–output accounts are given in Polenske and Skolka (1976, Chapter 1) and Stone (1984). A fairly complete history of applications of input–output analysis since Leontief's introduction of it is provided in Rose and Miernyk (1989). In the present text we cover many of the developments in input–output since its widespread application as an analysis tool began in the early 1950s. Leontief himself participated in a number of these developments and applications, as will be evident throughout this text (see also Polenske, 1999, 2004).

The widespread availability of high-speed digital computers has made Leontief's input–output analysis a widely applied and useful tool for economic analysis at many geographic levels – local, regional, national, and even international. Prior to the appearance of modern computers, the computational requirements of input–output models made them very difficult and even impractical to implement. Today, in the USA alone, input–output is routinely applied in national economic analysis by the US Department of Commerce, and in regional economic planning and analysis by states, industry, and the research community. The model is widely applied throughout the world; the United Nations has promoted input–output as a practical planning tool for developing countries and has sponsored a standardized system of economic accounts for constructing input–output tables.

Input–output has been also extended to be part of an integrated framework of employment and social accounting metrics associated with industrial production and other economic activity, as well as to accommodate more explicitly such topics as international and interregional flows of products and services or accounting for energy consumption and environmental pollution associated with interindustry activity. In this text, we present the foundations of the input–output model as originally developed by Leontief, as well as the evolution of many methodological extensions to the basic framework. In addition, we illustrate many of the applications of input–output and its usefulness for practical policy questions. Throughout the text, we will review some of the current research frontiers.

1.2 Input–Output Analysis: The Basic Framework

The basic Leontief input–output model is generally constructed from observed economic data for a specific geographic region (nation, state, county, etc.). One is concerned with the activity of a group of industries that both produce goods (outputs) and consume goods from other industries (inputs) in the process of producing each industry's own output. In practice, the number of industries considered may vary from only a few to hundreds or even thousands. For instance, an industrial sector title might read "manufactured products," or that same sector might be broken down into many different specific products.

The fundamental information used in input–output analysis concerns the flows of products from each industrial sector, considered as a producer, to each of the sectors, itself and others, considered as consumers. This basic information from which an input–output model is developed is contained in an interindustry transactions table. The rows of such a table describe the distribution of a producer's output throughout the economy. The columns describe the composition of inputs required by a particular industry to produce its output. These interindustry exchanges of goods constitute the shaded portion of the table depicted in Figure 1.1. The additional columns, labeled *Final Demand*, record the sales by each sector to final markets for their production, such as personal consumption purchases and sales to the federal government. For example, electricity is sold to businesses in other sectors as an input to production (an interindustry transaction)

		PRODUCERS AS CONSUMERS								FINAL DEMAND			
		Agric.	Mining	Const.	Manuf.	Trade	Transp.	Services	Other	Personal Consumption Expenditures	Gross Private Domestic Investment	Govt. Purchases of Goods & Services	Net Exports of Goods & Services
PRODUCERS	Agriculture												
	Mining												
	Construction												
	Manufacturing												
	Trade												
	Transportation												
	Services												
	Other Industry												
VALUE ADDED	Employees	Employee compensation								GROSS DOMESTIC PRODUCT			
	Business Owners and Capital	Profit-type income and capital consumption allowances											
	Government	Indirect business taxes											

Figure 1.1 Input–Output Transactions Table

and also to residential consumers (a final-demand sale). The additional rows, labeled *Value Added*, account for the other (non-industrial) inputs to production, such as labor, depreciation of capital, indirect business taxes, and imports.

The formulation of analytical models using the basic input–output data as just described is the principal purpose of this text. There is a considerable literature devoted to assembling the basic data used in input–output models from surveys or interpretation of other primary and secondary sources of economic data. Some of this literature is referenced in Chapter 4, but, for the most part, in this text we focus on the formulation of models using available data or on methods to compensate for the lack of available data.

1.3 Outline for this Text

This text is organized into 14 chapters, beginning with the theory and assumptions of the basic input–output framework, then exploring many of the extensions developed over the last half century. The text deals mostly with methodological developments, but also covers some of the practical issues associated with implementation of input–output models, including many references to the applied literature. Chapters 2–6 cover the main methodological considerations in input–output analysis. Chapters 7–13 cover many issues associated with the application of input–output analysis to practical problems. The concluding chapter, Chapter 14, sketches a number of relevant topics for which available space did not permit a more detailed treatment or that were beyond the scope of this text. The following describes the main topics covered in each chapter:

- Chapter 2 introduces Leontief's conceptual input–output framework and explains how to develop the fundamental mathematical relationships from the interindustry transactions table. The key assumptions associated with the basic Leontief model and implications of those assumptions are recounted and the economic interpretation of the basic framework is explored. The basic framework is illustrated with a highly aggregated model of the US economy. In addition, the "price model" formulation of the input–output framework is introduced to explore the role of prices in input–output models. Appendices to this chapter include a fundamental set of mathematical conditions for input–output models, known as the Hawkins–Simon conditions.

- Chapter 3 extends the basic input–output framework to analysis of regions and the relationships between regions. First, "single-region" models are presented and the various assumptions employed in formulating regional models versus national models are explored. Next, the structure of an interregional input–output (IRIO) model, designed to expand the basic input–output framework to capture transactions between industrial sectors in regions, is presented. An important simplification of the IRIO model designed to deal with the most common of data limitations in constructing such models is known as the multiregional input–output (MRIO) model. The basic MRIO formulation is presented and the implications of the simplifying assumptions explored. Next the balanced regional model is presented, which is mathematically identical to the IRIO framework, but is designed conceptually to capture the distinction between industrial production for regional versus national markets as opposed to delivery to specific regions as in the IRIO framework. In the final section a number of applied studies are cited in order to illustrate the extraordinary range of geographic scale reflected in real-world studies – from sub-city neighborhoods to so-called "world" models. Appendices to this chapter provide additional development of mathematical tools helpful for conceptualizing and implementing regional models.
- Chapter 4 deals with the construction of input–output tables from standardized conventions of national economic accounts, such as the widely used System of National Accounts (SNA) promoted by the United Nations, including a basic introduction to the so-called commodity-by-industry or supply-use input–output framework developed in additional detail in Chapter 5. A simplified SNA is derived from fundamental economic concepts of the circular flow of income and expenditure, that, as additional sectoral details are defined for businesses, households, government, foreign trade, and capital formation, ultimately result in the basic commodity-by-industry formulation of input–output accounts. The process is illustrated with the US input–output model and some of the key traditional conventions widely applied for such considerations as secondary production (multiple products or commodities produced by a business), competitive imports (commodities that are also produced domestically) versus non-competitive imports (commodities not produced domestically), trade and transportation margins on interindustry transactions, or the treatment of scrap and secondhand goods. Finally, the chapter concludes with an examination of issues associated with the level of sectoral and spatial detail in input–output models, e.g., the potential bias introduced by the level of aggregation of industries or regions.

The appendices illustrate the implications of aggregation bias using IRIO and MRIO models for Japan and the USA.

- Chapter 5 explores variations to the commodity-by-industry input–output framework introduced in Chapter 4, expanding the basic input–output framework to include distinguishing between commodities and industries, i.e., the supply of specific commodities in the economy and the use of those commodities by collections of businesses defined as industries. The chapter introduces the fundamental commodity-by-industry accounting relationships and how they relate to the basic input–output framework. Alternative assumptions are defined for handling the common accounting issue of secondary production, and economic interpretations of those alternative assumptions are presented. The formulations of commodity-driven and industry-driven models are also presented along with illustrations of variants on combining alternative assumptions for secondary production. Finally, the chapter illustrates a variety of special circumstances encountered with commodity-by-industry models, such as nonsquare commodity–industry systems or the interpretation of negative elements. Appendices to this chapter provide some alternative derivations of commodity-by-industry transactions matrices, methods for eliminating negative entries in specific types of commodity-by-industry models where appearance of such entries is most common, and additional observations on nonsquare commodity-by-industry systems are provided in an appendix on this text's website (www.cambridge.org/millerandblair).
- Chapter 6 examines a number of key summary analytical measures known as multipliers that can be derived from input–output models to estimate the effects of exogenous changes on (1) new outputs of economic sectors, (2) income earned by households resulting from new outputs, and (3) employment generated from new outputs or (4) value-added generated by production. The general structure of multiplier analysis and special considerations associated with regional, IRIO, and MRIO models are developed. Extensions to capture the effects of income generation for various household groups are explored, as well as additional multiplier variants and decomposition into meaningful economic components. Chapter appendices expand on a number of mathematical formulations of household and income multipliers.
- Chapter 7 introduces approaches designed to deal with the major challenge in input–output analysis that the kinds of information-gathering surveys needed to collect input–output data for an economy can be expensive and very time consuming, resulting in tables of input–output coefficients that are outdated before they are produced. These techniques, known as partial survey and nonsurvey approaches to input–output table construction, are central to modern applications of input–output analysis. The chapter begins by reviewing the basic factors contributing to the stability of input–output data over time, such as changing technology, prices, and the scale and scope of business enterprises. Several techniques for updating input–output data are developed and the economic implications of each described. The bulk of the chapter is concerned with the biproportional scaling (or RAS) technique and some "hybrid model" variants.

- Chapter 8 surveys a range of partial survey and nonsurvey estimation approaches for creating input–output tables at the regional level. Variants of the commonly used class of estimating procedures using location quotients are reviewed, which presume a regional estimate of input–output data can be derived using some information about a target region. The RAS technique developed in Chapter 7 is applied to developing regional input–output tables using a base national table or a table for another region and some available data for the target region. These are illustrated using data from a three-region model for China. Techniques for partial survey estimation of commodity flows between regions are also presented along with discussions of several real-world multinational applications, including the China–Japan Transnational Interregional Model and Leontief's World Model.
- Chapter 9 explores the extension of the input–output framework to more detailed analysis of energy consumption associated with industrial production, including some of the complications that can arise when measuring input–output transactions in physical units of production rather than in monetary terms of the value of production. Early approaches to energy input–output analysis are reviewed and compared with contemporary approaches and the strengths and limitations of alternative approaches are examined. Special methodological considerations such as adjusting for energy conversion efficiencies are developed and a number of illustrative applications are presented, including estimation of the energy costs of goods and services, impacts of new energy technologies, and energy taxes. Finally, the role of structural change of an input–output economy associated with changing patterns of energy use is introduced (more general approaches to structural decomposition analysis using input–output models are covered in Chapter 13). The appendix to this chapter develops more formally the strengths and limitations of alternative energy input–output formulations.
- Chapter 10 reviews the extensions of the input–output framework to incorporate activities of environmental pollution and elimination associated with economic activities as well as the linkages of input–output to models of ecosystems. The chapter begins with a "generalized" input–output framework which assumes that pollution generation (as well as other measurable factors associated with industrial production, such as energy or material consumption measured in physical units or employment measured in person-years) simply vary in direct proportion to the level of industrial production. Applications are presented of the generalized input–output formulation to measuring impacts of specified changes to industrial activity and to planning problems where the objective is to seek an optimal mix of industrial production subject to input–output relationships between industrial sectors and to constraints on factors associated with industrial production, such as pollution, energy use and employment. In exploring the application of the generalized input–output framework to planning problems, basic concepts of linear and multiobjective programming are introduced. The chapter also explores augmenting a basic Leontief input–output model with pollution generation and elimination sectors. Finally, expansion of the input–output framework to include ecologic sectors to more comprehensively trace

economic–ecosystem relationships is presented along with a variety of illustrative applications.

- Chapter 11 expands the input–output framework to a broader class of economic analysis tools known as social accounting matrices (SAM) and other so-called "extended" input–output models to capture activities of income distribution in the economy in a more comprehensive and integrated way, including especially employment and social welfare features of an economy. The basic concepts of SAMs are explored and derived from the SNA introduced in Chapters 4 and 5, and the relationships between SAMs and input–output accounts are presented. The concept of SAM multipliers as well as the decomposition of SAM multipliers into components with specific economic interpretations are introduced and illustrated. Finally, techniques for balancing SAM accounts for internal accounting consistency are discussed and a number of illustrative applications of the use of SAMs are presented.
- Chapter 12 presents the so-called supply side input–output model, with which the name Ghosh is most often associated. It is discussed both as a quantity model (the early interpretation) and as a price model (the more modern interpretation). Relationships to the standard Leontief quantity and price models are also explored. In addition, the fast growing literature on quantification of economic linkages and analysis of the overall structure of economies using input–output data is examined. Finally, approaches for identifying key or important coefficients in input–output models and alternative measures of coefficient importance are presented.
- Chapter 13 introduces and illustrates the basic concepts of structural decomposition analysis (SDA) within an input–output framework. The concept of decomposition of multipliers introduced in Chapter 6 and in Chapter 10 as applied to SAMs is revisited as a way to analyze economic structure. The application of SDA to MRIO is developed to introduce a spatial context, many applications are cited and summaries of their results are presented. Next, mixed endogenous–exogenous models are explored. These models expand upon the standard input–output model by allowing for exogenous specification of both (some) final demands and (some) outputs. This chapter also introduces dynamic input–output models that more explicitly capture the role of capital investment and utilization in the production process. Appendices develop extended presentations of additional decomposition and mixed-model results.
- Chapter 14 briefly describes some additional extensions to input–output analysis for which space does not permit a detailed treatment, including linkages to econometric models, computable general equilibrium models, and measuring economic productivity.
- Appendix A is an introductory review of matrix algebra concepts and methods used throughout this text.
- Appendix B presents a highly aggregated series of the US input–output tables referenced and used in end-of-chapter problems in a number of chapters or in supplementary problems included on the Internet website associated with this book (www.cambridge.org/millerandblair).

Table 1.1 Illustrative Real Input–Output Data Locations

Data	Location
US Domestic Direct Requirements Matrix, 2003	Table 2.7
US Domestic Total Requirements Matrix, 2003	Table 2.8
Chinese Interregional and Intraregional Transactions, 2000	Table 3.7
Direct Input Coefficients for the Chinese Multiregional Economy, 2000	Table 3.8
Leontief Inverse Matrix for the Chinese Multiregional Economy, 2000	Table 3.9
Four-Region, Three-Sector IRIO Model for the USA and Asia	Prob. 3.9
Three-Region, Five-Sector IRIO Model for Japan, 1965	Table A4.1.1
Three-Region, Five-Sector MRIO Model for the USA, 1963	Table A4.1.3
Components of US Total Commodity Final Demand, 2003	Table 5.11
Seven-Sector US Input–Output Tables for 1997, 2003, and 2005	Prob. 7.1
Seven-Sector Direct Input Coefficients Outputs for Washington State, 1997	Prob. 8.10
Input–Output Transactions for the US Economy in Hybrid Units, 1967	Table 9.5
Technical Coefficients for the US Economy in Hybrid Units, 1967	Table 9.6
Leontief Inverse for the US Economy in Hybrid Units, 1967	Table 9.7
Nine-Sector Hybrid Units US Technical Coefficients, 1963 and 1980	Prob. 9.10
Macro SAM for Sri Lanka, 1970	Prob. 11.5
Macro SAM for the US Economy, 1988	Prob. 11.8
SAM with Expanded Interindustry Detail for the USA, 1988	Table 11.22
Selected US Input–Output Tables, 1919–2006	Appendix B

- Appendix C provides an historical account of the early development of input–output analysis, including a "pre-history" of the concepts that led to Leontief's work as well as the many methodological developments and applications since.

1.4 Internet Website and Text Locations of Real Datasets

A website associated with this text, www.cambridge.org/millerandblair, includes supplementary information in three general areas: (1) additional text (appendices) in selected areas that were not possible to include in the printed text for a variety of reasons, (2) solutions to end-of-chapter problems as well as supplementary problems, case studies, and suggested input–output analysis experiments and study projects and (3) downloadable datasets of many of the examples and problems printed in the text as well as a library of supplementary real-world datasets and references to additional data that have come to our attention.

Throughout this text, in various illustrative examples and problems, we employ real but highly aggregated input–output related data for various regions and nations as well as illustrative interregional input–output (IRIO) and multiregional input–output (MRIO) data and social accounting matrices (SAM). For convenience, Table 1.1 shows a listing of these sets of data and their locations in this text.

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2 Foundations of Input–Output Analysis

2.1 Introduction

In this chapter we begin to explore the fundamental structure of the input–output model, the assumptions behind it, and also some of the simplest kinds of problems to which it is applied. Later chapters will examine the special features that are associated with regional models and some of the extensions that are necessary for particular kinds of problems – for example, in energy or environmental studies or as part of a broader system of social accounts.

The mathematical structure of an input–output system consists of a set of n linear equations with n unknowns; therefore, matrix representations can readily be used. In this chapter we will start with more detailed algebraic statements of the fundamental relationships and then go on to use matrix notation and manipulations more and more frequently. Appendix A contains a review of matrix algebra definitions and operations that are essential for input–output models. While solutions to the input–output equation system, via an inverse matrix, are straightforward mathematically, we will discover that there are interesting economic interpretations to some of the algebraic results.

2.2 Notation and Fundamental Relationships

An input–output model is constructed from observed data for a particular economic area – a nation, a region (however defined), a state, etc. In the beginning, we will assume (for reasons that will become clear in the next chapter) that the economic area is a country. The economic activity in the area must be able to be separated into a number of segments or producing sectors. These may be industries in the usual sense (e.g., steel) or they may be much smaller categories (e.g., steel nails and spikes) or much larger ones (e.g., manufacturing). The necessary data are the flows of products from each of the sectors (as a producer/seller) to each of the sectors (as a purchaser/buyer); these *interindustry* flows, or transactions (or intersectoral flows – the terms *industry* and *sector* are often used interchangeably in input–output analysis) are measured for a particular time period (usually a year) and in monetary terms – for example, the dollar value of steel sold to automobile manufacturers last year.[1]

The exchanges of goods between sectors are, ultimately, sales and purchases of physical goods – tons of steel bought by automobile manufacturers last year. In accounting for transactions between and among all sectors, it is possible in principle to record all exchanges either in physical or in monetary terms. While the physical measure is perhaps a better reflection of one sector's use of another sector's product, there are substantial measurement problems when sectors actually sell more than one good (a Cadillac CTS and a Ford Focus are distinctly different products with different prices; in physical units, however, both are cars). For these and other reasons, then, accounts are generally kept in monetary terms, even though this introduces problems due to changes in prices that do not reflect changes in the use of physical inputs. (In section 2.6 we will explore the implications of a data set in which transactions are expressed in physical units – for example, tons of steel sold to the automobile sector last year.)

One essential set of data for an input–output model are monetary values of the transactions between pairs of sectors (from each sector i to each sector j); these are usually designated as z_{ij}. Sector j's demand for inputs from other sectors during the year will have been related to the amount of goods produced by sector j over that same period. For example, the demand from the automobile sector for the output of the steel sector is very closely related to the output of automobiles, the demand for leather by the shoe-producing sector depends on the number of shoes being produced, etc.

In addition, in any country there are sales to purchasers who are more external or *exogenous* to the industrial sectors that constitute the producers in the economy – for example, households, government, and foreign trade. The demands of these units – and hence the magnitudes of their purchases from each of the industrial sectors – are generally determined by considerations that are relatively unrelated to the amount being produced. For example, government demand for aircraft is related to broad changes in national policy, budget levels, or defense needs; consumer demand for small cars is related to gasoline availability, and so on. The demand of these external units, since it tends to be much more for goods to be used as such and not to be used as an input to an industrial production process, is generally referred to as *final demand*.

Assume that the economy can be categorized into n sectors. If we denote by x_i the total output (production) of sector i and by f_i the total final demand for sector i's product, we may write a simple equation accounting for the way in which sector i distributes its product through sales to other sectors and to final demand:

$$x_i = z_{i1} + \cdots + z_{ij} + \cdots + z_{in} + f_i = \sum_{j=1}^{n} z_{ij} + f_i \tag{2.1}$$

[1] In Chapters 4 and 5 we will explore more recent distinctions between "commodities" and "industries" and see how these observations lead to alternative representations of the input–output model.

The z_{ij} terms represent *interindustry* sales by sector i (also known as *intermediate* sales) to all sectors j (including itself, when $j = i$). Equation (2.1) represents the distribution of sector i *output*. There will be an equation like this that identifies sales of the output of each of the n sectors:

$$
\begin{aligned}
x_1 &= z_{11} + \cdots + z_{1j} + \cdots + z_{1n} + f_1 \\
&\vdots \\
x_i &= z_{i1} + \cdots + z_{ij} + \cdots + z_{in} + f_i \\
&\vdots \\
x_n &= z_{n1} + \cdots + z_{nj} + \cdots + z_{nn} + f_n
\end{aligned}
\tag{2.2}
$$

Let

$$
\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nn} \end{bmatrix} \text{ and } \quad \mathbf{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \tag{2.3}
$$

Here and throughout this text we use lower-case bold letters for (column) vectors, as in $\mathbf{f}$ and $\mathbf{x}$ (so $\mathbf{x}'$ is the corresponding row vector) and upper case bold letters for matrices, as in $\mathbf{Z}$. With this notation, the information in (2.2) on the distribution of each sector's sales can be compactly summarized in matrix notation as

$$
\mathbf{x} = \mathbf{Zi} + \mathbf{f} \tag{2.4}
$$

We use $\mathbf{i}$ to represent a column vector of 1's (of appropriate dimension – here n). This is known as a "summation" vector (Section A.8, Appendix A). The important observation is that post-multiplication of a matrix by $\mathbf{i}$ creates a column vector whose elements are the row sums of the matrix. Similarly, $\mathbf{i}'$ is a row vector of 1's, and premultiplication of a matrix by $\mathbf{i}'$ creates a row vector whose elements are the column sums of the matrix. We will use summation vectors often in this and subsequent chapters.

Consider the information in the jth column of z's on the right-hand side:

$$
\begin{bmatrix} z_{1j} \\ \vdots \\ z_{ij} \\ \vdots \\ z_{nj} \end{bmatrix}
$$

These elements are sales to sector j – j's purchases of the products of the various producing sectors in the country; the column thus represents the sources and magnitudes of sector j's *inputs*. Clearly, in engaging in production, a sector also pays for other items – for example, labor and capital – and uses other inputs as well, such as inventoried items.

Table 2.1 Input–Output Table of Interindustry Flows of Goods

		Buying Sector				
		1	$\cdots$	j	$\cdots$	n
Selling Sector	1	z_{11}	$\cdots$	z_{1j}	$\cdots$	z_{1n}
	$\vdots$	$\vdots$		$\vdots$		$\vdots$
	i	z_{i1}	$\cdots$	z_{ij}	$\cdots$	z_{in}
	$\vdots$	$\vdots$		$\vdots$		$\vdots$
	n	z_{n1}	$\cdots$	z_{nj}	$\cdots$	z_{nn}

All of these *primary inputs* together are termed the *value added* in sector j. In addition, imported goods may be purchased as inputs by sector j. All of these inputs (value added and imports) are often lumped together as purchases from what is called the *payments* sector, whereas the z's on the right-hand side of (2.2) serve to record the purchases from the *processing* sector, the *interindustry inputs* (or *intermediate inputs*). Since each equation in (2.2) includes the possibility of purchases by a sector of its own output as an input to production, these *inter*industry inputs include *intra*industry transactions as well.

The magnitudes of these interindustry flows can be recorded in a table, with sectors of origin (producers) listed on the left and the same sectors, now destinations (purchasers), listed across the top. From the column point of view, these show each sector's inputs; from the row point of view the figures are each sector's outputs; hence the name *input–output table*. These figures are the core of input–output analysis.

2.2.1 Input–Output Transactions and National Accounts

As was suggested by Table 1.1, an input–output transactions (flow) table, such as that shown in Table 2.1, constitutes part of a complete set of income and product accounts for an economy. To emphasize the other elements in a full set of accounts, we consider a small, two-sector economy. We present an expanded flow table for this extremely simple economy in Table 2.2. (We examine more of the details of a system of national accounts in Chapter 4.)

The component parts of the final demand vector for sectors 1 and 2 represent, respectively, consumer (household) purchases, purchases for (private) investment purposes, government (federal, state, and local) purchases, and sales abroad (exports). These are often grouped into *domestic* final demand $(C+I+G)$ and *foreign* final demand (exports, E). Then $f_1 = c_1 + i_1 + g_1 + e_1$ and similarly $f_2 = c_2 + i_2 + g_2 + e_2$.

The component parts of the payments sector are payments by sectors 1 and 2 for employee compensation (labor services, l_1 and l_2) and for all other value-added items – for example, government services (paid for in taxes), capital (interest payments), land

Table 2.2 Expanded Flow Table for a Two-Sector Economy

		Processing Sectors		Final Demand				Total Output (**x**)
		1	2					
Processing Sectors	1	z_{11}	z_{12}	c_1	i_1	g_1	e_1	x_1
	2	z_{21}	z_{22}	c_2	i_2	g_2	e_2	x_2
Payments Sectors	Value Added ($\mathbf{v}'$)	l_1	l_2	l_C	l_I	l_G	l_E	L
		n_1	n_2	n_C	n_I	n_G	n_E	N
	Imports	m_1	m_2	m_C	m_I	m_G	m_E	M
Total Outlays ($\mathbf{x}'$)		x_1	x_2	C	I	G	E	X

(rental payments), entrepreneurship (profit), and so on. Denote these other value-added payments by n_1 and n_2; then total value-added payments are $v_1 = l_1 + n_1$, and $v_2 = l_2 + n_2$, for the two sectors.

Finally, assume that some (or perhaps all) sectors use imported goods in producing their outputs. One approach is to record these import amounts in an imports row in the payments sector as m_1 and m_2.[2] Total expenditures in the payments sector by sectors 1 and 2 are $l_1 + n_1 + m_1 = v_1 + m_1$ and $l_2 + n_2 + m_2 = v_2 + m_2$, respectively. However, it is often the case that the exports part of the final demand column is expressed as *net* exports so that the sum of all final demands is equal to traditional definitions of gross domestic product, i.e., net of imports. In that case a distinction is often made between imports of goods that are also domestically produced (competitive imports) and those for which there is no domestic source (noncompetitive imports), and all the competitive imports in the imports row will have been netted out of the appropriate elements in a *gross* exports column. Under these circumstances it is possible for one or more elements in the net export column to be negative, if the value of imports of those goods exceeds the value of exports. (For example, if an economy exported €300 million of agricultural products last year but imported €350 million, the net exports figure for the agricultural sector would be €50 million.) Also, if the federal government *sells* more of a stockpiled item (e.g., wheat) than it buys, a negative entry in the government column of the final demand part of the table could result. If the negative number is large enough, it could swamp the other (positive) final demand purchases of that good, leaving a negative total final demand figure.

The elements in the intersection of the value-added rows and the final demand columns represent payments by final consumers for labor services (for example, l_C includes household payments for, say, domestic help; l_G represents payments to

[2] The treatment of imports in input–output accounts is much more complicated than this, but for the present we prefer to concentrate on the overall structure of a transactions table. We return to imports in section 2.3.4 below, and in more detail in Chapter 4.

government workers) and for other value added (for example, n_C includes tax payments by households). In the imports row and final demand columns are, for example, m_G, which represents government purchases of imported items, and m_E, which represents imported items that are re-exported.

Summing down the total output column, total gross output throughout the economy, X, is found as

$$X = x_1 + x_2 + L + N + M$$

This same value can be found by summing across the total outlays row; namely

$$X = x_1 + x_2 + C + I + G + E$$

These are simply two alternative ways of summing all the elements in the table.

In national income and product accounting, it is the value of total *final* product that is of interest – goods available for consumption, export, and so on. Equating the two expressions for X and subtracting x_1 and x_2 from both sides leaves

$$L + M + N = C + I + G + E$$

or

$$L + N = C + I + G + (E - M)$$

The left-hand side represents gross national income – the total factor payments in the economy – and the right-hand side represents gross national product – the total spent on consumption and investment goods, total government purchases, and the total value of net exports from the economy. Again, national accounts are examined in more detail in Chapter 4.

In most developed economies, consumption is the largest individual component of final demand. For example, in the USA in 2003 the percentages of total final demand were as follows: personal consumption expenditure (PCE), 71 percent; gross private domestic investment (including producers' durable equipment, plant construction, residential construction, and net inventory change), 15 percent; government purchases (federal, state and local), 19 percent; net foreign exports, −5 percent (the value of imports exceeded the value of exports). [However, in the USA during the 1942–1945 period (World War II), PCE was between 40 and 48 percent and for much of the 1950s and 1960s it was under 60 percent.]

2.2.2 Production Functions and the Input–Output Model

In input–output work, a fundamental assumption is that the interindustry flows from i to j – recall that these are for a given period, say a year – depend entirely on the total output of sector j for that same time period. Clearly, no one would argue against the idea that the more cars produced in a year, the more steel will be needed during that year by automobile producers. Where argument *does* arise is over the exact nature of this relationship. In input–output analysis it is as follows: Given z_{ij} and x_j – for example, input of aluminum (i) bought by aircraft producers (j) last year and total aircraft production last year – form the ratio of aluminum input to aircraft output, z_{ij}/x_j [the units are (\$/\$)], and denote it by a_{ij}:

$$a_{ij} = \frac{z_{ij}}{x_j} = \frac{\text{value of aluminum bought by aircraft producers last year}}{\text{value of aircraft production last year}} \tag{2.5}$$

This ratio is called a technical coefficient; the terms input–output coefficient and direct input coefficient are also often used. For example, if $z_{14} = \$300$ and $x_4 = \$15,000$ (sector 4 used \$300 of goods from sector 1 in producing \$15,000 of sector 4 output), $a_{14} = z_{14}/x_4 = \$300/\$15,000 = 0.02$. Since a_{14} is actually \$0.02/\$1, the 0.02 is interpreted as the "dollars' worth of inputs from sector 1 per dollar's worth of output of sector 4."

From (2.5), $a_{ij}x_j = z_{ij}$. This is trivial algebra, but it presents the operational form in which the technical coefficients are used. In input–output analysis, once a set of observations has given us the result $a_{14} = 0.02$, this technical coefficient is assumed to be unchanging in the sense that if one asked how much sector 4 would buy from sector 1 if sector 4 were to produce a total output (x_4) of \$45,000, the input–output answer would be $z_{14} = a_{14}x_4 = (0.02)(\$45,000) = \$900$ – when output of sector 4 is tripled, the input from sector 1 is tripled. The a_{ij} are viewed as measuring fixed relationships between a sector's output and its inputs. Economies of scale in production are thus ignored; production in a Leontief system operates under what is known as constant returns to scale.

In addition, input–output analysis requires that a sector use inputs in *fixed proportions*. Suppose, to continue the previous example, that sector 4 also buys inputs from sector 2, and that, for the period of observation, $z_{24} = \$750$. Therefore $a_{24} = z_{24}/x_4 = \$750/\$15,000 = 0.05$. For $x_4 = \$15,000$, inputs from sector 1 and from sector 2 were used in the proportion $p_{12} = z_{14}/z_{24} = \$300/\$750 = 0.4$. If x_4 were \$45,000, z_{24} would be (0.05)(\$45,000) = \$2250; since $z_{14} = \$900$ for $x_4 = \$45,000$, the proportion between inputs from sector 1 and from sector 2 is \$900/\$2250 = 0.4, as before. This reflects the fact that

$$p_{12} = z_{14}/z_{24} = a_{14}x_4/a_{24}x_4 = a_{14}/a_{24} = 0.02/0.05 = 0.4;$$

the proportion is the ratio of the technical coefficients, and since the coefficients are fixed, then the input proportion is fixed.

For the reader with some background in basic microeconomics, we can identify the form of production function inherent in the input–output system and compare it with that in the general neoclassical microeconomic approach. Production functions relate the amounts of inputs used by a sector to the maximum amount of output that could be produced by that sector with those inputs. An illustration is

$$x_j = f(z_{1j}, z_{2j}, \ldots, z_{nj}, v_j, m_j)$$

Using the definition of the technical coefficients in (2.5), we can see that in the Leontief model this becomes

$$x_j = \frac{z_{1j}}{a_{1j}} = \frac{z_{2j}}{a_{2j}} = \cdots = \frac{z_{nj}}{a_{nj}}$$

(This ignores, for the moment, the contributions of v_j and m_j.)

A problem with this extremely simple formulation is that it is meaningless if a particular input i is not used in production of j, since then $a_{ij} = 0$ and hence z_{ij}/a_{ij} is infinitely large. Thus, the more usual specification of the kind of production function that is embodied in the input–output model is

$$x_j = \min\left(\frac{z_{1j}}{a_{1j}}, \frac{z_{2j}}{a_{2j}}, \cdots, \frac{z_{nj}}{a_{nj}}\right)$$

where min (x, y, z) denotes the smallest of the numbers x, y and z. In the input–output model, for those a_{ij} coefficients that are not zero, these ratios will all be the same, and equal to x_j – from the fundamental definition of a_{ij} in (2.5). For those a_{ij} coefficients that are zero, the ratio z_{ij}/a_{ij} will be infinitely large and hence will be overlooked in the process of searching for the smallest among the ratios. This specification of the production function in the input–output model reflects the assumption of constant returns to scale; multiplication of $z_{1j}, z_{2j}, \ldots, z_{nj}$ by any constant will multiply x_j by the same constant. (Tripling all inputs will triple output; cutting inputs in half will halve output, etc.)

For the reader who is acquainted with the economist's production function geometry, we show four alternative representations of production functions in input space for a two-sector economy in Figure 2.1. A *linear production function*, depicted in Figure 2.1(a) assumes that output is a simple linear function of inputs, which means that the inputs are infinitely substitutable for each other for any level of output. The figure shows a set of isoquants (constant output lines) depicting higher and higher levels of output.

A *classical production function*, depicted in Figure 2.1(b), also shows a set of isoquants (now constant output curves) depicting higher and higher levels of output. For a given value of z_{1j} in Figure 2.1(b), increasing z_{2j} leads to increases in x_j – intersections with higher-value isoquants. In this case input substitution is also possible but not linearly, as indicated by the isoquants showing alternative input combinations that generate the same level of output. For example, moving rightward along a particular isoquant in Figure 2.1(b) can be accomplished by reducing the amount of input 2 and increasing the amount of input 1, or leftward by reducing z_{1j} and increasing z_{2j}.

The shape of the isoquants in Figure 2.1(b) reflects two specific classical assumptions about how inputs are combined to produce outputs. The negative slopes of the isoquants represent the fact that as the amount of one input is decreased, the amount of the other input must be increased in order to maintain the level of production indicated by a specific isoquant. The fact that the curves bulge toward the origin (mathematically

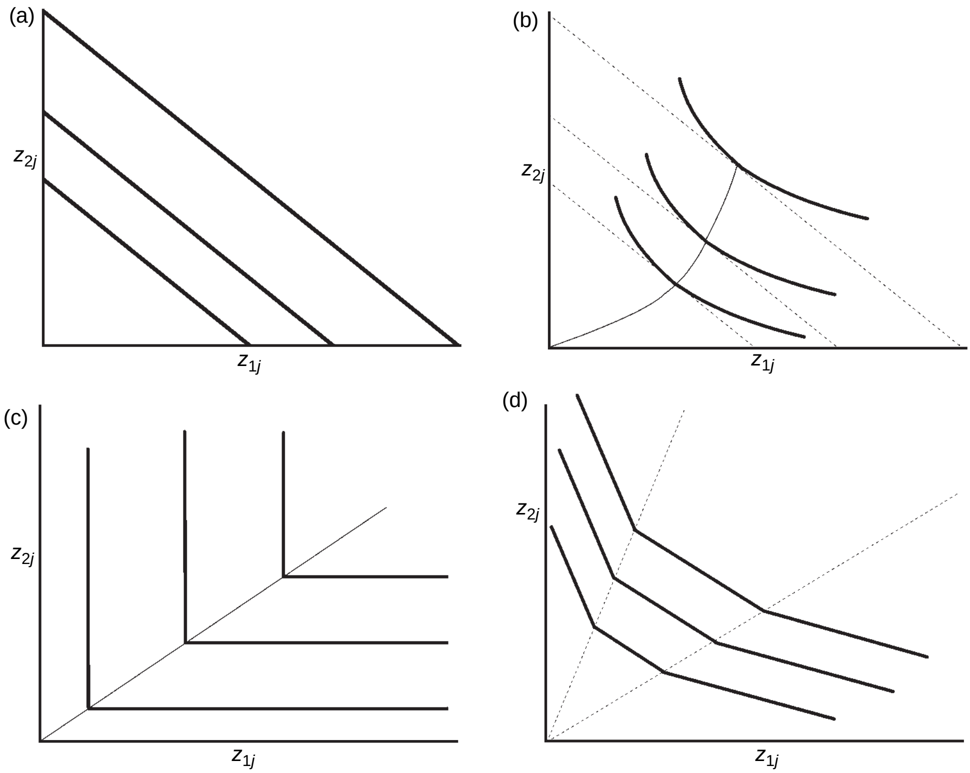

Figure 2.1a–d. Production Functions in Input Space. (a) Linear production function. (b) Classical production function. (c) Leontief production function. (d) Activity analysis production function.

their convexity) reflects the economist's law of diminishing marginal productivity.[3] The "expansion path" representing input combinations that are used for various levels of output is a curve from the origin through the points of tangency between isocost (constant cost) lines – dashed in Figure 2.1(b) – and the isoquants.

In the Leontief model, the isoquant "curves" of constant output appear as in Figure 2.1(c). Once the observed proportion of inputs 1 and 2 is known, as $p_{12} = z_{1j}/z_{2j}$, then additional amounts of either input 1 or input 2 alone are useless from the point of view of increasing the output of j. Only when availabilities of *both* input 1 and input 2 are increased can x_j increase; and only if the amounts of increase of 1 and 2 are in the proportion p_{12} will all the available amounts of both be used up. Of course, the "true" geometric representation should be in n-dimensional input space, with a separate axis for each of the n possible inputs, but the principles are the same when only

[3] From basic microeconomics concepts, recall that the slope of an isoquant (assuming that these are smooth functions) at any point is the ratio of the marginal productivities of inputs 1 and 2. These marginal productivities, in turn, are the partial derivatives of the production function (also assumed smooth) with respect to each of the inputs – thus the slope is $\frac{\partial f/\partial x_1}{\partial f/\partial x_2}$. As we move rightward along an isoquant, the amount of input 2 used decreases and the amount of input 1 used increases. By diminishing marginal productivity, then, $\partial f/\partial x_1$ decreases and $\partial f/\partial x_2$ increases; hence the slope decreases, as is true for the isoquants in Figure 2.1(b).

two inputs are considered. From the Leontief production function, if $z_{1j}, z_{2j}, \ldots, z_{(n-1)j}$ were all doubled but z_{nj} were only increased by 50 percent (multiplied by 1.5), then the minimum of the new ratios would be z_{nj}/a_{nj} and the new output of sector j would be 50 percent larger. There would be excess and unused amounts of inputs from sectors 1, 2,..., $(n-1)$. But since inputs are not free goods, sector j will not buy more from any sector than is needed for its production, and thus the input combinations chosen by sector j will lie along the ray as represented in Figure 2.1(c). In short, Leontief production functions require inputs in fixed proportions where a fixed amount of each input is required to produce one unit of output.

Figure 2.1(d) shows an *activity analysis production function*, which is a generalization of the Leontief production function and is a piece-wise linear approximation of the classical production function. Each isoquant is represented by a connected set of line segments. Each segment is a linear production function applicable over a limited range of combinations of inputs to produce a given level of output.

Once the notion of a set of fixed technical coefficients is accepted, (2.2) can be rewritten, replacing each z_{ij} on the right by $a_{ij}x_j$:

$$
\begin{aligned}
x_1 &= a_{11}x_1 + \cdots + a_{1i}x_i + \cdots + a_{1n}x_n + f_1 \\
&\vdots \\
x_i &= a_{i1}x_1 + \cdots + a_{ii}x_i + \cdots + a_{in}x_n + f_i \\
&\vdots \\
x_n &= a_{n1}x_1 + \cdots + a_{ni}x_i + \cdots + a_{nn}x_n + f_n
\end{aligned}
\tag{2.6}
$$

These equations serve to make explicit the dependence of interindustry flows on the total outputs of each sector. They also bring us closer to the form needed in input–output *analysis*, in which the following kind of question is asked: If the demands of the exogenous sectors were forecast to be some specific amounts next year, how much output from each of the sectors would be necessary to supply these final demands? From the point of view of this equation, the $f_1, \ldots, f_n$ are known numbers, the a_{ij} are known coefficients, and the $x_1, \ldots, x_n$ are to be found. Therefore, bringing all x terms to the left,

$$
\begin{aligned}
& x_1 - a_{11}x_1 - \cdots - a_{1i}x_i - \cdots - a_{1n}x_n = f_1 \\
& \vdots \\
& x_i - a_{i1}x_1 - \cdots - a_{ii}x_i - \cdots - a_{in}x_n = f_i \\
& \vdots \\
& x_n - a_{n1}x_1 - \cdots - a_{ni}x_i - \cdots - a_{nn}x_n = f_n
\end{aligned}
$$

and, grouping the x_1 together in the first equation, the x_2 in the second, and so on,

$$\begin{aligned}
(1-a_{11})x_1 - \cdots - a_{1i}x_i - \cdots - a_{1n}x_n &= f_1 \\
\vdots \\
-a_{i1}x_1 - \cdots + (1-a_{ii})x_i - \cdots - a_{in}x_n &= f_i \\
\vdots \\
-a_{n1}x_1 - \cdots - a_{ni}x_i - \cdots + (1-a_{nn})x_n &= f_n
\end{aligned} \tag{2.7}$$

These relationships can be represented compactly in matrix form. In matrix algebra notation, a "hat" over a vector denotes a diagonal matrix with the elements of the vector along the main diagonal, so, for example, $\hat{\mathbf{x}} = \begin{bmatrix} x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & x_n \end{bmatrix}$. From the basic definition of an inverse, $(\hat{\mathbf{x}})(\hat{\mathbf{x}})^{-1} = \mathbf{I}$, it follows that $\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 1/x_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1/x_n \end{bmatrix}$. Also, postmultiplication of a matrix, $\mathbf{M}$, by a diagonal matrix, $\hat{\mathbf{d}}$, creates a matrix in which each element in column j of $\mathbf{M}$ is multiplied by d_j in $\hat{\mathbf{d}}$ (Appendix A, section A.7). Therefore the $n \times n$ matrix of technical coefficients can be represented as

$$\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} \tag{2.8}$$

Using the definitions in (2.3) and (2.8), the matrix expression for (2.6) is

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{f} \tag{2.9}$$

Let $\mathbf{I}$ be the $n \times n$ identity matrix – ones on the main diagonal and zeros elsewhere;

$$\mathbf{I} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \text{ so then } (\mathbf{I}-\mathbf{A}) = \begin{bmatrix} (1-a_{11}) & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & (1-a_{22}) & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & (1-a_{nn}) \end{bmatrix}.$$

Then the complete $n \times n$ system shown in (2.7) is just[4]

$$(\mathbf{I}-\mathbf{A})\mathbf{x} = \mathbf{f} \tag{2.10}$$

For a given set of f's, this is a set of n linear equations in the n unknowns, $x_1, x_2, \ldots, x_n$ and hence it may or may not be possible to find a unique solution. In fact, whether or

[4] This is parallel to the form $\mathbf{Ax} = \mathbf{b}$ that is usually used to denote a set of linear equations. The difference is purely notational; since it is standard in input–output analysis to define the technical coefficients matrix as $\mathbf{A}$, then the matrix of coefficients in the input–output equation system becomes $(\mathbf{I}-\mathbf{A})$. Similarly, convention is responsible for denoting the right-hand sides of the input–output equations by $\mathbf{f}$ (for final demand) instead of $\mathbf{b}$.

not there is a unique solution depends on whether or not $(\mathbf{I}-\mathbf{A})$ is singular; that is, whether or not $(\mathbf{I}-\mathbf{A})^{-1}$ exists. The matrix $\mathbf{A}$ is known as the technical (or input–output, or direct input) coefficients matrix. From the basic definition of an inverse for a square matrix (Appendix A), $(\mathbf{I}-\mathbf{A})^{-1} = (1/|\mathbf{I}-\mathbf{A}|)[\mathrm{adj}(\mathbf{I}-\mathbf{A})]$. If $|\mathbf{I}-\mathbf{A}| \neq 0$, then $(\mathbf{I}-\mathbf{A})^{-1}$ can be found, and using standard matrix algebra results for linear equations the unique solution to (2.10) is given by

$$\mathbf{x} = (\mathbf{I}-\mathbf{A})^{-1}\mathbf{f} = \mathbf{L}\mathbf{f} \tag{2.11}$$

where $(\mathbf{I}-\mathbf{A})^{-1} = \mathbf{L} = [l_{ij}]$ is known as the *Leontief inverse* or the *total requirements matrix*.

In more detail, the equations summarized in (2.11) are

$$\begin{aligned}
x_1 &= l_{11}f_1 + \cdots + l_{1j}f_j + \cdots + l_{1n}f_n \\
&\vdots \\
x_i &= l_{i1}f_1 + \cdots + l_{ij}f_j + \cdots + l_{in}f_n \\
&\vdots \\
x_n &= l_{n1}f_1 + \cdots + l_{nj}f_j + \cdots + l_{nn}f_n
\end{aligned} \tag{2.12}$$

This makes clear the dependence of each of the gross outputs on the values of each of the final demands. Readers familiar with differential calculus and partial derivatives will recognize that $\partial x_i/\partial f_j = l_{ij}$.

2.3 An Illustration of Input–Output Calculations

2.3.1 Numerical Example: Hypothetical Figures – Approach I

Impacts on Industry Outputs We now turn to a small numerical example, as presented in Table 2.3. For the moment, the final demand elements and the value-added elements have not been disaggregated into their component parts.

The corresponding table of input–output coefficients, Table 2.4, is found by dividing each flow in a particular column of the producing sectors in Table 2.3 by the total output (row sum) of that sector. Thus, $a_{11} = 150/1000 = 0.15$; $a_{21} = 200/1000 = 0.2$; $a_{12} = 500/2000 = 0.25$; $a_{22} = 100/2000 = 0.05$. In particular,

$$\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 150 & 500 \\ 200 & 100 \end{bmatrix} \begin{bmatrix} 1/1000 & 0 \\ 0 & 1/2000 \end{bmatrix}$$

The $\mathbf{A}$ matrix is shown in Table 2.4. To add specificity for the remainder of this example, we assume sector 1 represents "Agriculture" and sector 2 "Manufacturing."

The principal way in which input–output coefficients are used for *analysis* is as follows. We assume that the numbers in Table 2.4 represent the structure of production in the economy; the columns are, in effect, the production recipes for each of the sectors, in terms of inputs from all the sectors. To produce one dollar's worth of manufactured goods, for example, 25 cents' worth of agricultural products and 5 cents' worth of

Table 2.3 Flows (z_{ij}) for the Hypothetical Example

		To Processing Sectors 1	2	Final Demand (f_i)	Total Output (x_i)
From	1	150	500	350	1000
Processing Sectors	2	200	100	1700	2000
Payments Sector		650	1400	1100	3150
Total Outlays (x_j)		1000	2000	3150	6150

Table 2.4 Technical Coefficients (the **A** Matrix) for the Hypothetical Example

	Sector 1 (Agriculture)	Sector 2 (Manufacturing)
Sector 1 (Agriculture)	.15	.25
Sector 2 (Manufacturing)	.20	.05

manufactures are needed as intermediate ingredients. These are, of course, only the inputs needed from other producing sectors; there will be inputs of a more "nonproduced" nature as well, such as labor, from the payments sectors. For an analysis of interrelationships among productive sectors, these are not of major importance.

We can now ask the question: If *final demand* for agriculture output were to increase to \$600 next year and that for manufactures were to decrease to \$1500 – for example, because of changes in government spending, consumers' tastes, and so on – how much total output from the two sectors would be necessary in order to meet this new demand? We denote this new demand as $\mathbf{f}^{new} = \begin{bmatrix} f_1^{new} \\ f_2^{new} \end{bmatrix} = \begin{bmatrix} 600 \\ 1500 \end{bmatrix}$. In the year of observation, when $\mathbf{f} = \begin{bmatrix} 350 \\ 1700 \end{bmatrix}$, we saw that $\mathbf{x} = \begin{bmatrix} 1000 \\ 2000 \end{bmatrix}$, precisely because, in producing to satisfy final demands, each sector must also produce to satisfy the demands for inputs into the processes of production themselves. Now we are asking, for $f_1^{new} = 600$ and $f_2^{new} = 1500$, what are the elements of $\mathbf{x}^{new} = \begin{bmatrix} x_1^{new} \\ x_2^{new} \end{bmatrix}$? To satisfy the demands, x_1^{new} can be no less than \$600 and x_2^{new} no less than \$1500. These would be the necessary outputs – the "direct effects" – if neither product were used in production and all output were directly available for final demand. But since both products serve as inputs, in a manner that is reflected in the technical coefficients of Table 2.4, it seems clear that in the end, more than \$600 worth of agriculture goods and more than \$1500 worth of manufactures will have to have been produced in order to meet the new final demands. That is, there will be "indirect effects" as well. Both of these effects are captured in the input–output model.

In the 2 × 2 case, $|\mathbf{I} - \mathbf{A}| = (1 - a_{11})(1 - a_{22}) - a_{12}a_{21}$ (Appendix A) and

$$\operatorname{adj}(\mathbf{I} - \mathbf{A}) = \begin{bmatrix} (1 - a_{22}) & a_{12} \\ a_{21} & (1 - a_{11}) \end{bmatrix}$$

For this example, $\mathbf{A} = \begin{bmatrix} .15 & .25 \\ .20 & .05 \end{bmatrix}$ so $(\mathbf{I} - \mathbf{A}) = \begin{bmatrix} .85 & -.25 \\ -.20 & .95 \end{bmatrix}$; hence $|\mathbf{I} - \mathbf{A}| = 0.7575 \neq 0$ and we know that $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$can be found. Here we have

$$\mathbf{L} = \begin{bmatrix} 1.2541 & .3300 \\ .2640 & 1.1221 \end{bmatrix}$$

Assuming that technology (as represented in **A**), does not change, the needed total outputs caused by $\mathbf{f}^{new}$ are then found as in (2.11):

$$\mathbf{x}^{new} = \mathbf{L}\mathbf{f}^{new} = \begin{bmatrix} 1.2541 & .3300 \\ .2640 & 1.1221 \end{bmatrix} \begin{bmatrix} 600 \\ 1500 \end{bmatrix} = \begin{bmatrix} 1247.52 \\ 1841.58 \end{bmatrix} \tag{2.13}$$

These values – $x_1^{new} = \$1247.52$ and $x_2^{new} = \$1841.58$ – are one measure of the *impact* on the economy of the new final demands.[5]

With this result for $\mathbf{x}^{new}$, it is straightforward to examine the changes in all elements in the interindustry flows table (as in Table 2.3) caused by $\mathbf{f}^{new}$. From the definition of coefficients in (2.8), $\mathbf{Z} = \mathbf{A}\hat{\mathbf{x}}$. With a constant **A** matrix and new outputs, $\mathbf{x}^{new}$, we find $\mathbf{Z}^{new} = \mathbf{A}\hat{\mathbf{x}}^{new} = \begin{bmatrix} 187.13 & 460.40 \\ 249.50 & 92.08 \end{bmatrix}$; along with $\mathbf{f}^{new} = \begin{bmatrix} 600 \\ 1500 \end{bmatrix}$, we have the results shown in Table 2.5.

The elements in the Payments Sector are found as the difference between new total outputs (total outlays) and new total interindustry inputs for each sector. (For the example we assume no change in payments sector transactions with final demand.) Notice that sector 1's purchases are larger (reflecting an increase in final demand for that sector) and sector 2's purchases are smaller (reflecting smaller demand for that sector).

The input–output model allows us to deal equally easily with *changes* in demands and outputs instead of *levels*. Here and throughout, we use superscripts "0" to represent the initial (base year) situation and "1" for values of variables after the change in demands (instead of "*new*" as we did above). Assuming that technology is unchanged means $\mathbf{A}^0 = \mathbf{A}^1 = \mathbf{A}$ and $\mathbf{L}^0 = \mathbf{L}^1 = \mathbf{L}$, so $\mathbf{x}^0 = \mathbf{L}\mathbf{f}^0$ and $\mathbf{x}^1 = \mathbf{L}\mathbf{f}^1$; letting $\Delta\mathbf{x} = \mathbf{x}^1 - \mathbf{x}^0$ and

[5] Here x_1^{new} and x_2^{new} are shown to two decimals for comparison with results from an alternative approach in section 2.3.2. These $\mathbf{x}^{new}$ values reflect computer calculations carried out with more than four significant digits and hence often will (as here) differ (to the right of the decimal point) from what the reader will produce with a hand calculator using the four-digit elements shown for **A**. In any actual analysis, such detail might be questionable because of the much less accurate data from which the technical coefficients are derived (compare the figures in Table 2.3).

Table 2.5 Flows (z_{ij}) for the Hypothetical Example Associated with $\mathbf{x}^{new}$

		To Processing Sectors			
		1	2	Final Demand (f_i)	Total Output (x_i)
From	1	187.13	460.40	600	1247.52
Processing Sectors	2	249.50	92.08	1500	1841.58
Payments Sector		810.89	1289.11	1100	3200.00
Total Outlays (x_i)		1247.52	1841.58	3200	6289.10

$\Delta\mathbf{f} = \mathbf{f}^1 - \mathbf{f}^0$

$$\Delta\mathbf{x} = \mathbf{L}\mathbf{f}^1 - \mathbf{L}\mathbf{f}^0 = \mathbf{L}\Delta\mathbf{f} \tag{2.14}$$

In this example, $\Delta\mathbf{f} = \begin{bmatrix} 250 \\ -200 \end{bmatrix}$, giving $\Delta\mathbf{x} = \begin{bmatrix} 247.5 \\ -158.4 \end{bmatrix}$ and so

$$\mathbf{x}^1 = \mathbf{x}^0 + \Delta\mathbf{x} = \begin{bmatrix} 1000 \\ 2000 \end{bmatrix} + \begin{bmatrix} 247.5 \\ -158.4 \end{bmatrix} = \begin{bmatrix} 1247.5 \\ 1841.6 \end{bmatrix}$$

This corresponds to the result in (2.13), except for rounding.

Other Impacts In many cases, the dollar value of each sector's gross output may not ultimately be the most important measure of the economic impact following a change in exogenous demands. Gross output requirements could be translated into employment effects (in either dollars or physical terms – for example, person-years), or effects on value-added, or energy consumption (of a particular type, e.g., petroleum), or pollution emissions (again, of a particular type, e.g., CO_2), and so forth. In each instance, we need a set of appropriate coefficients with which to convert outputs into associated effects. For illustration we consider employment in monetary terms. Let the value of employment in the two sectors be denoted as[6]

$$\mathbf{e}' = \begin{bmatrix} e_1 & e_2 \end{bmatrix}$$

A vector of employment *coefficients* contains the base-year employment in each sector divided by that sector's base-year gross output, x_1^0 and x_2^0,

$$\mathbf{e}'_c = \begin{bmatrix} e_1/x_1^0 & e_2/x_2^0 \end{bmatrix} = \begin{bmatrix} e_{c1} & e_{c2} \end{bmatrix}$$

Then $\boldsymbol{\varepsilon} = \hat{\mathbf{e}}'_c\mathbf{x}^1 = \hat{\mathbf{e}}'_c\mathbf{L}\mathbf{f}^1$ produces a vector whose elements are the total labor income in each sector that accompanies the new exogenous final demand;

$$\boldsymbol{\varepsilon} = \begin{bmatrix} e_{c1} & 0 \\ 0 & e_{c2} \end{bmatrix} \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix} = \begin{bmatrix} e_{c1}x_1^1 \\ e_{c2}x_2^1 \end{bmatrix}$$

[6] Later in this chapter (and still later, in Chapter 6 on multipliers) we will need to alter this notation to be able to accommodate additional possibilities.

To continue with the numerical example, suppose that $e_{c1} = 0.30$ and $e_{c2} = 0.25$ give the dollars' worth of labor inputs per dollar's worth of output of the two sectors. (We will examine the role of labor inputs and household consumption in an input–output model in some detail in section 2.5, below.) Then

$$\boldsymbol{\varepsilon} = \hat{\mathbf{e}}'_c \mathbf{x}^1 = \begin{bmatrix} 0.30 & 0 \\ 0 & 0.25 \end{bmatrix} \begin{bmatrix} 1247.52 \\ 1841.58 \end{bmatrix} = \begin{bmatrix} 374.26 \\ 460.40 \end{bmatrix}$$

This indicates the values of labor inputs purchased by the two sectors.

If, additionally, we have an occupation-by-industry matrix, $\mathbf{P}$, where p_{ij} is the *proportion* of sector j employment that is in occupation i, then $\tilde{\boldsymbol{\varepsilon}} = \mathbf{P}\hat{\boldsymbol{\varepsilon}}$ gives a matrix of employment by sector by occupation type. For example, with k occupation types and two sectors,

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} \\ \vdots & \vdots \\ p_{k1} & p_{k2} \end{bmatrix}$$

and

$$\tilde{\boldsymbol{\varepsilon}} = \mathbf{P}\hat{\boldsymbol{\varepsilon}} = \begin{bmatrix} p_{11}e_{c1}x_1^1 & p_{12}e_{c2}x_2^1 \\ \vdots & \vdots \\ p_{k1}e_{c1}x_1^1 & p_{k2}e_{c2}x_2^1 \end{bmatrix}$$

Column sums would give total labor use by sector; row sums give total employment of a particular occupational category across all sectors. (The vector $\mathbf{P}\boldsymbol{\varepsilon}$ shows employment by occupational category, aggregated across all sectors.)

Suppose that our economy has three occupational groups: (1) engineers, (2) bankers and (3) farmers, and

$$\mathbf{P} = \begin{bmatrix} 0 & 0.8 \\ 0.6 & 0.2 \\ 0.4 & 0 \end{bmatrix}$$

(For example, this says that 40 percent of the agricultural labor force is farmers; 80 percent of manufacturing labor force is made up of engineers, etc.) Then

$$\tilde{\boldsymbol{\varepsilon}} = \mathbf{P}\hat{\boldsymbol{\varepsilon}} = \begin{bmatrix} 0 & 0.8 \\ 0.6 & 0.2 \\ 0.4 & 0 \end{bmatrix} \begin{bmatrix} 374.26 & 0 \\ 0 & 460.40 \end{bmatrix} = \begin{bmatrix} 0 & 368.32 \\ 224.56 & 92.08 \\ 149.70 & 0 \end{bmatrix}$$

Column sums of $\tilde{\boldsymbol{\varepsilon}}$ are 374.26 and 460.40, as expected (the elements of $\boldsymbol{\varepsilon}$). Row sums give the economy-wide (across both sectors) employment of engineers, farmers and bankers, respectively. If sectoral disaggregation is not necessary, then

$$\mathbf{P}\boldsymbol{\varepsilon} = \begin{bmatrix} 0 & 0.8 \\ 0.6 & 0.2 \\ 0.4 & 0 \end{bmatrix} \begin{bmatrix} 374.26 \\ 460.40 \end{bmatrix} = \begin{bmatrix} 368.32 \\ 316.64 \\ 149.70 \end{bmatrix}$$

gives employment by occupational type, across sectors.

A wide variety of such conversion coefficients vectors (as in $\mathbf{e}'_c$) or matrices (as in $\mathbf{P}$) is possible. For example, in arid regions, water-use coefficients, $\mathbf{w}'_c = \begin{bmatrix} w_{c1} & w_{c2} \end{bmatrix}$, could be used in $\mathbf{w}'_c\mathbf{x}'$ to assess the water consumption associated with new outputs generated by new final demands. We explore these kinds of alternative impacts again in Chapter 6 on input–output multipliers, and in Chapters 9 and 10, some of the energy and environmental repercussions of final demand impacts are discussed in detail.

2.3.2 Numerical Example: Hypothetical Figures – Approach II

Consider the same economy, whose 2×2 technical coefficients matrix is given in Table 2.4 and for which the projected $\mathbf{f}^1$ vector is $\begin{bmatrix} 600 \\ 1500 \end{bmatrix}$. We can examine the question of outputs necessary to satisfy this final demand in a more intuitive way that is less mechanical than finding elements in an inverse matrix.

1. Initially, it is clear that agriculture needs to produce \$600 and manufacturing, \$1500. If the sectors are going to meet the new final demands, they could not get away with producing less than these amounts.
2. However, to produce \$600, agriculture needs, as inputs to that productive process, (0.15)(\$600) = \$90 from itself and (0.20)(\$600) = \$120 from manufacturing. These figures come from the coefficients in column 1 of the **A** matrix – the production recipe for agriculture. Similarly, to produce its \$1500, manufacturing will have to buy (0.25)(\$1500) = \$375 from agriculture and (0.05)(\$1500) = \$75 from itself. Thus agriculture must, in fact, produce the \$600 noted in 1, above, plus another \$(90 + 375) = \$465 more, to satisfy the needs for inputs that it has itself and also that come from manufacturing. Similarly, manufacturing will have to produce an additional \$(120 + 75) = \$195 to satisfy its own need plus that of agriculture for inputs to produce the "original" \$600 and \$1500.
3. In item 2, above, we found the interindustry needs that resulted from production of \$600 in agriculture and \$1500 in manufacturing. These were \$465 and \$195, respectively. But now we realize that this "extra" production, above the \$600 and \$1500, will also generate interindustry needs – in order to engage in the production of \$465, agriculture will need (0.15)(\$465) = \$69.75 from itself and (0.20)(\$465) = \$93 from manufacturing. Similarly, manufacturing will now additionally need (.025)(\$195) = \$48.75 from agriculture and (0.05)(\$195) = \$9.75 from itself. The total new demands for the two sectors are thus \$(69.75 + 48.75) = \$118.50 and \$(93 + 9.75) = \$102.75.
4. At this point we realize that it is necessary to treat the additional \$118.50 for agriculture and \$102.75 for manufacturing in the same fashion as the \$465 and \$195 in item 3. Hence we find additional required outputs of \$43.46 and \$28.84 from the two sectors.
5. Continuing in this way, we find that eventually the numbers become so small that they can be ignored (less than \$0.005).

Table 2.6 Round-by-Round Impacts (in dollars) of $f_1^1 = \$600$ and $f_2^1 = \$1500$

Round	0	1	2	3	4	5	6	7	8 + 9+ 10 + 11	$\mathbf{Lf}^1$
Sec. 1	600	465.00	118.50	43.46	13.73	4.60	1.50	0.50	0.24	1247.52
Sec. 2	1500	195.00	102.75	28.84	10.13	3.25	1.08	0.35	0.17	1841.58
Cumulative Total										
Sec. 1		1065.00	1183.50	1226.96	1240.64	1245.29				1247.52
Sec. 2		1695.00	1797.75	1826.59	1836.72	1839.97				1841.58
Percent of Total Effect Captured										
Sec. 1		85.40	94.90	98.40	99.50	99.80				1247.52
Sec. 2		92.00	97.60	99.20	99.70	99.90				1841.58

Looking at the total impact of a particular set of final demands this way is described as looking at the "round-by-round" effects. The initial demands generate a need for inputs from the productive sectors; this is the "first round" of effects, as found in item 2, above. But these outputs themselves generate a need for additional inputs – "second round" effects – as found in item 3, above; and so forth. For the present example, these figures have been collected in Table 2.6.

For agriculture, the sum of these round-by-round effects, \$647.53, plus the original demand of \$600, is \$1247.53; for manufacturing, the total is $\$341.57 + \$1500 = \$1841.57$. These total outputs (except for small rounding errors) are the same as those found by using the Leontief inverse, where $x_1^1 = \$1247.52$ and $x_2^1 = \$1841.58$. (It was for this comparison that the two-decimal accuracy was kept in the Leontief-inverse approach to this example.)

In this second view of the numerical example we have developed something of a feeling for the way in which external (final) demands are transmitted through the productive sectors of the economic system. In fact, we see that the elements of $(\mathbf{I}-\mathbf{A})^{-1}$ are really very useful and important numbers. Each captures, in a single *number*, an entire *series* of direct and indirect effects. (The equivalence between Approaches I and II is examined for the general case in Appendix 2.1.)

2.3.3 Numerical Example: Mathematical Observations

The inverse in this small example, $\mathbf{L} = \begin{bmatrix} 1.2541 & .3300 \\ .2640 & 1.1221 \end{bmatrix}$, illustrates a general feature of Leontief inverses for input–output models of any size – the diagonal elements are larger than 1. This is entirely consistent with the economic logic of the round-by-round approach. From (2.13)

$$x_1^1 = (1.2541)(600) + (0.3300)(1500)$$

Looking at the first product on the right, the new final demand of \$600 for agriculture output is multiplied by 1.2541. This can be thought of as (1 + 0.2541)(600). The (1)(600) reflects the fact that the \$600 new agriculture demand must be met by producing \$600 more agriculture output. The additional (0.2541)(600) captures the additional agriculture output required because this output is also used as an input to production activity in both agriculture and also manufacturing. Similarly, from (2.13),

$$x_2^1 = (0.2640)(600) + (1.1221)(1500)$$

and the same logic explains why the coefficient (1.1221) relating manufacturing output to new final demand for manufacturing goods, \$1500, must be greater than 1.

We examine why both of the diagonal elements in $\mathbf{L}$ will be greater than 1 in the two-sector case. (A more complicated derivation can be used for the general n-sector input–output model, and it is also apparent from the power series discussion in section 2.4.) For this 2×2 example, as we saw in section 2.3.1, above,

$$\mathbf{L} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} = \frac{1}{|\mathbf{I} - \mathbf{A}|}[\mathrm{adj}(\mathbf{I} - \mathbf{A})]$$
$$= \frac{1}{(1 - a_{11})(1 - a_{22}) - a_{12}a_{21}} \begin{bmatrix} (1 - a_{22}) & a_{12} \\ a_{21} & (1 - a_{11}) \end{bmatrix}$$

So, for example,

$$l_{11} = \frac{(1 - a_{22})}{(1 - a_{22})\left[(1 - a_{11}) - \frac{a_{12}a_{21}}{(1-a_{22})}\right]} = \frac{1}{1 - \left[a_{11} + \frac{a_{12}a_{21}}{(1-a_{22})}\right]}$$

Assuming that $(1 - a_{22}) > 0$, $l_{11} > 1$ if the denominator on the right-hand side is less than 1, which it will be when $a_{11} > 0$ and/or $a_{12}a_{21} > 0$ – since $(1 - a_{22}) > 0$. Similar reasoning shows that $l_{22} = (1 - a_{11})/\,|\mathbf{I} - \mathbf{A}|\, > 1$ under similar reasonable conditions on the a_{ij}.

Whether or not the off-diagonal elements are larger than 1 depends entirely on the sizes of a_{12} and a_{21}, relative to $|\mathbf{I} - \mathbf{A}|$. In most actual input–output tables, with a rather detailed breakdown of sectors, the off-diagonal elements in $\mathbf{L}$ will be less than 1, as in (2.13). However, for example, if a_{21} in Table 2.4 had been 0.70 instead of 0.20, so that the coefficients matrix had been

$$\mathbf{A} = \begin{bmatrix} .15 & .25 \\ .70 & .05 \end{bmatrix}$$

then

$$\mathbf{L} = \begin{bmatrix} 1.5020 & .3953 \\ 1.1067 & 1.3439 \end{bmatrix}$$

Notice that a coefficient as large as $a_{21} = 0.7$ – which says that there is 70 cents' worth of sector 2 output in a dollar's worth of sector 1 output – is not likely to be seen

Table 2.7 The 2003 US Domestic Direct Requirements Matrix, **A**

	Sector	1	2	3	4	5	6	7
1	Agriculture	.2008	.0000	.0011	.0338	.0001	.0018	.0009
2	Mining	.0010	.0658	.0035	.0219	.0151	.0001	.0026
3	Construction	.0034	.0002	.0012	.0021	.0035	.0071	.0214
4	Manufacturing	.1247	.0684	.1801	.2319	.0339	.0414	.0726
5	Trade, Transportation & Utilities	.0855	.0529	.0914	.0952	.0645	.0315	.0528
6	Services	.0897	.1668	.1332	.1255	.1647	.2712	.1873
7	Other	.0093	.0129	.0095	.0197	.0190	.0184	.0228

Table 2.8 The 2003 US Domestic Total Requirements Matrix, $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$

	Sector	1	2	3	4	5	6	7
1	Agriculture	1.2616	.0058	.0131	.0576	.0037	.0069	.0072
2	Mining	.0093	1.0748	.0122	.0343	.0193	.0033	.0073
3	Construction	.0075	.0034	1.0047	.0064	.0065	.0111	.0250
4	Manufacturing	.2292	.1192	.2615	1.3419	.0692	.0856	.1261
5	Trade, Transportation & Utilities	.1493	.0850	.1371	.1563	1.0887	.0598	.0853
6	Services	.2383	.2931	.2700	.2918	.2712	1.4116	.3138
7	Other	.0243	.0239	.0231	.0367	.0280	.0297	1.0338

often in real tables. The sizes of the between-sector technical coefficients, a_{ij} $(i \neq j)$, and of the off-diagonal elements in **L**, are related to the level of sectoral detail (that is, the number of sectors) in the model. We will return to this topic in Chapter 4, when we consider the effects of aggregating (combining) sectors in an input–output model. (In Appendix 2.2 we examine the conditions under which a Leontief inverse matrix will always contain only non-negative elements, as logic suggests should always be the case.)

2.3.4 Numerical Example: The US 2003 Data

We present a highly aggregated, seven-sector version of the 2003 US input–output coefficients matrix and its associated Leontief inverse in Tables 2.7 and 2.8. (Appendix B contains a series of such tables over time for the US economy at the seven-sector level of aggregation.) It is important to note that these data for the US represent *domestically produced* inputs; this requires explanation.

Imports are generally divided into two categories: "competitive" and "non-competitive" imports (or "competing" and "non-competing").

Competitive imports are goods that have a domestic counterpart (that is, are also produced in the USA). For example, grapes from Chile that are used to make grape jelly in the USA, where domestically grown grapes are also used in grape jelly recipes.

Non-competitive imports have no domestic counterpart. For example, coffee beans from Brazil used by US coffee roasting firms (coffee beans are not grown in the USA).

Some national tables (the USA is one example) show competitive imports within the transactions table, so that sales of grapes to jelly producers include both domestic and foreign sources. This correctly reflects the total amount of grapes needed by domestic producers. However, it causes problems when input–output models are used for impact analysis. Briefly put, this is because an analyst is usually interested in the economic consequences *on the domestic* (or regional or local) *economy* of an exogenous demand change. With Chilean grapes in a transactions matrix, and hence in the associated **A** and **L** matrices, some of the demand repercussions measured by the model would in fact be felt by Chilean grape growers. For this reason, we present here US data based on a *domestic* transactions matrix ($\mathbf{Z}^D$) in which the transactions matrix ($\mathbf{Z}$) has been purged of "competitive" (or "competing") imports. In matrix terms, $\mathbf{Z}^D = \mathbf{Z} - \mathbf{M}$, where **M** is a matrix of competitive imports. This "scrubbing" of the matrix is not always easy to do if the data are lumped together in a published **Z** table (as is the case in the USA), but it is very important when the question is one of impacts of final demand changes on the domestic economy (and this is usually the question of interest).[7]

Spending on non-competitive imports usually appears in a row in the payments sector (a single value indicating a sector's payments for all non-competitive imports). We return to these issues in Chapter 4.

The effects on US output of various final-demand vectors can be easily quantified using **L** in Table 2.8. For example, suppose that there were increased foreign demand (the export component of the final-demand vector) for agricultural and manufactured items of \$1.2 million and \$6.8 million, respectively. Here (in millions of dollars)

$$\Delta\mathbf{f} = \begin{bmatrix} 1.2 \\ 0 \\ 0 \\ 6.8 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

[7] By contrast, if one is interested in the structure of production ("production recipes") and if or how they have changed over time (structural analysis), it may be more useful to have competitive imports included in the **Z** matrix and hence reflected in **A** and **L**, since such imports are certainly part of those recipes.

and, using (2.14), we find from $\mathbf{L}$ in Table 2.8 that (in millions of dollars)

$$\Delta\mathbf{x} = \begin{bmatrix} 1.9114 \\ 0.2444 \\ 0.0526 \\ 9.1249 \\ 1.2421 \\ 2.2709 \\ 0.2788 \end{bmatrix}$$

As might be expected, the greatest effect, \$9.125 million, is felt in the manufacturing sector. The next-greatest effect, \$2.271 million, is felt in services. Also, agriculture output would increase by \$1.911 million and trade, transportation and utilities would increase by \$1.242 million. Effects on the remaining three sectors are less than \$1 million. The total new output effect throughout the country, obtained by summing the elements in $\Delta\mathbf{x}$, is \$15.125 million; this is generated by a total new exogenous demand of \$8 million. This again illustrates the multiplicative effect in an economy of an exogenous stimulus via an increase in one or more components of final demand. These multiplier effects will be discussed in further detail in Chapter 6.

2.4 The Power Series Approximation of $(\mathbf{I}-\mathbf{A})^{-1}$

In preparing input–output tables for many real-world applications of the model, in which one wants to maintain a reasonable distinction between sectors (e.g., so that sectors producing aluminum storm windows and women's apparel are not lumped together as a single sector labeled "manufacturing"), tables with hundreds of sectors are not unusual. However, early in the history of input–output studies, computer speed and capacity posed real problems for implementation of input–output models – inversion of large matrices was simply not possible.[8] The amount of computer capacity and time needed to invert, say, a 150×150 $(\mathbf{I}-\mathbf{A})$ matrix will vary with the particular computer and the inversion program that is used, and it is quite possible that in some cases the number of sectors that can be accommodated may be limited. One approach is then to aggregate the data into a smaller number of sectors. We will say more about such sectoral aggregation later, but clearly industrial (sectoral) detail is lost in the process. In addition, the inversion calculations themselves can be carried out sequentially on a series of smaller submatrices of $(\mathbf{I}-\mathbf{A})$.[9] However, there is a useful matrix algebra result generally applicable to $(\mathbf{I}-\mathbf{A})$ matrices that makes possible an approximation to $(\mathbf{I}-\mathbf{A})^{-1}$ requiring no inverses at all; moreover, this approximation procedure has a useful economic interpretation.

[8] In 1939 it reportedly took 56 hours to invert a 42-sector table (on Harvard's Mark II computer; see Leontief, 1951a, p. 20). In 1947, 48 hours were needed to invert a 38-sector input–output matrix. However, by 1953 the same operation took only 45 minutes. (Morgenstern, 1954, p. 496; also, see Lahr and Stevens, 2002, p. 478.) By 1969 a 100-sector matrix could be inverted in between 10 and 36 seconds, depending on the computer used. (Polenske, 1980, p. 15.)

[9] This is possible using a partitioned matrix approach; the details need not concern us at this point.

By definition, we know that $\mathbf{A}$ is a non-negative matrix with $a_{ij} \geq 0$ for all i and j. (This characteristic is often written as $\mathbf{A} \geq \mathbf{0}$, where it is understood that not all $a_{ij} = 0$.)[10] The sum of the elements in the jth column of $\mathbf{A}$ indicates the dollars' worth of inputs from other sectors that are used in making a dollar's worth of output of sector j. In an open model, given the economically reasonable assumption that each sector uses some inputs from the payments sector (labor, other value added, etc.), then each of these column sums will be less than one ($\sum_{i=1}^{n} a_{ij} < 1$ for all j). (We will see below, in section 2.6, that this column sum condition need not apply to tables based on physical, not monetary, measures of transactions and outputs.) For input–output coefficients matrices with these two characteristics – $a_{ij} \geq 0$ and $\sum_{i=1}^{n} a_{ij} < 1$ for all j – it is possible to approximate the gross output vector $\mathbf{x}$ associated with any final demand vector $\mathbf{f}$ without finding $(\mathbf{I} - \mathbf{A})^{-1}$.

Consider the matrix product

$$(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots + \mathbf{A}^n)$$

where, for square matrices, $\mathbf{A}^2$ denotes $\mathbf{AA}$, $\mathbf{A}^3 = \mathbf{AAA} = \mathbf{AA}^2$, and so on. Premultiplication of the series in parentheses by $(\mathbf{I} - \mathbf{A})$ can be accomplished by first multiplying all terms in the right-hand parentheses by $\mathbf{I}$ and then multiplying all terms by $(-\mathbf{A})$. This leaves only $(\mathbf{I} - \mathbf{A}^{n+1})$; all other terms cancel – for $\mathbf{A}^2$ there is a $-\mathbf{A}^2$, for $\mathbf{A}^3$ there is a $-\mathbf{A}^3$, and so on. Thus

$$(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots + \mathbf{A}^n) = (\mathbf{I} - \mathbf{A}^{n+1}) \tag{2.15}$$

If it were true that for large n (more formally, as $n \to \infty$), the elements in $\mathbf{A}^{n+1}$ all become zero, or close to zero (i.e., $\mathbf{A}^{n+1} \to \mathbf{0}$), then the right-hand side of (2.15) would be simply $\mathbf{I}$, and the matrix series that postmultiplies $(\mathbf{I} - \mathbf{A})$ in (2.15) would constitute the inverse to $(\mathbf{I} - \mathbf{A})$, from the fundamental defining property of an inverse.

For any matrix, $\mathbf{M}$, if we sum the absolute values of the elements in each column, the largest sum is called the norm of $\mathbf{M}$ – denoted $N(\mathbf{M})$ or $\|\mathbf{M}\|$.[11] For example, for the coefficients matrix $\mathbf{A}$ given in Table 2.4, $N(\mathbf{A}) = 0.35$, the sum of the elements in the first column. (The sum of the elements in column 2 is 0.30.) For a pair of matrices, $\mathbf{A}$ and $\mathbf{B}$, that are conformable for the multiplication $\mathbf{AB}$, there is a theorem stating that the product of the norms of $\mathbf{A}$ and $\mathbf{B}$ is no smaller than the norm of the product $\mathbf{AB}$ – $N(\mathbf{A})N(\mathbf{B}) \geq N(\mathbf{AB})$. By replacing $\mathbf{B}$ with $\mathbf{A}$, it follows that $N(\mathbf{A})N(\mathbf{A}) \geq N(\mathbf{A}^2)$

[10] A more exact characterization of vectors and matrices is often needed for more advanced matrix algebra results. See section A.9 in Appendix A, where $\mathbf{A} > \mathbf{0}$ is used for the case when $\mathbf{A} \geq \mathbf{0}$ and $\mathbf{A} \neq \mathbf{0}$.

[11] A *norm* is just a measure of the general size of the elements in a matrix. (A measure of the size of the matrix itself is given by the dimensions of the matrix.) For example, a non-negative $m \times n$ matrix that has all elements smaller than 0.1 will have a smaller norm than one that has all elements larger than 10. There are many possible definitions of the norm of a matrix. The one used here (maximum column sum of absolute values) is one of the simplest.

or $[N(\mathbf{A})]^2 \geq N(\mathbf{A}^2)$ and finally, continuing similarly,

$$[N(\mathbf{A})]^n \geq N(\mathbf{A}^n) \tag{2.16}$$

As was noted above, all column sums of an open and "reasonable" value-based $\mathbf{A}$ matrix will be less than one, so we know that $N(\mathbf{A}) < 1$. Moreover, since $a_{ij} \geq 0$, we also know that $a_{ij} \leq N(\mathbf{A})$; no element in a non-negative matrix can be larger than the largest column sum. Thus: (1) since $N(\mathbf{A}) < 1$, $[N(\mathbf{A})]^n \to 0$ as $n \to \infty$; (2) from (2.16), this means that $N(\mathbf{A}^n) \to 0$ also as $n \to \infty$; (3) finally, then, all elements in $\mathbf{A}^n$ must approach zero, since no single element in a non-negative matrix can be larger than the norm of that matrix. This is the result that we are interested in. The right-hand side of (2.15) becomes simply $\mathbf{I}$ as n gets large and so

$$\mathbf{L} = (\mathbf{I}-\mathbf{A})^{-1} = (\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\mathbf{A}^3+\cdots) \tag{2.17}$$

[This is analogous to the series result in ordinary algebra that $1/(1-a) = 1 + a + a^2 + a^3 + \cdots$, for $|a| < 1$.] Notice that the terms on the right-hand side of (2.17) are all positive. Even if some a_{ij} are zero, the increasing number of products of $\mathbf{A}$ virtually guarantees that no zeros will be in evidence at the end of the summation.[12] This means that $\mathbf{L}$ will contain only positive elements. (Appendix 2.2 looks into the issue of positivity of $\mathbf{L}$ in more detail.)

Then $\mathbf{x} = (\mathbf{I}-\mathbf{A})^{-1}\mathbf{f}$ can be found as

$$\mathbf{x} = (\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\mathbf{A}^3+\cdots)\mathbf{f} \tag{2.18}$$

Removing parentheses, this is

$$\mathbf{x} = \mathbf{f}+\mathbf{Af}+\mathbf{A}^2\mathbf{f}+\mathbf{A}^3\mathbf{f}+\cdots = \mathbf{f}+\mathbf{Af}+\mathbf{A}(\mathbf{Af})+\mathbf{A}(\mathbf{A}^2\mathbf{f})+\cdots \tag{2.19}$$

Each term after the first can be found as the preceding term premultiplied by $\mathbf{A}$. In many applications it has been found that after about $\mathbf{A}^7$ or $\mathbf{A}^8$, the terms multiplying $\mathbf{f}$ become insignificantly different from zero. Even with modern-day computer capacity and speed, there still may be times when the approximation in (2.18) or (2.19) may prove useful (for example, since matrix multiplications are much more straightforward than inversion, especially of a large matrix).[13]

Returning to the original $\mathbf{A}$ matrix and the $\mathbf{f}$ vector of the example in section 2.3 (and dropping the "0" superscripts for simplicity), where $\mathbf{A} = \begin{bmatrix} .15 & .25 \\ .20 & .05 \end{bmatrix}$ and

[12] As mentioned, the elements in any particular $\mathbf{A}^k$ *do* approach zero – which is the whole point.

[13] Alternatively, some analysts have used the power series approximation as a framework for introducing "dynamic" concepts into input–output models. We explore these ideas briefly in section 13.4.7.

$\mathbf{f} = \begin{bmatrix} 600 \\ 1500 \end{bmatrix}$, we have

$$\mathbf{If} = \begin{bmatrix} 600 \\ 1500 \end{bmatrix}$$

$$\mathbf{Af} = \begin{bmatrix} .15 & .25 \\ .20 & .05 \end{bmatrix} \begin{bmatrix} 600 \\ 1500 \end{bmatrix} = \begin{bmatrix} 465 \\ 195 \end{bmatrix}$$

$$\mathbf{A}^2\mathbf{f} = \begin{bmatrix} .0725 & .0500 \\ .0400 & .0525 \end{bmatrix} \begin{bmatrix} 600 \\ 1500 \end{bmatrix} = \begin{bmatrix} 118.50 \\ 102.75 \end{bmatrix}$$

$$\mathbf{A}^3\mathbf{f} = \begin{bmatrix} .0209 & .0206 \\ .0165 & .0126 \end{bmatrix} \begin{bmatrix} 600 \\ 1500 \end{bmatrix} = \begin{bmatrix} 43.44 \\ 28.80 \end{bmatrix}$$

$$\mathbf{A}^4\mathbf{f} = \begin{bmatrix} .0073 & .0063 \\ .0050 & .0048 \end{bmatrix} \begin{bmatrix} 600 \\ 1500 \end{bmatrix} = \begin{bmatrix} 13.83 \\ 10.20 \end{bmatrix}$$

$$\mathbf{A}^5\mathbf{f} = \begin{bmatrix} .0024 & .0021 \\ .0017 & .0015 \end{bmatrix} \begin{bmatrix} 600 \\ 1500 \end{bmatrix} = \begin{bmatrix} 4.59 \\ 3.27 \end{bmatrix}$$

$$\mathbf{A}^6\mathbf{f} = \begin{bmatrix} .0008 & .0007 \\ .0006 & .0005 \end{bmatrix} \begin{bmatrix} 600 \\ 1500 \end{bmatrix} = \begin{bmatrix} 1.53 \\ 1.11 \end{bmatrix}$$

$$\mathbf{A}^7\mathbf{f} = \begin{bmatrix} .0003 & .0002 \\ .0002 & .0002 \end{bmatrix} \begin{bmatrix} 600 \\ 1500 \end{bmatrix} = \begin{bmatrix} 0.48 \\ 0.42 \end{bmatrix}$$

We see that the individual terms in the power series approximation (except for rounding errors) simply represent the magnitudes of the round-by-round effects, as recorded in Table 2.6. (The reader should reconsider the algebra of the round-by-round calculations to be convinced that, in fact, they were equivalent to premultiplication of **f** by a series of powers of the **A** matrix.) Thus it is possible that one may capture "most" of the effects associated with a given final demand by using the first few terms in the power series. As illustrated in Table 2.6, for our small example more than 98 percent of the total effects in both sectors was captured in three rounds.

2.5 Open Models and Closed Models

The model that we have dealt with thus far, $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$, depends on the existence of an exogenous sector, disconnected from the technologically interrelated productive sectors, since it is here that the important final demands for outputs originate. The basic kinds of transactions that constitute the activity of this sector, as we have seen, are consumption purchases by households, sales to government, gross private domestic investment, and shipments in foreign trade (either gross exports or net exports – exports from a sector less the value of imports of the same goods). In the case of households, especially, this "exogenous" categorization is something of a strain on basic economic theory. Households (consumers) earn incomes (at least in part) in payment for their

Table 2.9 Input–Output Table of Interindustry Flows with Households Endogenous

		Buying Sector					
		1	$\cdots$	j	$\cdots$	n	*Households (Consumers)*
Selling Sector	1	z_{11}	$\cdots$	z_{1j}	$\cdots$	z_{1n}	$z_{1,n+1}$
	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
	i	z_{i1}	$\cdots$	z_{ij}	$\cdots$	z_{in}	$z_{i,n+1}$
	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
	n	z_{n1}	$\cdots$	z_{nj}	$\cdots$	z_{nn}	$z_{n,n+1}$
	Households (Labor)	$z_{n+1,1}$	$\cdots$	$z_{n+1,j}$	$\cdots$	$z_{n+1,n}$	$z_{n+1,n+1}$

labor inputs to production processes, and, as consumers, they spend their income in rather well patterned ways. And in particular, a *change* in the amount of labor needed for production in one or more sectors – say an increase in labor inputs due to increased output – will lead to a change (here an increase) in the amounts spent by households as a group for consumption. Although households tend to purchase goods for "final" consumption, the amount of their purchases is related to their income, which depends on the outputs of each of the sectors. Also, as we have seen, consumption expenditures constitute possibly the largest single element of final demand; at least in the US economy they have frequently constituted more than two-thirds of the total final-demand figure.

Thus one could move the household sector from the final-demand column and labor-input row and place it inside the technically interrelated table, making it one of the *endogenous* sectors. This is known as closing the model with respect to households. Input–output models can be "closed" with respect to other exogenous sectors as well (for example, government sales and purchases); however, closure with respect to households is more usual. It requires a row and a column of transactions for the new household sector – the former showing the distribution of its output (labor services) among the various sectors and the latter showing the structure of its purchases (consumption) distributed among the sectors. It is customary to add the household row and column at the bottom and to the right of the transactions and coefficients tables. Dollar flows *to* consumers, representing wages and salaries received by households from the n sectors in payment for their labor services, would fill an $(n+1)$st row – $[z_{n+1,1}, \ldots, z_{n+1,n}]$. Dollar flows *from* consumers, representing the values of household purchases of the goods of the n sectors, would fill an $(n+1)$st column: $\begin{bmatrix} z_{1,n+1} \\ \vdots \\ z_{n,n+1} \end{bmatrix}$. Finally, the element in the $(n+1)$st row and the $(n+1)$st column, $z_{n+1,n+1}$, would represent household purchases of labor services. Thus Table 2.1 would have one new row, at the bottom, and one new column, at the right, as indicated in Table 2.9.

The ith equation, as shown in (2.1), would now be modified to

$$x_i = z_{i1} + \cdots + z_{ij} + \cdots + z_{in} + z_{i,n+1} + f_i^* \tag{2.20}$$

where f^* is understood to represent the remaining final demand for sector i output – exclusive of that from households, which is now captured in $z_{i,n+1}$. In addition to this kind of modification on each of the equations in set (2.2), there would be one new equation for the total "output" of the household sector, defined to be the total value of its sale of labor services to the various sectors – total earnings. Thus

$$x_{n+1} = z_{n+1,1} + \cdots + z_{n+1,j} + \cdots + z_{n+1,n} + z_{n+1,n+1} + f_{n+1}^* \tag{2.21}$$

The last term on the right in (2.21) would include, for example, payments to government employees.

Household input coefficients are found in the same manner as any other element in an input–output coefficients table: The value of sector j purchases of labor (for a given period), $z_{n+1,j}$, divided by the value of total output of sector j (for the same period), x_j, gives the value of household services (labor) used per dollar's worth of j's output; $a_{n+1,j} = z_{n+1,j}/x_j$. For the elements of the household purchases (consumption) column, the value of sector i sales to households (for a given period), $z_{i,n+1}$, is divided by the total output (measured by income earned) of the household sector, x_{n+1}. Thus, household "consumption coefficients" are $a_{i,n+1} = z_{i,n+1}/x_{n+1}$. A drawback to this approach is that now household behavior is "frozen" in the model in the same way as producer behavior (constant coefficients).

The ith equation in the fundamental set given in (2.6), above, becomes

$$x_i = a_{i1}x_1 + \cdots + a_{in}x_n + a_{i,n+1}x_{n+1} + f_i^* \tag{2.22}$$

and the added equation which relates household output to output of all of the sectors is

$$x_{n+1} = a_{n+1,1}x_1 + \cdots + a_{n+1,n}x_n + a_{n+1,n+1}x_{n+1} + f_{n+1}^* \tag{2.23}$$

Similarly, parallel to the equations in (2.7), we now have, rewriting (2.22) for the ith equation,

$$-a_{i1}x_1 - \cdots + (1 - a_{ii})x_i - \cdots - a_{in}x_n - a_{i,n+1}x_{n+1} = f_i^*$$

And, for the household equation, rewriting (2.23),

$$-a_{n+1,1}x_1 - \cdots - a_{n+1,n}x_n + (1 - a_{n+1,n+1})x_{n+1} = f_{n+1}^*$$

Let the row vector of labor input coefficients, $a_{n+1,j} = z_{n+1,j}/x_j$, be denoted by $\mathbf{h}_R = [a_{n+1,1}, \ldots, a_{n+1,n}]$, the column vector of household consumption coefficients, $a_{i,n+1} = z_{i,n+1}/x_{n+1}$, be $\mathbf{h}_C = \begin{bmatrix} a_{1,n+1} \\ \vdots \\ a_{n,n+1} \end{bmatrix}$ and let $h = a_{n+1,n+1}$.[14] Denote by $\bar{\mathbf{A}}$

[14] In the initial numerical illustration in section 2.3.1, above, for simplicity we used $\mathbf{e}'_c$ for the vector of employment coefficients. These are seen to be the elements in $\mathbf{h}_R$, which is the notation frequently used in closed models. Strictly speaking, we should use a "prime" to denote a row vector, but the subscript "R" reminds us that this is a *row* of coefficients.

the $(n+1)\times(n+1)$ technical coefficients matrix with households included. Using partitioning to separate the old $\mathbf{A}$ matrix from the new sector,

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{h}_C \\ \mathbf{h}_R & h \end{bmatrix}$$

Let $\bar{\mathbf{x}}$ denote the $(n+1)$-element column vector of gross outputs

$$\bar{\mathbf{x}} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{x} \\ x_{n+1} \end{bmatrix}$$

Also, let $\mathbf{f}^*$ be the n-element vector of remaining final demands for output of the original n sectors and $\bar{\mathbf{f}}$ the $(n+1)$-element vector of final demands, including that for the output of households

$$\bar{\mathbf{f}} = \begin{bmatrix} f_1^* \\ \vdots \\ f_n^* \\ f_{n+1}^* \end{bmatrix} = \begin{bmatrix} \mathbf{f}^* \\ f_{n+1}^* \end{bmatrix}$$

Then the new system of $n+1$ equations, with households endogenous, can be represented as

$$(\mathbf{I}-\bar{\mathbf{A}})\bar{\mathbf{x}} = \bar{\mathbf{f}} \tag{2.24}$$

or

$$\begin{bmatrix} \mathbf{I}-\mathbf{A} & -\mathbf{h}_C \\ -\mathbf{h}_R & (1-h) \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^* \\ f_{n+1}^* \end{bmatrix} \tag{2.25}$$

That is, we have the set of n equations

$$(\mathbf{I}-\mathbf{A})\mathbf{x} - \mathbf{h}_C x_{n+1} = \mathbf{f}^*$$

[a matrix rearrangement of (2.22)] and the added one for households

$$-\mathbf{h}_R\mathbf{x} + (1-h)x_{n+1} = f_{n+1}^*$$

[a matrix rearrangement of (2.23)]. Together these determine the values of outputs for the n original sectors – $x_1,\ldots,x_n$ – and the value of household services used (wages paid) to produce those outputs – x_{n+1}. If the $(n+1)\times(n+1)$ coefficients matrix is nonsingular, the unique solution can be found using an inverse matrix in the usual way:

$$\begin{bmatrix} \mathbf{x} \\ x_{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{I}-\mathbf{A} & -\mathbf{h}_C \\ -\mathbf{h}_R & (1-h) \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}^* \\ f_{n+1}^* \end{bmatrix} \tag{2.26}$$

Table 2.10 Flows (z_{ij}) for Hypothetical Example, with Households Endogenous

From \ To	1	2	Household Consumption (C)	Other Final Demand (f^*)	Total Output ($\mathbf{x}$)
1	150	500	50	300	1000
2	200	100	400	1300	2000
Labor Services (L)	300	500	50	150	1000
Other Domestic Payments (N)	325	800	300	250	1675
Imports (M)	25	100	200	150	475
Total Outlays ($\mathbf{x}'$)	1000	2000	1000	2150	6150

or

$$\bar{\mathbf{x}} = (\mathbf{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{f}} = \bar{\mathbf{L}}\bar{\mathbf{f}}$$

Consider again the information given in Table 2.3. Suppose that the household consumption part of final demand and the household labor input part of the payments sector are as shown in Table 2.10. Of the \$650 bought by sector 1 from the payments sectors (Table 2.3), \$300 was for labor services; of the \$1400 bought by sector 2, \$500 was for labor inputs. Also, of the \$1100 which represented purchases of final-demand sectors from the payments sectors, \$50 was paid out by households for labor services (e.g., domestic help); government purchases of labor was \$150. The \$300 would record household payments to government (taxes), and so forth.

The total output of the household sector, as in (2.16), is (here $n+1=3$), $x_3 = z_{31} + z_{32} + z_{33} + f_3^* = 300 + 500 + 50 + 150 = 1000$. The household input coefficients, $a_{n+1,j} = z_{n+1,j}/x_j$, are: $a_{31} = 300/1000 = 0.3$, $a_{32} = 500/2000 = 0.25$ and $a_{33} = 50/1000 = 0.05$; $\mathbf{h}_R = \begin{bmatrix} 0.3 & 0.25 \end{bmatrix}$ and $h = 0.05$. Similarly, household consumption coefficients, $a_{i,n+1} = z_{i,n+1}/x_{n+1}$ are $a_{13} = 50/1000 = 0.05$ and $a_{23} = 400/1000 = 0.4$; thus $\mathbf{h}_C = \begin{bmatrix} 0.05 \\ 0.4 \end{bmatrix}$. Therefore,

$$\bar{\mathbf{A}} = \begin{bmatrix} .15 & .25 & .05 \\ .2 & .05 & .4 \\ .3 & .25 & .05 \end{bmatrix}, \quad (\mathbf{I} - \bar{\mathbf{A}}) = \begin{bmatrix} .85 & -.25 & -.05 \\ -.2 & .95 & -.4 \\ -.3 & -.25 & .95 \end{bmatrix}$$

and

$$\bar{\mathbf{L}} = (\mathbf{I} - \bar{\mathbf{A}})^{-1} = \begin{bmatrix} 1.3651 & .4253 & .2509 \\ .5273 & 1.3481 & .5954 \\ .5698 & .4890 & 1.2885 \end{bmatrix} \tag{2.27}$$

Consider again the numerical example in section 2.3 (again we ignore the "0" and "1" superscripts for simplicity). There we assumed a change in the final-demand vector such that f_1 went from 350 to 600 and f_2 from 1700 to 1500. Referring now to Table 2.10, simply for illustration, suppose that this entire final-demand change was concentrated in the Other Final Demand sector. In fact, let it represent a change in the demands of the federal government [which are a part of the Other Final Demand column (f_i^*) in Table 2.10]. These new demands of \$600 and \$1500 represent increases in both cases, from the current levels of \$300 and \$1300 for all nonhousehold final-demand categories.

The most straightforward comparison is now to use the 3×3 Leontief inverse $(\mathbf{I} - \bar{\mathbf{A}})^{-1}$ in (2.27) in conjunction with $\bar{\mathbf{f}} = \begin{bmatrix} 600 \\ 1500 \\ 0 \end{bmatrix}$ to find the impact of these changes in the final demands for the outputs of sectors 1 and 2 on the two original sectors plus the added impact due to closure of the model with respect to households. We have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \bar{\mathbf{x}} = \begin{bmatrix} 1.3651 & .4253 & .2509 \\ .5273 & 1.3481 & .5954 \\ .5698 & .4890 & 1.2885 \end{bmatrix} \begin{bmatrix} 600 \\ 1500 \\ 0 \end{bmatrix} = \begin{bmatrix} 1456.94 \\ 2338.51 \\ 1075.48 \end{bmatrix}$$

In the earlier example of section 2.3, with households exogenous to the model, the new outputs were $x_1 = \$1247.46$ and $x_2 = \$1841.55$. The new (larger) values – \$1456.94 and \$2338.51, respectively – reflect the fact that *additional* outputs are necessary to satisfy the anticipated increase in consumer spending, as reflected in the household consumption coefficients column, expected because of the increased household earnings due to increased outputs from sectors 1 and 2 and hence increased wage payments. Using the labor input coefficients $a_{31} = 0.3$ and $a_{32} = 0.25$, the necessary household inputs for the original gross outputs (when households were exogenous) would be

$$a_{31}x_1 + a_{32}x_2 = (0.3)\,(1247.46) + (0.25)(1841.55) = 834.63$$

As would be expected, outputs are increased for all three sectors, due to the introduction of the formerly exogenous household sector into the model. The example serves to illustrate an expected outcome – namely that when the added impact of more household consumption spending due to increased wage income is explicitly taken into consideration in the model, the outputs of the original sectors in the interindustry model (here sectors 1 and 2) are larger than is the case when consumer spending is ignored.

In this section we have introduced the basic considerations involved in moving households from final demand into the model as an endogenous sector – closing the model with respect to households. Similar kinds of data and algebraic extensions would be needed if other exogenous sectors – for example, federal, or state and/or local government activities – were to be made endogenous in the model. However, because the value of consumption tends to be the largest component of final demand and because of the relatively direct linkage between earned income and consumption and between consumption and output, the household sector is the one final-demand sector that is most often moved inside the model.

In practice, however, the issue is far more subtle, and the procedure is more complicated than might be suggested by the discussion in this section. All of the previous reservations about the a_{ij} apply here as well, if not with greater force. For *each* additional dollar of received earnings, households are assumed to spend 5 cents on the output of sector 1, 40 cents on the output of sector 2, and so on. Those coefficients, which reflect *average* behavior during the observation period when household income was \$1000 ($a_{13} = 50/1000$ and $a_{23} = 400/1000$), are assumed to hold for the additional, or *marginal*, amounts of household earnings associated with the new outputs of sectors 1 and 2. One approach, particularly at the regional level, is to divide consumers into two groups: established residents, for whom the new income associated with new production would represent an addition to current earnings, and new residents (in-migrants), who may have moved in search of employment and for whom the new income represents total earnings. For the former group, a set of marginal consumption coefficients might be appropriate, while for the latter group average consumption coefficients would be relevant.

In addition, spending patterns of consumers, especially out of additions to (or reductions in) disposable income, will depend on the income category in which a particular consumer is located. An addition of \$100 to the spendable income of a worker earning \$20,000 per year is likely to be spent differently than an additional \$100 in the hands of an engineer with an annual income of \$150,000, and both will no doubt differ from the way in which the \$100 would be spent by a previously unemployed person. In effect, this is simply noting that inputs to the household sector (consumption) per dollar of output (household income) will not be independent of the level of that output. Yet such independence is assumed in the way that the direct input coefficients are used in an input–output model; each sector's production function (column of direct input coefficients) is assumed to represent inputs per dollar's worth of output, regardless of the amount (level) of that output.

Another approach, then, is to disaggregate "the" household sector into several sectors, distinguished by total income. For example, \$0–\$10,000, \$10,001–\$20,000, \$20,001–\$30,000; and so on. Consumption coefficients, by sector, could then be derived for each income class. We will return to this issue in a regional context in Chapter 3 and in Chapter 10 when examining social accounting matrices. A very thorough discussion of an approach for incorporating a disaggregated household sector into the endogenous part of an input–output model, using a good deal of matrix algebra, can be found in Miyazawa (1976). We explore that model in more detail in Chapter 6.

Further disaggregations of the household sector have been proposed and incorporated in input–output analysis. These frameworks fall into the category of what are known as "extended" input–output models. (For a concise overview see Batey, Madden and Weeks, 1987 or Batey and Weeks, 1989.) The idea is to separate income payments to and consumption patterns of different household groups – for example, established vs. new residents (noted above) and currently employed vs. unemployed.

One could imagine a process of moving, one by one, each of the remaining sectors from the final-demand vector into the interindustry coefficients matrix, constructing rows of input coefficients and columns of purchase coefficients until there were no exogenous sectors at all. This is termed a *completely closed model*. However, the economic logic behind fixed coefficients in the case, say, of a government sector is less easy to accept than for the productive sectors, and completely closed models are seldom implemented in practice.[15]

2.6 The Price Model

2.6.1 Overview

Leontief originally developed the input–output model in physical units (bushels of wheat, yards of cloth, man-years of labor, etc.).[16] In particular, he assumed that direct input coefficients, **A**, are based on *physical quantities* of inputs divided by *physical quantities* of output. These data were then converted to a table of (base year) transactions in value terms by using (base year) unit prices – for a bushel of wheat, a yard of cloth and a man-year of labor. He writes (Leontief, 1986, pp. 22–23):

> All figures [in the *value* transactions table]…can also be interpreted as representing *physical quantities* of the goods or services to which they refer. This only requires that the physical unit in which the entries…are measured be redefined as being equal to that amount of output of that particular sector that can be purchased for $1 at [base year] prices… In practice the structural matrices are usually computed from input-output tables described in value terms…In any case, the input coefficients [**A**] – for analytical purposes…must be interpreted as ratios of two quantities measured in *physical units* [emphasis added].

As already noted, input–output data are usually assembled and input–output studies are generally carried out in monetary (value) units.

However, with the emergence of energy and environmental concerns, mixed-units models have been developed, where economic transactions are recorded in monetary terms and ecological and/or energy transactions are recorded in physical terms (tons, BTUs, joules, etc.).[17] Another line of inquiry has led to input–output tables in common physical units (e.g., all transactions and outputs measured in tons). Stahmer (2000) gives an overview of this kind of work, including tables for Germany for 1990 in both monetary and physical units – sometimes designated MIOTs and PIOTs, respectively. (There are problems in trying to measure outputs of services in physical units.)[18] We explore a small illustration in section 2.6.8, below, using an aggregation of the German data.

[15] The original work done by Leontief, however, was in the framework of a completely closed model of the United States for 1919. See Leontief (1951b).
[16] See, for example, Leontief (1951a, 1951b, 1986), Leontief *et al.* (1953).
[17] These issues are explored further in Chapters 9 and 10.
[18] Stahmer (2000) also introduces the notion of data measured in time units, leading to TIOTs.

Table 2.11 Transactions in Physical Units

	1	2	d_i	q_i	Physical units of measure
1	75	250	175	500	bushels
2	40	20	340	400	tons

Table 2.12 Transactions in Monetary Units (see Table 2.3)

	1	2	f_i	x_i	\$ Price per physical unit
1	150	500	350	1000	2
2	200	100	1700	2000	5

Table 2.13 Transactions in Revised Physical Units

	1	2	d_i	q_i	Revised physical units of measure
1	150	500	350	1000	1/2 bushels
2	200	100	1700	2000	1/5 tons

2.6.2 Physical vs. Monetary Transactions

We return to the illustration in section 2.3. Suppose the *physical* unit measures for outputs are bushels for sector 1 (agriculture) and tons for sector 2 (manufacturing) and that transactions measured in these physical units are shown in Table 2.11, where we now use d_i for physical amounts delivered to final demand and q_i for physical amounts of total output.

If we know the per-unit prices of the two products, the information in Table 2.11 can be converted to monetary units. For example, if the price per bushel is \$2.00 and the price per ton is \$5.00, then the *monetary* transactions table is exactly as shown in Table 2.3. Now, redefine the physical units of measurement for each sector to be the amount that can be bought for \$1.00; that is, so that the per-unit price for each sector's output is \$1.00. This simply means that we measure the physical output of sector 1 in *half bushel* units and the physical output of sector 2 in *fifths of a ton*. Then, in these revised units, the information in Table 2.12 can be reinterpreted as recording transactions in physical units, as in Table 2.13 – for example, 500 half-bushels of sector 1 output were bought by sector 2 (for \$500), 2000 fifths of a ton of sector 2 output were delivered to final demand (for \$2000), etc.

Table 2.14 Transactions in Monetary Terms

	Sectors						
Sectors	1	$\cdots$	j	$\cdots$	n	Final Demand	Total Output
1	z_{11}	$\cdots$	z_{1j}	$\cdots$	z_{1n}	f_1	x_1
2	z_{21}	$\cdots$	z_{2j}	$\cdots$	z_{2n}	f_2	x_2
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$
n	z_{n1}	$\cdots$	z_{nj}	$\cdots$	z_{nn}	f_n	x_n
Labor	v_1	$\cdots$	v_j	$\cdots$	v_n	f_{n+1}	x_{n+1}

In practice, sectors produce more than one good, and the assumption of one price for a sector's output is unrealistic. And in any case, monetary tables are assembled on the basis of recorded values of transactions; price and quantity are generally not recorded separately.

2.6.3 *The Price Model based on Monetary Data*

Monetary transactions are arranged as usual, where for notational simplicity we assume that all value added is represented by labor (Table 2.14). As we saw in section 2.2.1, when *all* inputs are accounted for in the processing *and* payments sectors, then the jth column sum (total outlays) is equal to the jth row sum (total output). Thus, summing down the jth column in Table 2.14,

$$x_j = \sum_{i=1}^{n} z_{ij} + v_j \tag{2.28}$$

or

$$\mathbf{x}' = \mathbf{i}'\mathbf{Z} + \mathbf{v}' \tag{2.29}$$

where, as earlier, $\mathbf{v}' = [v_1, \ldots, v_n]$, total value-added expenditures by each sector.

Substituting $\mathbf{Z} = \mathbf{A}\hat{\mathbf{x}}$, $\mathbf{x}' = \mathbf{i}'\mathbf{A}\hat{\mathbf{x}} + \mathbf{v}'$, and postmultiplying by $\hat{\mathbf{x}}^{-1}$,

$$\mathbf{x}'\hat{\mathbf{x}}^{-1} = \mathbf{i}'\mathbf{A}\hat{\mathbf{x}}\hat{\mathbf{x}}^{-1} + \mathbf{v}'\hat{\mathbf{x}}^{-1}$$

or

$$\mathbf{i}' = \mathbf{i}'\mathbf{A} + \mathbf{v}'_c \tag{2.30}$$

where $\mathbf{v}'_c = \mathbf{v}'\hat{\mathbf{x}}^{-1} = [v_1/x_1, \ldots, v_n/x_n]$. The right-hand side of (2.30) is the cost of inputs per unit of output. Output prices are set equal to total cost of production (in the general case, this will include an allocation for profit and other primary inputs in $\mathbf{v}'$ and hence in $\mathbf{v}'_c$), so each price is equal to 1 [the left-hand side of (2.30)]. This illustrates the unique measurement units in the base year table – amounts that can be purchased for \$1.00. If we denote these base year index prices by $\tilde{p}_j$, so $\tilde{\mathbf{p}}' = [\tilde{p}_1, \ldots, \tilde{p}_n]$, then the input–output price model is:

$$\tilde{\mathbf{p}}' = \tilde{\mathbf{p}}'\mathbf{A} + \mathbf{v}'_c \tag{2.31}$$

which leads to $\tilde{\mathbf{p}}'(\mathbf{I} - \mathbf{A}) = \mathbf{v}'_c$ and

$$\tilde{\mathbf{p}}' = \mathbf{v}'_c(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{v}'_c\mathbf{L} \tag{2.32}$$

Frequently the model is transposed and expressed in terms of column vectors rather than row vectors. In that case,

$$\tilde{\mathbf{p}} = (\mathbf{I} - \mathbf{A}')^{-1}\mathbf{v}_c = \mathbf{L}'\mathbf{v}_c \tag{2.33}$$

[The interested reader can show that, given $(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{L}$, then $(\mathbf{I} - \mathbf{A}')^{-1} = \mathbf{L}'$.]

From (2.32), index prices, $\tilde{\mathbf{p}}$, are determined by the exogenous values (costs) of primary inputs. For a two-sector model,

$$\tilde{p}_1 = l_{11}v_{c1} + l_{21}v_{c2}$$
$$\tilde{p}_2 = l_{12}v_{c1} + l_{22}v_{c2}$$

The logic is that changes in labor input prices (or, more generally, primary input price changes) lead to changes in sectoral unit costs (and therefore output prices, not output quantities) via the fixed production recipes in $\mathbf{A}$, and hence in $\mathbf{L}$ and $\mathbf{L}'$. For example, cost increases are passed along completely as intermediate input price increases to all purchasers, who in turn pass on these increases by raising their output prices accordingly, etc. As opposed to the *demand-pull input–output quantity* model earlier in this chapter, the price model in (2.32) or (2.33) is more completely known as the *cost-push input–output price model* (Oosterhaven, 1996; Dietzenbacher, 1997). In it, quantities are fixed and prices change. Table 2.15 summarizes the two (dual) models where, again, superscripts "0" and "1" indicate values before and after accounting for the exogenous change. Examples in the following section illustrate the workings of this model.

2.6.4 *Numerical Examples Using the Price Model based on Monetary Data*

Example 1: Base Year Prices Table 2.16 contains data from Table 2.10 to construct an added row to reflect labor as the only primary input. The corresponding direct inputs matrix is

$$\bar{\mathbf{A}} = \begin{bmatrix} .15 & .25 & .11 \\ .20 & .05 & .54 \\ .65 & .70 & .35 \end{bmatrix} \tag{2.34}$$

Using $\mathbf{A}$ for the 2×2 submatrix of sector 1 and sector 2 coefficients,

$$(\mathbf{L}^0)' = (\mathbf{I} - \mathbf{A}')^{-1} = \begin{bmatrix} 1.254 & .264 \\ .330 & 1.122 \end{bmatrix} \tag{2.35}$$

Table 2.15 The Leontief Quantity and Price Models

Leontief Quantity Model (Demand-pull) [Prices fixed; quantities change]	Exogenous Variables	$\mathbf{f}^1 = [f_i^1]$ or $\Delta\mathbf{f} = [\Delta f_i]$
	Endogenous Variables	$\mathbf{x}^1 = \mathbf{L}^0\mathbf{f}^1$ or $\Delta\mathbf{x} = \mathbf{L}^0(\Delta\mathbf{f})$
Leontief Price Model (Cost-push) [Quantities fixed; prices change]	Exogenous Variables	$\mathbf{v}_c^1 = (\hat{\mathbf{x}}^0)^{-1}\mathbf{v}^1 = [v_j^1/x_j^0]$ or $\Delta\mathbf{v}_c = (\hat{\mathbf{x}}^0)^{-1}(\Delta\mathbf{v}) = [\Delta v_j/x_j^0]$
	Endogenous Variables	$\tilde{\mathbf{p}}^1 = (\mathbf{L}^0)'\mathbf{v}_c^1$ or $\Delta\tilde{\mathbf{p}} = (\mathbf{L}^0)'(\Delta\mathbf{v}_c)$

Table 2.16 Transactions for Hypothetical Example with One Primary Input

	1	2	f_i	x_i
1	150	500	350	1000
2	200	100	1700	2000
3 (Labor)	650	1400	1100	3150

From the base year data, $\mathbf{v}_c^0 = \begin{bmatrix} .65 \\ .70 \end{bmatrix} = \begin{bmatrix} \bar{a}_{31} \\ \bar{a}_{32} \end{bmatrix}$ – from the bottom row of $\bar{\mathbf{A}}$ in (2.34). Thus, in (2.33),

$$\tilde{\mathbf{p}}^0 = (\mathbf{L}^0)'\mathbf{v}_c^0 = \begin{bmatrix} 1.254 & .264 \\ .330 & 1.122 \end{bmatrix}\begin{bmatrix} .65 \\ .70 \end{bmatrix} = \begin{bmatrix} 1.00 \\ 1.00 \end{bmatrix} \tag{2.36}$$

This reproduces the base year index prices, as expected.

Example 2: Changed Base Year Prices The value-based cost-push price model is generally used to measure the impact on prices throughout the economy of new primary-input costs (or a change in those costs) in one or more sectors. Again, suppose that these costs consist entirely of wage payments and that wages in sector 1 increase by 30 percent (from 0.65 to 0.845) while those in sector 2 remain unchanged.

The vector of new labor costs is

$$\mathbf{v}_c^1 = \begin{bmatrix} .845 \\ .700 \end{bmatrix}$$

and, from (2.33),

$$\tilde{\mathbf{p}}^1 = (\mathbf{L}^0)'\mathbf{v}_c^1 = \begin{bmatrix} 1.254 & .264 \\ .330 & 1.122 \end{bmatrix} \begin{bmatrix} .845 \\ .700 \end{bmatrix} = \begin{bmatrix} 1.245 \\ 1.064 \end{bmatrix} \tag{2.37}$$

Relative to the original index prices ($\tilde{p}_1^0 = 1.00$ and $\tilde{p}_2^0 = 1.00$), sector 1's price has gone up to 1.245 (a 24.5 percent increase), and sector 2's price has increased by 6.4 percent.

As with the demand-driven input–output model in earlier sections, this exercise can just as well be carried out in the "Δ" form of the model, namely

$$\Delta\tilde{\mathbf{p}} = (\mathbf{L}^0)'\Delta\mathbf{v}_c \tag{2.38}$$

In this case, $\Delta\mathbf{v}_c = \begin{bmatrix} .195 \\ 0 \end{bmatrix}$, where (0.195) = (0.30)(0.65), and using (2.38),

$$\Delta\tilde{\mathbf{p}} = (\mathbf{L}^0)'\Delta\mathbf{v}_c = \begin{bmatrix} 1.254 & .264 \\ .330 & 1.122 \end{bmatrix} \begin{bmatrix} .195 \\ 0 \end{bmatrix} = \begin{bmatrix} .245 \\ .064 \end{bmatrix} \tag{2.39}$$

The results in either (2.37) or (2.39) convey the same information – the economy-wide effect of the 30 percent wage increase in sector 1 is that the price of sector 1 output goes up by 24.5 percent and that of sector 2 increases by 6.4 percent. In this cost-push input–output price model, we find *relative* price impacts – the absolute values of those prices, even in the base year, are not explicit in the model.

Notice that if labor costs are only a part of the value-added component for sector 1, then a 30 percent increase in wages in sector j will generate a less than 30 percent increase in v_{cj} – for example, if wages comprise 40 percent of sector j's value-added payments, and no other value-added costs increase, a 30 percent wage increase translates into a 12 percent increase in v_{cj}. The effects of primary input price *decreases* can also be quantified in the same way by the models in (2.32) [or (2.33)] or (2.38).

2.6.5 Applications

An early example of the use of this input–output price model is provided by Melvin (1979), where the price effects of changes in corporate income taxes are estimated for both the United States and Canada, using an 82-sector US table for 1965 and a 110-sector Canadian table for 1966. Another illustration is provided by Duchin and Lange (1995) who use the price model framework to assess price effects of alternative technologies in the US economy. Based on US 1963 and 1977 data, they use 1977 technology with 1963 factor prices to assess the price effects of the change in technology over that period. Similarly, using projections to 2000, they examine the price effects of technology change over 1977 to 2000. (They also change technology in the **A** matrix one column

Table 2.17 Flows in Physical Units

	Sectors					
Sectors	1	2	$\cdots$	n	Final Demand	Total Output
1	s_{11}	s_{12}	$\cdots$	s_{1n}	d_1	q_1
2	s_{21}	s_{22}	$\cdots$	s_{2n}	d_2	q_2
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
n	s_{n1}	s_{n2}	$\cdots$	s_{nn}	d_n	q_n
Labor	$s_{n+1,1}$	$s_{n+1,2}$	$\cdots$	$s_{n+1,n}$	d_{n+1}	q_{n+1}

at a time; and this is done in the context of a *dynamic* price model. We explore dynamic input–output models in Chapter 13.) A few additional examples include Lee, Blakesley and Butcher (1977) at a regional level, Polenske (1978) for a multiregional example, Marangoni (1995) for Italy and Dietzenbacher and Velázquez (2007) who include an analysis of cost-push effects of changes in water prices.

2.6.6 The Price Model based on Physical Data

In this section we examine the implications of an input–output model based on a set of data in *physical* units, as was shown in Table 2.11. Here, in Table 2.17, we let s_{ij} represent the physical quantity of i goods shipped to j [e.g., bushels of agricultural products (i) sold to manufacturers (j)], d_i is deliveries to final demand (e.g., in bushels for agricultural demand) and q_i is total sector i production (e.g., total bushels produced by agriculture). Again, for simplicity, let the exogenous payments (value added) sector consist exclusively of labor inputs (measured in person-days).

Reading across any row in Table 2.17 we have the basic accounting relationships in physical units:

$$q_i = s_{i1} + \cdots + s_{ij} + \cdots + s_{in} + d_i = \sum_{j=1}^{n} s_{ij} + d_i \tag{2.40}$$

[Compare with (2.1) in value terms.] Using obvious matrix definitions, this is

$$\mathbf{q} = \mathbf{Si} + \mathbf{d} \tag{2.41}$$

This is the physical-units parallel to (2.4).

Direct input coefficients in *physical* terms are defined as

$$c_{ij} = \frac{s_{ij}}{q_j} \quad \text{or} \quad \mathbf{C} = \mathbf{S}\hat{\mathbf{q}}^{-1} \tag{2.42}$$

For the example of agricultural input into manufacturing (Table 2.11), this would be 250/400 = 0.625 (bushels per ton). Then, in a series of steps that parallel the earlier development of the value-based model in section 2.2, substitution into (2.41) gives

$$\mathbf{q} = \mathbf{C}\hat{\mathbf{q}}\mathbf{i} + \mathbf{d} = \mathbf{C}\mathbf{q} + \mathbf{d}$$

from which

$$\mathbf{q} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{d} \tag{2.43}$$

This is the physical-units model parallel to (2.11).

Introduction of Prices Suppose also that we know the per-unit price for each sector's output, p_i, and the labor cost per person-hour, p_{n+1}. Then, as Leontief observes in the quotation above, we can easily convert the basic data to the value units from earlier in this chapter:

$$x_i = p_i q_i \tag{2.44}$$

$$z_{ij} = p_i s_{ij} \tag{2.45}$$

$$f_i = p_i d_i \tag{2.46}$$

Multiplying (2.40) on both sides by p_i gives

$$x_i = p_i q_i = \sum_{j=1}^{n} p_i s_{ij} + p_i d_i = \sum_{j=1}^{n} z_{ij} + f_i \tag{2.47}$$

or $\mathbf{x} = \mathbf{Z}\mathbf{i} + \mathbf{f}$. These are, of course, the original accounting relationships in (2.1) and (2.3) in value terms.

In section 2.6.1, the representation of total outputs in terms of column sums of Table 2.14 was given in monetary terms in (2.28), namely $x_j = \sum_{i=1}^{n} z_{ij} + v_j$. Column sums are not meaningful in Table 2.17 since elements in each row are measured in different units. The objective now is to introduce the results from (2.44) and (2.45) into (2.28). Assume, for now, that the wage rate is p_{n+1} (dollars per person-hour) across all sectors. Then $z_{n+1,j} = p_{n+1}s_{n+1,j} = v_j$; this represents sector j's total expenditure on labor – the price, p_{n+1}, times total person-hours of labor, $s_{n+1,j}$. Then (2.28) becomes

$$p_j q_j = \sum_{i=1}^{n} p_i s_{ij} + p_{n+1} s_{n+1,j} \tag{2.48}$$

Dividing by q_j (which we assume is not zero),

$$p_j = \sum_{i=1}^{n} p_i s_{ij}/q_j + p_{n+1} s_{n+1,j}/q_j = \sum_{i=1}^{n} p_i c_{ij} + p_{n+1} c_{n+1,j} \tag{2.49}$$

In matrix form, this is

$$\mathbf{p}' = \mathbf{p}'\mathbf{C} + \boldsymbol{\upsilon}'_c \tag{2.50}$$

where $\mathbf{p}' = [p_1, \ldots, p_n]$, $\mathbf{C}$ is defined in (2.42) and $\boldsymbol{\upsilon}'_c = p_{n+1}[c_{n+1,1}, \ldots, c_{n+1,n}]$. So $\boldsymbol{\upsilon}'_c$ represents the labor cost (price) per unit of physical output – for example, labor costs per ton of output [\$/ton = (\$/person-hour) × (person-hours/ton)].

Labor costs were assumed to be uniform across all sectors; thus we have only p_{n+1} and not $p_{n+1,j}$. This can easily be extended to encompass differing labor costs (perhaps reflecting labor of differing skills) among sectors. Equation (2.50) defines the unit price for each sector's output as equal to the total costs (interindustry plus primary inputs) of producing a unit of that output. (In general there will be more than one component to primary input costs for each sector, but the principles remain the same.)

From (2.50),

$$\mathbf{p}' = \boldsymbol{\upsilon}'_c(\mathbf{I} - \mathbf{C})^{-1} \tag{2.51}$$

As before, we can transpose both sides of (2.50) and (2.51) to have the prices in a column vector instead of a row vector,

$$\mathbf{p} = \mathbf{C}'\mathbf{p} + \boldsymbol{\upsilon}_c \quad \text{and} \quad \mathbf{p} = (\mathbf{I} - \mathbf{C}')^{-1}\boldsymbol{\upsilon}_c \tag{2.52}$$

This is the Leontief price model based on physical units. These structures are completely parallel to those in (2.32) and (2.33) for the monetary-based index-price model, For the $n = 2$ case, we have

$$\begin{aligned} p_1 &= p_1 c_{11} + p_2 c_{21} + \upsilon_{c1} \\ p_2 &= p_1 c_{12} + p_2 c_{22} + \upsilon_{c2} \end{aligned} \tag{2.53}$$

and

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} (1 - c_{11}) & -c_{21} \\ -c_{12} & (1 - c_{22}) \end{bmatrix}^{-1} \begin{bmatrix} \upsilon_{c1} \\ \upsilon_{c2} \end{bmatrix} \tag{2.54}$$

Relationship between **A** *and* **C** Direct input coefficients in value terms are

$$a_{ij} = \frac{z_{ij}}{x_j} \quad \text{or} \quad \mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$$

Therefore [from (2.44) and (2.45)]

$$a_{ij} = \frac{p_i s_{ij}}{p_j q_j} = c_{ij}\left(\frac{p_i}{p_j}\right) \tag{2.55}$$

In matrix terms[19]

$$\mathbf{A} = \hat{\mathbf{p}}\mathbf{S}(\hat{\mathbf{p}}\hat{\mathbf{q}})^{-1} = \hat{\mathbf{p}}\mathbf{C}\hat{\mathbf{q}}(\hat{\mathbf{q}}^{-1}\hat{\mathbf{p}}^{-1}) = \hat{\mathbf{p}}\mathbf{C}\hat{\mathbf{p}}^{-1} \tag{2.56}$$

[19] When two matrices, **M** and **N**, satisfy the relationship $\mathbf{M} = \hat{\mathbf{v}}\mathbf{N}\hat{\mathbf{v}}^{-1}$, they are said to be *similar*.

Either the value-based coefficients, **A**, or the physical coefficients, **C**, are assumed fixed in applications of the input–output model. However, assuming fixed c_{ij} (in effect, a fixed "engineering" production function) has been seen by many as less restrictive than fixed a_{ij} (a fixed "economic" production function), because in the latter case both a physical coefficient, c_{ij}, and a price ratio, p_i/p_j, are assumed unchanging.[20]

2.6.7 Numerical Examples Using the Price Model based on Physical Data

Example 1: Base Year Prices Consider again the two-sector economy (agriculture and manufacturing) in Table 2.11 closed with an added row showing labor inputs and final demand (consumption). From that table we can find the physical technical coefficients [as in (2.42)]

$$\bar{\mathbf{C}} = \begin{bmatrix} .15 & .625 & .556 \\ .08 & .05 & 1.079 \\ .13 & .35 & .349 \end{bmatrix} \tag{2.57}$$

We use $\bar{\mathbf{C}}$ for the (closed) technical coefficients matrix that includes households; **C** will represent the 2×2 matrix in the upper-left corner – technical coefficients connecting the two producing sectors in the economy. Note that $\bar{c}_{23} > 1$; column sums in $\bar{\mathbf{C}}$ are meaningless, since each row is measured in different units.

The relationships in (2.54) are

$$\begin{aligned} 2 &= (2)(.15) + (5)(.08) + (10)(.13) = .30 + .40 + 1.30 \\ 5 &= (2)(.625) + (5)(.05) + (10)(.35) = 1.25 + .25 + 3.50 \end{aligned} \tag{2.58}$$

If we use the base-period value-added-per-unit-of-output figures,

$$\upsilon_{c1}^0 = p_3\bar{c}_{31} = (10)(.13) = 1.30 \text{ and } \upsilon_{c2}^0 = p_3\bar{c}_{32} = (10)(.35) = 3.50$$

along with

$$(\mathbf{I} - \mathbf{C}')^{-1} = \begin{bmatrix} 1.243 & .106 \\ .825 & 1.122 \end{bmatrix} \tag{2.59}$$

(from the 2×2 upper-left submatrix of $\bar{\mathbf{C}}$) in $\mathbf{p} = (\mathbf{I} - \mathbf{C}')^{-1}\boldsymbol{\upsilon}_c$,

$$\begin{bmatrix} p_1^0 \\ p_2^0 \end{bmatrix} = (\mathbf{I} - \mathbf{C}')^{-1}\boldsymbol{\upsilon}_c^0 = \begin{bmatrix} 1.254 & .106 \\ .825 & 1.122 \end{bmatrix}\begin{bmatrix} 1.3 \\ 3.5 \end{bmatrix} = \begin{bmatrix} 2.00 \\ 5.00 \end{bmatrix} \tag{2.60}$$

This generates the base year prices, as expected.

[20] Economists have held differing opinions on this question of the plausibility of the assumption of stability for physical vs. value-based coefficients. For early examples, see Klein (1953) who suggests that a_{ij}'s may be more stable than c_{ij}'s, and Moses (1974) who argues the opposite.

Example 2: Changed Base Year Prices Continuing with this physical coefficients model, suppose that the wage costs in sector 1 increase from \$10.00 to \$13.00 (a 30 percent increase) while those in sector 2 remain unchanged ($p_{31}^1 = \$13.00$ and $p_{32}^1 = p_{32}^0 = 10.00$), so $\boldsymbol{v}_c^1 = \begin{bmatrix} (13)(.13) \\ (10)(.35) \end{bmatrix} = \begin{bmatrix} 1.69 \\ 3.50 \end{bmatrix}$. Then

$$\begin{bmatrix} p_1^0 \\ p_2^0 \end{bmatrix} = (\mathbf{I} - \mathbf{C}')^{-1}\boldsymbol{v}_c^0 = \begin{bmatrix} 1.254 & .106 \\ .825 & 1.122 \end{bmatrix} \begin{bmatrix} 1.69 \\ 3.50 \end{bmatrix} = \begin{bmatrix} 2.49 \\ 5.32 \end{bmatrix} \tag{2.61}$$

Specifically, $p_1^1 = \$2.49$ (an increase of 24.5 percent over $p_1^0 = \$2.00$) and $p_2^1 = \$5.32$ (a 6.4 percent increase over $p_2^0 = \$5.00$). This illustrates the operation of the cost-push input–output price model based on physical input coefficients. It generates the new prices directly (from which percentage changes can easily be found). In section 2.6.4 we found these percentage increases directly from the index-price model in (2.37).

2.6.8 The Quantity Model based on Physical Data

Data in physical units can also form the core of an input–output quantity model, as in $\mathbf{q} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{d}$ in (2.43) – before the introduction of prices. Using the data from the two numerical examples immediately above,

$$\mathbf{C} = \begin{bmatrix} .150 & .625 \\ .080 & .050 \end{bmatrix} \quad \text{and} \quad (\mathbf{I} - \mathbf{C})^{-1} = \begin{bmatrix} 1.254 & .825 \\ .106 & 1.122 \end{bmatrix}$$

Base year outputs are correctly generated by

$$\mathbf{q}^0 = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{d}^0 \Rightarrow \begin{bmatrix} 1.254 & .825 \\ .106 & 1.122 \end{bmatrix} \begin{bmatrix} 175 \\ 340 \end{bmatrix} = \begin{bmatrix} 500 \\ 400 \end{bmatrix}$$

and, for example, doubling demand doubles outputs,

$$\mathbf{q}^1 = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{d}^1 \Rightarrow \begin{bmatrix} 1.254 & .825 \\ .106 & 1.122 \end{bmatrix} \begin{bmatrix} 350 \\ 680 \end{bmatrix} = \begin{bmatrix} 1000 \\ 800 \end{bmatrix}$$

This is completely parallel to the demand-driven model in monetary terms, except that units of measurement are consistent only across each row. This means that the new demands (350 bushels and 680 tons) lead to production of 1000 bushels and 800 tons. Notice the units in $(\mathbf{I} - \mathbf{C})^{-1}$. For example, in the first column, 1.254 represents direct and indirect bushels of output per bushel of final demand, and 0.106 is direct and indirect output of tons per bushel of final demand.

A real-world illustration of this kind of model based on physical units appears in Stahmer (2000)[21]. This consists of a 12-sector input–output data set in physical terms for Germany in 1990 (an aggregation of a 91-sector model), where all transactions and outputs are measured in a common physical unit – tons. Hubacek and Giljum (2003)

[21] Also available as: "The Magic Triangle of Input–Output Tables," paper presented to the 13th International Input–Output Association Conference on Input–Output Techniques, Macerata, Italy, August, 2000.

Table 2.18 Transactions in Physical Terms (Germany, 1990) (millions of tons)

	Primary	Secondary	Tertiary	Final Demand	Total Output
Primary	2248	1442	336	84	4110
Secondary	27	1045	206	708	1986
Tertiary	5	69	51	36	161

generate a three-sector aggregation of these data for the illustrations in their study. In particular, transactions are shown in Table 2.18, above.

As in the illustration in (2.57), the associated direct inputs matrix, here $\mathbf{C}$, has coefficients that are larger than 1:

$$\mathbf{C} = \begin{bmatrix} .5470 & .7261 & 2.0870 \\ .0066 & .5262 & 1.2795 \\ .0012 & .0347 & .3168 \end{bmatrix}$$

As we saw above, this does not pose any problems for the usual input–output calculations; here the Leontief inverse is easily found to be

$$(\mathbf{I} - \mathbf{C})^{-1} = \begin{bmatrix} 2.3185 & 4.7204 & 15.9220 \\ .0502 & 2.5486 & 4.9262 \\ .0067 & .1380 & 1.7425 \end{bmatrix}$$

As we see, some elements are large; these are associated with the large elements in $\mathbf{C}$, but they are not inappropriate in the context of this PIOT. The reader can easily check the validity of this inverse from the base-case data, namely

$$\mathbf{x} = \begin{bmatrix} 4110 \\ 1986 \\ 161 \end{bmatrix} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{f} = \begin{bmatrix} 2.3185 & 4.7204 & 15.9220 \\ .0502 & 2.5486 & 4.9262 \\ .0067 & .1380 & 1.7425 \end{bmatrix} \begin{bmatrix} 84 \\ 708 \\ 36 \end{bmatrix}$$

Despite the unusual elements in $\mathbf{C}$, the power series approximation to the Leontief inverse – $\mathbf{I} + \mathbf{C} + \mathbf{C}^2 + \mathbf{C}^3 + \cdots$ – works just fine, although slowly; it requires 37 terms to come within four-digit accuracy. Here are some of the terms:

$$\mathbf{C}^{10} = \begin{bmatrix} 0.0077 & 0.0971 & 0.3551 \\ 0.0011 & 0.0167 & 0.0616 \\ 0.0001 & 0.0018 & 0.0067 \end{bmatrix}, \quad \mathbf{C}^{20} = \begin{bmatrix} 0.0002 & 0.0030 & 0.0111 \\ 0.0000 & 0.0005 & 0.0018 \\ 0.0000 & 0.0001 & 0.0002 \end{bmatrix},$$

$$\mathbf{C}^{30} = \begin{bmatrix} 0 & .0001 & .0003 \\ 0 & 0 & .0001 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{C}^{37} = \mathbf{0}$$

and

$$\left(\mathbf{I} + \sum_{k=1}^{37} \mathbf{C}^k\right) = \begin{bmatrix} 2.3185 & 4.7204 & 15.9220 \\ .0502 & 2.5486 & 4.9262 \\ .0067 & .1380 & 1.7425 \end{bmatrix} = (\mathbf{I} - \mathbf{C})^{-1}$$

The interested reader with access to combinatorial algebra software on a computer might check that for this illustration, with $(\mathbf{I}-\mathbf{C}) = \begin{bmatrix} .4530 & -.7261 & -2.0870 \\ -.0066 & .4738 & -1.2795 \\ -.0012 & -.0347 & .6832 \end{bmatrix}$, the Hawkins–Simon conditions are satisfied, meaning that all seven principal minors of $(\mathbf{I}-\mathbf{C})$ are positive (Appendix 2.2).

2.6.9 A Basic National Income Identity

From (2.43), $\mathbf{q} = (\mathbf{I}-\mathbf{C})^{-1}\mathbf{d}$; from (2.51), $\mathbf{p}' = \boldsymbol{\upsilon}'_c(\mathbf{I}-\mathbf{C})^{-1}$, and postmultiplying this by $\mathbf{d}$,

$$\mathbf{p}'\mathbf{d} = \boldsymbol{\upsilon}'_c(\mathbf{I}-\mathbf{C})^{-1}\mathbf{d} = \boldsymbol{\upsilon}'_c\mathbf{q}$$

The total value of spending (exogenous final demand, $\mathbf{p}'\mathbf{d}$) equals the total value of earnings (payments to exogenous primary inputs, $\boldsymbol{\upsilon}'_c\mathbf{q}$), or national income spent equals national income received.

2.7 Summary

We have introduced the basic structure of the input–output model in this chapter. After investigating the special features of sectoral production functions that are assumed in the Leontief system, we examined its mathematical features. Importantly, the model is expressed in a set of linear equations, and we have tried to indicate the connection between the purely algebraic solution to the input–output equations, using the Leontief inverse matrix, and the logical, economic content of the round-by-round view of production interrelationships in an economy. Both the algebraic details as well as the economic assumptions needed to close the model with respect to households were discussed. Some of the special problems associated with the concept of household consumption coefficients have been addressed in applications, especially at the regional level. We also introduced the Leontief price model, a logical (and mathematical) companion to the quantity model, and we explored alternatives to both models when the underlying data are measured in physical rather than monetary terms. Table 2.19 summarizes the alternatives. (Information in the monetary row is in Table 2.15.)

We turn to regional input–output models in the next chapter. It is important to add the regional dimension; many if not most important policy questions are not purely national in scope. Rather, analysts (even at the national level) are interested in differential regional effects of, say, a change in national government policy regarding exports. It is important to know not only the total magnitudes of the new outputs, by sector, that come about because of stimulation of exports, but also to know something of their geographical incidence – is a particularly depressed area helped by such export stimulation, or does the increased output occur largely in regions that are economically more healthy? Extensions of the basic model to deal with issues of this sort will occupy us in Chapter 3.

Table 2.19 Alternative Input–Output Price and Quantity Models

Measurement Units	Quantity Model	Price Model
Monetary	$\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$ (2.11)	$\tilde{\mathbf{p}}' = \mathbf{v}'_c(\mathbf{I} - \mathbf{A})^{-1}$ (2.32) or $\tilde{\mathbf{p}} = (\mathbf{I} - \mathbf{A}')^{-1}\mathbf{v}_c$ (2.33)
Physical	$\mathbf{q} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{d}$ (2.43)	$\mathbf{p}' = \boldsymbol{\upsilon}'_c(\mathbf{I} - \mathbf{C})^{-1}$ (2.51) or $\mathbf{p} = (\mathbf{I} - \mathbf{C}')^{-1}\boldsymbol{\upsilon}_c$ (2.52)

Appendix 2.1 The Relationship between Approaches I and II

To examine the connection between the two alternative approaches to the numerical example in section 2.3, we consider a general two-sector economy with $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and let f_1 and f_2 represent values of the new final demands.[22]

A2.1.1 Approach I

Using the Leontief-inverse, we find $(\mathbf{I} - \mathbf{A}) = \begin{bmatrix} (1 - a_{11}) & -a_{12} \\ -a_{21} & (1 - a_{22}) \end{bmatrix}$ and, provided that $|\mathbf{I} - \mathbf{A}| \neq 0$, which means that $(1-a_{11})(1-a_{22})-(-a_{12})(-a_{21}) \neq 0$ (Appendix A)

$$(\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{|\mathbf{I} - \mathbf{A}|}[\text{adj}(\mathbf{I} - \mathbf{A})] = \begin{bmatrix} \dfrac{(1 - a_{22})}{|\mathbf{I} - \mathbf{A}|} & \dfrac{a_{12}}{|\mathbf{I} - \mathbf{A}|} \\ \dfrac{a_{21}}{|\mathbf{I} - \mathbf{A}|} & \dfrac{(1 - a_{11})}{|\mathbf{I} - \mathbf{A}|} \end{bmatrix} \tag{A2.1.1}$$

The associated gross outputs are found from $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$, namely

$$\begin{aligned} x_1 &= \left[\frac{(1 - a_{22})}{|\mathbf{I} - \mathbf{A}|}\right]f_1 + \left[\frac{a_{12}}{|\mathbf{I} - \mathbf{A}|}\right]f_2 \\ x_2 &= \left[\frac{a_{21}}{|\mathbf{I} - \mathbf{A}|}\right]f_1 + \left[\frac{(1 - a_{11})}{|\mathbf{I} - \mathbf{A}|}\right]f_2 \end{aligned} \tag{A2.1.2}$$

[22] As elsewhere in this chapter, we ignore the "0" and "1" superscripts for notational simplicity when the intended meaning is clear from the context.

A2.1.2 Approach II

The round-by-round calculation of total impacts requires only the elements of the **A** matrix. The first-round impact on sector 1 – in terms of what it must produce to satisfy its own and sector 2's needs for inputs – is $\underbrace{a_{11}f_1 + a_{12}f_2}_{\text{Sector 1, Round 1}}$. For sector 2, the first-round impact is $\underbrace{a_{21}f_1 + a_{22}f_2}_{\text{Sector 2, Round 1}}$. (These were \$465 and \$195 in the numerical example.)

The second-round impacts result from production that is required to take care of first-round needs. These are easily seen to be

$$\text{For sector 1: } a_{11}\underbrace{(a_{11}f_1 + a_{12}f_2)}_{\text{Sector 1, Round 1}} + a_{12}\underbrace{(a_{21}f_1 + a_{22}f_2)}_{\text{Sector 2, Round 1}}$$

$$\text{For sector 2: } a_{21}\underbrace{(a_{11}f_1 + a_{12}f_2)}_{\text{Sector 1, Round 1}} + a_{22}\underbrace{(a_{21}f_1 + a_{22}f_2)}_{\text{Sector 2, Round 1}}$$

(These were \$118.50 and \$102.75 in the numerical example.)

The nature of the expansion is now clear. For sector 1 in round 3, we will have

$$a_{11}\underbrace{[a_{11}(a_{11}f_1 + a_{12}f_2) + a_{12}(a_{21}f_1 + a_{22}f_2)]}_{\text{Sector 1, Round 2}}$$
$$+ a_{12}\underbrace{[a_{21}(a_{11}f_1 + a_{12}f_2) + a_{22}(a_{21}f_1 + a_{22}f_2)]}_{\text{Sector 2, Round 2}}$$

and for sector 2 in round 3:

$$a_{21}\underbrace{[a_{11}(a_{11}f_1 + a_{12}f_2) + a_{12}(a_{21}f_1 + a_{22}f_2)]}_{\text{Sector 1, Round 2}}$$
$$+ a_{22}\underbrace{[a_{21}(a_{11}f_1 + a_{12}f_2) + a_{22}(a_{21}f_1 + a_{22}f_2)]}_{\text{Sector 2, Round 2}}$$

(These were \$43.46 and \$28.84 in the numerical example.)

Without going further, we can develop an expression for an approximation to x_1 in terms of f_1 and f_2 and the technical coefficients on the basis of only three rounds of effects. Collecting the terms for round-by-round effects on sector 1, we have

$$\begin{aligned} x_1 \cong{} & f_1 + a_{11}f_1 + a_{11}^2 f_1 + a_{12}a_{21}f_1 + a_{11}^3 f_1 + a_{11}a_{12}a_{21}f_1 \\ & + a_{12}a_{21}a_{11}f_1 + a_{12}f_2 + a_{11}a_{12}f_2 + a_{12}a_{22}f_2 + a_{11}a_{11}a_{12}f_2 \\ & + a_{11}a_{12}a_{22}f_2 + a_{12}a_{21}a_{12}f_2 + a_{12}a_{22}a_{22}f_2 \end{aligned}$$

or

$$\begin{aligned} x_1 \cong\ & (1 + a_{11} + a_{11}^2 + a_{12}a_{21} + a_{11}^3 + a_{11}a_{12}a_{21} + a_{12}a_{21}a_{11})f_1 \\ & + (a_{12} + a_{11}a_{12} + a_{12}a_{22} + a_{11}a_{11}a_{12} + a_{11}a_{12}a_{22} \\ & + a_{12}a_{21}a_{12} + a_{12}a_{22}a_{22})f_2 \end{aligned} \tag{A2.1.3}$$

A similar expression can be derived for x_2.

The object of this algebra is to make clear that in round 2, the effect is found in products of *pairs* of coefficients (e.g., a_{11}^2 and $a_{11}a_{12}$); in round 3, the effect comes from products of *triples* of coefficients (e.g., a_{11}^3 and $a_{11}a_{12}a_{21}$). Similarly, in round 4, sets of four coefficients will be multiplied together, ... and in round n, sets of n coefficients will be multiplied. In monetary terms, all $a_{jj} < 1$ and $a_{ij} < 1$ since producer j must buy, from himself and each supplier i, less than one dollar's worth of inputs per dollar's worth of output. Therefore it is clear that eventually the effects in the "next" round will be essentially negligible. Mathematically, the expression for x_1 has the form

$$\begin{aligned} x_1 = & (1 + \text{infinite series of terms involving products of pairs, triples, \ldots, of } a_{ij})f_1 \\ & + (\text{similar infinite series})f_2 \end{aligned} \tag{A2.1.4}$$

There would be a parallel expression for x_2. If we denote these two parenthetical series terms for x_1 by s_{11} and s_{12}, and in the similar expression for x_2 by s_{21} and s_{22}, we have gross outputs related to final demands by

$$\begin{aligned} x_1 &= s_{11}f_1 + s_{12}f_2 \\ x_2 &= s_{21}f_1 + s_{22}f_2 \end{aligned} \tag{A2.1.5}$$

The evaluation of the s terms as four different infinite series would be a difficult and tedious task.

Alternatively, we could think of the new total output x_1 as composed of two parts: (a) the new final demands for sector 1's output, f_1, and (b) all direct and indirect effects on sector 1 generated by f_1 and f_2. (This approach was suggested in Dorfman, Samuelson and Solow, 1958, section 9.3.) To this end, define $F_1 = a_{11}f_1 + a_{12}f_2$, the first-round response from sector 1, and, similarly, let $F_2 = a_{21}f_1 + a_{22}f_2$ for sector 2. These first-round outputs will similarly generate second-round outputs, and so on, exactly as did f_1 and f_2 above. The suggestion is that the final outputs can be looked at as (1) a series of round-by-round effects on f_1 and f_2 *or* as (2) f_1 and f_2, plus a series of round-by-round effects on F_1 and F_2. In this alternative view, a complete derivation similar to that preceding (A2.1.5) would lead to

$$\begin{aligned} x_1 &= f_1 + s_{11}F_1 + s_{12}F_2 \\ x_2 &= f_2 + s_{21}F_1 + s_{22}F_2 \end{aligned} \tag{A2.1.6}$$

Substituting $F_1 = a_{11}f_1 + a_{12}f_2$ and $F_2 = a_{21}f_1 + a_{22}f_2$ and collecting terms,

$$\begin{aligned} x_1 &= (1 + s_{11}a_{11} + s_{12}a_{21})f_1 + (s_{11}a_{12} + s_{12}a_{22})f_2 \\ x_2 &= (s_{21}a_{11} + s_{22}a_{21})f_1 + (1 + s_{21}a_{12} + s_{22}a_{22})f_2 \end{aligned} \tag{A2.1.7}$$

Both (A2.1.5) and (A2.1.7) show x_1 and x_2 as linear functions of f_1 and f_2, so the coefficients in corresponding positions must be equal. That is,

$$\begin{array}{ll} s_{11} = 1 + s_{11}a_{11} + s_{12}a_{21} & s_{12} = s_{11}a_{12} + s_{12}a_{22} \\ s_{21} = s_{21}a_{11} + s_{22}a_{21} & s_{22} = 1 + s_{21}a_{12} + s_{22}a_{22} \end{array}$$

The top two are linear equations in the unknowns s_{11} and s_{12}, and the bottom two are linear equations in s_{21} and s_{22}. Rearranging to emphasize that the s are unknowns and the a are known coefficients,

$$\begin{aligned} (1 - a_{11})s_{11} - a_{21}s_{12} &= 1 \\ -a_{12}s_{11} + (1 - a_{22})s_{12} &= 0 \\ (1 - a_{11})s_{21} - a_{21}s_{22} &= 0 \\ -a_{12}s_{21} + (1 - a_{22})s_{22} &= 1 \end{aligned}$$

or

$$\begin{bmatrix} (1 - a_{11}) & -a_{21} \\ -a_{12} & (1 - a_{22}) \end{bmatrix} \begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{A2.1.8a}$$

$$\begin{bmatrix} (1 - a_{11}) & -a_{21} \\ -a_{12} & (1 - a_{22}) \end{bmatrix} \begin{bmatrix} s_{21} \\ s_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{A2.1.8b}$$

Both sets of equations have the same coefficient matrix. Since

$$\begin{bmatrix} (1 - a_{11}) & -a_{21} \\ -a_{12} & (1 - a_{22}) \end{bmatrix}^{-1} = \frac{1}{(1 - a_{11})(1 - a_{22}) - a_{12}a_{21}} \begin{bmatrix} (1 - a_{22}) & a_{21} \\ a_{12} & (1 - a_{11}) \end{bmatrix}$$

and since $(1-a_{11})(1-a_{22})-a_{12}a_{21} = |\mathbf{I} - \mathbf{A}|$ [in (A2.1.1) and (A2.1.2)], the solutions to the two pairs of linear equations in (A2.1.8) are

$$\begin{bmatrix} s_{11} \\ s_{12} \end{bmatrix} = \begin{bmatrix} \dfrac{(1 - a_{22})}{|\mathbf{I} - \mathbf{A}|} & \dfrac{a_{21}}{|\mathbf{I} - \mathbf{A}|} \\ \dfrac{a_{12}}{|\mathbf{I} - \mathbf{A}|} & \dfrac{(1 - a_{11})}{|\mathbf{I} - \mathbf{A}|} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and}$$

$$\begin{bmatrix} s_{21} \\ s_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{(1 - a_{22})}{|\mathbf{I} - \mathbf{A}|} & \dfrac{a_{21}}{|\mathbf{I} - \mathbf{A}|} \\ \dfrac{a_{12}}{|\mathbf{I} - \mathbf{A}|} & \dfrac{(1 - a_{11})}{|\mathbf{I} - \mathbf{A}|} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

That is,

$$s_{11} = \frac{(1 - a_{22})}{|\mathbf{I} - \mathbf{A}|}, \quad s_{12} = \frac{a_{12}}{|\mathbf{I} - \mathbf{A}|}, \quad s_{21} = \frac{a_{21}}{|\mathbf{I} - \mathbf{A}|}, \quad s_{22} = \frac{(1 - a_{11})}{|\mathbf{I} - \mathbf{A}|}$$

These algebraic expressions equate the four infinite series terms, whose complex form was suggested in (A2.1.3) and (A2.1.4), to very simple functions of the elements of **A**.

Moreover, these four simple functions are precisely the four elements of the Leontief inverse, as found in (A2.1.1). In economic terms, the $(\mathbf{I}-\mathbf{A})^{-1}$ matrix captures in each of its elements all of the infinite series of round-by-round direct and indirect effects that the new final demands have on the outputs of the two sectors. (A demonstration along these lines is much more complex for a three-sector input–output model and unwieldy for more than three sectors.)

The elements of this Leontief inverse matrix are often termed *multipliers*. With $(\mathbf{I}-\mathbf{A})^{-1}=\mathbf{L}=\begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix}$ and forecasts for f_1 and f_2, the total effect on x_1 is given by $l_{11}f_1+l_{12}f_2$, the sum of the multiplied effects of each of the individual final demands. And similarly for x_2. Input–output multipliers are explored in Chapter 6.

Appendix 2.2 The Hawkins–Simon Conditions

No matter how many terms we use in the series approximation to $(\mathbf{I}-\mathbf{A})^{-1}$in (2.17), it is clear that each of the terms contains only non-negative elements, since all $a_{ij}\geq 0$. As noted in section 2.4, not only is $\mathbf{A}\geq\mathbf{0}$, but $\mathbf{A}^2\geq\mathbf{0},\ldots,\mathbf{A}^n\geq\mathbf{0}$; therefore $(\mathbf{I}+\mathbf{A}+\mathbf{A}^2+\cdots)$ is a matrix of non-negative terms. If the elements of $\mathbf{f}$ are all non-negative, then the associated $\mathbf{x}$ will contain non-negative elements also. This is what one would expect; when faced with a set of non-negative final demands it would be meaningless in an economy to find that one or more of the necessary gross outputs were negative.[23] For a Leontief system with $\mathbf{A}\geq\mathbf{0}$ and $N(\mathbf{A})<1$ [so that the results in (2.17) hold], we know that negative outputs will never be required from any sector to satisfy non-negative final demands.

One could also explore conditions under which $\mathbf{f}\geq\mathbf{0}$ would always generate $\mathbf{x}\geq\mathbf{0}$ by examining the general definition $(\mathbf{I}-\mathbf{A})^{-1}=\frac{1}{|\mathbf{I}-\mathbf{A}|}[\mathrm{adj}(\mathbf{I}-\mathbf{A})]$ (Appendix A). For the simplest, two-sector case,

$$(\mathbf{I}-\mathbf{A})^{-1}=\begin{bmatrix} \dfrac{(1-a_{22})}{|\mathbf{I}-\mathbf{A}|} & \dfrac{a_{12}}{|\mathbf{I}-\mathbf{A}|} \\ \dfrac{a_{21}}{|\mathbf{I}-\mathbf{A}|} & \dfrac{(1-a_{11})}{|\mathbf{I}-\mathbf{A}|} \end{bmatrix}$$

and all of the elements in $(\mathbf{I}-\mathbf{A})^{-1}$ must be non-negative – the numerators must all be non-negative and the denominator must be positive (the denominator must not be zero, either). Or, all numerators could be non-positive and the denominator negative.

We have already noted that $a_{ij}\geq 0$ and that $N(\mathbf{A})<1$ and (also by their definition) all $a_{ij}<1$.[24] Thus all numerators in $(\mathbf{I}-\mathbf{A})^{-1}$ are non-negative. Therefore, if $|\mathbf{I}-\mathbf{A}|>0$, all elements in the 2×2 Leontief inverse will be non-negative.

[23] In some models, as we have seen, negative values could have meaning. When both $\mathbf{x}$'s and $\mathbf{f}$'s are defined as "changes in", namely $\Delta\mathbf{x}$ and $\Delta\mathbf{f}$, then a result like $\Delta x_3=-400$ is interpreted as a *decrease* of \$400 in sector 3's output.

[24] As we saw in section 2.6, this need not be the case in input–output tables denominated in *physical* rather than monetary terms – for example, liters of input per kilogram of output. See also Chapters 9 and 10.

Hawkins and Simon (1949) investigated the issue of non-negative solutions to more general equation systems. For a system in which $\mathbf{A} \geq \mathbf{0}$ (as in the input–output case) but in which no restriction is placed on the column sums of $\mathbf{A}$, they found for the 2×2 case that necessary and sufficient conditions to assure $\mathbf{x} \geq \mathbf{0}$ are[25]

$$\begin{array}{l}\text{(a) } (1 - a_{11}) > 0 \text{ and } (1 - a_{22}) > 0 \\ \text{(b) } |\mathbf{I} - \mathbf{A}| > 0\end{array} \tag{A2.2.1}$$

These conditions have a straightforward geometrical interpretation for the 2×2 case. We examine the solution-space representation. The fundamental relations

$$\begin{array}{l}\text{(a) } (1 - a_{11})x_1 - a_{12}x_2 = f_1 \\ \text{(b) } -a_{21}x_1 + (1 - a_{22})x_2 = f_2\end{array} \tag{A2.2.2}$$

define a pair of linear equations in x_1x_2 space. By setting one variable at a time equal to zero in each equation, it is easy to find the intercepts of each line on each axis. These are shown in Figure A2.2.1a, for arbitrary (but positive) f_1 and f_2. (Assume that both a_{12} and a_{21} are strictly positive, i.e., that each sector sells some inputs to the other. In a highly aggregated model this is virtually certain to be the case.)

As long as $(1 - a_{11}) > 0$ and $(1 - a_{22}) > 0$ – the first Hawkins–Simon condition in the 2×2 case – for $f_1 > 0$ and $f_2 > 0$, the intercept of (A2.2.2)(a) on the x_1-axis will be to the right of the origin and the intercept of (A2.2.2)(b) on the x_2-axis will be above the origin. Therefore, for non-negative total outputs, it is required that these two equations intersect in the first quadrant; this means that the slope of equation (a) must be greater

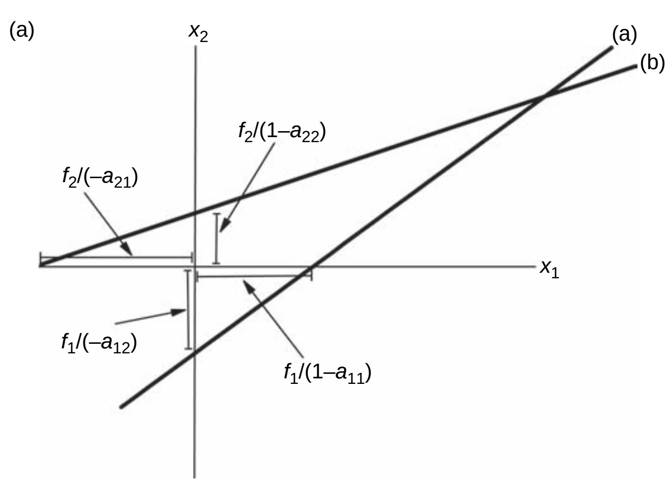

Figure A2.2.1a Solution Space Representation of (A2.2.2); $a_{12} > 0$ and $a_{21} > 0$

[25] The matrix algebra requirement for a unique solution to $(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{f}$ is that $|\mathbf{I} - \mathbf{A}| \neq 0$. Now we are further restricting this determinant to only positive values.

than the slope of equation (b). These slopes are:

$$\text{For equation (a)}\ \frac{\dfrac{f_1}{a_{12}}}{\dfrac{f_1}{(1-a_{11})}} = \frac{(1-a_{11})}{a_{12}}$$

$$\text{For equation (b)}\ \frac{\dfrac{f_2}{(1-a_{22})}}{\dfrac{f_2}{a_{21}}} = \frac{a_{21}}{(1-a_{22})}$$

and thus the slope requirement is $(1-a_{11})/a_{12} > a_{21}/(1-a_{22})$. Multiplying both sides of the inequality by $(1-a_{22})$ and by a_{12} – both of which are assumed to be strictly positive – does not alter the direction of the inequality, giving $(1-a_{11})(1-a_{22}) > a_{12}a_{21}$ or $(1-a_{11})(1-a_{22}) - a_{12}a_{21} > 0$, which is just $|\mathbf{I} - \mathbf{A}| > 0$, the second Hawkins–Simon condition in the 2×2 case.

The effects of less interdependence in the two-sector economy are illustrated in Figures A2.2.1b and A2.2.1c. If $a_{21} = 0$, meaning that $z_{21} = 0$ (sector 1 uses no inputs from sector 2), then the slope of the line labeled (b) is zero. It is a horizontal line intersecting the x_2-axis at the height $f_2/(1-a_{22})$. This is to be expected; the gross output necessary from sector 2 depends only on final demand for the output of sector 2, f_2, and the amount of *intra*industry input that sector 2 buys from itself, a_{22} (Figure A2.2.1b). Similarly, if $a_{12} = 0$ – sector 2 buys no inputs from sector 1 – line (a) in the figure will have an infinite slope; it will be vertical through the point $f_1/(1-a_{11})$ on the x_1-axis (Figure A2.2.1c).

The geometry of the 2×2 case does not generalize easily, at least for $n > 3$. For this, we need some matrix terminology. The *minor* of an element a_{ij} in an $n \times n$ square

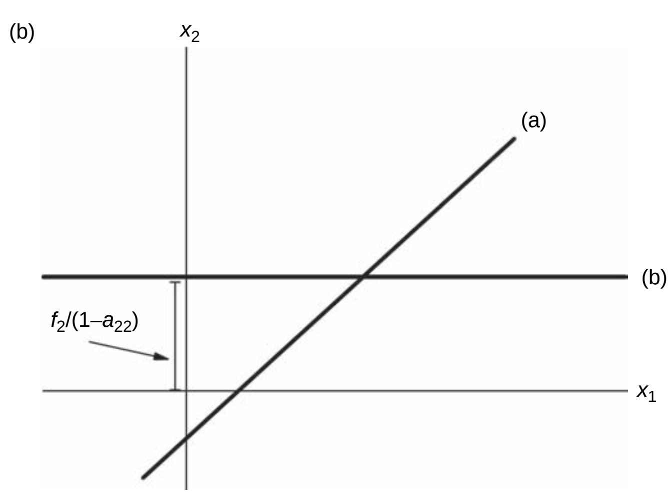

Figure A2.2.1b Solution Space Representation of (A2.2.2); $a_{21} = 0$

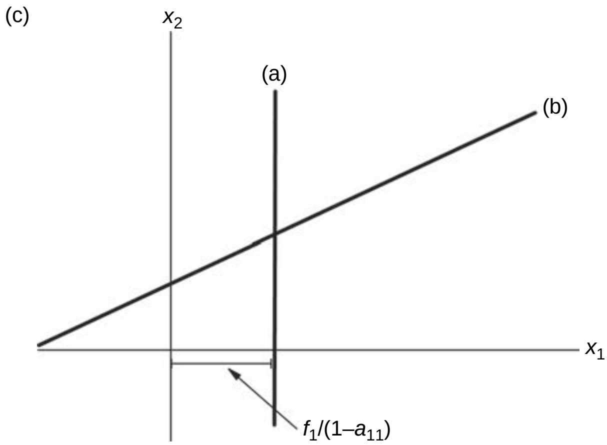

Figure A2.2.1c Solution Space Representation of (A2.2.2); $a_{12} = 0$

matrix, **A**, is defined as the determinant of the $(n-1) \times (n-1)$ matrix remaining when row i and column j are removed from **A** (Appendix A). Another kind of minor that is associated with a matrix (not with a particular element in a matrix) is a *principal minor*. If none, or one, or more than one row *and* the same columns are removed from **A**, the determinant of the remaining square matrix is a principal minor of **A**. Using the concept of principal minors, the Hawkins–Simon conditions for the 2×2 case in (A2.2.1) can be expressed compactly as the requirement that *all principal minors* of $(\mathbf{I} - \mathbf{A})$ be strictly positive – (a) in (A2.2.1) results from removing row and column 1 or row and column 2, (b) in (A2.2.1) results from removing no rows and columns. (It is impossible to remove more than $n-1$ rows and columns; if all n are gone, there is no matrix left.)

For a 3×3 matrix **A**, removal of row and column 1, *or* row and column 2, *or* row and column 3 leaves, in each case, a square 2×2 matrix. The determinants of those three matrices are all principal minors of **A** (sometimes called second-order principal minors, because they are determinants of 2×2 matrices). Moreover, removal of rows and columns 1 and 2, *or* rows and columns 1 and 3, *or* rows and columns 2 and 3 leaves, in each case, a square 1×1 matrix (the determinant of a 1×1 matrix is defined simply as the value of the element itself); these are the three first-order principal minors of **A**. By extension, the third-order principal minor in this case is just the determinant of the entire 3×3 matrix, when no rows and columns are removed. Thus there are seven principal minors in a 3×3 matrix.

This principal minor rule can be generalized; namely, regardless of the size of n, the parallel to (A2.2.1) is that *all* principal minors of $(\mathbf{I}-\mathbf{A})$ – first-order, second-order, . . ., nth-order – should be positive. The interested reader might try writing out the seven principal minors of a 3×3 $(\mathbf{I}-\mathbf{A})$ matrix. In the 4×4 case there are 15 principal minors. (For the reader familiar with the mathematics of combinations, this number is found as $C_0^4 + C_1^4 + C_2^4 + C_3^4 = 1 + 4 + 6 + 4 = 15$.) This gives some idea of the way in which the complexity of these rules increases with the number of sectors in the input–output model, and extension and application of the results in (A2.2.1) to conditions for an $n \times n$ system with n even modestly large would be cumbersome and tedious, even though the definition of principal minors of a matrix presents a simple way of expressing the rule for the general case. These conditions are totally impractical to check for large, real-world input–output systems. [For example, for a 10-sector model, the number of principal minors is 1023 (!).]

However, there is a large amount of published work on alternative sets of conditions on $\mathbf{A}$ and $\mathbf{f}$ that serve to identify when non-negative final demands will generate non-negative outputs. Dietzenbacher (2005) provides an extremely simple sufficient condition. If the original data are $\mathbf{Z}^0 > \mathbf{0}$ and $\mathbf{f}^0 \geq \mathbf{0}$ (with at least one $f_i^0 > 0$), then $\mathbf{L}^0 = (\mathbf{I} - \mathbf{A}^0)^{-1} > \mathbf{0}$ and $\mathbf{x}^1 = \mathbf{L}^0\mathbf{f}^1 \geq \mathbf{0}$ for any $\mathbf{f}^1 \geq \mathbf{0}$. These requirements on $\mathbf{Z}^0$ and $\mathbf{f}^0$ are easily checked by inspection, bypassing the need for the Hawkins–Simon principal minors. In fact, as noted in Dietzenbacher (2005), the positivity condition, $\mathbf{Z}^0 > \mathbf{0}$, can be relaxed to the requirement of non-negativity, $\mathbf{Z}^0 \geq \mathbf{0}$, using an assumption that allows $\mathbf{Z}^0$ to contain many zeros.[26] This allows for the more realistic case, especially in highly disaggregated models, of zero-valued intermediate flows between some sectors. An additional benefit is that derivation of these results does not depend on $a_{ij}^0 < 1$. When tables are based on transactions measured in physical terms it is entirely possible that some coefficients will be larger than 1 and hence that $N(\mathbf{A}) > 1$ – as we saw in section 2.6, above.

Problems

2.1 Dollar values of last year's interindustry transactions and total outputs for a two-sector economy (agriculture and manufacturing) are as shown below:

$$\mathbf{Z} = \begin{bmatrix} 500 & 350 \\ 320 & 360 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1000 \\ 800 \end{bmatrix}$$

a. What are the two elements in the final-demand vector $\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$?

b. Suppose that f_1 increases by \$50 and f_2 decreases by \$20. What new gross outputs would be necessary to satisfy the new final demands?

i. Find an approximation to the answer by using the first five terms in the power series, $\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^n$.

ii. Find the exact answer using the Leontief inverse.

[26] Both the assumption and the further analysis are considerably more complex and beyond the scope of this text – involving, for example, Frobenius theorems, indecomposable (irreducible) matrices, eigenvectors, and eigenvalues. The interested reader is referred to the thorough discussion of these and other mathematical issues in input–output analysis in Takayama (1985, Chapter 4).

2.2 Interindustry sales and total outputs in a small three-sector national economy for year t are given in the following table, where values are shown in thousands of dollars. (S_1, S_2 and S_3 represent the three sectors.)

	Interindustry Sales			
	S_1	S_2	S_3	Total Output
S_1	350	0	0	1000
S_2	50	250	150	500
S_3	200	150	550	1000

a. Find the technical coefficients matrix, **A**, and the Leontief inverse matrix, **L**, for this economy.
b. Suppose that because of government tax policy changes, final demands for the outputs of sectors 1, 2 and 3 are projected for next year (year $t+1$) to be 1300, 100 and 200, respectively (also measured in thousands of dollars). Find the total outputs that would be necessary from the three sectors to meet this projected demand, assuming that there is no change in the technological structure of the economy (that is, assuming that the **A** matrix does not change from year t to year $t+1$).
c. Find the original (year t) final demands from the information in the table of data. Compare with the projected (year $t+1$) final demands. Also, compare the original total outputs with the outputs found in part b. What basic feature of the input–output model do these two comparisons illustrate?

2.3 Using the data of Problem 2.1, above, suppose that the household (consumption) expenditures part of final demand is \$90 from sector 1 and \$50 from sector 2. Suppose, further, that payments from sectors 1 and 2 for household labor services were \$100 and \$60, respectively; that total household (labor) income in the economy was \$300; and that household purchases of labor services were \$40. Close the model with respect to households and find the impacts on sectors 1 and 2 of a final demand of \$200 and \$1000 for sectors 1 and 2, respectively, using the Leontief inverse for the new 3×3 coefficient matrix. Compare the outputs of sectors 1 and 2 with those obtained without closing the model to households. How do you explain the differences?

2.4 Consider an economy organized into three industries: lumber and wood products, paper and allied products, and machinery and transportation equipment. A consulting firm estimates that last year the lumber industry had an output valued at \$50 (assume all monetary values are in units of \$100,000), 5 percent of which it consumed itself; 70 percent was consumed by final demand; 20 percent by the paper and allied products industry; 5 percent by the equipment industry. The equipment industry consumed 15 percent of its own products, out of a total of \$100; 25 percent went to final demand; 30 percent to the lumber industry; 30 percent to the paper and allied products industry. Finally, the paper and allied products industry produced \$50, of which it consumed

10 percent; 80 percent went to final demand; 5 percent went to the lumber industry; and 5 percent to the equipment industry.

a. Construct the input–output transactions matrix for this economy on the basis of these estimates from last year's data. Find the corresponding matrix of technical coefficients, and show that the Hawkins–Simon conditions are satisfied.
b. Find the Leontief inverse for this economy.
c. A recession in the economy this year is reflected in decreased final demands, reflected in the following table:

Industry	% Decrease in Final Demand
Lumber & Wood Products	25
Machinery & Transportation Equipment	10
Paper & Allied Products	5

d. What would be the total production of all industries required to supply this year's decreased final demand? Compute the value-added and intermediate output vectors for the new transactions table.

2.5 Consider a simple two-sector economy containing industries A and B. Industry A requires \$2 million worth of its own product and \$6 million worth of Industry B's output in the process of supplying \$20 million worth of its own product to final consumers. Similarly, Industry B requires \$4 million worth of its own product and \$8 million worth of Industry A's output in the process of supplying \$20 million worth of its own product to final consumers.

a. Construct the input–output transactions table describing economic activity in this economy.
b. Find the corresponding matrix of technical coefficients and show that the Hawkins–Simon conditions are satisfied.
c. If in the year following the one in which the data for this model was compiled there were no changes expected in the patterns of industry consumption, and if a final demand of \$15 million worth of good A and \$18 million worth of good B were presented to the economy, what would be the total production of all industries required to supply this final demand as well as the interindustry activity involved in supporting deliveries to this final demand?

2.6 Consider the following transactions table, $\mathbf{Z}$, and total outputs vector, $\mathbf{x}$, for two sectors, A and B:

$$\mathbf{Z} = \begin{bmatrix} 6 & 2 \\ 4 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 20 \\ 15 \end{bmatrix}$$

a. Compute the value-added and final-demand vectors. Show that the Hawkins–Simon conditions are satisfied.

b. Consider the r-order round-by-round approximation of $\mathbf{x} = \mathbf{Lf}$ to be $\tilde{\mathbf{x}} = \sum_{i=0}^{r} A^i f$ (remember that $\mathbf{A}^0 = \mathbf{I}$). For what value of r do all the elements of $\tilde{\mathbf{x}}$ come within 0.2 of the actual values of $\mathbf{x}$?

c. Assume that the cost of performing impact analysis on the computer using the round-by-round method is given by $C_r = c_1 r + c_2(r - 1.5)$ where r is the order of the approximation (c_1 is the cost of an addition operation and c_2 is the cost of a multiplication operation). Also, assume that $c_1 = 0.5c_2$, that the cost of computing $(\mathbf{I}-\mathbf{A})^{-1}$ exactly is given by $C_e = 20c_2$ and the cost of using this inverse in impact analysis (multiplying it by a final-demand vector) is given by $C_f = c_2$. If we wish to compute the impacts (total outputs) of a particular (arbitrary) final-demand vector to within at least 0.2 of the actual values of $\mathbf{x} = \mathbf{Lf}_a$, where $\mathbf{f}_a$ is an arbitrary final-demand vector, should we use the round-by-round method or should we compute the exact inverse and then perform impact analysis? The idea is to find the least-cost method for computing the solution.

d. Suppose we had five arbitrary final-demand vectors whose impact we wanted to assess. How would you now answer part c?

e. For what number of final-demand vectors does it not make any difference which method we use (in answer to the question in part c)?

2.7 Given the following transactions table for industries a, b, and c, and the total output as shown, compute the final-demand vectors and show that the inverse of $(\mathbf{I} - \mathbf{A})$ exists.

Industries	a	b	c	Total Output
a	3	8	6	22
b	2	4	5	18
c	7	3	9	31

Use the power series to approximate $\mathbf{x}$ to within 0.1 of the actual output values shown above. What was the highest power of $\mathbf{A}$ required?

2.8 Consider the following transactions and total output data for an eight-sector economy.

$$\mathbf{Z} = \begin{bmatrix} 8,565 & 8,069 & 8,843 & 3,045 & 1,124 & 276 & 230 & 3,464 \\ 1,505 & 6,996 & 6,895 & 3,530 & 3,383 & 365 & 219 & 2,946 \\ 98 & 39 & 5 & 429 & 5,694 & 7 & 376 & 327 \\ 999 & 1,048 & 120 & 9,143 & 4,460 & 228 & 210 & 2,226 \\ 4,373 & 4,488 & 8,325 & 2,729 & 29,671 & 1,733 & 5,757 & 14,756 \\ 2,150 & 36 & 640 & 1,234 & 165 & 821 & 90 & 6,717 \\ 506 & 7 & 180 & 0 & 2,352 & 0 & 18,091 & 26,529 \\ 5,315 & 1,895 & 2,993 & 1,071 & 13,941 & 434 & 6,096 & 46,338 \end{bmatrix}$$

$$\mathbf{x}' = \begin{bmatrix} 37,610 & 45,108 & 46,323 & 41,059 & 209,403 & 11,200 & 55,992 & 161,079 \end{bmatrix}$$

a. Compute $\mathbf{A}$ and $\mathbf{L}$.
b. If final demands in sectors 1 and 2 increase by 30 percent while that in sector 5 decreases by 20 percent (while all other final demands are unchanged), what new total outputs will be necessary from each of the eight sectors in this economy?

2.9 Consider the following two-sector input–output table measured in millions of dollars:

	Manuf.	Services	Final Demand	Total Output
Manufacturing	10	40	50	100
Services	30	25	85	140
Value Added	60	75	135	
Total Output	100	140		240

If labor costs in the services sector increase, causing a 25 percent increase in value added inputs required per unit of services and labor costs in manufacturing decrease by 25 percent, what are the resulting changes in relative prices of manufactured goods and services?

2.10 For the US direct requirements table given in Table 2.7, what would be the impact on relative prices if a national corporate income tax increased total value added of primary industries (agriculture and mining) by 10 percent, construction and manufacturing by 15 percent, and all other sectors by 20 percent?

2.11 Consider an input–output economy with three sectors: agriculture, services, and personal computers. The matrix of interindustry transactions and vector of total outputs are given, respectively, by $\mathbf{Z} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$ so that $\mathbf{f} = \mathbf{x} - \mathbf{Zi} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. Notice that this is a closed economy where all industry outputs become inputs. In other words, with the given $\mathbf{x}$, the vector of total value added is found by $\mathbf{v}' = \mathbf{x}' - \mathbf{i}'\mathbf{Z} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$ and, of course, gross domestic product is $\mathbf{v}'\mathbf{i} = \mathbf{i}'\mathbf{f} = 0$. Does $\mathbf{L}$ exist for this economy? Suppose we determine that all of the inputs for the personal computers sector are imported and we seek to create a domestic transactions matrix by "opening" the economy to imports, i.e., transfer the value of all inputs to personal computers to final demand. What are the modified values of $\mathbf{Z}$, $\mathbf{f}$ and $\mathbf{v}$? What is the new value of gross domestic product? Does $\mathbf{L}$ exist for this modified representation of the economy? If so, compute it.

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3 Input–Output Models at the Regional Level

3.1 Introduction

Originally, applications of the input–output model were carried out at national levels – for example, to assess the impact on the individual sectors of the US economy of a change from war to peacetime production as the end of World War II approached. Over time, interest in economic analysis at the regional level – whether for a group of states (as in a federal reserve district), an individual state, a county or a metropolitan area – has led to modifications of the input–output model which attempt to reflect the peculiarities of a regional (subnational) problem. There are at least two basic features of a regional economy that influence the characteristics of a regional input–output study.

First, although the data in a national input–output coefficients table are obviously some kind of averages of data from individual producers located in specific regions, the structure of production in a particular region may be identical to or it may differ markedly from that recorded in the national input–output table. Soft drinks of a particular brand that are bottled in Boston probably incorporate basically the same ingredients in the same proportions as are present in that brand of soft drink produced in Kansas City or Atlanta or in any other bottling plant in the United States. On the other hand, electricity produced in eastern Washington by water power (Coulee Dam) represents quite a different mix of inputs from electricity that is produced from coal in Pennsylvania or by means of nuclear power or "wind farms" elsewhere. For these reasons, the early methodology for regional input–output applications – which used national input coefficients with some minor modifications – has given way to coefficients tables that are tailored to a particular region on the basis of data specific to that region.

Secondly, it is generally true that the smaller the economic area, the more dependent that area's economy is on trade with "outside" areas – transactions that cross the region's borders – both for sales of regional outputs and purchases of inputs needed for production. That is, one of the elements that contributed to the exogenous final-demand sector in the model described in Chapter 2 – exports – now will generally be relatively much more important and a higher proportion of inputs will be imported from producers located outside of the region. To exaggerate, a one-world economy would have no "foreign trade," since all sales and purchases would be internal to the worldwide "region," whereas an urban area depends very much on imports and exports (imports of components to aircraft production and exports of Boeing airliners from the Seattle area).

In this chapter we will explore some of the attempts that have been made to incorporate these features of a regional economy into an input–output framework. Such regional input–output models may deal with a single region or with two or more regions and their interconnections. The several-region case is termed *interregional input–output analysis* (in one version) or *multiregional input–output analysis* (in another version). We will examine each of these kinds of regionalized input–output models, as well as what is known as the *balanced regional* model.

There has been an enormous amount of input–output work at the regional level. Examples of some of the earliest single-region applications are found in Moore and Petersen (1955), Isard and Kuenne (1953), Miller (1957), and Hirsch (1959). A very thorough discussion and documentation of the details involved in producing a regional input–output table during the early period in the development of this area of application is provided by Isard and Langford (1971) – in this case the region was the Philadelphia Standard Metropolitan Statistical Area – and in Miernyk *et al.* (1967) for Boulder, Colorado, and Miernyk *et al.* (1970) for West Virginia. Overviews of early regional input–output models can be found in Polenske (1980, Chapter 3) and in Miernyk (1982). For an idea of the large amount of continuing work in this area, the reader is referred to annual indexes in such journals as *Economic Systems Research, Journal of Regional Science, International Regional Science Review* and *Papers in Regional Science*.[1] In addition, many regional input–output tables and studies using these tables have been published by the appropriate sub-national agencies (state and local governments or their counterparts outside the USA) for whom the analysis was done, or by universities where the work was done.

In section 3.6 we indicate some examples of how the geographic scale of connected-region models has evolved in both the micro- and macroscopic directions from these earliest applications – down to models of as small an area as an inner-city neighborhood and up to what are often referred to as "world" models, encompassing several blocs of mega-nations. Examples of regional applications will also be discussed in Chapter 6 on multipliers and in Chapter 8 on estimating regional data. Much of the material on regional and interregional input–output models in this chapter and several chapters later in this book is covered (in less detail) in Miller (1998).

3.2 Single-Region Models

3.2.1 National Coefficients

Generally, regional input–output studies attempt to quantify the impacts on the producing sectors located in a particular region that are caused by new final demands for

[1] Other relevant journals include *Environment and Planning A*, *Annals of Regional Science*, *Regional Studies*, *Growth and Change*, *Urban Studies*, *Land Economics*, *Regional Science and Urban Economics*, *Regional Science Perspectives*, and *Economic Geography*.

products made in the region. Early regional studies (Isard and Kuenne, 1953; Miller, 1957) used a national table of technical coefficients in conjunction with an adjustment procedure that was designed to capture some of the characteristics of the regional economies, since specific coefficients tables for the particular regions did not exist.[2]

We use a superscript r to designate "region r" in the same way that subscript i denoted "sector i" in the discussion in Chapter 2. Thus, just as x_i was used to denote the gross output of sector i, let $\mathbf{x}^r = [x_i^r]$ denote the vector of gross outputs of sectors in region r. Similarly, $\mathbf{f}^r = [f_i^r]$ represents the vector of exogenous demands for goods made in region r. For example, if r denotes Washington State, one element of $\mathbf{f}^r$ could be an order from a foreign airline for commercial aircraft from Boeing in Washington.

The problem in these early regional studies was that only a national technical coefficients matrix, $\mathbf{A}$, was available, but what was needed, essentially, was a matrix showing inputs from firms *in the region* to production *in that region*. Denote this unknown matrix by $\mathbf{A}^{rr} = [a_{ij}^{rr}]$, where a_{ij}^{rr} is the amount of input from sector i in r per dollar's worth of output of sector j in r. (This anticipates notation later for many-region models, where we will need two superscripts to identify origin and destination regions, just as i and j are origin and destination sectors.) Assume, in the absence of evidence to the contrary, that local producers use the same production recipes as are shown in the national coefficients table, meaning that the *technology* of production in each sector in region r is the same as in the nation as a whole. Nonetheless, in order to translate regional final demands into outputs of *regional* firms ($\mathbf{x}^r$), the national coefficients matrix must be modified to produce $\mathbf{A}^{rr}$ (*locally produced* goods in local production).

Early studies carried out this modification through the use of estimated *regional supply percentages*, one for each sector in the regional economy, designed to show the percentage of the total required outputs from each sector that could be expected to originate within the region. One straightforward way to estimate these percentages, using data that may often be obtainable at the regional level, requires knowledge of (1) total regional output of each sector i, x_i^r, (2) exports of the product of each sector i from region r, e_i^r, and (3) imports of good i into region r, m_i^r. Then, one can form an expression for the *proportion* of the total amount of good i available in region r that was produced in r (the *regional supply proportion* of good i). We denote this by p_i^r, where

$$p_i^r = \frac{(x_i^r - e_i^r)}{(x_i^r - e_i^r + m_i^r)}$$

The numerator is the *locally produced* amount of i that is available to purchasers in r; the denominator is the *total* amount of i available in r, either produced locally or imported. (Thus $p_i^r \times 100$ is an estimate of the regional supply percentage for sector i in region r – the percentage of good i available in r that was produced there.)

Assuming that we can estimate such proportions for each sector in the economy, each element in the ith row of the national coefficients matrix could be multiplied by

[2] The "regions" were the Greater New York–Philadelphia urban-industrial region (consisting of 2 counties in Connecticut, 11 in New York, 19 in New Jersey, and 5 in Pennsylvania) in the first case and the states of Washington, Oregon, and Idaho in the second.

p_i^r to generate a row of locally produced direct input coefficients of good i to each local producer. If we arrange these proportions in an n-element column vector, $\mathbf{p}^r$, then our working estimate of the regional matrix will be $\mathbf{A}^{rr} = \hat{\mathbf{p}}^r \mathbf{A}$. For a two-sector model, this is

$$\mathbf{A}^{rr} = \hat{\mathbf{p}}^r \mathbf{A} = \begin{bmatrix} p_1^r & 0 \\ 0 & p_2^r \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} p_1^r a_{11} & p_1^r a_{12} \\ p_2^r a_{21} & p_2^r a_{22} \end{bmatrix}$$

For any $\mathbf{f}^r$ we could then find $\mathbf{x}^r = (\mathbf{I} - \hat{\mathbf{p}}^r \mathbf{A})^{-1} \mathbf{f}^r$. This uniform modification of the elements in a row of $\mathbf{A}$ is a strong assumption. It means, for example, that if the aircraft, kitchen equipment, and pleasure boat sectors in Washington all use aluminum (sector i) as an input, all three sectors buy the same percentage, p_i^r, of their total aluminum needs from firms located within the state.

In the two-sector example in Chapter 2 we had $\mathbf{A} = \begin{bmatrix} .15 & .25 \\ .20 & .05 \end{bmatrix}$. Assume that this is a national table, and that we want to create $\mathbf{A}^{rr}$ from it, and that there is no evidence that the basic structure of production in the region differs from the national average structure reflected in $\mathbf{A}$. The unique features of the region, however, are captured in the regional supply percentages. Using regional output, export and import data, suppose we estimate that 80 percent of sector 1 goods will come from firms in that sector within the region, but only 60 percent of sector 2 goods can be expected to be supplied by regional firms in sector 2, so $\mathbf{p}^r = \begin{bmatrix} .8 \\ .6 \end{bmatrix}$. Suppose that the projected (new) final demand in the region is $\mathbf{f}^r = \begin{bmatrix} 600 \\ 1500 \end{bmatrix}$ (this is the final demand vector that was used for some of the numerical examples in Chapter 2). Then

$$\hat{\mathbf{p}}^r = \begin{bmatrix} .8 & 0 \\ 0 & .6 \end{bmatrix}, \ \mathbf{A}^{rr} = \hat{\mathbf{p}}^r \mathbf{A} = \begin{bmatrix} .8 & 0 \\ 0 & .6 \end{bmatrix} \begin{bmatrix} .15 & .25 \\ .20 & .05 \end{bmatrix} = \begin{bmatrix} .12 & .20 \\ .12 & .03 \end{bmatrix},$$

$$(\mathbf{I} - \mathbf{A}^{rr})^{-1} = \begin{bmatrix} 1.169 & 0.241 \\ 0.145 & 1.061 \end{bmatrix}$$

and using this regional inverse directly,

$$\mathbf{x}^r = (\mathbf{I} - \mathbf{A}^{rr})^{-1} \mathbf{f}^r = \begin{bmatrix} 1.169 & 0.241 \\ 0.145 & 1.061 \end{bmatrix} \begin{bmatrix} 600 \\ 1500 \end{bmatrix} = \begin{bmatrix} 1062.90 \\ 1678.50 \end{bmatrix} \tag{3.1}$$

This tells us that the total output that will need to be produced in the region by sectors 1 and 2 is \$1062.90 and \$1678.50, respectively.

In more recent regional input–output analyses, attempts have been made to model the characteristics of a regional economy more precisely. We examine these briefly in the following section, and we return to the "regionalization" problem in Chapter 8.

3.2.2 Regional Coefficients

We noted above that electricity produced in Washington will most likely have a different production recipe (column of technical coefficients) from electricity made in

Pennsylvania. These regionally produced electricities are really two different products – "hydroelectric power" and "coal-fired electrical power." As another example, consider the aircraft sector. In a national table, this would include the manufacture of a mix of commercial, business, and personal aircraft. One input to this sector would be the huge jet engines used on Boeing commercial airliners. On the other hand, the aircraft sector in a regional table for the state of Florida might reflect the manufacture of small personal aircraft, for which the jumbo jet engines are not an input at all; in a Washington table, however, jet engines are an extremely important input.

Sectors in even very disaggregated national input–output tables will be made up of a variety of products – as in the aircraft sector example. And firms within that sector, located in various regions of the country, will generally produce only a small number of those products – Boeing in Washington does not produce small propeller-driven airplanes; Piper in Florida does not produce jet airliners that can carry upwards of 300 passengers. This illustrates the so-called *product-mix* problem in input–output; firms classified in the same sector actually produce different sets of products. The most straightforward way to avoid this problem is to survey firms in the region and construct what is called a survey-based regional input–output table. In conducting such a survey, one can pose essentially two variants of the basic question. In asking firms in sector j in a particular region about their use of various inputs, the question can be:

1. How much sector i product did you buy last year in making your output? (For example, how much aluminum did aircraft manufacturers in Washington State buy last year?), or
2. How much sector i product did you buy last year from firms located in the region? (For example, how much aluminum used by aircraft producers in Washington was purchased from producers in Washington?)[3]

In the former case a truly *regional technical coefficients* table would be produced; this would better reflect production practices in the region than does the national table – it would eliminate the input of large jet engines into the manufacture of private aircraft in Florida, for example. But it would not address the question of how much of each required input came from within the region and how much was imported. On the other hand, a set of coefficients based on inputs supplied from firms within the region for outputs of firms in the region would reflect regional production technology. These might be termed *regional input coefficients*. They are to be distinguished from regional technical coefficients since they do not always accurately describe the technology of regional firms, but rather only the way in which local firms use local inputs. (*Intraregional input coefficients* would be an even more precise, although cumbersome, description.[4])

Rather than adapt a national coefficients table through application of regional supply proportions, some regional analysts have tried to derive true regional input coefficient

[3] If it is also possible to determine how much came from firms located outside the state then one has the beginnings of an interregional or multiregional model. These are discussed below in sections 3.3 and 3.4.

[4] Tiebout (1969, p. 335) used "direct intraregional interindustry coefficient," which is completely precise but also rather cumbersome.

tables through surveys of regional establishments using variants of question 2. A series of tables for Washington State illustrates this kind of survey-based modeling effort, specifically for the state for 1963, 1967, 1972, 1982, 1987, and 2002. (There is also a Washington table for 1997 produced mainly by a nonsurvey estimating technique; nonsurvey approaches are explored in Chapters 7 and 8. The 1997 and 2002 tables are available at www.ofm.wa.gov/economy/io.) The Washington tables can be found in Bourque and Weeks (1969), Beyers *et al.* (1970), Bourque and Conway (1977), Bourque (1987), and Chase, Bourque and Conway (1993). These data have been the basis of many comparative studies.

To examine this kind of extension, we need more complicated notation. We continue to use a superscript r for the region in question. Then let z_{ij}^{rr} denote the dollar flow of goods from sector i in region r to sector j in region r.[5] Just as the order of subscripts is "from–to" with respect to sectors, the order of superscripts indicates "from–to" with respect to geographic locations. If we had a complete set of data on z_{ij}^{rr} for all n sectors in the regional economy, and also data on gross outputs (x_j^r) of each sector in the region, a set of regional input coefficients could be derived as

$$a_{ij}^{rr} = \frac{z_{ij}^{rr}}{x_j^r} \tag{3.2}$$

Let $\underset{(n\times n)}{\mathbf{Z}^{rr}} = [z_{ij}^{rr}]$ and $\underset{(n\times 1)}{\mathbf{x}^r} = [x_j^r]$; then the regional input coefficients matrix is

$$\mathbf{A}^{rr} = \mathbf{Z}^{rr}(\hat{\mathbf{x}}^r)^{-1} \tag{3.3}$$

(This is what was approximated in the early regional studies described above by $\hat{\mathbf{p}}^r\mathbf{A}$.) Then the impacts on *regional* production of a final-demand change in region r would be found as

$$\mathbf{x}^r = (\mathbf{I} - \mathbf{A}^{rr})^{-1}\mathbf{f}^r \tag{3.4}$$

3.2.3 Closing a Regional Model with respect to Households

The Washington State models noted above were closed with respect to households in the manner described in Chapter 2 – by adding a household consumption column and a labor input row. One extension to the process of endogenizing households in an input–output model is to add more than one row and column to the direct input coefficients matrix. This approach is frequently implemented at a regional level, although it can apply equally well to national models. As usual, the impacts of projected increases in final demand will be increased sectoral outputs and therefore increased payments for labor services. The basic idea is that a distinction should be made between consumption habits of various kinds of consumers – for example, at a sub-national level, those of established residents of the region, who may experience an increase in their incomes (for example, due to productivity increases) and the consumption patterns of new residents, who may

[5] We need double superscripts because later we will also measure interindustry flows between regions – as in z_{ij}^{sr}.

move into the region in anticipation of employment (new income). This distinction apparently originated with Tiebout (1969), where they are designated *intensive* and *extensive* income growth, respectively.

The reason for the distinction is that current residents may spend each dollar of new income according to a set of *marginal* consumption coefficients, while new residents may distribute their purchases according to a set of *average* consumption coefficients.

> The presumption should be clear: as new residents move in to fill jobs at the same wage rate as established residents, average consumption propensities are relevant. Insofar as regional income rises because of increased per capita incomes, marginal consumption propensities apply (Tiebout, 1969, p. 336).

If sales, by sector, could be broken down into those to new residents and those to existing residents, and if labor payments, by sector, could be similarly disaggregated, then marginal and average household consumption coefficients could be derived. Similarly, knowing each sector's outputs, "old" and "new" labor inputs per dollar's worth of output could be found. These would form two additional rows and columns with which to close the model.

In practice, such data are not so conveniently available. Tiebout (1969) describes the derivation of extensive and intensive coefficients in a regional model for the state of Washington. Miernyk *et al.* (1967) investigate essentially the same issue for their pioneering Boulder, Colorado, input–output study.[6] In addition, an attempt was made in the Boulder study to disaggregate the income increases to existing residents by income class, with lower marginal consumption propensities in higher income classes. (See Miernyk *et al.* 1967, esp. Chapter V.)

Instead of disaggregating households into "old" and "new" residents, Blackwell (1978) proposes a tripartite division into intensive and extensive (current residents and new residents, respectively, as above) and also redistributive, which is that portion of any new income that goes to previously unemployed local residents. The distinction between currently employed and currently unemployed workers is also explored in some detail by Madden and Batey (1983, and elsewhere).[7] The considerable work of Madden and Batey and their colleagues on "extended" input–output models is representative of a large body of research linking population and economic models. It is summarized in Batey and Madden (1999), which also contains references to a great deal of earlier work by them and by others. Miyazawa (1976) also investigates extensions to multiple categories of consumption spending and income recipients. We further explore various model closures (including the Miyawaza formulation) in Chapter 6 when we investigate input–output multipliers.

[6] Tiebout's contribution in formulating this distinction between extensive and intensive consumption propensities in a region is noted by Miernyk *et al.* (1967, p. 104, n. 9). A draft of Tiebout's paper was completed by 1967 and was published posthumously in 1969, following his death in January, 1968.

[7] Other early examples of "extended" models with households included (by no means an exhaustive list) include Schinnar (1976), Beyers (1980), Gordon and Ledent (1981), Ledent and Gordon (1981), and Joun and Conway (1983). These combined models are sometimes referred to as demo-economic – or also as eco-demographic. The demo-economic components reflect inputs from various labor (household) groups, and the eco-demographic components capture activity such as consumption by various household types.

3.3 Many-Region Models: The Interregional Approach

Single-region models of the sort described in the previous section represent one approach to modeling a regional economy in input–output terms. What they fail to do, however, is to recognize in an operational way the interconnections between regions. The one region of interest (in the above, this was region r) was essentially "disconnected" from the rest of the country within which it is located, in the sense that its production recipes are reflected in an intraregional matrix, $\mathbf{A}^{rr}$. For a country made up of several regions, a number of important questions have several-region implications. Next year's national defense budget might include a large order for a certain type of aircraft built in California, the overhaul of one or more ships in Virginia, and modernization and upgrading of an army base in New Jersey. Each of these activities can be expected to have ramifications not only within the region (state, in this example) where the activity takes place, but also in other states. The total economic effect is therefore likely to be larger than the sum of the regional effects in California, Virginia, and New Jersey. Firms outside California will produce goods that will be imported to California for aircraft production; those firms, in turn, may import goods from other states for *their* production. Materials for ship overhaul will come to Virginia from suppliers outside that state. Electronic parts for the base upgrading in New Jersey may be imported from elsewhere and the electronics firms, in turn, will need both local (wherever they are located) and imported inputs, and so on.

A fundamental problem in many-region input–output modeling is therefore the estimation of the transactions between regions. One approach, the *interregional* model, requires a complete (ideal) set of both intra- and interregional data. For the two-region case, this means knowing $\mathbf{x}^r = [x_i^r]$, $\mathbf{x}^s = [x_i^s]$, $\mathbf{Z}^{rr} = [z_{ij}^{rr}]$ and $\mathbf{Z}^{ss} = [z_{ij}^{ss}]$ along with $\mathbf{Z}^{rs} = [z_{ij}^{rs}]$ – recording transactions from sector i in region r to sector j in region s – and $\mathbf{Z}^{sr} = [z_{ij}^{sr}]$ – in which flows from s to r are captured. It is the last two matrices that cause the most trouble. In practice, it is never the case that one has such detailed information, and the requirements grow quickly with the number of regions – a three-region model has six interregional matrices, a four-region model has 12, and so on.

Alternative forms of many-region input–output models were created and elaborated by members of the Harvard Economic Research Project (HERP) under Leontief's direction, from its inception through the 1960s.[8] Taken chronologically, the interregional input–output model (IRIO) structure was first described by Isard (1951) and elaborated in Isard *et al.* (1960). (This is often labeled the "Isard model".) Leontief *et al.* (1953) sketched the framework of an intranational input–output model (often referred to as a "balanced regional model;" section 3.5, below). This was later applied to assess the sectoral and regional impact of a cut in US arms spending in Leontief *et al*. (1965). The multiregional input–output model (MRIO) was (almost simultaneously) described in Chenery (1953) (a two-region model for Italy) and in Moses (1955) (a nine-region US model) – thus the label "Chenery–Moses model." Finally, Leontief and Strout (1963)

[8] HERP was started at Harvard by Leontief in 1948 and continued until 1972. Thorough accounts of this formative work can be found in Polenske (1995, 2004).

Table 3.1 Interindustry, Interregional Flows of Goods

		Purchasing Sector				
		Region r			Region s	
Selling Sector		1	2	3	1	2
Region r	1	z_{11}^{rr}	z_{12}^{rr}	z_{13}^{rr}	z_{11}^{rs}	z_{12}^{rs}
	2	z_{21}^{rr}	z_{22}^{rr}	z_{23}^{rr}	z_{21}^{rs}	z_{22}^{rs}
	3	z_{31}^{rr}	z_{32}^{rr}	z_{33}^{rr}	z_{31}^{rs}	z_{32}^{rs}
Region s	1	z_{11}^{sr}	z_{12}^{sr}	z_{13}^{sr}	z_{11}^{ss}	z_{12}^{ss}
	2	z_{21}^{sr}	z_{22}^{sr}	z_{23}^{sr}	z_{21}^{ss}	z_{22}^{ss}

proposed a gravity-model approach to estimation of interregional flows in a connected-region input–output model.[9] In this section we explore the interregional input–output (IRIO) model.

3.3.1 Basic Structure of Two-Region Interregional Input–Output Models

For purposes of illustration, we consider a two-region economy (for example, in Italy, northern Italy and southern Italy; or, in the United States, New England and the rest of the United States). Using r and s, as before, for the two regions, let there be three producing sectors (1, 2, 3) in region r and two (1, 2) in region s. Suppose that one has information for region r on both *intra*regional flows, z_{ij}^{rr}, and *inter*regional flows, z_{ij}^{sr}. There will be nine of the former and six of the latter. Suppose, further, that the same kind of information is available (perhaps through a survey) on the use of inputs by firms located in region s, z_{ij}^{rs} and z_{ij}^{ss}. This complete table of intraregional and interregional data can be represented as

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}^{rr} & \mathbf{Z}^{rs} \\ \mathbf{Z}^{sr} & \mathbf{Z}^{ss} \end{bmatrix}$$

Table 3.1 indicates the full set of data.[10]

In the regional models of section 3.2, we utilized intraregional information only – as in (3.2), (3.3), and (3.4). We now want to incorporate much more explicitly the interregional linkages, as represented by information in $\mathbf{Z}^{rs}$ and $\mathbf{Z}^{sr}$.

[9] Isard *et al.* (1960, esp. Chapter 11) described gravity models and explored their potential for estimating interregional interactions (including commodity flows) in detail. We explore the gravity approach and others in section 8.6, below, on estimating interregional flows.

[10] To be more consistent with already-familiar subscript notation, one could denote the regions by 1 and 2, respectively. Then an element such as z_{13}^{sr} would be denoted z_{13}^{21}. However, for purposes of exposition it seems clearer to use lowercase letters to designate regions; for example, so as to avoid having z's with four different numbers attached to them.

These off-diagonal matrices need not be square. Here $\mathbf{Z}^{rs}$ has dimensions 3×2 and $\mathbf{Z}^{sr}$ is a 2×3 matrix. The on-diagonal matrices are always square; for this example, $\mathbf{Z}^{rr}$ and $\mathbf{Z}^{ss}$ are 3×3 and 2×2, respectively. While the elements in $\mathbf{Z}^{rs}$ represent "exports" from region r and simultaneously "imports" to region s, it is usual in regional input–output work to refer to these as *interregional trade* (or simply *trade*) flows and to use the terms *export* and *import* when dealing with foreign trade that crosses national, not just regional, boundaries.

By surveying firms in both regions on their purchases of locally produced inputs and inputs from the other region, one would accumulate the data shown in the various *columns* of Table 3.1. On the other hand, the data in Table 3.1 could also be gathered by asking firms in each region how much they sold to each sector in their region and how much they sold to sectors in the other region. This would generate the figures shown in the various *rows* of Table 3.1.[11]

Consider again the basic equation for the distribution of sector i's product, as given in equation (2.1) of Chapter 2:

$$x_i = z_{i1} + z_{i2} + \cdots + z_{ij} + \cdots + z_{in} + f_i$$

One of the components recorded in the final-demand term was exports of sector i goods. In the two-region interregional input–output model, that part of f_i that represents sales of sector i's product to the productive sectors in the other region (but not to consumers in the other region) is removed from the final-demand category and specified explicitly. For our two-region example, the output of sector 1 in region r would be expressed as

$$x_1^r = \underbrace{z_{11}^{rr} + z_{12}^{rr} + z_{13}^{rr}}_{\substack{\text{Sector 1 intraregional,} \\ \text{interindustry sales}}} + \underbrace{z_{11}^{rs} + z_{12}^{rs}}_{\substack{\text{Sector 1 interregional,} \\ \text{interindustry sales}}} + \underbrace{f_1^r}_{\substack{\text{Sector 1 intraregional} \\ \text{sales to final demand}}} \tag{3.5}$$

There will be similar equations for x_2^r and x_3^r, and also for x_1^s and x_2^s. The regional input coefficients for region r were given in (3.2). There will also be a set for region s,

$$a_{ij}^{ss} = \frac{z_{ij}^{ss}}{x_j^s} \tag{3.6}$$

Interregional trade coefficients are found in the same manner, where the denominators are gross outputs of sectors in the receiving region. Here these are

$$a_{ij}^{rs} = \frac{z_{ij}^{rs}}{x_j^s} \text{ and } a_{ij}^{sr} = \frac{z_{ij}^{sr}}{x_j^r} \tag{3.7}$$

Using these regional input and trade coefficients, (3.5) can be re-expressed as

$$x_1^r = a_{11}^{rr}x_1^r + a_{12}^{rr}x_2^r + a_{13}^{rr}x_3^r + a_{11}^{rs}x_1^s + a_{12}^{rs}x_2^s + f_1^r \tag{3.8}$$

[11] Usually, one has some (not complete) information on purchases and also some (not complete) information on sales. The problem then is to produce a table from possibly inconsistent data. This reconciliation problem is discussed in section 8.9.

Again, there will be similar expressions for x_2^r, x_3^r, x_1^s, and x_2^s. [Compare the equations (2.4) in Chapter 2, where there was no regional dimension – no superscripts r and s – and where there were n sectors.] Following the same development as in Chapter 2, by moving all terms involving $\mathbf{x}^r$ or $\mathbf{x}^s$ to the left (3.8) becomes

$$(1 - a_{11}^{rr})x_1^r - a_{12}^{rr}x_2^r - a_{13}^{rr}x_3^r - a_{11}^{rs}x_1^s - a_{12}^{rs}x_2^s = f_1^r \tag{3.9}$$

There are similar equations with f_2^r, f_3^r, f_1^s, and f_2^s on the right-hand sides.

For the present example, $\mathbf{A}^{rr}$ [(3.3)] is

$$\mathbf{A}^{rr} = \begin{bmatrix} a_{11}^{rr} & a_{12}^{rr} & a_{13}^{rr} \\ a_{21}^{rr} & a_{22}^{rr} & a_{23}^{rr} \\ a_{31}^{rr} & a_{32}^{rr} & a_{33}^{rr} \end{bmatrix}$$

Also, for this example, $\mathbf{A}^{ss} = \mathbf{Z}^{ss}(\hat{\mathbf{x}}^s)^{-1}$, and the two trade coefficients matrices are $\mathbf{A}^{rs} = \mathbf{Z}^{rs}(\hat{\mathbf{x}}^s)^{-1}$ and $\mathbf{A}^{sr} = \mathbf{Z}^{sr}(\hat{\mathbf{x}}^r)^{-1}$. Using these four matrices, the five equations of which (3.9) is the first can be represented compactly as

$$\begin{aligned} (\mathbf{I} - \mathbf{A}^{rr})\mathbf{x}^r \; - \; \mathbf{A}^{rs}\mathbf{x}^s &= \mathbf{f}^r \\ - \mathbf{A}^{sr}\mathbf{x}^r \; + (\mathbf{I} - \mathbf{A}^{ss})\mathbf{x}^s &= \mathbf{f}^s \end{aligned} \tag{3.10}$$

where $\mathbf{f}^r$ is the three-element vector of final demands for region r goods, and $\mathbf{f}^s$ is the two-element vector of final demands for region s goods.

We define the complete coefficients matrix for a two-region interregional model as consisting of the four submatrices

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^{rr} & \mathbf{A}^{rs} \\ \mathbf{A}^{sr} & \mathbf{A}^{ss} \end{bmatrix}$$

For the current example, this will be a 5×5 matrix. Similarly, let

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}^r \\ \mathbf{x}^s \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} \mathbf{f}^r \\ \mathbf{f}^s \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} \underset{(3\times3)}{\mathbf{I}} & \underset{(3\times2)}{\mathbf{0}} \\ \underset{(2\times3)}{\mathbf{0}} & \underset{(2\times2)}{\mathbf{I}} \end{bmatrix}$$

Then (3.10) can be expressed as

$$(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{f} \tag{3.11}$$

as in (2.10) in Chapter 2. To highlight the structure of (3.11), it can be expressed less compactly as

$$\left\{ \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} - \begin{bmatrix} \mathbf{A}^{rr} & \mathbf{A}^{rs} \\ \mathbf{A}^{sr} & \mathbf{A}^{ss} \end{bmatrix} \right\} \begin{bmatrix} \mathbf{x}^r \\ \mathbf{x}^s \end{bmatrix} = \begin{bmatrix} \mathbf{f}^r \\ \mathbf{f}^s \end{bmatrix} \tag{3.12}$$

Note that in using an interregional model of this kind for analysis, not only is stability of the (intra)regional input coefficients necessary (the elements of $\mathbf{A}^{rr}$ and $\mathbf{A}^{ss}$), but also interregional input coefficients in $\mathbf{A}^{rs}$ and $\mathbf{A}^{sr}$ are assumed unvarying over time. Thus both the structure of production in each region and interregional trade patterns are "frozen" in the model. For a given level of final demands in either or both regions, the necessary gross outputs in both regions can be found in the usual input–output fashion as $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$. As is clear from (3.12), this complete $(\mathbf{I} - \mathbf{A})$ matrix will be larger than that for the single-region model – if both regions are divided into n sectors, the single-region matrix would be of size $n \times n$ and the full two-region interregional model would be $2n \times 2n$, which means four times as many (possible) elements of information are needed (many of which may be zero, of course). However, aside from these dimensionality effects, the analysis proceeds along similar lines.

The advantage is that the model captures the magnitude of effects on each sector in each region; interregional linkages are made specific by sector in the supplying region and by sector in the receiving region. The accompanying disadvantages are primarily the greatly increased data needs and the necessary assumptions of constancy of interregional trading relationships. If it is not always easy to accept the idea of constant input coefficients in general, in the national input–output model, it may be even more difficult to believe that imports of good i per dollar's worth of sector j output in a specific region remain constant, no matter how much sector j's output changes.

3.3.2 Interregional Feedbacks in the Two-Region Model

Consider an increase in the demand by a foreign airline for commercial aircraft produced in Washington State (region r). Certain subassemblies and parts will be purchased from sectors outside the region (for example, jet engines from Connecticut, region s). This stimulus of new output in Connecticut because of new output in Washington is often called an interregional spillover. The increased demand for aircraft will increase the demand for engines and consequently for all of the direct and indirect inputs to the manufacture of jet engines, one of which might be extruded aluminum components made in Washington. This idea is illustrated in Figure 3.1.

The downward arrow connecting Washington output to Connecticut output represents an interregional spillover effect; the upward arrow from Connecticut to Washington

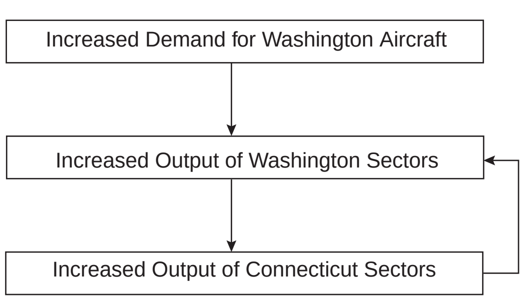

Figure 3.1 Increases in Washington Final Demands Affecting Washington Outputs via Connecticut

is also an interregional spillover – the first originates in Washington ($r \rightarrow s$), the second originates in Connecticut ($s \rightarrow r$). The loop (two arrows) connecting Washington output to itself, via Connecticut output, represents an interregional feedback effect ($r \rightarrow r$); in other words, Washington needs more inputs from Connecticut and therefore Connecticut needs more inputs from everywhere, including Washington. The interregional model in its two-matrix-equation form [in (3.10)] allows one to isolate exactly the magnitude of such interregional feedbacks.

Suppose, in (3.10), that we read $\mathbf{x}^r$, $\mathbf{x}^s$, $\mathbf{f}^r$ and $\mathbf{f}^s$ as "changes in" – that is, $\Delta\mathbf{x}^r$, $\Delta\mathbf{x}^s$, $\Delta\mathbf{f}^r$, and $\Delta\mathbf{f}^s$. Given a vector of changes in final demands in the two regions, we can find the consequent changes in gross outputs in both regions. Assume, for simplicity, that $\Delta\mathbf{f}^s = \mathbf{0}$; we are assessing the impacts in both regions of a change in final demands in region r only. Under these conditions, solving the second equation in (3.10) for $\mathbf{x}^s$ gives

$$\mathbf{x}^s = (\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr}\mathbf{x}^r$$

and putting this into the first equation, we have

$$(\mathbf{I} - \mathbf{A}^{rr})\mathbf{x}^r - \mathbf{A}^{rs}(\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr}\mathbf{x}^r = \mathbf{f}^r \tag{3.13}$$

Note that a single-region model (for region r), as in (3.4), would be $(\mathbf{I} - \mathbf{A}^{rr})\mathbf{x}^r = \mathbf{f}^r$. The "extra" (second) term, subtracted on the left in (3.13),

$$\mathbf{A}^{rs}(\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr}\mathbf{x}^r \tag{3.14}$$

represents exactly the added demands made on the output of region r because of interregional trade linkages; it is an interregional feedback term. Consider the various parts, starting at the right: (a) $\mathbf{A}^{sr}\mathbf{x}^r$ captures the magnitude of flows from s to r because of increased output in r [the value of engines that are shipped from Connecticut to Washington for installation in the new airplanes], (b) $(\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr}\mathbf{x}^r$ then translates these flows into total direct and indirect needs in s to produce the required shipments from s (Connecticut production in all sectors needed to supply the engines for shipment to Washington), (c) $\mathbf{A}^{rs}(\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr}\mathbf{x}^r$ indicates the magnitude of the additional sales from r to s that will be necessary to sustain the total s-based production found in (b) [new outputs from Washington sectors to satisfy Connecticut demand for inputs to Connecticut production quantified in (b)].[12]

Thus the strength and importance of interregional linkages depend not only on the elements of the interregional input coefficients matrices – $\mathbf{A}^{rs}$ and $\mathbf{A}^{sr}$, in this example – but also on the full set of regional input coefficients in the other region, as represented by $(\mathbf{I} - \mathbf{A}^{ss})^{-1}$. It is precisely these kinds of spatial linkages that distinguish complete interregional models from single-region models. Since the feedback term is subtracted from $(\mathbf{I} - \mathbf{A}^{rr})\mathbf{x}^r$ in (3.13), a given value of $\mathbf{f}^r$ will generate a larger $\mathbf{x}^r$ than in a single-region analysis in order that the required shipments to region s can be met, as well

12 The arrows in Figure 3.1 indicate the directions of transmission of demands to producers. The output responses to those demands travel in the opposite direction along the arrows.

Table 3.2 Flow Data for a Hypothetical Two-Region Interregional Case

Selling Sector		Purchasing Sector: Region r 1	2	3	Region s 1	2
Region r	1	150	500	50	25	75
	2	200	100	400	200	100
	3	300	500	50	60	40
Region s	1	75	100	60	200	250
	2	50	25	25	150	100

as the usual intraregional shipments, $\mathbf{A}^{rr}\mathbf{x}^r$. In terms of outputs, the single- and two-region models will generate $x^r = (\mathbf{I} - \mathbf{A}^{rr})^{-1}\mathbf{f}^r$ and $\mathbf{x}^r = (\mathbf{I} - \mathbf{A}^{rr} - \mathbf{A}^{rs}\mathbf{L}^{ss}\mathbf{A}^{sr})^{-1}\mathbf{f}^r$, respectively.

3.3.3 Numerical Example: Hypothetical Two-Region Interregional Case

To illustrate for the two-region case, suppose that the figures in Table 3.2 represent the data in Table 3.1. Also, let

$$\mathbf{f}^r = \begin{bmatrix} 200 \\ 1000 \\ 50 \end{bmatrix} \text{ and } \mathbf{f}^s = \begin{bmatrix} 515 \\ 450 \end{bmatrix}, \text{ so that } \mathbf{f} = \begin{bmatrix} \mathbf{f}^r \\ \mathbf{f}^s \end{bmatrix} = \begin{bmatrix} 200 \\ 1000 \\ 50 \\ \\ 515 \\ 450 \end{bmatrix}$$

Thus

$$\mathbf{x}^r = \begin{bmatrix} 1000 \\ 2000 \\ 1000 \end{bmatrix}, \mathbf{x}^s = \begin{bmatrix} 1200 \\ 800 \end{bmatrix}, \text{ and } \mathbf{x} = \begin{bmatrix} \mathbf{x}^r \\ \mathbf{x}^s \end{bmatrix} = \begin{bmatrix} 1000 \\ 2000 \\ 1000 \\ \\ 1200 \\ 800 \end{bmatrix}$$

and $\mathbf{A}^{rr}$ is found to be

$$\mathbf{A}^{rr} = \begin{bmatrix} .150 & .250 & .050 \\ .200 & .050 & .400 \\ .300 & .250 & .050 \end{bmatrix}$$

Similarly,

$$\mathbf{A}^{ss} = \begin{bmatrix} .1667 & .3125 \\ .1250 & .1250 \end{bmatrix}, \mathbf{A}^{rs} = \begin{bmatrix} .0208 & .0938 \\ .1667 & .1250 \\ .0500 & .0500 \end{bmatrix}, \mathbf{A}^{sr} = \begin{bmatrix} .0750 & .0500 & .0600 \\ .0500 & .0125 & .0250 \end{bmatrix}$$

so

$$\mathbf{A}=\begin{bmatrix}\mathbf{A}^{rr} & \mathbf{A}^{rs}\\ \mathbf{A}^{sr} & \mathbf{A}^{ss}\end{bmatrix}=\begin{bmatrix}0.1500 & 0.2500 & 0.0500 & 0.0208 & 0.0938\\ 0.2000 & 0.0500 & 0.4000 & 0.1667 & 0.1250\\ 0.3000 & 0.2500 & 0.0500 & 0.0500 & 0.0500\\ 0.0750 & 0.0500 & 0.0600 & 0.1667 & 0.3125\\ 0.0500 & 0.0125 & 0.0250 & 0.1250 & 0.1250\end{bmatrix}$$

and define

$$\mathbf{L}=\begin{bmatrix}\mathbf{L}_{11} & \mathbf{L}_{12}\\ \mathbf{L}_{21} & \mathbf{L}_{22}\end{bmatrix}=\begin{bmatrix}1.4234 & 0.4652 & 0.2909 & 0.1917 & 0.3041\\ 0.6346 & 1.4237 & 0.6707 & 0.4092 & 0.4558\\ 0.6383 & 0.5369 & 1.3363 & 0.2501 & 0.3108\\ 0.2672 & 0.2000 & 0.1973 & 1.3406 & 0.5473\\ 0.1468 & 0.0908 & 0.0926 & 0.2155 & 1.2538\end{bmatrix}$$

We use $\mathbf{L}_{11}$, $\mathbf{L}_{12}$ and so on because later it will be necessary to refer to these individual submatrices in $\mathbf{L}$, and they are to be distinguished from $\mathbf{L}^{rr}=(\mathbf{I}-\mathbf{A}^{rr})^{-1}$ and $\mathbf{L}^{ss}=(\mathbf{I}-\mathbf{A}^{ss})^{-1}$, which are often used to denote Leontief inverses associated with regional direct input coefficients matrices.

Impacts on the sectors in both regions of various new final-demand vectors in either or both regions can now be found. For example, with new demand of 100 for the output of sector 1 in region r, $(\mathbf{f}^{new})'=\begin{bmatrix}100 & 0 & 0 & 0 & 0\end{bmatrix}$, and, using $\mathbf{L}$, above,

$$\mathbf{x}^{new}=\begin{bmatrix}(\mathbf{x}^{r})^{new}\\ (\mathbf{x}^{s})^{new}\end{bmatrix}=\mathbf{L}\mathbf{f}^{new}=\begin{bmatrix}142.34\\ 63.46\\ 63.83\\ 26.72\\ 14.68\end{bmatrix}$$

The new outputs in region s of sectors 1 (26.72) and 2 (14.68) that result from the new demand in region r reflect *interregional spillovers* – economic stimulus in a region other than the one in which the exogenous change occurs (in this case spillovers from region r to region s).

It is to be emphasized that the final demands in the interregional input–output model are for outputs produced in a particular region. That is, $f_1^r=100$ means that there is a final demand of 100 for sector 1 goods that are produced in region r. If sector 1 were aircraft production and region r were Washington, new orders from a foreign airline for Boeing commercial airliners would be represented in the value for f_1^r.

Using these hypothetical data, we can illustrate the differences between the results from a single-region model for region r alone and the results from this two-region interregional model. From the information on $\mathbf{A}^{rr}$ alone we find

$$\mathbf{L}^{rr}=(\mathbf{I}-\mathbf{A}^{rr})^{-1}=\begin{bmatrix}1.3651 & .4253 & .2509\\ .5273 & 1.3481 & .5954\\ .5698 & .4890 & 1.2885\end{bmatrix}$$

Using this single-region model with $(\mathbf{f}^r)^{new} = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$ and ignoring interregional linkages, as in (3.4), we have

$$\mathbf{x}_S^r = \mathbf{L}^{rr}\mathbf{f}^r = \begin{bmatrix} 1.3651 & .4253 & .2509 \\ .5273 & 1.3481 & .5954 \\ .5698 & .4890 & 1.2885 \end{bmatrix} \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 136.51 \\ 52.73 \\ 56.99 \end{bmatrix}$$

We use a subscript S to make clear that these are outputs in the *single*-region model, and we drop the superscript "*new*." With the complete two-region model we had, for region r,

$$\mathbf{x}_T^r = \begin{bmatrix} 142.34 \\ 63.46 \\ 63.83 \end{bmatrix}$$

Here, $\mathbf{x}_T^r$ reminds us that these are outputs in the *two*-region interregional model. The difference in results for region r is seen to be

$$\mathbf{x}_T^r - \mathbf{x}_S^r = \begin{bmatrix} 142.34 \\ 63.46 \\ 63.83 \end{bmatrix} - \begin{bmatrix} 136.51 \\ 52.73 \\ 56.99 \end{bmatrix} = \begin{bmatrix} 5.83 \\ 10.73 \\ 6.84 \end{bmatrix}$$

Each region r output is larger in the interregional model because the interregional feedbacks are captured in that model. One measure of the "error" that would be involved in ignoring these feedbacks – in using a single-region model as opposed to an interregional model – would be given by the percentage of total output in region r that one fails to capture when using a single-region model only. Total output over all sectors in region r in the two-region model is $\mathbf{i}'\mathbf{x}_T^r = 269.63$. Total output estimated in the single-region model is $\mathbf{i}'\mathbf{x}_S^r = 246.23$. By this measure, the underestimate that occurs in using the single-region model is $\mathbf{i}'\mathbf{x}_T^r - \mathbf{i}'\mathbf{x}_S^r = 23.40$, or (23.40/269.63) $\times$ 100 = 8.7 percent of the total true (two-region model) output. Formally, this *overall percentage error* measure is found as

$$OPE = [(\mathbf{i}'\mathbf{x}_T^r - \mathbf{i}'\mathbf{x}_S^r)/\mathbf{i}'\mathbf{x}_T^r] \times 100 = [\mathbf{i}'(\mathbf{x}_T^r - \mathbf{x}_S^r)/\mathbf{i}'\mathbf{x}_T^r] \times 100$$

It thus becomes an interesting empirical question to try to assess the importance of interregional feedbacks in real-world regional input–output models. If it turned out that the error caused by ignoring interregional linkages when assessing the impact of new region r final demands on region r outputs was quite small, then one might argue that (at least for such questions) the apparatus of an interregional model would be unnecessary. The answer will depend, in part, upon the relative strengths of the interregional linkages; in the two-region model this means on the magnitudes of the elements in $\mathbf{A}^{rs}$ and $\mathbf{A}^{sr}$. Precisely this question has been investigated; however, the results are inconclusive. The conclusion from an early set of experiments was that interregional feedback effects were likely to be very small (less than one half of one percent, using the overall percentage error measure presented above for illustration). (See Miller, 1966, 1969.) Other studies have tended to confirm the relative smallness of interregional feedback effects by comparing output multipliers from single- and many-region input–output models. (Chapter 6.) There has been work on derivation of upper limits on the percentage error that could be expected in certain interregional input–output models when the interregional feedbacks are ignored (in particular, Gillen and Guccione, 1980; Miller, 1986; Guccione *et al.*, 1988).

The error caused by ignoring interregional feedbacks is strongly influenced by the level of self-sufficiency in region r – whether or not region r is relatively dependent on inputs from region s. This is because higher dependence is reflected in larger coefficients in $\mathbf{A}^{sr}$ which, again as in (3.14), generate a larger feedback term. Self-sufficiency is also a function of the geographic size of the region. In a two-region model with Nebraska (region r) and the rest of the United States (region s), the average element in $\mathbf{A}^{sr}$ will be larger than in a two-region model in which region r is the United States west of the Mississippi and region s is the United States east of the Mississippi. However, in the Nebraska (r)/rest-of-the-United States (s) example, the elements in $\mathbf{A}^{rs}$ (reflecting rest-of-the-United States dependence on inputs from Nebraska) will be generally much smaller than in the United States West (r)/United States East (s) example. Thus it is not easy to generalize on how the geographical size of the respective regions ultimately influences the size of the interregional feedbacks.

In any case, a single-region model, by definition, cannot capture effects outside of that region (spillovers) in regional/sectoral detail, and there are many kinds of economic impact questions that have important ramifications in more than one region of a national economy. In these cases, some kind of connected-region model is essential. The interregional input–output framework provides one such approach. Feedbacks and spillovers in input–output models will be examined again in Chapter 6 when we discuss multiplier decompositions.

Some analysts (for example, Oosterhaven, 1981) suggest that measurement of feedback effects should be based not on total impacts (direct and indirect) but rather should be found as percentages of indirect impacts only – without the first term in the power series or with $\mathbf{f}$ netted out from gross outputs in $OPE = [(\mathbf{i}'\mathbf{x}_T^r - \mathbf{i}'\mathbf{x}_S^r)/\mathbf{i}'\mathbf{x}_T^r] \times 100$. This means

$$\begin{aligned} OPE^n &= \{[(\mathbf{i}'\mathbf{x}_T^r - \mathbf{i}'\mathbf{f}) - (\mathbf{i}'\mathbf{x}_S^r - \mathbf{i}'\mathbf{f})]/(\mathbf{i}'\mathbf{x}_T^r - \mathbf{i}'\mathbf{f})\} \times 100 \\ &= [(\mathbf{i}'\mathbf{x}_T^r - \mathbf{i}'\mathbf{x}_S^r)/(\mathbf{i}'\mathbf{x}_T^r - \mathbf{i}'\mathbf{f})] \times 100 \end{aligned}$$

This "net" measure is larger than OPE (except in the trivial case when $\mathbf{f} = \mathbf{0}$); namely $OPE^n = (OPE)\left(\frac{\mathbf{i}'\mathbf{x}_T^r}{\mathbf{i}'\mathbf{x}_T^r - \mathbf{i}'\mathbf{f}}\right)$. In our numerical example, $\left(\frac{\mathbf{i}'\mathbf{x}_T^r}{\mathbf{i}'\mathbf{x}_T^r - \mathbf{i}'\mathbf{f}}\right) = 1.59$ and $OPE^n = 13.8$. Alternatively, $100 \times (OPE/OPE^n) = 10 \times \left(\frac{\mathbf{i}'\mathbf{x}_T^r - \mathbf{i}'\mathbf{f}}{\mathbf{i}'\mathbf{x}_T^r}\right)$ indicates the percentage of the net measure that is captured by the original measure. In the example, this is 63 percent.

3.3.4 Interregional Models with more than Two Regions

The fundamental structure of models with more than two regions is identical to the two-region case in section 3.3.1, although the numbers of matrices and their sizes increase. The objective is to capture explicitly the various economic connections between and among the several regions in a multiregional economy. For example, in a three-region model (regions 1, 2, and 3), the complete coefficients matrix would be

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^{11} & \mathbf{A}^{12} & \mathbf{A}^{13} \\ \mathbf{A}^{21} & \mathbf{A}^{22} & \mathbf{A}^{23} \\ \mathbf{A}^{31} & \mathbf{A}^{32} & \mathbf{A}^{33} \end{bmatrix} \tag{3.15}$$

and the parallel to (3.10) is

$$\begin{aligned} (\mathbf{I} - \mathbf{A}^{11})\mathbf{x}^1 \;-\; \mathbf{A}^{12}\mathbf{x}^2 \;-\; \mathbf{A}^{13}\mathbf{x}^3 &= \mathbf{f}^1 \\ -\mathbf{A}^{21}\mathbf{x}^1 + (\mathbf{I} - \mathbf{A}^{22})\mathbf{x}^2 \;-\; \mathbf{A}^{23}\mathbf{x}^3 &= \mathbf{f}^2 \\ -\mathbf{A}^{31}\mathbf{x}^1 \;-\; \mathbf{A}^{32}\mathbf{x}^2 + (\mathbf{I} - \mathbf{A}^{33})\mathbf{x}^3 &= \mathbf{f}^3 \end{aligned} \tag{3.16}$$

With $\mathbf{x} = \begin{bmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \mathbf{x}^3 \end{bmatrix}$, $\mathbf{f} = \begin{bmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \end{bmatrix}$ and $\mathbf{I} = \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}$, the complete three-region interregional input–output model is still represented as $(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{f}$. The underlying logic is the same as that for the two-region model, and the equations in (3.16) can be built up in the same way as were those in (3.10). Also, the magnitudes of the interregional feedback effects can be made specific.

The extension to a p-region model is straightforward. (For example, there are nine-region models for Japan, noted in section 3.3.5, below.) The parallel to (3.16) is

$$\begin{aligned} &(\mathbf{I} - \mathbf{A}^{11})\mathbf{x}^1 - \mathbf{A}^{12}\mathbf{x}^2 \;-\; \cdots \;-\; \mathbf{A}^{1p}\mathbf{x}^p = \mathbf{f}^1 \\ &\vdots \\ &-\mathbf{A}^{p1}\mathbf{x}^1 - \mathbf{A}^{p2}\mathbf{x}^2 - \cdots + (\mathbf{I} - \mathbf{A}^{pp})\mathbf{x}^p = \mathbf{f}^p \end{aligned} \tag{3.17}$$

(The interested reader can construct the parallel expressions for $\mathbf{A}$, $\mathbf{I}$, $\mathbf{f}$ and $\mathbf{x}$.)

The data requirements increase quickly with the number of regions. Assuming that all regions are divided into n sectors (not a necessary requirement at all – each region could have a different number of sectors), a complete two-region interregional model requires data for four coefficients matrices of size $n \times n$, a three-region model contains nine $n \times n$ matrices, a four-region model has sixteen such matrices, and a p-region model has p^2 such $n \times n$ matrices. However, interregional models with a relatively small number of regions may be useful, since one region can always be defined as the "rest of the country" or the "rest of the world." A three-region model might concentrate on a particular county, region 2 could be the "rest of the state" and region 3 the "rest of the nation" (outside the state).

3.3.5 *Implementation of the IRIO Model*

Clearly, the interregional input–output model requires a large amount of detailed data. For this reason, there have been few real-world applications. Perhaps the most ambitious attempts at implementation are contained in the impressive series of Japanese survey-based interregional tables, with nine regions and (ultimately) 25 sectors, beginning with 1960 and updated every five years. [See Ministry of International Trade and Industry (MITI), various years; this was reorganized as the Ministry of Economy, Trade and Industry (METI) in 2001.] This very rich data source has generated a number of Japanese comparative regional studies (see, for example, Akita, 1994, 1999; Akita and Kataoka, 2002).

3.4 Many-Region Models: The Multiregional Approach

While a complete interregional model of the sort described in section 3.3 is generally impossible to implement for very many regions and/or sectors because of the enormous amounts of data that it requires, the approach has inspired modifications and simplifications in the direction of a more operational framework. One attempt in this direction uses the "Chenery–Moses" approach (noted in section 3.3, above) for consistent estimation of the intra- and interregional transactions required in the IRIO model. It has come to be known as a multiregional input–output model. It contains counterparts to the regional input coefficients matrices – as in $\mathbf{A}^{rr}$ – and the interregional input (trade) coefficients matrices – as in $\mathbf{A}^{rs}$. In both cases the attempt has been to specify a model in which the data are more easily obtained.

Polenske examined and implemented three versions of the MRIO model – the Chenery–Moses version (also known as a "column-coefficient" model for reasons that will become clear below), an alternative row-coefficient version, and one using the gravity model approach of Leontief and Strout (1963).[13] Problems with the latter two approaches ultimately precluded their use, and the column-coefficient model was chosen as the structure on which to develop the US MRIO model. [Polenske, 1970a, 1970b, 1980, 1995 (section 2), 2004 (section 8); Bon, 1984.]

3.4.1 *The Regional Tables*

The multiregional input–output model uses a regional *technical* coefficients matrix, $\mathbf{A}^r$, in place of the regional *input* coefficients matrix, $\mathbf{A}^{rr}$. These regional technical coefficients, a_{ij}^r, can be produced from responses to the question "How much sector i product did you buy last year in making your output?" [Question (1) in section 3.2], where they were contrasted with the regional input coefficients, a_{ij}^{rr}. Information regarding the region of origin of a given input is ignored; one only needs information on the dollars' worth of input from sector i used by sector j in region r. These transactions are usually

[13] Leontief and Strout (1963) "devised the multiregional input–output (MRIO) accounts" (Polenske and Hewings, 2004, p. 274).

denoted by $z_{ij}^{\cdot r}$, where the dot indicates that all possible geographical locations for sector i are lumped together.[14] These coefficients are defined as $a_{ij}^r = \frac{z_{ij}^{\cdot r}}{x_j^r}$ and $\mathbf{A}^r = [a_{ij}^r]$.

In practice, when actual regional data on technology are not available, estimates of regional technical coefficients matrices are sometimes made using what is known as the product-mix approach. The basic assumption is that input requirements per unit of output are constant from region to region at a very fine level of industrial classification, but that an important distinguishing characteristic of production at the regional level is the composition of sector outputs, when one is dealing with more aggregate sectors. To return to our earlier illustration of the product-mix problem, when two-engine commercial jets are made in Washington (or anywhere else), they use, among other things, two jet engines as inputs; when single-engine propeller-driven private aircraft are made in Florida or in any other state, they use one propeller engine as one of the inputs to production. But the important fact to capture is that the output of the sector designated "aircraft" in a Washington table is composed of a vastly different mix of products (commercial jets) than the "aircraft" sector in Florida (private/corporate airplanes).

To illustrate, assume that sector 2 is food and kindred products, and that it contains only three subsectors, which can be designated by their outputs: tomato soup (sector 2.1), chocolate bars (sector 2.2), and guava jelly (sector 2.3). Assume that the *national* technical coefficients from sector 8, paper and allied products, to each of these subsectors are: 0.005, 0.009, and 0.003. (These represent various aspects of packaging – labels, wrappers, etc.) Suppose that we want to derive coefficients for inputs from sector 8 to sector 2, a_{82}, for New Jersey (region J) and for Florida (region F). The data that we would need are shown in Table 3.3, where N designates *national* data. The food and kindred products sector was composed of only tomato soup ($700,000) and chocolate bars ($300,000) output (no guava jelly) in New Jersey; in Florida it was made up of tomato soup ($80,000) and guava jelly ($420,000) – no chocolate bars.

Purchases of paper and allied products as inputs to New Jersey food and kindred products production over the period covered by the output figures in Table 3.3 are then assumed to be the sum of

$$\begin{aligned} a_{8,2.1}^N\, x_{2.1}^J &= (.005)(700,000) = 3500 \\ a_{8,2.2}^N\, x_{2.2}^J &= (.009)(300,000) = 2700 \\ a_{8,2.3}^N\, x_{2.3}^J &= (.003)(0) = 0 \end{aligned}$$

for a total of $6200 in necessary inputs from sector 8 to production in sector 2 in New Jersey. Since $x_2^J = x_{2.1}^J + x_{2.2}^J + x_{2.3}^J = 1,000,000$,

$$a_{82}^J = 6200/1,000,000 = .0062$$

[14] Sometimes a small $\circ$ or a larger dot is used, primarily because it is easier to read.

Table 3.3 Data Needed for Conversion of National to Regional Coefficients via the Product-Mix Approach

National Data			
To sector 2: Food and Kindred Products			
Subsectors	2.1 (tomato soup)	2.2 (chocolate bars)	2.3 (guava jelly)
From sector 8: Paper and Allied Products			
	$a^N_{8,2.1} = .005$	$a^N_{8,2.2} = .009$	$a^N_{8,2.3} = .003$
Regional Data			
Outputs (in 1000 dollars) by subsector of sector 2			
	(New Jersey)	(Florida)	
	$x^J_{2.1} = 700$	$x^F_{2.1} = 80$	
	$x^J_{2.2} = 300$	$x^F_{2.2} = 0$	
	$x^J_{2.3} = 0$	$x^F_{2.3} = 420$	
Total Outputs (Sector 2)			
	$x^J_2 = 1000$	$x^F_2 = 500$	

Similarly, for Florida,

$$\begin{aligned} a^N_{8,2.1}x^F_{2.1} &= (.005)(80,000) = 400 \\ a^N_{8,2.2}x^F_{2.2} &= (.009)(0) = 0 \\ a^N_{8,2.1}x^F_{2.3} &= (.003)(420,000) = 1260 \end{aligned}$$

The total Florida inputs from sector 8 would be estimated as \$1660. Since $x^F_2 = 500{,}000$, we have

$$a^F_2 = 1660/500,000 = .0033$$

Formally,

$$a^J_{82} = \frac{(a^N_{8,2.1}x^J_{2.1} + a^N_{8,2.2}x^J_{2.2} + a^N_{8,2.3}x^J_{2.3})}{x^J_2} = a^N_{8,2.1}\left(\frac{x^J_{2.1}}{x^J_2}\right) + a^N_{8,2.2}\left(\frac{x^J_{2.2}}{x^J_2}\right) + a^N_{8,2.3}\left(\frac{x^J_{2.3}}{x^J_2}\right)$$

$$a^F_{82} = \frac{(a^N_{8,2.1}x^F_{2.1} + a^N_{8,2.2}x^F_{2.2} + a^N_{8,2.3}x^F_{2.3})}{x^F_2} = a^N_{8,2.1}\left(\frac{x^F_{2.1}}{x^F_2}\right) + a^N_{8,2.2}\left(\frac{x^F_{2.2}}{x^F_2}\right) + a^N_{8,2.3}\left(\frac{x^F_{2.3}}{x^F_2}\right)$$

The *regional* coefficients derived in this way are *weighted* averages of the national detailed coefficients, where the weights are the proportions of subsector outputs to total output of the sector (e.g., $x^J_{2.1}/x^J_2$) in each state.

3.4.2 The Interregional Tables

The interconnections among regions in the multiregional input–output model are captured in an entirely different way from the interregional input–output framework. Trade

Table 3.4 Interregional Shipments of Commodity i

	Receiving Region					
Shipping Region	1	2	$\cdots$	s	$\cdots$	p
1	z_i^{11}	z_i^{12}	$\cdots$	z_i^{1s}	$\cdots$	z_i^{1p}
2	z_i^{21}	z_i^{22}	$\cdots$	z_i^{2s}	$\cdots$	z_i^{2p}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$
r	z_i^{r1}	z_i^{r2}	$\cdots$	z_i^{rs}	$\cdots$	z_i^{rp}
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
p	z_i^{p1}	z_i^{p2}	$\cdots$	z_i^{ps}	$\cdots$	z_i^{pp}
Total	T_i^1	T_i^2	$\cdots$	T_i^s	$\cdots$	T_i^p

flows in the multiregional model are estimated by sector, again to take advantage of the kinds of data likely to be available. For sector i, let z_i^{rs} denote the dollar flow of good i from region r to region s, irrespective of the sector of destination in the receiving region.[15] These flows will include shipments to the producing sectors in region s as well as to final demand in s. Thus there is, for each sector, a shipments matrix of the sort shown in Table 3.4.

Note that each of the column sums in this table represents the total shipments of good i into that region from all of the regions in the model; this total, for column s, is denoted in the table for good i by T_i^s:

$$T_i^s = z_i^{1s} + z_i^{2s} + \cdots + z_i^{rs} + \cdots + z_i^{ps} \tag{3.18}$$

If each element in column s is divided by this total, we have coefficients denoting the *proportion* of all of good i used in s that comes from each region r ($r = 1, \ldots, p$). These proportions are denoted c_i^{rs}:

$$c_i^{rs} = \frac{z_i^{rs}}{T_i^s}$$

For later use, these coefficients are rearranged as follows. For each possible origin-destination pair of regions, denote by $\mathbf{c}^{rs}$ the n-element column vector

$$\mathbf{c}^{rs} = \begin{bmatrix} c_1^{rs} \\ \vdots \\ c_n^{rs} \end{bmatrix}$$

[15] To be consistent with the notation $z_{ij}^{\cdot r}$ or $z_{ij}^{\circ r}$, above, this should properly be $z_{i\cdot}^{rs}$ or $z_{i\circ}^{rs}$. However, when the blank space is in the second subscript position, it is easier to distinguish than when it is in the first superscript position, and so we avoid the double subscript option.

These elements show, for region s, the proportion of the total amount of each good used in s that comes from region r. Finally, construct $\hat{\mathbf{c}}^{rs}$,

$$\hat{\mathbf{c}}^{rs} = \begin{bmatrix} c_1^{rs} & 0 & \cdots & 0 \\ 0 & c_2^{rs} & & \\ \vdots & & & \\ 0 & 0 & \cdots & c_n^{rs} \end{bmatrix} \tag{3.19}$$

for $r, s = 1, \ldots, p$. Note that there will be *intra*regional matrices in this set. For example, there will be a matrix $\hat{\mathbf{c}}^{ss}$, namely

$$\hat{\mathbf{c}}^{ss} = \begin{bmatrix} c_1^{ss} & 0 & \cdots & 0 \\ 0 & c_2^{ss} & & \\ \vdots & & & \\ 0 & 0 & \cdots & c_n^{ss} \end{bmatrix} \tag{3.20}$$

whose elements, $c_i^{ss} = z_i^{ss}/T_i^s$, indicate the proportion of good i used in region s that came from within region s.

3.4.3 The Multiregional Model[16]

Consider a small two-sector, two-region example, where

$$\mathbf{A}^r = \begin{bmatrix} a_{11}^r & a_{12}^r \\ a_{21}^r & a_{22}^r \end{bmatrix}, \quad \mathbf{A}^s = \begin{bmatrix} a_{11}^s & a_{12}^s \\ a_{21}^s & a_{22}^s \end{bmatrix}$$

$$\hat{\mathbf{c}}^{rs} = \begin{bmatrix} c_1^{rs} & 0 \\ 0 & c_2^{rs} \end{bmatrix}, \quad \hat{\mathbf{c}}^{ss} = \begin{bmatrix} c_1^{ss} & 0 \\ 0 & c_2^{ss} \end{bmatrix}$$

Then the multiregional input–output model uses the matrix

$$\hat{\mathbf{c}}^{rs}\mathbf{A}^s = \begin{bmatrix} c_1^{rs}a_{11}^s & c_1^{rs}a_{12}^s \\ c_2^{rs}a_{21}^s & c_2^{rs}a_{22}^s \end{bmatrix}$$

as an estimate of $\mathbf{A}^{rs}$ in the interregional input–output model. Similarly,

$$\hat{\mathbf{c}}^{ss}\mathbf{A}^s = \begin{bmatrix} c_1^{ss}a_{11}^s & c_1^{ss}a_{12}^s \\ c_2^{ss}a_{21}^s & c_2^{ss}a_{22}^s \end{bmatrix}$$

in the multiregional model replaces $\mathbf{A}^{ss}$ in the interregional model. Therefore the multiregional input–output model embodies the same assumption as was used in the earlier regional models with estimated supply percentages. Looking at the top rows of the

[16] In this section we emphasize the structural parallels between the multiregional model and the interregional model. In Appendix 3.1 to this chapter the basic relationships in the multiregional model are derived from standard economic and input–output theory.

$\hat{\mathbf{c}}^{rs}\mathbf{A}^s$ and $\hat{\mathbf{c}}^{ss}\mathbf{A}^s$ matrices, note that both sectors 1 and 2 in region s are assumed to have the same proportion of their total use of commodity 1 supplied from region r, namely c_1^{rs}, and the same proportion supplied from within region $s - c_1^{ss}$.

Suppose that sector 1 in both regions r and s is electricity production and sector 2 in region s is automobile production, then if $c_1^{rs} = 0.6$, this means that 60 percent of all electricity used in making electricity in region s comes from region r and 60 percent of all electricity used in automobile manufacture in region s also comes from region r. And similarly, since in this two-region model it would be true that $c_1^{ss} = 0.4$, 40 percent of the electricity used in both electricity production and automobile production in s comes from within that region.

Since the interregional shipments recorded in Table 3.4 include sales to both producing sectors and final-demand users in the receiving region, the final demands in region s are met in part by firms within the region ($\hat{\mathbf{c}}^{ss}\mathbf{f}^s$) and in part by purchases from firms in region r ($\hat{\mathbf{c}}^{rs}\mathbf{f}^s$). To continue the illustration with $c_1^{rs}= 0.6$, where sector 1 is electricity production, 60 percent of the final demand for electricity in region s will *also* be satisfied by producers in region r.

The multiregional input–output counterpart to (3.10) for the interregional model is therefore

$$\begin{aligned}(\mathbf{I} - \hat{\mathbf{c}}^{rr}\mathbf{A}^r)\mathbf{x}^r \; - \; \hat{\mathbf{c}}^{rs}\mathbf{A}^s\mathbf{x}^s &= \hat{\mathbf{c}}^{rr}\mathbf{f}^r + \hat{\mathbf{c}}^{rs}\mathbf{f}^s \\ - \hat{\mathbf{c}}^{sr}\mathbf{A}^r\mathbf{x}^r + (\mathbf{I} - \hat{\mathbf{c}}^{ss}\mathbf{A}^s)\mathbf{x}^s &= \hat{\mathbf{c}}^{sr}\mathbf{f}^r + \hat{\mathbf{c}}^{ss}\mathbf{f}^s\end{aligned} \tag{3.21}$$

Let

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^r & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^s \end{bmatrix}, \; \mathbf{C} = \begin{bmatrix} \hat{\mathbf{c}}^{rr} & \hat{\mathbf{c}}^{rs} \\ \hat{\mathbf{c}}^{sr} & \hat{\mathbf{c}}^{ss} \end{bmatrix}, \; \mathbf{x} = \begin{bmatrix} \mathbf{x}^r \\ \mathbf{x}^s \end{bmatrix}, \text{ and } \mathbf{f} = \begin{bmatrix} \mathbf{f}^r \\ \mathbf{f}^s \end{bmatrix}$$

so that (3.21) can be represented as

$$(\mathbf{I} - \mathbf{CA})\mathbf{x} = \mathbf{Cf} \tag{3.22}$$

and the solution will be given by

$$\mathbf{x} = (\mathbf{I} - \mathbf{CA})^{-1}\mathbf{Cf} \tag{3.23}$$

The extension to more than two regions is straightforward. Equations for the three-region model would be

$$\begin{aligned}(\mathbf{I} - \hat{\mathbf{c}}^{11}\mathbf{A}^1)\mathbf{x}^1 \; - \; \hat{\mathbf{c}}^{12}\mathbf{A}^2\mathbf{x}^2 \; - \; \hat{\mathbf{c}}^{13}\mathbf{A}^3\mathbf{x}^3 &= \hat{\mathbf{c}}^{11}\mathbf{f}^1 + \hat{\mathbf{c}}^{12}\mathbf{f}^2 + \hat{\mathbf{c}}^{13}\mathbf{f}^3 \\ - \hat{\mathbf{c}}^{21}\mathbf{A}^1\mathbf{x}^1 + (\mathbf{I} - \hat{\mathbf{c}}^{22}\mathbf{A}^2)\mathbf{x}^2 \; - \; \hat{\mathbf{c}}^{23}\mathbf{A}^3\mathbf{x}^3 &= \hat{\mathbf{c}}^{21}\mathbf{f}^1 + \hat{\mathbf{c}}^{22}\mathbf{f}^2 + \hat{\mathbf{c}}^{23}\mathbf{f}^3 \\ - \hat{\mathbf{c}}^{31}\mathbf{A}^1\mathbf{x}^1 \; - \; \hat{\mathbf{c}}^{32}\mathbf{A}^2\mathbf{x}^2 + (\mathbf{I} - \hat{\mathbf{c}}^{33})\mathbf{A}^3\mathbf{x}^3 &= \hat{\mathbf{c}}^{31}\mathbf{f}^1 + \hat{\mathbf{c}}^{32}\mathbf{f}^2 + \hat{\mathbf{c}}^{33}\mathbf{f}^3\end{aligned}$$

[Compare (3.16), for the three-region interregional model.] By appropriate extension of matrices **A**, **C**, **x**, and **f** to incorporate three regions, the fundamental model is still $(\mathbf{I} - \mathbf{CA})\mathbf{x} = \mathbf{Cf}$, as in (3.22), with solution $\mathbf{x} = (\mathbf{I} - \mathbf{CA})^{-1}\mathbf{Cf}$, as in (3.23).

Table 3.5 Flow Data for a Hypothetical Two-Region Multiregional Case

	Purchasing Sector					
	Region r			Region s		
Selling Sector	1	2	3	1	2	3
1	225	600	110	225	325	125
2	250	125	425	350	200	270
3	325	700	150	360	240	200

Finally, when there are p regions, let

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^2 & \cdots & \mathbf{0} \\ \vdots & \vdots & & \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}^p \end{bmatrix}, \mathbf{C} = \begin{bmatrix} \hat{\mathbf{c}}^{11} & \cdots & \hat{\mathbf{c}}^{1p} \\ \hat{\mathbf{c}}^{21} & \cdots & \hat{\mathbf{c}}^{2p} \\ \vdots & & \\ \hat{\mathbf{c}}^{p1} & \cdots & \hat{\mathbf{c}}^{pp} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \vdots \\ \mathbf{x}^p \end{bmatrix}, \text{ and } \mathbf{f} = \begin{bmatrix} \mathbf{f}^1 \\ \mathbf{f}^2 \\ \vdots \\ \mathbf{f}^p \end{bmatrix}$$

Then $(\mathbf{I} - \mathbf{CA})\mathbf{x} = \mathbf{Cf}$ and $\mathbf{x} = (\mathbf{I} - \mathbf{CA})^{-1}\mathbf{Cf}$ still represents the system and its solution; only the dimensions of the matrices have changed.

3.4.4 Numerical Example: Hypothetical Two-Region Multiregional Case

Assume that we have the flow data in Table 3.5, representing total inputs purchased by producing sectors in each region, regardless of whether these are locally produced or imported from the other region. These are the $\mathbf{Z}^r = [z_{ij}^{\cdot r}]$ and $\mathbf{Z}^s = [z_{ij}^{\cdot s}]$ data.

Suppose, further, that $\mathbf{x}^r = \begin{bmatrix} 1000 \\ 2000 \\ 1000 \end{bmatrix}$ and $\mathbf{x}^s = \begin{bmatrix} 1200 \\ 800 \\ 1500 \end{bmatrix}$, so that the regional technical coefficients matrices, $\mathbf{A}^r = [a_{ij}^r]$ and $\mathbf{A}^s = [a_{ij}^s]$, are

$$\mathbf{A}^r = \begin{bmatrix} .225 & .300 & .110 \\ .250 & .063 & .425 \\ .325 & .350 & .150 \end{bmatrix}, \mathbf{A}^s = \begin{bmatrix} .188 & .406 & .083 \\ .292 & .250 & .180 \\ .300 & .300 & .133 \end{bmatrix}$$

For the trade proportions, we need measures of the total amount of each good, i, that is available in each region – T_i^r and T_i^s, in (3.18). Table 3.6 provides an example of these data. (Note that the row sums for each sector in each region must be the total output for that sector in that region, as recorded in the appropriate $\mathbf{x}$ vector.) The proportions – $c_i^{rs} = z_i^{rs}/T_i^s$ – are easily found. Here

$$\mathbf{c}^{rr} = \begin{bmatrix} .721 \\ .812 \\ .735 \end{bmatrix}, \mathbf{c}^{rs} = \begin{bmatrix} .183 \\ .583 \\ .078 \end{bmatrix}, \mathbf{c}^{sr} = \begin{bmatrix} .279 \\ .188 \\ .265 \end{bmatrix}, \text{ and } \mathbf{c}^{ss} = \begin{bmatrix} .817 \\ .417 \\ .922 \end{bmatrix}$$

Table 3.6 Interregional Commodity Shipments for the Hypothetical Two-Region Multiregional Case

	Commodity 1		Commodity 2		Commodity 3	
	r	s	r	s	r	s
r	800	200	1300	700	900	100
s	310	890	300	500	325	1175
T	$T_1^r = 1110$	$T_1^s = 1090$	$T_2^r = 1600$	$T_2^s = 1200$	$T_3^r = 1225$	$T_3^s = 1275$

Thus the building blocks in this example for the two-region multiregional input–output model are

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^r & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^s \end{bmatrix} = \begin{bmatrix} .225 & .300 & .110 & 0 & 0 & 0 \\ .250 & .063 & .425 & 0 & 0 & 0 \\ .325 & .350 & .150 & 0 & 0 & 0 \\ 0 & 0 & 0 & .188 & .406 & .083 \\ 0 & 0 & 0 & .292 & .250 & .180 \\ 0 & 0 & 0 & .300 & .300 & .133 \end{bmatrix}$$

and

$$\mathbf{C} = \begin{bmatrix} \hat{\mathbf{c}}^{rr} & \hat{\mathbf{c}}^{rs} \\ \hat{\mathbf{c}}^{sr} & \hat{\mathbf{c}}^{ss} \end{bmatrix} = \begin{bmatrix} .721 & 0 & 0 & .183 & 0 & 0 \\ 0 & .812 & 0 & 0 & .583 & 0 \\ 0 & 0 & .735 & 0 & 0 & .078 \\ .279 & 0 & 0 & .817 & 0 & 0 \\ 0 & .188 & 0 & 0 & .417 & 0 \\ 0 & 0 & .265 & 0 & 0 & .922 \end{bmatrix}$$

Therefore

$$(\mathbf{I} - \mathbf{CA})^{-1}\mathbf{C} = \begin{bmatrix} 1.127 & .447 & .300 & .478 & .418 & .153 \\ .628 & 1.317 & .606 & .552 & 1.115 & .323 \\ .512 & .526 & 1.101 & .335 & .470 & .247 \\ .625 & .369 & .250 & 1.224 & .456 & .216 \\ .238 & .385 & .205 & .278 & .650 & .167 \\ .472 & .445 & .589 & .594 & .529 & 1.232 \end{bmatrix} \quad (3.24)$$

and, for example, the impacts of new final demands of 100 for sector 1 outputs by consumers in each region – that is, with $\mathbf{f}' = \begin{bmatrix} 100 & 0 & 0 & 100 & 0 & 0 \end{bmatrix}$ – are found, as in (3.23),

$$\mathbf{x} = (\mathbf{I} - \mathbf{CA})^{-1}\mathbf{Cf} = \begin{bmatrix} 160.50 \\ 118.00 \\ 84.70 \\ 184.90 \\ 51.60 \\ 106.60 \end{bmatrix}$$

So, $\mathbf{x}^r = \begin{bmatrix} 160.50 \\ 118.00 \\ 84.70 \end{bmatrix}$ and $\mathbf{x}^s = \begin{bmatrix} 184.90 \\ 51.60 \\ 106.60 \end{bmatrix}$.

Similarly, if $\mathbf{f}' = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, which represents new final demands of 100 for sector 1 output by consumers in region r only, we find

$$\mathbf{x} = \begin{bmatrix} 112.70 \\ 62.80 \\ 51.20 \\ 62.50 \\ 23.80 \\ 47.20 \end{bmatrix}$$

Exactly as in an interregional model, $\mathbf{x}^s = \begin{bmatrix} 62.50 \\ 23.80 \\ 47.20 \end{bmatrix}$ reflects interregional spillovers in the multiregional system, in this case from region r (the location of the final demand change) to region s.

It is important to bear in mind, from the general statement of the multiregional input–output model in (3.22) or (3.23), that both intermediate demands, $\mathbf{Ax}$, and final demand, $\mathbf{f}$, are premultiplied by the matrix $\mathbf{C}$; this distributes these demands to supplying sectors across regions. Thus $\mathbf{f}^r$ and $\mathbf{f}^s$ represent demands by (shipments to) the final-demand sectors in regions r and s respectively, not final demands for the products of regions r and s (as in the interregional input–output model). The operation $\mathbf{Cf}$ converts these demands into a set of shipments by each region to contribute toward satisfaction of the final demands. In the two-region model here, $\mathbf{f}^r$ is satisfied in part by shipments from sectors in region r, $\hat{\mathbf{c}}^{rr}\mathbf{f}^r$ and in part by shipments from sectors in region s, $\hat{\mathbf{c}}^{sr}\mathbf{f}^r$. An example of a typical element in $\mathbf{f}^r$ might be new energy demands by a state government resulting from a new state office building in region r in that state. Depending upon the particular region, some or all of that energy demand will be met from within region r, the rest from outside the region. This is reflected in the appropriate elements in $\hat{\mathbf{c}}^{rr}$ and $\hat{\mathbf{c}}^{sr}$.

Thus, if one wants to assess the impacts of new *region-specific* final demands (such as from a foreign airline for Boeing airliners, as in the interregional example in section 3.3) it is necessary to replace $\mathbf{Cf}$ by, say, $\mathbf{f}^*$, which represents the new final demands already distributed appropriately to the region or regions of interest, and then to find

$$\mathbf{x} = (\mathbf{I} - \mathbf{CA})^{-1}\mathbf{f}^* \tag{3.25}$$

This is to be contrasted with (3.23). Continuing with the data for this example,

$$(\mathbf{I} - \mathbf{CA})^{-1} = \left[\begin{array}{ccc|ccc} 1.463 & .471 & .359 & .258 & .345 & .135 \\ .668 & 1.483 & .720 & .526 & .600 & .290 \\ .604 & .572 & 1.445 & .274 & .327 & .145 \\ \hline .314 & .298 & .263 & 1.428 & .676 & .212 \\ .216 & .167 & .221 & .292 & 1.326 & .162 \\ .409 & .376 & .329 & .636 & .734 & 1.308 \end{array}\right] \tag{3.26}$$

If $(f^*)_1^r = 100$ represents the value of new foreign airline orders for aircraft produced in region r, we would find, using (3.25)

$$\mathbf{x} = \left[\begin{array}{c} 146.30 \\ 66.80 \\ 60.40 \\ \hline 31.40 \\ 21.60 \\ 40.90 \end{array}\right]$$

3.4.5 The US MRIO Models

The first large-scale implementation of the MRIO framework was initiated at the Harvard Economic Research Project (HERP) and was further developed by Professor Karen Polenske and her associates at MIT. In its most detailed form, this is a model for 1963 with 51 regions (the 50 states and Washington, DC) and 79 sectors in each region. A thorough description of the model and its construction is provided in Polenske (1980). There was a second estimation and implementation of the MRIO framework for the 1977 US economy involving researchers at MIT and also Jack Faucett Associates, Inc., an economics consulting firm (see Jack Faucett Associates, Inc., 1981–1983). Since then there have been some additional attempts at creating multiregional input–output models for the USA. Because of widespread use, this system is viewed as an *alternative* to the IRIO model; as we will see below, it could as well be seen as an approach to *estimating* the intra- and interregional elements of an IRIO framework.[17]

Most implementations of interregional/multiregional input–output structures in recent decades have been generated through a combination of techniques and estimating procedures, all designed to estimate the numbers (especially the interregional transactions/coefficients) needed for the MRIO framework. These are generally known as "hybrid" techniques; they are a blend of some survey information, expert opinion and mechanical approaches. Some of these are explored in more detail in Chapter 8.

[17] An early comparison of the MRIO and IRIO models is provided in Hartwick, 1971.

3.4.6 Numerical Example: The Chinese Multiregional Model for 2000

In 2003 the Institute of Developing Economies (Tokyo) in conjunction with the Japanese External Trade Organization published an ambitious set of multiregional input–output data for China in 2000, with 30 sectors and eight regions. (See Okamoto and Ihara, 2005, for detailed discussions of table construction and a number of comparative regional economic analyses that use the Chinese multiregional input–output framework.)

Tables 3.7–3.9 contain data for a highly aggregated version of the Chinese work, with three sectors and three regions (this is for illustration purposes only).[18] The transactions are denominated in 10,000 yuan (CYN) [also known as renminbi, meaning "people's currency" (RMB)].[19] We can easily trace the effects of hypothesized changes in final demands throughout the sectors and regions of the Chinese economy in this three-regional illustration. For example, assume that there is an increase of ¥100,000 in export demand for manufactured goods from the North. We would use

$$(\Delta \mathbf{f}^N)' = \begin{bmatrix} 0 & 100 & 0 & \quad 0 & 0 & 0 & \quad 0 & 0 & 0 \end{bmatrix}$$

in conjunction with the total requirements matrix in Table 3.9 to assess the impacts of this final demand change throughout the economy. We can examine similar implications of the same amount of increased export demand for manufactured goods in each of the other regions, using in turn $(\Delta \mathbf{f}^S)' = \begin{bmatrix} 0 & 0 & 0 & \quad 0 & 100 & 0 & \quad 0 & 0 & 0 \end{bmatrix}$ for export demands in the South and $(\Delta \mathbf{f}^R)' = \begin{bmatrix} 0 & 0 & 0 & \quad 0 & 0 & 0 & \quad 0 & 100 & 0 \end{bmatrix}$ for export demands in the Rest of China.

Premultiplying each of these vectors, in turn, by the total requirements matrix in Table 3.9 produces the results shown in Table 3.10. The new export demand generates differing own-region economic effects, depending on the region in which the manufacturing sector experiences the new export demand. When the demand is for manufactured goods made in the North, the total output of all sectors in that region increases by ¥215,300. If the demand is for Southern manufactured goods, the total value of new outputs in that region is ¥236,100, and when the new demand is for manufactured goods from the rest of China, output of all sectors there increases by ¥203,900. Interregional spillovers to each of the other regions are indicated by the other entries in the bottom row of Table 3.10. Adding spillovers to own-region impacts, we see that total national effects of the ¥100,000 stimulus for manufactures are ¥259,800, ¥268,500, and ¥240,200, respectively, when the stimulus is in the North, the South, and the Rest of China, respectively.

Many other observations can be made with the aid of results like these in Table 3.10. For example, in terms of interregional spillovers, it is clear that the largest external effect occurs when the demand is in the North; the ¥40,700 increase in Southern outputs is the largest effect of any in the bottom row of the table. In this highly aggregated example from China, it is clear from both the within-South effects (¥268,500) and the North-to-South spillover effect (¥40,700) that Southern manufacturing occupies a dominant

[18] These data are from the Institute of Developing Economies-Japan External Trade Organization (IDE-JETRO), 2003. Details of the regional and sectoral aggregations can be found in Appendix 3.2.

[19] The symbol usually seen is ¥, although sometimes with just one horizontal stroke. With two lines it is the same as the symbol for the Japanese yen.

Table 3.7 Chinese Interregional and Intraregional Transactions, 2000 (in ¥10,000)

	North			South			Rest of China		
	Nat. Res.	Manuf. & Const.	Services	Nat. Res.	Manuf. & Const.	Services	Nat. Res.	Manuf. & Const.	Services
North									
Natural Resources	1,724	6,312	406	188	1,206	86	14	49	4
Manuf. & Const.	2,381	18,458	2,987	301	3,331	460	39	234	57
Services	709	3,883	1,811	64	432	138	5	23	5
South									
Natural Resources	149	656	42	3,564	8,828	806	103	178	15
Manuf. & Const.	463	3,834	571	3,757	34,931	5,186	202	1,140	268
Services	49	297	99	1,099	6,613	2,969	31	163	62
ROC									
Natural Resources	9	51	3	33	254	18	1,581	3,154	293
Manuf. & Const.	32	272	41	123	1,062	170	1,225	6,704	1,733
Services	4	25	7	25	168	47	425	2,145	1,000
Total Output	16,651	49,563	15,011	27,866	81,253	23,667	11,661	21,107	8,910

Table 3.8 Direct Input Coefficients for the Chinese Multiregional Economy, 2000

	North			South			Rest of China		
	Nat. Res.	Manuf. & Const.	Services	Nat. Res.	Manuf. & Const.	Services	Nat. Res.	Manuf. & Const.	Services
North									
Natural Resources	0.1035	0.1273	0.0270	0.0067	0.0148	0.0036	0.0012	0.0023	0.0005
Manuf. & Const.	0.1430	0.3724	0.1990	0.0108	0.0410	0.0194	0.0034	0.0111	0.0064
Services	0.0426	0.0783	0.1206	0.0023	0.0053	0.0058	0.0004	0.0011	0.0006
South									
Natural Resources	0.0089	0.0132	0.0028	0.1279	0.1087	0.0340	0.0089	0.0084	0.0017
Manuf. & Const.	0.0278	0.0774	0.0381	0.1348	0.4299	0.2191	0.0173	0.0540	0.0301
Services	0.0029	0.0060	0.0066	0.0394	0.0814	0.1255	0.0026	0.0077	0.0070
ROC									
Natural Resources	0.0006	0.0010	0.0002	0.0012	0.0031	0.0008	0.1356	0.1494	0.0329
Manuf. & Const.	0.0019	0.0055	0.0027	0.0044	0.0131	0.0072	0.1050	0.3176	0.1945
Services	0.0002	0.0005	0.0004	0.0009	0.0021	0.0020	0.0364	0.1016	0.1122

Table 3.9 Leontief Inverse Matrix for the Chinese Multiregional Economy, 2000

	North			South			Rest of China		
	Nat. Res.	Manuf. & Const.	Services	Nat. Res.	Manuf. & Const.	Services	Nat. Res.	Manuf. & Const.	Services
North									
Natural Resources	1.1631	0.2561	0.0965	0.0227	0.0582	0.0268	0.0064	0.0161	0.0085
Manuf. & Const.	0.3008	1.7275	0.4080	0.0537	0.1596	0.0849	0.0191	0.0529	0.0314
Services	0.0840	0.1686	1.1794	0.0115	0.0306	0.0202	0.0035	0.0093	0.0054
South									
Natural Resources	0.0325	0.0681	0.0321	1.1919	0.2504	0.1114	0.0245	0.0459	0.0232
Manuf. & Const.	0.1194	0.2943	0.1588	0.3258	1.9193	0.5036	0.0742	0.2010	0.1187
Services	0.0193	0.0447	0.0284	0.0848	0.1920	1.1965	0.0142	0.0375	0.0252
ROC									
Natural Resources	0.0034	0.0079	0.0039	0.0062	0.0164	0.0082	1.1958	0.2793	0.1061
Manuf. & Const.	0.0098	0.0245	0.0133	0.0176	0.0478	0.0272	0.2068	1.5681	0.3532
Services	0.0021	0.0051	0.0030	0.0045	0.0114	0.0075	0.0730	0.1916	1.1716

Table 3.10 Region- and Sector-Specific Effects (in ¥1000) of a ¥100,000 Increase in Final Demand for Manufacturing Goods, China, 2000

	Produced in the North			Produced in the South			Produced in ROC		
Sector	North	South	ROC	North	South	ROC	North	South	ROC
Nat. Res.	25.6	6.8	0.8	5.8	25.0	1.6	1.6	4.6	27.9
Mfg. &Const.	172.8	29.4	2.5	16.0	191.9	4.8	5.3	20.1	156.8
Services	16.9	4.5	0.5	3.1	19.2	1.1	0.9	3.8	19.2
Total	215.3	40.7	3.8	24.9	236.1	7.5	7.8	28.5	203.9

position in the economy. We will explore measures of intra- and interregional impacts in more detail in Chapter 6.

3.5 The Balanced Regional Model

3.5.1 Structure of the Balanced Regional Model

A model that has a different sort of "regional" character was proposed in Leontief *et al*. (1953, Ch. 4) and has been implemented in specific applications, including an analysis of the effects in the US economy, on both sectors and regions, of a diversion of production away from military goods and to nonmilitary consumer goods (Leontief *et al.*, 1965). This has been called a *balanced regional model* (or *intranational* model). The basic mathematical structure of this model is identical to that of the interregional input–output model, but the interpretation of each of the components of the model is rather different. The entire analytical structure is based on the observation that in any national economy there are goods with different kinds of market areas. There are some goods for which production and consumption are equal ("balance") only at the national level. These are goods that have essentially a national (or, indeed, international) market area – sectors such as automobiles, aircraft (total airliner production in Washington $\neq$ total demand for aircraft in Washington), furniture, and agriculture. On the other hand, there are other sectors for which production and consumption tend to balance at a lower geographical level; they serve a regional or local rather than a national market. Examples might be electricity, real estate, warehousing, and personal and repair services (the number of shoeshines produced in an urban area equals the demand for shoeshines in that area). Clearly there is in reality an entire spectrum of possibilities, from sectors that serve extremely small local markets (shoe repair) to large national and international markets (aircraft). To illustrate the model structure with a simple example, we suppose that all sectors can be assigned to either a national (N) or a regional (R) category. (One possible criterion for classification of sectors would be the percentage of interregional as opposed to intraregional shipments of the products of that sector.)

Then, from a table of national input coefficients, one can rearrange the sectors so that, for example, all the regional sectors are listed first and all the national sectors follow.

Let sectors $1, 2, \ldots, r$ represent the regionally balanced sectors and let sectors $r+1$, $\ldots, n$ represent nationally balanced sectors. Then, the rearranged table of national input coefficients will be

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^{RR} & \mathbf{A}^{RN} \\ \mathbf{A}^{NR} & \mathbf{A}^{NN} \end{bmatrix} \tag{3.27}$$

Let $\mathbf{x}^R$ and $\mathbf{f}^R$ (r-element column vectors) represent total output and final demand for the regional sectors, and let $\mathbf{x}^N$ and $\mathbf{f}^N$, which are $(n-r)$-element column vectors, represent output and final demand for the national sectors. Define

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}^R \\ \mathbf{x}^N \end{bmatrix} \text{ and } \mathbf{f} = \begin{bmatrix} \mathbf{f}^R \\ \mathbf{f}^N \end{bmatrix}$$

Then, in exactly the same spirit as the two-region interregional input–output model, we have $(\mathbf{I}-\mathbf{A})\mathbf{x} = \mathbf{f}$. Here this is

$$\begin{aligned} (\mathbf{I}-\mathbf{A}^{RR})\mathbf{x}^R \; - \; \mathbf{A}^{RN}\mathbf{x}^N &= \mathbf{f}^R \\ -\;\mathbf{A}^{NR}\mathbf{x}^N + (\mathbf{I}-\mathbf{A}^{NN})\mathbf{x}^N &= \mathbf{f}^N \end{aligned} \tag{3.28}$$

It is important to notice that the R and N superscripts do not refer here to specific geographic *locations* of sectors, as in the interregional model. Rather, they serve to partition the sectors into two types – those whose market areas are national and those whose market areas are regional.[20] For example, a typical element $a_{ij}^{RN}x_j^N$ of the vector $\mathbf{A}^{RN}\mathbf{x}^N$ in (3.28) records inputs from sector i (in the regionally balanced set of sectors) to sector j (in the nationally balanced set of sectors). This will become clearer in the numerical example below.

More compactly, in partitioned matrix form,

$$\begin{bmatrix} (\mathbf{I}-\mathbf{A}^{RR}) & -\mathbf{A}^{RN} \\ -\mathbf{A}^{NR} & (\mathbf{I}-\mathbf{A}^{NN}) \end{bmatrix} \begin{bmatrix} \mathbf{x}^R \\ \mathbf{x}^N \end{bmatrix} = \begin{bmatrix} \mathbf{f}^R \\ \mathbf{f}^N \end{bmatrix}$$

and so

$$\begin{bmatrix} \mathbf{x}^R \\ \mathbf{x}^N \end{bmatrix} = \begin{bmatrix} (\mathbf{I}-\mathbf{A}^{RR}) & -\mathbf{A}^{RN} \\ -\mathbf{A}^{NR} & (\mathbf{I}-\mathbf{A}^{NN}) \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}^R \\ \mathbf{f}^N \end{bmatrix} \tag{3.29}$$

Using regular solution procedures, we find the total outputs of each sector in each of the two categories, due to an exogenous change in final demand for the outputs of one or more national sectors and/or one or more regional sectors. For example, in the arms-reduction study, there was assumed to be a 20 percent across-the-board decrease in government demand for the output of military-related goods, some of which were produced by national sectors (e.g., aircraft) and some of which were produced

[20] Partitioning of this sort can be done for a wide variety of purposes. For example, if one is particularly interested in energy-producing sectors, one might want to divide all sectors into two groups – those that produce energy and those that do not produce energy. Partitioned matrices will be employed frequently in the remainder of this book. Important results on inverses of partitioned matrices are presented in Appendix A.

by regional sectors (e.g., warehousing), and an assumed across-the-board increase in nonmilitary final demands. Hence, elements in both $\mathbf{f}^R$ and $\mathbf{f}^N$ experienced change.

Thus far, there is nothing explicitly *spatial* in the model. The categorization of either nationally balanced or regionally balanced sectors deals only with the size of the market areas involved. For regional sectors, we need to have the new final demands, $\mathbf{f}^R$, distributed across regions. That is, we need to have $\mathbf{f}^{R(s)}$, the final demand for regionally balanced goods in region s, where $\sum_s \mathbf{f}^{R(s)} = \mathbf{f}^R$. In addition, we need, for each region, s, an estimate of the proportion of the output of each nationally balanced sector that is produced in region s, namely

$$\mathbf{p}^s = \begin{bmatrix} p^s_{r+1} \\ \vdots \\ p^s_n \end{bmatrix}$$

The vector $\hat{\mathbf{p}}^s\mathbf{x}^N$ indicates that part of the output of new national goods, $\mathbf{x}^N$, that must be produced by sectors $r+1$ through n in region s. Since the elements of $\mathbf{p}^s$ are the proportions of total national output that occur in region s, $\sum_s p^s_i = 1$ for $i = r+1, \ldots, n$, or $\sum_s \hat{\mathbf{p}}^s = \mathbf{I}$.

Total output in region s is an n-element vector

$$\mathbf{x}^{(s)} = \begin{bmatrix} \mathbf{x}^{R(s)} \\ \mathbf{x}^{N(s)} \end{bmatrix} \tag{3.30}$$

where $\mathbf{x}^{R(s)}$ contains the outputs of the r regionally balanced goods that are made in region s, and $\mathbf{x}^{N(s)} (= \hat{\mathbf{p}}^s\mathbf{x}^N)$ indicates production of nationally balanced goods that occurs in region s.

The $\mathbf{x}^{R(s)}$ term involves two components: (1) production in region s to meet region-specific final demand for regionally balanced goods, $\mathbf{f}^{R(s)}$ (e.g., production in Michigan to satisfy interindustry needs and new final demand in Michigan for electricity produced in that state) and (2) production in region s to turn out that region's share of nationally balanced goods, $\mathbf{x}^{N(s)}$ (e.g., Michigan electricity used as an input to Michigan production of automobiles to satisfy part of the nationwide demand for automobiles). That is,

$$\begin{aligned} \mathbf{x}^{R(s)} &= (\mathbf{I} - \mathbf{A}^{RR})^{-1}\mathbf{f}^{R(s)} + (\mathbf{I} - \mathbf{A}^{RR})^{-1}\mathbf{A}^{RN}\mathbf{x}^{N(s)} \\ &= (\mathbf{I} - \mathbf{A}^{RR})^{-1}\mathbf{f}^{R(s)} + (\mathbf{I} - \mathbf{A}^{RR})^{-1}\mathbf{A}^{RN}\hat{\mathbf{p}}^s\mathbf{x}^N \end{aligned} \tag{3.31}$$

Remember, from (3.27), that all the coefficients in the $\mathbf{A}$ matrices reflect *national* technology; the "R" and "N" serve to partition this national technology into two types of sectors. Production in each particular region is assumed to utilize this same technology, as reflected in the $(\mathbf{I} - \mathbf{A}^{RR})$ matrix and its inverse. In Appendix 3.3, these results are derived directly from observations on the inverse of the partitioned matrix in (3.29).

For the allocation of region R's share of production of nationally balanced goods, found in (3.29), we have

$$\mathbf{x}^{N(s)} = \hat{\mathbf{p}}^s\mathbf{x}^N \tag{3.32}$$

In this way, then, the balanced regional model allocates the impacts of new $\mathbf{f}^R$ and $\mathbf{f}^N$ demand to the various sectors in each region.

3.5.2 Numerical Example

An example will illustrate more exactly how this works. Let

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^{RR} & \mathbf{A}^{RN} \\ \mathbf{A}^{NR} & \mathbf{A}^{NN} \end{bmatrix} = \begin{bmatrix} .10 & .15 & .05 & .03 \\ .03 & .10 & .02 & .10 \\ .12 & .03 & .20 & .10 \\ .10 & .02 & .25 & .15 \end{bmatrix} \tag{3.33}$$

and

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}^R \\ \mathbf{f}^N \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \\ 200 \\ 200 \end{bmatrix}$$

Then $\mathbf{x}$ is found as in (3.29)

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}^R \\ \mathbf{x}^N \end{bmatrix} = \begin{bmatrix} 168.30 \\ 163.40 \\ 325.70 \\ 354.70 \end{bmatrix} \tag{3.34}$$

These figures represent total outputs, throughout the nation, of the four sectors.

Assume that there are three regions in the country and that the region-specific distribution of final demands $\mathbf{f}^R$ is

$$\mathbf{f}^{R(1)} = \begin{bmatrix} 40 \\ 30 \end{bmatrix}, \ \mathbf{f}^{R(2)} = \begin{bmatrix} 50 \\ 30 \end{bmatrix}, \ \mathbf{f}^{R(3)} = \begin{bmatrix} 10 \\ 40 \end{bmatrix}$$

and that $\mathbf{p}^1 = \begin{bmatrix} 0.6 \\ 0.3 \end{bmatrix}$, $\mathbf{p}^2 = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}$, and $\mathbf{p}^3 = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}$. We find $(\mathbf{I} - \mathbf{A}^{RR})^{-1}$ from the data in $\mathbf{A}$;

$$(\mathbf{I} - \mathbf{A}^{RR})^{-1} = \begin{bmatrix} 1.117 & .186 \\ .037 & 1.117 \end{bmatrix}$$

Using (3.31),

$$\mathbf{x}^{R(1)} = \begin{bmatrix} 67.47 \\ 51.75 \end{bmatrix}, \ \mathbf{x}^{R(2)} = \begin{bmatrix} 72.73 \\ 52.97 \end{bmatrix}, \ \mathbf{x}^{R(3)} = \begin{bmatrix} 28.05 \\ 58.65 \end{bmatrix} \tag{3.35}$$

[Note, as must be the case in a consistent model, that $\mathbf{x}^R$, in (3.34), is indeed $\mathbf{x}^{R(1)} + \mathbf{x}^{R(2)} + \mathbf{x}^{R(3)}$.] Using $\hat{\mathbf{p}}^1$, $\hat{\mathbf{p}}^2$, and $\hat{\mathbf{p}}^3$, the distribution of nationally balanced goods across regions is found as

$$\mathbf{x}^{N(1)} = \hat{\mathbf{p}}^1\mathbf{x}^N = \begin{bmatrix} 195.40 \\ 106.40 \end{bmatrix},\ \mathbf{x}^{N(2)} = \hat{\mathbf{p}}^2\mathbf{x}^N = \begin{bmatrix} 65.14 \\ 141.90 \end{bmatrix},\ \mathbf{x}^{N(3)} = \hat{\mathbf{p}}^3\mathbf{x}^N = \begin{bmatrix} 65.14 \\ 106.40 \end{bmatrix} \tag{3.36}$$

where $\mathbf{x}^N$ must equal $\mathbf{x}^{N(1)} + \mathbf{x}^{N(2)} + \mathbf{x}^{N(3)}$, because of the way in which the $\mathbf{p}$ are defined.

Putting the results in (3.35) and (3.36) together, as in (3.30), we have

$$\mathbf{x}^{(1)} = \begin{bmatrix} 67.47 \\ 51.75 \\ \\ 195.40 \\ 106.40 \end{bmatrix},\quad \mathbf{x}^{(2)} = \begin{bmatrix} 72.73 \\ 52.97 \\ \\ 65.14 \\ 141.90 \end{bmatrix},\quad \mathbf{x}^{(3)} = \begin{bmatrix} 28.05 \\ 58.65 \\ \\ 65.14 \\ 106.40 \end{bmatrix} \tag{3.37}$$

The entire outputs in (3.34) have been allocated across the three regions. As noted, production in each region is assumed to utilize the same technology, as reflected in $(\mathbf{I} - \mathbf{A}^{RR})$. But the model does recognize that production, whether of goods with a national market area or with a subnational market area, occurs in geographically specific locations, and the information in the distribution of the $\mathbf{f}^N$ elements and in the $\mathbf{p}^s$ vectors reflects this spatial distribution of production.

3.6 The Spatial Scale of Regional Models

To give the reader a feeling for the vast variety of geographic scales that have been modeled in "regional" input–output applications, we list a few (of very many) references, starting at the micro-spatial end of the spectrum.

- Cole (1987) describes a model for the city of Buffalo, New York, and Cole (1999) looks at an inner-city neighborhood in Buffalo.
- Robison and Miller (1988, 1991) consider small Idaho timber economies (logging/sawmills) – in the latter reference consisting of six communities (five containing sawmills; combined population around 20,000). They term these "community" input–output models. In Robison (1997) the model is for a rural two-county region in central Idaho (total population less than 12,000) which was disaggregated into seven community-centered sub-county regions.
- Hewings, Okuyama and Sonis (2001) present a four-region metropolitan area model. Three of the regions are sub-divisions of the City of Chicago, and the fourth is composed of the remaining counties making up the Chicago metropolitan area (six counties in all).
- Jackson *et al.* (2006) and Schwarm, Jackson and Okuyama (2006) suggest a new approach to generating data for the 51-state US model (as in the US MRIO model discussed above in section 3.4.5).

- Richardson, Gordon and Moore (2007, and numerous other citations) create a 51-state US MRIO model.
- Boomsma and Oosterhaven (1992) describe a variety of two-region Dutch models made up of one region of interest and the rest of The Netherlands as the second region.
- West (1990) contains a summary of Australian input–output models in single-region and connected-region frameworks.
- Eurostat (2002), Hoen (2002). These references deal with the construction (and application) of a kind of many-region (or many-nation) model for the EC that lies between the IRIO and MRIO styles.
- IDE-JETRO (2006). Here the focus of attention is the Asian "multinational" or "multilateral" tables connecting ten countries (China, Indonesia, Japan, Korea, Malaysia, Philippines, Singapore, Taiwan, Thailand, and the USA). These are produced at five-year intervals.
- Leontief (1974), Leontief, Carter and Petri (1977), Fontana (2004) and Duchin (2004). These references discuss various aspects of what has come to be called the Leontief world model. Originally this was structured in terms of two "mega-regions" (developed and less developed countries). In Duchin and Lange (1994) the application uses a framework of 16 world regions (aggregations of countries) covering 189 countries.
- Inomata and Kuwamori (2007) and Development Studies Center, IDE-JETRO (2007). These references discuss a ten-sector model that combines a multinational character – China, Japan, ASEAN5 (Indonesia, Malaysia, the Philippines, Singapore, and Thailand), East Asia (Korea and Taiwan) and the USA – with *regional* disaggregations of China into seven regions and Japan into eight regions. Thus there are 18 geographic areas; some are true sub-national regions (the 15 in China and Japan), one is a nation (the USA) and two are multinational areas (ASEAN5, East Asia). The originators have called it a transnational interregional input–output (TIIO) model.

Many of these applications are discussed in Chapter 8.

3.7 Summary

In this chapter we have explored some of the most important modifications that need to be made to the basic input–output model (Chapter 2) when analysis is to be carried out at a regional level. We have seen that the input–output framework can be used either to study one single region in isolation, or it can be employed in studying one or more regions whose economic connections are made explicit in the model. While the representations of these connected regional models appear quite complicated, the models are logical extensions of the basic input–output structure that are designed to (1) reflect possibly differing production practices for the same sectors in different regions and (2) capture the trade relationships between sectors in different regions.

In more recent decades, work has been carried out with multi*national* input–output models, where "region" is replaced by "nation" in the framework. These have come about as a result of the increasing economic interdependence of nations – as exemplified, for example, in the European Union. We will explore some of these models in Chapter 8, because they generally involve "hybrid" approaches to estimation of the necessary data. Finally, a "global" model has been proposed as an interconnected set of broad groups of national economies. In this kind of framework, impacts of alternative development policies in less-developed countries can be studied for global impacts. (For example, Leontief, 1974; Leontief, Carter and Petri, 1977.) This will be explored briefly in Chapter 8 also.

Appendix 3.1 Basic Relationships in the Multiregional Input–Output Model

In standard input–output fashion, the total demand for commodity i in region s is given by

$$\sum_{j=1}^{n} a_{ij}^{s} x_{j}^{s} + f_{j}^{s} \tag{A3.1.1}$$

The total supply of commodity i in region s is the total that is shipped in from other regions,

$$\sum_{r=1}^{p} z_{i}^{rs} \ (r \neq s)$$

plus the amount that is supplied from within the region, z_i^{ss}. This is just T_i^s, the sum of the elements in column s in Table 3.8, as defined in (3.18). Since shipments (supplies) occur only to satisfy needs (demands), we have, for each commodity i

$$T_{i}^{s} = \sum_{j=1}^{n} a_{ij}^{s} x_{j}^{s} + f_{i}^{s} \tag{A3.1.2}$$

Total production of i in region r is equivalent to the total amount of i shipped from r, including that kept within the region

$$x_{i}^{r} = \sum_{s=1}^{p} z_{i}^{rs} \tag{A3.1.3}$$

From the definition of the interregional proportions in section 3.4.2, $c_i^{rs} = z_i^{rs}/T_i^s$, (A3.1.3) can be rewritten as

$$x_{i}^{r} = \sum_{s=1}^{p} c_{i}^{rs} T_{i}^{s} \tag{A3.1.4}$$

Putting T_i^s, as defined in (A3.1.2), into (A3.1.4)

$$x_{i}^{r} = \sum_{s=1}^{p} c_{i}^{rs} \left(\sum_{j=1}^{n} a_{ij}^{s} x_{j}^{s} + f_{i}^{s} \right) \quad (i = 1, \ldots, n) \tag{A3.1.5}$$

Using familiar matrix notation, let

$$\mathbf{x}^r = \begin{bmatrix} x_1^r \\ \vdots \\ x_n^r \end{bmatrix}, \ \mathbf{x}^s = \begin{bmatrix} x_1^s \\ \vdots \\ x_n^s \end{bmatrix}, \ \mathbf{f}^s = \begin{bmatrix} f_1^s \\ \vdots \\ f_n^s \end{bmatrix}$$

$$\mathbf{A}^s = \begin{bmatrix} a_{11}^s & \cdots & a_{1n}^s \\ \vdots & & \vdots \\ a_{n1}^s & & a_{nn}^s \end{bmatrix}, \ \hat{\mathbf{c}}^{rs} = \begin{bmatrix} c_1^{rs} & 0 & \cdots & 0 \\ 0 & c_2^{rs} & & \\ \vdots & & & \\ 0 & & & c_n^{rs} \end{bmatrix}$$

The reader should be convinced that the entire set of n equations for outputs of goods in region r can be expressed as

$$\mathbf{x}^r = \sum_{s=1}^{p} \hat{\mathbf{c}}^{rs}(\mathbf{A}^s\mathbf{x}^s + \mathbf{f}^s) = \sum_{s=1}^{p} \hat{\mathbf{c}}^{rs}\mathbf{A}^s\mathbf{x}^s + \sum_{s=1}^{p} \hat{\mathbf{c}}^{rs}\mathbf{f}^s \tag{A3.1.6}$$

There will be p such matrix equations, one for each region r ($r = 1, \ldots, p$). Again using matrix notation, as in section 3.4, we can construct

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}^1 \\ \vdots \\ \mathbf{x}^s \\ \vdots \\ \mathbf{x}^p \end{bmatrix}, \ \mathbf{f} = \begin{bmatrix} \mathbf{f}^1 \\ \vdots \\ \mathbf{f}^s \\ \vdots \\ \mathbf{f}^p \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} \mathbf{A}^1 & \cdots & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & & & & \vdots \\ \mathbf{0} & & \mathbf{A}^s & & \mathbf{0} \\ \vdots & & & & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{A}^p \end{bmatrix},$$

and

$$\mathbf{C} = \begin{bmatrix} \hat{\mathbf{c}}^{11} & \cdots & \hat{\mathbf{c}}^{1s} & \cdots & \hat{\mathbf{c}}^{1p} \\ \vdots & & \vdots & & \vdots \\ \hat{\mathbf{c}}^{r1} & \cdots & \hat{\mathbf{c}}^{rs} & \cdots & \hat{\mathbf{c}}^{rp} \\ \vdots & & \vdots & & \vdots \\ \hat{\mathbf{c}}^{p1} & \cdots & \hat{\mathbf{c}}^{ps} & \cdots & \hat{\mathbf{c}}^{pp} \end{bmatrix}$$

Then the p matrix equations in (A3.1.6) can be compactly expressed as

$$\mathbf{x} = \mathbf{C}(\mathbf{A}\mathbf{x} + \mathbf{f}) = \mathbf{C}\mathbf{A}\mathbf{x} + \mathbf{C}\mathbf{f}$$

from which

$$(\mathbf{I} - \mathbf{C}\mathbf{A})\mathbf{x} = \mathbf{C}\mathbf{f} \tag{A3.1.7}$$

and

$$\mathbf{x} = (\mathbf{I} - \mathbf{C}\mathbf{A})^{-1}\mathbf{C}\mathbf{f} \tag{A3.1.8}$$

as in (3.22) and (3.23) in the text.

Appendix 3.2 Sectoral and Regional Aggregation in the 2000 Chinese Multiregional Model

Figure A3.2.1 Regional Aggregation in the 2000 Chinese Multiregional Model

Table A3.2.1 Regional Classifications in the 2000 Chinese Multiregional Model

3-Region Aggregation	Regions	Provinces and Municipalities
North	Northeast	Heilongjiang, Jilin, Liaoning
	North	Beijing, Tianjin, Hebei, Shandong
South	South	Hainan, Guangdong, Fujian
	Central	Hunan, Jiangxi, Hubei, Henan, Anhui, Shanxi
	East	Jiangsu, Shanghai, Zhejiang
Rest of China	Northwest	Xinjiang, Qinghai, Gansu, Ningxia, Shaanxi, Inner Mongolia
	Southwest	Tibet, Sichuan, Yunnan, Guizhou, Guangxi, Chongqing

Table A3.2.2 Sectoral Aggregation in the 2000 Chinese Multiregional Model

3-Sector Aggregation	Industry Sectors
Natural Resources	agriculture
	mining & processing
Manufacturing & Construction	light industry
	energy industry
	heavy industry & chemical industry
	construction
Services & Other Sectors	transportation & telecommunications services
	commercial services
	other

Appendix 3.3 The Balanced Regional Model and the Inverse of a Partitioned $(\mathbf{I}-\mathbf{A})$ Matrix

We use the results from Appendix A on the inverse of a partitioned matrix. For the balanced regional model, let

$$(\mathbf{I}-\mathbf{A}) = \begin{bmatrix} (\mathbf{I}-\mathbf{A}^{RR}) & -\mathbf{A}^{RN} \\ -\mathbf{A}^{NR} & (\mathbf{I}-\mathbf{A}^{NN}) \end{bmatrix} = \begin{bmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{bmatrix} \text{ and } (\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} \mathbf{S} & \mathbf{T} \\ \mathbf{U} & \mathbf{V} \end{bmatrix}.$$

Then, from (3.29)

$$\begin{aligned} \mathbf{x}^R &= \mathbf{S}\mathbf{f}^R + \mathbf{T}\mathbf{f}^N \\ \mathbf{x}^N &= \mathbf{U}\mathbf{f}^R + \mathbf{V}\mathbf{f}^N \end{aligned} \tag{A3.3.1}$$

This generates total output throughout the nation of both regionally balanced goods $(\mathbf{x}^R)$ and nationally balanced goods $(\mathbf{x}^N)$.

In this case, using the partitioned inverse results above, we have

$$\begin{aligned} &\mathbf{S} = (\mathbf{I}-\mathbf{A}^{RR})^{-1}(\mathbf{I}+\mathbf{A}^{RN}\mathbf{U}) \quad \mathbf{T} = (\mathbf{I}-\mathbf{A}^{RR})^{-1}\mathbf{A}^{RN}\mathbf{V} \\ &\mathbf{U} = \mathbf{V}\mathbf{A}^{NR}(\mathbf{I}-\mathbf{A}^{RR})^{-1} \quad \mathbf{V} = [(\mathbf{I}-\mathbf{A}^{NN}) - \mathbf{A}^{NR}(\mathbf{I}-\mathbf{A}^{RR})^{-1}\mathbf{A}^{RN}]^{-1} \end{aligned} \tag{A3.3.2}$$

Substituting for $\mathbf{S}$ and $\mathbf{T}$ in (A3.3.2), from (A3.3.1),

$$\mathbf{x}^R = (\mathbf{I}-\mathbf{A}^{RR})^{-1}\mathbf{f}^R + (\mathbf{I}-\mathbf{A}^{RR})^{-1}\mathbf{A}^{RN}(\mathbf{U}\mathbf{f}^R + \mathbf{V}\mathbf{f}^N) \tag{A3.3.3}$$

But $\mathbf{x}^N$, as in (A3.3.2), is just the $(\mathbf{U}\mathbf{f}^R + \mathbf{V}\mathbf{f}^N)$ term on the right-hand side of (A3.3.3), so

$$\mathbf{x}^R = (\mathbf{I}-\mathbf{A}^{RR})^{-1}\mathbf{f}^R + (\mathbf{I}-\mathbf{A}^{RR})^{-1}\mathbf{A}^{RN}\mathbf{x}^N \tag{A3.3.4}$$

To distribute both $\mathbf{x}^R$ and $\mathbf{x}^N$ production to individual regions, we need the regional distribution of final demands for regional goods – $\mathbf{f}^{R(s)}$, for each region s – and we need the regional distribution of production of each of the nationally balanced goods – $\mathbf{p}^s$ – for each region. Then, to add the spatial dimension, for a specific region s, $\mathbf{f}^R$ becomes $\mathbf{f}^{R(s)}$ and $\mathbf{x}^N$ becomes $\mathbf{x}^{N(s)}$, which is $\hat{\mathbf{p}}^s\mathbf{x}^N$. Therefore

$$\mathbf{x}^{R(s)} = (\mathbf{I} - \mathbf{A}^{RR})^{-1}\mathbf{f}^{R(s)} + (\mathbf{I} - \mathbf{A}^{RR})^{-1}\mathbf{A}^{RN}\hat{\mathbf{p}}^s\mathbf{x}^N \qquad \text{(A3.3.5)}$$

This is (3.31) in the text.

Problems

3.1 The data in problem 2.2 described a small national economy. Consider a region within that national economy that contains firms producing in each of the three sectors. Suppose that the technological structure of production of firms within the region is estimated to be the same as that reflected in the national data, but that there is need to import into the region (from producers elsewhere in the country) some of the inputs used in production in each of the regional sectors. In particular, the percentages of required inputs from sectors 1, 2, and 3 that come from within the region are 60, 90, and 75, respectively. If new final demands for the outputs of the regional producers are projected to be 1300, 100, and 200, what total outputs of the three regional sectors will be needed in order to meet this demand?

3.2 The following data represent sales (in dollars) between and among two sectors in regions r and s.

	r		s	
r	40	50	30	45
	60	10	70	45
s	50	60	50	80
	70	70	50	50

In addition, sales to final demand purchasers were $\mathbf{f}^r = \begin{bmatrix} 200 \\ 200 \end{bmatrix}$ and $\mathbf{f}^s = \begin{bmatrix} 300 \\ 400 \end{bmatrix}$. These data are sufficient to create a two-region interregional input–output model connecting regions r and s. If, because of a stimulated economy, household demand increased by \$280 for the output of sector 1 in region r and by \$360 for the output of sector 2 in region r, what are the new necessary gross outputs from each of the sectors in each of the two regions to satisfy this new final demand? That is, find $\Delta\mathbf{x} = \begin{bmatrix} \Delta\mathbf{x}^r \\ \Delta\mathbf{x}^s \end{bmatrix}$ associated with $\Delta\mathbf{f}$.

3.3 Suppose that you have assembled the following information on the dollar values of purchases of each of two goods in each of two regions, and also on the shipments of each of the two goods between regions:

Purchases in Region r		*Purchases in Region s*	
$z_{11}^r = 40$	$z_{12}^r = 50$	$z_{11}^s = 30$	$z_{12}^s = 45$
$z_{21}^r = 60$	$z_{22}^r = 10$	$z_{21}^s = 70$	$z_{22}^s = 45$
Shipments of Good 1		*Shipments of Good 2*	
$z_1^{rr} = 50$	$z_1^{rs} = 60$	$z_2^{rr} = 50$	$z_2^{rs} = 80$
$z_1^{sr} = 70$	$z_1^{ss} = 70$	$z_2^{sr} = 50$	$z_2^{ss} = 50$

These data are sufficient to generate the necessary matrices for a two-region multiregional input–output model connecting regions r and s. There will be six necessary matrices – $\mathbf{A}^r$, $\mathbf{A}^s$, $\hat{\mathbf{c}}^{rr}$, $\hat{\mathbf{c}}^{rs}$, $\hat{\mathbf{c}}^{sr}$ and $\hat{\mathbf{c}}^{ss}$. All of these will be 2×2 matrices. If the projected demands for the coming period are $\mathbf{f}^r = \begin{bmatrix} 50 \\ 50 \end{bmatrix}$ and $\mathbf{f}^s = \begin{bmatrix} 40 \\ 60 \end{bmatrix}$, find the gross outputs for each sector in each region necessary to satisfy this new final demand; that is, find $\mathbf{x}^r$ and $\mathbf{x}^s$.

3.4 A federal government agency for a three-region country has collected the following data on input purchases for two sectors, (1) manufacturing and (2) agriculture, for last year, in dollars. These flows are not specific with respect to region of origin; that is, they are of the $z_{ij}^{\cdot s}$ sort. Denote the three regions by A, B, and C.

	Region A		Region B		Region C	
	1	2	1	2	1	2
1	200	100	700	400	100	0
2	100	100	100	200	50	0

Also, gross outputs for each of the two sectors in each of the three regions are known. They are:

$$\mathbf{x}^A = \begin{bmatrix} 600 \\ 300 \end{bmatrix}, \ \mathbf{x}^B = \begin{bmatrix} 1200 \\ 700 \end{bmatrix} \text{ and } \mathbf{x}^C = \begin{bmatrix} 200 \\ 0 \end{bmatrix}$$

The agency hires you to advise them on potential uses for this information.

a. Your first thought is to produce a regional technical coefficients table for each region. Is it possible to construct such tables? If so, do it; if not, why not?
b. You also consider putting the data together to generate a national technical coefficients table. Is this possible? If so, do it; if not, why not?
c. Why is it not possible to construct from the given data a three-region multiregional input–output model?

d. If the federal government is considering spending \$5,000 on manufactured goods and \$4,500 on agricultural products next year, what would you estimate as the national gross outputs necessary to satisfy this government demand?
e. Compare the national gross outputs for sectors 1 and 2 found in d, above, with the original gross outputs, given in the data set from last year. What feature of the input–output model does this comparison illustrate?

3.5 Consider the following two-region interregional input–output transactions table:

	North			South			
	Agric. (1)	Mining (2)	Constr. & Manuf.(3)	Agric.(1)	Mining (2)	Const. & Manuf. (3)	Total Output
North							
Agriculture (1)	277, 757	3, 654	1, 710, 816	8, 293	26	179, 483	3, 633, 382
Mining (2)	319	2, 412	598, 591	15	112	30, 921	743, 965
Construction & Manufacturing (3)	342, 956	39, 593	6, 762, 703	45, 770	3, 499	1, 550, 298	10, 931, 024
South							
Agriculture (1)	7, 085	39	98, 386	255, 023	3, 821	1, 669, 107	3, 697, 202
Mining (2)	177	92	15, 966	365	3, 766	669, 710	766, 751
Construction & Manufacturing (3)	71, 798	7, 957	2, 017, 905	316, 256	36, 789	8, 386, 751	14, 449, 941

a. Find the final-demand vectors and the technical coefficients matrices for each region.
b. Assume that the rising price of imported oil (upon which the economy is 99 percent dependent) has forced the construction and manufacturing industry (sector 3) to reduce total output by 10 percent in the South and 5 percent in the North. What are the corresponding amounts of output available for final demand? (Assume interindustry relationships remain the same, that is, the technical coefficients matrix is unchanged.)
c. Assume that tough import quotas imposed in Western Europe and the USA on this country's goods have reduced the final demand for output from the country's construction and manufacturing industries by 15 percent in the North. What is the impact on the output vector for the North region? Use a full two-region interregional model.
d. Answer the question in part c, above, ignoring interregional linkages, that is, using the Leontief inverse for the North region only. What do you conclude about the importance of interregional linkages in this aggregated version of this economy?

3.6 Consider the MRIO transactions table for China given in Table 3.7. Suppose all of the inputs to the North region from the South region were replaced with corresponding industry production from the Rest of China region. How would you reflect such a situation in the MRIO model? What would be the impact on total outputs of all regions and sectors for a final demand of ¥100,000 on export demand for manufactured goods produced in the North?

3.7 A three-region, five-sector version of the US multiregional input–output economy is given in Table A4.1.3 in the next chapter. Suppose that a new government military project is initiated in the western United States which stimulates new final demand in that region of (in millions of dollars) $\Delta\mathbf{f}^W = \begin{bmatrix} 0 & 0 & 100 & 50 & 25 \end{bmatrix}'$. What is the impact on total production of all sectors in all three regions of the United States economy stimulated by this final demand in the West?

3.8 Consider the three-region, five-sector version of an interregional input–output economy of Japan for 1965 given in Table A4.1.1 of Appendix 4.1. Suppose the same final demand vector given in problem 3.7 is placed on goods and services produced in Japan's South region. What is the impact on total production of all sectors in all three regions of Japan of this final demand in the South?

3.9 Consider the year 2000 IRIO model for China, Japan, the United States and an aggregation of other Asian nations including Indonesia, Malaysia, the Philippines, Singapore, and Thailand provided in the table below. Assume that annual final demand growth in China is 8 percent, growth in the USA and Japan is 4 percent, and that of other Asian nations is 3 percent. Compute the percentage growth in total output corresponding to the growth in final demand.

	United States			Japan			China			Rest of Asia		
2000	Nat. Res.	Manuf. & Const.	Services	Nat. Res.	Manuf. & Const.	Services	Nat. Res.	Manuf. & Const.	Services	Nat. Res.	Manuf. & Const.	Services
USA												
Nat. Res.	75,382	296,016	17,829	351	4,764	473	174	403	17	103	2,740	83
Manuf. & Const.	68,424	1,667,042	960,671	160	21,902	3,775	587	8,863	1,710	383	45,066	4,391
Services	95,115	1,148,999	3,094,357	118	6,695	807	160	1,466	296	197	7,393	953
Japan												
Nat. Res.	7	52	53	8,721	78,936	11,206	13	66	2	14	180	27
Manuf. & Const.	859	41,484	11,337	28,088	1,414,078	484,802	764	20,145	2,809	462	72,258	4,108
Services	97	4,390	1,424	24,901	662,488	1,001,832	107	2,763	335	270	7,816	1,189
China												
Nat. Res.	72	343	147	50	2,316	229	49,496	183,509	15,138	102	2,430	99
Manuf. & Const.	331	15,657	6,442	93	10,199	1,989	89,384	892,227	181,932	157	15,093	1,237
Services	38	2,218	1,099	17	1,780	280	25,391	210,469	136,961	23	2,078	132
ROA												
Nat. Res.	322	1,068	203	64	11,906	266	64	1,475	14	12,153	92,647	6,402
Manuf. & Const.	503	56,287	18,129	278	35,418	3,562	1,141	41,496	4,685	23,022	566,274	144,417
Services	152	4,578	1,921	41	3,982	447	138	3,669	422	15,163	213,470	239,053
TOTAL OUTPUT	468,403	5,866,935	11,609,307	140,622	3,883,455	4,658,191	408,153	2,000,741	702,248	173,080	1,727,367	1,225,460

3.10 Assume that you have a very limited computer that can directly determine the inverse of matrices no larger than 2×2. Given this limited computer, explain how you could go about determining $\mathbf{L}$ for

$$\mathbf{A} = \begin{bmatrix} 0 & 0.1 & 0.3 & 0.2 & 0.2 \\ 0.1 & 0.1 & 0.1 & 0 & 0 \\ 0.2 & 0 & 0.1 & 0.3 & 0.1 \\ 0.3 & 0 & 0 & 0.1 & 0.3 \\ 0.3 & 0.2 & 0.1 & 0.1 & 0.2 \end{bmatrix}$$

a. Compute the Leontief inverse in this manner.
b. What implications does such a procedure have for the computation of very large matrices (e.g., $n > 1000$)?

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4 Organization of Basic Data for Input–Output Models

4.1 Introduction

Among the most formidable challenges in using input–output analysis in practice is assembling the detailed basic data needed to construct input–output tables characterizing the economic area of interest – national, regional, or perhaps multiple-regions. There are a variety of means by which these data are compiled, either for the specific area and time period of interest or adapted from other data sources.

In many cases, the data needed for construction of input–output tables are part of a larger collection of data assembled for a wide variety of socio-economic reasons. This is the case particularly in national economies where well-developed systems and standards for collecting economic data exist for such purposes as analyzing economic impacts of government policy that affects the economy, accountability for distribution of government revenues, or simply measuring the ongoing health of the economy. These basic data are often derived from social accounting data assembled in the form of a system of national (or regional) economic information[1] which is often routinely collected by means of a periodic census or some other survey.

4.2 Observations on Ad Hoc Survey-Based Input–Output Tables

Sometimes data collected for an input–output table result from an ad hoc survey designed specifically for that purpose. This is especially common at the regional level, but has seldom led to such data collection becoming a routine part of the region's annual collection of socio-economic data for the region. The methods, conventions, and standards for carrying out ad hoc surveys designed specifically for constructing input–output tables vary widely depending upon circumstances. Some of the classic historical efforts in the United States include Leontief's original work for the US

[1] For example, the US National Economic Accounts are routinely compiled by the US Department of Commerce. The national input–output tables are derived from these accounts periodically. In 1968 the United Nations published a standardized system of national accounts (SNA) which is consistent with the discussion presented here; this system is widely applied in the literature (originally in United Nations, 1968, and more recently in United Nations, 1993). Viet (1994) surveys the common practices adopted by many nations compared with the SNA and many of the considerations are explored in Chapter 5.

national input–output tables (summarized in Appendix C), and regional efforts such as the State of Utah (Moore and Peterson, 1955), the St. Louis Metropolitan Area (Hirsch, 1959), the state of Kansas (Emerson, 1969), the Philadelphia Regional Input–Output Study (Isard and Langford, 1971) and the Washington State series of input–output tables (e.g., Chase, Bourque and Conway, 1993). There have been many other cases, of course, in many regions and nations around the world, and the researchers referenced above along with many others have published chronicles of their experiences, including elaboration of alternative mechanisms for dealing with data shortages, methods for reconciling inconsistent information, and many other best practices in construction of these so-called *survey-based* input–output tables.

4.3 Observations on Common Methods for Generating Input–Output Tables

In planning for virtually any input–output modeling effort, the existence of a statistically robust data source for precisely the geographic area under consideration, for precisely the time period of interest, and for precisely the level of sectoral detail of interest is both the most desirable and the least likely situation. The construction of full *survey-based* input–output tables is a major undertaking that is both complex and expensive, often prohibitively so.

Far more common is the situation where one is faced with adapting previously constructed input–output tables to reflect more current conditions or to make assumptions about the similarity of the geographic area of interest to that of an area where an input–output table already exists. Techniques for adapting existing tables either over time or across space are often referred to as *nonsurvey* methods. Sometimes surveys of selected industry sectors or other institutions may exist that are only part of what would constitute a complete survey of the economy of interest, in which case many methods have been developed to incorporate the selected new information into a strategy for constructing a new table. Such techniques are often referred to as *partial survey* methods. These *nonsurvey* and *partial survey* alternatives to full survey-based construction of input–output tables have been an active area of research for over thirty years and are explored in Chapters 7 and 8.

As noted above, in many cases where use of input–output models has become a part of a government's analysis of economic structure and performance, the collection of data needed for construction of input–output tables has become part of a larger and more routine collection of national economic statistics assembled for a wide variety of socio-economic planning and policy analysis reasons. Since the 1950s many nations have increasingly adopted common conventions for collecting national economic data. In the balance of this chapter we examine many of these conventions.

In developing the most common conventions for national economic accounting that are particularly relevant to input–output analysis, we can also set the stage for some key enhancements to the input–output framework to deal with complications arising from collection of data. For example, for practical reasons the data for input–output tables (and many other purposes) must be collected from business firms or establishments, which are subsequently aggregated to characterize a specific industrial sector. However, it is quite common for a particular firm to produce multiple products or services that are associated with differently defined industrial sectors. Hence, it is important to have standardized ways for our accounting conventions to accommodate such situations. Many of the conventions used in national economic accounting are included in what has become known as the *System of National Accounts* (SNA). In Chapter 5 we deal with many of the important ways to use these conventions to address the complications of secondary production and other features of the economic accounts, but for now we develop the basic concepts of the SNA.

4.4 A System of National Economic Accounts

In an appendix to a landmark 1947 United Nations (UN) report entitled *Measurement of National Income and the Construction of Social Accounts*, British economist Richard Stone set forth the basic framework for the standardized system of national economic accounts that is most commonly used around the world today (United Nations, 1947, and Stone, 1947). These concepts were formalized in the subsequent 1950 UN report, *A System of National Accounts and Supporting Tables*, and, finally, in 1968 Stone led the team that produced an integrated input–output framework and system of national accounts (United Nations, 1968), work coupled with his related and subsequent contributions (e.g., Stone, 1961) for which he received the 1984 Nobel Prize in Economic Science "for having made fundamental contributions to the development of the system of national accounts and hence greatly improved the basis for empirical economic analysis."[2]

In Stone's work on national income and production accounting, he describes input–output transactions tables as a "bridge between statistics that can actually be collected about the productive process and the requirements of applied economic analysis" (Stone, 1961). Fashioning an ability to "build" this bridge in an organized way, however, was a key development in making input–output the practical tool it has become. In the following we describe the relationship between input–output tables and national (or regional) economic accounts and, in the process, show how input–output tables can be derived from such accounts. We focus on the SNA noted above, which continues to be developed to refine the methodology and broadly standardize the definitions and accounting rules that are used widely today. Published updates to the SNA standards and conventions have appeared in United Nations (1993, 1999, 2004).

A key concept in the development of the SNA throughout its history has been the adoption of an accounting structure in which not only economic production could be subdivided so as to display the commodity flows between industries, the traditional basis for Leontief's model, but also that such information could be reconciled with all the relevant information tracing the flows of income and wealth ultimately associated with those flows. This is accomplished by means of "balance sheets" for the key sectors

[2] From the citation announcing the award of *The Bank of Sweden 1984 Prize in Economic Sciences in Memory of Alfred Nobel*, awarded December 8, 1984, as reported in Sveriges Riksbank (1984).

of the economy. For purposes of this text, and in particular in this chapter, we focus on the use of the SNA to facilitate construction of input–output models, but it is important to recognize that the SNA provides the basis for providing a national "balance sheet" and for describing and analyzing economic change for many forms of economic decision making.[3]

The data included in the SNA also enables expansion of the basic input–output framework to handle systematically such issues as secondary production in the economy, as noted above. These so-called "commodity-by-industry" concepts, and their implications for input–output models, are developed in much more detail in Chapter 5 but are introduced in this chapter. In addition the SNA provides the basis for broader social accounting modeling, building once again on the Leontief model. These extensions are examined in Chapters 9, 10, and 11.

4.4.1 The Circular Flow of Income and Consumer Expenditure

As noted earlier the principal goal of the SNA is to provide "a framework within which the statistical information needed to analyze the economic process in all of its many aspects could be organized and related."[4] Conceptually, this takes us back to the fundamental roots of much of economic thought – the notion of a circular flow of economic resources in an economy introduced by Cantillon and Quésnay in the eighteenth century. These conceptual beginnings are described in Appendix C.

In the simplest of economies there are businesses that produce goods and services and household consumers that purchase them. The consumers also manage and work for the businesses and, hence, receive income from them, the value of which exactly equals the total value of their purchases (see Figure 4.1). This is the fundamental tenet of the circular flow of income and expenditures in that the total value of production can be measured either by the value of all goods and services delivered to households or by the payments for the factors of production delivered by consumer households, as depicted in Figure 4.1.

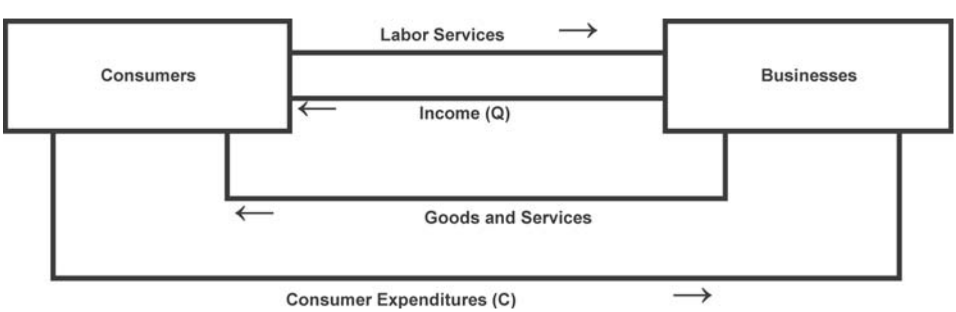


Figure 4.1 The Circular Flow of Income and Expenditures

[3] United Nations (1968), Chapter 1, p. 12.
[4] United Nations (1968), Chapter 1, p. 13.

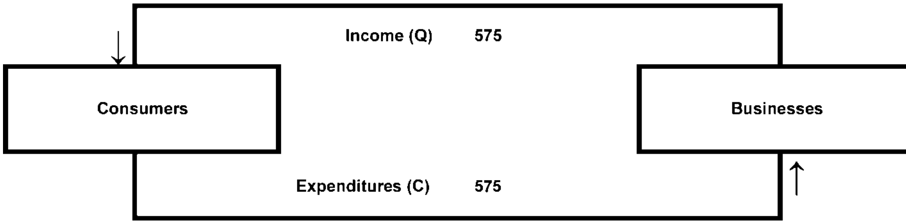


Figure 4.2 Circular Flow Example: Point of Departure

From now on in this chapter we will show only the flow of money associated with transactions, e.g., in Figure 4.1 the income received by consumers in return for their labor services and the expenditures made by consumers in return for goods and services delivered by businesses. We begin with a simple example, depicted in Figure 4.2, where consumers receive \$575 million dollars as income for the labor services that they provide to businesses (Q) and, in turn, they use that income to purchase the same value of goods and services from businesses (C). That is, the total of expenditures equals the total of all income, so in this simplest of economies the circular flow is described as the equally simple (4.1):

$$Q = C \tag{4.1}$$

4.4.2 Savings and Investment

As our first refinement of this most simple of economies, recall that businesses, in the course of producing the goods and services they deliver to final consumers, also consume inputs other than labor services, such as raw materials and capital equipment. So, once again as in Chapter 2, we refer to deliveries to final markets as *final demand*, deliveries of goods and services to other businesses as *intermediate output*, and purchases of goods and services by businesses that are not resold as intermediate goods to other firms or consumers, i.e., they become long-term depreciable assets, as *capital investments*.

For the time being, we will leave industrial production and use of intermediate output in the course of that production, the major focus of input–output analysis, within the block labeled "Businesses" in our flow diagram. Figure 4.3 depicts the addition of capital expenditures as a portion of total expenditures in the economy.

In general some portion of consumer income is not spent on final goods, but rather saved or invested for longer term financial gain. In Figure 4.3 the portion of consumer income that is not spent on final goods is referred to as *personal savings* (S) and, in our simple economy, we add a *capital market*, to hold personal savings on behalf of consumers and lend it to businesses to purchase *capital goods* as investments.

Note that in the example (Figure 4.3) the total income generated by businesses remains \$575 million, but current consumer expenditures are reduced to \$525 million, with a \$50 million residual that is not used for current consumption reserved as savings (S) in the capital markets, which in turn provides that same amount as investment resources (I) to businesses acquiring capital goods.

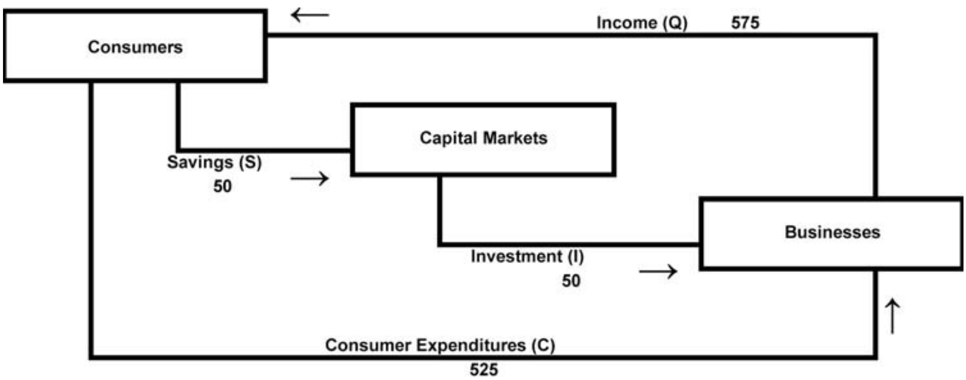


Figure 4.3 Introduction of Savings and Investment into the Circular Flow of Income and Expenditures

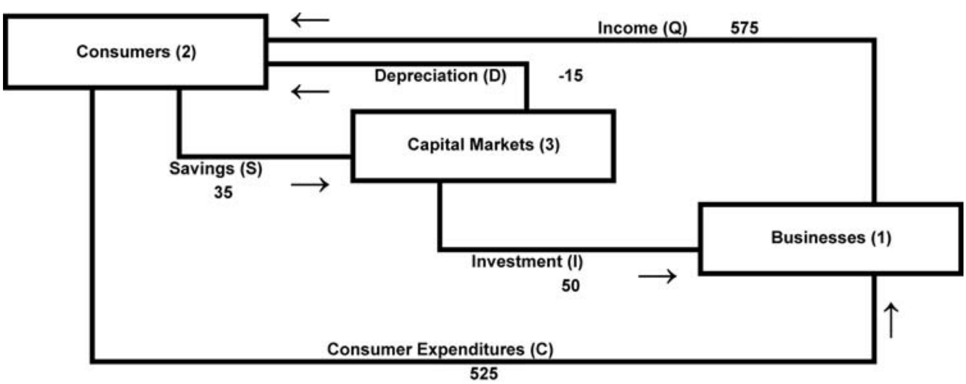


Figure 4.4 Introduction of Depreciation into the Circular Flow of Income and Expenditures

As we proceed, sequentially unbundling activities to reflect more realistic complexity in the economy, we will find we need to identify a way to include the valuation of not only the transactions or *flows* in the economy, but also we seek to reconcile those transactions with the valuation of the assets and liabilities or *stocks* in the economy, which we will accomplish by means of a national "balance sheet." For the moment, however, we will only account for transactions or flows that relate to *accumulations* of stocks, not the valuation of the stocks themselves.

Returning to the example, we are generally interested in measuring the key economic flows in our economy, income, and expenditures, over a standardized period, e.g., a year. Expenditures on capital, which we can loosely define as the use of goods and services that extend beyond the standardized period, accumulate in the economy as a stock, only a portion of which gets "consumed" during the current period. This is why, in national income accounting parlance, this depreciation of a capital investment is often referred to as a *capital consumption allowance*. Figure 4.4 includes a capital consumption allowance, labeled *depreciation* (*D*), to reflect the depreciation of the

Table 4.1 Basic National Accounts: Example Economy

Debits		Credits	
Production (Domestic Product Account)			
Income (Q)	575	Sales of consumption goods (C)	525
		Sales of capital goods (I)	50
Total	575	Total	575
Consumption (Income and Outlay Account)			
Purchases of consumption goods (C)	525	Income (Q)	575
Savings (S)	35	Depreciation (D)	−15
Total	560	Total	560
Accumulation (Capital Transactions Account)			
Purchase of capital goods (I)	50	Savings (S)	35
Depreciation (D)	−15		
	0		
Total	35	Total	35

accumulated stock of capital in the current period. Investment flows to businesses (I) are then drawn from the stock of capital.

We can begin to keep track of the transactions systematically in our simple economy by maintaining some traditional accounting balance sheets (sometimes called "T" accounts), one for each major type of economic activity: (1) production by businesses, often called the *Domestic Product Account*, (2) consumption by consumers, often called the *Income and Outlays Account*, and (3) capital accumulation in the capital markets, often called the *Capital Transactions Account*. For our example so far these accounts are shown in Table 4.1.[5]

We can also summarize these double entry bookkeeping accounts by the following simple balance equations:

$$Q = C + I \tag{4.2}$$

$$C + S = Q + D \tag{4.3}$$

$$I + D = S \tag{4.4}$$

[5] For more detailed expositions on the relationship between macroeconomics, national economic accounts, and input–output analysis, see Gordon (1978), Sommers (1985), and especially United Nations (1968, 1993, 1999, and 2004).

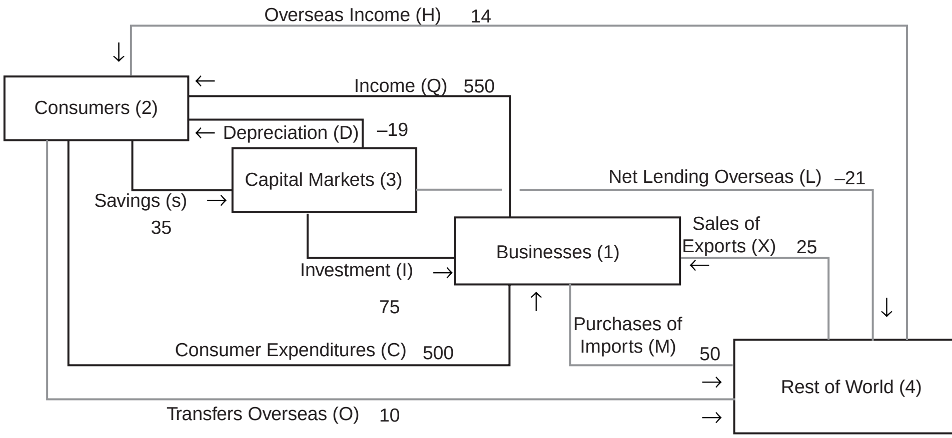


Figure 4.5 Addition of the Rest of World Account

4.4.3 Adding Overseas Transactions: Imports, Exports, and Other Transactions

Our next refinement recognizes that if some of the businesses in our simple economy are located overseas, it will be important for a variety of reasons to account for them separately. We can do this by defining and adding the various transactions with overseas businesses as imports (M) and exports (X) of goods and services, consumer expenditures overseas (called overseas transfers, O), the net of lending and borrowing of capital overseas (net overseas lending, L), and consumer income received from overseas (H). For our example, these additional transactions are depicted in Figure 4.5, along with the addition of a new category of economic activity and corresponding balance sheet called "Rest of World."

The corresponding balance sheet is often called the *Balance of Payments Account*, which is included in Table 4.2 along with the other revised accounts. We can also summarize these expanded double entry bookkeeping accounts by the following set of balance equations:

$$Q + M = C + I + X \tag{4.5}$$

$$C + S + O = Q + D + H \tag{4.6}$$

$$I + D + L = S \tag{4.7}$$

$$X + H = M + O + L \tag{4.8}$$

Conceptually, a useful way to visualize these balance equations is to draw a circle around each block in the flow diagram, then calculate the sum of all the transactions going into the block, which will equal the sum of all the transactions leaving the block or account.

Table 4.2 Basic National Accounts Including Rest of World

Debits		Credits	
Production (Domestic Product Account)			
Consumer Income Payments (Q)	550	Sales of consumption goods (C)	500
Purchases of Imports (M)	50	Sales of capital goods (I)	75
		Sales of Exports (X)	25
Total	600	Total	600
Consumption (Income and Outlay Account)			
Purchases of consumption goods (C)	500	Income (Q)	550
Net Transfers Overseas (O)	10	Depreciation (D)	−19
Savings (S)	35	Net Overseas Income (H)	14
Total	545	Total	545
Accumulation (Capital Transactions Account)			
Purchases of capital goods (I)	75	Savings (S)	35
Depreciation (D)	−19		
Net Lending Overseas (L)	−21		
Total	35	Total	35
Rest of World (Balance of Payments Account)			
Purchases of Exports (X)	25	Sales of Imports (M)	50
Net Overseas Income (H)	14	Net Transfers Overseas (O)	10
		Net Borrowing Overseas (L)	−21
Total	39	Total	39

4.4.4 The Government Sector

Finally, the major role of government in most economies suggests it should also be included explicitly as a major activity in our portfolio of national economic accounts, which we label the *Government Account*. This involves highlighting government transactions, including taxes paid by consumers (T), government purchases of goods and services (G), and government deficit spending (B). For our example these additions are reflected in Figure 4.6. The corresponding modified national accounts are shown in Table 4.3.

We can, once again, also summarize these double entry bookkeeping accounts by the following balance equations:

$$Q + M = C + I + X + G \tag{4.9}$$

$$C + S + O + T = Q + D + H \tag{4.10}$$

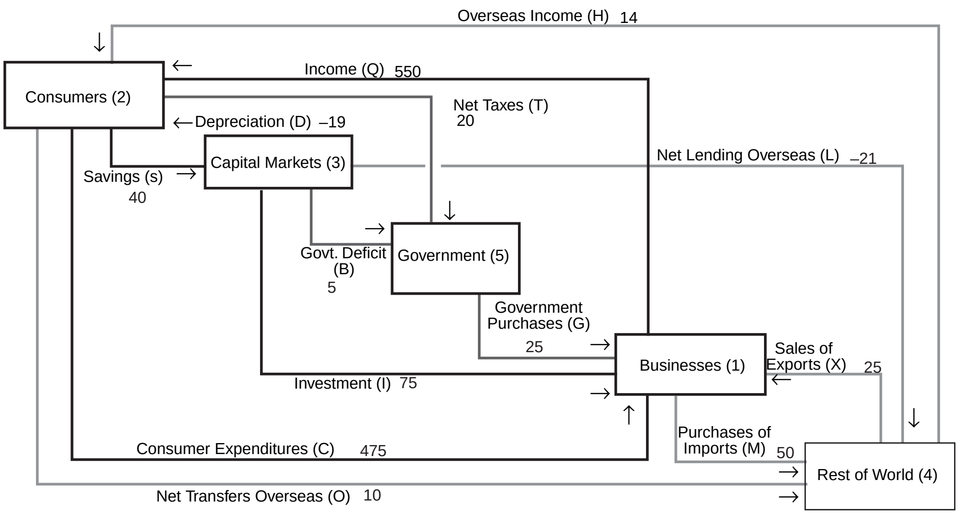


Figure 4.6 Addition of the Government Account

$$I + D + L + B = S \tag{4.11}$$

$$X + H = M + O + L \tag{4.12}$$

$$G = T + B \tag{4.13}$$

4.4.5 The Consolidated Balance Statement for National Accounts

For convenience we can summarize all the double entry bookkeeping accounts (and corresponding balance equations) we have accumulated so far much more succinctly in a single *Balance Statement*, provided in Table 4.4. Subsequently, and even more compactly, we can represent the transactions (each of which appears twice in our current tables) by entries in a matrix with the nature of the transaction (source and destination) to be inferred from the transaction's position in the matrix (Table 4.5).

Through this example, the reader may anticipate that we are gradually working our way to the more familiar input–output table format. This will be apparent ultimately by subdividing the Production and Consumption transactions to show activities and output in specific industries and the use of specific products. For example, the Production–Consumption transaction, instead of being a single number (475 in the example), will be represented by a matrix (often called the *Use* matrix) with rows indicating specific products or commodities and columns indicating specific industries.

At this point we have characterized the major economic activities in our simple economy by representing them in a matrix that captures the information in the following basic series of principal national economic accounts:

1. Production of Goods and Services or the *Domestic Product Account*
2. Consumption of Goods and Services or the *Income and Outlay Account*
3. Accumulation of Capital or the *Capital Transactions Account*
4. Imports and Exports or the *Balance of Payments Account*
5. Government or the *Government Account*

So far, with the partial exception of transactions associated with capital accumulation, we have accounted for only transactions or *flows* in the economy. We have largely ignored accounting for the total value of accumulated assets and liabilities or *stocks*

Table 4.3 Basic National Accounts Including the Government Sector

Debits		Credits	
Production (Domestic Product Account)			
Consumer Income Payments (Q)	550	Sales of consumption goods (C)	475
Purchases of Imports (M)	50	Sales of capital goods (I)	75
		Government Purchases (G)	25
		Sales of Exports (X)	25
Total	600	Total	600
Consumption (Income and Outlay Account)			
Purchases of consumption goods (C)	475	Income (Q)	550
Net Transfers Overseas (O)	10	Depreciation (D)	−19
Taxes (T)	20	Net Overseas Income (H)	14
Savings (S)	40		
Total	545	Total	545
Accumulation (Capital Transactions Account)			
Purchase of capital goods (I)	75	Savings (S)	40
Depreciation (D)	−19		
Government deficit spending (B)	5		
Net Lending Overseas (L)	−21		
Total	40	Total	40
Rest of World (Balance of Payments Account)			
Purchases of Exports (X)	25	Sales of Imports (M)	50
Net Overseas Income (H)	14	Net Transfers Overseas (O)	10
		Net Borrowing Overseas (L)	−21
Total	39	Total	39
Government (Government Account)			
Government purchases (G)	25	Taxes (T)	20
		Government deficit spending (B)	5
Total	25	Total	25

Table 4.4 Balance Statement for the Basic National Accounts

Debits							Credits				
Prod.	Cons.	Capital Accum.	Govt.	Rest of World	Var.	Economic Transaction	Prod.	Cons.	Capital Accum.	Govt.	Rest of World
	475				C	Consumption Goods	475				
		75			I	Capital Goods	75				
				25	X	Exports	25				
50					M	Imports					50
550					Q	Income		550			
		−19			D	Depreciation		−19			
				14	H	Overseas Income		14			
	10				O	Transfers Abroad					10
	40				S	Savings			40		
		−21			L	Net Lending Abroad					−21
			25		G	Govt. Expenditures	25				
	20				T	Taxes				20	
		5			B	Govt. Deficit Spending				5	
600	545	40	25	39		Totals	600	545	40	25	39

Table 4.5 The Basic National Accounts Balance Statement in Matrix Form

	Prod.	Cons.	Cap.	ROW	Govt.	Total
Production		475	75	25	25	600
Consumption	550		−19	14		545
Capital Accum.		40				40
Rest of World	50	10	−21			39
Government		20	5			25
Total	600	545	40	39	25	

in the economy. Conceptually we can accomplish this by incorporating the balance statement into an accounting balance sheet with a valuation of opening net assets in the economy, i.e., the depreciated value of tangible assets held plus the excess of any financial claims held as assets over financial claims issued as liabilities, which is defined as the "net worth" of the economy.

As always, we measure all of these activities in terms of the value of the transaction, but so far we have not specified the prices used in valuing these transactions or how we can accommodate price changes. Conceptually, we seek to incorporate all the economic activities we have described so far as well as market changes in prices of goods and services. As we learned in Chapter 2, in input–output analysis we will most often assume fixed prices in any given analysis, but we need to be able to account for year-to-year revaluations of assets and liabilities.

As we devise a "system" of national accounts to track the evolution of the economy from one period to the next, for any given period of time, we will trace the transformation of an "opening" balance sheet into a "closing" balance sheet in two equivalent ways: (1) the net assets at the end of the period are equal to the sum of net assets at the beginning of the period, net domestic and foreign investment during the period, and revaluations needed to adjust the value of assets previously acquired or liabilities previously issued to the prices in place at the closing date; and (2) the net worth at the end of the period is equal to the net worth at the beginning of the period plus new savings accumulated during the period and revaluations resulting from price changes.

4.4.6 Expressing Net Worth

Now we can add the notion of *Initial* and *Closing* asset values, which in moving from one to the other are transformed by interaction of consumption, production, capital accumulation, and net exports, as well as asset and liability revaluations resulting from price changes of goods and services. With two basic observations it follows that the opening and closing balance sheets are related by the effect of price revaluations and the net effect of capital transactions. The two observations are the following: (1) total saving in the economy is equal to net capital investment and (2) net worth can only be revalued by applying to it the revaluation determined from the price changes of tangible assets and financial claims.

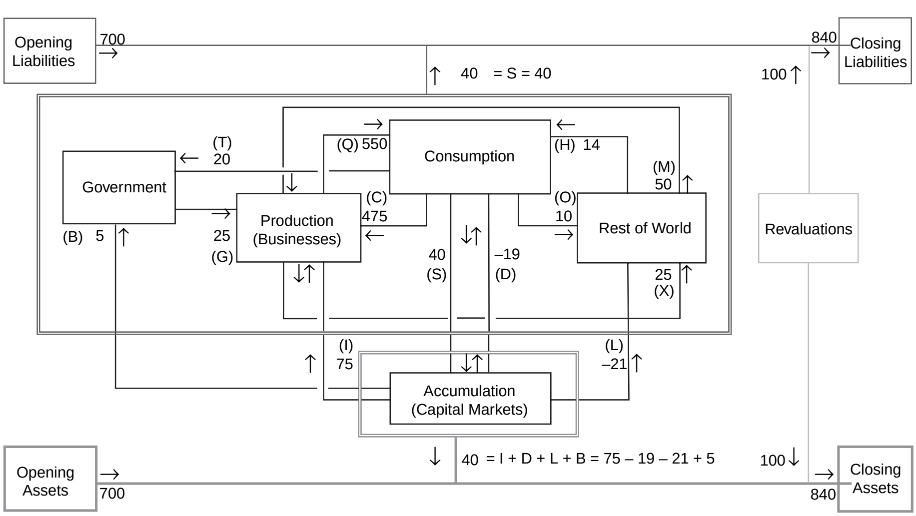


Figure 4.7 Net Worth

Table 4.6 Matrix of National Accounts Including Net Worth Calculations

	NOA	Prod.	Cons.	Cap.	ROW	Govt.	Reval	NCA
Net Opening Assets (NOA)				700				
Production			475	75	25	25		
Consumption		550		−19	14			
Capital Accum.	700		40				100	840
Rest of World (ROW)		50	10	−21				
Govt.			20	5				
Revaluations (Reval)				100				
Net Closing Assets (NCA)				840				

The relationship between opening and closing balance sheets for our example is depicted schematically in Figure 4.7. Note that this expanded schematic reflects two changes: (1) reformatting of Figure 4.6, specifically highlighting (4.11), which is the balance equation for allocation of savings, i.e., consumer income not spent on current consumption is otherwise invested either directly as private investments or indirectly as net taxes (the difference between gross payments for taxes and that portion of taxes that are returned to consumers, often referred to as *welfare transfers*). Savings during the period become additions to the stock of accumulated capital so the opening value of assets is increased by the amount of savings, or equivalently the level of investment diminished by depreciation, net lending abroad, and government deficit spending, which is

$$S = I + D + L + B \tag{4.14}$$

For our example, the accumulated saving is $S = 40$ and, if revaluations due to price changes amount to 100, then the opening asset value (net worth) of 700 is increased at the end of the period by sum of accumulated savings and revaluations, or $700 + 40 + 100 = 840$. We can express this much more succinctly by expanding the basic matrix presented in Table 4.5 to include opening net worth or net opening assets (NOA) closing net worth or net closing assets (NCA), which includes revaluations, as shown in Table 4.6.

We can recap the balance equations, now including the net worth balances, by defining R as the total of revaluations due to price changes, W_1 as the opening net worth, and W_2 as the closing net worth to yield the following:

$$Q + M = C + I + X + G \tag{4.15}$$

$$C + S + O + T = Q + D + H \tag{4.16}$$

$$I + D + L + B = S \tag{4.17}$$

$$X + H = M + O + L \tag{4.18}$$

$$G = T + B \tag{4.19}$$

$$W_2 = W_1 + S + R \tag{4.20}$$

4.5 National Income and Product Accounting Conventions

So far in this chapter we have developed the system of national income and product accounts sufficiently that we can review some of the traditional assumptions and conventions often used in compiling these accounts. For the most part these conventions are completely consistent with the basic input–output framework developed in Chapter 2, but some of the peculiarities of nomenclature are different and worth noting. Perhaps the most important characteristic is that the system is a closed system, i.e., the system taken together accounts for all activities in the economy such that the value of total production is equal to total consumption (production and consumption are balanced). This will be of more interest in Chapter 11 when the input–output table is expanded to a Social Accounting Matrix, which is formulated specifically to represent all components of the closed system of national accounts.

The following are some basic principles included in most common systems of national accounts:[6]

- **Double Entry Bookkeeping**. The national economic accounts are generally maintained as a traditional double entry bookkeeping accounting system, tabulating total economic output on the debit side and the total resulting income flows on the credit side. That is, by adopting the financial accounting convention of a "T" account we can characterize a business establishment's production account (BEPA) in two columns, with debits to the account (expenses and profits) recorded in the left-hand column and credits (sales and other revenue) recorded in the right-hand column. The two columns are usually separated and labeled with lines resembling a T; hence the name, "T" account. If net income before taxes is viewed as a payment to capital, then total income, recorded as employee compensation and earnings from real and financial property, is equal to total cost including depreciation.
- **Output Equals Demand**. Total output of the economic system is exactly equal to total demand or, equivalently, gross national product is the same as gross national expenditure. Perhaps the key concept is that inventories are essentially a reservoir that appears as either a slack or surplus variable for output. If demand in final markets (final demand) falls short of output, inventories would rise – an inventory accumulation. If demand exceeds output then the result is an inventory drawdown – a negative demand or inventory liquidation. The result is that output always equals demand with inventories providing the accounting convention for handling surpluses or shortages in any given period of time.
- **Requiring Total Expenditure to Equal Total Income**. We have presumed in our running example that total output in the economy is equivalent to both total expenditures as well as total income generated. This means if consumers spend less than their total generated income, then the unspent balance goes into savings, as we derived from the original concept of a circular flow. Conversely, in current terms businesses typically spend more than their generated income, which comes in the form of net

[6] United Nations (1968, 1993, and 1999), Sommers (1985), Gordon (1978).

investment. So, in general, the total of savings in the economy is equal to total investment, as in the example.

- **Avoiding Double Counting of Output of Goods and Services**. We measure economic output as the value of output delivered to final demand. If copper produced by a copper company is counted both when it is delivered to a circuit board maker and in the final sale of the circuit board that incorporates the original copper, then the copper is counted twice. Rather, we think of the value of the copper as embedded in the value of the circuit board and we measure the value added at each stage of production, the sum of which is total economic output.
- **Valuing Output at Market Value**. Generally we assume that all industry output is valued at the prevailing producers' market price or the sales price. Some output does not pass though a market, however, e.g., food produced and consumed on a family farm. In such cases, the market value is usually estimated and registered as *imputed income or expenditure*.
- **Measuring Gross Domestic Product**. The value of economic output includes the value of tangible goods as well as new construction and services. It is called the Gross Domestic Product (GDP) since contributions to GDP are compiled as *gross* measures, i.e., recorded prior to accounting for depreciation (or capital consumption allowances). Note that the GDP measures the total amount of goods and services that are produced within a country's geographic borders. A related measure, the Gross National Product (GNP), measures the total amount of goods and services that a country's citizens produce regardless of where they produce them. For example, GNP for the United States includes such items as corporate profits that multinational firms earn in overseas markets whereas, in GDP terms, such profits contribute to GDP overseas.
- **Excluding Revaluations**. As noted earlier, during any given time period, the national accounts do not include capital gains and losses – the creation or destruction of value not attributable to real economic output or income generated. That is, revaluations resulting from price changes provide the principal connection between periods but are excluded from valuation during any given period.

4.6 Assembling Input–Output Accounts: The US Case

Our final task in getting us from the basic concept of the circular flow of income and expenditures and tracing income and expense through the national economic accounts back to where we started in Chapter 1, namely the input–output transactions table, is to expand the national economic accounts to include industry and commodity levels of detail. As noted earlier, in practice compiling the data needed to provide this detail on interindustry transactions is often accomplished through a census or survey of all economic activity of establishments or firms involved in the economy. For example, in the United States, the so-called "benchmark" IO accounts (IOAs) are prepared every five years for years ending with 2 and 7 (e.g., 1992, 1997, 2002, and 2007) based on the quinquennial US economic census and other data sources.[7] The basic data are collected from various government agencies, such as the Department of Labor's surveys of prices and the Department of Commerce's Census Bureau surveys of retailers and manufacturers, but the IOAs are prepared by the Bureau of Economic Analysis (BEA), an agency of the Commerce Department.

Historically, a primary goal of the US economic census has been to estimate the nation's GDP (and GNP), as reported in the detailed series of Gross Domestic Product by Industry (GDPI) Accounts (see, for example, Yuskavage and Pho, 2004) and ultimately included in the formal national economic accounts recording the nation's overall generation of income and production, the National Income and Product Accounts (NIPA). The level of industry production and intermediate consumption detail collected during the census also provides a convenient basis for construction of the interindustry IOAs as well, but the sequence of preparation of these data is actually reversed from the order in which we have discussed them so far in this chapter. That is, since the 1950s, in order to provide as much consistency as possible between various economic accounting systems used by the BEA, the preparation of these systems has been coordinated among data collection processes such that the IOAs are generally assembled first and comprise the basic building blocks for constructing a wide variety of national economic accounts, including GDPI and NIPA (see Jaszi, 1986).

The US IOAs are published in the form of two tables defined in terms of production and consumption of defined goods and services or *commodities* by *industries* or groups of economic establishments that may produce more than one commodity. Both commodities and industries are grouped according to a standardized classification scheme, such as the North American Industrial Classification System (NAICS) adopted in 1997 as a replacement for the former Standard Industrial Classification (SIC) and other systems for organizing US economic data.[8]

The first key table of the IOAs is the so-called "Use" table, which provides information on the *consumption* or use of commodities by industries or by final demand sectors, such as households, government, investment or exports. A column of the Use table is an industry or final-demand sector and the rows indicate the use of commodities by that industry or final-demand sector. Also included in the Use table are rows corresponding to components of value added by industry, such as employee compensation, business taxes, and other value added. The basic organization of the Use table is provided in Table 4.7. Note that in this table the sectoral designations for industries and commodities are the same, i.e., the Use matrix is square with the labels of industries and commodities identical. This simply means that we organize accounting for industries

[7] A great deal of the data used in constructing the US benchmark IOAs comes from the quinquennial US economic census, but additional data is often utilized from other sources, particularly in some economic sectors such as natural resources and mining, financial activities and services; see Webb (1995), Lawson *et al.* (2002) and Moyer *et al.* (2004a and 2004b).

[8] The NAICS was adopted to better reflect the changing structure of North American economies since the early 1980s, including in particular the large growth in services-producing industries. For example, the NAICS defines 575 services-producing industries as opposed to 407 in the SIC. About 250 of the 358 new industries defined in the NAICS-based IO industry classification system are services-producing industries (see Horowitz and McCulla, 2001, and McCulla and Moylan, 2003).

Table 4.7 The Commodity-by-Industry Use Table

	Industries								Final Demand				Total Commodity
	Nat. Res.	Const.	Manuf.	Transp.	Util.	Inform.	Fin. Ser.	Other Ser.	Pers. Cons.	Invest.	Net Exports	Govt.	Output
Commodities													
Natural Resources													
Construction													
Manufacturing													
Transportation													
Utilities					Interindustry Transactions								
Information													
Financial Services													
Other Services													
Value Added													
Employee Compensation													
Indirect Business Tax										Gross Domestic Product			
Other Value Added													
Total Industry Output													

and commodities in the economy by the same sector definitions. This is a standard convention adopted for the US input–output tables and many others for a variety of reasons, but it need not be, as we will find later in this chapter and in much more detail in Chapter 5. The numbers of industries and commodities and their definitions can be different, in which case the Use table may be nonsquare.

The second key table of the IOAs is the so-called Make table, the transpose of which is also sometimes referred to as the Supply table. The Make table provides information on industry *production* of commodities. A column of the Make table corresponds to a commodity and rows indicate the production of that commodity by different industries. If there were a one-to-one correspondence between industries and commodities, i.e., each industry produces one and only one commodity, then the Make table would be square and contain nonzero elements only along its main diagonal. Later on in this chapter, and in much more detail in Chapter 5, we examine alternative ways of combining the information in Make and Use tables to fashion the interindustry transactions matrix used in input–output models. An illustration of the basic organization of the Make table is provided in Table 4.8.

Finally, it is common to present the Make and Use matrices as a consolidated set of IOAs in the format of Table 4.9. In Chapter 2 we noted that final demand and value added sectors of an input–output economy can be viewed as somewhat exogenous to the more closely interrelated system of industrial sectors. The principal components of final demand are usually taken to be personal consumption expenditures (purchases by households), government purchases, gross private domestic capital investment, and finally, net exports of goods and services – that is, exports of goods and services less the value of any imports of those same goods and services. In Table 4.9 all of these transactions are aggregated into the column labeled Total Final Demand, which specifies total final demand for commodities.

The income categories comprising value-added inputs to industries usually include wages and salaries paid to employees, rental and proprietors' income, profits, taxes, interest, adjustments to inventories, and noncompetitive imports and, in Table 4.9, all of these transactions are aggregated into the row labeled Total Value Added, which specifies the total of value-added inputs for each industry.

4.7 Additional Considerations

An important challenge in assembling IO accounts, beyond simply coping with the scale and expense of a comprehensive census or survey, has to do with timeliness for the various uses of these data. If an industry's technology is changing rapidly, such as the computer industry over the decades of the 1980s and 1990s, lengthy time lags in availability of the IO accounts can lead to very misleading results if used in economic modeling. In such situations, nonsurvey tools or partial surveys of the type discussed in detail in Chapters 7 and 8 are often used. Specifically, in the US, data for the years intervening the availability of the US benchmark IOAs are provided by the Annual Input–Output Accounts (AIOA), such as reported in Okubo, Lawson and

Table 4.8 The Industry-by-Commodity Make Table

	Commodities								Total Industry Output
	Nat. Res.	Const.	Manuf.	Transp.	Util.	Inform.	Fin. Ser.	Other Ser.	
Industries									
Natural Resources									
Construction									
Manufacturing									
Transportation									
Utilities									
Information									
Financial Services									
Other Services									
Total Commodity Output									

Table 4.9 Consolidated Make and Use Accounts

		Commodities Nat. Res. Const. Manuf. Transp. Util. Inform. Fin. Ser. Other Ser.	Industries Nat. Res. Const. Manuf. Transp. Util. Inform. Fin. Ser. Other Ser.	**Total Final Demand**	**Total Output**
Commodities	Natural Resources Construction Manufacturing Transportation Utilities Information Financial Services Other Services		**Use Matrix**	Final Demand	Total Commodity Output
Industries	Natural Resources Construction Manufacturing Transportation Utilities Information Financial Services Other Services	**Make Matrix**			Total Industry Output
Total Value Added			Value Added	GDP	
Total Output		Total Commodity Output	Total Industry Output		Total Output

Planting (2000), Kuhbach and Planting (2001), and Planting and Kuhbach (2001), where more frequent but more highly aggregated annual surveys are used to update earlier benchmark input–output accounts.

Conceptually, in constructing an input–output model, the most important components of a system of national economic accounts are: (1) the national income and product accounts (NIPAs) where we started and (2) the interindustry or input–output accounts (IOAs). The former, as we have found so far in this chapter, present the aggregated productive output of the national[9] economy, that is, the GDP both in terms of final products or final demands and in terms of income categories or value added inputs to industries.[10] The IOAs present interindustry flows of goods and services which, with a number of adjustments we describe later, ultimately become the interindustry transactions matrix.

The NIPAs and IOAs comprise perhaps the bulk of the basic data of the national economic accounts (at least those relevant to input–output analysis) and, as noted earlier, are often compiled as part of a census and, hence, they are usually collected by establishment or individual business unit. For the present, we will invoke three additional simplifying assumptions in order to facilitate our discussion of reconciling the IOAs with the NIPAs:

- **Inventory Adjustments**. We ignore, for the time being, the complications of inventory adjustments, that is, we assume no changes in inventories. All automobiles produced are purchased during the current year and are not held over until the next year. We saw earlier that inventory adjustments are essential to provide the balance between total consumption and total output.

[9] For the most part, the following discussion will apply to regional as well as to national accounts.

[10] Recall from Chapter 2 that the sum of all final demands equals GNP, which also equals the sum of all income types or "charges against" GNP.

- **Secondary Products**. By distinguishing between commodities and industries, we assume an industry can produce multiple products (called *secondary* production). In the basic framework we allowed each industry to produce one and only one distinct commodity (or service), e.g., automobile manufacturers make only automobiles, not additional automobile parts which may be classified as a different industry category. The implications of this distinction will become much clearer in Chapter 5, and ways of accommodating this distinction in modeling become straightforward by using the Make matrix to allocate secondary production in an organized way.
- **Capital Formation**. We ignore, for the present, transactions of capital goods *between* industries; they are instead assigned to final demand (gross private domestic capital formation). New-car assembly equipment purchased by automobile manufacturers is a capital good acquisition; it is recorded as a final demand for capital by the manufacturer and not as an interindustry transaction.

These assumptions are somewhat limiting and can be relaxed with some additional modifications that we address in later chapters. The first of these simplifications will require relatively minor adjustments later in this chapter, but the last two will require major changes in the way we construct input–output models. While there are a number of different ways to deal with the problem of secondary production, ultimately we will resort to alternative commodity-by-industry model formulations that we introduce later in this chapter and develop in more detail in Chapter 5. Modeling of capital formation is the principal concern of dynamic input–output models and is a subject of Chapter 13.

The basic input–output accounts we have focused on in this chapter, by themselves, are far from adequate for constructing a useful input–output model. In order to make the derived table a useful analytical tool, we must deal with the simplifications that were made earlier. We now discuss a number of conventions and modifications to the basic input–output accounting framework that are designed to deal with several of these simplifying assumptions (others are addressed in subsequent chapters).

4.7.1 Secondary Production: Method of Reallocation

As noted earlier, in the construction of IOAs we compiled data by establishment or individual business unit; we assigned an establishment to a defined "industry" category according to the output of the establishment which comprises the primary source of revenues (primary product). Many business units, however, may produce substantial amounts of products that do not belong to the primary product industry classification; such products are termed *secondary products*. For example, many automobile manufacturers may produce automobile parts in addition to fully assembled automobiles, or petroleum refiners may produce petrochemicals as a by-product to producing gasoline or other petroleum products.

Early input–output studies, such as the pre-1972 US national tables, treated secondary products in the following manner. First, selected secondary products were "reallocated," that is, the level of secondary production *and* its constituent inputs were assigned to the sector defining that product as primary output. Such treatment was used only for

Table 4.10 Input–Output Transactions: Example 1 (millions of dollars)

	Industry 1	Industry 2	Industry 3	Total Outputs
Industry 1	266	378	230	1000
Industry 2	267	110	224	1500
Industry 3	340	340	468	1200

industries where secondary production comprised a significant fraction of total output. All other secondary products in the economy were treated as if they had been sold by the producing sectors to the sectors for which those products were classified as primary. To accomplish this, a table of *transfers* was constructed recording these imaginary sales. This matrix of transfers was then added to the basic transactions matrix, thereby double counting the value of secondary products and consequently inflating total output. This was done in order to ensure that the secondary products were distributed correctly to consumers, at the expense of inflating the total outputs of some industries that are secondary producers.

Example 1: Reallocation of Secondary Production Reallocation (sometimes referred to as redefinition) of secondary production, as just mentioned, involves factoring out the amount of secondary product produced as well as the inputs used in that production and reassigning both to the industry for which the product is classified as primary. However, this requires that a firm allocate its inputs between the production of primary and secondary products; in effect, it is necessary to break the firm into two independent subfirms – one a producer of the primary product and the other a producer of the secondary product. Most firms do not record data in a form that permits this accounting easily, so a less desirable treatment of secondary production is often employed in input–output studies. For example, the US Department of Commerce (prior to preparation of the 1972 national input–output table) reassigned the output of secondary production to the sector for which the activity was considered primary, but did not reassign the inputs. This amounted to a double counting of the inputs required for secondary production, which we see in the following example. (*Survey of Current Business*, 1969, 1974; Vaccara, Shapiro and Simon, 1970.)

Consider a three-industry economy; the matrix of interindustry transactions and vector of total outputs are given in Table 4.10. Suppose that firms in both industries 1 and 3 are secondary producers of product 2, that is, industry 1 produces $100 million worth of product 2 in addition to $900 million worth of product 1, and industry 3 produces $10 million worth of product 2 as well as $1,190 million worth of product 3. As mentioned earlier, since the proper distribution of output is often desirable in input–output studies, the convention often employed is to treat the secondary product as if it were sold to the industry for which the product is classified as primary.

In this example, if the secondary production for all producers was "transferred" to the correct primary producer, then the revised transactions matrix, $\bar{\mathbf{Z}}$, and corresponding total outputs vector, $\bar{\mathbf{x}}$, would be $\bar{\mathbf{Z}} = \begin{bmatrix} 266 & 378 & 230 \\ 367 & 110 & 234 \\ 340 & 340 & 468 \end{bmatrix}$ and $\bar{\mathbf{x}} = \begin{bmatrix} 1000 \\ 1610 \\ 1200 \end{bmatrix}$. Note that this is accomplished by simply adding the amount of secondary production, termed a *transfer*, to the industry for which the secondary product is classified as primary, that is, the \$100 million worth of product 2 produced by industry 1 is added to the original z_{21} transaction as if it were sold by industry 2 to the secondary producer, industry 1. Similarly, the \$10 million worth of product 2 produced by industry 3 is added to the original z_{23} transaction. Finally, total output of industry 2 is increased by the sum of all secondary production of product 2. However, the total outputs of industries 1 and 3 are not decreased, since the inputs required in secondary production were not reallocated; this inflates total output, since secondary production is counted twice.

In many earlier input–output studies (prior to about the mid 1980s and especially prior to widespread adoption of the SNA framework) such transferring of secondary production was used except where secondary production comprised a large portion of total output of an industry, in which case both secondary products and inputs were reallocated.

4.7.2 Secondary Production: Commodity-by-Industry Accounting

A more realistic classification scheme that accounts for industrial production by commodity type rather than industry category eliminates the somewhat clumsy and biased accounting of reallocating secondary production. More recent studies, including the US National Tables complied for years since 1972, redefine all secondary production by establishing a set of "commodity-by-industry" accounts as described earlier. In Chapter 5 we examine the commodity-by-industry accounting framework in detail. For the present, we can introduce the basic features of accounting for secondary production in a commodity-by-industry framework as we expand the way we represent production and consumption in the national accounting system to include the IOAs.

Example 2: Commodity-by-Industry Accounts We illustrate the use of commodity-by-industry accounts in fashioning an input–output model with a two-industry and two-commodity example. Consider the consolidated commodity-by-industry accounts in Table 4.11. In this example, industries are defined as A and B corresponding to the primary products of the establishments included in the definition of these two industries; that is, industry A's primary product is commodity A and industry B's primary product is commodity B. Industry A produces only commodity A (all establishments included in defining industry A produce only commodity A; there is no secondary product). Establishments assigned to industry B, however, while primarily producing commodity B, also produce, as a secondary product, some amount of commodity A. In this simple economy industry A produces \$90 million

Table 4.11 Consolidated Input–Output Accounts: Example 2

Millions of Dollars		Commodities *A*	Commodities *B*	Industries *A*	Industries *B*	Final Demand	Total
Commodities	*A*			12	8	80	100
	B			10	7	83	100
Industries	*A*	90	0				90
	B	10	100				110
Value Added				68	95	163	
Total		100	100	90	110		

worth of commodity A and industry B produces \$100 million worth of commodity B and \$10 million worth of commodity A. We define the Make matrix for this example as $\mathbf{V} = \begin{bmatrix} 90 & 0 \\ 10 & 100 \end{bmatrix}$. We found earlier that while the Make matrix is a complete picture of the economy, it does not provide information about the interindustry activity in an economy, such as deliveries of commodities to other industries or to final demand. The interindustry activity is found in the Use matrix, which for the example is $\mathbf{U} = \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix}$.

Also from Table 4.11 we can define the vector of commodity final demands, $\mathbf{e} = \begin{bmatrix} 80 \\ 83 \end{bmatrix}$, the vector of total commodity outputs, $\mathbf{q} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$, the (row) vector of total value-added inputs, $\mathbf{v}' = \begin{bmatrix} 68 & 95 \end{bmatrix}$ and the vector of total industry outputs, $\mathbf{x} = \begin{bmatrix} 90 \\ 110 \end{bmatrix}$.

4.7.3 Reconciling with the National Accounts

Recall the matrix version of the summary of the national accounts for our example, shown in Table 4.5. We can expand our representation of consumption in the economy, which is currently reflected by a single number indicating the net value of total consumption (545 in our example). The expansion will capture the role of individual industries and specific products (goods and services). For the example, let us consider an economy with three industries (Natural Resources, Manufacturing, and Services) which produce five products (agricultural products, energy, manufactured goods, financial services, and other services).

We presume that industries consume products (*commodities*) as interindustry transactions in the course of delivering their own product(s) to other industries as well as final customers. For our example, in Table 4.11, we allocate total consumption transactions of commodities between interindustry transactions and sales to final customers as well as the government, the total of which is $C = 550$ (total domestic consumption) from Table 4.5. We also allocate capital accumulation and net foreign income, which total to $D = -19$ and $H = 14$, respectively, from Table 4.5. The total of all outputs then is 545. There are, of course, many possible interindustry transactions tables such that the relevant interindustry totals for C, D, and X are consistent with the totals provided in Table 4.5. One such transactions table is shown as Table 4.12.

The interindustry transactions portion of Table 4.12, shown shaded in the table, the reader should recognize as the Use matrix, as defined earlier, since it depicts the commodities used by each industry in producing its output. This is, of course, as before, analogous to the interindustry transactions matrix in the input–output analysis framework, except that in the basic framework both rows and columns refer to specific industries, rather than commodities and industries, respectively, for rows and columns. While one could easily express the transactions matrix in industry-by-industry terms, the SNA adopts a commodity level of detail in order to provide a more transparent picture of industrial production functions. There are other benefits as well that we will see in the following, and in more detail in Chapter 5. In national accounting parlance, the commodity-by-industry interindustry transactions tables are also often referred to as Supply and Use tables. Note that, as before, the Use matrix was constructed in dimensions of commodities (rows) by industries (columns) and, in matrix terms, the accounting identities are $\mathbf{q} = \mathbf{Ui} + \mathbf{e}$ and $\mathbf{x}' = \mathbf{i}'\mathbf{U} + \mathbf{v}'$.

The sources of production or supply in the economy are depicted as before in the Make matrix. The Make matrix is constructed in dimensions of industries by commodities, where as before the row entries indicate the production of commodities by a particular industry. Hence, recall that the row sums of the Make matrix form the vector of total industry production in the economy, while the column sums form the vector of total commodity production in the economy. For our earlier running national accounts example, there are many possible Make matrices, the only strict requirement being that the row sums equal total industry output and the column sums equal total commodity output. One such matrix is given in Table 4.13.

It is now possible to show for our running example a complete consolidated set of commodity-by-industry accounts that illustrates the relationships between Use and Make matrices and the measures of total industry value added, commodity final demand, and total industry and commodity output. This is shown in Table 4.14. Chapter 5 explores construction of input–output models from these accounts. For present purposes, we define the Make matrix as $\mathbf{V}$, the row sums of which comprise the vector of total industry output, $\mathbf{x} = \mathbf{Vi}$, and the column sums of which comprise total commodity output, $\mathbf{q}' = \mathbf{i}'\mathbf{V}$.

4.7.4 Producers' and Consumers' Prices

Most input–output studies value the entries in input–output accounts (and subsequently the transactions matrix) in *producers' prices*, that is, the prices at which the seller completes the transaction (sometimes called *free-on-board* or *FOB prices*). The purchaser incurs the producer's price plus trade and transportation margins (and often excise taxes). The convention in most input–output studies is to assign the margins on all

Table 4.12 Production Account Allocated to Individual Products and Sectors

Consumption–Production	Nat. Res.	Manuf.	Services	Total Intermediate Output	Personal Consumption Expenditures	Government Expenditures	Capital Accumu-lation	Net Overseas Income	Total Output
Agriculture	25	10	15	50	62	23	−5	3	133
Energy	13	7	9	29	35	25	−7	5	87
Manufacturing	10	20	7	37	65	9	−4	2	109
Financial Services	10	10	25	45	55	25	−1	3	127
Other Services	8	30	20	58	17	15	−2	1	89
Totals				219	234	97	$D = -19$	$H = 14$	545
Total Final Consumption	$331 = 234 + 97$								
Total Final Demand	$326 = 234 + 97 - 19 + 14$								
Total Domestic Consumption	$C = 550 = 219 + 234 + 97 = 219 + 331$								

Table 4.13 Industry-by-Commodity Make Matrix: Running Example

	Agriculture	Energy	Manufactured Products	Financial Services	Other Services	Total Industry Output
Natural Resources	88	68	0	0	0	156
Manufacturing	45	10	98	10	33	196
Services	0	9	11	117	56	193
Total Commodity Output	133	87	109	127	89	

interindustry transactions in a column to the industry responsible for the margin. That is, all wholesale and retail trade margins on all inputs to an industry are summed and recorded as the "trade" entry in that column. Similarly, all transportation margins on inputs are summed and recorded as the input entry for "transportation." Hence, the trade and transportation sectors are not really treated as producing and consuming sectors in the economy, but only as "pass-through sectors."

These conventions simply mean that the input–output table does not actually trace flows through the trade and transportation sectors, since this would depict an economy where industries and final customers would make most of their purchases from and sales to these two industries alone. Instead, transactions are depicted as flowing directly from producer to consumer, bypassing trade and transportation. This is done to show the links between producers, consumers, and final customers.

Since trade and transportation margins for all transactions into an industry are accumulated as single values for each industry, they in effect become service inputs to that industry. Hence, the sum of all inputs measured in producers' prices plus the value of all transportation and trade margins valued as service inputs (hence, valued in de facto producers' prices) is then the value of all inputs in consumers' prices (the column sums of the transactions matrix).

Example 3: Trade and Transportation Margins Suppose we have a four-sector input–output economy with two manufacturing sectors, A and B, and two service sectors, trade and transportation (for simplicity we return to the industry-by-industry accounting framework for the moment). The service sectors act as both interindustry sectors in their own right as well as a repository for all markups or margins. The interindustry transactions paid in millions of dollars including both trade and transportation margins – that is, in purchasers' or consumers' prices – are given by $\tilde{\mathbf{Z}}$, final demands including margins by $\bar{\mathbf{f}}$, and total outputs including margins by $\bar{\mathbf{x}}$ defined by

$$\tilde{\mathbf{Z}} = \begin{bmatrix} 36 & 46 & 83 & 24 \\ 76 & 78 & 94 & 35 \\ 8 & 7 & 8 & 4 \\ 3 & 1 & 5 & 1 \end{bmatrix}, \quad \bar{\mathbf{f}} = \begin{bmatrix} 475 \\ 263 \\ 120 \\ 150 \end{bmatrix}, \quad \text{and} \quad \bar{\mathbf{x}} = \begin{bmatrix} 664 \\ 546 \\ 147 \\ 160 \end{bmatrix}$$

Table 4.14 Consolidated Commodity-by-Industry Input–Output Accounts: Running Example

	Commodities					Industries				
	Agricultural Products	Energy	Manufactured Products	Financial Services	Other Services	Natural Resources	Manuf.	Services	Final Demand	Total Output
Commodities										
Agricultural Products						25	10	15	83	133
Energy						13	7	9	58	87
Manufactured Products						10	20	7	72	109
Financial Services						10	10	25	82	127
Other Services						8	30	20	31	89
Industries										
Natural Resources	88	68	0	0	0					156
Manufacturing	45	10	98	10	33					196
Services	0	9	11	117	56					193
Value Added						90	119	117		
Total Output	133	87	109	127	89	156	196	193		

Table 4.15 Example Trade and Transportation Margins: Example 2

	A	B	Trade	Transp.	Final Demand
			Trade Margins		
Industry A	9	10	11	6	50
Industry B	5	8	7	4	20
Transportation	3	1	5	1	20
Total Margins	17	19	23	11	90
			Transportation Margins		
Industry A	7	4	9	5	75
Industry B	6	8	7	6	13
Trade	8	7	8	4	50
Total Margins	21	19	24	15	138

Suppose that the trade and transportation margins in millions of dollars are given in Table 4.15.

The sum of the margins in these two tables is the difference between the purchasers' prices and producers' prices. For example, the transaction $\tilde{z}_{11} = \$36$ million incurs a trade markup of \$9 million and a transport markup of \$7 million, leaving a so-called *direct allocation* of \$20 million. Likewise, the transaction $\tilde{z}_{43} = 5$ is transport markup on trade services, for example, transport costs associated with transactions between wholesale and retail trade. The direct allocation for this transaction is zero. Similarly, the transaction $\tilde{z}_{34} = 4$ is the trade markup on transportation services, for example, the markup imposed by a principal carrier that subcontracts transport services from a secondary carrier. If we factor out (subtract) the margins from all the interindustry transactions in purchasers' prices, the result is the *direct allocations* matrix:

$$\boldsymbol{\Psi} = \begin{bmatrix} 20 & 32 & 63 & 13 \\ 65 & 62 & 80 & 25 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \mathbf{f}_d = \begin{bmatrix} 350 \\ 230 \\ 100 \\ 100 \end{bmatrix}, \text{ and } \mathbf{x}_d = \begin{bmatrix} 478 \\ 462 \\ 100 \\ 100 \end{bmatrix}$$

Note that we have also factored the margins out of final demand and total output and termed these vectors $\mathbf{f}_d$ and $\mathbf{x}_d$ respectively. The vector of column sums of the trade margins, labeled "total margins" in the table, represents the sums of all trade margins on inputs to industries; for example, the first element of this vector, \$17 million, is the sum of all trade margins on inputs to industry A. If we add this vector to the trade row of the direct allocations matrix we are, in effect, distributing the trade margins as a service of the trade sector. Similarly, if we assign the "total transportation margins" to the transportation row of the direct allocations matrix, we account for transportation margins as a service of the transportation industry. In this way, we do not trace the flows of goods and services through the trade and transportation sectors, but instead, treat them as service inputs to producing sectors and record the flows directly from producer to consumer. The result is an interindustry transactions matrix in producers' prices:

$$\mathbf{Z} = \begin{bmatrix} 20 & 32 & 63 & 13 \\ 65 & 62 & 80 & 25 \\ 17 & 19 & 23 & 11 \\ 21 & 19 & 24 & 15 \end{bmatrix}, \ \mathbf{f} = \begin{bmatrix} 350 \\ 230 \\ 190 \\ 238 \end{bmatrix}, \ \text{and } \mathbf{x} = \begin{bmatrix} 478 \\ 462 \\ 260 \\ 317 \end{bmatrix}$$

Methods of valuation in current use in the literature are discussed in more detail in Bulmer-Thomas (1982).

4.7.5 Accounting for Imports and Exports

As briefly mentioned earlier, imports in an input–output framework are usually divided into two basic groups: (1) imports of commodities that are also domestically produced (competitive imports) and (2) imports of commodities that are not domestically produced (noncompetitive imports).[11] The distinction, of course, is that competitive imports can be represented in a technical coefficients matrix, while noncompetitive imports cannot. Competitive imports are usually handled by adding transactions to the domestic transactions matrix (as in the case of transfers of secondary products) as if they were domestically produced. However, the "domestic port value" (in effect, value in producers' prices) of all imports of a particular commodity is included as a negative entry in final demand. The purpose of this adjustment is to assure that the total output of an industry, computed as the row sum of the interindustry transactions to other industries and final-demand allocation, is total *domestic* production, net of imports.

Noncompetitive imports are assigned to a new industry category, but the total value of all noncompetitive imports is given a negative value in final demand so that, as in the case of competitive imports, the total output – that is, the row sum of transactions and final demand – will be total domestic production, which in this case is zero. The negative final-demand entries in both classes of imports ensure the total production and GNP of the economy are not incorrectly biased by imports.

Example 4: Competitive and Noncompetitive Imports Table 4.16 shows a domestic transactions matrix, $\mathbf{Z}$; final demand vector, $\mathbf{f}$; and total outputs vector, $\mathbf{x}$; for a two-sector (industries A and B) input–output economy in millions of dollars.

In addition to these domestic transactions, industry A consumes \$10 million worth of B that is imported, in addition to the \$30 million worth of B that is domestically produced. Also, both industries A and B consume another product, C, that is only produced overseas (a noncompetitive import) – \$5 million and \$4 million worth for A

[11] In the literature the terms comparable and noncomparable imports are used interchangeably with competitive and noncompetitive imports, respectively. Note that the treatment of competitive imports outlined here has been adopted only in more recent input–output studies such as the 1972 US National Input–Output Table; earlier studies treated competitive imports in the same manner as secondary products; see Ritz (1979 and 1980).

Table 4.16 Domestic Interindustry Transactions: Example 4

Millions of Dollars	*A*	*B*	Final Demand	Total Output
Industry *A*	10	20	70	100
Industry *B*	30	40	30	100

Table 4.17 Modified Interindustry Transactions: Example 4

				Final Demand		
Millions of Dollars	*A*	*B*	*C*	Imports Adjustment	Other	Total Output
Industry *A*	10	20	0	0	70	100
Industry *B*	40	40	0	−10	30	100
Imports of *C*	5	3	0	−9	0	0

and *B*, respectively. The convention usually adopted in accounting for these imports is depicted in the modified table of transactions shown in Table 4.17.

Note that the competitive import transaction of *B* by *A* is transferred – that is, added – as in the convention for transferring secondary production, to the z_{BA} transaction and also recorded as a negative final demand. The noncompetitive imports of *C* by both *A* and *B* are recorded as a new row in the transactions matrix and the sum of all imports of *C*, \$9 million, is recorded as a negative final demand. Hence, the final demand, net of imports, and total outputs are unchanged from the domestic table; final demand and total outputs are customarily defined to include only domestic production.

4.7.6 Removing Competitive Imports from Total Transactions Tables

In Chapter 2, Tables 2.6 and 2.7, we presented direct and total requirements tables for the United States 2003 input–output tables where the underlying transactions were "scrubbed" of competitive imports so that impacts on the domestic economy could be analyzed. As noted above, the US input–output tables are routinely published with competitive imports included as part of interindustry transactions with negative entries for imports added to final demand so that the sum of intermediate production and final demand equals total domestic production and the sum of all final demands equal total gross domestic product. In this section we consider several approximation techniques for removing imports from tables prepared in such a manner.

First we assume that interindustry transactions can be divided into domestic transactions and imports, i.e., $\mathbf{Z} = \mathbf{D} + \mathbf{M}$ where $\mathbf{D}$ is the matrix of domestic transactions and

$\mathbf{M}$ is the matrix of competitive imports.[12] So the vector comprising the row sums of $\mathbf{M}$ is the vector of total imports, $\mathbf{m} = \mathbf{Mi}$. For the US tables the vector of imports, $\mathbf{m}$, is expressed as a negative final demand, so we define $\mathbf{g}$ as the vector of final demand other than imports such that $\mathbf{f} = \mathbf{g} + (-\mathbf{m})$. We can also define $\tilde{\mathbf{m}}'$ as the vector of column sums of $\mathbf{M}$, each element of which is the value of all imports to an industry, $\tilde{\mathbf{m}}' = \mathbf{i}'\mathbf{M}$.

If $\mathbf{M}$ is known then $\mathbf{x} = \mathbf{Zi} + \mathbf{f}$, so we can write $\mathbf{x} = (\mathbf{D} + \mathbf{M})\mathbf{i} + (\mathbf{g} - \mathbf{m})$ and separating terms, $\mathbf{x} = \mathbf{Di} + \mathbf{Mi} + \mathbf{g} - \mathbf{m}$. Since $\mathbf{m} = \mathbf{Mi}$, the terms $\mathbf{m}$ and $\mathbf{Mi}$ cancel each other out so it follows that $\mathbf{x} = \mathbf{Di} + \mathbf{g}$. Also, since total value added is computed as the residual of intermediate production and the value of total output, i.e., $\mathbf{v}' = \mathbf{x}' - \mathbf{i}'\mathbf{Z}$, then it follows that $\mathbf{x}' = \mathbf{i}'\mathbf{Z} + \mathbf{v}'$. Again separating terms we have $\mathbf{x}' = \mathbf{i}'(\mathbf{D} + \mathbf{M}) + \mathbf{v}'$ or $\mathbf{x}' = \mathbf{i}'\mathbf{D} + \mathbf{i}'\mathbf{M} + \mathbf{v}'$ so we can write $\mathbf{x}' = \mathbf{i}'\mathbf{D} + \tilde{\mathbf{m}}' + \mathbf{v}'$. If we define the new total valued added (row) vector as $\tilde{\mathbf{v}}' = \tilde{\mathbf{m}}' + \mathbf{v}'$, it is easily seen that the total value of imports has simply been transferred from the interindustry inputs, $\mathbf{Z}$, to value added since $\mathbf{x}' = \mathbf{i}'\mathbf{D} + \tilde{\mathbf{v}}'$.

Often we are faced with the situation where $\mathbf{m}$ is known but not $\mathbf{M}$. That is, we may know the total value of steel imports but not the value of steel imports to each industry individually. The following are the two approximation procedures for estimating the matrices of domestic transactions and interindustry imports. They both rely on an assumption commonly referred to as *import similarity* where for each product the mix of imports and domestically produced goods is the same across all consuming sectors, but may be different for each product. For example, the mix of imported and domestically produced agricultural goods is the same for all consumers of agricultural goods and the amount of steel imported for use in automobile production is the same fraction of total steel consumed in automobile production as is amount of imported steel as a fraction of total steel used in shipbuilding. This assumption may not be very realistic in many developed economies, but is often necessary due to the limits of available data. These limitations are discussed in NRC (2006).

Approximation Method I If $\mathbf{M}$ is not known, we can approximate it by allocating $\mathbf{m}$ proportionately to the distribution of intermediate output by the following. First create the matrix of intermediate output proportions as $\mathbf{B} = \hat{\mathbf{u}}^{-1}\mathbf{Z}$ where $\mathbf{u}$ is the vector of intermediate outputs, $\mathbf{u} = \mathbf{Zi}$. So we define $\tilde{\mathbf{M}} = \hat{\mathbf{m}}\mathbf{B}$ as an approximation of $\mathbf{M}$, for which we can guarantee that $\mathbf{m} = \mathbf{Mi} = \tilde{\mathbf{M}}\mathbf{i}$. Hence, we can define our approximation of domestic transactions, $\tilde{\mathbf{D}} = \mathbf{Z} - \tilde{\mathbf{M}}$ and from before $\mathbf{g} = \mathbf{f} + \mathbf{m}$, so that $\mathbf{x} = \tilde{\mathbf{D}}\mathbf{i} + \mathbf{g}$.

Approximation Method II Method I assumes (implicitly) that there are no imports consumed directly by final demand, i.e., $\mathbf{m} = \mathbf{Mi}$, which is probably seldom the case. If we make the simplifying assumption that for each industry the fraction of a given input supplied by imports is the same for each industry and that that fraction

[12] For the balance of this section we assume that all imports are competitive imports.

also applies to consumer and government expenditures, then that same fraction of total output is attributable to imports. That is, assume that for each industry that fraction is given by r_i so that $m_i = r_i x_i$. We multiply through the equation $x_i = \left(\sum_{j=1}^{n} z_{ij}\right) + f_i$ by r_i to yield $r_i x_i = \left(r_i \sum_{j=1}^{n} z_{ij}\right) + r_i f_i$ or, by recalling $u_i = \sum_{j=1}^{n} z_{ij}$, we can write $m_i = r_i x_i = r_i u_i + r_i f_i$ or $r_i = \frac{m_i}{u_i + f_i}$. We can use r_i to define an estimate of the domestic transactions matrix by $\bar{d}_{ij} = z_{ij} - r_i z_{ij} = (1 - r_i) z_{ij}$ or, in matrix terms, $\bar{\mathbf{D}} = \mathbf{Z} - \hat{\mathbf{r}}\mathbf{Z}$ and we can define $\bar{\mathbf{M}} = \hat{\mathbf{r}}\mathbf{Z}$ as the estimate of the matrix of interindustry imports. We can define $\bar{\mathbf{m}} = \bar{\mathbf{M}}\mathbf{i}$ as the vector of total interindustry imports. If we define $h_i = r_i f_i$ as the estimate of the vector of imports consumed directly by final demand, which in matrix terms is given by $\mathbf{h} = \hat{\mathbf{r}}\mathbf{f}$, then $\mathbf{m} = \mathbf{h} + \bar{\mathbf{m}}$, so we can use $\mathbf{x} = \mathbf{Zi} + \mathbf{f}$ to write $\mathbf{x} = (\bar{\mathbf{D}} + \bar{\mathbf{M}})\mathbf{i} + (\mathbf{g} - \mathbf{m})$. Again separating terms, $\mathbf{x} = \bar{\mathbf{D}}\mathbf{i} + \bar{\mathbf{M}}\mathbf{i} + \mathbf{g} - \mathbf{h} - \bar{\mathbf{m}}$ or $\mathbf{x} = \bar{\mathbf{D}}\mathbf{i} + \bar{\mathbf{M}}\mathbf{i} + \mathbf{g} - \hat{\mathbf{r}}\mathbf{f} - \bar{\mathbf{m}}$. As before, the terms $\bar{\mathbf{m}}$ and $\bar{\mathbf{M}}\mathbf{i}$ cancel each other out so that $\mathbf{x} = \bar{\mathbf{D}}\mathbf{i} + \mathbf{g} - \hat{\mathbf{r}}\mathbf{f}$. In method I we defined $\mathbf{g}$ as the vector of final demand other than imports, which was assumed to be only interindustry imports, so if we now define $\bar{\mathbf{g}}$ as the vector of final demand other than interindustry imports as well as imports consumed directly by final demand, i.e., $\mathbf{g}$ reduced by imports consumed directly in final demand ($\mathbf{h}$), then $\bar{\mathbf{g}} = \mathbf{g} - \mathbf{h}$ and we can write $\mathbf{x} = \bar{\mathbf{D}}\mathbf{i} + \bar{\mathbf{g}}$.

While it may not be obvious, it is important to observe that method I is equivalent to method II if we create the matrix of intermediate output proportions as $\bar{\mathbf{B}} = \hat{\mathbf{x}}^{-1}\mathbf{Z}$ instead of $\bar{\mathbf{B}} = \hat{\mathbf{u}}^{-1}\mathbf{Z}$ and, hence, $\bar{\mathbf{M}} = \hat{\mathbf{m}}\bar{\mathbf{B}}$. Method II by either calculation, of course, is a more realistic approximation of most economies. We illustrate the two methods in the following example.

Example 5: Import Scrubbing We define an input–output economy with $\mathbf{Z} = \begin{bmatrix} 350 & 0 & 0 \\ 50 & 250 & 150 \\ 200 & 150 & 550 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1000 \\ 500 \\ 1000 \end{bmatrix}$. From $\mathbf{Z}$ and $\mathbf{x}$, we compute the corresponding values of $\mathbf{f} = \mathbf{x} - \mathbf{Zi} = \begin{bmatrix} 650 \\ 50 \\ 100 \end{bmatrix}$, $\mathbf{u} = \mathbf{Zi} = \begin{bmatrix} 350 \\ 450 \\ 900 \end{bmatrix}$, $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 0.3500 & 0.0000 & 0.0000 \\ 0.0500 & 0.5000 & 0.1500 \\ 0.2000 & 0.3000 & 0.5500 \end{bmatrix}$, $\mathbf{v}' = \mathbf{x}' - \mathbf{i}'\mathbf{Z} = \begin{bmatrix} 400 & 100 & 300 \end{bmatrix}$ and $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.5385 & 0.0000 & 0.0000 \\ 0.4487 & 2.5000 & 0.8333 \\ 0.9829 & 1.6667 & 2.7778 \end{bmatrix}$. We presume this is a "US-type" table where the transactions matrix includes competitive imports, so $\mathbf{Z} = \mathbf{D} + \mathbf{M}$ and $\mathbf{f} = \mathbf{g} - \mathbf{m}$, and we define (arbitrarily for this example) $\mathbf{M} = \begin{bmatrix} 100 & 0 & 0 \\ 25 & 50 & 30 \\ 25 & 50 & 100 \end{bmatrix}$, which means

$\mathbf{D} = \mathbf{Z} - \mathbf{M} = \begin{bmatrix} 250 & 0 & 0 \\ 25 & 200 & 120 \\ 175 & 100 & 45 \end{bmatrix}$, $\mathbf{m} = \mathbf{Mi} = \begin{bmatrix} 100 \\ 105 \\ 175 \end{bmatrix}$, $\mathbf{g} = \mathbf{f} + \mathbf{m} = \begin{bmatrix} 750 \\ 155 \\ 275 \end{bmatrix}$

and the balance equation $\mathbf{x} = \mathbf{Di} + \mathbf{g}$ holds: $\mathbf{x} = \begin{bmatrix} 1000 \\ 500 \\ 1000 \end{bmatrix} = \mathbf{Di} + \mathbf{g} = \begin{bmatrix} 250 & 0 & 0 \\ 25 & 200 & 120 \\ 175 & 100 & 450 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 750 \\ 155 \\ 275 \end{bmatrix}$. Then the new total value added vector, $\tilde{\mathbf{v}}' = \tilde{\mathbf{m}}' + \mathbf{v}' = \mathbf{x}' - \mathbf{i}'\mathbf{D} = \begin{bmatrix} 550 & 200 & 430 \end{bmatrix}$, inflates the original vector of total valued added, $\mathbf{v}' = \begin{bmatrix} 400 & 100 & 300 \end{bmatrix}$, by the total value of all imports to each industry, $\tilde{\mathbf{m}}' = \begin{bmatrix} 150 & 100 & 130 \end{bmatrix}$.

Where $\mathbf{m}$ is known but not $\mathbf{M}$, in the following we apply the two approximation procedures outlined above for estimating the matrices of domestic transactions and interindustry imports.

For Approximation Method I, we first generate $\mathbf{B} = \hat{\mathbf{u}}^{-1}\mathbf{Z} = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 \\ 0.1111 & 0.5556 & 0.3333 \\ 0.2222 & 0.1667 & 0.6111 \end{bmatrix}$ to allocate $\mathbf{m}$ across interindustry output by $\tilde{\mathbf{M}} = \hat{\mathbf{m}}\mathbf{B} = \begin{bmatrix} 100.0000 & 0.0000 & 0.0000 \\ 11.6667 & 58.3333 & 35.0000 \\ 38.8889 & 29.1667 & 106.9444 \end{bmatrix}$.

In this technique, $\mathbf{m} = \mathbf{Mi} = \tilde{\mathbf{M}}\mathbf{i}$ so we can compute

$$\tilde{\mathbf{D}} = \mathbf{Z} - \tilde{\mathbf{M}} = \begin{bmatrix} 250.0000 & 0.0000 & 0.0000 \\ 38.3333 & 191.6667 & 115.0000 \\ 161.1111 & 120.8333 & 443.0556 \end{bmatrix}$$

and the balance equation, $\mathbf{x} = \tilde{\mathbf{D}}\mathbf{i} + \mathbf{g}$, holds:

$$\mathbf{x} = \begin{bmatrix} 1000 \\ 500 \\ 1000 \end{bmatrix} = \tilde{\mathbf{D}}\mathbf{i} + \mathbf{g} = \begin{bmatrix} 250.0000 & 0.0000 & 0.0000 \\ 38.3333 & 191.6667 & 115.0000 \\ 161.1111 & 120.8333 & 443.0556 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 750 \\ 155 \\ 275 \end{bmatrix}$$

Then, in this case, the new total value added vector, $\tilde{\mathbf{v}}' = \mathbf{x}' - \mathbf{i}'\tilde{\mathbf{D}} = [\ 550.5556 \ \ 187.5000 \ \ 441.9444\ ]$, inflates the original vector of total value added, $\mathbf{v}' = \begin{bmatrix} 400 & 100 & 300 \end{bmatrix}$, by the total value of all imports to each industry, $\tilde{\mathbf{m}}' = \begin{bmatrix} 150.5555 & 87.5 & 141.9444 \end{bmatrix}$, but as noted above this assumes that no imports are consumed directly in final demand.

For Approximation Method II we begin by calculating the scaling factors $r_i = \frac{m_i}{u_i + f_i}$, which for the example are the elements of $\mathbf{r} = \begin{bmatrix} .1 \\ .21 \\ .175 \end{bmatrix}$. We can then compute

$\bar{\mathbf{D}} = \mathbf{Z} - \hat{\mathbf{r}}\mathbf{Z} = \begin{bmatrix} 315 & 0 & 0 \\ 39.5 & 197.5 & 118.5 \\ 165 & 123.75 & 453.75 \end{bmatrix}$, $\bar{\mathbf{M}} = \hat{\mathbf{r}}\mathbf{Z} = \begin{bmatrix} 35 & 0 & 0 \\ 10.5 & 52.5 & 31.5 \\ 35 & 26.25 & 96.25 \end{bmatrix}$

and

$\mathbf{h} = \hat{\mathbf{r}}\mathbf{f} = \begin{bmatrix} 65 \\ 10.5 \\ 17.5 \end{bmatrix}$ so that $\bar{\mathbf{g}} = \mathbf{g} - \mathbf{h} = \begin{bmatrix} 685 \\ 144.5 \\ 257.5 \end{bmatrix}$. We can show that the balance equation $\mathbf{x} = \bar{\mathbf{D}}\mathbf{i} + \bar{\mathbf{g}}$ still holds: $\mathbf{x} = \begin{bmatrix} 1000 \\ 500 \\ 1000 \end{bmatrix} = \bar{\mathbf{D}}\mathbf{i} + \bar{\mathbf{g}} =$
$\begin{bmatrix} 315 & 0 & 0 \\ 39.5 & 197.5 & 118.5 \\ 165 & 123.75 & 453.75 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 685 \\ 144.5 \\ 257.5 \end{bmatrix}$.

This balance equation, $\mathbf{x} = \bar{\mathbf{D}}\mathbf{i} + \bar{\mathbf{g}}$, now accounts for only domestic transactions, but interindustry imports are reassigned to total value added.

The new total value added vector, $\bar{\mathbf{v}}' = \mathbf{x}' - \mathbf{i}'\bar{\mathbf{D}} = \begin{bmatrix} 480.5 & 178.75 & 427.75 \end{bmatrix}$, inflates the original vector of total value added, $\mathbf{v}' = \begin{bmatrix} 400 & 100 & 300 \end{bmatrix}$ by only all interindustry imports to each industry, i.e., this time, $\tilde{\mathbf{m}}' = [80.5\ \ 78.75\ \ 127.75]$, excluding the value of imports consumed directly in final demand. As noted above, while perhaps not intuitively obvious, this procedure is equivalent to creating the matrix of intermediate output proportions as $\bar{\mathbf{B}} = \hat{\mathbf{x}}^{-1}\mathbf{Z}$, instead of $\mathbf{B} = \hat{\mathbf{u}}^{-1}\mathbf{Z}$ defined for Method I. For the example,

$$\bar{\mathbf{B}} = \hat{\mathbf{x}}^{-1}\mathbf{Z} = \begin{bmatrix} 1/1000 & 0 & 0 \\ 0 & 1/500 & 0 \\ 0 & 0 & 1/1000 \end{bmatrix} \begin{bmatrix} 350 & 0 & 0 \\ 50 & 250 & 150 \\ 200 & 150 & 550 \end{bmatrix}$$
$$= \begin{bmatrix} .35 & 0 & 0 \\ .1 & .5 & .3 \\ .2 & .15 & .55 \end{bmatrix}$$

Hence,

$$\bar{\mathbf{M}} = \hat{\mathbf{m}}\bar{\mathbf{B}} = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 105 & 0 \\ 0 & 0 & 275 \end{bmatrix} \begin{bmatrix} .35 & 0 & 0 \\ .1 & .5 & .3 \\ .2 & .15 & .55 \end{bmatrix} = \begin{bmatrix} 35 & 0 & 0 \\ 10.5 & 52.5 & 31.5 \\ 35 & 26.25 & 96.25 \end{bmatrix}$$

and, as before,

$$\bar{\mathbf{D}} = \mathbf{Z} - \bar{\mathbf{M}} = \begin{bmatrix} 315 & 0 & 0 \\ 39.5 & 197.5 & 118.5 \\ 165 & 123.75 & 453.75 \end{bmatrix}$$

The application of the alternative estimation procedures for the example is summarized in Table 4.18.

Implications of the Estimating Assumptions Method II is perhaps the most commonly used method for removing imports to create $\mathbf{D}$ if $\mathbf{M}$ is not known directly. Examples of its application to the US economy are OTA (1988), Guo and Planting

Table 4.18 Approximation Methods for Scrubbing Interindustry Transactions of Competitive Imports: Example 5

	US Type Table			Industry Total	Final Demand	Total Output	Approximation Method I			Industry Total	Final Demand	Total Output	Approximation Method II			Industry Total	Final Demand	Total Output
Domestic	250	0	0	250	750	1000	250	0	0	250	750	1000	315	0	0	315	685	1000
Trans-	25	200	120	345	155	500	38.33	191.67	115	345	155	500	39.50	197.50	118.50	355.50	144.50	500
actions	175	100	450	725	275	1000	161.11	120.83	443.06	725	275	1000	165	123.75	453.75	742.50	257.50	1000
Interindustry Total	450	300	570				449.44	312.50	558.06				519.5	321.25	572.25			
Value Added	550	200	430		1180		550.56	187.50	441.94		1180		480.5	178.75	427.75		1087	
Total Output	1000	500	1000			2500	1000	500	1000			2500	1000	500	1000			2500
Interindustry	100	0	0	100			100	0	0	100			35	0	0	35		
Imports	25	50	30	105			11.67	58.33	35	105			10.50	52.50	31.50	94.50		
	25	50	100	175			38.89	29.17	106.94	175			35	26.25	96.25	157.50		
Total	150	100	130	380			150.56	87.50	141.94	380			80.50	78.75	127.75	287		
Direct	0.2500	0.0000	0.0000				0.2500	0.0000	0.0000				0.3150	0.0000	0.0000			
Require-	0.0250	0.4000	0.1200				0.0383	0.3833	0.1150				0.0395	0.3950	0.1185			
ments	0.1750	0.2000	0.4500				0.1611	0.2417	0.4431				0.1650	0.2475	0.4538			
Total Direct	0.4500	0.6000	0.5700				0.4494	0.6250	0.5581				0.5195	0.6425	0.5723			
Value Added	0.5500	0.4000	0.4300				0.5506	0.3750	0.4419				0.4805	0.3575	0.4278			
Total	1.3333	0.0000	0.0000				1.3333	0.0000	0.0000				1.4599	0.0000	0.0000			
Require-	0.1514	1.7974	0.3922				0.1684	1.7644	0.3643				0.1994	1.8139	0.3935			
ments	0.4793	0.6536	1.9608				0.4588	0.7656	1.9536				0.5313	0.8218	2.0089			
Total	1.9641	2.4510	2.3529				1.9606	2.5300	2.3179				2.1905	2.6357	2.4024			

Table 4.18 (*cont.*)

	US Type Table			Industry Total	Final Demand	Total Output	Approximation Method I	Industry Total	Final Demand	Total Output	Approximation Method II	Industry Total	Final Demand	Total Output
Transactions	350	0	0	350	650	1000								
	50	250	150	450	50	500								
	200	150	550	900	100	1000								
Interindustry Total	600	400	700											
Value Added	400	100	300		800									
Total Output	1000	500	1000			2500								
Direct Require-	0.3500	0.0000	0.0000											
ments	0.0500	0.5000	0.1500											
	0.2000	0.3000	0.5500											
Total Direct	0.6000	0.8000	0.7000											
Value Added	0.4000	0.2000	0.3000											
Total Require-	1.5385	0.0000	0.0000											
ments	0.4487	2.5000	0.8333											
	0.9829	1.6667	2.7778											
Total	2.9701	4.1667	3.6111											

(2000), and NRC (2006). Recall that the key assumption for both Methods I and II is of import similarity, i.e., for each industry product the mix of imports and domestically produced goods is the same across all consuming sectors for that product, but are or may be different for each product.

NRC (2006) uses this assumption to analyze the US content of imports and the foreign content of US exports as one possible way of gauging the implications of the globalization of industry for the overall health of the US economy. Dietzenbacher, Albino and Kühtz (2005) propose another alternative to both Methods I and II.

4.7.7 Adjustments for Inventory Change

Inventories in an input–output model are not quite equivalent to the conventional definition of that term. In input–output models, inventory change is usually taken to mean the change in inventories of an industry's primary product, regardless of which industry or industries hold the inventories. For example, coal inventories held by electric power plants are classified as coal inventory. The traditional definition is usually restricted to the inventory actually held by the industry producing the product. This modified definition is adopted in input–output to ensure that the row total of the transactions and final demands is equal to total current output of the industry. If we ignore inventory depletion or addition, then the row totals are total consumption, not output.

4.7.8 Adjustments for Scrap

Input–output accounts typically deal with scrap as a production by-product. That is, it is assumed that no industry produces scrap on demand, so scrap is the result of production to meet other demands. This is typically accomplished by calculating the ratio of nonscrap output to industry output for each industry and then applying these ratios to the market shares matrix in order to account for total industry output. In Chapter 5 we develop a variety of methods for handling secondary production and by-products, but one commonly applied technique to adjust for scrap is the following.

First, we recall that $\mathbf{g} = \mathbf{Vi}$ (the row sums of the Make matrix) and assume some portion of total commodity production, $\mathbf{g}$, is scrap, $\mathbf{h}$, so we can write $\mathbf{g} = \mathbf{Vi} + \mathbf{h}$. If, as noted above, we assume that scrap production is related to total production by a constant ratio, we can write $h_i = c_i g_i$ where c_i is the ratio of the value of scrap produced in industry i to the total output. In matrix terms this is expressed as $\mathbf{h} = \hat{\mathbf{c}}\mathbf{g}$. We can rewrite $\mathbf{g} = \mathbf{Vi} + \mathbf{h}$ as $\mathbf{g} - \mathbf{h} = \mathbf{Vi}$ and substitute $\mathbf{h} = \hat{\mathbf{c}}\mathbf{g}$ to yield $\mathbf{g} - \hat{\mathbf{c}}\mathbf{g} = \mathbf{Vi}$ and, hence $(\mathbf{I} - \hat{\mathbf{c}})\mathbf{g} = \mathbf{Vi}$. Finally, we multiply through both sides by $(\mathbf{I} - \hat{\mathbf{c}})^{-1}$ to yield $\mathbf{g} = \left[(\mathbf{I} - \hat{\mathbf{c}})^{-1}\mathbf{V}\right]\mathbf{i}$ where we can define the bracketed quantity as $\bar{\mathbf{V}}$, the Make matrix adjusted for scrap.

4.8 Valuation and Double Deflation

In comparing input–output data for different years, it is often important to distinguish changes attributable to prices from other sources of difference. This essentially involves converting tables originally valued at nominal prices for the year in which the data for the table were collected (current prices) to corresponding tables valued at constant prices for some established base time period, usually a base year. A very common method for accomplishing this is called *double deflation*, which refers to a two-step process (hence the "double") by which (1) intermediate inputs, final demands, and total outputs valued at current prices in the accounting period are "deflated" by using (multiplying by) commodity price indices for all intermediate inputs, final demand, and total output and then (2) deriving a value added price index that balances the fundamental identity that the value of total outputs must always be equal to the value of total inputs. All the output of a particular industry, i.e., deliveries to other industries and to final demand, is adjusted by a price index for that industry's output. The price index is simply a ratio of the price of a commodity in the year for which valuation is sought to the corresponding price in the base year.

We illustrate the process of double deflation by recalling that interindustry transactions in value terms refer to a physical transaction and a corresponding price, i.e.,

$$z_{ij} = p_i s_{ij} \tag{4.21}$$

where z_{ij} is the dollar transaction of industry i's output consumed by industry j; p_i is the price per physical unit of industry i's output; and s_{ij} is the physical units transaction of industry i's output consumed by industry j. We can rearrange terms in (4.21) to $s_{ij} = \frac{z_{ij}}{p_i}$. If we define a superscript to denote the accounting period, we can write $s_{ij} = \frac{z_{ij}^1}{p_i^1} = \frac{z_{ij}^2}{p_i^2} = \cdots = \frac{z_{ij}^t}{p_i^t} = \cdots = \frac{z_{ij}^n}{p_i^n}$ for $1, 2, \ldots, n$ accounting periods (usually years). If we choose any arbitrary year to be the reference or base year (b), we can write $s_{ij} = \frac{z_{ij}^b}{p_i^b} = \frac{z_{ij}^t}{p_i^t}$ or $z_{ij}^b = \left(\frac{p_i^b}{p_i^t}\right) z_{ij}^t$. We refer to the term $\left(\frac{p_i^b}{p_i^t}\right)$ as the price index for industry i in year t relative to base or reference year b. Similarly, we define final demand and total output of industry i's output as d_i and q_i corresponding to those quantities in value terms, f_i and x_i. Hence, $f_i = p_i d_i$ and $q_i = p_i x_i$ and, once again introducing superscripts to denote accounting periods, we can write $d_i = \frac{f_i^1}{p_i^1} = \frac{f_i^2}{p_i^2} = \cdots = \frac{f_i^t}{p_i^t} = \cdots = \frac{f_i^n}{p_i^n}$ and $q_i = \frac{x_i^1}{p_i^1} = \frac{x_i^2}{p_i^2} = \cdots = \frac{x_i^t}{p_i^t} = \cdots = \frac{x_i^n}{p_i^n}$. By introducing a base year, b, we can write $d_i = \frac{f_i^b}{p_i^b} = \frac{f_i^t}{p_i^t}$ and $q_i = \frac{x_i^b}{p_i^b} = \frac{x_i^t}{p_i^t}$. Rearranging terms, $f_i^b = \left(\frac{p_i^b}{p_i^t}\right) f_i^t$ and $x_i^b = \left(\frac{p_i^b}{p_i^t}\right) x_i^t$.

This and the earlier expression for transactions allows us to "deflate" transactions, final demand, and total output from year t to base year b or if we do this for multiple years we can express all years' values for these quantities in year b's prices, or, so-called *constant* prices. In sum, if we define $\pi_i^t = \left(\frac{p_i^b}{p_i^t}\right)$ as the price index or deflator for industry i, then we can write $z_{ij}^b = \pi_i^t z_{ij}^t$, $f_i^b = \pi_i^t f_i^t$ and $x_i^b = \pi_i^t x_i^t$. In matrix terms, we define the vector of price indices as $\boldsymbol{\pi}^t = \left[\pi_1^t \; \pi_2^t \ldots \pi_n^t\right]$ so that we can write $\mathbf{Z}^b = (\hat{\boldsymbol{\pi}}^t)\mathbf{Z}^t$, $\mathbf{f}^b = (\hat{\boldsymbol{\pi}}^t)\mathbf{f}^t$ and $\mathbf{x}^b = (\hat{\boldsymbol{\pi}}^t)\mathbf{x}^t$.

Since we have deflated $\mathbf{Z}$, $\mathbf{f}$ and $\mathbf{x}$ all by the same price index, we can be assured the basic identity, $\mathbf{Z}^t\mathbf{i} + \mathbf{f}^t = \mathbf{x}^t$ and $\mathbf{Z}^b\mathbf{i} + \mathbf{f}^b = \mathbf{x}^b$ both hold, since for each industry we have simply multiplied through the distribution of all output to intermediate consumers and to final demand by the same price. However, we need to ensure that total outputs

Table 4.19 Double Deflation: Example 6

	Industry Transactions			Final	Total	Price	Price
	1	2	3	Demand	Output	Year 1	Year 2
1	10	20	30	65	125	2	7
2	5	25	12	40	82	2	6
3	22	3	7	104	136	3	5
Value Added	88	34	87	209			

are equal to total inputs, as well, i.e., the fundamental identity $\mathbf{i}'\mathbf{Z}^b + (\mathbf{v}^b)' = (\mathbf{x}^b)'$ must hold as well, where $(\mathbf{v}^b)'$ is the as yet undetermined deflated vector of value added inputs – undetermined since we have not yet specified a price index for value added. We only have a deflator for interindustry inputs. Here we should observe that if an industry sector experiences price changes for all of its intermediate inputs (including the price of its own output), then the value added is the only term left that can change if the value of total inputs is to remain equal to the value of total outputs. Hence, in order to maintain this identity we can compute the new value added as the residual, i.e., $(\mathbf{v}^b)' = (\mathbf{x}^b)' - \mathbf{i}'\mathbf{Z}^b$; we can define the deflator for value added for each industry simply as the ratio $r_i^t = \left(\frac{v_i^b}{v_i^t}\right)$ since $v_i^b = r_i^t v_i^t$ or, in matrix terms, $\hat{\mathbf{v}}^b = \hat{\mathbf{r}}^t \hat{\mathbf{v}}^t$ and, rearranging terms, we have $\hat{\mathbf{r}}^t = \hat{\mathbf{v}}^b(\hat{\mathbf{v}}^t)^{-1}$.

Example 6: Double Deflation We define a three-sector economy for year 2 with industry prices for two different years, years 1 and 2, in Table 4.19. To express the transactions, final demand, and total output in year 1 (which we define as the base year) prices, first we compute $\boldsymbol{\pi}^t$ as $\boldsymbol{\pi}^t = \begin{bmatrix} 2/7 \\ 2/6 \\ 3/5 \end{bmatrix} = \begin{bmatrix} .286 \\ .333 \\ .600 \end{bmatrix}$. Hence, we can compute $\mathbf{Z}^b = \hat{\boldsymbol{\pi}}^t \mathbf{Z}^t$, $\mathbf{f}^b = \hat{\boldsymbol{\pi}}^t \mathbf{f}^t$, $\mathbf{x}^b = \hat{\boldsymbol{\pi}}^t \mathbf{x}^t$ as

$$\mathbf{Z}^b = \hat{\boldsymbol{\pi}}^t \mathbf{Z}^t = \begin{bmatrix} .286 & 0 & 0 \\ 0 & .333 & 0 \\ 0 & 0 & .600 \end{bmatrix} \begin{bmatrix} 10 & 20 & 30 \\ 5 & 25 & 12 \\ 22 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 2.9 & 5.7 & 8.6 \\ 1.7 & 8.3 & 4.0 \\ 13.2 & 1.8 & 4.2 \end{bmatrix}$$

$$\mathbf{f}^b = \hat{\boldsymbol{\pi}}^t \mathbf{f}^t = \begin{bmatrix} .286 & 0 & 0 \\ 0 & .333 & 0 \\ 0 & 0 & .600 \end{bmatrix} \begin{bmatrix} 65 \\ 40 \\ 104 \end{bmatrix} = \begin{bmatrix} 18.6 \\ 13.3 \\ 62.4 \end{bmatrix}$$

$$\mathbf{x}^b = \hat{\boldsymbol{\pi}}^t \mathbf{x}^t = \begin{bmatrix} .286 & 0 & 0 \\ 0 & .333 & 0 \\ 0 & 0 & .600 \end{bmatrix} \begin{bmatrix} 125 \\ 82 \\ 136 \end{bmatrix} = \begin{bmatrix} 35.7 \\ 27.3 \\ 81.6 \end{bmatrix}$$

From the original data, $(\mathbf{v}^t)' = \begin{bmatrix} 88 & 34 & 87 \end{bmatrix}$ and we can compute the necessary value added to ensure that total inputs remain equal to total outputs as

$$(\mathbf{v}^b)' = (\mathbf{x}^b)' - \mathbf{i}'\mathbf{Z}^b = [18.0 \;\; 11.5 \;\; 64.8]$$

Hence, we can find the value added deflator as

$$\hat{\mathbf{r}}^t = \hat{\mathbf{v}}^b(\hat{\mathbf{v}}^t)^{-1} = \begin{bmatrix} 18 & 0 & 0 \\ 0 & 11.5 & 0 \\ 0 & 0 & 64.5 \end{bmatrix} \begin{bmatrix} 1/88 & 0 & 0 \\ 0 & 1/34 & 0 \\ 0 & 0 & 1/87 \end{bmatrix}$$
$$= \begin{bmatrix} .204 & 0 & 0 \\ 0 & .338 & 0 \\ 0 & 0 & .745 \end{bmatrix}$$

The method of double deflation, while widely used (as in United Nations, 1993), has many disadvantages for deflating input–output tables, perhaps not the least of which is that all elements in a row of the transactions matrix are deflated by the same index. In many economies interindustry prices may vary considerably and, hence, deflating by the same index can be misleading or even wrong. Even without variation in interindustry prices for a commodity, a single price index for that commodity may only be plausible at very high levels of sectoral disaggregation where products are more distinct. Such an assumption can be very misleading at higher levels of aggregation where multiple products are represented. These and other problems with double deflation are discussed in Hoen (2002) and Dietzenbacher and Hoen (1999). A perhaps preferable alternative to double deflation is biproportional scaling (also known as the RAS technique), which will be explored in detail in Chapter 7.

4.9 The Aggregation Problem: Level of Detail in Input–Output Tables

The number of industrial sectors defined in an input–output table (often referred to as the level of *sectoral* aggregation) is usually decided in the context of the problem being considered, for example, whether or not it is important to distinguish between fully assembled automobiles and automobile parts produced separately by a specific automobile manufacturer; a more aggregated sector labeled "automobiles and parts" may be sufficient. Other factors such as computational expense or availability of data may also be considerations in such a decision. Similarly, in multiple-region models – that is, interregional or multiregional models as defined in Chapter 3 – the number of regions considered in the model (the level of *spatial* aggregation) must also be selected in the problem being considered; for example, if we are interested in the impacts of increased coal development on regions in the United States, how should states be grouped into regions (assuming the basic data are state-specific) to construct an applicable model? An additional and often important consideration is what information, if any, is lost in performing either a sectoral or spatial aggregation?

Since the early 1950s considerable attention has been given in the literature to establishing criteria for and measuring the effects of aggregation of sectors in input–output models. Representative earlier examples include Ara (1959), Balderston and Whitin (1954), Hatanaka (1952), McManus (1956), Malinvaud (1956), Theil (1957), Morimoto (1970), and, more recently, Kymn (1990), Cabrer, Contreras and Miravete (1991) and Olsen (1993). Many of these efforts were aimed at compensating for limited computing capabilities at the time. Today the issues center more around bias introduced by sectoral or, in the case of multiregional or interregional models, spatial aggregation or the definition of regions in input–output models. The questions of the level of aggregation (number of sectors or regions) is likely to be even more important at the regional level, where good data are often unavailable or difficult and prohibitively expensive to obtain (see Doeksen and Little, 1968; Williamson, 1970; Hewings, 1972; and Stevens and Lahr, 1993). The subject of spatial aggregation is examined in more detail for interregional and multiregional input–output models in Miller and Blair (1981) and Blair and Miller (1983).

In this section we examine the basic effects of aggregation on input–output models. In particular, we investigate several measures of the bias or error introduced by aggregation.

4.9.1 The Aggregation Matrix

Before examining the effects of aggregation, let us develop a systematic way of accomplishing aggregation of sectors in an input–output table. First, define a matrix **S**, the aggregation matrix, to be a $k \times n$ matrix of ones and zeros, where k is the number of sectors in the to-be-created aggregated version of the input–output table and n is the number of sectors in the existing unaggregated version of the table. The locations of ones in row i of **S** indicate which sectors of the unaggregated table will be grouped together as sector i in the aggregated table.

For example, let $n = 4$ and $k = 3$; suppose that sectors 2 and 3 of the disaggregated table are to be combined. Then the aggregation matrix that accomplishes this is $\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Let **Z** denote the unaggregated 4×4 transactions matrix and $\mathbf{Z}^*$ be the corresponding aggregated 3×3 transactions matrix. Similarly, **f** and $\mathbf{f}^*$ are the unaggregated and aggregated vectors of final demand, respectively. Recall that our aim is to aggregate sectors 2 and 3 of the unaggregated model; for **f** this can easily be accomplished by premultiplying by **S**:

$$\mathbf{f}^* = \mathbf{Sf} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 + f_3 \\ f_4 \end{bmatrix} \tag{4.22}$$

For $\mathbf{Z}$, this can be accomplished by

$$\mathbf{Z}^* = \mathbf{S}\mathbf{Z}\mathbf{S}' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} \\ z_{21} & z_{22} & z_{23} & z_{24} \\ z_{31} & z_{32} & z_{33} & z_{34} \\ z_{41} & z_{42} & z_{43} & z_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.23}$$

$$\mathbf{Z}^* = \begin{bmatrix} z_{11} & z_{12} + z_{13} & z_{14} \\ z_{21} + z_{31} & z_{22} + z_{23} + z_{32} + z_{33} & z_{24} + z_{34} \\ z_{41} & z_{42} + z_{43} & z_{44} \end{bmatrix}$$

The new corresponding vector of total outputs $\mathbf{x}^*$ can be computed as

$$\mathbf{x}^* = \mathbf{Z}^*\mathbf{i} + \mathbf{f}^* \tag{4.24}$$

where, as before, $\mathbf{i}$ is a column vector of ones.

We can also use the aggregation matrix to reorder sectors. For example, the matrix $\mathbf{S}$ given above is the one that introduces the least sector labeling rearrangement into the aggregated matrix; that is, the original first sector remains sector 1 and the original "last" sector, 4, becomes the "last" sector, 3, in the aggregated model. Alternatively, $\mathbf{S} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ groups original sectors 2 and 3 together and labels them sector 1 in the aggregated matrix, labels original sector 1 as sector 2 in the aggregated matrix, and the original sector 4 as sector 3 in the aggregated matrix.

If we are given a new set of final demands, $\mathbf{f}$, for which we wish to compute the corresponding total output needed to support that final demand, we can compute the Leontief inverse matrices for both unaggregated and aggregated versions of the model: $(\mathbf{I} - \mathbf{A})^{-1}$ and $(\mathbf{I} - \mathbf{A}^*)^{-1}$ where $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$ and $\mathbf{A}^* = \mathbf{Z}^*(\hat{\mathbf{x}}^*)^{-1}$.

As in the case of the initial set of final demands, the aggregated vector of new final demands is $\tilde{\mathbf{f}}^* = \mathbf{S}\tilde{\mathbf{f}}$. Hence, impact analysis yields $\tilde{\mathbf{x}} = (\mathbf{I} - \mathbf{A})^{-1}\tilde{\mathbf{f}}$ and $\tilde{\mathbf{x}}^* = (\mathbf{I} - \mathbf{A}^*)^{-1}\tilde{\mathbf{f}}^*$. Note that, except under very special circumstances which we describe later, $\tilde{\mathbf{x}}^* \neq \mathbf{S}\tilde{\mathbf{x}}$; the difference between $\tilde{\mathbf{x}}^*$ and $\mathbf{S}\tilde{\mathbf{x}}$ is one indication of the bias introduced by aggregating the input–output table from four to three sectors.

Example 7: Sectoral Aggregation We begin with a four-sector input–output model defined by

$$\mathbf{Z} = \begin{bmatrix} 26.5 & 75.0 & 46.0 & 53.0 \\ 34.0 & 5.0 & 68.0 & 68.0 \\ 41.5 & 38.0 & 52.0 & 83.0 \\ 33.5 & 6.0 & 53.0 & 67.0 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} 659.5 \\ 1835.0 \\ 2515.5 \\ 1560.5 \end{bmatrix}, \text{ and } \mathbf{x} = \begin{bmatrix} 860 \\ 2010 \\ 2730 \\ 1720 \end{bmatrix}$$

Let us consider two alternative sectoral aggregations of this model, given respectively by the aggregation matrices $\mathbf{S}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ and $\mathbf{S}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$.

$\mathbf{S}_1$ combines sectors 3 and 4 of the four-sector model into sector 3 of a three-sector model, leaving sectors 1 and 2 unaggregated. $\mathbf{S}_2$ combines sectors 1 and 4 of the four-sector model into sector 3 of a three-sector model and assigns sectors 2 and 3 of the four-sector model to sectors 1 and 2, respectively, in a three-sector model.

From (4.22), (4.23), and (4.24) we can compute the corresponding aggregated values of $\mathbf{f}$, $\mathbf{Z}$, and $\mathbf{x}$ for the two alternative aggregation schemes. For the $\mathbf{S}_1$ aggregation scheme, we have,

$$\mathbf{f}_1^* = \mathbf{S}_1\mathbf{f} = \begin{bmatrix} 659.5 \\ 1835.0 \\ 4076.0 \end{bmatrix}$$

$$\mathbf{Z}_1^* = \mathbf{S}_1\mathbf{Z}\mathbf{S'}_1 = \begin{bmatrix} 26.5 & 75.0 & 99.0 \\ 34.0 & 5.0 & 136.0 \\ 75.0 & 44.0 & 255.0 \end{bmatrix}$$

and

$$\mathbf{x}_1^* = \mathbf{Z}_1^*\mathbf{i} + \mathbf{f}_1^* = \begin{bmatrix} 860 \\ 2010 \\ 4450 \end{bmatrix}.$$

Similarly, for the $\mathbf{S}_2$ aggregation scheme, we have $\mathbf{f}_2^* = \mathbf{S}_2\mathbf{f} = \begin{bmatrix} 1835.0 \\ 2515.5 \\ 2220.0 \end{bmatrix}$, $\mathbf{Z}_2^* = \mathbf{S}_2\mathbf{Z}\mathbf{S'}_2 = \begin{bmatrix} 5.0 & 68.0 & 102.0 \\ 38.0 & 52.0 & 124.5 \\ 81.0 & 99.0 & 180.0 \end{bmatrix}$ and $\mathbf{x}_2^* = \mathbf{Z}_2^*\mathbf{i} + \mathbf{f}_2^* = \begin{bmatrix} 2010 \\ 2730 \\ 2580 \end{bmatrix}$.

Let us now compute the technical coefficients matrix and Leontief inverse for each of the aggregation schemes. For $\mathbf{S}_1$, we have

$$\mathbf{A}_1^* = \mathbf{Z}_1^*(\hat{\mathbf{x}}_1^*)^{-1} = \begin{bmatrix} 0.031 & 0.037 & 0.022 \\ 0.040 & 0.003 & 0.031 \\ 0.087 & 0.022 & 0.057 \end{bmatrix}$$

and

$$(\mathbf{I} - \mathbf{A}_1^*)^{-1} = \begin{bmatrix} 1.036 & 0.039 & 0.026 \\ 0.044 & 1.005 & 0.034 \\ 0.097 & 0.041 & 1.064 \end{bmatrix}$$

and for $\mathbf{S}_2$ we have

$$\mathbf{A}_2^* = \mathbf{Z}_2^*(\hat{\mathbf{x}}_2^*)^{-1} = \begin{bmatrix} 0.002 & 0.025 & 0.040 \\ 0.019 & 0.019 & 0.048 \\ 0.040 & 0.036 & 0.070 \end{bmatrix}$$

and

$$(\mathbf{I} - \mathbf{A}_2^*)^{-1} = \begin{bmatrix} 1.005 & 0.027 & 0.044 \\ 0.022 & 1.022 & 0.054 \\ 0.044 & 0.041 & 1.079 \end{bmatrix}$$

Suppose we are given a new final demand, $\tilde{\mathbf{f}}$, which is presented to the economy as $\tilde{\mathbf{f}} = \begin{bmatrix} 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}$. For the two alternative aggregations, the corresponding final-demand vectors are $\tilde{\mathbf{f}}_1^* = \mathbf{S}_1\tilde{\mathbf{f}} = \begin{bmatrix} 10 \\ 10 \\ 20 \end{bmatrix}$ and $\tilde{\mathbf{f}}_2^* = \mathbf{S}_2\tilde{\mathbf{f}} = \begin{bmatrix} 10 \\ 10 \\ 20 \end{bmatrix}$. The corresponding total output vectors are $\tilde{\mathbf{x}}_1^* = (\mathbf{I} - \mathbf{A}_1^*)^{-1}\tilde{\mathbf{f}}_1^* = \begin{bmatrix} 11.26 \\ 11.16 \\ 22.52 \end{bmatrix}$ and $\tilde{\mathbf{x}}_2^* = (\mathbf{I} - \mathbf{A}_2^*)^{-1}\tilde{\mathbf{f}}_2^* = \begin{bmatrix} 11.20 \\ 11.51 \\ 22.43 \end{bmatrix}$.

If we use the unaggregated model in impact analysis, the total output vector is $\tilde{\mathbf{x}} = (\mathbf{I} - \mathbf{A})^{-1}\tilde{\mathbf{f}} = \begin{bmatrix} 11.30 \\ 11.20 \\ 11.51 \\ 11.13 \end{bmatrix}$ where $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$ from the original unaggregated matrix of transactions, $\mathbf{Z}$, and vector of total outputs, $\mathbf{x}$. If we aggregate the vector $\tilde{\mathbf{x}}$ by the two aggregation schemes, we obtain $\mathbf{S}_1\tilde{\mathbf{x}} = \begin{bmatrix} 11.30 \\ 11.20 \\ 22.64 \end{bmatrix}$ and $\mathbf{S}_2\tilde{\mathbf{x}} = \begin{bmatrix} 11.20 \\ 11.51 \\ 22.43 \end{bmatrix}$.

Note that while $\tilde{\mathbf{x}}_1^*$ and $\mathbf{S}_1\tilde{\mathbf{x}}$ are quite different, $\tilde{\mathbf{x}}_2^*$ and $\mathbf{S}_2\tilde{\mathbf{x}}$ are identical. That is, no error is introduced in the second aggregation scheme, $\mathbf{S}_2$. We will see more formally later why this is true, but for the time being, we can examine the original unaggregated matrix of technical coefficients, $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 0.031 & 0.037 & 0.017 & 0.031 \\ 0.040 & 0.003 & 0.025 & 0.040 \\ 0.048 & 0.019 & 0.019 & 0.048 \\ 0.039 & 0.003 & 0.019 & 0.039 \end{bmatrix}$. Note that the first and last columns of $\mathbf{A}$ are identical, that is, the two industries have identical production characteristics. In the $\mathbf{S}_2$ aggregation scheme, these two industries are aggregated into one; this was not the case in the $\mathbf{S}_1$ aggregation scheme. This should not be surprising, however, since two industries with the same production function are by definition the same industry and, hence, there should be no bias introduced by aggregation.

4.9.2 Measures of Aggregation Bias

Total aggregation bias has been defined – for example, in Morimoto (1970) – as the difference between the vector of total outputs in the aggregated system and the vector obtained by aggregating the total outputs in the original unaggregated system. As in the last example, for some new vector of final demands, $\mathbf{f}$, the total output vector in the unaggregated model is $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$. The total output vector in the aggregated model is $\mathbf{x}^* = (\mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{f}^*$, and the total aggregation bias is defined as

$$\boldsymbol{\tau} = \mathbf{x}^* - \mathbf{S}\mathbf{x} \tag{4.25}$$

That is, $\boldsymbol{\tau} = (\mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{f}^* - \mathbf{S}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$, or $\boldsymbol{\tau} = [(\mathbf{I} - \mathbf{A}^*)^{-1}\mathbf{S} - \mathbf{S}(\mathbf{I} - \mathbf{A})^{-1}]\mathbf{f}$. Using the power series results,

$$\begin{aligned} \boldsymbol{\tau} &= [(\mathbf{I} + \mathbf{A}^* + \mathbf{A}^{*2} + \cdots)\mathbf{S} - \mathbf{S}(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots)]\mathbf{f} \\ &= [(\mathbf{A}^*\mathbf{S} - \mathbf{S}\mathbf{A}) + (\mathbf{A}^{*2}\mathbf{S} - \mathbf{S}\mathbf{A}^2) + \cdots]\mathbf{f} \end{aligned} \tag{4.26}$$

The first term in this series has been defined as the "first-order" aggregation bias (Theil, 1957); that is,

$$\boldsymbol{\varphi} = (\mathbf{A}^*\mathbf{S} - \mathbf{S}\mathbf{A})\mathbf{f} \tag{4.27}$$

We present two basic theorems regarding aggregation bias and, in particular, when it will vanish. One has to do with the nature of the $\mathbf{A}$ and $\mathbf{A}^*$ matrices, that is, with the structural characteristics of the economy; the other has to do with the nature of the final-demand vectors, $\mathbf{f}$ and $\mathbf{f}^*$, being studied. The former is

Theorem 4.1. The total aggregation bias vanishes (i.e., $\boldsymbol{\tau} = \mathbf{0}$) for any $\boldsymbol{\varphi}$ if and only if $\mathbf{A}^*\mathbf{S} = \mathbf{S}\mathbf{A}$.

This follows from the expression for $\boldsymbol{\tau}$ in (4.26) since, if $\mathbf{A}^*\mathbf{S} = \mathbf{S}\mathbf{A}$, then

$$\mathbf{A}^{*2}\mathbf{S} - \mathbf{S}\mathbf{A}^2 = \mathbf{A}^*\mathbf{A}^*\mathbf{S} - \mathbf{S}\mathbf{A}\mathbf{A} = \mathbf{A}^*(\mathbf{S}\mathbf{A}) - (\mathbf{A}^*\mathbf{S})\mathbf{A} = \mathbf{0}$$

and similarly, for higher-order terms in the series. This theorem suggests that if two (or more) sectors have identical interindustry structures (i.e., equal columns in the $\mathbf{A}$ matrix, as we found in the example), then aggregation of these sectors will result in zero total aggregation bias. For example, consider a three-sector economy in which sectors 1 and 3 have the same interindustry input structure:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The corresponding transactions matrix is found by

$$\mathbf{Z} = \mathbf{A}\hat{\mathbf{x}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 & a_{12}x_2 & a_{13}x_3 \\ a_{21}x_1 & a_{22}x_2 & a_{23}x_3 \\ a_{31}x_1 & a_{32}x_2 & a_{33}x_3 \end{bmatrix}$$

The proper aggregation matrix for combining sectors 1 and 3 is $\mathbf{S} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Hence, the aggregated transactions matrix and total outputs vector are

$$\mathbf{Z}^* = \mathbf{SZS}' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11}x_1 & a_{12}x_2 & a_{13}x_3 \\ a_{21}x_1 & a_{22}x_2 & a_{23}x_3 \\ a_{31}x_1 & a_{32}x_2 & a_{33}x_3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

or

$$\mathbf{Z}^* = \begin{bmatrix} a_{11}x_1 + a_{31}x_1 + a_{11}x_3 + a_{31}x_3 & a_{12}x_2 + a_{32}x_2 \\ a_{21}x_1 + a_{21}x_3 & a_{22}x_2 \end{bmatrix}$$

and $\mathbf{x}^* = \mathbf{Sx} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 \\ x_2 \end{bmatrix}$. Hence, the aggregated technical coefficients matrix is found by

$$\mathbf{A}^* = \mathbf{Z}^*(\hat{\mathbf{x}}^*)^{-1} = \begin{bmatrix} \dfrac{(a_{11} + a_{31})(x_1 + x_3)}{x_1 + x_3} & \dfrac{(a_{12} + a_{32})x_2}{x_2} \\ \dfrac{a_{21}(x_1 + x_3)}{x_1 + x_3} & \dfrac{a_{22}x_2}{x_2} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} + a_{31} & a_{12} + a_{32} \\ a_{21} & a_{22} \end{bmatrix}$$

Theorem 4.1 asserts that there will be no aggregation bias when two columns are identical, that is, when $\mathbf{A}^*\mathbf{S} = \mathbf{SA}$. For our general example this can be shown by

$$\mathbf{A}^*\mathbf{S} = \begin{bmatrix} a_{11} + a_{31} & a_{12} + a_{32} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} + a_{31} & a_{12} + a_{32} & a_{11} + a_{31} \\ a_{21} & a_{22} & a_{21} \end{bmatrix}$$

and

$$\mathbf{SA} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} a_{11} + a_{31} & a_{12} + a_{32} & a_{11} + a_{31} \\ a_{21} & a_{22} & a_{21} \end{bmatrix},$$

which are the same.

The second theorem on aggregation bias is as follows.

Theorem 4.2. If some sectors are not aggregated and the new final demands occur only in unaggregated sectors, the first-order aggregation bias will vanish.

For a general three-sector economy, the unaggregated and aggregated technical coefficients matrices, $\mathbf{A}$ and $\mathbf{A}^*$, respectively, are

$$\mathbf{A} = \begin{bmatrix} \dfrac{z_{11}}{x_1} & \dfrac{z_{12}}{x_2} & \dfrac{z_{13}}{x_3} \\ \dfrac{z_{21}}{x_1} & \dfrac{z_{22}}{x_2} & \dfrac{z_{23}}{x_3} \\ \dfrac{z_{31}}{x_1} & \dfrac{z_{32}}{x_2} & \dfrac{z_{33}}{x_3} \end{bmatrix} \text{ and } \mathbf{A}^* = \begin{bmatrix} \dfrac{z_{11}}{x_1} & \dfrac{z_{12} + z_{13}}{x_2 + x_3} \\ \dfrac{z_{21} + z_{31}}{x_1} & \dfrac{z_{22} + z_{23} + z_{32} + z_{33}}{x_2 + x_3} \end{bmatrix}$$

The unaggregated sector is sector 1 (in both the aggregated and unaggregated models). Consider final-demand vectors for which only the unaggregated elements are nonzero: $\mathbf{f} = \begin{bmatrix} f_1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{f}^* = \mathbf{Sf} = \begin{bmatrix} f_1 \\ 0 \end{bmatrix}$. This theorem asserts that the first-order aggregation bias, $\boldsymbol{\varphi} = (\mathbf{A}^*\mathbf{S} - \mathbf{SA})\mathbf{f}$ is zero for final demands such as those given as $\mathbf{f}$ and $\mathbf{f}^*$ above. For the example:

$$\mathbf{SA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{z_{11}}{x_1} & \frac{z_{12}}{x_2} & \frac{z_{13}}{x_3} \\ \frac{z_{21}}{x_1} & \frac{z_{22}}{x_2} & \frac{z_{23}}{x_3} \\ \frac{z_{31}}{x_1} & \frac{z_{32}}{x_2} & \frac{z_{33}}{x_3} \end{bmatrix} = \begin{bmatrix} \frac{z_{11}}{x_1} & \frac{z_{12}}{x_2} & \frac{z_{13}}{x_3} \\ \frac{z_{21}+z_{31}}{x_1} & \frac{z_{22}+z_{32}}{x_2} & \frac{z_{23}+z_{33}}{x_3} \end{bmatrix}$$

and

$$\mathbf{A}^*\mathbf{S} = \begin{bmatrix} \frac{z_{11}}{x_1} & \frac{z_{12}+z_{13}}{x_2+x_3} & \frac{z_{12}+z_{13}}{x_2+x_3} \\ \frac{z_{21}+z_{31}}{x_1} & \frac{z_{22}+z_{23}+z_{32}+z_{33}}{x_2+x_3} & \frac{z_{22}+z_{23}+z_{32}+z_{33}}{x_2+x_3} \end{bmatrix}.$$

Hence, the first-order bias, $\boldsymbol{\varphi}$, as defined earlier, is $\boldsymbol{\varphi} = (\mathbf{A}^*\mathbf{S} - \mathbf{SA})\mathbf{f} =$

$$\begin{bmatrix} 0 & \left(\frac{z_{12}+z_{13}}{x_2+x_3} - \frac{z_{12}}{x_2}\right) & \left(\frac{z_{12}+z_{13}}{x_2+x_3} - \frac{z_{13}}{x_3}\right) \\ 0 & \left(\frac{z_{22}+z_{23}+z_{32}+z_{33}}{x_2+x_3} - \frac{z_{22}+z_{32}}{x_2}\right) & \left(\frac{z_{22}+z_{23}+z_{32}+z_{33}}{x_2+x_3} - \frac{z_{23}+z_{33}}{x_3}\right) \end{bmatrix} \begin{bmatrix} f_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, if one is studying the effect of new final demand only for sector 1's output in an n-sector model, any and all combinations of sectors 2 through n into fewer sectors will generate no first-order aggregation bias. Although these theorems are stated in terms of sectoral aggregation, they also have implications for spatial aggregation in interregional models. In general, the conditions of Theorem 4.1 are almost certain not to be met as one combines regions in an interregional input–output model, but the conditions of Theorem 4.2 will be met in many cases.

Aggregation bias in interregional and multiregional input–output models is discussed in detail in Miller and Blair (1981) and Blair and Miller (1983). Additional theorems on sectoral aggregation bias based on statistical properties are discussed in Gibbons, Wolsky and Tolley (1982). Examples of sectoral aggregation for the three-region Japanese interregional and the US multiregional input–output models are included for the interested reader in Appendix 4.1.

4.10 Summary

In this chapter we have explored some of the most important practical issues associated with applying input–output analysis, namely construction of the basic input–output tables as part of a system of national accounting conventions and data collection. The chapter focuses primarily on a System of National Accounts (SNA), including the integral Input–Output Accounts (IOAs), derived from basic concepts of the circular flow of income and expenditure. In order to define interindustry production and consumption within the SNA, the framework includes conventions for distinguishing between commodities and industries, i.e., production and consumption of defined goods and services or *commodities* and *industries* or groups of economic establishments that produce those commodities, with an individual industry perhaps producing more than one commodity. This commodity-by-industry framework lays the foundation for more detailed examination of commodity-by-industry models in Chapter 5, alternatives to full survey-based construction of input–output tables in Chapters 7 and 8, and extensions to the basic input–output framework in later chapters, such as the SNA as the basis for broader social accounting in Chapter 11. Finally, this chapter examines some of the key considerations in defining the level of sectoral detail in input–output models, especially measures of bias introduced by aggregation of sectors.

Appendix 4.1 Spatial Aggregation in IRIO and MRIO Models

We consider two examples of spatial aggregation for two multiple region input–output models: (1) a three-region interregional (IRIO) model for Japan and (2) the US multiregional (MRIO) model and using the basic measures of aggregation bias introduced in Section 4.8.2.

A4.1.1 Spatial Aggregation of IRIO Models

Spatial aggregation of IRIO models is in many respects identical to sectoral aggregation. As an example for the IRIO case we consider a highly aggregated, three-region, five-sector version of the Japanese IRIO model defined in Table A4.1.1. In the following we consider the case of aggregating this model to two regions, the first being region 1 (Central), unaggregated, of the three-region model. The second aggregated model region is to be composed by combining regions 2 (North) and 3 (South) of the three-region model. Hence, using the notation of Chapter 3 for IRIO transactions and denoting the regions of the aggregated model by a (Central) and b (North plus South), the new transactions matrix is found by (for $i, j = 1, 2, \ldots, 5$ in all cases) $z_{ij}^{aa} = z_{ij}^{11}$, $z_{ij}^{ab} = z_{ij}^{12} + z_{ij}^{13}$, $z_{ij}^{ba} = z_{ij}^{21} + z_{ij}^{31}$, $z_{ij}^{bb} = z_{ij}^{22} + z_{ij}^{23} + z_{ij}^{32} + z_{ij}^{33}$. Similarly, total outputs are found by $\mathbf{x}_i^a = \mathbf{x}_i^1$ and $\mathbf{x}_i^b = \mathbf{x}_i^2 + \mathbf{x}_i^3$.

Table A4.1.1 Input Coefficients for the Five-Sector, Three-Region Interregional Input–Output Table for Japan (1965)

		Central					North					South					Total
		1	2	3	4	5	1	2	3	4	5	1	2	3	4	5	Output*
	Central																
1	Agriculture	0.053	0.000	0.009	0.011	0.009	0.001	0.000	0.007	0.000	0.001	0.001	0.000	0.001	0.000	0.000	1, 307
2	Mining	0.000	0.001	0.001	0.001	0.002	0.000	0.000	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	123
3	Const. & Manuf.	0.428	0.723	0.250	0.240	0.180	0.012	0.004	0.052	0.001	0.013	0.017	0.005	0.044	0.000	0.014	16, 400
4	Transportation	0.000	0.001	0.010	0.090	0.012	0.000	0.000	0.002	0.015	0.001	0.000	0.000	0.001	0.007	0.001	1, 342
5	Other	0.012	0.029	0.042	0.117	0.125	0.000	0.001	0.015	0.001	0.010	0.000	0.000	0.007	0.001	0.014	8, 591
	North																
1	Agriculture	0.004	0.000	0.000	0.000	0.000	0.089	0.001	0.017	0.039	0.021	0.002	0.000	0.000	0.000	0.000	1, 308
2	Mining	0.000	0.000	0.000	0.000	0.000	0.002	0.005	0.002	0.007	0.011	0.000	0.000	0.000	0.000	0.000	201
3	Const. & Manuf.	0.068	0.041	0.020	0.000	0.002	0.362	0.521	0.160	0.233	0.129	0.034	0.028	0.012	0.000	0.001	4, 167
4	Transportation	0.000	0.002	0.000	0.014	0.000	0.000	0.008	0.010	0.025	0.011	0.000	0.000	0.000	0.023	0.000	394
5	Other	0.003	0.034	0.001	0.000	0.001	0.010	0.033	0.027	0.095	0.103	0.002	0.008	0.000	0.000	0.001	2, 759
	South																
1	Agriculture	0.002	0.000	0.002	0.000	0.000	0.002	0.000	0.006	0.000	0.000	0.072	0.000	0.011	0.016	0.010	2, 131
2	Mining	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.001	0.000	0.000	0.001	0.004	0.001	0.002	0.004	267
3	Const. & Manuf.	0.036	0.021	0.082	0.000	0.013	0.012	0.012	0.056	0.000	0.007	0.473	0.719	0.303	0.264	0.196	22, 053
4	Transportation	0.000	0.000	0.001	0.024	0.000	0.000	0.000	0.001	0.022	0.000	0.000	0.003	0.009	0.068	0.012	1, 546
5	Other	0.001	0.005	0.006	0.000	0.003	0.000	0.001	0.009	0.000	0.003	0.012	0.050	0.037	0.112	0.110	9, 968

*Total output measured in billions of Japanese Yen.

Note that we can easily accomplish this spatial aggregation by constructing an aggregation matrix, **S**, as we did in the case of sectoral aggregation:

$$\mathbf{S} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&1 \end{bmatrix}$$

We can use **S** to create $\mathbf{x}^* = \mathbf{Sx}$, $\mathbf{Z}^* = \mathbf{SZS}'$ where $\mathbf{x}^*$ is the 10×1 aggregated vector of final demands (the unaggregated vector, **x**, is 15×1); $\mathbf{Z}^*$ is the aggregated 10×10 interindustry transactions matrix (the unaggregated transactions matrix, **Z**, is 15×15).

We can subsequently compute the new aggregated total outputs vector as $\mathbf{x}^* = \mathbf{Sx} = \begin{bmatrix}1307 & 123 & 16400 & 1342 & 8591 & 3440 & 468 & 26220 & 1940 & 12727\end{bmatrix}'$ The new aggregated matrix of IRIO input coefficients is

$$\mathbf{A}^* = \mathbf{Z}^*(\hat{\mathbf{x}}^*)^{-1} = \begin{bmatrix} .053&0&.009&.011&.009&.001&0&.002&0&0\\ 0&.001&.001&.001&.002&0&0&0&0&0\\ .428&.723&.25&.24&.18&.015&.005&.045&0&.014\\ 0&.001&.01&.09&.012&0&0&.001&.009&.001\\ .012&.029&.042&.117&.125&0&0&.008&.001&.013\\ .006&0&.002&0&0&.08&0&.013&.021&.012\\ 0&0&0&0&0&.001&.004&.001&.003&.006\\ .104&.062&.102&0&.015&.456&.655&.299&.258&.184\\ 0&.002&.001&.038&0&0&.005&.009&.082&.012\\ .004&.039&.007&0&.004&.012&.048&.037&.109&.110 \end{bmatrix}$$

The corresponding Leontief inverse is

$$(\mathbf{I} - \mathbf{A}^*)^{-1} = \begin{bmatrix} 1.063&.012&.015&.019&.014&.004&.004&.005&.002&.002\\ .001&1.002&.001&.002&.003&0&.001&.001&0&0\\ .639&1.016&1.380&.413&.299&.075&.081&.101&.041&.050\\ .008&.013&.016&1.107&.019&.002&.002&.003&.012&.002\\ .050&.088&.071&.170&1.161&.012&.016&.021&.011&.023\\ .013&.008&.007&.004&.002&1.099&.018&.023&.033&.021\\ .001&.001&.001&0&0&.003&1.007&.003&.005&.007\\ .267&.267&.217&.092&.076&.754&1.050&1.480&.477&.335\\ .005&.009&.006&.049&.003&.009&.018&.017&1.098&.018\\ .021&.064&.020&.015&.010&.049&.105&.065&.155&1.140 \end{bmatrix}$$

Table A4.1.2 Spatial Aggregation of IRIO Models: Results for Japanese IRIO Table

		Aggregated Gross Outputs from the Three-Region Model $\mathbf{S\tilde{x}}$	Outputs from the Aggregated Two-Region Model $\tilde{\mathbf{x}}^*$	Aggregation Error $\mathbf{S\tilde{x}} - \tilde{\mathbf{x}}^*$	Aggregation Error as a Percent of Gross Outputs of the Three-Region Model $100\left(\frac{\lvert\mathbf{S\tilde{x}}-\tilde{\mathbf{x}}^*\rvert}{\mathbf{S\tilde{x}}}\right)$
Region *a*	Sector				
	1	116.801	115.749	1.052	.901
	2	101.649	101.394	.255	.251
	3	443.529	444.330	−.801	−.181
	4	121.260	120.363	.896	.739
	5	171.553	170.789	.764	.446
Region *a* Total (Absolute)		954.792	952.625	3.768	
Region *b*	Sector				
	1	246.876	242.116	4.769	1.928
	2	206.519	205.343	1.176	.570
	3	853.242	911.145	−57.904	−6.786
	4	235.381	238.800	−3.418	−1.452
	5	309.717	315.778	−6.061	−1.957
Region *b* Total (Absolute)		1851.735	1913.182	73.319	
Total (Absolute)		2806.527	2865.807	77.087	

Let us now compute the aggregation bias introduced by grouping regions 2 and 3. Consider the following vector of final demands for the unaggregated (three-region, five-sector) model: $\tilde{\mathbf{f}} = \begin{bmatrix} 100 & 100 & \cdots & 100 \end{bmatrix}'$. The corresponding aggregated (two-region, five-sector) version is $\tilde{\mathbf{f}}^* = [\,100 \ \ 100 \ \ 100 \ \ 100 \ \ 100 \quad 200 \ \ 200 \ \ 200 \ \ 200 \ \ 200\,]'$.

We can compute $\tilde{\mathbf{x}}^* = (\mathbf{I} - \mathbf{A}^*)^{-1}\tilde{\mathbf{f}}^*$ and $\mathbf{x} = (\tilde{\mathbf{I}} - \mathbf{A})^{-1}\tilde{\mathbf{f}}$ where $\mathbf{A}$ is the original unaggregated technical coefficients matrix. In order to compare $\tilde{\mathbf{x}}^*$ and $\tilde{\mathbf{x}}$, we must aggregate $\tilde{\mathbf{x}}$, which can be accomplished with the sectoral aggregation matrix, $\mathbf{S}$, given earlier, that is, $\mathbf{S\tilde{x}}$. Table A4.1.2 gives the vectors $\tilde{\mathbf{x}}^*$, $\mathbf{S\tilde{x}}$, and the differences between the corresponding elements. The sum of absolute differences $\lvert\mathbf{S\tilde{x}} - \tilde{\mathbf{x}}^*\rvert$ for the unaggregated region *a* (Central) as a percentage of the total outputs in that region, that is, $\mathbf{S\tilde{x}i}$, is $100\left(\frac{\lvert\mathbf{S\tilde{x}}-\tilde{\mathbf{x}}^*\rvert\mathbf{i}}{\mathbf{S\tilde{x}i}}\right) = 100\left(\frac{3.768}{954.792}\right) = 0.395\%$ and the corresponding value for region *b* (North and South) is $100\left(\frac{73.319}{1851.735}\right) = 3.959\%$. This indicates, as expected, that more error is introduced into the prediction of outputs in the aggregated region than in the unaggregated region. The overall error (for both regions) is $100\left(\frac{77.087}{2806.527}\right) = 2.747\%$.

Notice from the table that the aggregation bias is quite small in all three calculations, that is, region *a*, region *b*, and overall, particularly in the unaggregated region. Miller and Blair (1981) show that spatial aggregation of IRIO models generally seems to introduce only modest bias. This suggests, for example, that if one is interested in the impacts in one region in an interconnected interregional system of a change in final demands for some of the sectors in that region (e.g., effects on the California economy of new federal spending in California, which is one of the interconnected 48 continental states), then a "two-region" model of California and the rest of the United States may be sufficient.

A4.1.2 Spatial Aggregation of MRIO Models

Consider a highly aggregated (three-region, five-sector) MRIO input–output model of the United States given in Table A4.1.3. We consider the case of aggregating regions 2 (Central) and 3 (West) of the basic three-region model, leaving region 1 (East) unaggregated. We designate the regions in the aggregated model by superscripts a (East) and b (Central plus West) so that the new *intraregional* flow matrices are found by (for i, $j = 1, \ldots, 5$ in all cases) the following: $z_{ij}^a = z_{ij}^1$, $z_{ij}^b = z_{ij}^2 + z_{ij}^3$. Similarly, total regional outputs are $x_i^a = x_i^1$, $x_i^b = x_i^2 + x_i^3$. Hence the input coefficients for the aggregated model are found by $a_{ij}^a = \frac{z_{ij}^a}{x_j^a}$, $a_{ij}^b = \frac{z_{ij}^b}{x_j^b}$.

The resulting block diagonal aggregated technical coefficients matrix, which we denote $\mathbf{A}^*$, is given by

$$\mathbf{A}^* = \begin{bmatrix} .082 & .003 & .012 & .005 & .61 & 0 & 0 & 0 & 0 & 0 \\ 0 & .196 & .043 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .156 & .211 & .302 & .076 & .110 & 0 & 0 & 0 & 0 & 0 \\ .096 & .133 & .131 & .220 & .101 & 0 & 0 & 0 & 0 & 0 \\ .012 & .001 & .061 & .002 & .234 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .046 & .002 & .030 & .007 & .075 \\ 0 & 0 & 0 & 0 & 0 & 0 & .302 & .057 & .001 & 0 \\ 0 & 0 & 0 & 0 & 0 & .103 & .143 & .281 & .075 & .115 \\ 0 & 0 & 0 & 0 & 0 & .207 & .151 & .127 & .216 & .101 \\ 0 & 0 & 0 & 0 & 0 & .010 & .001 & .075 & .002 & .230 \end{bmatrix}$$

The *interregional* commodity flow matrices for the original unaggregated model are $z_i = z_i^{rs}$ for $r, s = 1, 2, 3$ regions and $i = 1, \ldots, 5$ sectors, a total of five 3×3 matrices. Aggregation from three to two regions for the commodity flows can be accomplished by constructing a spatial aggregation matrix $\mathbf{R}$, as in the case of sectoral aggregation; for this example, $\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. We define $\mathbf{R}$ to be distinct from the sectoral aggregation matrix, $\mathbf{S}$, defined earlier. The aggregated (2×2) interregional flow matrices, $\mathbf{Z}_i^*$, are found by $\mathbf{Z}_i^* = \mathbf{R}\mathbf{Z}_i\mathbf{R}'$ for $i = 1, \ldots, 5$ industries.

Table A4.1.3 Five-Sector, Three-Region Multiregional Input–Output Tables for the United States (1963)

	Agriculture	Mining	Const. & Manuf.	Services	Transport & Utilities
Regional Transactions (millions of dollars)					
East					
Agriculture	2,013	0	7,863	44	0
Mining	35	335	3,432	44	843
Const. & Manuf.	2,029	400	78,164	11,561	2,333
Services	1,289	294	19,699	26,574	2,301
Transport & Util.	225	384	7,232	4,026	3,534
Central					
Agriculture	10,303	0	13,218	97	0
Mining	82	472	8,686	15	1,271
Const. & Manuf.	4,422	1,132	93,816	10,155	2,401
Services	4,952	2,378	21,974	22,358	2,473
Transport & Util.	667	406	9,296	3,468	4,513
West					
Agriculture	2,915	0	3,452	65	0
Mining	4	292	2,503	0	353
Const. & Manuf.	1,214	466	27,681	4,925	1,015
Services	1,307	721	8,336	10,809	991
Transport & Util.	338	160	2,936	1,659	1,576
Commodity Trade Flows and Total Outputs (millions of dollars)					
Agriculture	East	West	Central		
East	6,007	2,124	208		
West	3,845	28,885	2,521		
Central	403	2,922	7,028		
Mining	East	West	Central		
East	2,904	415	53		
West	1,108	10,942	271		
Central	71	772	3,996		
Const. & Manuf.	East	West	Central		
East	158,679	42,150	8,368		
West	44,589	201,025	11,778		
Central	4,702	6,726	61,385		
Services	East	West	Central		
East	146,336	16,116	2,955		
West	9,328	121,079	3,185		
Central	1,939	3,643	58,663		
Transport & Util.	East	West	Central		
East	21,434	4,974	263		
West	4,396	23,811	1,948		
Central	1,009	1,334	9,635		
Total Output	East	West	Central		
Agriculture	10,259	33,939	9,753		
Mining	4,084	12,129	4,319		
Const. & Manuf.	207,948	249,840	81,512		
Services	157,468	140,850	64,803		
Transport & Util.	26,847	30,130	11,841		

We can then construct the aggregated trade coefficients $c_i^{ab} = \frac{z_i^{ab}}{T_i^b}$. The trade coefficients matrix for the aggregated MRIC model, $\mathbf{C}^*$, is

$$
\mathbf{C}^* = \begin{bmatrix} \hat{\mathbf{c}}^{aa} & \hat{\mathbf{c}}^{ab} \\ \hat{\mathbf{c}}^{ba} & \hat{\mathbf{c}}^{bb} \end{bmatrix} = \begin{bmatrix}
.621 & 0 & 0 & 0 & 0 & .047 & 0 & 0 & 0 & 0 \\
0 & .586 & 0 & 0 & 0 & 0 & .053 & 0 & 0 & 0 \\
0 & 0 & .738 & 0 & 0 & 0 & 0 & .144 & 0 & 0 \\
0 & 0 & 0 & .824 & 0 & 0 & 0 & 0 & .121 & 0 \\
0 & 0 & 0 & 0 & .721 & 0 & 0 & 0 & 0 & .157 \\
.379 & 0 & 0 & 0 & 0 & .953 & 0 & 0 & 0 & 0 \\
0 & .414 & 0 & 0 & 0 & 0 & .947 & 0 & 0 & 0 \\
0 & 0 & .262 & 0 & 0 & 0 & 0 & .856 & 0 & 0 \\
0 & 0 & 0 & .176 & 0 & 0 & 0 & 0 & .879 & 0 \\
0 & 0 & 0 & 0 & .279 & 0 & 0 & 0 & 0 & .843
\end{bmatrix}
$$

The corresponding matrix of MRIO multipliers is

$$
(\mathbf{I} - \mathbf{C}^*\mathbf{A}^*)^{-1}\mathbf{C}^* = \begin{bmatrix}
.658 & .004 & .012 & .005 & .039 & .053 & .002 & .006 & .002 & .015 \\
.004 & .680 & .032 & .003 & .004 & .002 & .088 & .014 & .002 & .003 \\
.124 & .180 & 1.007 & .084 & .129 & .045 & .078 & .271 & .037 & .071 \\
.103 & .142 & .161 & 1.031 & .127 & .055 & .068 & .077 & .193 & .065 \\
.017 & .015 & .063 & .008 & .895 & .008 & .010 & .036 & .005 & .243 \\
.425 & .013 & .028 & .007 & .061 & 1.008 & .015 & .048 & .012 & .095 \\
.013 & .678 & .066 & .008 & .014 & .013 & 1.358 & .100 & .010 & .017 \\
.118 & .202 & .493 & .064 & .128 & .153 & .264 & 1.213 & .109 & .189 \\
.138 & .176 & .131 & .281 & .111 & .237 & .274 & .218 & 1.115 & .180 \\
.021 & .022 & .066 & .009 & .433 & .025 & .025 & .105 & .013 & 1.083
\end{bmatrix}
$$

We now compute the aggregation bias introduced by this spatial consolidation. Consider the following 15-element vector of hypothesized final demands for the unaggregated (three-region, five-sector) model $\tilde{\mathbf{f}} = \begin{bmatrix} 100 & 100 & \cdots & 100 \end{bmatrix}'$. The corresponding aggregated (two-region, five-sector) version is $\tilde{\mathbf{f}}^* = [100\ 100\ 100\ 100\ 100\ \ 200\ 200\ 200\ 200\ 200]'$. We can compute $\tilde{\mathbf{x}}^* = (\mathbf{I} - \mathbf{C}^*\mathbf{A}^*)^{-1}\mathbf{C}^*\tilde{\mathbf{f}}^*$ and $\tilde{\mathbf{x}} = (\mathbf{I} - \mathbf{CA})^{-1}\mathbf{C}\tilde{\mathbf{f}}$ where $\mathbf{A}$ and $\mathbf{C}$ are from the original unaggregated model. In order to compare $\tilde{\mathbf{x}}^*$ and $\tilde{\mathbf{x}}$, we must aggregate $\tilde{\mathbf{x}}$, which we can accomplish with the following sectoral aggregation matrix, **S**:

$$\mathbf{S}\tilde{\mathbf{x}} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&0&1 \end{bmatrix} \begin{bmatrix} 117\\127\\254\\240\\155\\139\\192\\293\\277\\175\\126\\145\\191\\219\\135 \end{bmatrix}$$

Table A4.1.4 Spatial Aggregation of MRIO Models: Results for US MRIO Model

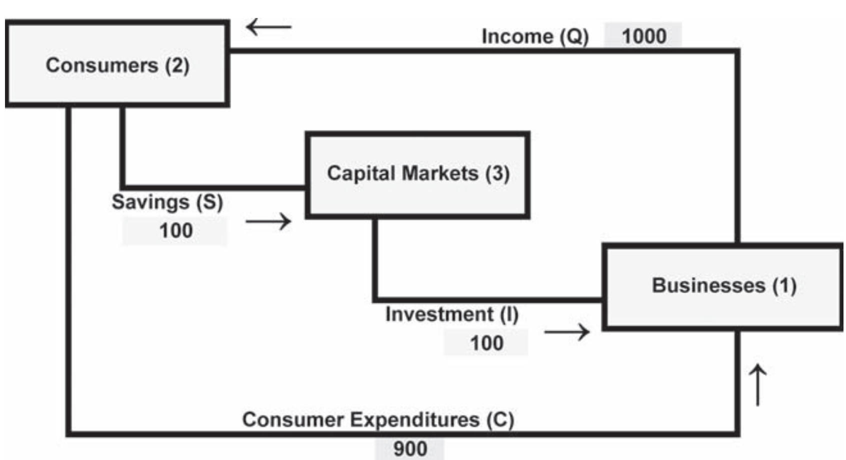
		Aggregated Gross Outputs from the Three-Region Model $\mathbf{S}\tilde{\mathbf{x}}$	Outputs from the Aggregated Two-Region Model $\tilde{\mathbf{x}}^*$	Aggregation Error $\mathbf{S}\tilde{\mathbf{x}} - \tilde{\mathbf{x}}^*$	Aggregation Error as a Percent of Gross Outputs of the Three-Region Model $100\left(\frac{\lvert\mathbf{S}\tilde{\mathbf{x}}-\tilde{\mathbf{x}}^*\rvert}{\mathbf{S}\tilde{\mathbf{x}}}\right)$
Region *a*	Sector				
	1	131.718	135.265	.547	.405
	2	109.863	110.036	.173	.157
	3	352.078	358.354	6.276	1.751
	4	133.305	134.171	.866	.645
	5	215.715	216.205	.490	.226
Region *a* Total (Absolute)		954.679	954.031	8.352	
Region *b*	Sector				
	1	311.061	318.149	7.088	2.228
	2	229.036	229.359	.324	.141
	3	658.678	633.958	−24.720	−3.899
	4	262.909	257.744	−5.164	−2.004
	5	392.772	388.443	−4.329	−1.115
Region *b* Total (Absolute)		1854.456	1827.653	41.625	
Total (Absolute)		2800.135	2781.684	49.977	

Table A4.1.4 gives the vectors $\tilde{\mathbf{x}}^*$, $\mathbf{S}\tilde{\mathbf{x}}$, and the differences between corresponding elements. The sum of absolute differences $\mathbf{S}\tilde{\mathbf{x}} - \tilde{\mathbf{x}}^*$ for the unaggregated region a (East) as a percentage of the total outputs in that region, that is, $\mathbf{S}\tilde{\mathbf{x}}\mathbf{i}$, is $100\left(\frac{|\mathbf{S}\tilde{\mathbf{x}}-\tilde{\mathbf{x}}^*|\mathbf{i}}{\mathbf{S}\tilde{\mathbf{x}}\mathbf{i}}\right) = 100\left(\frac{8.352}{945.679}\right) = 0.883\%$ and the corresponding value for region b (Central plus West) is $100\left(\frac{41.625}{1854.456}\right) = 2.245\%$. This indicates, as expected, that more error is introduced into the prediction of outputs in the aggregated region than in the unaggregated region. The overall error (for both regions) is $100\left(\frac{49.977}{2800.135}\right) = 1.785\%$.

As we found with the IRIO model, it appears that spatial aggregation in MRIO models produces only modest aggregation bias, at least judging from the results of the example (see Blair and Miller, 1983, for a more detailed discussion). Hence, for questions pertaining to one or more specific regions, it appears that an MRIO model in which those regions are distinct, while the rest of the economy is aggregated into the "remaining" region, is likely to be entirely adequate.

Problems

4.1 Consider a macroeconomy provided in the figure below where transactions are measured in millions of dollars. Create the corresponding set of "T" accounts for production, income and capital transactions. Write the account balance equations.



4.2 For the macroeconomy shown in Problem 4.1, add a capital consumption allowance to account for depreciation of capital investments of 10 percent of total investment (I). Also add a "rest of world" account to accommodate purchases of imports of \$75 million, sales of exports of \$50 million, and savings made available to capital markets from overseas lenders of \$25 million (resulting in a new total amount of capital available for businesses of $125 million). Construct the modified set of "T" accounts and the corresponding balance equations.

4.3 The national economic balance sheet for an economy is given by the following:

Debits						Credits				
Prod.	Cons.	Capital Accum.	Govt.	Rest of World	Economic Transaction	Prod.	Cons.	Capital Accum.	Govt.	Rest of World
	475				Consumption Goods (C)	475				
		54			Capital Goods (I)	54				
				46	Exports (X)	46				
46					Imports (M)					46
554					Income (Q)		554			
		−29			Depreciation (D)		−29			
	30				Savings (S)			30		
			25		Govt. Expenditures (G)	25				
	20				Taxes (T)				20	
		5			Govt. Deficit Spending (B)				5	
600	525	30	25	46	Totals	600	525	30	25	46

a. Write the compete set of macro balance equations for this economy.
b. Construct the matrix representation of the consolidated national accounts.

4.4 Consider the following four-sector input–output transactions table for the year 2005 along with industry prices for 2000 and 2005.

	Industry Transactions						
	1	2	3	4	Total Output	Price Year 2000	Price Year 2005
1	24	86	56	64	398	2	5
2	32	15	78	78	314	3	6
3	104	49	62	94	469	5	9
4	14	16	63	78	454	7	12

Compute the matrices of interindustry transactions and technical coefficients as well as the vector of total outputs deflated to year 2000 value terms.

4.5 Consider the transactions data given in Problem 2.8. One way of assessing the effects of aggregation is as follows. Using a final-demand vector of all 1's, determine the effect on total outputs throughout the entire economy (i.e., summed over all the sectors) of the following set of increasingly aggregated models. (Remember to aggregate the final-demand vector appropriately each time you aggregate the sectors.)

- Case 1 (8×8) No sectoral aggregation
- Case 2 (7×7) Combine sector 6 with sector 2
- Case 3 (6×6) Also combine sector 5 with sector 1
- Case 4 (5×5) Also combine sector 8 with sector 3
- Case 5 (4×4) Also combine sector 7 with previously combined 6 and 2
- Case 6 (3×3) Also combine sector 4 with previously combined 5 and I

4.6 Consider the seven-sector input–output table of technical coefficients for the US economy (1972) given in Appendix B. Given a vector final demands of

$$\Delta\mathbf{f} = [100\ 100\ 100\ 100\ 100\ 100\ 100]$$

compute the first-order and total aggregation bias associated with combining agriculture with mining, construction with manufacturing, and transportation-utilities with services and other sectors to yield a new three-sector model.

4.7 Consider the following national accounting equations:

$$\begin{aligned} Q + M &= C + I + X + G \\ C + S + T &= Q + D \\ L + I + D + B &= S \\ X &= M + L \\ G &= T + B \end{aligned}$$

where Q = total consumer income payments; M = purchases of imports; C = total sales of consumption goods; S = total consumer savings; T = total taxes paid to government; I = total purchases of capital goods; D = total capital consumption allowance (depreciation); L = net lending from overseas; B = total government deficit spending; X = total sales of exports; G = total government purchases and the following are known: $Q = 500$, $M = 75$, $S = 60$, $T = 20$, $D = 10$, $L = 20$, and $B = 10$. Write the consolidated table of national accounts represented in matrix form.

4.8 Consider the following table of national accounts.

	Prod.	Cons.	Cap.	ROW	Govt.	Total
Prod.		410	80	55	30	575
Cons.	500		−10			490
Cap.		60				60
ROW	75		−20			55
Govt.		20	10			30
Total	575	490	60	55	30	

Suppose the following tables become available providing the interindustry supply and use detail for this economy.

Use of commodities by industries:

	Industry			Total Intermed. Output
	Nat. Res.	Manuf.	Serv.	
Commodity				
Agriculture	20	12	18	50
Mining	5	30	12	47
Manufacturing	10	13	11	34
Services	12	17	40	69

Final uses of commodity production:

	Households	Government	Investment	Exports
Agriculture	30	6	16	5
Mining	60	9	16	17
Manufacturing	50	3	40	22
Services	70	12	8	11
Totals	210	30	80	55

Supply of commodities by industries:

	Commodity				Total Industry Output
	Agric.	Mining	Manuf.	Services	
Industry					
Natural Resources	99			10	109
Manufacturing	8	143	137	10	298
Services		6	12	150	168
Total Commodity Output	107	149	149	170	575

Construct a consolidated set of supply and use accounts including the sector detail for interindustry transactions.

4.9 We define an input–output economy with $\mathbf{Z} = \begin{bmatrix} 500 & 0 & 0 \\ 50 & 300 & 150 \\ 200 & 150 & 550 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1000 \\ 750 \\ 1000 \end{bmatrix}$. Suppose this is a "US style" input–output table in which interindustry transactions include competitive imports but the sum of all imports across all industries of a particular product is included as a negative component of final demand.

a. If the vector of total value of competitive imports is found to be $\mathbf{m} = \begin{bmatrix} 150 \\ 105 \\ 210 \end{bmatrix}$, using the assumption of import similarity, compute the domestic transactions matrix where competitive imports are removed from interindustry transactions. Compute the corresponding **A** and **L**.

b. If we subsequently learn that $\mathbf{M} = \begin{bmatrix} 100 & 0 & 0 \\ 25 & 50 & 30 \\ 25 & 50 & 100 \end{bmatrix}$, compute the domestic transactions matrix and the corresponding **A** and **L**.

c. Now compute the mean absolute deviation (the average of the absolute value differences) between the total requirements matrices computed in (a) and (b).

4.10 Consider the three-region, three-sector 2000 Chinese interregional model specified in Tables 3.7, 3.8, and 3.9, which are **Z**, **A**, and **L**, respectively. Aggregate regions 1 and 2 and leave region 3 unaggregated to yield a two-region model. Calculate the aggregation bias measured as a percent of gross outputs with a reference vector of final demands given by $\tilde{\mathbf{f}} = [100\ 100 \ldots 100]'$ for the unaggregated model.

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5 The Commodity-by-Industry Approach in Input–Output Models

5.1 Introduction

In this chapter we explore a variation in the underlying data sets from which an input–output model is constructed. Using a "commodity–industry" format, we are able to account for the fact that an industry may produce more than one commodity (product). This was a major reason for the introduction of the commodity–industry accounting system – to explicitly account for "non-characteristic" production such as secondary products and by-products. In addition, data organized in this way are more easily integrated with a broader system of national accounts (SNA) for a country, as we saw in Chapter 4. These commodity–industry accounts lead to input–output models that have more complicated structures than those in Chapters 2 and 3; commodity–industry models are the concern of this chapter. The large Eurostat manual (Eurostat/European Commission, 2008) provides an excellent and comprehensive discussion of this framework. There, as in many other publications, "product" is used instead of "commodity." We will use "commodity" in this text because that is the predominant terminology associated with the early derivations and discussions of this system, in the 1960s and 1970s, and it continues to be used by many analysts.

The commodity-by-industry accounting framework originated largely in the work of Sir Richard Stone and his associates (Stone, 1961; Cambridge University, 1963). It was proposed in 1968 by the United Nations as a standard for data gathering in countries throughout the world (United Nations, 1968), and it subsequently has become a feature of data collection and input–output statistics virtually everywhere. (United Nations *et al.*, 1993. Also, Viet, 1994, reviews input–output data collection and assembly practices in 53 countries during the 1970s and 1980s.) Specific examples include Canada, where this framework has been used at both a national and a regional scale since the early 1960s (Statistics Canada, 1981) and the USA, where national data have been collected and presented in commodity-by-industry form starting with the 1972 tables.[1] It has also become the template for countries in the European Community (Eurostat, 1996, describing the *European System of Accounts, ESA*, 1995) – Denmark (annual tables

[1] For extensive discussions of data collection and modeling efforts and conventions in a number of countries using some version of commodity–industry accounts, see Franz and Rainer (1989) and Viet (1994).

since 1966), the Netherlands and Norway provide examples. The commodity–industry approach indeed provides a framework in which secondary products, by-products, etc. can be much more explicitly accounted for; however, it also introduces new problems (including the possibility of negative coefficients or transactions), as we will see below.

The underlying observation is that *industries* use *commodities* to make *commodities*. It is commodities that are the inputs to industrial processes and that are used to satisfy final demands. An industry can be thought of as defined by its primary product (commodity) but some industries also produce additional commodities as secondary products. (There are several kinds of non-characteristic or secondary products – joint products, by-products, subsidiary products; we will investigate some of these distinctions later.) In order to highlight the differentiation between commodities and industries, assume that the commodity index, i, runs from 1 to m and the industry index, j, runs from 1 to n. If every commodity produced in an economy is primary to some industry in that economy, then the number of commodities and the number of industries will be the same, $m = n$. Initially we will investigate this case in some detail. Complications can arise when $m \neq n$; we explore these in section 5.6.[2]

5.1.1 The Use Matrix

In ordinary input–output models with n sectors, an element of the $n \times n$ transactions matrix $\mathbf{Z} = [z_{ij}]$ represents the value of purchases of industry i output by industry j. In addition, there is an n-element vector of total *industry* outputs, $\mathbf{x} = [x_j]$, where

$$x_j = z_{j1} + \cdots + z_{jn} + f_j \tag{5.1}$$

and f_j is *industry* j's sales to final demand. In matrix form,

$$\mathbf{x} = \mathbf{Zi} + \mathbf{f} \tag{5.2}$$

and direct input (technical) coefficients, $\mathbf{A} = [a_{ij}]$, are defined as

$$\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} \tag{5.3}$$

Throughout, the adjectives "ordinary" and "original" will be used with "input–output" to refer to the model that is derived from (5.1)–(5.3), as in Chapter 2 and in (5.11) and (5.12), below. These are the relationships as Leontief first articulated them and that are reflected, in the case of the USA, in pre-1972 published input–output data.

In the commodity-by-industry approach, the interindustry transactions matrix, $\mathbf{Z}$, is replaced, initially, by the *Use* matrix, $\underset{(c\times i)}{\mathbf{U}} = [u_{ij}]$, where u_{ij} is the value of purchases of *commodity* i by industry j.[3] Thus the "industries use commodities" part of "industries

[2] If $m > n$, aggregation of commodity accounts could proceed until $m = n$; similarly, if $m < n$, industry accounts could be aggregated. This is often done in practice (again, see the papers in Franz and Rainer, 1989). But of course aggregation covers up information from the originally more detailed data sets.

[3] Normally, the parentheses below a matrix indicate its dimensions – number of rows and number of columns. In this section we will sometimes use expressions like $(c \times i)$ to help us remember which dimension enumerates commodities (in this case, rows) and which enumerates industries (in this case, columns). Thus we will write that $\mathbf{U}$ has "commodity-by-industry dimensions."

Table 5.1 The Use Matrix (**U**) and Other Data for a Two-Commodity, Two-Industry Hypothetical Example (in Dollars)

	Industry 1	Industry 2	Final Demand for Commodities (**e**)	Total Commodity Output (**q**)
Commodity 1	12	8	80	100
Commodity 2	10	7	83	100
Value Added ($\mathbf{v}'$)	68	95		
Total Industry Outputs ($\mathbf{x}'$)	90	110		

use commodities to make commodities" is quantified in **U**. (**U** is sometimes called the *absorption* or *input* matrix.) In conjunction with total industry output, **x**, the parallel to ordinary technical coefficients, a_{ij}, would appear to be

$$b_{ij} = u_{ij}/x_j$$

or

$$\mathbf{B} = \mathbf{U}\hat{\mathbf{x}}^{-1} \tag{5.4}$$

in which column j represents the value of inputs of each *commodity* per dollar's worth of *industry* j's output.[4] The dimensions of **B** are therefore commodities-by-industries. However, we will see that among the other matrices that emerge in this system, some will have the dimensions "commodity-by-commodity;" others will be of "industry-by-commodity" or "industry-by-industry" structure. For this reason, in what remains we will use the general term "commodity–industry" to characterize this accounting framework and the models that are derived from it. If we also have information on *commodity* sales to final demand, this can be arranged as in Table 5.1.

5.1.2 The Make Matrix

As might be expected, the matrix showing how industries *make* commodities is termed the *Make* matrix, usually denoted **V** (it is also called the *output* matrix).[5] Table 5.2 provides an example.

An element of **V**, v_{ij}, shows the value of the output of *commodity* j that is produced by *industry* i. (Thus, the dimensions of **V** are industries-by-commodities.) In this example, industry 1 produces only its primary product, commodity 1, but the output of industry 2 consists of \$100 worth of its primary product, commodity 2, and also \$10 worth of commodity 1, which is a secondary product in industry 2. (In an economy in which there

[4] The notation $\mathbf{B} = [b_{ij}]$ is also used for the coefficients matrix in a supply-side model (Chapter 12) and for capital coefficients in a dynamic input–output model (Chapter 13). Its use in (5.4) in the commodity-by-industry literature is fairly widespread, and in general the context of any discussion should make clear which meaning is intended.

[5] Use of **V** for the Make matrix and $\mathbf{v}'$ for the row *vector* of value added elements is also standard in the input–output literature and, again, should not lead to confusion when read in appropriate context. As we will see later, the transpose of the Make matrix, $\mathbf{V}'$, is also known as the *supply* matrix.

Table 5.2 The Make Matrix (**V**) and Other Data for a Two-Commodity, Two-Industry Hypothetical Example (in Dollars)

		Commodity 1	Commodity 2	Total Industry Output (**x**)
Industry	1	90	0	90
	2	10	100	110
Total Commodity Output ($\mathbf{q}'$)		100	100	

Table 5.3 The Complete Set of Commodity–Industry Data

		Commodities 1	Commodities 2	Industries 1	Industries 2	Final Demand	Total Output
Commodities	1			12	6	80	100
	2			10	7	83	100
				U		**e**	**q**
Industries	1	90	0				90
	2	10	100				110
		V					**x**
Value Added				60	95	163	
				$\mathbf{v}'$			
Total Inputs		100	100	90	110		
		$\mathbf{q}'$		$\mathbf{x}'$			

is no secondary production, the Make matrix will be diagonal and, as we will see below, all of the commodity–industry results reduce to the original Leontief industry-based approach.) Table 5.3 shows one way of presenting all of the data in a commodity–industry framework.

5.2 The Basic Accounting Relationships

In the ordinary input–output model, the basic accounting relationship for total (industry) output is given in (5.1) and (5.2). In the commodity–industry framework, both total industry output (**x**) and total commodity output (**q**) are accounted for. From the data in the Make matrix, total output of any industry is found by summing over all commodities produced by that industry. These totals are the *row* sums of **V**,

$$x_j = v_{j1} + \cdots + v_{jm} \tag{5.5}$$

or

$$\mathbf{x} = \mathbf{V}\mathbf{i} \tag{5.6}$$

Similarly, total output of any commodity can be found by summing over all industries that produce the commodity. These totals are the *column* sums of $\mathbf{V}$ (or the row sums of $\mathbf{V}'$)

$$q_j = v_{1j} + \cdots + v_{nj} \quad \text{and} \quad \mathbf{q}' = \mathbf{i}'\mathbf{V} \tag{5.7}$$

or

$$\mathbf{q} = (\mathbf{V}')\mathbf{i} \tag{5.8}$$

Alternatively, from the illustration in Table 5.1,

$$q_j = u_{j1} + \cdots + u_{jn} + e_j \tag{5.9}$$

or

$$\mathbf{q} = \mathbf{U}\mathbf{i} + \mathbf{e} \tag{5.10}$$

The original input–output model combines (5.2) and (5.3). From (5.3), $\mathbf{Z} = \mathbf{A}\hat{\mathbf{x}}$; substituting into (5.2) gives

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{f} \tag{5.11}$$

(since $\hat{\mathbf{x}}\mathbf{i} = \mathbf{x}$), and the operational form of the model becomes

$$\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f} = \mathbf{L}\mathbf{f} \tag{5.12}$$

The driving force is the exogenous vector of final demand for industry outputs. In conjunction with the Leontief inverse (total requirements) matrix, industry outputs necessary to sustain the final demand are determined.

The commodity–industry approach uses (5.10) and (5.4) in the same way as (5.2) and (5.3), respectively. From (5.4), $\mathbf{U} = \mathbf{B}\hat{\mathbf{x}}$, and substituting into (5.10) gives

$$\mathbf{q} = \mathbf{B}\mathbf{x} + \mathbf{e} \tag{5.13}$$

as a parallel to (5.11) in the ordinary input–output model. The problem is that, unlike (5.11), one cannot generate a total requirements matrix, as in (5.12), because (5.13) contains commodity output ($\mathbf{q}$) on the left-hand side and industry output ($\mathbf{x}$) on the right-hand side.

5.3 Technology and Total Requirement Matrices in the Commodity–Industry Approach

One solution to this problem in (5.13) is to find an expression transforming industry outputs, $\mathbf{x}$, to commodity outputs, $\mathbf{q}$ – or, alternatively, to transform commodity outputs (and commodity final demand, $\mathbf{e}$) into industry terms. The data needed for such transformations are to be found in the Make matrix, whose row sums are industry outputs and whose column sums are commodity outputs. Two alternative ways of using the information in the Make matrix are described below. These algebraic alternatives have quite different economic interpretations.

5.3.1 Industry Source of Commodity Outputs

Define $d_{ij} = v_{ij}/q_j$ (each element in column j of $\mathbf{V}$ is divided by the jth column sum, q_j), so that d_{ij} denotes the fraction of total commodity j output that was produced by industry i. Forming a matrix of these *commodity output proportions*, $\underset{(i\times c)}{\mathbf{D}} = [d_{ij}]$, we have

$$\mathbf{D} = \mathbf{V}\hat{\mathbf{q}}^{-1} \tag{5.14}$$

For the numerical example,

$$\mathbf{D} = \begin{bmatrix} .9 & 0 \\ .1 & 1 \end{bmatrix}$$

From column 1, for example, we see that 90 percent of the total amount of commodity 1 made in the economy was produced by industry 1 and 10 percent was produced by industry 2. ($\mathbf{D}$ is often called the *market shares* matrix.) By definition, each column sum in $\mathbf{D}$ is unity.

5.3.2 Commodity Composition of Industry Outputs

Define $c_{ij} = v_{ij}/x_i$ (each element in row i of $\mathbf{V}$ is divided by the ith row sum, x_i), so that c_{ij} denotes the fraction of total industry i output that is in the form of commodity j. For later purposes it will turn out to be convenient to have these *industry output proportions* arranged in a matrix with commodities-by-industries dimensions (remember that $\mathbf{V}$ has industry-by-commodity dimensions). Define $\mathbf{V}'$ as the *supply* matrix, with commodity-by-industry dimensions; then the matrix of these industry output proportions is found as[6]

$$\mathbf{C} = \mathbf{V}'\hat{\mathbf{x}}^{-1} \tag{5.15}$$

For the numerical example,

$$\mathbf{C} = \begin{bmatrix} 1 & .0909 \\ 0 & .9091 \end{bmatrix}$$

The second column, for example, says that 90.9 percent of the value of industry 2's output consisted of commodity 2 and 9.1 percent was accounted for by commodity 1. ($\mathbf{C}$ is sometimes called the *product mix* matrix or the *commodity mix* matrix.) By definition, each column sum in $\mathbf{C}$ is unity.

5.3.3 Generating Total Requirements Matrices

The results in (5.14) and (5.15) – in conjunction with (5.6) and (5.8) – provide two alternative linear transformations between commodity and industry outputs. Using (5.14),

$$\mathbf{D} = \mathbf{V}\hat{\mathbf{q}}^{-1} \Rightarrow \mathbf{D}\hat{\mathbf{q}} = \mathbf{V} \Rightarrow \mathbf{D}\hat{\mathbf{q}}\mathbf{i} = \mathbf{V}\mathbf{i}$$

and from (5.6)

$$\mathbf{D}\mathbf{q} = \mathbf{x} \tag{5.16}$$

6 $\mathbf{C}$ is another letter that serves more than one purpose in the input–output literature. Recall from Chapter 3 that it is also used for the matrix of regional trade proportions.

This also means

$$\mathbf{q} = \mathbf{D}^{-1}\mathbf{x} \tag{5.17}$$

if $\mathbf{D}$ is square and nonsingular.[7]

A compact statement of the relationships in (5.13) and (5.16) is provided as follows:[8] from (5.16), $\mathbf{x} - \mathbf{Dq} = \mathbf{0}$; from (5.13), $-\mathbf{Bx} + \mathbf{q} = \mathbf{e}$. These are two matrix equations in $\mathbf{x}$ and $\mathbf{q}$; in partitioned matrix form they can be represented as

$$\begin{bmatrix} \mathbf{I} & -\mathbf{D} \\ -\mathbf{B} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{e} \end{bmatrix}$$

Alternatively, using (5.15),

$$\mathbf{C} = \mathbf{V}'\hat{\mathbf{x}}^{-1} \Rightarrow \mathbf{C}\hat{\mathbf{x}} = \mathbf{V}' \Rightarrow \mathbf{C}\hat{\mathbf{x}}\mathbf{i} = \mathbf{Cx} = (\mathbf{V}')\mathbf{i}$$

and from (5.8)

$$\mathbf{Cx} = \mathbf{q} \tag{5.18}$$

so

$$\mathbf{x} = \mathbf{C}^{-1}\mathbf{q} \tag{5.19}$$

again provided that $\mathbf{C}$ is square and nonsingular.

A compact statement of the results in (5.13) and (5.18) is: from (5.18), $\mathbf{Cx} - \mathbf{q} = \mathbf{0}$, and from (5.13), again, $-\mathbf{Bx} + \mathbf{q} = \mathbf{e}$. This pair of relationships in $\mathbf{x}$ and $\mathbf{q}$ can be represented in partitioned matrix form as

$$\begin{bmatrix} \mathbf{C} & -\mathbf{I} \\ -\mathbf{B} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{e} \end{bmatrix}$$

*Using **D*** One solution to the dilemma posed by the presence of both $\mathbf{x}$ and $\mathbf{q}$ in (5.13) is provided by (5.16). Substitute $\mathbf{Dq}$ for $\mathbf{x}$ in (5.13),

$$\mathbf{q} = \mathbf{B}(\mathbf{Dq}) + \mathbf{e} = (\mathbf{BD})\mathbf{q} + \mathbf{e}$$

from which

$$\mathbf{q} = (\mathbf{I} - \mathbf{BD})^{-1}\mathbf{e} \tag{5.20}$$

The inverse on the right-hand side, which is called a *commodity-by-commodity total requirements matrix*, connects commodity final demand to commodity output. It thus plays the role of $(\mathbf{I} - \mathbf{A})^{-1}$ in the ordinary input–output model, (5.12). It is to be noted that the "parallel" to the $\mathbf{A}$ matrix (direct input requirements) in the ordinary model appears now to be $\mathbf{BD}$ [and not simply $\mathbf{B}$ alone, as seemed initially the case when $\mathbf{B}$ was defined in (5.4)].

7 For the moment we will assume that **D** (and **C**) are nonsingular. Later we will explore how important (and how likely) these assumptions are.

8 This parallels the representation in Jack Faucett Associates, Inc. (1981-1983, Vol. 5, pp. 11–4 and 11–5), which was developed in the context of the US multiregional model for 1977.

Using (5.20), and since $\mathbf{Dq} = \mathbf{x}$,

$$\mathbf{x} = [\mathbf{D}(\mathbf{I} - \mathbf{BD})^{-1}]\mathbf{e} \tag{5.21}$$

The bracketed matrix on the right connects commodity final demand to industry output. It is an *industry-by-commodity total requirements matrix*.

There are alternative possible expressions for total requirements matrices. For example, premultiplying both sides of (5.13) by $\mathbf{D}$ gives, since $\mathbf{Dq} = \mathbf{x}$,

$$\mathbf{x} = \mathbf{DBx} + \mathbf{De}$$

and

$$\mathbf{x} = [(\mathbf{I} - \mathbf{DB})^{-1}\mathbf{D}]\mathbf{e} \tag{5.22}$$

so the bracketed expression on the right-hand side is also an industry-by-commodity total requirements matrix.[9]

Using $\boldsymbol{C}$ A second transformation of (5.13) is easily accomplished using (5.19) – as long as $\mathbf{C}^{-1}$exists. Substitute $\mathbf{C}^{-1}\mathbf{q}$ for $\mathbf{x}$ in (5.13),

$$\mathbf{q} = \mathbf{B}(\mathbf{C}^{-1}\mathbf{q}) + \mathbf{e} = (\mathbf{BC}^{-1})\mathbf{q} + \mathbf{e}$$

from which

$$\mathbf{q} = (\mathbf{I} - \mathbf{BC}^{-1})^{-1}\mathbf{e} \tag{5.23}$$

It is apparent that the inverse on the right-hand side is also a *commodity-by-commodity total requirements matrix*, connecting commodity final demand to commodity output, and it differs from the expression in (5.20) which has the same name. Thus another "parallel" to the $\mathbf{A}$ matrix in the ordinary Leontief inverse is $\mathbf{BC}^{-1}$.

Using (5.23), and since $\mathbf{C}^{-1}\mathbf{q} = \mathbf{x}$,

$$\mathbf{x} = [\mathbf{C}^{-1}(\mathbf{I} - \mathbf{BC}^{-1})^{-1}]\mathbf{e} \tag{5.24}$$

Here we have an *industry-by-commodity total requirements matrix* (in brackets) on the right, and this differs from the expression in (5.21) with the same name.[10]

To introduce some order into this apparent profusion of alternatives to the Leontief inverse in the ordinary input–output model, it is instructive to go behind the matrix algebra and investigate the basic assumptions that underpin these results, as in (5.20) and (5.21) as compared with (5.23) and (5.24). Recall that transformations of the data in the Make matrix gave us the industry output proportions, in $\mathbf{C}$, and the commodity output proportions, in $\mathbf{D}$. In the remainder of this section (5.3) we follow the classification approach that has traditionally been used since the outset of the commodity–industry discussion, for example as in the system of national accounts (SNA) described in United Nations (1968); later (section 5.5.3) we present an alternative and more recent view.

[9] This reflects a general matrix algebra result (for nonsingular $\mathbf{D}$). For example, starting at (5.22) $(\mathbf{I} - \mathbf{DB})^{-1}\mathbf{D} = [\mathbf{D}^{-1}(\mathbf{I} - \mathbf{DB})]^{-1} = (\mathbf{D}^{-1} - \mathbf{B})^{-1} = (\mathbf{D}^{-1} - \mathbf{BDD}^{-1})^{-1} = [(\mathbf{I} - \mathbf{BD})\mathbf{D}^{-1}]^{-1} = \mathbf{D}(\mathbf{I} - \mathbf{BD})^{-1}$ This is the total requirements matrix in (5.21).

[10] Using the same algebra as in footnote 9, this can be expressed as $\mathbf{x} = [(\mathbf{I} - \mathbf{C}^{-1}\mathbf{B})^{-1}]\mathbf{C}^{-1}\mathbf{e}$.

5.3.4 "Industry-Based" Technology

The commodity-by-commodity total requirements matrix in (5.20) was derived from

$$\mathbf{q} = (\mathbf{BD})\mathbf{q} + \mathbf{e}$$

The matrix **BD** plays the role of a technical coefficients matrix, showing *commodity* inputs per dollar's worth of *commodity* output. For our example,

$$\mathbf{B} = \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} \begin{bmatrix} 1/90 & 0 \\ 0 & 1/110 \end{bmatrix} = \begin{bmatrix} .1333 & .0727 \\ .1111 & .0636 \end{bmatrix}$$

and so

$$\mathbf{BD} = \begin{bmatrix} .1333 & .0727 \\ .1111 & .0636 \end{bmatrix} \begin{bmatrix} .9 & 0 \\ .1 & 1 \end{bmatrix} = \begin{bmatrix} .1273 & .0727 \\ .1064 & .0636 \end{bmatrix}$$

Using $\mathbf{B}_1$ and $\mathbf{B}_2$ for the two columns in **B**, this product can be shown as

$$\mathbf{BD} = \begin{bmatrix} \mathbf{B}_1(.9) + \mathbf{B}_2(.1) & \mathbf{B}_1(0) + \mathbf{B}_2(1) \end{bmatrix}$$

The columns in **BD** are seen to be convex combinations of the columns in **B**, where the weights come from the elements in each column of **D**. (This simply means that $\mathbf{BD} = \alpha_1\mathbf{B}_1 + \alpha_2\mathbf{B}_2$, where $\alpha_1, \alpha_2 \geq 0$ and $\alpha_1 + \alpha_2 = 1$.) Thus **BD** embodies the assumption that commodity inputs to commodity *j* production are weighted averages of commodity inputs to each industry that produces commodity *j* (from the **B** matrix), and the weights are the proportions of each industry's contribution to total commodity *j* output (from the **D** matrix). A given commodity can have differing input structures if it is produced by more than one industry. In this example, the first column of **BD** reflects the fact that 90 percent of the total amount of commodity 1 that is available in the economy is produced by industry 1 (using the production recipe embodied in $\mathbf{B}_1$) and 10 percent of total commodity 1 output is produced in industry 2 (using that industry's production recipe, as embodied in $\mathbf{B}_2$).

All commodities produced by an industry are assumed to have the same input structure, as given by that industry's column in the **B** matrix. This is shown in the example by the fact that $\mathbf{B}_2$, the recipe for industry 2 production, appears in both of the columns of **BD**. That part (10 percent) of commodity 1 that is produced in industry 2 and that part of commodity 2 (100 percent) that is produced in industry 2 are both made according to the industry 2 production technology, given in $\mathbf{B}_2$.[11] For this reason, BD is said to embody *industry-based technology* (or simply *industry technology* hereafter), and since its dimensions are *commodities-by-commodities*, it is sometimes denoted $\underset{(c\times c)}{\mathbf{A}_I}$:[12]

$$\underset{(c\times c)}{\mathbf{A}_I} = \mathbf{BD}$$

[11] It has been argued that this may be an appropriate assumption for commodities that are by-products of an industry's production process.

[12] This notation may seem cumbersome, but in view of the various alternative direct requirements matrices that will emerge in this and subsequent sections, it is essential to identify precisely both the dimensions and the technology assumptions that underpin these matrices.

The inverse $(\mathbf{I} - \mathbf{BD})^{-1}$ in (5.20) is therefore referred to, more completely, as the *commodity-by-commodity total requirements matrix under industry technology*.

A matrix of technical coefficients that is more parallel to **A** in the original input–output model (connecting *industry* inputs per unit of *industry* output) arose in the derivation of (5.22), where

$$\mathbf{x} = (\mathbf{DB})\mathbf{x} + \mathbf{De}$$

and it is clear that **DB** shows inputs from industries per dollar's worth of industry production. Its dimensions are industries-by-industries, and it is thus seen to be comparable to the technological coefficients matrices, **A**, in the original industry-by-industry input–output models; we will denote it by $\underset{(i\times i)}{\mathbf{A}_I}$.

Carrying out the pre-multiplication of **B** by **D** in the small numerical example shows exactly how the commodity inputs (in **B**) are distributed back to the industries where they are made:

$$\underset{(i\times i)}{\mathbf{A}_I} = \mathbf{DB} = \begin{bmatrix} .9 & 0 \\ .1 & 1 \end{bmatrix}\begin{bmatrix} .1333 & .0727 \\ .1111 & .0636 \end{bmatrix} = \begin{bmatrix} .1200 & .0655 \\ .1244 & .0709 \end{bmatrix}$$

Using $\mathbf{D}_1$ and $\mathbf{D}_2$ for the two columns in **D**,

$$\mathbf{DB} = \left[\mathbf{D}_1(.1333) + \mathbf{D}_2(.1111) \quad \mathbf{D}_1(.0727) + \mathbf{D}_2(.0636)\right]$$

Consider, for example, the second column in **DB**. It disaggregates $b_{12} = .0727$ (commodity 1 input per dollar's worth of industry 2 output) and $b_{22} = .0636$ (commodity 2 input per dollar's worth of industry 2 output) into two components (vectors). The first,

$$\begin{bmatrix} .9 \\ .1 \end{bmatrix}(.0727) = \begin{bmatrix} .0655 \\ .0073 \end{bmatrix}$$

shows the industry 1 (90 percent) and industry 2 (10 percent) contributions to the total .0727 needed of commodity 1. Similarly, industry 1 and 2 proportions of the .0636 of commodity 2 used by industry 2 are 0 and 1 (the elements of $\mathbf{D}_2$), and so the vector showing industry origins of commodity 2 input to industry 2 is

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}(.0636) = \begin{bmatrix} 0 \\ .0636 \end{bmatrix}$$

The sum of these two vectors indicates the inputs from industry 1 and 2, respectively, per dollar's worth of industry 2 output. This is the second column in $\underset{(i\times i)}{\mathbf{A}_I}$; there is a similar interpretation for the first column.

5.3.5 "Commodity-Based" Technology

With the industry technology assumption, industry input structures (in the columns of **B**) are the basic data, and commodity input structures are found as weighted averages of these columns. An alternative point of view would suggest that a given commodity should have the same input structure in all of the industries that produce it.[13] In this case, commodity inputs to industry j production (the elements of the jth column of $\mathbf{B}$) are viewed as weighted averages of commodity inputs to commodity production for each of the commodities that industry j makes, and the weights are the proportions of each commodity in industry j's total output. This is known as the *commodity-based technology*, or simply *commodity technology*, assumption.

From our small example, we found

$$\mathbf{C} = \begin{bmatrix} 1 & .0909 \\ 0 & .9091 \end{bmatrix}$$

with dimensions commodities-by-industries. We know $\mathbf{B}$, with dimensions commodities-by-industries also. The (presently unknown) commodity-by-commodity technological coefficients matrix can be denoted $\underset{(c\times c)}{\mathbf{A}_C}$. The commodity technology assumption is that $\mathbf{B} = (\underset{(c\times c)}{\mathbf{A}_C})\mathbf{C}$. For the small example, letting $\underset{(c\times c)}{\mathbf{A}_C} = \begin{bmatrix} \mathbf{A}_{C1} & \mathbf{A}_{C2} \end{bmatrix}$,

$$\mathbf{B} = \begin{bmatrix} \mathbf{A}_{C1} & \mathbf{A}_{C2} \end{bmatrix} \begin{bmatrix} 1 & .0909 \\ 0 & .9091 \end{bmatrix}$$

or

$$\begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \end{bmatrix} = \begin{bmatrix} (\mathbf{A}_{C1})(1) + (\mathbf{A}_{C2})(0) & (\mathbf{A}_{C1})(.0909) + (\mathbf{A}_{C2})(.9091) \end{bmatrix}$$

From $\mathbf{B} = (\underset{(c\times c)}{\mathbf{A}_C})\mathbf{C}$, the (unknown) matrix of commodity inputs per dollar's worth of commodity production is found as $\underset{(c\times c)}{\mathbf{A}_C} = \mathbf{B}\mathbf{C}^{-1}$ (again, provided $\mathbf{C}$ is square and nonsingular). Here

$$\underset{(c\times c)}{\mathbf{A}_C} = \mathbf{B}\mathbf{C}^{-1} = \begin{bmatrix} .1333 & .0727 \\ .1111 & .0636 \end{bmatrix} \begin{bmatrix} 1 & -0.1 \\ 0 & 1.1 \end{bmatrix} = \begin{bmatrix} .1333 & .0667 \\ .1111 & .0589 \end{bmatrix}$$

Hence, for example, $(\mathbf{I} - \mathbf{B}\mathbf{C}^{-1})^{-1}$ in (5.23) is properly described as the *commodity-by-commodity total requirements matrix under commodity technology*.

The matrix of direct commodity inputs per dollar's worth of commodity output under the industry technology assumption was

$$\underset{(c\times c)}{\mathbf{A}_I} = \mathbf{B}\mathbf{D} = \begin{bmatrix} .1333 & .0727 \\ .1111 & .0636 \end{bmatrix} \begin{bmatrix} .9 & 0 \\ .1 & 1 \end{bmatrix} = \begin{bmatrix} .1273 & .0727 \\ .1064 & .0636 \end{bmatrix}$$

Clearly, the two technology assumptions can and generally will lead to different direct commodity input matrices, as in this example. The "size" of this difference and, perhaps more importantly, the resulting differences in the corresponding total requirements

[13] This may be an appropriate assumption for subsidiary products that are produced by an industry in a separate facility, employing a similar technology to that used by the industry to which the commodity is primary.

matrices – $(\mathbf{I}-\mathbf{BD})^{-1}$ and $(\mathbf{I}-\mathbf{BC}^{-1})^{-1}$ – is a topic of continuing research and empirical examination with real-world data sets.

From (5.13) and (5.19),

$$\mathbf{x}=\mathbf{C}^{-1}\mathbf{Bx}+\mathbf{C}^{-1}\mathbf{e} \tag{5.25}$$

so under the assumption of commodity technology it is clear that the matrix $\mathbf{C}^{-1}\mathbf{B}$ plays the role of $\mathbf{DB}$ in an industry technology model and $\mathbf{A}$ in the ordinary input–output models; namely, it records *industry* inputs per dollar of *industry* outputs. Let $\underset{(i\times i)}{\mathbf{A}_C}=\mathbf{C}^{-1}\mathbf{B}$; from the numerical example

$$\underset{(i\times i)}{\mathbf{A}_C}=\mathbf{C}^{-1}\mathbf{B}=\begin{bmatrix}1 & -0.1\\ 0 & 1.1\end{bmatrix}\begin{bmatrix}.1333 & .0727\\ .1111 & .0636\end{bmatrix}=\begin{bmatrix}.1222 & .0664\\ .1222 & .0700\end{bmatrix}$$

As expected, this differs from $\underset{(i\times i)}{\mathbf{A}_I}=\mathbf{DB}$, calculated earlier. (In this particular numerical example the difference is not great, but in general there is no reason to expect that $\mathbf{DB}=\mathbf{C}^{-1}\mathbf{B}$.)

We explore the economic content of the operation $\mathbf{C}^{-1}\mathbf{B}$. Let $\mathbf{C}^{-1}\mathbf{B}=\mathbf{T}$ (instead of $\underset{(i\times i)}{\mathbf{A}_C}$, to simplify notation). For the general two-industry case,

$$\mathbf{T}=\begin{bmatrix}t_{11} & t_{12}\\ t_{21} & t_{22}\end{bmatrix}$$

Then

$$\mathbf{B}=\mathbf{CT}=\begin{bmatrix}1 & .0909\\ 0 & .9091\end{bmatrix}\begin{bmatrix}t_{11} & t_{12}\\ t_{21} & t_{22}\end{bmatrix}$$

and, for example,

$$\mathbf{B}_2=\begin{bmatrix}b_{12}\\ b_{22}\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}(t_{12})+\begin{bmatrix}.0909\\ .9091\end{bmatrix}(t_{22})$$

The vector multiplying t_{12} disaggregates industry 1 input into commodity inputs – commodity 1 (100 percent) and commodity 2 (0 percent) – reflecting the commodity composition of industry 1 output (column 1 of $\mathbf{C}$). Similarly, the vector multiplying t_{22} distinguishes industry 2 input as composed of commodity 1 (9.1 percent) and commodity 2 (90.9 percent), from column 2 of $\mathbf{C}$. Adding these together gives $\mathbf{B}_2$, showing commodity 1 and 2 inputs per dollar of industry 2 output. A similar analysis of the composition of $\mathbf{B}_1$ can be carried out.

5.3.6 Direct Requirements (Technical Coefficients) Matrices Derived from Basic Data

In the ordinary input–output model, the direct requirements matrix is derived directly from interindustry flows, $\mathbf{Z}$, and industry outputs, $\mathbf{x}$, as in $\mathbf{A}=\mathbf{Z}\hat{\mathbf{x}}^{-1}$ in (5.3). In (5.4), we saw how commodity-to-industry flows, $\mathbf{U}$, and industry outputs, $\mathbf{x}$, were used to calculate direct requirements in terms of commodity inputs per dollar's worth of industry output – $\mathbf{B} = \mathbf{U}\hat{\mathbf{x}}^{-1}$. In the commodity technology models, the matrix that relates commodity inputs per dollar's worth of *commodity* output is $\underset{(c\times c)}{\mathbf{A}_C} = \mathbf{B}\mathbf{C}^{-1}$. Since $\mathbf{C} = \mathbf{V}'\hat{\mathbf{x}}^{-1}$, $\underset{(c\times c)}{\mathbf{A}_C}$ is found directly from the basic data in Tables 5.2 and 5.3 as

$$\underset{(c\times c)}{\mathbf{A}_C} = \mathbf{B}\mathbf{C}^{-1} = [\mathbf{U}\hat{\mathbf{x}}^{-1}][\mathbf{V}'\hat{\mathbf{x}}^{-1}]^{-1} = [\mathbf{U}\hat{\mathbf{x}}^{-1}][\hat{\mathbf{x}}(\mathbf{V}')^{-1}] = \mathbf{U}(\mathbf{V}')^{-1} \tag{5.26}$$

and the matrix that relates industry inputs per dollar's worth of industry output is

$$\underset{(i\times i)}{\mathbf{A}_C} = \mathbf{C}^{-1}\mathbf{B} = [\hat{\mathbf{x}}(\mathbf{V}')^{-1}][\mathbf{U}\hat{\mathbf{x}}^{-1}]$$

In contrast to (5.26), further simplifications are not possible.

In the industry technology models, $\underset{(c\times c)}{\mathbf{A}_I} = \mathbf{B}\mathbf{D}$ relates commodity inputs to each dollar's worth of commodity output. Since $\mathbf{D} = \mathbf{V}\hat{\mathbf{q}}^{-1}$,

$$\underset{(c\times c)}{\mathbf{A}_I} = \mathbf{B}\mathbf{D} = [\mathbf{U}\hat{\mathbf{x}}^{-1}][\mathbf{V}\hat{\mathbf{q}}^{-1}] \tag{5.27}$$

Finally, industry inputs per dollar's worth of industry output under the industry technology assumption are found from basic data as

$$\underset{(i\times i)}{\mathbf{A}_I} = \mathbf{D}\mathbf{B} = [\mathbf{V}\hat{\mathbf{q}}^{-1}][\mathbf{U}\hat{\mathbf{x}}^{-1}]$$

If one wants to compare direct requirements matrices for, say, the US economy both before and after 1972, it is $\underset{(i\times i)}{\mathbf{A}_I} = \mathbf{D}\mathbf{B}$ and $\underset{(i\times i)}{\mathbf{A}_C} = \mathbf{C}^{-1}\mathbf{B}$ that are the comparable to $\mathbf{A}$, since they have industry-by-industry dimensions of the earlier tables. Notice that these four definitions of direct requirements matrices will all be equal in the case of no secondary production in any industry. This means that $\mathbf{V}$ is diagonal, $\mathbf{V} = \hat{\mathbf{x}} = \hat{\mathbf{q}}$, $\mathbf{V}\mathbf{i} = \mathbf{x} = \mathbf{i}'\mathbf{V} = \mathbf{q}$, and so in all four cases, above, $\mathbf{A} = \mathbf{U}\mathbf{V}^{-1} = \mathbf{U}(\mathbf{V}')^{-1}$.

5.3.7 Total Requirements Matrices

Approach I: Starting with Technical Coefficients Results thus far for total requirements matrices [from (5.20)–(5.24)] are collected together in Table 5.4. Since in each case the exogenous force driving the model is final demand for commodities, these are called *commodity-demand driven* models. We continue, for now, to assume that $\mathbf{C}$ is nonsingular. An alternative presentation of the four cases is explored briefly in section 5.5.3, below.

These commodity-by-commodity results are derived from (5.13) through transformations that generate either

$$\mathbf{q} = \underset{(c\times c)}{\mathbf{A}_I}\,\mathbf{q} + \mathbf{e} \quad \text{or} \quad \mathbf{q} = \underset{(c\times c)}{\mathbf{A}_C}\,\mathbf{q} + \mathbf{e}$$

Table 5.4 Total Requirements Matrices, Commodity-Demand Driven Models

	Industry Technology	Commodity Technology
Commodity-by-Commodity	$(\mathbf{I}-\mathbf{BD})^{-1}$	$(\mathbf{I}-\mathbf{BC}^{-1})^{-1}$
Industry-by-Commodity	$[\mathbf{D}(\mathbf{I}-\mathbf{BD})^{-1}]$	$[\mathbf{C}^{-1}(\mathbf{I}-\mathbf{BC}^{-1})^{-1}]$

Table 5.5 Total Requirements Matrices, Industry-Demand Driven Models

	Industry Technology	Commodity Technology
Industry-by-Industry	$(\mathbf{I}-\mathbf{DB})^{-1}$	$(\mathbf{I}-\mathbf{C}^{-1}\mathbf{B})^{-1}$
Commodity-by-Industry	$[\mathbf{D}^{-1}(\mathbf{I}-\mathbf{DB})^{-1}]$	$[\mathbf{C}(\mathbf{I}-\mathbf{C}^{-1}\mathbf{B})^{-1}]$

and then

$$\mathbf{q}=(\mathbf{I}-\underset{(c\times c)}{\mathbf{A}_I})^{-1}\mathbf{e} \quad \text{or} \quad \mathbf{q}=(\mathbf{I}-\underset{(c\times c)}{\mathbf{A}_C})^{-1}\mathbf{e}$$

These total requirements matrices have exactly the same structure as the Leontief inverse in the original input–output model – namely, the inverse of a matrix containing technical coefficients subtracted from an identity matrix.

It is also possible to derive total requirements matrices for *industry-demand driven* models, replacing $\mathbf{e}$ by an equivalent expression involving $\mathbf{f}$ in appropriate equations [from among (5.20)–(5.25)]. In commodity–industry models, one of the basic premises is that commodities are the products of industries, and therefore it is commodities that are used to satisfy final demand. Hence the notion of "industry final demand," $\mathbf{f}$ (the exogenous driving force in ordinary input–output models), is not very meaningful in commodity–industry models. However, for analyses of structural change in an economy it is necessary to have consistent data sets for two or more years. For example, for comparisons of US input–output tables across time (in which some of the data are pre-1972), it is useful to have an industry-by-industry format, since this was inherent in the original input–output models, and their inverses, as in (5.12).

For industry technology models, in which $\mathbf{Dq}=\mathbf{x}$, the assumption can be made that the same commodity-to-industry transformation is valid for final demands, that is, $\mathbf{De}=\mathbf{f}$. Similarly, for commodity technology models, in which $\mathbf{Cx}=\mathbf{q}$, the same industry-to-commodity transformation can be used for final demands – $\mathbf{Cf}=\mathbf{e}$. For example, from (5.22), since $\mathbf{De}=\mathbf{f}$, $\mathbf{x}=(\mathbf{I}-\mathbf{DB})^{-1}\mathbf{f}$ and, using $\mathbf{q}=\mathbf{D}^{-1}\mathbf{x}$, we have $\mathbf{q}=\mathbf{D}^{-1}(\mathbf{I}-\mathbf{DB})^{-1}\mathbf{f}$. The latter is the only industry technology result that requires $\mathbf{D}^{-1}$. Parallel results can be derived for commodity technology models. These are collected together in Table 5.5. (As in Table 5.4, it is clear that the transformation from industry technology to commodity technology involves replacement of $\mathbf{D}$ by $\mathbf{C}^{-1}$ throughout.)

It is worth re-emphasizing that most real-world applications of the commodity–industry input–output model assume that final demand for *commodities* is the exogenous driving force, so the results in Table 5.4 are of primary interest. In Table 5.5 the industry-by-industry case (first row) is useful principally for studies in which commodity–industry tables are compared with earlier data in the original input–output format, as in (5.11) and (5.12). Thus, for example, both $(\mathbf{I}-\mathbf{DB})^{-1}$and $(\mathbf{I}-\mathbf{C}^{-1}\mathbf{B})^{-1}$ are candidates if one is making comparisons with total requirements matrices for the pre-1972 US economy. The commodity-by-industry results (second row) are included in Table 5.5 primarily for completeness – they are of little practical use.

Approach II: Avoiding $\mathbf{C}^{-1}$ *in Commodity Technology Cases* The only case in which $\mathbf{D}^{-1}$ appears in a total requirements matrix in an industry technology model is in the relatively unimportant commodity-by-industry format. On the other hand, in the commodity technology model, $\mathbf{C}^{-1}$ is everywhere, and this presents a problem if $\mathbf{C}$ is singular. (It also presents a problem if, as is usual, it contains negative elements, as we will see below.) However, there is an alternative derivation that circumvents the singularity issue, although it does not create parallels to a technical coefficients matrix, as in Approach I. Starting again with (5.13) and (5.18),

$$\mathbf{Cx}=\mathbf{Bx}+\mathbf{e}\Rightarrow(\mathbf{C}-\mathbf{B})\mathbf{x}=\mathbf{e}\Rightarrow\mathbf{x}=(\mathbf{C}-\mathbf{B})^{-1}\mathbf{e} \tag{5.28}$$

Thus $(\mathbf{C}-\mathbf{B})^{-1}$ also serves as an industry-by-commodity total requirements matrix.[14] Also, premultiplying both sides of (5.28) by $\mathbf{C}$, and since $\mathbf{Cx}=\mathbf{q}$,

$$\mathbf{q}=\mathbf{C}(\mathbf{C}-\mathbf{B})^{-1}\mathbf{e} \tag{5.29}$$

This is an alternative to the commodity-by-commodity total requirements matrix in Table 5.4, $(\mathbf{I}-\mathbf{BC}^{-1})^{-1}$, that does not require a nonsingular $\mathbf{C}$.

Substituting $\mathbf{Cf}$ for $\mathbf{e}$ on the right-hand sides of (5.28) and (5.29) generates total requirements matrices with dimensions industry-by-industry and commodity-by-industry, comparable to the results in Table 5.5:

$$\mathbf{x}=(\mathbf{C}-\mathbf{B})^{-1}\mathbf{Cf} \tag{5.30}$$

and

$$\mathbf{q}=\mathbf{C}(\mathbf{C}-\mathbf{B})^{-1}\mathbf{Cf} \tag{5.31}$$

The important point is that all four of these results for total requirements matrices under commodity technology – in (5.28) through (5.31) – do not require that $\mathbf{C}$ be nonsingular. (There is a numerical illustration in the next subsection.)

These results are collected together in Table 5.6, along with their counterparts for industry technology.[15] These latter are included primarily for completeness; they are

[14] Simple matrix algebra converts $(\mathbf{C}-\mathbf{B})^{-1}$to $[\mathbf{C}^{-1}(\mathbf{I}-\mathbf{BC}^{-1})^{-1}]$ (or vice versa), but only if $\mathbf{C}^{-1}$exists. The point is that the total requirements matrix – $(\mathbf{C}-\mathbf{B})^{-1}$ in (5.28) or (5.29) – does not depend on an inverse for $\mathbf{C}$.

[15] The derivations are similar to those for the commodity technology model cases and are left as an exercise for the interested reader.

Table 5.6 Rewritten Forms of Total Requirements Matrices

	Industry Technology	Commodity Technology
Commodity-Demand Driven Models		
Commodity-by-Commodity	$\mathbf{D}^{-1}(\mathbf{D}^{-1} - \mathbf{B})^{-1}$	$\mathbf{C}(\mathbf{C} - \mathbf{B})^{-1}$
Industry-by-Commodity	$(\mathbf{D}^{-1} - \mathbf{B})^{-1}$	$(\mathbf{C} - \mathbf{B})^{-1}$
Industry-Demand Driven Models		
Industry-by-Industry	$(\mathbf{D}^{-1} - \mathbf{B})^{-1}\mathbf{D}^{-1}$	$(\mathbf{C} - \mathbf{B})^{-1}\mathbf{C}$
Commodity-by-Industry	$\mathbf{D}^{-1}(\mathbf{D}^{-1} - \mathbf{B})^{-1}\mathbf{D}^{-1}$	$\mathbf{C}(\mathbf{C} - \mathbf{B})^{-1}\mathbf{C}$

of little practical interest since they all require $\mathbf{D}^{-1}$, the very inverse that was avoided in three out of the four industry technology results in Tables 5.4 and 5.5.

Looking down either column in Table 5.6, it is clear that there is an inverse matrix that is common to all of the total requirements matrices in that column. For industry technology this is $(\mathbf{D}^{-1} - \mathbf{B})^{-1}$, for commodity technology it is $(\mathbf{C} - \mathbf{B})^{-1}$. These are the complete total requirements matrices for the industry-by-commodity case. In the first column of the table (industry technology), it is clear that the other total requirements matrices differ from $(\mathbf{D}^{-1} - \mathbf{B})^{-1}$ through pre- or postmultiplication (or both) by $\mathbf{D}^{-1}$. As we have seen, under the industry technology assumption, premultiplication by $\mathbf{D}^{-1}$ serves to convert the row dimension of a matrix (or vector) from industries to commodities. Thus $\mathbf{D}^{-1}(\mathbf{D}^{-1} - \mathbf{B})^{-1}$ changes the industry-by-commodity total requirements matrix to commodity-by-commodity form. This is the first matrix in Table 5.6.

Postmultiplication of a total requirements matrix by $\mathbf{D}^{-1}$ is equivalent to premultiplication of a final demand vector by $\mathbf{D}^{-1}$; in an industry-demand driven model, we saw that the conversion of final demand to commodity terms is provided by $\mathbf{e} = \mathbf{D}^{-1}\mathbf{f}$. This explains the last two matrices in the first column of Table 5.6. A similar relationship holds for the matrices in the second column (commodity technology). Recall that under commodity technology premultiplication by $\mathbf{C}$ transforms the rows from industry terms to commodity terms; $\mathbf{e} = \mathbf{C}\mathbf{f}$.

Is Singularity Likely to be a Problem in Real-World Models? In the original numerical example in Table 5.3 we had

$$\mathbf{V} = \begin{bmatrix} 90 & 0 \\ 10 & 100 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} 90 \\ 110 \end{bmatrix} \text{ and } \mathbf{q} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$$

From this, we found

$$\mathbf{C} = \mathbf{V}'\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 90 & 10 \\ 0 & 100 \end{bmatrix} \begin{bmatrix} 1/90 & 0 \\ 10 & 1/110 \end{bmatrix} = \begin{bmatrix} 1 & .0909 \\ 0 & .9091 \end{bmatrix}$$

$$\mathbf{D} = \mathbf{V}\hat{\mathbf{q}}^{-1} = \begin{bmatrix} 90 & 0 \\ 10 & 100 \end{bmatrix} \begin{bmatrix} 1/100 & 0 \\ 0 & 1/100 \end{bmatrix} = \begin{bmatrix} .9 & 0 \\ .1 & 1 \end{bmatrix}$$

and both $\mathbf{C}$ and $\mathbf{D}$ are nonsingular.

For $\mathbf{C}$ to be singular, we must have $|\mathbf{C}| = 0$, which means $|\mathbf{V}'| = 0$ (or $|\mathbf{V}| = 0$). Similarly, for $\mathbf{D}$ to be singular, the requirement is $|\mathbf{D}| = 0$; this also means $|\mathbf{V}| = 0$. As an example, suppose that the second column of $\mathbf{V}$ is the same as the first, so $\tilde{\mathbf{V}} = \begin{bmatrix} 90 & 90 \\ 10 & 10 \end{bmatrix}$ with an associated $\tilde{\mathbf{x}} = \tilde{\mathbf{V}}\mathbf{i} = \begin{bmatrix} 180 \\ 20 \end{bmatrix}$ and $\tilde{\mathbf{C}} = \tilde{\mathbf{V}}'(\hat{\tilde{\mathbf{x}}})^{-1} = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$.[16] Clearly, $\tilde{\mathbf{C}}$ is singular [as is the associated $\tilde{\mathbf{D}} = \tilde{\mathbf{V}}\hat{\mathbf{q}}^{-1}$, which the reader can easily check], and so the total requirements matrices under commodity technology, as expressed in Tables 5.4 and 5.5, cannot be found.

Since industry output has changed from the original example, so has $\mathbf{B}$, and we now have

$$\tilde{\mathbf{B}} = \mathbf{U}(\hat{\tilde{\mathbf{x}}})^{-1} = \begin{bmatrix} .0667 & .4 \\ .0556 & .35 \end{bmatrix} \text{ and } (\tilde{\mathbf{C}} - \tilde{\mathbf{B}}) = \begin{bmatrix} .4333 & .1 \\ .4444 & .15 \end{bmatrix}$$

Since $(\tilde{\mathbf{C}} - \tilde{\mathbf{B}})$ is nonsingular, we can find

$$(\tilde{\mathbf{C}} - \tilde{\mathbf{B}})^{-1} = \begin{bmatrix} 7.2973 & -4.8649 \\ -21.6216 & 21.0811 \end{bmatrix}$$

which appears to serve as the industry-by-commodity total requirements matrix under commodity technology, as expressed in Table 5.6.

This can be easily checked. For example, using the unchanged commodity final demand (Table 5.3), we find that

$$(\tilde{\mathbf{C}} - \tilde{\mathbf{B}})^{-1}\mathbf{e} = \begin{bmatrix} 7.2973 & -4.8649 \\ -21.6216 & 21.0811 \end{bmatrix} \begin{bmatrix} 80 \\ 83 \end{bmatrix} = \begin{bmatrix} 180 \\ 20 \end{bmatrix} = \tilde{\mathbf{x}}$$

exactly as would be expected (industry output required for commodity final demand). Similarly,

$$\tilde{\mathbf{C}}(\tilde{\mathbf{C}} - \tilde{\mathbf{B}})^{-1}\mathbf{e} = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} \begin{bmatrix} 7.2973 & -4.8649 \\ -21.6216 & 21.0811 \end{bmatrix} \begin{bmatrix} 80 \\ 83 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \end{bmatrix} = \mathbf{q}$$

and $\tilde{\mathbf{C}}(\tilde{\mathbf{C}} - \tilde{\mathbf{B}})^{-1}$ is the commodity-by-commodity total requirements matrix under commodity technology (Table 5.6).

The trouble with $(\tilde{\mathbf{C}} - \tilde{\mathbf{B}})^{-1}$ is that it contains negative elements. These are implausible; for example, an increase in final demand for commodity 1 leads to a *decrease* in industry 2 output. We will explore the issue of negative elements in total requirements matrices in more detail in section 5.5. Here we simply illustrate the problems that they create. For example, suppose we were to use $(\tilde{\mathbf{C}} - \tilde{\mathbf{B}})^{-1}$ in the standard way – namely to assess the impact on industry outputs ($\Delta\tilde{\mathbf{x}}$) of some change in final demand for commodities ($\Delta\mathbf{e}$). As the reader can easily check, for $\Delta\mathbf{e} = \begin{bmatrix} 100 \\ 85 \end{bmatrix}$, $\Delta\tilde{\mathbf{x}} = \begin{bmatrix} 316 \\ -370 \end{bmatrix}$, which is difficult if not impossible to interpret meaningfully. Remember that originally

[16] In a 2×2 matrix both rows and columns must be proportional for singularity. Here, for simplicity, we use the only possible illustration that leaves commodity output (column sums of $\mathbf{V}$ and of $\tilde{\mathbf{V}}$) unchanged at $\mathbf{q} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$.

$\mathbf{e} = \begin{bmatrix} 80 \\ 83 \end{bmatrix}$ and $\tilde{\mathbf{x}} = \begin{bmatrix} 180 \\ 20 \end{bmatrix}$, so an increase in demands to $\mathbf{e}^{new} = \begin{bmatrix} 180 \\ 168 \end{bmatrix}$ generates $\tilde{\mathbf{x}}^{new} = \begin{bmatrix} 496 \\ -350 \end{bmatrix}$. As we will see in section 5.5, negative elements are a problem with commodity technology models even when $\mathbf{C}$ is nonsingular.

In any event, how likely is it that $\mathbf{C}$ (or $\mathbf{D}$ – or $\mathbf{V}$) will be singular in any real-world model? Not very. In this small illustration, the implication of $\underset{(i \times c)}{\tilde{\mathbf{V}}} = \begin{bmatrix} 90 & 90 \\ 10 & 10 \end{bmatrix}$ is that industry 1 produces 90 percent of the output of commodity 1 and also 90 percent of the output of commodity 2. But if industries are named on the basis of their *primary* product, there will only be one primary product per industry, and industry 1 could not produce 90 percent of the output of commodity 2 which is, by definition, primary to industry 2. In fact, each industry should produce more than one-half of the output of its primary commodity, if the commodity is truly "primary" to that industry.

There are matrix algebra results that are very pertinent here. A matrix $\mathbf{M}$ is said to have a *dominant diagonal* if

$$\left| m_{jj} \right| > \sum_{\substack{i=1 \\ i \neq j}}^{n} \left| m_{ij} \right| \text{ for } j = 1, \ldots, n$$

In words, and for a matrix (like $\mathbf{V}$) with non-negative elements (so the absolute value bars are not needed), in each column the element on the main diagonal is larger than the sum of all the other elements in that column.[17] The important point is that it can be shown that an $n \times n$ matrix with a dominant diagonal is always nonsingular. In the case of a $\mathbf{V}$ matrix, a dominant diagonal means that more than one-half of the output of each commodity (each column sum in $\mathbf{V}$) would be made by the corresponding industry (row) to which that commodity is primary. And, as just noted, since industries are named for their primary commodity, diagonal dominance of $\mathbf{V}$ is to be expected. This means that singularity of $\mathbf{C}$ (or $\mathbf{D}$) may not be a problem in most real-world commodity–industry input–output models.[18]

5.4 Numerical Examples of Alternative Direct and Total Requirements Matrices

From the numerical example in Table 5.3, we had

$$\mathbf{B} = \begin{bmatrix} .1333 & .0727 \\ .1111 & .0636 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & .0909 \\ 0 & .9091 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} .9 & 0 \\ .1 & 1 \end{bmatrix}$$

[17] There are several alternative definitions of dominant diagonal matrices but these are not necessary for us at this point. See, for example, Takayama (1985, Chapter 4) or Lancaster (1968, Chapter R7). Both of these include discussions of related concepts, including Frobenius theorems and the notion of indecomposable matrices; these topics are also beyond our needs here.

[18] The reader might think about whether diagonal dominance will be more or less likely as the number of commodities/industries increases. The simplicity of the two-commodity, two-industry case may be misleading.

$$\mathbf{C}^{-1} = \begin{bmatrix} 1 & -.1 \\ 0 & 1.1 \end{bmatrix} \; \mathbf{D}^{-1} = \begin{bmatrix} 1.1111 & 0 \\ -.1111 & 1 \end{bmatrix}$$

(Notice the negative element in $\mathbf{C}^{-1}$.) We collect together the associated direct and total input requirements matrices in this section.

5.4.1 Direct Requirements Matrices

$$\underset{(c \times c)}{\mathbf{A}_I} = \mathbf{BD} = \begin{bmatrix} .1273 & .0727 \\ .1064 & .0636 \end{bmatrix} \; \underset{(i \times i)}{\mathbf{A}_I} = \mathbf{DB} = \begin{bmatrix} .1200 & .0655 \\ .1244 & .0709 \end{bmatrix}$$

$$\underset{(c \times c)}{\mathbf{A}_C} = \mathbf{BC}^{-1} = \begin{bmatrix} .1333 & .0667 \\ .1111 & .0589 \end{bmatrix} \; \underset{(i \times i)}{\mathbf{A}_C} = \mathbf{C}^{-1}\mathbf{B} = \begin{bmatrix} .1222 & .0664 \\ .1222 & .0700 \end{bmatrix}$$

5.4.2 Total Requirements Matrices

Commodity-Demand-Driven Models

Industry Technology **Commodity Technology**

Commodity-by-Commodity

$$(\mathbf{I} - \mathbf{BD})^{-1} = \begin{bmatrix} 1.1568 & .0898 \\ .1314 & 1.0782 \end{bmatrix} \qquad (\mathbf{I} - \mathbf{BC}^{-1})^{-1} = \begin{bmatrix} 1.1644 & .0825 \\ .1375 & 1.0723 \end{bmatrix}$$

Industry-by-Commodity

$$\mathbf{D}(\mathbf{I} - \mathbf{BD})^{-1} = \begin{bmatrix} 1.0411 & .0809 \\ .2471 & 1.0871 \end{bmatrix} \qquad \mathbf{C}^{-1}(\mathbf{I} - \mathbf{BC}^{-1})^{-1} = \begin{bmatrix} 1.1507 & -.0247 \\ .1512 & 1.1795 \end{bmatrix}$$

Industry-Demand-Driven Models

Industry Technology **Commodity Technology**

Industry-by-Industry

$$(\mathbf{I} - \mathbf{DB})^{-1} = \begin{bmatrix} 1.1478 & .0809 \\ .1537 & 1.0871 \end{bmatrix} \qquad (\mathbf{I} - \mathbf{C}^{-1}\mathbf{B})^{-1} = \begin{bmatrix} 1.1507 & .0821 \\ .1512 & 1.0861 \end{bmatrix}$$

Commodity-by-Industry

$$\mathbf{D}^{-1}(\mathbf{I} - \mathbf{DB})^{-1} = \begin{bmatrix} 1.2753 & .0898 \\ .0262 & 1.0782 \end{bmatrix} \qquad \mathbf{C}(\mathbf{I} - \mathbf{C}^{-1}\mathbf{B})^{-1} = \begin{bmatrix} 1.1644 & .1808 \\ .1375 & .9873 \end{bmatrix}$$

Notice that a negative element appears in one of these total requirements matrices. This reflects the negative element in $\mathbf{C}^{-1}$. (In fact, $\mathbf{C}^{-1}$ appears in the other three commodity technology total requirements matrices also, but the influence of the negative element is mitigated in the products $\mathbf{BC}^{-1}$ and $\mathbf{C}^{-1}\mathbf{B}$.) We will look into negative elements in commodity–industry models in some detail in the next section.

5.5 Negative Elements in the Commodity–Industry Framework

In the original input–output model one does not expect to find negative elements, either in an interindustry transactions matrix (**Z**) or in a total outputs vector (**x**). This means that there will not be any negative elements in the technical coefficients matrix (**A**) or in the Leontief inverse.[19] However, the commodity–industry format, designed to improve on the original Leontief framework in accounting for secondary products, introduces a new problem of its own – the possibility of negatives.

5.5.1 Commodity Technology

Direct Requirements Matrices Consider the structure of $\underset{(c\times c)}{\mathbf{A}_C} = \mathbf{B}\mathbf{C}^{-1} = \mathbf{U}(\mathbf{V}')^{-1}$ for the general 2 × 2 case:

$$\underset{(c\times c)}{\mathbf{A}_C} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{bmatrix}^{-1} = (1/|\mathbf{V}|) \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} v_{22} & -v_{21} \\ -v_{12} & v_{11} \end{bmatrix}$$

$$= (1/|\mathbf{V}|) \begin{bmatrix} u_{11}v_{22} - u_{12}v_{12} & -u_{11}v_{21} + u_{12}v_{11} \\ u_{21}v_{22} - u_{22}v_{12} & -u_{21}v_{21} + u_{22}v_{11} \end{bmatrix}$$

If **V** is a dominant diagonal matrix, as it is expected to be, then $|\mathbf{V}| = v_{11}v_{22} - v_{12}v_{21} > 0$, and the signs of the elements in $\underset{(c\times c)}{\mathbf{A}_C}$ will depend on the relative sizes of the u_{ij} and v_{ij}.

As an illustration, suppose that **U** is as shown in Table 5.3 and that all elements in **V** remain the same except for v_{21}. For what values of v_{21} would at least one element in $\underset{(c\times c)}{\mathbf{A}_C}$ be negative? (Notice that v_{21} only appears in the second column of $\underset{(c\times c)}{\mathbf{A}_C}$.) This means, at what value of v_{21} would v_{21} become larger than either $(u_{12}v_{11}/u_{11})$ or $(u_{22}v_{11}/u_{21})$? In this case, we have $v_{21} > 60$ or $v_{21} > 63$, respectively, so the point at which $(\underset{(c\times c)}{a_C})_{12}$ becomes negative is when $v_{21} > 60$ and $(\underset{(c\times c)}{a_C})_{22}$ becomes negative when $v_{21} > 63$. For example, as the reader can easily check, if $v_{21} = 60$,

$$\underset{(c\times c)}{\mathbf{A}_C} = \mathbf{U}(\mathbf{V}')^{-1} = \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} \begin{bmatrix} 90 & 60 \\ 0 & 100 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} \begin{bmatrix} .0111 & -.0067 \\ 0 & .01 \end{bmatrix} = \begin{bmatrix} .1333 & 0 \\ .1111 & .0033 \end{bmatrix}$$

while for $v_{21} = 61$, $\underset{(c\times c)}{\mathbf{A}_C} = \begin{bmatrix} .1333 & -.0013 \\ .1111 & .0022 \end{bmatrix}$ and for $v_{21} = 64$, $\underset{(c\times c)}{\mathbf{A}_C} = \begin{bmatrix} .1333 & -.0053 \\ .1111 & -.0011 \end{bmatrix}$. So $\underset{(c\times c)}{\mathbf{A}_C}$ exhibits a kind of unsatisfactory instability; there is no obvious reason why $v_{21} = 60$ is any more economically plausible than $v_{21} = 61$,

[19] Extensions of the original framework *could* accommodate negative elements, as in a pollution-generation model in which a negative z_{ij} might indicate the amount of pollutant i generated in conjunction with production activity in industry j. The associated a_{ij} would also be negative (amount of pollutant i released per unit of industry j output).

yet this one-unit variation means the difference between a reasonable direct requirements matrix and a much less reasonable one. For example, the implication of $\underset{(c\times c)}{\mathbf{A}_C} = \begin{bmatrix} .1333 & -.0013 \\ .1111 & .0022 \end{bmatrix}$ is that production of commodity 2 *releases* rather than consumes an amount of commodity 1, even though, as shown in $\mathbf{U}$, *industry* 2 consumes positive amounts of that commodity as a production input. And $\underset{(c\times c)}{\mathbf{A}_C} = \begin{bmatrix} .1333 & -.0053 \\ .1111 & -.0011 \end{bmatrix}$, when $v_{21} = 64$, is even more implausible.

From $\underset{(c\times c)}{\mathbf{A}_C} = \mathbf{U}(\mathbf{V}')^{-1}$ and the basic definition of the inverse of a 2×2 matrix – in this case$(\mathbf{V}')^{-1} = (1/\,|\mathbf{V}|)[\mathrm{adj}(\mathbf{V}')]$ – we recognize that negative elements in $\underset{(c\times c)}{\mathbf{A}_C}$ mean that at least one of the off-diagonal elements in $\mathbf{V}'$ will be negative.[20] As the examples above illustrate, a negative element in $(\mathbf{V}')^{-1}$ may or may not translate into one or more negative elements in $\underset{(c\times c)}{\mathbf{A}_C}$ – for $v_{21} = 60$ it does not but for $v_{21} = 61$ and larger it does.

It is worth carefully examining the operations involved in $\underset{(c\times c)}{\mathbf{A}_C}$ for this small 2×2 case where $\underset{(c\times c)}{\mathbf{A}_C} = (1/\,|\mathbf{V}|)\begin{bmatrix} u_{11}v_{22} - u_{12}v_{12} & -u_{11}v_{21} + u_{12}v_{11} \\ u_{21}v_{22} - u_{22}v_{12} & -u_{21}v_{21} + u_{22}v_{11} \end{bmatrix}$. To simplify the exposition, suppose $v_{12} = 0$; that is, industry 1 produces commodity 1 only, whereas industry 2 produces some of both commodities. In this case, as the reader can check, $\underset{(c\times c)}{\mathbf{A}_C}$ becomes

$$\underset{(c\times c)}{\mathbf{A}_C} = \begin{bmatrix} u_{11}/v_{11} & u_{12}/v_{22} - (u_{11}/v_{11})(v_{21}/v_{22}) \\ u_{21}/v_{11} & u_{22}/v_{22} - (u_{21}/v_{11})(v_{21}/v_{22}) \end{bmatrix}$$

Consider the element that measures commodity 1 input per unit of commodity 2 output – $(\underset{(c\times c)}{a_C})_{12} = u_{12}/v_{22} - (u_{11}/v_{11})(v_{21}/v_{22})$. First of all, u_{12}/v_{22} normalizes the input of commodity 1 to industry 2, u_{12}, *as if* all output of industry 2 were in the form of commodity 2. But some u_{12} went to industry 2 for production there of commodity 1. Under the commodity technology assumption, the recipe for commodity 1 production is the same in both industries, and from the first column in $\underset{(c\times c)}{\mathbf{A}_C}$ we know that (u_{11}/v_{11}) represents commodity 1 input per unit of commodity 1 output, *wherever produced*. From the second row of $\mathbf{V}$, we know that industry 2 made v_{21} units of commodity 1 while also producing v_{22} units of commodity 2 – so (v_{21}/v_{22}) represents commodity 1 production in industry 2 *per unit* of commodity 2 production there. Hence the per unit recipe for commodity 1 times the number of units – $(u_{11}/v_{11})(v_{21}/v_{22})$ – must be netted

[20] Of course if $\mathbf{V}$ is diagonal, there will be no off-diagonal elements in $(\mathbf{V}')^{-1}$. A diagonal $\mathbf{V}$ means that all production in the economy is primary, none secondary, and there is no need for the entire commodity–industry apparatus. (See section 5.6.) These observations strictly hold only for a two-commodity/two-industry example. A more general argument is needed for the case of m commodities and n industries, where $m = n > 2$ and where both $|\mathbf{V}'|$ and $[\mathrm{adj}(\mathbf{V}')]$ have more complicated structures.

out of u_{12}/v_{22} to account for the fact that industry 2 used u_{12} to make both commodity 2 and commodity 1. And what we want in $(\underset{(c\times c)}{a_C})_{12}$ is just that part of commodity 1 input that was used for commodity 2 production. A similar argument holds for $(\underset{(c\times c)}{a_C})_{22}$.[21] From column 2 of $\underset{(c\times c)}{\mathbf{A}_C}$ it is clear that if the negative term in either element exceeds the positive term a negative coefficient will result.

Transactions Matrices In the original input–output model, the underlying interindustry transactions matrix is retrieved from $\mathbf{A}$ and $\mathbf{x}$ as $\mathbf{Z} = \mathbf{A}\hat{\mathbf{x}}$ [for example, from (5.3)]. Similarly, an inter*commodity* transactions matrix (commodity inputs to support commodity outputs) in a commodity technology model can be derived; denote it by $\underset{(c\times c)}{\mathbf{Z}_C}$. In section 5.3 we saw $\mathbf{q} = (\mathbf{B}\mathbf{C}^{-1})\mathbf{q} + \mathbf{e}$ in which $\mathbf{B}\mathbf{C}^{-1}$ serves as a direct inputs matrix – $\underset{(c\times c)}{\mathbf{A}_C} = \mathbf{B}\mathbf{C}^{-1}$. Then $\underset{(c\times c)}{\mathbf{Z}_C} = \underset{(c\times c)}{\mathbf{A}_C}\,\hat{\mathbf{q}}\ (= \mathbf{B}\mathbf{C}^{-1}\hat{\mathbf{q}})$. The implication of a negative element in $\underset{(c\times c)}{\mathbf{A}_C}$ is that the underlying transaction is negative, and this is generally viewed as implausible. Since $\underset{(c\times c)}{\mathbf{A}_C}$ is postmultiplied by a (positive) *diagonal* matrix, any negative element in $\underset{(c\times c)}{\mathbf{A}_C}$ will immediately translate into a negative element in the corresponding location in $\underset{(c\times c)}{\mathbf{Z}_C}$.

For the modified example in which $v_{21} = 64$, so that $\mathbf{V} = \begin{bmatrix} 90 & 0 \\ 64 & 100 \end{bmatrix}$, we found $\underset{(c\times c)}{\mathbf{A}_C} = \begin{bmatrix} .1333 & -.0053 \\ .1111 & -.0011 \end{bmatrix}$. In this case, the associated vector of commodity outputs is $\mathbf{q} = \begin{bmatrix} 90 \\ 164 \end{bmatrix}$ and so the resulting transactions matrix is $\underset{(c\times c)}{\mathbf{Z}_C} = \underset{(c\times c)}{\mathbf{A}_C}\,\hat{\mathbf{q}} = \begin{bmatrix} 20.53 & -.53 \\ 17.11 & -.11 \end{bmatrix}$ with negative flows exactly where expected.

Using the definitions of $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ in terms of $\mathbf{U}$, $\mathbf{V}$, $\mathbf{q}$ and $\mathbf{x}$, along with matrix algebra facts on transposes and inverses of products and of diagonal matrices, it is easy to show also that $\underset{(c\times c)}{\mathbf{Z}_C} = \mathbf{U}(\mathbf{D}')^{-1}$.[22] In this form, the original commodity-to-industry transactions matrix, $\mathbf{U}$, is converted to a commodity-to-commodity transactions matrix via postmultiplication by a "conversion" matrix. We examine the notion of generating $\underset{(c\times c)}{\mathbf{Z}}$ or $\underset{(i\times i)}{\mathbf{Z}}$ via modifications of $\underset{(c\times i)}{\mathbf{U}}$ under commodity technology ($\underset{(c\times c)}{\mathbf{Z}_C}$ or $\underset{(i\times i)}{\mathbf{Z}_C}$) or industry technology ($\underset{(c\times c)}{\mathbf{Z}_I}$ or $\underset{(i\times i)}{\mathbf{Z}_I}$) assumptions in Appendix 5.1. In Appendix 5.2,

[21] If both $v_{12} \neq 0$ and $v_{21} \neq 0$, the economic logic behind the more complicated expressions that will make up $\underset{(c\times c)}{\mathbf{A}_C}$ is much more difficult to sort out. And for cases larger than 2×2 it is a lot worse.

[22] The steps from $\mathbf{B}\mathbf{C}^{-1}$ to $\mathbf{U}(\mathbf{D}')^{-1}$ are purely algebraic. Readers who are interested in this kind of matrix algebra should work through the details.

building on a result in Appendix 5.1, we explore an approach to eliminating negative elements if they should appear in $\underset{(c\times c)}{\mathbf{Z}_C}$ (as in Almon, 2000).

From (5.25) in section 5.3, we saw that $\mathbf{C}^{-1}\mathbf{B}$ plays the role of an industry-to-industry direct inputs matrix in the commodity technology model, and therefore the underlying transactions matrix is $\underset{(i\times i)}{\mathbf{Z}_C} = \underset{(i\times i)}{\mathbf{A}_C}\hat{\mathbf{x}} = \mathbf{C}^{-1}\mathbf{B}\hat{\mathbf{x}} = \mathbf{C}^{-1}\mathbf{U}$. Again, any (implausible) negative element in $\underset{(i\times i)}{\mathbf{A}_C}$ must reflect a corresponding (equally implausible) negative transaction. Since

$$\underset{(i\times i)}{\mathbf{A}_C} = \mathbf{C}^{-1}\mathbf{B} = \hat{\mathbf{x}}(\mathbf{V}')^{-1}\mathbf{U}\hat{\mathbf{x}}^{-1}$$

the influence on the direct requirements matrix of negative elements in $(\mathbf{V}')^{-1}$ is a little less straightforward than in the case of $\underset{(c\times c)}{\mathbf{A}_C}$. However, using just a bit more algebra, one can find for this example with $v_{12} = 0$ that the possible negative elements will be located in the top row of $\underset{(i\times i)}{\mathbf{A}_C}$, and that they occur if (1) $v_{21} > u_{11}v_{22}/u_{21}$ or (2) $v_{21} > u_{12}v_{22}/u_{22}$. These work out to be (1) $v_{21} > 120$ and (2) $v_{21} > 114.3$, respectively. In terms of our example, either of these larger values for v_{21} is highly improbable because each of them exceeds $v_{11} = 90$ – and we expect diagonal dominance in $\mathbf{V}$.[23] In particular, using the $v_{21} = 64$ case, as above, and the associated new industry output vector $\mathbf{x} = \begin{bmatrix} 154 \\ 100 \end{bmatrix}$ (column sums of $\mathbf{V}$), we find $\underset{(i\times i)}{\mathbf{A}_C} = \begin{bmatrix} .0622 & .0215 \\ .1822 & .0700 \end{bmatrix}$, with no negative elements.

Total Requirements Matrices Negative elements also appear in total requirements matrices. With $\underset{(c\times c)}{\mathbf{A}_C}$ as above when $v_{21} = 60$, one of the total requirements matrices in the commodity-demand driven model (Table 5.4) contains a negative element:

$$(\mathbf{I} - \mathbf{B}\mathbf{C}^{-1})^{-1} = \begin{bmatrix} 1.1538 & 0 \\ .1286 & 1.0033 \end{bmatrix} \text{ and } \mathbf{C}^{-1}(\mathbf{I} - \mathbf{B}\mathbf{C}^{-1})^{-1} = \begin{bmatrix} 1.0767 & -.6020 \\ .2058 & 1.6054 \end{bmatrix}$$

However, when $v_{21} = 61$, there are negative elements in both matrices

$$(\mathbf{I} - \mathbf{B}\mathbf{C}^{-1})^{-1} = \begin{bmatrix} 1.1536 & -.0015 \\ .1285 & 1.0021 \end{bmatrix} \text{ and } \mathbf{C}^{-1}(\mathbf{I} - \mathbf{B}\mathbf{C}^{-1})^{-1} = \begin{bmatrix} 1.0753 & -.6128 \\ .2068 & 1.6133 \end{bmatrix}$$

The same is true for $v_{21} = 64$.

The first of these matrices, $(\mathbf{I} - \mathbf{B}\mathbf{C}^{-1})^{-1}$, connects commodity final demands to commodity outputs, so the negative element in the $v_{21} = 61$ case means that an *increase* in final demand for commodity 2 generates a *decrease* in the output of commodity 1. The second matrix, $\mathbf{C}^{-1}(\mathbf{I} - \mathbf{B}\mathbf{C}^{-1})^{-1}$, connects commodity final demands to industry outputs, and so increases in final demand for commodity 2 create a decrease in the

23 These results are of course completely dependent on the specific values in **U** and **V** in our small numerical example.

output of industry 1. By contrast, as the reader can easily verify, both total requirements matrices in the industry-demand driven model (Table 5.5) – $(\mathbf{I}-\mathbf{C}^{-1}\mathbf{B})^{-1}$ and $\mathbf{C}(\mathbf{I}-\mathbf{C}^{-1}\mathbf{B})^{-1}$ – are non-negative under any of the assumptions about v_{21}. As mentioned, however, the commodity-demand driven model is generally the one of interest, since demands for commodities are usually preferred as the exogenous stimuli in models built on commodity–industry data sets.

5.5.2 Industry Technology

Direct Requirements Matrices In contrast to the situation under commodity technology, the direct requirements matrix under industry technology – $\underset{(c\times c)}{\mathbf{A}_I}=\mathbf{BD}=\mathbf{U}\hat{\mathbf{x}}^{-1}\mathbf{V}\hat{\mathbf{q}}^{-1}$, as in (5.27) – can never contain negative elements (as long as there are none in $\mathbf{U}$ and $\mathbf{V}$) since inversion of the *diagonal* matrices of (positive) industry and commodity outputs will never generate negative elements. For example, under the assumption that $v_{21}=64$ – when both elements in the second column of $\underset{(c\times c)}{\mathbf{A}_C}$ turn out to be negative – there are no negative elements in $\underset{(c\times c)}{\mathbf{A}_I}$:

$$\underset{(c\times c)}{\mathbf{A}_I}=\mathbf{U}\hat{\mathbf{x}}^{-1}\mathbf{V}\hat{\mathbf{q}}^{-1}=\mathbf{BD}$$
$$=\begin{bmatrix}12 & 8\\ 10 & 7\end{bmatrix}\begin{bmatrix}1/90 & 0\\ 0 & 1/164\end{bmatrix}\begin{bmatrix}90 & 0\\ 64 & 100\end{bmatrix}\begin{bmatrix}1/154 & 0\\ 0 & 1/100\end{bmatrix}$$
$$=\begin{bmatrix}.1333 & .0485\\ .1111 & .0424\end{bmatrix}\begin{bmatrix}.5844 & 0\\ .4156 & 1\end{bmatrix}=\begin{bmatrix}.0982 & .0488\\ .0827 & .0427\end{bmatrix}$$

Similarly, $\underset{(i\times i)}{\mathbf{A}_I}=\mathbf{DB}=[\mathbf{V}\hat{\mathbf{q}}^{-1}][\mathbf{U}\hat{\mathbf{x}}^{-1}]$, and negative elements will never be present. Again, for the example with $v_{21}=64$,

$$\underset{(i\times i)}{\mathbf{A}_I}=\mathbf{DB}=\begin{bmatrix}.0779 & .0283\\ .1665 & .0626\end{bmatrix}$$

Under industry technology we will never have to deal with the problem of possible negative transactions – either in $\underset{(c\times c)}{\mathbf{Z}_I}$ or $\underset{(i\times i)}{\mathbf{Z}_I}$.

Total Requirements Matrices The fact that direct requirements matrices will be non-negative under industry technology assures that all but one of the associated total requirements matrices will also be non-negative. Continuing with the data from the example with $v_{21}=64$, these matrices are

$$(\mathbf{I}-\mathbf{BD})^{-1}=\begin{bmatrix}1.1139 & .0564\\ .0960 & 1.0726\end{bmatrix}\quad \mathbf{D}(\mathbf{I}-\mathbf{BD})^{-1}=\begin{bmatrix}.6510 & .0330\\ .5590 & 1.0726\end{bmatrix}$$

$$(\mathbf{I}-\mathbf{DB})^{-1} = \begin{bmatrix} 1.0905 & .0330 \\ .1937 & 1.0726 \end{bmatrix} \quad \mathbf{D}^{-1}(\mathbf{I}-\mathbf{DB})^{-1} = \begin{bmatrix} 1.8659 & .0564 \\ -.5817 & 1.0492 \end{bmatrix}$$

and despite the positivity of the direct requirements matrices, we find that $\mathbf{D}^{-1}(\mathbf{I}-\mathbf{DB})^{-1}$ contains a negative element, reflecting the influence of the negative element in $\mathbf{D}^{-1}$. It is worth noting, however, that $\mathbf{D}^{-1}(\mathbf{I}-\mathbf{DB})^{-1}$ is the *only* total requirements matrix under industry technology that might ever contain negative elements and, as noted earlier, this is the least interesting or important of these matrices.

5.5.3 Making a Model Choice

Which Model to Choose? There is a large literature discussing the merits and drawbacks of various models in a commodity–industry framework. There is, however, no consensus of which should be preferred. For example, ten Raa, Chakraborty and Small (1984) rule out an industry technology model in favor of commodity technology. Later, ten Raa (1988) rejects the commodity technology model [and hence also the mixed technology model (section 5.7, below)], leaving "frustration." Then Kop Jansen and ten Raa (1990) examine the alternative technical coefficients matrices that are created from Make and Use matrices – $\mathbf{A}(\mathbf{U},\mathbf{V})$ – under the assumptions of commodity technology [$\mathbf{A}(\mathbf{U},\mathbf{V}) = \mathbf{U}(\mathbf{V}')^{-1}$, as in (5.26)], industry technology [$\mathbf{A}(\mathbf{U},\mathbf{V}) = \mathbf{U}\langle\mathbf{Vi}\rangle^{-1}\mathbf{V}\langle\mathbf{V}'\mathbf{i}\rangle^{-1}$, as in (5.27), and recalling that $\mathbf{Vi} = \mathbf{x}$ and $\mathbf{V}'\mathbf{i} = \mathbf{q}$] and various mixed technologies. They evaluate these against the backdrop of a set of four "desirable properties:" Material Balance [$\mathbf{x} = \mathbf{Ax} + \mathbf{y}$ becomes $\mathbf{A}(\mathbf{U},\mathbf{V})\mathbf{V}'\mathbf{i} = \mathbf{Ui}$], Financial Balance [revenues = costs for each sector, expressed as $\mathbf{i}'\mathbf{A}(\mathbf{U},\mathbf{V})\mathbf{V}' = \mathbf{i}'\mathbf{U}$], Price Invariance [new relative prices, $\mathbf{p} > \mathbf{0}$, imply new values in the Use and Make matrices, $\hat{\mathbf{p}}\mathbf{U}$ and $\mathbf{V}\hat{\mathbf{p}}$, and the resulting coefficients matrix should be $\mathbf{A}(\hat{\mathbf{p}}\mathbf{U},\mathbf{V}\hat{\mathbf{p}}) = \hat{\mathbf{p}}\mathbf{A}(\mathbf{U},\mathbf{V})\hat{\mathbf{p}}^{-1}$], and Scale Invariance [multiplying all inputs and outputs of each sector, i, by a constant, s_i, should leave the coefficients unchanged, so, for $\mathbf{s} > \mathbf{0}, \mathbf{A}(\mathbf{U}\hat{\mathbf{s}},\hat{\mathbf{s}}\mathbf{V}) = \mathbf{A}(\mathbf{U},\mathbf{V})$]. Only the commodity technology model satisfies all four criteria (among seven different models examined).

The wide variety of opinion continues. The Eurostat Manual (Eurostat/European Commission, 2008), which sets out recommended standards for data collection for member countries of the European Union, supports an industry-technology model – for example, as in $\underset{(c\times c)}{\mathbf{A}_I} = \mathbf{BD} = [\mathbf{U}\hat{\mathbf{x}}^{-1}][\mathbf{V}\hat{\mathbf{q}}^{-1}]$ [in (5.27)] or $\underset{(i\times i)}{\mathbf{A}_I} = \mathbf{DB} = [\mathbf{V}\hat{\mathbf{q}}^{-1}][\mathbf{U}\hat{\mathbf{x}}^{-1}]$. The Manual recommends a different classification scheme that reflects observations like those made by Konijn and Steenge (1995, pp. 34–35):

> It can relatively easily be understood that these technology assumptions [commodity technology and industry technology] are not used to construct industry-by-industry tables. In constructing industry-by-industry tables, assumptions are made on the origins and destinations of products [commodities] and not on the technology of production. Hence we find the traditional presentation of methods…to be incorrect.

This leads to an alternative presentation (adapted from Eurostat/European Commission, 2008, Figure 11.3, p. 310):

Table 5.7 Alternative Classifications, Total Requirements Matrices, Commodity-Demand Driven Models

	Commodity Technology	Industry Technology	Fixed Industry Sales Structure	Fixed Commodity Sales Structure
Commodity-by-Commodity	$(\mathbf{I} - \mathbf{BC}^{-1})^{-1}$ Model A	$(\mathbf{I} - \mathbf{BD})^{-1}$ Model B		
Industry-by-Industry			$(\mathbf{I} - \mathbf{C}^{-1}\mathbf{B})^{-1}\mathbf{C}^{-1}$ Model C	$(\mathbf{I} - \mathbf{DB})^{-1}\mathbf{D}$ Model D

Model A: Each commodity is produced in its own specific way, irrespective of the industry where it is produced. Negative elements may occur.
Model B: Each industry has its own specific way of production, irrespective of its product mix. No negative elements.
Model C: Each industry has its own specific sales structure, irrespective of its product mix. Negative elements may occur.
Model D: Each product has its own specific sales structure, irrespective of the industry where it is produced. No negative elements.

The case is then made for Model D (p. 310):

Industry-by-industry tables which are based on the fixed product sales structure (Model D) do not involve any technology assumptions (A and B), and do not require the application of sometimes arbitrary methods to adjust for negatives (A and C).

This Manual contains a wealth of numerical examples illustrating the consequences of alternative assumptions. It emphasizes the issues surrounding the compilation of *symmetric input–output tables* (SIOTs) – meaning tables with dimensions commodity-by-commodity or industry-by-industry – from the data in Supply and Use tables (SUTs). The interested reader is referred to the Manual or to Thage (2002, 2005) or Thage and ten Raa (2006) for details. (There are useful numerical examples in the Manual and in Thage, 2005.)

The issue is primarily with the negatives that can appear in the commodity technology model; we turn to these next.

Dealing with Negative Values Researchers working with real-world input–output data repeatedly find that the commodity technology model generates negative direct input coefficients and transactions (frequently relatively small). For example: "There are numerous examples of the [commodity technology] method leading to negative coefficients which are clearly nonsensical from an economic point of view" (United Nations *et al.*, 1993, Section 15.147; quoted in Almon, 2000, p. 28). Table 5.8 provides a few examples.

Table 5.8 Examples of Negative Elements in Real-World Commodity-Technology Direct Requirements Matrices [$\underset{(c\times c)}{\mathbf{A}_C} = \mathbf{BC}^{-1}$]

Reference	Dimensions	Country, year	Number of negative elements	Negative as percentage of total
ten Raa, Chakraborty and Small (1984)	43×43	Canada, 1977	10^a [in $(\mathbf{I} - \mathbf{A}_{CB})^{-1}$]	0.5
ten Raa and van der Ploeg (1989)	39×39	UK, 1975	22^b	1.4
Rainer (1989)	175×175	Austria, 1976	Negative elements accounted for 1.4 percent of total value of intermediate flows in $c \times c$ model	
Steenge (1989)	79×79	USA, 1977	116^b	1.9
Steenge (1990)	14×14	USA, 1977	7^b	3.6

[a] Criterion for selection: $> |0.03|$.
[b] Criterion for selection: $\geq |0.001|$.

These kinds of "nonsensical" results have generated two reactions. One is to abandon the commodity technology model entirely – for example, de Mesnard (2004)[24] or Eurostat/European Commission (2008) along with the references by Thage, above. Others find that rejection of the commodity-technology model is much too harsh a judgment (for example, Rainer and Richter, 1992). The other reaction is to propose "adjustments" to commodity-technology models to avoid negative elements and to deal with such elements when they occur in practice. Examples of additional literature dealing with both of these reactions, and others, include (in approximately chronological order): van Rijckeghem (1967), Stahmer (1985), ten Raa and van der Ploeg (1989), Steenge (1989, 1990), Rainer (1989), Rainer and Richter (1989), Mattey and ten Raa (1997), Londero (1990, 1999, 2001), ten Raa (1995, 2005), Almon (2000) and ten Raa and Rueda-Cantuche (2003, 2007). Clearly this continues to be very much an open question.

When the negative elements are relatively small, they have sometimes simply been changed to zeros [e.g., in work at the Cambridge (UK) Growth Project, under Stone's direction, cited in Armstrong, 1975]. One problem with this approach is deciding what constitutes a "relatively small" element. Alternatively, in some studies a negative element has been replaced by a small positive element, with "compensating adjustments in other entries in the matrix so that the overall row and column accounting constraints were still met" (Armstrong, 1975, p. 80). The "compensating adjustments" will be somewhat ad hoc, and different researchers might make differing sets of such

[24] de Mesnard (2004) argues against any version of a model in which $\mathbf{C}^{-1}$ (with its negative elements) appears. He does so by viewing various models in terms of economic circuits (directed impulses). The interested reader is referred to the article for details.

adjustments. In Appendix 5.2, as an illustration of one approach to dealing with negatives in commodity-based technology, we investigate a procedure that has been used successfully in many INFORUM studies for decades (Almon, 2000).

5.6 Nonsquare Commodity–Industry Systems

If the number of commodities in the input–output accounts is not the same as the number of industries, then various matrices in the commodity–industry modeling system will be "rectangular" rather than square.[25] In this section we explore some of the implications of this $m \neq n$ possibility. In principle, this can mean either $m > n$ or $m < n$.

We consider these in turn. The case of more commodities than industries ($m > n$) is seen often in real-world input–output accounts. Less usual is the case of fewer commodities than industries ($m < n$), although it can sometimes be found in disaggregated versions of data sets – that is, at the data-collection phase, for instance, when "dummy" industries are used to account for such products as scrap, used/second-hand goods or import duties. Since it is general practice to aggregate to $m \geq n$ levels before implementing an input–output model, we will consider only the $m > n$ case in this section. In Web Appendix 5W.1, brief attention is given to the much less important $m < n$ situation.

As an illustration, let $m = 3$ (commodities) and $n = 2$ (industries). The dimensions of the matrices that are the building blocks of the commodity–industry model are $\underset{(3\times2)}{\mathbf{U}}$, $\underset{(2\times3)}{\mathbf{V}}$, $\underset{(2\times1)}{\mathbf{x}}$, and $\underset{(3\times1)}{\mathbf{q}}$. This leads to

$$\underset{(3\times2)}{\mathbf{B}} = \underset{(3\times2)}{\mathbf{U}}\ \underset{(2\times2)}{\hat{\mathbf{x}}^{-1}},\quad \underset{(3\times2)}{\mathbf{C}} = \underset{(3\times2)}{\mathbf{V}'}\ \underset{(2\times2)}{\hat{\mathbf{x}}^{-1}},\quad \text{and}\quad \underset{(2\times3)}{\mathbf{D}} = \underset{(2\times3)}{\mathbf{V}}\ \underset{(3\times3)}{\hat{\mathbf{q}}^{-1}}$$

Later in this section we will use the following three-commodity, two-industry data for illustration (Table 5.9). In this case, one can easily find that

$$\mathbf{B} = \begin{bmatrix} .1667 & .1446 \\ .1852 & .1928 \\ .0185 & .0723 \end{bmatrix},\ \mathbf{C} = \begin{bmatrix} .8333 & .1205 \\ .0926 & .7952 \\ .0741 & .0843 \end{bmatrix} \text{ and } \mathbf{D} = \begin{bmatrix} .9 & .1316 & .5333 \\ .1 & .8684 & .4667 \end{bmatrix}$$

5.6.1 Commodity Technology

Under commodity technology, there is an immediate problem when trying to convert $\mathbf{x}$ into a function of $\mathbf{q}$ on the right-hand side in (5.13). Recall that the sequence goes from $\mathbf{Cx} = \mathbf{q}$ [in (5.18)] to $\mathbf{x} = \mathbf{C}^{-1}\mathbf{q}$ [in (5.19)], in order to convert $\mathbf{q} = \mathbf{Bx} + \mathbf{e}$ [in (5.13)] into $\mathbf{q} = \mathbf{BC}^{-1}\mathbf{q} + \mathbf{e}$, from which we had $\underset{(c\times c)}{\mathbf{A}_C} = \mathbf{BC}^{-1}$. Also, again using (5.13), but now premultiplying both sides by $\mathbf{C}^{-1}$, we had $\mathbf{x} = \mathbf{C}^{-1}\mathbf{q} = \mathbf{C}^{-1}\mathbf{Bx} + \mathbf{C}^{-1}\mathbf{e}$, and so

[25] Strictly speaking, a square is a rectangle all of whose sides are the same length. In general, however, in the commodity–industry literature, "rectangular" is used to indicate a "nonsquare" system in which the number of commodities does not equal the number of industries.

Table 5.9 A Three-Commodity, Two-Industry Example

		Commodities			Industries			
		1	2	3	1	2	Final Demand	Total Output
Commodities	1				18	12	70	100
	2				20	16	40	76
	3				2	6	7	15
					U		**e**	**q**
Industries	1	90	10	8				108
	2	10	66	7				83
			V					**x**
Value Added					68	49	117	
					$\mathbf{v}'$			
Total Inputs		100	76	15	108	83		
			$\mathbf{q}'$		$\mathbf{x}'$			

$\underset{(i\times i)}{\mathbf{A}_C} = \mathbf{C}^{-1}\mathbf{B}$. Clearly, these operations are in trouble without a well defined $\mathbf{C}^{-1}$. We explore in Web Appendix 5W.1 (for the reader interested in mathematical detail) why it is that the mathematical notions of inverses for rectangular matrices are of no help here. The conclusion is that commodity technology models cannot generate a plausible direct requirements matrix in the case of more commodities than industries. Consequently, total requirements matrices also cannot be found. The usual solution is to aggregate commodities (and perhaps also industries) until some level at which $m = n$.

Notice that Approach II in section 5.3.7 (avoiding direct requirements matrices completely) is of no help here. In particular, since **B** and **C** are both $m \times n$ (here 3×2) matrices, the inverse that is required in (5.28) – $(\mathbf{C} - \mathbf{B})^{-1}$ – is just as problematic as $\mathbf{C}^{-1}$ alone.

5.6.2 *Industry Technology*

Direct Requirements Matrices A rectangular format presents no problems under industry technology. Substitution of (5.16) – $\mathbf{Dq} = \mathbf{x}$ – into (5.13) – $\mathbf{q} = \mathbf{Bx} + \mathbf{e}$ – is straightforward and requires no inverse; $\mathbf{q} = \mathbf{BDq} + \mathbf{e}$. Using **B** and **D** from Table 5.9, $\underset{(c\times c)}{\mathbf{A}_I} = \underset{(3\times 2)}{\mathbf{B}}\ \underset{(2\times 3)}{\mathbf{D}} = \begin{bmatrix} .1645 & .1475 & .1564 \\ .1859 & .1918 & .1887 \\ .0239 & .0652 & .0436 \end{bmatrix}$ [as in (5.27)] has the correct 3×3 dimensions (commodity-by-commodity) and $\underset{(i\times i)}{\mathbf{A}_I} = \underset{(2\times 3)}{\mathbf{D}}\ \underset{(3\times 2)}{\mathbf{B}} = \begin{bmatrix} .1842 & .1940 \\ .1861 & .2156 \end{bmatrix}$ is, appropriately, a 2×2 matrix relating industry inputs to industry outputs.

Total Requirements Matrices As to total requirements matrices, we know from Tables 5.4 and 5.5 and the previous section that the only problem under industry technology occurs with the commodity-by-industry matrix in Table 5.5, where $\mathbf{D}^{-1}$plays a role – in $\mathbf{D}^{-1}(\mathbf{I}-\mathbf{DB})^{-1}$. The problem is exactly the same as in the case when $\mathbf{D}$ is singular. It occurs when trying to move from (5.16), $\mathbf{Dq}=\mathbf{x}$, to (5.17), $\mathbf{q}=\mathbf{D}^{-1}\mathbf{x}$, only now it is because $\mathbf{D}$ is rectangular, not because of singularity. We explore this problem briefly also in Web Appendix 5W.1.

The remaining total requirements matrices, from Tables 5.4 and 5.5, are unhampered by a rectangular format. The reader can check that for the data in Table 5.9 these are

$$(\mathbf{I}-\mathbf{BD})^{-1}=\begin{bmatrix} 1.2599 & .2505 & .2554 \\ .3020 & 1.3173 & .3093 \\ .0521 & .0961 & 1.0731 \end{bmatrix}$$

$$\mathbf{D}(\mathbf{I}-\mathbf{BD})^{-1}=\begin{bmatrix} 1.2014 & .4500 & .8429 \\ .4126 & 1.2139 & .7949 \end{bmatrix}$$

$$(\mathbf{I}-\mathbf{DB})^{-1}=\begin{bmatrix} 1.2992 & .3214 \\ .3083 & 1.3511 \end{bmatrix}$$

5.7 Mixed Technology in the Commodity–Industry Framework

At even a very detailed level of industry and commodity disaggregation, there are certain to be many industries that produce more than one commodity. For example, Danish annual tables (1966–1998) are based on supply and use tables with about 2,750 commodities and 130 industries, and annual tables for the Netherlands start from data on 800 commodities and 250 industries (Thage, 2002, pp. 3 and 13).

In the USA, correspondences between input–output commodities/industries at the six-digit level and the four-digit 1987 US Standard Industrial Classification Code (SIC)[26] indicate under SIC 2211, "Broadwoven fabric mills, cotton," almost four pages (single spaced) of some 147 individual commodities, among them "Sheets and sheetings, cotton-mitse ("manufactured in the same establishment"), along with such diverse items as diaper fabrics, mosquito netting, and typewriter ribbon cloth.[27] The output of SIC 2211 (along with four other six-digit SIC industries) is classified under I-O *industry* 16.0100, "Broadwoven fabric mills and fabric finishing plants." On the other hand, under SIC 2392 "House furnishing, except curtains and draperies," there are 43 commodities, including "Sheets, fabric-mfpm ("manufactured from purchased materials"), along with such products as boat cushions, dust cloths, and shoe bags. The output of SIC 2392 is counted under I-O *industry* 19.0200, "House furnishings, except curtains and draperies."

26 This has since been redefined as the North American Industrial Classification (NAIC) system, but the principles remain the same. Correspondences between the 1997 NAIC system and the 1987 SIC system are shown at www.census.gov:80/epcd/www/naicstab.htm.

27 It is very informative to investigate the contents of various US SIC "industries." This is easily done at the website mentioned in the previous footnote and also at www.osha.gov/oshastats/sicser.html.

Of course, as data are aggregated into fewer commodity and industry classifications, all of these diverse products get lumped together, making for even more heterogeneity in an industry's output – for example, SIC 2392 is combined with SIC 2391, 2393–2397, and 2399 into three-digit SIC industry 239, "Miscellaneous fabricated textile products," which, in turn, is part of two-digit SIC industry 23 "Apparel and other finished products made from fabrics and similar materials."

There is an extensive and often contradictory literature on alternative definitions for classifying, among others, secondary products, subsidiary products, joint products, and by-products. Generally, the "principal" (or "primary") output of a multiproduct production process is the one that accounts for the maximum value of production (sometimes maximum value added is used); remaining outputs (if any) are classified as "secondary." Primary and secondary together are sometimes called "joint products" and a "by-product" is then sometimes defined as a joint product that is of "distinctly lesser importance …" than the other joint product(s) [United Nations, 1966, 2.60 as quoted in Londero, 2001, p. 39[28]]. These distinctions can play a role in helping to establish whether a commodity or an industry technology model is more appropriate for a particular secondary product.

A variation on these classifications identifies several kinds of secondary products in the following way (for example, Bulmer-Thomas, 1982, Chapter 9).

1. A product whose output level is *independent* of the level of primary production in an establishment where it is produced and which is:
   a. produced according to the technology used in the industry that produces it as a primary product. Examples: computer hard drives made by a computer manufacturer (e.g., IBM) according to the production "recipe" used by other computer hard drive manufacturers and sold to other computer manufacturers; cotton sheets that are made in a cotton mill (as noted above). Such secondary products would be logical candidates for the commodity technology assumption.
   b. produced according to the technology used in the industry where it is produced as a secondary product. Example: computer services developed by an aircraft establishment (e.g., Boeing, for aircraft design) then marketed as a secondary product. This would appropriately be treated under an industry technology assumption.
2. A product whose output level is *not independent* of that of the primary product in an establishment and for which:
   a. there is another industry that makes it as a primary product. This is classified as a "by-product." Example: ethylene generated during petroleum refining (but also produced in "natural gas" plants). This kind of secondary product appears not to conform well to either technology assumption.
   b. there is not another industry making it as a primary product. This is classified as a joint product. Examples: wool produced in conjunction with sheep ranching; hides from cattle raising; radioactive waste generated in producing electrical

[28] This reference also includes some discussion of the confusing and contradictory language employed in various publications – including several from the United Nations.

power at a nuclear plant; ash, generated by coal burning electric power plants, that is used as a hardening agent in some types of road surfaces (e.g., airport runways). Here it is at least clear that a commodity technology model is *not* appropriate.

In any event, the issue is how to "assign" each of the various secondary products to a particular production technology in a commodity–industry input–output system. In practice, there are about as many approaches as there are real-world accounts and models.[29]

In years prior to the adoption of commodity–industry accounting systems, input–output modelers recognized this obvious fact of secondary production in many industries ("sectors" in pre-commodity–industry days) and dealt with it in a number of ways. Suppose that industry j produces not only commodity j (its primary product) but also some of commodity k (which is primary to industry k), as a secondary product. One general approach was to subtract from the value of industry j's total output the amount that represents the value of commodity k production and add this amount to the value of output of industry k. Do the same for inputs; subtract from industry j's total input vector those inputs that were used for commodity k production, and add these to industry k's inputs column. This is known as *redefinition* (or, for reasons that we will see in a minute, *specific redefinition*). Usually, this industry redefinition is easier said than done, especially as regards inputs. An alternative approach created *transfers* of secondary production – in input–output parlance this meant that industry j's output of commodity k was treated as if sold by industry j to industry k and added to industry k's total output.[30]

The commodity–industry accounting approach introduces a wider range of options. It generates what has been called a *mechanical redefinition* of secondary production – through use of the commodity technology assumption (for example, as in $\underset{(c\times c)}{\mathbf{A}_C} = \mathbf{B}\mathbf{C}^{-1}$) or the industry technology assumption (as in $\underset{(c\times c)}{\mathbf{A}_I} = \mathbf{B}\mathbf{D}$). And these two technology approaches can be used in a variety of ways. For example, in the US models, starting with 1972, secondary products that are "obviously" [however defined; for example, as in (1)(a)] produced under commodity technology are specifically redefined to their primary industries and for the remainder the mechanical redefinition of an industry technology model is employed.[31]

Another refinement that is available with commodity–industry accounts is to use both commodity technology and industry technology mechanical redefinitions in the

[29] For a comprehensive account of the US procedures in the early commodity–industry years, see Ritz (1980). Rainer and Richter (1992) examine aspects of the Austrian experience. For an overview in a number of other countries, see Franz and Rainer (1989) and for a comprehensive classification of approaches see ten Raa and Rueda-Cantuche (2003, esp. Table 2).

[30] Fukui and Seneta (1985) examine and classify four "conventional" methods for dealing with joint products as of the mid-1980s.

[31] In speaking of the adoption of the commodity–industry approach for the US, starting with the 1972 table, Ritz writes: "The use of the mechanical redefinition for all secondary products other than those which have been specifically redefined is a substantial improvement over the transfer treatment used in earlier I-O studies" (Ritz, 1980, p. 51).

same model, thereby bypassing an exclusively commodity technology model or an exclusively industry technology model. This is accomplished in what are known as "mixed-technology" or "hybrid" models.

The essential idea is to divide the Make matrix, $\mathbf{V}$, into two components, so that

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 \tag{5.32}$$

where $\mathbf{V}_1$ records the making of commodities that are best identified with one of the two technology assumptions and $\mathbf{V}_2$ records the making of commodities associated with the other technology assumption.[32] We can define vectors of total industry outputs under the two technology assumptions as

$$\mathbf{x}_1 = \mathbf{V}_1\mathbf{i} \quad \text{and} \quad \mathbf{x}_2 = \mathbf{V}_2\mathbf{i} \tag{5.33}$$

(where, as usual, $\mathbf{i}$ is a column vector of 1's). Total industry output then is just

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 = (\mathbf{V}_1 + \mathbf{V}_2)\mathbf{i}$$

Similarly, total commodity outputs are identified as

$$\mathbf{q}_1 = (\mathbf{V}'_1)\mathbf{i} \quad \text{and} \quad \mathbf{q}_2 = (\mathbf{V}'_2)\mathbf{i} \tag{5.34}$$

so that

$$\mathbf{q} = \mathbf{q}_1 + \mathbf{q}_2 = (\mathbf{V}'_1 + \mathbf{V}'_2)\mathbf{i}$$

5.7.1 Commodity Technology in $\mathbf{V}_1$

We illustrate the general idea by attaching commodity technology to production recorded in $\mathbf{V}_1$ and industry technology to $\mathbf{V}_2$. (This is an arbitrary assignment; we could equally well decide to reflect industry technology in $\mathbf{V}_1$ and commodity technology in $\mathbf{V}_2$. This is explored in the next section.) Using $\mathbf{V}_1$, define

$$\mathbf{C}_1 = (\mathbf{V}'_1)(\hat{\mathbf{x}}_1)^{-1} \tag{5.35}$$

This creates a commodity-by-industry matrix whose i, jth element records the proportion of industry j's (commodity technology) output that is in the form of commodity i. This is identical in spirit to the definition of $\mathbf{C}$ in (5.15). From (5.35), $\mathbf{C}_1\hat{\mathbf{x}}_1\mathbf{i} = (\mathbf{V}'_1)\mathbf{i}$ and using (5.34), $\mathbf{C}_1\mathbf{x}_1 = \mathbf{q}_1$, or

$$\mathbf{x}_1 = \mathbf{C}_1^{-1}\mathbf{q}_1 \tag{5.36}$$

assuming that $\mathbf{C}_1$ is nonsingular. This provides a transformation between $\mathbf{x}_1$ and $\mathbf{q}_1$. [Without the subscripts, this is completely parallel to the connection between $\mathbf{x}$ and $\mathbf{q}$ given by $\mathbf{C}$ in the pure commodity technology case, in (5.18) and (5.19).]

[32] Detailed derivations, examples, variations, and discussions can be found in, among others, United Nations (1968), Aidenoff (1970), Gigantes (1970), Cressy (1976), Armstrong (1975), and ten Raa, Chakraborty and Small (1984).

To account for the industry technology character of the outputs recorded in $\mathbf{V}_2$, define the industry-by-commodity matrix $\mathbf{D}_2$ as

$$\mathbf{D}_2 = \mathbf{V}_2\hat{\mathbf{q}}^{-1} \tag{5.37}$$

Notice that the "normalization" of $\mathbf{V}_2$ is done using *total* commodity outputs, not just those recorded in $\mathbf{V}_2$. [This is similar to the definition of $\mathbf{D}$, in (5.14), in the pure industry technology case.][33] The i,jth element in $\mathbf{D}_2$ identifies that fraction of all commodity j production that is made by industry i under an industry technology assumption. Since $\mathbf{x}_2 = \mathbf{V}_2\mathbf{i}$ [as in (5.33)], a line of argument similar to that used above for $\mathbf{C}_1$ leads to a transformation between $\mathbf{x}_2$ and $\mathbf{q}$ (not $\mathbf{q}_2$); namely

$$\mathbf{x}_2 = \mathbf{D}_2\mathbf{q} \tag{5.38}$$

If we put the two pieces of $\mathbf{x}$ together, from (5.36) and (5.38), we have

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 = \mathbf{C}_1^{-1}\mathbf{q}_1 + \mathbf{D}_2\mathbf{q}$$

Our interest is in the transformation between all of $\mathbf{x}$ and all of $\mathbf{q}$; that means that we need to replace $\mathbf{q}_1$ on the right with some function of $\mathbf{q}$.

To do this, we examine the components of $\mathbf{q}$ in some detail. As above, $\mathbf{q} = \mathbf{q}_1 + \mathbf{q}_2$. We know that $\mathbf{q}_2 = (\mathbf{V}'_2)\mathbf{i}$ and $\mathbf{D}_2 = \mathbf{V}_2\hat{\mathbf{q}}^{-1}$, so that $\mathbf{q}_2 = (\mathbf{D}_2\hat{\mathbf{q}})'\mathbf{i}$. From matrix algebra rules on transposes of products of appropriately dimensioned matrices – $(\mathbf{MN})' = \mathbf{N}'\mathbf{M}'$ – and on the product of two diagonal matrices (order of multiplication makes no difference), it follows that[34]

$$\mathbf{q}_2 = (\mathbf{D}_2\hat{\mathbf{q}})'\mathbf{i} = \langle \mathbf{D}'_2\mathbf{i}\rangle\mathbf{q} = \langle \mathbf{i}'\mathbf{D}_2\rangle\mathbf{q}$$

Since $\mathbf{q}_1 = \mathbf{q} - \mathbf{q}_2 = \mathbf{q} - \langle \mathbf{i}'\mathbf{D}_2\rangle\mathbf{q}$, this allows $\mathbf{x}$ to be expressed as

$$\mathbf{x} = \mathbf{C}_1^{-1}(\mathbf{q} - \langle \mathbf{i}'\mathbf{D}_2\rangle\mathbf{q}) + \mathbf{D}_2\mathbf{q} = [\mathbf{C}_1^{-1}(\mathbf{I} - \langle \mathbf{i}'\mathbf{D}_2\rangle) + \mathbf{D}_2]\mathbf{q}$$

If we define

$$\mathbf{R} = [\mathbf{C}_1^{-1}(\mathbf{I} - \langle \mathbf{i}'\mathbf{D}_2\rangle) + \mathbf{D}_2] \tag{5.39}$$

we see the parallel with earlier results. Here the transformation between industry outputs and commodity outputs is given by $\mathbf{x} = \mathbf{Rq}$. Previously, under pure industry technology, it was $\mathbf{x} = \mathbf{Dq}$, and under pure commodity technology, it was $\mathbf{x} = \mathbf{C}^{-1}\mathbf{q}$. Notice that $\mathbf{R}$ contains elements that are reminiscent of both previous transformations; namely $\mathbf{C}_1^{-1}$ and also $\mathbf{D}_2$. Total requirements matrices under this particular mixed technology assumption are found as the exact parallels to those under pure commodity technology or pure industry technology, with $\mathbf{C}^{-1}$ or with $\mathbf{D}$ replaced by $\mathbf{R}$.

[33] This definition is not "parallel" to that for $\mathbf{C}_1$ in the sense that the divisors here are not elements of $\mathbf{q}_2$ but rather of $\mathbf{q}$. The reason for this will become clear as the algebra is worked out. Other definitions for $\mathbf{D}_2$ could be (and have been) used, with different algebraic consequences.

[34] The specific steps are: $(\mathbf{D}_2\hat{\mathbf{q}})'\mathbf{i} = \hat{\mathbf{q}}'\mathbf{D}_2'\mathbf{i}$ (transpose of a product) $= \hat{\mathbf{q}}\mathbf{D}_2'\mathbf{i}$ (transpose of a diagonal matrix) $= \hat{\mathbf{q}}\langle \mathbf{D}'_2\mathbf{i}\rangle\mathbf{i}$ (the vector that is created from row sums of a matrix is the same as the vector that is created from row sums of the diagonal matrix formed from that vector) $= \langle \mathbf{D}'_2\mathbf{i}\rangle\hat{\mathbf{q}}\mathbf{i}$ (order of multiplication of two diagonal matrices makes no difference) $= \langle \mathbf{D}'_2\mathbf{i}\rangle\mathbf{q}$.

5.7.2 *Industry Technology in* $\mathbf{V}_1$

In order to invoke the industry technology assumption for $\mathbf{V}_1$, we define

$$\mathbf{D}_1 = \mathbf{V}_1\hat{\mathbf{q}}_1^{-1} \tag{5.40}$$

[compare (5.14) and now also (5.37).] This identifies an industry-by-commodity matrix whose i, jth element records the fraction of total commodity j produced under the industry technology assumption that is made by industry i using that technology. Along with the definition $\mathbf{x}_1 = \mathbf{V}_1\mathbf{i}$ [as in (5.33)], we see that

$$\mathbf{x}_1 = \mathbf{D}_1\mathbf{q}_1 \tag{5.41}$$

which provides the transformation between $\mathbf{x}_1$ and $\mathbf{q}_1$. [Without the subscripts, this is completely parallel to the definition of $\mathbf{D}$ in the pure industry technology case, shown in (5.16).]

To account for the commodity technology character of the outputs in $\mathbf{V}_2$, define a commodity-by-industry matrix, $\mathbf{C}_2$, as

$$\mathbf{C}_2 = \mathbf{V}'_2\hat{\mathbf{x}}^{-1} \tag{5.42}$$

[Compare (5.15) and now also (5.35).] Here the "normalization" of $\mathbf{V}_2$ has been carried out using *total* industry outputs, not just those attributable to production in $\mathbf{V}_2$.[35] The i, jth element in $\mathbf{C}_2$ represents the fraction of *all* industry j output that takes the form of commodity i production under commodity technology. An argument similar to that above shows that this definition of $\mathbf{C}_2$, along with the fact that $\mathbf{q}_2 = (\mathbf{V}'_2)\mathbf{i}$, leads to a transformation between $\mathbf{q}_2$ and $\mathbf{x}$ (not $\mathbf{x}_2$); namely

$$\mathbf{q}_2 = \mathbf{C}_2\mathbf{x} \tag{5.43}$$

From (5.33) and the definition of $\mathbf{C}_2$, $\mathbf{x}_2 = (\mathbf{C}_2\hat{\mathbf{x}})'\mathbf{i}$ and, following an argument parallel to that in footnote 29, we find

$$\mathbf{x}_2 = \langle \mathbf{C}'_2\mathbf{i}\rangle\mathbf{x} = \langle \mathbf{i}'\mathbf{C}_2\rangle\mathbf{x} \tag{5.44}$$

Using $\mathbf{q}_1 = \mathbf{q} - \mathbf{q}_2 = \mathbf{q} - \mathbf{C}_2\mathbf{x}$ allows the components of $\mathbf{x}$ to be expressed as

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2 = \mathbf{D}_1(\mathbf{q} - \mathbf{C}_2\mathbf{x}) + \langle \mathbf{i}'\mathbf{C}_2\rangle\mathbf{x}$$

and rearrangement, putting $\mathbf{x}$ alone on the left, gives

$$\mathbf{x} = [(\mathbf{I} + \mathbf{D}_1\mathbf{C}_2 - \langle \mathbf{i}'\mathbf{C}_2\rangle)^{-1}\mathbf{D}_1]\mathbf{q}$$

We can define

$$\mathbf{T} = [(\mathbf{I} + \mathbf{D}_1\mathbf{C}_2 - \langle \mathbf{i}'\mathbf{C}_2\rangle)^{-1}\mathbf{D}_1] \tag{5.45}$$

[35] As in the case of $\mathbf{C}_1$ and $\mathbf{D}_2$, this definition of $\mathbf{C}_2$ is not "parallel" to that for $\mathbf{D}_1$ since the divisors are not elements of $\mathbf{x}_2$ but rather of $\mathbf{x}$. Again, as noted in footnote 33, other definitions could be used, with differing algebraic results.

and we see the parallel, again, with earlier results. Now the transformation between industry outputs and commodity outputs is provided by $\mathbf{x} = \mathbf{Tq}$. Thus $\mathbf{T}$ plays the role of $\mathbf{R}$ from the previous mixed technology case, and of $\mathbf{D}$ and $\mathbf{C}^{-1}$ earlier. Again, too, there are elements of both industry technology, here in $\mathbf{D}_1$, and commodity technology, here in $\mathbf{C}_2$, embedded in the transformation matrix, $\mathbf{T}$. Total requirements matrices under this different mixed technology assumption are found by replacing $\mathbf{C}^{-1}$, $\mathbf{D}$ or $\mathbf{R}$ by $\mathbf{T}$ in each case.

5.7.3 Numerical Examples with Mixed Technology Assumptions

We continue with the same set of hypothetical data derived from Table 5.3. In particular

$$\mathbf{B} = \begin{bmatrix} .1333 & .0727 \\ .1111 & .0636 \end{bmatrix} \quad \mathbf{V} = \begin{bmatrix} 90 & 0 \\ 10 & 100 \end{bmatrix}$$

Suppose that we decompose $\mathbf{V}$ as follows:[36]

$$\mathbf{V}_1 = \begin{bmatrix} 90 & 0 \\ 3 & 100 \end{bmatrix} \text{ and } \mathbf{V}_2 = \begin{bmatrix} 0 & 0 \\ 7 & 0 \end{bmatrix}$$

Example 1: Commodity Technology in $\mathbf{V}_1$ Here we assume that $\mathbf{V}_1$ reflects commodity technology and $\mathbf{V}_2$ embodies industry technology. The necessary pieces of information are

$$\mathbf{x}_1 = \begin{bmatrix} 90 \\ 103 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 0 \\ 7 \end{bmatrix}, \mathbf{q}_1 = \begin{bmatrix} 93 \\ 100 \end{bmatrix}, \mathbf{q}_2 = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$$

so that

$$\mathbf{C}_1 = \begin{bmatrix} 90 & 3 \\ 0 & 100 \end{bmatrix} \begin{bmatrix} 90 & 0 \\ 0 & 103 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & .0291 \\ 0 & .9709 \end{bmatrix}$$

$$\mathbf{D}_2 = \begin{bmatrix} 0 & 0 \\ 7 & 0 \end{bmatrix} \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 \\ .07 & 0 \end{bmatrix}$$

and, from (5.39),

$$\mathbf{R} = \begin{bmatrix} .93 & -.03 \\ .07 & 1.03 \end{bmatrix}$$

Notice that, for this numerical illustration, $\mathbf{i}'\mathbf{C}_1 = \mathbf{i}'$ and $\mathbf{i}'\mathbf{R} = \mathbf{i}'$. [Problem 5.6 asks the reader to show that this is always the case, using the definitions in (5.35) and (5.39).]

[36] With such small (2 × 2) examples, there is relatively little flexibility in the ways that the elements of $\mathbf{V}$ can be split between $\mathbf{V}_1$ and $\mathbf{V}_2$, especially since it is likely that all of the production represented in v_{11} and v_{22} would be assigned to either $\mathbf{V}_1$ or $\mathbf{V}_2$. (Of course, at such a high level of aggregation as we have in 2 × 2 examples, it is impossible to imagine that either commodity is in fact just one product.)

From this information, we can find the four total requirements matrices as

$$\begin{array}{cc} \textit{Commodity-by-Commodity} & \textit{Commodity-by-Industry} \\ (\mathbf{I}-\mathbf{BR})^{-1} = \begin{bmatrix} 1.1591 & .0876 \\ .1332 & 1.0764 \end{bmatrix} & \mathbf{R}^{-1}(\mathbf{I}-\mathbf{RB})^{-1} = \begin{bmatrix} 1.2372 & .1211 \\ .0644 & 1.0469 \end{bmatrix} \\ \textit{Industry-by-Commodity} & \textit{Industry-by-Industry} \\ \mathbf{R}(\mathbf{I}-\mathbf{BR})^{-1} = \begin{bmatrix} 1.0739 & .0492 \\ .2184 & 1.1148 \end{bmatrix} & (\mathbf{I}-\mathbf{RB})^{-1} = \begin{bmatrix} 1.1487 & .0812 \\ .1530 & 1.0868 \end{bmatrix} \end{array}$$

If we compare the matrices in section 5.4.2, we see that these results lie somewhere between those for the "pure" industry technology model and those for the "pure" commodity technology model.

Example 2: Industry Technology in $\mathbf{V}_1$ Now we attribute production in $\mathbf{V}_1$ to the industry technology assumption. In this case, we have

$$\mathbf{D}_1 = \begin{bmatrix} 90 & 0 \\ 3 & 100 \end{bmatrix} \begin{bmatrix} 93 & 0 \\ 0 & 100 \end{bmatrix}^{-1} = \begin{bmatrix} .9677 & 0 \\ .0323 & 1 \end{bmatrix}$$

$$\mathbf{C}_2 = \begin{bmatrix} 0 & 7 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 90 & 0 \\ 0 & 110 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & .0636 \\ 0 & 0 \end{bmatrix}$$

from which

$$\mathbf{T} = \begin{bmatrix} .9656 & -.0656 \\ .0344 & 1.0656 \end{bmatrix}$$

These results illustrate that $\mathbf{i}'\mathbf{D}_1 = \mathbf{i}'$ and $\mathbf{i}'\mathbf{T} = \mathbf{i}'$. [Again, problem 5.6 asks for a general proof of this, using in this case (5.40) and (5.45).] The four total requirements matrices are found to be

$$\begin{array}{cc} \textit{Commodity-by-Commodity} & \textit{Commodity-by-Industry} \\ (\mathbf{I}-\mathbf{BT})^{-1} = \begin{bmatrix} 1.1617 & .0850 \\ .1354 & 1.0743 \end{bmatrix} & \mathbf{T}^{-1}(\mathbf{I}-\mathbf{TB})^{-1} = \begin{bmatrix} 1.1976 & .1535 \\ .1041 & 1.0146 \end{bmatrix} \\ \textit{Industry-by-Commodity} & \textit{Industry-by-Industry} \\ \mathbf{T}(\mathbf{I}-\mathbf{BT})^{-1} = \begin{bmatrix} 1.1129 & .0116 \\ .1842 & 1.1477 \end{bmatrix} & (\mathbf{I}-\mathbf{TB})^{-1} = \begin{bmatrix} 1.1496 & .0816 \\ .1521 & 1.0864 \end{bmatrix} \end{array}$$

Here, also, the total requirements matrices lie between those for the two "pure" technology model illustrations in section 5.4.2.

5.7.4 Additional Mixed Technology Variants

Yet another variant is what is known as the *by-product technology* model. Here *all* secondary production is categorized as by-products, and these are treated as negative inputs. Since u_{ij} is total input of commodity i for production by industry j, the *net* input of commodity i into industry j becomes $u_{ij} - v_{ji}$ (for $i \neq j$). Using a hat on a *matrix* $\mathbf{M}$ (square or otherwise) to denote a diagonal matrix whose elements are the m_{ii} elements in $\mathbf{M}$, and using an "upside down" hat to denote the matrix $\mathbf{M}$ but with its m_{ii} elements replaced by zeros (so that $\mathbf{M} = \hat{\mathbf{M}} + \check{\mathbf{M}}$), the *net* use matrix is seen to be

$$\mathbf{U} - \check{\mathbf{V}}' = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} - \begin{bmatrix} 0 & v_{21} \\ v_{12} & 0 \end{bmatrix}$$

and the technology matrix relating commodity inputs to commodity outputs, under the by-product technology assumption, can be represented as

$$\mathbf{A}_B = (\mathbf{U} - \check{\mathbf{V}}')(\hat{\mathbf{V}})^{-1} \tag{5.46}$$

For our numerical example,

$$\mathbf{A}_B = \left\{ \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} - \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} 90 & 0 \\ 0 & 100 \end{bmatrix}^{-1} = \begin{bmatrix} .1333 & -.0200 \\ .1111 & .0700 \end{bmatrix}$$

and we see that, as in the commodity technology models, negative elements are possible. The associated total requirements matrix (with a not-surprising negative element) is

$$(\mathbf{I} - \mathbf{A}_B)^{-1} = \begin{bmatrix} 1.1507 & -.0247 \\ .1375 & 1.0723 \end{bmatrix}$$

Other variants are also possible. For example, ten Raa, Chakraborty and Small (1984) suggest a combination of a mixed technology assumption with the by-product technology assumption. Letting $\mathbf{V}_1$ contain primary and *ordinary* secondary products (defined as those for which the commodity technology will be invoked), ten Raa, Chakraborty and Small propose the by-product technology assumption rather than the industry technology assumption for $\mathbf{V}_2$. In this case, the direct inputs matrix, call it $\mathbf{A}_{CB}$, is

$$\mathbf{A}_{CB} = (\mathbf{U} - \mathbf{V}'_2)(\mathbf{V}'_1)^{-1} \tag{5.47}$$

For the numerical example, this works out to be

$$\mathbf{A}_{CB} = \left\{ \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} - \begin{bmatrix} 0 & 7 \\ 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} 90 & 3 \\ 0 & 100 \end{bmatrix}^{-1} = \begin{bmatrix} .1333 & .0060 \\ .1111 & .0667 \end{bmatrix}$$

and

$$(\mathbf{I} - \mathbf{A}_{CB})^{-1} = \begin{bmatrix} 1.1548 & .0074 \\ .1375 & 1.0723 \end{bmatrix}$$

If the commodity technology model is invoked for *all* secondary products, then $\mathbf{V}_1 = \mathbf{V}$, $\mathbf{V}_2 = 0$ and so $\mathbf{A}_{CB} = \mathbf{A}_B = \mathbf{A}_C$, as in (5.26). On the other hand, if all secondary products are by-products for which industry technology is used, then $\mathbf{V}_1 = \hat{\mathbf{V}}$, $\mathbf{V}_2 = \check{\mathbf{V}}$ and so $\mathbf{A}_{CB} = \mathbf{A}_B$ in (5.46). The distinction between which secondary products are ordinary or by-products is an empirical one; ten Raa, Chakraborty and Small investigate the question using regression analysis to examine whether primary and secondary output of an industry are proportional. If they are, the secondary output is classified as a by-product; if they are not, it is classified as an ordinary secondary product.

Negative numbers in the direct inputs matrix will arise in the by-product technology model when an industry produces more of a particular commodity (as a by-product) than it uses in production. One brute force approach changes these negatives to zeros. This is equivalent to setting negative elements in the "net transactions" matrix, $\mathbf{U}-\check{\mathbf{V}}'$, in (5.46), to zero. The logic is: if a particular element $u_{ij} - v_{ji}$ $(i \neq j)$ is non-negative, then the by-product approach correctly records the net use of commodity i by industry j; if $u_{ij} - v_{ji} < 0$, then as a first approximation one could assume that all of j's needs for i goods are met from j's own production and so j's net use of commodity i is zero. Following this approach for our example,

$$\mathbf{A}_{B_1} = \begin{bmatrix} .1333 & 0 \\ .1111 & .0700 \end{bmatrix} \quad \text{and} \quad (\mathbf{I} - \mathbf{A}_{B_1})^{-1} = \begin{bmatrix} 1.1538 & 0 \\ .1379 & 1.0753 \end{bmatrix}$$

An even more radical approach would be to just ignore all secondary production, after netting it out of an industry's primary production. In this case, $\check{\mathbf{V}}$ becomes a null matrix (primary production is the only production that is accounted for), so, from (5.46),

$$\mathbf{A}_{B_2} = \mathbf{U}\hat{\mathbf{V}}^{-1} = \begin{bmatrix} .1333 & .0800 \\ .1111 & .0700 \end{bmatrix} \text{ and } (\mathbf{I} - \mathbf{A}_{B_2})^{-1} = \begin{bmatrix} 1.1667 & .1004 \\ .1394 & 1.0873 \end{bmatrix}$$

The most radical approach of all would be to force the assumption of no secondary production at all by collecting all of the elements in each row of $\mathbf{V}$ into the on-diagonal element. This then simply reduces to the original Leontief system. In this example, $\mathbf{V}$ would be replaced by $\boldsymbol{\mathcal{V}} = \langle \mathbf{Vi} \rangle = \begin{bmatrix} 90 & 0 \\ 0 & 110 \end{bmatrix}$. Using $\boldsymbol{\mathcal{V}}$ in (5.46) (note that $\hat{\boldsymbol{\mathcal{V}}} = \boldsymbol{\mathcal{V}}$ and $\check{\boldsymbol{\mathcal{V}}} = \mathbf{0}$)

$$\mathbf{A}_L = \mathbf{U}\boldsymbol{\mathcal{V}}^{-1} = \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} \begin{bmatrix} 90 & 0 \\ 0 & 110 \end{bmatrix}^{-1} = \begin{bmatrix} .1333 & .0727 \\ .1111 & .0636 \end{bmatrix} \tag{5.48}$$

and

$$(\mathbf{I} - \mathbf{A}_L)^{-1} = \begin{bmatrix} 1.1655 & .0905 \\ .1383 & 1.0787 \end{bmatrix}$$

In this case, $\boldsymbol{\mathcal{V}} = \hat{\mathbf{x}} = \hat{\mathbf{q}}$, and all four direct requirements matrices in Section 5.3.6 – $\underset{(c\times c)}{\mathbf{A}_C}$, $\underset{(i\times i)}{\mathbf{A}_C}$, $\underset{(c\times c)}{\mathbf{A}_I}$, and $\underset{(i\times i)}{\mathbf{A}_I}$ – will be the same [and equal to $\mathbf{A}_L$ in (5.48)], and the commodity–industry accounting has been completely swept under the rug.

5.8 Summary

In this chapter we have explored the issues that are introduced by a commodity-by-industry accounting approach. This framework was introduced primarily as an attempt to accommodate the real-world fact that sectors (industries) generally produce more

Table 5.10 Share of Secondary Product Output in Total Industry Output (European Union Countries, 60-sector level)

	1995	1999	2003
Highest	19.0 (Czech Republic)	16.7 (Belgium)	12.0 (Hungary)
Lowest	1.8 (France)	1.7 (Greece)	3.1 (Luxembourg)
EU Average	6.1	6.2	7.4

Source: Selected from Eurostat/European Commission, 2008, Table 11.8, p. 308.

than one product, thereby violating the "one industry/one product" assumption of the original input–output model (as in Chapter 2). This leads to the possibility of "rectangular" input–output systems, where the number of commodities (products) need not be the same as the number of industries. In turn, this rectangularity leads to computational problems insofar as inverse matrices are concerned. However, the commodity-by-industry approach is also completely valid when the number of commodities and industries is the same; that is, it still allows for the more realistic representation of economies in which some industries produce more than one product.

We conclude with Table 5.10, extracted from a larger table in the Eurostat Manual (Eurostat/Economic Commission, 2008) that records the actual amount of secondary production in some 24 countries of the EU annually from 1995–2003 (not all countries are reported for all years). These figures may help to put into perspective how significant secondary activity is (or is not) in these countries. The Manual concludes (p. 309):

> In most European countries the reported level of secondary products of industries as well as the production of products in secondary industries is relatively low. [Thus] the difference between product-by-product input–output tables and industry-by-industry input–output tables is relatively small. Both transformations can be regarded as valid options for impact analysis.

Appendix 5.1 Alternative Approaches to the Derivation of Transactions Matrices

The generation of commodity-by-commodity or industry-by-industry transactions matrices from the commodity-by-industry transaction data in the Use matrix, **U**, can be visualized as a process of "adjusting" **U** to convert it to the proper commodity-by-commodity or industry-by-industry dimensions. Since the dimensions of **U** are commodities-by-industries, the adjustment must be one that either (1) replaces commodity rows by industry rows or (2) replaces industry columns by commodity columns. In what follows, we assume that the appropriate technical coefficients matrix is known and we examine the consequent structure of the corresponding transactions matrix.

A5.1.1 Industry Technology

Commodity-by-Commodity Requirements Here, $\underset{(c\times c)}{\mathbf{A}_I} = \mathbf{BD}$; therefore (parallel to $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} \Rightarrow \mathbf{A}\hat{\mathbf{x}} = \mathbf{Z}$ from earlier chapters)

$$\underset{(c\times c)}{\mathbf{Z}_I} = (\mathbf{BD})\hat{\mathbf{q}} = [\mathbf{U}\hat{\mathbf{x}}^{-1}][\mathbf{V}\hat{\mathbf{q}}^{-1}]\hat{\mathbf{q}} = \mathbf{U}\hat{\mathbf{x}}^{-1}\mathbf{V} \tag{A5.1.1}$$

Since $\mathbf{C} = \mathbf{V}'\hat{\mathbf{x}}^{-1}$, then $\mathbf{C}' = \hat{\mathbf{x}}^{-1}\mathbf{V}$, and hence, from (A5.1.1),

$$\underset{(c\times c)}{\mathbf{Z}_I} = \underset{(c\times i)}{\mathbf{U}}\ \underset{(i\times c)}{\mathbf{C}'} \tag{A5.1.2}$$

The use of $\mathbf{D}$ in defining $\mathbf{A} = \mathbf{BD}$ and the consequent definition of $\underset{(c\times c)}{\mathbf{Z}_I}$ as $\mathbf{BD}(\hat{\mathbf{q}})$ in (A5.1.1) identifies the industry technology assumption. The matrix $\mathbf{C}$ is associated with the commodity technology assumption; its appearance ($\mathbf{C}'$) in the representation of $\underset{(c\times c)}{\mathbf{Z}_I}$ in (A5.1.2) is a matter of algebraic convenience; it is simply a result of the algebraic definition of $\mathbf{C}$ and the algebraic rearrangement of (A5.1.1) that this definition makes possible.

Postmultiplication of $\mathbf{U}$ by $\mathbf{C}'$, as in (A5.1.2), serves to rearrange the "destinations" (columns) of commodity sales (rows) to commodity categories of purchasers rather than to industries as purchasers. Since, by definition, columns in $\mathbf{C}$ all sum to one, row sums in $\mathbf{C}'$ are also unity. Thus, $\underset{(c\times c)}{\mathbf{Z}_I}\ \mathbf{i} = \mathbf{UC}'\mathbf{i} = \underset{(c\times i)}{\mathbf{U}}\ \mathbf{i}$; row sums in $\underset{(c\times c)}{\mathbf{Z}_I}$ and $\underset{(c\times i)}{\mathbf{U}}$ are the same. This is as it should be; the redistribution accomplished by $\underset{(c\times c)}{\mathbf{Z}_I} = \mathbf{UC}'$ does not change the total intermediate sales of any commodity, only the names that are given to the purchasers.

Ritz (1980, p. 41) has observed that (A5.1.2) can be re-expressed as

$$\underset{(c\times c)}{\mathbf{Z}_I} = \mathbf{U}(\mathbf{I} + \mathbf{C}' - \mathbf{I}) = \mathbf{U} + \mathbf{U}(\mathbf{C}' - \mathbf{I}) \tag{A5.1.3}$$

and that the operation in the $\mathbf{U}(\mathbf{C}' - \mathbf{I})$ term "incorporates the 'mechanical redefinitions' required to shift inputs and create a commodity-by-commodity use matrix (or, in other words, to make the industry classification scheme conform precisely to the commodity classification scheme)." [Since $\mathbf{C}'\mathbf{i} = \mathbf{i}$ and $\mathbf{Ii} = \mathbf{i}$, $(\mathbf{C}' - \mathbf{I})\mathbf{i} = \mathbf{0}$ and in this form it remains true that $\underset{(c\times c)}{\mathbf{Z}_I}\ \mathbf{i} = \mathbf{Ui}$.] The numerical example that follows illustrates the logic of this adjustment technique.

Using the data from the Table 5.3 example, $\underset{(c\times c)}{\mathbf{Z}_I}$ in (A5.1.3) is found as

$$\begin{aligned}
\underset{(c\times c)}{\mathbf{Z}_I} &= \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} + \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} \left\{ \begin{bmatrix} 1 & 0 \\ .0909 & .9091 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \\
&= \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} + \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ .0909 & -.0909 \end{bmatrix} \\
&= \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} + \begin{bmatrix} .7272 & -.7272 \\ .6363 & -.6363 \end{bmatrix} \\
&= \begin{bmatrix} 12.7272 & 7.2728 \\ 10.6363 & 6.3637 \end{bmatrix} \qquad \text{(A5.1.4)}
\end{aligned}$$

Consider, for illustration, the upper-right element; $z_{12} = 8 + (12)(0) + (8)(-.0909) = 7.2728$. The original 8 is modified to reflect the fact that commodity 2 comprises only 90.9 percent of industry 2's total output. Thus $100 - 90.91 = 9.09$ percent of the input of commodity 1 to industry 2 must have been used for the production of commodity 1 and $(8)\times(.0909) = .7272$ is therefore netted out of the original u_{12} (commodity-by-industry) transaction of 8. On the other hand, 12 units of commodity 1 are used as inputs to industry 1 production (u_{11}). We know from the **C** matrix that industry 1 produces no commodity 2, so none of this transaction $[(12)\times(0) = 0]$ should be added in to produce the estimate of z_{12}. The logic behind other elements in **Z** in (A5.1.4) is similar.

Industry-by-Industry Requirements Here, $\mathbf{A} = \mathbf{DB}$; therefore, since $\mathbf{U} = \mathbf{B}\hat{\mathbf{x}}$,

$$\underset{(i\times i)}{\mathbf{Z}_I} = (\mathbf{DB})\hat{\mathbf{x}} = \underset{(i\times c)}{\mathbf{D}}\ \underset{(c\times i)}{\mathbf{U}} \qquad \text{(A5.1.5)}$$

Premultiplication of **U** by **D** serves to rearrange the "origins" (rows) of industry purchases (columns) to industry categories of sellers rather than commodity categories. By definition, columns in **D** all sum to one, $\mathbf{i}'\underset{(i\times i)}{\mathbf{Z}_I} = \mathbf{i}'\mathbf{DU} = \mathbf{i}'\underset{(c\times i)}{\mathbf{U}}$. Column sums in $\underset{(i\times i)}{\mathbf{Z}_I}$ and $\underset{(c\times i)}{\mathbf{U}}$ are the same, which is also as it should be; the redistribution accomplished by $\underset{(i\times i)}{\mathbf{Z}_I} = \mathbf{DU}$ should not change the total intermediate purchases by any industry, only the names given to the sellers (rows).

Alternatively, since $(\mathbf{I} + \mathbf{D} - \mathbf{I}) = \mathbf{D}$, $\underset{(i\times i)}{\mathbf{Z}_I}$ in (A5.1.5) can be expressed as

$$\underset{(i\times i)}{\mathbf{Z}_I} = (\mathbf{I} + \mathbf{D} - \mathbf{I})\mathbf{U} = \mathbf{U} + (\mathbf{D} - \mathbf{I})\mathbf{U} \qquad \text{(A5.1.6)}$$

and in this case the $(\mathbf{D} - \mathbf{I})\mathbf{U}$ term provides the adjustment elements to convert **U** to an industry-by-industry Use matrix. Since $\mathbf{i}'\mathbf{D} = \mathbf{i}'$ and $\mathbf{i}'\mathbf{I} = \mathbf{i}'$, $\mathbf{i}'(\mathbf{D} - \mathbf{I}) = \mathbf{0}$, and column sums in $\underset{(i\times i)}{\mathbf{Z}_I}$ and **U** remain equal.

From the numerical example in the text,

$$\underset{(i\times i)}{\mathbf{Z}_I} = \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} + \left\{ \begin{bmatrix} .9 & 0 \\ .1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix}$$
$$= \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} + \begin{bmatrix} -.1 & 1 \\ .1 & 0 \end{bmatrix} \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} = \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} + \begin{bmatrix} -1.2 & -.8 \\ 1.2 & .8 \end{bmatrix}$$
$$= \begin{bmatrix} 10.8 & 7.2 \\ 11.2 & 7.8 \end{bmatrix} \tag{A5.1.7}$$

Again, consider the upper-right element; $z_{12} = 8 + (-0.1)(8) + (0)(7) = 7.2$. The original 8, which is commodity 1 inputs to industry 2, now needs to be converted to industry 1 inputs to industry 2. From the **D** matrix, we know that 90 percent of commodity 1 is produced by industry 1. Therefore $100 - 90 = 10$ percent of the original 8 must be netted out, since it represents industry 2's production of commodity 1. On the other hand, 7 units of commodity 2 are also used by industry 2 (u_{22}). But we know from the **D** matrix that none of this comes from industry 1, so the second (potential) adjustment to the original 8 is $(0)(7) = 0$. Other elements in $\underset{(i\times i)}{\mathbf{Z}_I}$ in (A5.1.7) can be interpreted similarly.

A5.1.2 Commodity Technology

Commodity-by-Commodity Requirements Here $\mathbf{A} = \mathbf{B}\mathbf{C}^{-1}$; therefore

$$\underset{(c\times c)}{\mathbf{Z}_C} = (\mathbf{B}\mathbf{C}^{-1})\hat{\mathbf{q}} = [\mathbf{U}\hat{\mathbf{x}}^{-1}][\hat{\mathbf{x}}(\mathbf{V}')^{-1}]\hat{\mathbf{q}} = \mathbf{U}(\mathbf{V}')^{-1}\hat{\mathbf{q}} \tag{A5.1.8}$$

Since $\mathbf{D} = \mathbf{V}\hat{\mathbf{q}}^{-1}$, $\mathbf{D}' = \hat{\mathbf{q}}^{-1}\mathbf{V}'$ and $(\mathbf{D}')^{-1} = (\mathbf{V}')^{-1}\hat{\mathbf{q}}$; therefore

$$\underset{(c\times c)}{\mathbf{Z}_C} = \underset{(c\times i)}{\mathbf{U}} \; (\underset{(i\times c)}{\mathbf{D}'})^{-1} \tag{A5.1.9}$$

This requires that **V** and hence **V**′ be nonsingular, so that **C** and **D**′ are nonsingular also. The fact that this is a commodity-by-commodity transactions matrix under the commodity technology assumption is emphasized by the fact that the matrix **C** (more precisely, $\mathbf{C}^{-1}$) is used in defining $\mathbf{A} = \mathbf{B}\mathbf{C}^{-1}$ and $\underset{(c\times c)}{\mathbf{Z}_C} = (\mathbf{B}\mathbf{C}^{-1})\hat{\mathbf{q}}$ in (A5.1.8). The matrix **D** (industry technology assumption) appears only because of the algebraic rearrangement which the fact that $\mathbf{D} = \mathbf{V}\hat{\mathbf{q}}^{-1}$ makes possible. This is parallel to the way in which the **C** matrix crept into the commodity-by-commodity requirements expressions under the industry technology assumption, in (A5.1.1) and (A5.1.2).

As in (A5.1.2), postmultiplication of **U** serves to rearrange the destinations (columns) of the commodity sales (rows) – again, the relabeling is from industries to commodities as purchasers. We know that column sums of **D** are all unity, so $\mathbf{D}'\mathbf{i} = \mathbf{i}$ from which it follows that $(\mathbf{D}')^{-1}\mathbf{i} = \mathbf{i}$. Therefore, from (A5.1.9), $\underset{(c\times c)}{\mathbf{Z}_C}\mathbf{i} = \underset{(c\times i)}{\mathbf{U}}\mathbf{i}$ – row sums of $\underset{(c\times c)}{\mathbf{Z}_C}$

and $\underset{(c\times i)}{\mathbf{U}}$ are the same, as is to be expected and as we also saw was true for $\underset{(c\times c)}{\mathbf{Z}_I}$ and $\underset{(c\times i)}{\mathbf{U}}$, in (A5.1.2), above, under the industry technology assumption.

Alternatively, using the same kind of algebraic reasoning as previously, (A5.1.9) can be written as

$$\underset{(c\times c)}{\mathbf{Z}_C} = \mathbf{U}[\mathbf{I} + (\mathbf{D}')^{-1} - \mathbf{I}] = \mathbf{U} + \mathbf{U}[(\mathbf{D}')^{-1} - \mathbf{I}] \tag{A5.1.10}$$

and in this case the matrix $[(\mathbf{D}')^{-1} - \mathbf{I}]$ supplies the adjustment terms to convert the original Use matrix to commodity-by-commodity terms, but now under the assumption of a commodity-based technology [as opposed to the adjustment in (A5.1.3), again to commodity-by-commodity terms, but under an industry technology assumption]. It is easily shown that $[(\mathbf{D}')^{-1} - \mathbf{I}]\mathbf{i} = \mathbf{0}$, so again row sums are not altered by the relabeling of the columns of $\mathbf{U}$.

From the numerical example,

$$\begin{aligned}
\underset{(c\times c)}{\mathbf{Z}_C} &= \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} + \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix}\left\{\begin{bmatrix} 1.1111 & -.1111 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} \\
&= \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} + \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix}\begin{bmatrix} 1.1111 & -.1111 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} + \begin{bmatrix} 1.3332 & -1.3332 \\ 1.1111 & -1.1111 \end{bmatrix} = \begin{bmatrix} 13.3332 & 6.6668 \\ 11.1111 & 5.8889 \end{bmatrix}
\end{aligned} \tag{A5.1.11}$$

Note that this differs from $\underset{(c\times c)}{\mathbf{Z}_I}$ in (A5.1.4), which was derived under the assumption of an industry technology. While the numerical example in (A5.1.11) illustrates the parallels with the adjustments in (A5.1.4) and (A5.1.7), interpretation of the elements in $[(\mathbf{D}')^{-1} - \mathbf{I}]$ is not as straightforward as in the case of $(\mathbf{C}' - \mathbf{I})$ in (A5.1.4) or $(\mathbf{D} - \mathbf{I})$ in (A5.1.7). The operations in this case are more easily understood by going back to (A5.1.8) or (A5.1.9) and noting that

$$\mathbf{U} = \underset{(c\times c)}{\mathbf{Z}_C}[(\mathbf{V}')^{-1}\hat{\mathbf{q}}]^{-1} \text{ or } \mathbf{U} = \underset{(c\times c)}{\mathbf{Z}_C}\mathbf{D}' \tag{A5.1.12}$$

That is, *if* the commodity-by-commodity Use matrix, $\underset{(c\times c)}{\mathbf{Z}_C}$, were known, then post-multiplication by $\mathbf{D}'$ would serve to rearrange the column labels (purchasers) from commodity groups to industry groups. From the numerical example,

$$\underset{(c\times i)}{\mathbf{D}'} = \begin{bmatrix} .9 & .1 \\ 0 & 1 \end{bmatrix}$$

and so

$$\underset{(c\times i)}{\mathbf{U}} = \underset{(c\times c)}{\mathbf{Z}_C}\underset{(c\times i)}{\mathbf{D}'} = \begin{bmatrix} (z_{11})(.9) + (z_{12})(0) & (z_{11})(.1) + (z_{12})(1) \\ (z_{21})(.9) + (z_{22})(0) & (z_{21})(.1) + (z_{22})(1) \end{bmatrix} \tag{A5.1.13}$$

Consider element u_{12} in this matrix (total input of commodity 1 to industry 2). The second term, $(z_{12})(1)$, reflects the fact that 100 percent of the output of commodity 2 is made by industry 2, and so *all* of the sales of commodity 1 to commodity 2 production (z_{12}) can be thought of equally well as purchases by industry 2. In addition, however, since 10 percent of the output of commodity 1 is produced by industry 2, 10 percent of the sales of commodity 1 for commodity 1 production (z_{11}) would be purchased by industry 2. Thus $u_{12} = (z_{11})(.1)+(z_{12})(1)$. Other elements can be interpreted similarly.

Industry-by-Industry Requirements Here $\mathbf{A} = \mathbf{C}^{-1}\mathbf{B}$ and

$$\underset{(i\times i)}{\mathbf{Z}_C} = (\mathbf{C}^{-1}\mathbf{B})\hat{\mathbf{x}} = \underset{(i\times c)}{\mathbf{C}^{-1}}\underset{(c\times i)}{\mathbf{U}} \tag{A5.1.14}$$

Since $\mathbf{i}'\mathbf{C} = \mathbf{i}'$, $\mathbf{i}'\mathbf{C}^{-1} = \mathbf{i}'$ also, column sums of $\underset{(i\times i)}{\mathbf{Z}_C}$ and $\underset{(c\times i)}{\mathbf{U}}$ are the same. The premultiplication of $\mathbf{U}$ by $\mathbf{C}^{-1}$ serves to relabel the rows from commodities to industries as sellers. Following the earlier examples, $\underset{(i\times i)}{\mathbf{Z}_C}$ in (A5.1.14) can be re-expressed as

$$\underset{(i\times i)}{\mathbf{Z}_C} = (\mathbf{I} + \mathbf{C}^{-1} - \mathbf{I})\mathbf{U} = \mathbf{U} + (\mathbf{C}^{-1} - \mathbf{I})\mathbf{U} \tag{A5.1.15}$$

and now it is clear that the matrix $(\mathbf{C}^{-1} - \mathbf{I})$ represents the adjustment mechanism for converting $\mathbf{U}$ to an industry-by-industry basis, under the commodity-based technology assumption. All column sums in the adjustment (second) term in (A5.1.15) are zero, so the row relabeling leaves column sums of $\underset{(i\times i)}{\mathbf{Z}_C}$ and $\underset{(c\times i)}{\mathbf{U}}$ equal.

Using the numerical example

$$\begin{aligned}
\underset{(i\times i)}{\mathbf{Z}_C} &= \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} + \left\{ \begin{bmatrix} 1 & -.1 \\ 0 & 1.1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} \\
&= \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} + \begin{bmatrix} 0 & -.1 \\ 0 & .1 \end{bmatrix} \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} = \begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix} + \begin{bmatrix} -1 & -.7 \\ 1 & .7 \end{bmatrix} \\
&= \begin{bmatrix} 11 & 7.3 \\ 11 & 7.7 \end{bmatrix}
\end{aligned} \tag{A5.1.16}$$

This differs (although in this case only slightly) from the $\underset{(i\times i)}{\mathbf{Z}_I}$ matrix in (A5.1.7), derived under the industry technology assumption.

As with the commodity-by-commodity requirements case immediately above, the logic of this adjustment is easily seen by rewriting (A5.1.14) as

$$\underset{(c\times i)}{\mathbf{C}}\ \underset{(i\times i)}{\mathbf{Z}_C} = \underset{(c\times i)}{\mathbf{U}} \tag{A5.1.17}$$

If the industry-by-industry Use matrix, $\underset{(i\times i)}{\mathbf{Z}_C}$, were known, premultiplication by $\underset{(c\times i)}{\mathbf{C}}$ would rearrange row labels (sellers) from industries to commodities. Here

$$\mathbf{C} = \begin{bmatrix} 1 & .0909 \\ 0 & .9091 \end{bmatrix}$$

and therefore

$$\underset{(c\times i)}{\mathbf{U}} = \underset{(c\times i)}{\mathbf{C}}\,\underset{(i\times i)}{\mathbf{Z}_C} = \begin{bmatrix} (1)(z_{11}) + (.0909)(z_{21}) & (1)(z_{12}) + (.0909)(z_{22}) \\ (0)(z_{11}) + (.9091)(z_{21}) & (0)(z_{12}) + (.9091)(z_{22}) \end{bmatrix} \quad \text{(A5.1.18)}$$

Again, we consider u_{12}, which represents total inputs of commodity 1 into industry 2 production. From the first row of **C**, we know that 100 percent of industry 1's output consists of commodity 1 and that 9.09 percent of industry 2's output is commodity 1. Thus all of the industry-to-industry transactions z_{12} can be viewed as commodity 1 sales to industry 2, while the commodity 1 composition of the industry-to-industry transaction z_{22} is represented by (.0909) (z_{22}). Thus $u_{12} = (1)(z_{12}) + (.0909)(z_{22})$. Other elements in (A5.1.18) have similar interpretations.

Appendix 5.2 Elimination of Negatives in Commodity Technology Models

A5.2.1 The Problem

A major practical problem with the commodity technology assumption is the very real possibility of negative entries appearing in the direct requirements matrix and hence also in the associated transactions matrix. We explored this in section 5.5. Table A5.2.1 summarizes some of the results from the 2 × 2 numerical illustrations in that section.

These examples might be viewed as overly simplistic (too small), or exaggerated, since 40 percent and more of commodity 1 is produced in industry 2 (v_{21} is large relative to v_{11}). Here are some larger examples that also generate negative elements in direct inputs matrices (not shown) and therefore also in the associated transactions matrices (shown).

3 × 3 *Example*

$$\mathbf{U} = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 5 & 2 \\ 6 & 1 & 3 \end{bmatrix} \quad \mathbf{V} = \begin{bmatrix} 20 & 2 & 1 \\ 5 & 25 & 7 \\ 3 & 2 & 15 \end{bmatrix} \quad \underset{(c\times c)}{\mathbf{Z}_C} = \begin{bmatrix} 5.340 & -.667 & 5.327 \\ 2.225 & 4.556 & 2.219 \\ 8.364 & -1.778 & 3.414 \end{bmatrix}$$

4 × 4 *Example*

$$\mathbf{U} = \begin{bmatrix} 1 & 4 & 5 & 6 \\ 3 & 1 & 2 & 5 \\ 10 & 6 & 1 & 7 \\ 15 & 3 & 4 & 2 \end{bmatrix} \quad \mathbf{V} = \begin{bmatrix} 30 & 4 & 10 & 10 \\ 8 & 20 & 5 & 8 \\ 5 & 1 & 50 & 2 \\ 5 & 5 & 5 & 60 \end{bmatrix}$$

Table A5.2.1 Summary of Two Commodity/Two Industry Results

$\mathbf{U}$	$\mathbf{V}$	$\underset{(c\times c)}{\mathbf{A_C}}$	$\underset{(c\times c)}{\mathbf{Z_C}}$
$\begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix}$	$\begin{bmatrix} 90 & 0 \\ 60 & 100 \end{bmatrix}$	$\begin{bmatrix} .1333 & 0 \\ .1111 & .0033 \end{bmatrix}$	$\begin{bmatrix} 20 & 0 \\ 16.67 & .33 \end{bmatrix}$
$\begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix}$	$\begin{bmatrix} 90 & 0 \\ 61 & 100 \end{bmatrix}$	$\begin{bmatrix} .1333 & -.0013 \\ .1111 & .0022 \end{bmatrix}$	$\begin{bmatrix} 20.13 & -.13 \\ 16.78 & .22 \end{bmatrix}$
$\begin{bmatrix} 12 & 8 \\ 10 & 7 \end{bmatrix}$	$\begin{bmatrix} 90 & 0 \\ 64 & 100 \end{bmatrix}$	$\begin{bmatrix} .1333 & -.0053 \\ .1111 & -.0011 \end{bmatrix}$	$\begin{bmatrix} 20.53 & -.53 \\ 17.11 & -.11 \end{bmatrix}$

$$\underset{(c\times c)}{\mathbf{Z_C}} = \begin{bmatrix} -2.327 & 4.856 & 6.882 & 6.589 \\ 3.181 & -.443 & 2.143 & 6.119 \\ 13.961 & 4.656 & -1.079 & 6.462 \\ 23.949 & -1.648 & 2.202 & -.503 \end{bmatrix}$$

5×5 *Example (from Almon, 2000)*

$$\mathbf{U} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 37 & 0 & 0 & 0 \\ 15 & 5 & 0 & 0 & 0 \\ 28 & 72 & 30 & 5 & 0 \end{bmatrix} \quad \mathbf{V} = \begin{bmatrix} 70 & 20 & 0 & 0 & 0 \\ 30 & 180 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 20 & 0 \\ 0 & 0 & 0 & 0 & 535 \end{bmatrix}$$

$$\underset{(c\times c)}{\mathbf{Z}_C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1.67 & 41.67 & 0 & 0 & 0 \\ 21.67 & -1.67 & 0 & 0 & 0 \\ 30 & 70 & 30 & 5 & 0 \end{bmatrix}$$

A5.2.2 Approaches to Elimination of Negative Elements

Whether or not the reader finds these examples compelling, input–output practitioners repeatedly find that the commodity technology model generates negative elements in real-world applications (section 5.5.1, above). Clearly, if one wants to use a commodity technology input–output model, this is an issue that needs to be addressed. In section 5.5.1 we noted a large amount of literature that addresses this issue in a variety of ways.

Here we examine an approach to negatives that is reported in Almon (2000). He and his associates have used it repeatedly and successfully to transform an observed

Use matrix, $\underset{(c\times i)}{\mathbf{U}}$, into a *non-negative* commodity technology based commodity-by-commodity transactions matrix, $\underset{(c\times c)}{\mathbf{Z}_C}$.[37] The building blocks are the usual commodity–industry accounts – a Use matrix and a Make matrix, $\mathbf{V}$, from which $\mathbf{D} = \mathbf{V}\hat{\mathbf{q}}^{-1}$, as usual. Initially, we will explore the approach for a 3×3 model.[38]

Given

$$\underset{(c\times i)}{\mathbf{U}} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} \text{ and } \underset{(i\times c)}{\mathbf{D}} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$$

we want to find an associated

$$\underset{(c\times c)}{\mathbf{Z}_C} = \mathbf{Z} = \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix}$$

that contains no negative elements.

We have simplified the representation of the elements in $\underset{(c\times c)}{\mathbf{Z}_C}$ (showing them only as z_{ij}) to keep the notation as uncluttered as possible. Equation (A5.1.9) from Appendix 5.1 is our point of departure – $\mathbf{Z} = \mathbf{U}(\mathbf{D}')^{-1}$ or

$$\mathbf{Z}\mathbf{D}' = \mathbf{U} \tag{A5.2.1}$$

Rewriting as $\mathbf{U} - \mathbf{Z}\mathbf{D}' = \mathbf{0}$ and adding $\mathbf{Z}$ to both sides, we have

$$\mathbf{Z} = \mathbf{U} + \mathbf{Z}(\mathbf{I} - \mathbf{D}') \tag{A5.2.2}$$

This suggests an *iterative* approach that could be used to construct an estimate of $\mathbf{Z}$. Let the *next* [$(k+1)$st] estimate, $\mathbf{Z}^{(k+1)}$, depend on the *current* (kth) estimate, $\mathbf{Z}^{(k)}$, and on the elements in $\mathbf{U}$ in the following way:

$$\mathbf{Z}^{(k+1)} = \mathbf{U} + \mathbf{Z}^{(k)}(\mathbf{I} - \mathbf{D}') \tag{A5.2.3}$$

Given the characteristics of $\mathbf{D}$, and hence of $(\mathbf{I} - \mathbf{D}')$, it is possible to show that this kind of sequential procedure will in fact converge.[39] The process begins ($k = 0$) by assuming

$$\mathbf{Z}^{(0)} = \mathbf{U} \tag{A5.2.4}$$

We know that this will turn out to be wrong, since $\mathbf{U}$ has dimensions commodity-by-industry and our transactions matrix $\mathbf{Z}$ must have commodity-by-commodity

[37] The procedure has been in use for decades at the University of Maryland's INFORUM project. It was mentioned in print at least as early as Almon (1970) and also in Almon *et al.* (1974). The associated non-negative direct input coefficients matrix can easily be derived from the non-negative transactions.

[38] As will become clear, a two-commodity and -industry illustration is too small to properly illustrate the intricacies of the technique.

[39] Details are beyond what we need here. See Almon (2000) for further discussion.

dimensions. But notice that because both **U** and our final **Z** have commodities in the row dimension, the transformation from the former to the latter must preserve row sums. Next, from (A5.2.3)

$$\mathbf{Z}^{(1)} = \mathbf{U} + \mathbf{Z}^{(0)}(\mathbf{I} - \mathbf{D}')$$

and so on:

$$\begin{aligned} \mathbf{Z}^{(2)} &= \mathbf{U} + \mathbf{Z}^{(1)}(\mathbf{I} - \mathbf{D}') \\ \vdots \quad & \qquad \vdots \\ \mathbf{Z}^{(n)} &= \mathbf{U} + \mathbf{Z}^{(n-1)}(\mathbf{I} - \mathbf{D}') \end{aligned} \tag{A5.2.5}$$

Consider the process one row at a time. Let ${}_i\mathbf{U} = \begin{bmatrix} u_{i1} & u_{i2} & u_{i3} \end{bmatrix}$ (the ith row of **U**, which is known) and ${}_i\mathbf{Z} = \begin{bmatrix} z_{i1} & z_{i2} & z_{i3} \end{bmatrix}$ (the ith row of **Z**, which we want to find). Then (A5.2.2) can be expressed as

$${}_i\mathbf{Z} = {}_i\mathbf{U} + {}_i\mathbf{Z}(\mathbf{I} - \mathbf{D}') \tag{A5.2.6}$$

for $i = 1, 2, 3$. More explicitly,

$$\begin{bmatrix} z_{i1} & z_{i2} & z_{i3} \end{bmatrix} = \begin{bmatrix} u_{i1} & u_{i2} & u_{i3} \end{bmatrix} + \begin{bmatrix} z_{i1} & z_{i2} & z_{i3} \end{bmatrix} \begin{bmatrix} 1 - d_{11} & -d_{21} & -d_{31} \\ -d_{12} & 1 - d_{22} & -d_{32} \\ -d_{13} & -d_{23} & 1 - d_{33} \end{bmatrix} \tag{A5.2.7}$$

The iterative process in (A5.2.3) can be carried out on each row. That is,

$${}_i\mathbf{Z}^{(k+1)} = {}_i\mathbf{U} + {}_i\mathbf{Z}^{(k)}(\mathbf{I} - \mathbf{D}') \tag{A5.2.8}$$

with

$${}_i\mathbf{Z}^{(0)} = {}_i\mathbf{U} \tag{A5.2.9}$$

and then (A5.2.8), starting with $k = 0$

$${}_i\mathbf{Z}^{(1)} = {}_i\mathbf{U} + {}_i\mathbf{Z}^{(0)}(\mathbf{I} - \mathbf{D}') \tag{A5.2.10}$$

and so on, as in (A5.2.5).

Specifically, for the 3×3 illustration, here are the three linear equations in (A5.2.8) written out explicitly for the $i = 1$ case [elements of the first row of $\mathbf{Z}^{(k+1)}$]:

$$\begin{aligned} z_{11}^{(k+1)} &= u_{11} + (1 - d_{11})z_{11}^{(k)} - d_{12}z_{12}^{(k)} - d_{13}z_{13}^{(k)} \\ z_{12}^{(k+1)} &= u_{12} - d_{21}z_{11}^{(k)} + (1 - d_{22})z_{12}^{(k)} - d_{23}z_{13}^{(k)} \\ z_{13}^{(k+1)} &= u_{13} - d_{31}z_{11}^{(k)} - d_{32}z_{12}^{(k)} + (1 - d_{33})z_{13}^{(k)} \end{aligned}$$

To add further specificity, we use the **D** matrix from the 3×3 numerical example, above:

$$\mathbf{D} = \mathbf{V}(\hat{\mathbf{q}})^{-1} = \begin{bmatrix} 20 & 2 & 1 \\ 5 & 25 & 7 \\ 3 & 2 & 15 \end{bmatrix} \begin{bmatrix} 28 & 0 & 0 \\ 0 & 29 & 0 \\ 0 & 0 & 23 \end{bmatrix}^{-1} = \begin{bmatrix} .7143 & .0690 & .0435 \\ .1786 & .8621 & .3043 \\ .1071 & .0690 & .6522 \end{bmatrix}$$

and so

$$\begin{aligned} z_{11}^{(k+1)} &= 4 + .2857z_{11}^{(k)} - .0690z_{12}^{(k)} - .0435z_{13}^{(k)} \\ z_{12}^{(k+1)} &= 2 - .1786z_{11}^{(k)} + .1379z_{12}^{(k)} - .3043z_{13}^{(k)} \\ z_{13}^{(k+1)} &= 4 - .1071z_{11}^{(k)} - .0690z_{12}^{(k)} + .3478z_{13}^{(k)} \end{aligned} \tag{A5.2.11}$$

In general it is not easy to come up with a convincing set of names for commodities and industries in a small (say 3×3) numerical example – "Agriculture," "Manufacturing," and "Services" are too aggregate to make much sense as "commodities." And at a finer level of detail – e.g., "Cheese," "Ice Cream," and "Other Foodstuffs" – it is usually difficult to create a reasonable numerical illustration unless many of the elements in the Use matrix are zero; for example, ice cream is an unlikely input to cheese manufacturing.[40] At the same time, it is helpful in sorting out what is going on in Equations (A5.2.11) to have some specificity, so (with apologies to Almon, 2000) we opt for 1 = cheese, 2 = ice cream, and 3 = other foodstuffs, without going into a careful analysis of the plausibility of each and every element in **U**.

Consider how the $(k + 1)$st estimate of the input of cheese $(i = 1)$ into production of the commodity other foodstuffs $(j = 3)$ is built up on the basis of the current estimate (iteration k), i.e. $z_{13}^{(k+1)}$, the third of the three equations in (A5.2.11).

We start on the right with u_{13} – the original observed input of cheese into the other foodstuffs *industry* (4 units). Since the other foodstuffs industry made secondary products – cheese $(v_{31} = 3)$ and ice cream $(v_{32} = 2)$ – we need to net out the cheese that was used by other foodstuffs for both of those *non-primary* products. (Remember that we are building a commodity-to-commodity transactions table.) First we deal with the cheese produced secondarily by other foodstuffs. $z_{11}^{(k)}$ is the "current" estimate of *commodity* 1 input into *commodity* 1 production (cheese into cheese). But 10.71 percent of cheese production occurs in other foodstuffs. So $(.1071)z_{11}^{(k)}$ accounts for the cheese used by other foodstuffs to make a secondary product, cheese, and it is netted out of the u_{13} transaction: $(-.1071)z_{11}^{(k)}$. Similarly, $z_{12}^{(k)}$ is the current estimate of commodity 1 input into commodity 2 production (cheese into ice cream). But 6.9 percent of the ice cream produced is as a secondary product for other foodstuffs, so $(-.0690)z_{12}^{(k)}$ nets out from u_{13} other foodstuffs' use of cheese for another of its secondary products.

Finally, we have $(+34.78)z_{13}^{(k)}$. This reflects the fact that 34.78 percent of the *commodity* other foodstuffs is not made by the other foodstuffs industry but rather by the cheese industry (4.35 percent; $d_{13} = .0435$) and by the ice cream industry (30.43 percent; $d_{23} = .3043$). So we need to add to u_{13} these amounts of cheese used elsewhere to make other foodstuffs as a secondary product: $(.0435)z_{13}^{(k)}$ for what is made by cheese and $(.3043)z_{13}^{(k)}$ for what is made by ice cream.

By the property of **D** matrices that column sums are 1, we recognize that the amount added in $z_{13}^{(k+1)}$ – $(.3478)z_{13}^{(k)}$ – is exactly right, namely $(.0435)z_{13}^{(k)} + (.3043)z_{13}^{(k)}$. The absolute amount of cheese to other foodstuffs that is netted out of the first two equations

[40] With larger examples, it is easier, as we will see below in the 5×5 story.

in (A5.2.11) is exactly what is added back into that transaction in the third equation. And the same is true for cheese to cheese and cheese to ice cream adjustments in these equations. The amounts netted out in two of the equations are added back in the other equation – transactions are simply "rearranged" and nothing is "lost."

A5.2.3 Results of the Iterative Procedure

3×3 *Example* Here is the commodity technology based commodity-by-commodity transactions matrix ($\underset{(c\times c)}{\mathbf{Z}^A}$) that results from using the iterative procedure on the 3×3 illustration in section A5.2.1:

$$\underset{(c\times c)}{\mathbf{Z}^A} = \begin{bmatrix} 5.087 & 0 & 4.913 \\ 2.225 & 4.556 & 2.219 \\ 7.270 & 0 & 2.730 \end{bmatrix}$$

Table A5.2.2 indicates a number of the steps in the procedure. Iterations stop for any particular row when some criterion is met – for example, when the differences between successive values for all $z_{ij}^{(k)}$ are less than some prespecified level. We show the results to four decimal places in order to give a feeling for how things develop. (In practice, each row went through additional iterations, until there were changes in the sixth decimal place only, but that level of detail is unnecessary for this simple illustration.)

As we would expect, for rows in $\underset{(c\times c)}{\mathbf{Z}_C}$ in which there are no negative elements, the iterative approach generates exactly the same vector (here row 2) in $\underset{(c\times c)}{\mathbf{Z}^A}$. As the reader can verify, in moving from one iteration to the next, the amount(s) that are subtracted from the original u's in a given row are exactly balanced (except perhaps for rounding) by the amount(s) that are added to other u's in that same row (preservation of row sums from the original $\mathbf{U}$ matrix). For example, for row 1, moving from $k = 0$ to $k = 1$, $(4.8310 - 4) + (4.8248 - 4) = 1.6558 = (2 - .3442)$.[41]

4×4 *Example* The original transactions matrix

$$\underset{(c\times c)}{\mathbf{Z}_C} = \begin{bmatrix} -2.327 & 4.856 & 6.882 & 6.589 \\ 3.181 & -.443 & 2.143 & 6.119 \\ 13.961 & 4.656 & -1.079 & 6.462 \\ 23.949 & -1.648 & 2.202 & -.503 \end{bmatrix}$$

[41] There are additional aspects to the Almon procedure that we need not explore here. They influence the "speed" at which convergence is approached – for example, how much is subtracted, at each iteration, for an element that turns out to be negative in $\underset{(c\times c)}{\mathbf{Z}_C}$.

is transformed into

$$\underset{(c\times c)}{\mathbf{Z}^A} = \begin{bmatrix} 0 & 4.031 & 5.966 & 6.003 \\ 2.983 & 0 & 2.118 & 5.899 \\ 12.859 & 4.730 & 0 & 6.412 \\ 21.646 & 0 & 2.354 & 0 \end{bmatrix}$$

Table A5.2.2 Steps in the Iterative Procedure for the 3 × 3 Example

k	$z_{i1}^{(k)}$	$z_{i2}^{(k)}$	$z_{i3}^{(k)}$
${}_1\mathbf{Z}^{(k)}$ (Row 1)			
0	4	2	4
1	4.8310	.3442	4.8248
2	5.0242	.0475	4.9283
3	5.0693	.0065	4.9241
4	5.0817	.0009	4.9174
5	5.0856	.0001	4.9143
6	5.0868	.0000	4.9132
${}_2\mathbf{Z}^{(k)}$ (Row 2)			
0	2	5	2
1	2.1396	4.7238	2.1365
2	2.1927	4.6192	2.1881
3	2.2128	4.5796	2.2076
4	2.2204	4.5647	2.2149
5	2.2233	4.5590	2.2177
6	2.2244	4.5569	2.2188
7	2.2248	4.5560	2.2191
${}_3\mathbf{Z}^{(k)}$ (Row 3)			
0	6	1	3
1	6.9834	.1379	2.8787
2	7.2009	.0190	2.7801
3	7.2525	.0026	2.7449
4	7.2654	.0004	2.7342
5	7.2688	.0001	2.7312
6	7.2697	.0000	2.7303

5×5 *Example* This example is from Almon (2000), where the unsatisfactory **Z** matrix is shown (with its negative elements) but the modified **Z** that results from the iterative procedure (without negatives) is not. We saw above, in the 5×5 example, that those **U** and **V** matrices generated

$$\underset{(c\times c)}{\mathbf{Z}_C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1.7 & 41.7 & 0 & 0 & 0 \\ 21.7 & -1.7 & 0 & 0 & 0 \\ 30 & 70 & 30 & 5 & 0 \end{bmatrix}$$

containing "the infamous negative flows" (Almon, 2000, p. 31). After using the iterative procedure, we have

$$\underset{(c\times c)}{\mathbf{Z}^A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 40 & 0 & 0 & 0 \\ 20 & 0 & 0 & 0 & 0 \\ 30 & 70 & 30 & 5 & 0 \end{bmatrix}$$

In the relatively plausible story that goes along with this example, the commodities are: (1) cheese, (2) ice cream, (3) chocolate, (4) rennet,[42] and (5) other. Notice that in $\underset{(c\times c)}{\mathbf{Z}^A}$, in particular, there is (thankfully) no chocolate used to make cheese ($z^A_{31} = 0$) nor is there rennet going into (what would turn out to be curdled) ice cream ($z^A_{42} = 0$).

As a final illustration, here is a 5×5 Use matrix that differs very little from the original one that we used:

$$\tilde{\mathbf{U}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 36 & 0 & 0 & 0 \\ 14 & 6 & 0 & 0 & 0 \\ 28 & 72 & 30 & 5 & 0 \end{bmatrix}$$

There is only a one-unit difference in four elements from their original counterparts in $\mathbf{U}$ – $\tilde{u}_{31}$ and $\tilde{u}_{42}$ are one unit larger, $\tilde{u}_{32}$ and $\tilde{u}_{41}$ are one unit smaller. In this case, in conjunction with the original $\mathbf{V}$,

$$\underset{(c\times c)}{\tilde{\mathbf{Z}}_C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 40 & 0 & 0 & 0 \\ 20 & 0 & 0 & 0 & 0 \\ 30 & 70 & 30 & 5 & 0 \end{bmatrix}$$

This is exactly the same as the $\underset{(c\times c)}{\mathbf{Z}^A}$ that was derived from the original 5×5 $\mathbf{U}$ and $\mathbf{V}$, which makes sense. The one-unit variations that differentiated $\mathbf{U}$ from $\tilde{\mathbf{U}}$ were essentially ignored when the iterative procedure created a no-nonsense commodity-to-commodity transactions matrix from $\mathbf{U}$.

This example contains one further (possible) surprise. If we elect to use the industry technology assumption instead of commodity technology, then we find that the transactions matrix is

$$\underset{(c\times c)}{\mathbf{Z}_I} = \mathbf{BD\hat{q}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 8.25 & 31.75 & 0 & 0 & 0 \\ 11.75 & 8.25 & 0 & 0 & 0 \\ 32.06 & 67.94 & 30 & 5 & 0 \end{bmatrix}$$

[42] Rennet is a milk curdling agent, used in making cheese but not ice cream.

This is (appropriately and appealingly) characterized as "massive nonsense," and to call this a commodity-to-commodity table "… would be little short of scandalous" (Almon, 2000, p. 31). Why? Because now we find that chocolate is being used as an input to make cheese and rennet is part of the recipe for ice cream. Bad cooking; and a nice illustration that neither technology assumption is flawless.

Problems

5.1 In a system of commodity-by-industry accounts, suppose we have defined three commodities and two industries. The Make matrix, $\mathbf{V}$, and the Use matrix, $\mathbf{U}$, are given below.

$$\mathbf{U} = \begin{bmatrix} 3 & 5 \\ 2 & 7 \\ 2 & 3 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} 15 & 5 & 10 \\ 5 & 25 & 0 \end{bmatrix}$$

a. Compute the vector of commodity final demands, the vector of industry value added inputs, the vector of total commodity outputs, and the vector of total industry outputs.
b. Assuming an "industry-based" technology, compute the industry-by-commodity total requirements matrix.

5.2 Consider the following system of commodity and industry accounts for a region:

		Commodities		Industries			
		1	2	1	2	Final Demand	Total Output
Commodities	1			1	2	7	10
	2			3	4	3	10
Industries	1	10	2				12
	2	0	8				8
Value Added				8	2	10	
Total Inputs		10	10	12	8		

a. Compute the commodity-by-industry matrix of direct requirements.
b. Compute the industry-by-commodity total requirements matrices under both assumptions of industry-based and commodity-based technology.
c. If a new naval facility is being constructed in the region, represented by commodity final demands $\Delta\mathbf{e} = [6 \quad 5]'$, what would be the total production of each industry in the region required to support this facility? Do this for both technology assumptions.

5.3 Consider again the system of accounts given in problem 5.1. Suppose we can split $\mathbf{V}$ into two components, $\mathbf{V}_1 = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 0 \end{bmatrix}$ and $\mathbf{V}_2 = \begin{bmatrix} 10 & 0 & 5 \\ 0 & 20 & 0 \end{bmatrix}$ such that $\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2$. Which of the two "mixed technology" assumptions that were covered in sections 5.7.1 and 5.7.2 can we invoke in computing the industry-by-commodity total requirements matrix for this system of accounts? Compute the matrix. Why can we not invoke the other assumption? Can we invoke either the commodity-based or industry-based technology assumptions?

5.4 Use both mixed technology assumptions in deriving industry-by-commodity total requirements matrices for the system of accounts given in problem 5.2.

5.5 In a system of commodity-by-industry accounts, suppose we have defined four commodities and three industries. The Make matrix, **V**, and the Use matrix, **U**, are given below:

$$\mathbf{U} = \begin{bmatrix} 20 & 12 & 18 \\ 5 & 30 & 12 \\ 10 & 13 & 11 \\ 12 & 17 & 40 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} 99 & 0 & 0 & 10 \\ 8 & 143 & 137 & 10 \\ 0 & 6 & 12 & 150 \end{bmatrix}$$

a. Is it possible to compute the total commodity-by-industry total requirements matrix using the assumption of industry-based technology? If not, why not. If so, calculate that matrix.

b. Using the assumption of an industry-based technology, calculate the industry-by-commodity requirements matrix for commodity-driven demand.

c. Aggregate the first two commodities to one in the Make and Use matrices. Assume that you can decompose the resulting aggregated **V** into $\mathbf{V}_1$ and $\mathbf{V}_2$, where $\mathbf{V}_1 = \begin{bmatrix} 99 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 30 \end{bmatrix}$. Assume a commodity-based technology for $\mathbf{V}_1$ and an industry-based technology for $\mathbf{V}_2$. Calculate the four total requirements matrices (i.e., commodity-by-commodity, industry-by-commodity, commodity-by-industry, and industry-by-industry) to be used with commodity-driven demand calculations.

5.6 The numerical results in section 5.7.3 illustrate that column sums of both the **R** and **T** matrices are one.

a. Prove that $\mathbf{i}'\mathbf{C}_1 = \mathbf{i}'$ and $\mathbf{i}'\mathbf{R} = \mathbf{i}'$.

b. Prove that $\mathbf{i}'\mathbf{D}_1 = \mathbf{i}'$ and $\mathbf{i}'\mathbf{T} = \mathbf{i}'$.

5.7 For the Make and Use matrices specified in problem 5.5, assume that the three industries are: Agriculture, Oil Production, and Manufacturing. The four commodities are Agricultural Products, Crude Oil, Natural Gas, and Manufactured Products. We can interpret this as meaning in this case that natural gas is considered a secondary product of the oil industry. For a final demand of 100 of manufactured products what levels of oil industry output are generated and how much natural gas production is generated to satisfy this final demand?

5.8 Consider the following Make and Use matrices:

$$\mathbf{U} = \begin{bmatrix} 20 & 15 & 18 \\ 5 & 30 & 12 \\ 10 & 16 & 11 \end{bmatrix}, \mathbf{V} = \begin{bmatrix} 30 & 0 & 0 \\ 10 & 50 & 35 \\ 0 & 25 & 150 \end{bmatrix}$$

Compute the corresponding commodity-by-commodity transactions table using the assumption of commodity-based technology. Notice that there are negative elements.

Table 5.11 Commodity Final Demands for US 2003 Input–Output Tables

Commodity/Final Demand	Personal consumption expenditures	Private fixed investment	Change in private inventories	Exports of goods and services	Imports of goods and services	Government consumption expenditures and gross investment	Total Final Demand
Agriculture	47,922	–	175	24,859	(26,769)	(1,136)	45,050
Mining	72	35,698	1,912	4,739	(125,508)	702	(82,384)
Construction	–	704,792	–	71	–	224,468	929,331
Manufacturing	1,301,616	573,197	8,983	506,780	(1,075,128)	94,705	1,410,152
Trade, Transportation & Utilities	1,549,792	125,271	2,994	131,884	8,065	10,289	1,828,294
Services	4,780,516	303,426	461	175,546	(44,060)	30,256	5,246,145
Other	80,963	(75,404)	(15,748)	98,989	(177,578)	1,716,238	1,627,459
Total	7,760,881	1,666,980	(1,224)	942,868	(1,440,979)	2,075,522	11,004,047

Use the iterative procedure developed in Appendix 5.2 to generate a revised commodity-by-commodity transactions table that includes no negative entries.

5.9 Consider the Use and Make matrices for the US input–output tables for 2003 provided in Appendix B and used to construct an industry-based technology commodity-by-industry **A** and **L**. Table 5.11 gives the detail of the components of total final demand. Note that the total final demand entry for mining is negative due to a negative trade balance, i.e., the value of net exports (exports minus imports) is negative and is sufficiently large to offset other components of final demand to render total final demand negative. Suppose that the value for total imports of manufactured goods is projected to increase by \$1 trillion from its 2003 value with, for simplicity, all other elements of total final demand remaining identical to those for 2003. What is the impact on gross national product and on total output of all sectors of the economy?

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6 Multipliers in the Input–Output Model

6.1 Introduction

One of the major uses of the information in an input–output model is to assess the effect on an economy of changes in elements that are exogenous to the model of that economy. For example,

> Leontief input–output economics derive their significance largely from the fact that output multipliers measuring the combined effects of the direct and indirect repercussions of a change in final demand were readily calculated. (Steenge, 1990, p. 377.)

In Chapters 2 and 3 we presented several numerical illustrations of the ways in which assumed changes in final-demand elements (e.g., federal government spending, household consumption, exports) were translated, via the appropriate Leontief inverse, to corresponding output changes in the industrial sectors of the economy. When the exogenous changes occur because of the actions of only one "impacting agent" (or a small number of such agents) and when the changes are expected to occur in the short run (e.g., next year), this is usually called *impact analysis*. Examples are a change in federal government defense spending or in consumer demand for recreation vehicles.

On the other hand, when longer-term and broader changes are examined, then we are dealing with projections and forecasting. If we project the levels of final demand for outputs of *all* sectors in an economy five years hence, and estimate, using the Leontief inverse, the outputs from all sectors that will be needed to satisfy this demand, this is an exercise in *forecasting*. As the period of projection gets longer, the accuracy of such an exercise tends to decrease, both because our ability to forecast the new final demands accurately (the elements of **f**) will diminish and also because the coefficients matrix – the elements of **A** and hence of **L** – may have become outdated. (The issue of temporal stability of input–output coefficients is examined in Chapter 7.) If the model is built from commodity–industry accounts, then it is the matrices **B**, **C** and/or **D** that may become out of date.

In either impact analysis or forecasting, the general form of the model is $\mathbf{x} = \mathbf{Lf}$ [or $\Delta\mathbf{x} = \mathbf{L}\Delta\mathbf{f}$], and the usefulness of the result, $\mathbf{x}$ (or $\Delta\mathbf{x}$), will depend on the "correctness" of both the Leontief inverse and the final-demand vector. Our primary concern in this section is with the elements a_{ij}, and hence with $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$. The $\mathbf{f}$ (or $\Delta\mathbf{f}$) vector incorporates the assumed or projected behavior of one or more final-demand elements, and accuracy in the estimation of these elements is also of paramount importance to generating an accurate result. When the question is one of impact, then the final-demand value or values are usually completely specified – for example, what is the impact, by sector, of a new order for \$2.5 million worth of sector j output by the federal government? Then $\Delta\mathbf{f}$ contains 2.5 (million) in the jth row and zeros elsewhere.

Alternatively, to find $\mathbf{x}$ for some future year requires a projection of both $\mathbf{A}$ and $\mathbf{f}$ to that year. We will investigate some of the approaches for changing $\mathbf{A}$ over time in Chapter 7. The projection of $\mathbf{f}$ is a problem that is often approached via econometric models. The input–output forecasts of 1985 industrial outputs (and employment) for the US economy in Almon *et al.* (1974, Chapters 8 and 9) depend on detailed and painstaking projections of each of the components of final demand – personal consumption expenditures, investment in capital equipment, construction, inventories, imports and exports, and government expenditures (1974, Chapters 2 through 7, respectively). In some but by no means all "joined" input–output and econometric models, the econometric model provides a forecast of the final demands, which then "drive" the input–output model. (There is a growing literature on this issue of the interactions between input–output models and econometric models, particularly at the regional level. Some of this is explored in Chapter 14.)

A number of summary measures, derived from the elements of $\mathbf{L}$, are often employed in impact analysis; these are input–output multipliers. We examine multipliers in this chapter.

6.2 General Structure of Multiplier Analysis

Several of the most frequently used types of multipliers are those that estimate the effects of exogenous changes on (a) outputs of the sectors in the economy, (b) income earned by households in each sector because of the new outputs, (c) employment (jobs, in physical terms) that is expected to be generated in each sector because of the new outputs and (d) the value added that is created by each sector in the economy because of the new outputs. We examine these in this section.

The notion of multipliers rests upon the difference between the *initial* effect of an exogenous change and the *total* effects of that change. The total effects can be defined either as the *direct* and *indirect* effects (found from an input–output model that is open with respect to households) or as *direct, indirect* and *induced* effects (found from a model that is closed with respect to households).[1] The multipliers that incorporate direct and indirect effects are also known as *simple* multipliers. When direct, indirect and induced effects are captured, they are often called *total* multipliers.

[1] In some discussions of multipliers in an input–output model, what we have called the *initial* effect is termed the *direct* effect. For later exposition – for example, in looking at shortcut methods for finding multipliers – when the power series approximation

$$(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots$$

will be used, it seems to us preferable to associate "initial" with the $\mathbf{I}$ term, "direct" with $\mathbf{A}$, and "indirect" with the remaining terms, $\mathbf{A}^2 + \mathbf{A}^3 + \cdots$

6.2.1 Output Multipliers

An output multiplier for sector j is defined as the total value of production in all sectors of the economy that is necessary in order to satisfy a dollar's worth of final demand for sector j's output.

Simple Output Multipliers For the simple output multiplier, this total production is obtained from a model with households exogenous. The initial output effect on the economy is defined to be just the initial dollar's worth of sector j output needed to satisfy the additional final demand. Then, formally, the output multiplier is the ratio of the direct and indirect effect to the initial effect alone.

We continue with the small example in Chapter 2, section 2.3, where

$$\mathbf{A} = \begin{bmatrix} .15 & .25 \\ .20 & .05 \end{bmatrix}$$

and

$$\mathbf{L} = \begin{bmatrix} 1.254 & .330 \\ .264 & 1.122 \end{bmatrix}$$

(In the remainder of this book we will sometimes keep three figures to the right of the decimal point and sometimes four, depending on the purposes of the numerical illustration.) Note that $\Delta\mathbf{f}(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ indicates an additional dollar's worth of final demand for the output of sector 1 only, and $\Delta\mathbf{f}(2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ indicates, similarly, an additional dollar's worth of final demand for the output of sector 2 only. Consider $\Delta\mathbf{f}(1)$; the implications for sectors 1 and 2 are found as $\mathbf{L}\Delta\mathbf{f}(1)$. Denote this by $\Delta\mathbf{x}(1)$, so

$$\Delta\mathbf{x}(1) = \begin{bmatrix} 1.254 & .330 \\ .264 & 1.122 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.254 \\ .264 \end{bmatrix} \tag{6.1}$$

This is, of course, just the first column of $\mathbf{L}$ – $\begin{bmatrix} l_{11} \\ l_{21} \end{bmatrix}$.

The additional outputs of \$1.254 from sector 1 and \$0.264 from sector 2 are required for a dollar of new final demand for the output of sector 1 *only*. The \$1.254 from sector 1 represents \$1.00 to satisfy the original new dollar of final demand plus an additional \$0.254 for intra- and interindustry use. The \$0.264 from sector 2 is for intra- and interindustry use only. The sector 1 output multiplier, $m(o)_1$, is defined as the sum of the elements in the $\Delta\mathbf{x}(1)$ column, namely \$1.518, divided by \$1; $m(o)_1 = \$1.518/\$1 = 1.518$, a dimensionless number. The \$1 in the denominator is the initial effect on sector 1 output of the new dollar's worth of final demand for sector 1's product; the dollar's worth of final demand becomes an additional dollar's worth of sector 1 output as the first term in the series assessment of total direct and indirect effects on sector 1 production. Formally, using $\mathbf{i}' = [1\ 1]$ as usual to generate column sums

$$m(o)_1 = \mathbf{i}'\Delta\mathbf{x}(1) = \sum_{i=1}^{n} l_{i1} \tag{6.2}$$

where $n = 2$ in this example.

Similarly,

$$\Delta\mathbf{x}(2) = \begin{bmatrix} 1.254 & .330 \\ .264 & 1.122 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} .330 \\ 1.122 \end{bmatrix} = \begin{bmatrix} l_{12} \\ l_{22} \end{bmatrix}$$

and

$$m(o)_2 = \mathbf{i}'\Delta\mathbf{x}(2) = \sum_{i=1}^{n} l_{i2} \tag{6.3}$$

Here $m(o)_2 = 1.452$. In general, the simple output multiplier for sector j is

$$m(o)_j = \sum_{i=1}^{n} l_{ij} \tag{6.4}$$

Thus, for example, if a government agency were trying to determine the differential effects of spending an additional dollar (or \$100, or \$1,000,000, or whatever amount) on the output of a sector, comparison of output multipliers would show where this spending would have the greatest impact in terms of total dollar value of output generated throughout the economy. Note that when maximum total output effects are the exclusive goal of government spending, it would always be rational to spend all the money in the sector with the largest output multiplier. Even with anticipated expenditures of \$1,000,000, there would be no reason, on the basis of output multipliers alone, to divide that spending between the sectors.

Of course, there might well be other reasons – taking into account strategic factors, equity, capacity constraints for sectoral production, and so on – for using some of the new final-demand dollars on the output of the other sector (or sectors, when $n > 2$). Note also that multipliers of this sort may overstate the effect on the economy in question if some sectors are operating at or near capacity and hence some of the needed new inputs would have to be imported to the economy and/or outputs from some sectors would be shifted from exports and kept in the economy for use as inputs. Phenomena such as these will assume even more importance in regional models.

We see that $\mathbf{L}$ is a matrix of sector-to-sector multipliers, l_{ij}, relating final demand in sector j to output in sector i. Output multipliers (column sums of $\mathbf{L}$) represent sector-to-economy multipliers, relating final demand in sector j to economy-wide output. For an n-sector model, denote the row vector of these multipliers by $\mathbf{m}(o) = [m(o)_1, \ldots, m(o)_n]$.[2]

[2] Strictly speaking, one expects a row vector to include a "prime" in its designation, as with $\mathbf{x}$ and $\mathbf{x}'$ in earlier chapters. However, here and throughout this discussion of multipliers we simply define various rows of multipliers without the prime to save on notational complexity.

With $\underset{(1\times n)}{\mathbf{i}'} = [1, \ldots, 1]$, we have

$$\mathbf{m}(o) = \underbrace{\mathbf{i}' \overbrace{\mathbf{L}}^{\substack{\textit{Sector-demand-}\\ \textit{to-sector-output}\\ \textit{multipliers}}}}_{\substack{\textit{Sector-demand-}\\ \textit{to-economy-wide-}\\ \textit{output multipliers}}} \tag{6.5}$$

We will see that many additional input–output multiplier variations build on this representation. All that is required is to alter the elements in the multiplier matrix so that instead of $(\Delta f_j = 1) \rightarrow (\Delta x_i)$ they represent $(\Delta f_j = 1) \rightarrow$ (some function of Δx_i), such as employment or energy use or pollution emissions.

Total Output Multipliers If we consider the input coefficients matrix closed with respect to households (as described in section 2.5) we capture in the model the additional *induced* effects of household income generation through payments for labor services and the associated consumer expenditures on goods produced by the various sectors. Continuing with the example from section 2.5, the augmented coefficient matrix, with an added household row and column, was

$$\bar{\mathbf{A}} = \begin{bmatrix} .15 & .25 & .05 \\ .20 & .05 & .40 \\ \\ .30 & .25 & .05 \end{bmatrix}$$

and the Leontief inverse, with elements $\bar{l}_{ij}$, was

$$\bar{\mathbf{L}} = (\mathbf{I} - \bar{\mathbf{A}})^{-1} = \begin{bmatrix} 1.365 & 0.425 & 0.251 \\ 0.527 & 1.348 & 0.595 \\ \\ 0.570 & 0.489 & 1.289 \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{L}}_{11} & \bar{\mathbf{L}}_{12} \\ \\ \bar{\mathbf{L}}_{21} & \bar{\mathbf{L}}_{22} \end{bmatrix} \tag{6.6}$$

as in (2.7) but rounded here to three decimals. We have added the partitioned matrix representation because it will be useful in much of what follows in this chapter. Clearly, the elements in $\bar{\mathbf{L}} = [\bar{l}_{ij}]$ also relate final-demand changes to sector outputs, only now these are in a model with households endogenous, and hence the effects tend to be larger.

To assess the impact of a new dollar's worth of final demand for sector 1 output, we would now form the three-element vector $\Delta\bar{\mathbf{f}}(1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ (meaning no exogenous change in demand for sector 2 output or for labor services), and find exactly the first column of $\bar{\mathbf{L}}$, namely

$$\Delta\bar{\mathbf{x}}(1) = \bar{\mathbf{L}}\Delta\bar{\mathbf{f}}(1) = \begin{bmatrix} 1.365 \\ 0.527 \\ 0.570 \end{bmatrix}$$

[Compare (6.1) above.] Adding these elements gives a parallel to (6.2),

$$\bar{m}(o)_1 = \mathbf{i}'\Delta\bar{\mathbf{x}}(1) = \sum_{i=1}^{n+1} \bar{l}_{i1} = 2.462 \tag{6.7}$$

with $n = 2$, as before but now with $\mathbf{i}' = [1, 1, 1]$. (In what follows we assume that $\mathbf{i}$ or $\mathbf{i}'$ always has appropriate dimensions for the multiplication in which it is involved.)

Sums of the first n elements in each of the columns of $\bar{\mathbf{L}}$ ($n = 2$ for our example) represent the total output multiplier effects over the original n sectors only – the *truncated* output multipliers. They can be found as $\mathbf{i}'\bar{\mathbf{L}}_{11}$. When interest is centered on the total output multipliers for the original n sectors (for example, to be compared with the simple output multipliers for these same n sectors), these *truncated* output multipliers are of interest. Denote these *truncated* total output multipliers by $\bar{m}[o(t)]_j$; here $\bar{m}[o(t)]_1 = 1.892$.

The total output multiplier for sector 2 is

$$\bar{m}(o)_2 = \sum_{i=1}^{n+1} \bar{l}_{i2} = 2.262 \tag{6.8}$$

and $\bar{m}[o(t)]_2 = 1.773$. In general, for sector j, the total output multiplier is given by

$$\bar{m}(o)_j = \sum_{i=1}^{n+1} \bar{l}_{ij} \tag{6.9}$$

and the truncated total output multiplier is $\bar{m}[o(t)]_j = \sum_{i=1}^{n} \bar{l}_{ij}$. In compact matrix terms,

$$\bar{\mathbf{m}}(o) = \mathbf{i}'\bar{\mathbf{L}} \text{ and } \bar{\mathbf{m}}[o(t)] = \mathbf{i}'\bar{\mathbf{L}}_{11} \tag{6.10}$$

Example: The US Input–Output Model for 2003 We again use the seven-sector 2003 US model. The Leontief inverse was shown as Table 2.7 in Chapter 2 and is not repeated here. The simple output multipliers are easily found to be

$$\mathbf{m}(o) = [1.9195\ 1.6051\ 1.7218\ 1.9250\ 1.4868\ 1.6081\ 1.5985]$$

In this case, the largest multipliers are associated with manufacturing (4) and agriculture (1). This is hardly surprising, considering the seven-sector level of aggregation.

Table 6.1 Total Requirements Matrices in Commodity–Industry Models

	Industry Technology	Commodity Technology
Commodity-Demand Driven Models		
Commodity-by-Commodity	$(\mathbf{I}-\mathbf{BD})^{-1}$	$(\mathbf{I}-\mathbf{BC}^{-1})^{-1}$
Industry-by-Commodity	$[\mathbf{D}(\mathbf{I}-\mathbf{BD})^{-1}]$	$[\mathbf{C}^{-1}(\mathbf{I}-\mathbf{BC}^{-1})^{-1}]$
Industry-Demand Driven Models		
Industry-by-Industry	$(\mathbf{I}-\mathbf{DB})^{-1}$	$(\mathbf{I}-\mathbf{C}^{-1}\mathbf{B})^{-1}$
Commodity-by-Industry	$[\mathbf{D}^{-1}(\mathbf{I}-\mathbf{DB})^{-1}]$	$[\mathbf{C}(\mathbf{I}-\mathbf{C}^{-1}\mathbf{B})^{-1}]$

Output Multipliers in Commodity–Industry Models With commodity-by-industry models, no new principles are involved. As usual, output multipliers would be found as column sums of the relevant total requirements matrices (open or closed with respect to households). In Table 6.1 we collect the results for total requirements matrices from Tables 5.4 and 5.5 in Chapter 5.

For example, for the commodity-by-commodity total requirements matrix under industry technology, the row vector of these output multipliers is $\mathbf{i}'(\mathbf{I}-\mathbf{BD})^{-1}$. Notice that since $\mathbf{i}'\mathbf{D}=\mathbf{i}'$ (column sums of $\mathbf{D}$ are all 1), the same output multipliers will be found for the industry-by-commodity total requirements matrix: $\mathbf{i}'[\mathbf{D}(\mathbf{I}-\mathbf{BD})^{-1}]=\mathbf{i}'(\mathbf{I}-\mathbf{BD})^{-1}$. The same will be true for any other pair of matrices (vertically) in the table. This is because (1) $\mathbf{i}'\mathbf{C}=\mathbf{i}'$ ($\mathbf{C}$ is constructed so that is true), (2) $\mathbf{i}'\mathbf{D}^{-1}=\mathbf{i}'$ (this is easy to show, given $\mathbf{i}'\mathbf{D}=\mathbf{i}'$) and (3) similarly, $\mathbf{i}'\mathbf{C}^{-1}=\mathbf{i}'$. This result is what we would expect – summing down the columns in a total requirements matrix (over all rows) should give the same result, irrespective of the row labels ("commodities" or "industries").

The results below are for the total requirements matrices in the numerical examples from Chapter 5. They illustrate the identical results for pairs of matrices.

Commodity-Demand-Driven Models

Industry Technology **Commodity Technology**

Commodity-by-Commodity

$$(\mathbf{I}-\mathbf{BD})^{-1}=\begin{bmatrix}1.1568 & .0898\\ .1314 & 1.0782\end{bmatrix} \qquad (\mathbf{I}-\mathbf{BC}^{-1})^{-1}=\begin{bmatrix}1.1644 & .0825\\ .1375 & 1.0723\end{bmatrix}$$

Output Multipliers $[\,1.2882 \;\; 1.1680\,]$ $[\,1.3019 \;\; 1.1548\,]$

Industry-by-Commodity

$$\mathbf{D}(\mathbf{I}-\mathbf{BD})^{-1}=\begin{bmatrix}1.0411 & .0809\\ .2471 & 1.0871\end{bmatrix} \qquad \mathbf{C}^{-1}(\mathbf{I}-\mathbf{BC}^{-1})^{-1}=\begin{bmatrix}1.1507 & -.0247\\ .1512 & 1.1795\end{bmatrix}$$

Output Multipliers $[\,1.2882 \;\; 1.1680\,]$ $[\,1.3019 \;\; 1.1548\,]$

Industry-Demand-Driven Models

	Industry Technology	**Commodity Technology**
	Industry-by-Industry	
	$(\mathbf{I}-\mathbf{DB})^{-1}=\begin{bmatrix}1.1478 & .0809\\ .1537 & 1.0871\end{bmatrix}$	$(\mathbf{I}-\mathbf{C}^{-1}\mathbf{B})^{-1}=\begin{bmatrix}1.1507 & .0821\\ .1512 & 1.0861\end{bmatrix}$
Output Multipliers	[1.3015 1.1680]	[1.3019 1.1682]
	Commodity-by-Industry	
	$\mathbf{D}^{-1}(\mathbf{I}-\mathbf{DB})^{-1}=\begin{bmatrix}1.2753 & .0898\\ .0262 & 1.0782\end{bmatrix}$	$\mathbf{C}(\mathbf{I}-\mathbf{C}^{-1}\mathbf{B})^{-1}=\begin{bmatrix}1.1644 & .1808\\ .1375 & .9873\end{bmatrix}$
Output Multipliers	[1.3015 1.1680]	[1.3019 1.1682[3]]

6.2.2 Income/Employment Multipliers

Generally an analyst is more likely to be concerned with the economic impacts of new final demand as measured by jobs created, increased household earnings, value added generated, etc., rather than simply gross output by sector. In this section we explore impacts on households; the approach is exactly the same whether we measure this impact in terms of jobs (physical) or earnings (monetary). In what follows, we illustrate using income, but this applies equally well to jobs.

Income Multipliers One straightforward approach is simply to convert the elements in $\mathbf{L}$ into dollars' worth of employment using labor-input coefficients – either monetary (wages earned per unit of output, as in $[a_{n+1,1},\ldots,a_{n+1,n}]$) or physical (person-years, or some such measure, per unit of output). We begin with transactions information; let $\mathbf{h}'$ (for households) denote the row vector of these data. In the monetary case, this is $\mathbf{h}'=[z_{n+1,1},\ldots,z_{n+1,n}]$; in physical terms it would be some measure of numbers of employees in each sector in the base period. Then $\mathbf{h}'_c=\mathbf{h}'\hat{\mathbf{x}}^{-1}$ is the row of associated household input *coefficients*.[4] Again, in monetary terms these are the elements in $[a_{n+1,1},\ldots,a_{n+1,n}]$, used in the example above to close the model with respect to households ($a_{n+1,j}=z_{n+1,j}/x_j$), indicating household income received per dollar's worth of sector output.

Associated with $\Delta\mathbf{f}=\begin{bmatrix}1\\0\end{bmatrix}$, we found output effects in the first column of $\mathbf{L}-\begin{bmatrix}l_{11}\\l_{21}\end{bmatrix}$, as in (6.1). The conversion of this first column to income terms is accomplished by weighting the first element by $a_{n+1,1}$ and the second element by $a_{n+1,2}$,

[3] This is not equal to 0.1808 + 0.9873 only because of rounding in the total requirements matrix.

[4] We denoted this as $\mathbf{h}_R$ earlier when closing the model with respect to households. Now we modify the notation to emphasize that this is a row vector of coefficients and to allow for generalization to other kinds of multipliers.

giving $\begin{bmatrix} a_{n+1,1}l_{11} \\ a_{n+1,2}l_{21} \end{bmatrix}$. In general, then, using $m(h)_j$ for the simple household income multiplier for sector j,

$$m(h)_j = \sum_{i=1}^{n} a_{n+1,i} l_{ij} \tag{6.11}$$

Again, "simple" refers to the fact that these multipliers are found using elements in the **L** matrix, with households exogenous.

Continuing the same example, we had $a_{n+1,1} = 0.3$ and $a_{n+1,2} = 0.25$. Thus

$$m(h)_1 = (0.3)(1.254) + (0.25)(0.264) = 0.376 + 0.066 = 0.442$$

and

$$m(h)_2 = (0.3)(0.33) + (0.25)(1.122) = 0.099 + 0.281 = 0.380$$

In this illustration, $m(h)_1 = 0.442$ indicates that an additional dollar of final demand for the sector 1 output would generate \$0.442 of new household income, when all direct and indirect effects are converted into dollar estimates of income. If earnings in individual sectors are of interest, we see that \$0.376 would be earned by employees in sector 1 and \$0.066 would be earned by sector 2 employees. And similarly, $m(h)_2 = 0.380$ could be disaggregated into earnings in each of the sectors. From this example, using this measure of effectiveness, dollars of final demand – for example, new government purchases – generate more dollars of new household income when they are spent on the output of sector 1 than when they are spent on the output of sector 2.

If the elements in $\bar{\mathbf{L}}$ are weighted similarly, *total* (direct plus indirect plus induced) income effects or household income multipliers are obtained. As before, using an overbar to denote a multiplier derived from $\bar{\mathbf{L}}$, the parallel to $m(h)_j$ in (6.11) is

$$\bar{m}(h)_j = \sum_{i=1}^{n+1} a_{n+1,i} \bar{l}_{ij} \tag{6.12}$$

For our numerical example, with $a_{n+1,3} = 0.05$,

$$\bar{m}(h)_1 = (0.3)(1.365) + (0.25)(0.527) + (0.05)(0.570) = 0.570$$

and

$$\bar{m}(h)_2 = (0.3)(0.425) + (0.25)(1.348) + (0.05)(0.489) = 0.489$$

These total income multipliers for sectors 1 and 2 are equal to $\bar{l}_{n+1,1}$ and $\bar{l}_{n+1,2}$, the elements of $\bar{\mathbf{L}}_{21}$ [in $\bar{\mathbf{L}}$ (6.6)]. Recall the interpretation of any element $\bar{l}_{ij}$; it measures the total (direct, indirect, and induced) effect on sector i output of a dollar's worth of new demand for sector j output. Thus $\bar{l}_{n+1,j}$ is the total effect on the output of the *household* sector (the total value of labor services needed) when there is a dollar's worth of new final demand for goods of sector j. This is precisely what we mean by the total household income effect or total household income multiplier. So

$$\bar{m}(h)_j = \bar{l}_{n+1,j} \tag{6.13}$$

(In Appendix 6.1, the relationship between the total household income multipliers and the bottom-row elements of $\bar{\mathbf{L}}$ is shown exactly, using matrix algebra results on the inverse of a partitioned matrix.) Again, if we are only interested in household income-generating effects originating in the n original sectors, we would calculate a truncated total household income multiplier, $\bar{m}[h(t)]_1$, by summing down the columns of $\bar{\mathbf{L}}_{11}$ only. For the example, $\bar{m}[h(t)]_1 = 0.541$ and $\bar{m}[h(t)]_2 = 0.465$.

In this and all subsequent discussions in this chapter, all results hold if $\mathbf{A}$ and $\mathbf{L}$ are understood to be direct and total requirements matrices in a commodity–industry model – as for example with $\underset{(c\times c)}{\mathbf{A}_I} = \mathbf{BD}$ and $\underset{(c\times c)}{\mathbf{L}_I} = (\mathbf{I}-\mathbf{BD})^{-1}$. We illustrated the case of output multipliers for various commodity–industry models in section 6.2.1, above.

Type I and Type II Income Multipliers With income multipliers, one has some choice regarding what should logically be termed the initial effect of new final demand. With output multipliers it was fairly clear that the initial effect of a new dollar's worth of final demand for sector j output is that sector j production must increase by one dollar (and eventually, of course, by more than that dollar). With income effects, the same dollar's worth of new demand for sector j becomes, initially, the same dollar's worth of new output by sector j; this is what we considered the initial effect in developing the household income multipliers, above. However, the initial dollar's worth of new output from sector j means an initial additional income payment of $a_{n+1,j}$ to workers in sector j. Hence $a_{n+1,j}$ could be viewed as the initial *income* effect of the new demand for sector j output.

Thus there is another kind of simple income multiplier, usually called the type I income multiplier, for any sector j. This has the direct and indirect income effect, or the simple household income multiplier [(6.11)] as a numerator, and uses as a denominator not the initial dollar's worth of output but rather its initial labor income effect, $a_{n+1,j}$.[5] Let $m(h)_j^I$ represent this type I income multiplier for sector j, so

$$m(h)_j^I = \frac{\sum_{i=1}^{n} a_{n+1,i}l_{ij}}{a_{n+1,j}} = \frac{m(h)_j}{a_{n+1,j}} \tag{6.14}$$

For our numerical example,

$$m(h)_1^I = 0.442/0.3 = 1.473$$

$$m(h)_2^I = 0.380/0.25 = 1.520$$

Again, if the coefficients matrix is closed with respect to households, income effects similar to these type I multipliers can be calculated; these are called type II income

[5] These have also been called "normalized" multipliers; for example, in Oosterhaven (1981).

multipliers:[6]

$$m(h)_j^{II} = \frac{\sum_{i=1}^{n+1} a_{n+1,i}\bar{l}_{ij}}{a_{n+1,j}} = \frac{\bar{m}(h)_j}{a_{n+1,j}} \tag{6.15}$$

Again, for the numerical example,

$$m(h)_1^{II} = \frac{0.570}{0.3} = 1.900$$

$$m(h)_2^{II} = \frac{0.489}{0.25} = 1.956$$

The parallel between this measure and the type I effect in (6.14) is the same as that between the total and simple household income multipliers – $\bar{m}(h)_j$ and $m(h)_j$. The numerator for $m(h)_j^I$ is $m(h)_j$ from (6.11); the numerator for $m(h)_j^{II}$ is $\bar{m}(h)_j$ from (6.12) or from (6.13). Thus, for exactly the same reasons as for $\bar{m}(h)_j$, we can alternatively define $m(h)_j^{II}$ as

$$m(h)_j^{II} = \bar{l}_{n+1,j}/a_{n+1,j} \tag{6.16}$$

These multipliers show by how much the initial *income* effects (0.3 and 0.25) are blown up when direct, indirect, and induced effects (due to household spending because of increased household income) are taken into account, via $\bar{\mathbf{L}}$. Truncated type II income multipliers would be found, as usual, by considering columns in $\bar{\mathbf{L}}_{11}$ only. In this example they are $m[h(t)]_1^{II} = 1.803$ and $m[h(t)]_2^{II} = 1.860$.

It is generally conceded that Type I multipliers probably underestimate economic impacts (since household activity is absent) and Type II multipliers probably give an overestimate (because of the rigid assumptions about labor incomes and attendant consumer spending). For example, Oosterhaven, Piek and Stelder (1986, p. 69) suggest

> These two multipliers [Type II and Type I] may be considered as upper and lower bounds on the true indirect effect of an increase in final demand; a realistic estimate generally lies roughly halfway between the Type I and Type II multipliers.

Relationship Between Simple and Total Income Multipliers or Between Type I and Type II Income Multipliers To the extent that the results of an input–output analysis with households exogenous tend to underestimate total effects, total or type II multipliers may be more useful than simple or type I multipliers in estimating potential impacts. Or some in-between figure might be more realistic, as noted above, but deciding exactly where between these two limits may be problematic. However, if one is primarily interested in *ranking* or ordering the sectors – which sector has the largest multiplier, which has the next largest, and so on – then type I multipliers are just as useful as type II (and usually easier to obtain), because the ratio of type II

[6] The designations "type I" and "type II" seem to have originated with Moore (1955). Calculation of these measures (in a regional setting) was pioneered by Moore and Petersen (1955) for Utah and later by Hirsch (1959) for St. Louis.

to type I income multipliers can be shown to be a constant across all sectors. Since $m(h)_j^{II} = \bar{m}(h)_j/a_{n+1,j}$ and $m(h)_j^I = m(h)_j/a_{n+1,j}$, $m(h)_j^{II}/m(h)_j^I = \bar{m}(h)_j/m(h)_j$. What is now claimed is that $m(h)_j^{II}/m(h)_j^I = k$ (a constant) for all j. Moreover, k can be easily found without any need for $\bar{\mathbf{L}}$. This represents a computational advantage. To show that this ratio is a constant requires that we apply some facts on the inverse of the partitioned matrix $\bar{\mathbf{L}}$. This is done in Appendix 6.2, for the interested reader. In our illustrative example we found $m(h)_1 = 0.442$, $\bar{m}(h)_1 = 0.570$, $m(h)_2 = 0.380$, $\bar{m}(h)_2 = 0.489$, $m(h)_1^I = 1.473$, $m(h)_1^{II} = 1.900$, $m(h)_2^I = 1.520$, and $m(h)_2^{II} = 1.956$. Therefore (to two decimals), $\bar{m}(h)_1/m(h)_1 = 0.570/0.442 = 1.29$, $m(h)_1^{II}/m(h)_1^I = 1.90/1.47 = 1.29$, and the same values can be found for $\bar{m}(h)_2/m(h)_2$, and $m(h)_2^{II}/m(h)_2^I$, so $k = 1.29$ for this example.

Which Multiplier to Use? As a practical matter, the choice between multiplier effects as measured by $m(h)_j$ [and $\bar{m}(h)_j$] or by $m(h)_j^I$ [and $m(h)_j^{II}$] depends on the nature of the exogenous change whose impact is being studied. If that change is, for example, an increase in federal government spending on output of the aircraft sector, then the most useful figures may be those that convert the total dollar value of new government spending into total new income earned by households in the economy – the income multipliers $m(h)_j$ and $\bar{m}(h)_j$. Using $m(h)_1 = 0.442$ and $m(h)_2 = 0.380$ from the example, we would estimate that a tariff policy that would increase foreign demand for sector 1 goods by \$100,000 would ultimately lead to an increase of (0.442)(\$100,000) = \$44,200 in new income earned, while a policy that increased export demand for sector 2 goods by \$100,000 would generate (0.380)(\$100,000) = \$38,000 in new household income earned. If we also attempt to capture the consumer spending that is associated with income earned, in a closed model, we would use $\bar{m}(h)_1$ and $\bar{m}(h)_2$ and find (0.570)(\$100,000) = \$57,000 and (0.489)(\$100,000) = \$48,900, respectively. In either case, we find that stimulation of export demand for sector 1 output generates the larger effect, as expected, because $\bar{m}(h)_j/m(h)_j = k$ (here 1.29), so the largest simple multiplier will be the largest total multiplier.

The impacts of decreases can be assessed just as easily. Suppose that management teams in two different industries, i and j, were considering moving a large assembly plant out of the country because of lower labor costs abroad. If these plants had annual payrolls of \$$p_i$ and \$$p_j$, respectively, then a measure of the total household income lost throughout the national economy because of the contemplated relocations would be given by $m(h)_i^I p_i$ and $m(h)_j^I p_j$ – or by $m(h)_i^{II} p_i$ and $m(h)_j^{II} p_j$, if one wants to include induced households consumption effects. For example, using $m(h)_1^I = 1.473$ and $m(h)_2^I = 1.520$ from our example, if a plant in industry 1 with an annual payroll of \$100,000 were to move out of the country, we would estimate a total income loss of (1.47)(\$100,000) = \$147,300 throughout the economy. Similarly, if a plant in industry 2, with an annual payroll of \$250,000, were to move out of the economy, we could estimate the total loss to household income throughout the economy because of this out-movement as (1.520)(\$250,000) = \$380,000. Again, if we capture consumer spending using a closed model, our estimates, using $m(h)_1^{II} = 1.900$ and $m(h)_2^{II} = 1.956$, would be a (1.900)($100,000) = $190,000 income loss from the out-movement of the plant in industry 1 and a (1.956)($250,000) = $489,000 income decrease from loss of the plant in industry 2.

Even More Income Multipliers As noted above (section 3.2.3), in an important early study of Boulder, Colorado, Miernyk *et al.* (1967) implement a model that distinguishes between consumption propensities of new residents in a region and those of established residents. In addition, current residents were divided into income classes (four in this study), and separate regional consumption functions were estimated for each income class. The results of this approach have been termed type III income multipliers, and they are smaller, sector by sector, than the type II income multipliers. This is to be expected, since *marginal* consumption coefficients, associated with current residents' consumption habits, were smaller than *average* consumption coefficients, associated with new residents' consumption habits and which are the exclusive basis of the type II multipliers.[7]

Although the ratio of type III to type II income multipliers is not constant across sectors, the range was only 0.87–0.91, with an average of 0.88. Since the (constant) ratio of type II to type I income multipliers in this study was 1.34, this means that the ratio of type III to type I income multipliers averaged 1.18. If a similar narrow range of ratios of type III to type II income multipliers were found in other regional studies in which households were similarly disaggregated, it would be possible to approximate type III income multipliers across all sectors by appropriate "inflation" of the type I multiplier. In the Boulder study, the inflating factor would be 1.18.

Further, Madden and Batey (1983 and elsewhere) derive a type IV income multiplier. Like the type III multipliers, these are (generally) larger than type I but smaller than type II income multipliers. The distinction here is between the spending patterns of currently employed local residents and the spending patterns of currently unemployed local residents.[8] The models giving rise to these four kinds of multipliers are discussed and summarized in Batey and Weeks (1989). Table 6.2 provides an overview.

Physical Employment Multipliers All of the above types of multipliers apply equally well if we are interested in counts of jobs, in physical terms. Our initial information, in $\mathbf{h}'$, would be in person-years or some similar unit of measure, and the results

[7] In the Boulder study, the *average* (aggregate) household consumption coefficient, for the products of all 31 sectors of the local economy, is 0.40. (This is $\mathbf{i}'\mathbf{h}_C$, using the household column in the Boulder study.) The *marginal* (aggregate) household consumption coefficients for the products of the same 31 sectors, are 0.31, 0.21, 0.16, and 0.02 for the four income classes; their average is 0.1730. (Calculated from Tables IV-2 and V-4a, respectively, in Miernyk *et al.*, 1967.) The type III multipliers in the Boulder study were found not from the Leontief inverse of a model that had been closed with respect to households in this disaggregated way but rather in an iterative, round-by-round fashion.

[8] Conway (1977) proposed applying the terms "type A" and "type B" multipliers to the numerators of "type I" and "type II" multipliers. The motivation is to facilitate studies of changes in multiplier values over time. When the multiplier is a ratio in which both numerator and denominator elements change over time, a change in a multiplier value can reflect changes in either the numerator or in the denominator or in both.

Table 6.2 Model Closures with Respect to Households

	Measured Effects			
Model	Direct + Indirect	Induced*	Model Closure	Income Multiplier
1	Direct + Indirect	None	None	Type I
2	Direct + Indirect	Intensive	Single household row and column	Type II
3	Direct + Indirect	Intensive + Extensive	Two household rows and columns	Type III
4	Direct + Indirect	Intensive + Extensive + Redistributive	Three household rows and columns	Type IV

*Intensive effects are associated with indigenous workers and marginal consumption coefficients. Extensive effects are associated with in-migrants and average consumption coefficients. Redistributive effects are associated with unemployed residents and their consumption propensities based on benefit payments.

in (6.11) through (6.16) remain valid, with the interpretation in physical rather than monetary terms.

6.2.3 Value-Added Multipliers

Another kind of multiplier relates the new value added created in each sector in response to the initial exogenous shock to that initial shock. The principles are identical, and the results in (6.11) through (6.16) again remain valid. The only new information required is a set of sectoral value-added coefficients – $\mathbf{v}'_c = \mathbf{v}'\hat{\mathbf{x}}^{-1}$. We leave it for the reader to fill in details. It is often argued that value added is a better measure of a sector's contribution to an economy than, say, total output, since it truly captures the value that is added by the sector in engaging in production – the difference between a sector's total output and the cost of its intermediate inputs.

6.2.4 Matrix Representations

Matrix representation provides a compact and efficient way to express multipliers. Output multipliers were represented in (6.5) as

$$\mathbf{m}(o) = \mathbf{i}'\mathbf{L}$$

For income multipliers (simple), with $\mathbf{h}'_c = \mathbf{h}'\hat{\mathbf{x}}^{-1}$, we have

$$\mathbf{m}(h) = [m(h)_1, \ldots, m(h)_n] = \mathbf{h}'_c\mathbf{L} \tag{6.17}$$

Here the summation row, $\mathbf{i}'$ in (6.5), has been replaced by the row of labor-input coefficients, $\mathbf{h}'_c$. We can deconstruct this in the following way:

$$\mathbf{m}(h) = \mathbf{h}'_c\mathbf{L} = \mathbf{h}'\hat{\mathbf{x}}^{-1}\mathbf{L} = \underbrace{\mathbf{i}' \quad \overbrace{\hat{\mathbf{h}}'\hat{\mathbf{x}}^{-1}\mathbf{L}}^{\substack{\textit{Sector-demand-}\\ \textit{to-sector income}\\ \textit{multipliers}\\ [\mathbf{M}(h)]}}}_{\substack{\textit{Sector-demand-}\\ \textit{to-economy-wide}\\ \textit{income multipliers}\\ [\mathbf{m}(h)]}} \tag{6.18}$$

In particular, $\hat{\mathbf{h}}'\hat{\mathbf{x}}^{-1}\mathbf{L}$ converts the inverse matrix of final demand-to-output multipliers in $\mathbf{L}$ into a *matrix* of final demand-to-income multipliers, $\mathbf{M}(h)$. Then $\mathbf{i}'\hat{\mathbf{h}}'\hat{\mathbf{x}}^{-1}\mathbf{L} = \mathbf{h}'\hat{\mathbf{x}}^{-1}\mathbf{L} = \mathbf{h}'_c\mathbf{L}$ generates a *vector* of economy-wide income multipliers, $\mathbf{m}(h)$, the column sums of the converted inverse. Notice that in this generic format, the simple output multipliers in (6.5) can be thought of as

$$\mathbf{m}(o) = \mathbf{i}'\mathbf{L} = \mathbf{x}'\hat{\mathbf{x}}^{-1}\mathbf{L} = \mathbf{i}'\hat{\mathbf{x}}'\hat{\mathbf{x}}^{-1}\mathbf{L}$$

For the closed model, the n-element vector of total income multipliers for the n sectors is

$$\bar{\mathbf{m}}(h) = [\bar{m}(h)_1, \ldots, \bar{m}(h)_n] = \underset{[1\times(n+1)]}{\begin{bmatrix} \mathbf{h}'_c & a_{n+1,n+1} \end{bmatrix}} \underset{[(n+1)\times n]}{\begin{bmatrix} \bar{\mathbf{L}}_{11} \\ \bar{\mathbf{L}}_{21} \end{bmatrix}} = \bar{\mathbf{h}}'_c \begin{bmatrix} \bar{\mathbf{L}}_{11} \\ \bar{\mathbf{L}}_{21} \end{bmatrix} \tag{6.19}$$

This makes clear that $\bar{m}(h)_j > m(h)_j$ for two reasons: (1) even though the weights in $\mathbf{h}'_c$ are the same for both models, the inverse elements in $\bar{\mathbf{L}}_{11}$ are consistently larger than those in $\mathbf{L}$, and (2) each $\bar{m}(h)_j$ includes the additional term $a_{n+1,n+1}\bar{l}_{n+1,j}$. [In the case of truncated multipliers, only (1) is relevant.]

The n-element row vector of type I income multipliers for each sector, $\mathbf{m}(h)^I$, can be compactly represented using $\mathbf{m}(h)$ from (6.18), namely

$$\mathbf{m}(h)^I = \mathbf{m}(h)(\hat{\mathbf{h}}'_c)^{-1} = \mathbf{h}'_c\mathbf{L}(\hat{\mathbf{h}}'_c)^{-1} \tag{6.20}$$

A row vector of type II income multipliers for the original n sectors can be defined using $\bar{\mathbf{L}}_{21} = [\bar{l}_{n+1,1}, \bar{l}_{n+1,2}, \ldots, \bar{l}_{n+1,n}]$, namely

$$\mathbf{m}(h)^{II} = \bar{\mathbf{L}}_{21}(\hat{\mathbf{h}}'_c)^{-1} \tag{6.21}$$

6.2.5 Summary

Table 6.3 presents a summary of the results in sections 6.2.1–6.2.3. Table 6.4 summarizes these multiplier results in a set of generic templates. We use "$\mathbf{z}'_c$" for the

Table 6.3 Input–Output Multipliers

	Output Effects	Income Effects[a]	
Exogenous Change	$\Delta f_j = 1$	$\Delta f_j = 1$	
Initial Effect (N) (sector j)	$\Delta x_j = 1$	$\Delta x_j = 1$	Δ in sector j payments to labor = $a_{n+1,j}$
Total Effect (T) in open model (Direct + Indirect)	$\sum_{i=1}^{n} l_{ij}$	$\sum_{i=1}^{n} a_{n+1,i} l_{ij}$	
Simple Multiplier (T/N) (open model)	Simple output multiplier $m(o)_j = \sum_{i=1}^{n} l_{ij}/\Delta f_j = \sum_{i=1}^{n} l_{ij}$ [(6.4)]	Simple income multiplier $m(h)_j = \sum_{i=1}^{n} a_{n+1,i} l_{ij}/\Delta f_j = \sum_{i=1}^{n} a_{n+1,i} l_{ij}$ [(6.11)]	Type I income multiplier $m(h)_j^I = \sum_{i=1}^{n} a_{n+1,i} l_{ij}/a_{n+1,j} = m(h)_j/a_{n+1,j}$ [(6.14)]
Total Effect ($\bar{T}$) in closed model (Direct + Indirect + Induced)	$\sum_{i=1}^{n+1} \bar{l}_{ij}$	$\sum_{i=1}^{n+1} a_{n+1,i} \bar{l}_{ij}$	
Total Multiplier ($\bar{T}/N$) (closed model)[b]	Total output multiplier $\bar{m}(o)_j = \sum_{i=1}^{n+1} \bar{l}_{ij}/\Delta f_j = \sum_{i=1}^{n+1} \bar{l}_{ij}$ [(6.9)]	Total income multiplier $\bar{m}(h)_j = \sum_{i=1}^{n+1} a_{n+1,i} \bar{l}_{ij}/\Delta f_j = \sum_{i=1}^{n+1} a_{n+1,i} \bar{l}_{ij}$ [(6.12)] $= \bar{l}_{n+1,j}$ [(6.13)]	Type II income multiplier $m(h)_j^{II} = \sum_{i=1}^{n+1} a_{n+1,i} \bar{l}_{ij}/a_{n+1,j}$ $= \bar{m}(h)_j/a_{n+1,j}$ [(6.15)] $= \bar{l}_{n+1,j}/a_{n+1,j}$ [(6.16)]

[a] For income effects, $a_{n+1,j} = z_{n+1,j}/x_j$, where $z_{n+1,j} =$ sector j's payments to households (labor). For employment effects, replace $z_{n+1,j}$ with sector j's employment measured in physical units. For value-added effects, replace $z_{n+1,j}$ with sector j's value-added payments.

[b] For truncated total multiplier effects, sum over $i = 1, \ldots, n$ rather than $i = 1, \ldots, n+1$.

appropriate row vector of coefficients, found from transactions ($\mathbf{z}'$) and output ($\mathbf{x}$) information; $\mathbf{z}'_c = \mathbf{z}'\hat{\mathbf{x}}^{-1}$. When $\mathbf{z}' = \mathbf{x}'$, $\mathbf{z}'_c = \mathbf{i}'$, and we have traditional output multipliers. [Contrary to subsequent notation, we denoted these as $\mathbf{m}(o)$, for "output," rather than $\mathbf{m}(x)$.] Note that Type I and II *output* multipliers are meaningless;

Table 6.4 General Multiplier Formulas

Multiplier	Matrix Definition
Simple	$\mathbf{m}(z) = \mathbf{z}'_c\mathbf{L}$
Total	$\bar{\mathbf{m}}(z) = \bar{\mathbf{z}}'_c \begin{bmatrix} \bar{\mathbf{L}}_{11} \\ \bar{\mathbf{L}}_{21} \end{bmatrix}$ where $\bar{\mathbf{z}}'_c = \begin{bmatrix} \mathbf{z}'_c & z_{n+1,n+1}/x_{n+1} \end{bmatrix}$
Truncated	$\bar{\mathbf{m}}[z(t)] = \mathbf{z}'_c\bar{\mathbf{L}}_{11}$
Type I	$\mathbf{m}(z)^I = \mathbf{z}'_c\mathbf{L}(\hat{\mathbf{z}}'_c)^{-1}$
Type II	$\mathbf{m}(z)^{II} = \bar{\mathbf{L}}_{21}(\hat{\mathbf{z}}'_c)^{-1}$

they are identical to simple and total output multipliers since $\hat{\mathbf{i}}' = \mathbf{I}$. When $\mathbf{z}' = \mathbf{h}'$or $\mathbf{z}' = \mathbf{v}'$ we have household (either income or employment) or value-added multipliers, respectively. Many other kinds of multipliers are possible. For example, if $\mathbf{z}' = \mathbf{e}'$ is a row measuring amounts of pollution emitted by production in each of the sectors, we would have an environmental (pollution-generation) multiplier, or if $\mathbf{z}' = \mathbf{n}'$ is a row indicating energy consumption by sector, we would have energy-use multipliers. Energy-use, pollution-generation, and other such multipliers are frequently found in truncated form, as $\mathbf{z}'_c\bar{\mathbf{L}}_{11}$, which is equivalent to setting $\bar{\mathbf{z}}'_c = \begin{bmatrix} \mathbf{z}'_c & 0 \end{bmatrix}$ in Table 6.4. Some of these energy and environmental extensions are discussed in Chapter 10.

6.3 Multipliers in Regional Models

In section 6.2 we presented the basic concepts of various input–output multipliers. All of these multipliers, which quantify impacts on the economy under study, rely on the fact that the $\mathbf{A}$ matrix (as well as the associated coefficients for income, employment, value added, etc.) must represent interindustry relationships *within that economy*. In particular, if sector i is agriculture and sector j is food processing, a_{ij} must represent the value of inputs of agricultural products *produced within the economy* (not imported) per dollar's worth of output of the food-processing sector in the same economy.

6.3.1 Regional Multipliers

Very often an analyst is interested in impacts at a regional level. For example, the federal government may be trying to decide where to award a new military contract and have as one of its concerns the stimulation of economic development in one or more less-developed regions. A state government may wish to allocate funds for labor skill training in one or more industries among several counties with currently above-average levels of unemployment, and so on. In a single-region input–output model, as in section 3.2, the $\mathbf{A}^r = \hat{\mathbf{p}}^r\mathbf{A}$ matrix represented one way of trying to capture regional interrelationships among sectors, and the various kinds of multipliers discussed above would acquire a spatial dimension by using the elements of $\mathbf{A}^r$ and its associated Leontief inverse.

For example, in section 3.2 a national table, $\mathbf{A} = \begin{bmatrix} .15 & .25 \\ .20 & .05 \end{bmatrix}$, was modified because of the assumption that in region r the basic technology of production in sectors 1 and 2 was essentially the same as that reflected in the two columns of **A**, but the *proportions* of inputs required from sectors 1 and 2 that could be expected to come from within the region were $p_1^r = 0.8$ and $p_2^r = 0.6$, so $\mathbf{p}^r = \begin{bmatrix} 0.8 \\ 0.6 \end{bmatrix}$, and

$$\mathbf{A}^r = \hat{\mathbf{p}}^r \mathbf{A} = \begin{bmatrix} .12 & .20 \\ .12 & .03 \end{bmatrix} \quad \text{and} \quad \mathbf{L}^r = (\mathbf{I} - \mathbf{A}^r)^{-1} = \begin{bmatrix} 1.169 & 0.241 \\ 0.145 & 1.061 \end{bmatrix}$$

Hence the regional *simple* output multipliers, as in (6.4), are $m(o)_1^r = 1.314$ and $m(o)_2^r = 1.302$. Recall from section 6.2.1 that the output multipliers in the original **A** matrix were $m(o)_1 = 1.518$ and $m(o)_2 = 1.452$. The difference, of course, is due to the fact that the elements of **A** have been reduced, using the regional percentages in $\mathbf{p}^r$, to reflect the need for imports to supply some of the necessary production. Similarly, external output multipliers (not regional – denoted $\tilde{r}$) are $m(o)_1^{\tilde{r}} = 1.518 - 1.314 = 0.204$ for sector 1 and $m(o)_2^{\tilde{r}} = 1.452 - 1.302 = 0.150$ for sector 2. The interpretation of these is similar to that for other output multipliers: for each dollar's worth of final demand in the region for sector 1 output, 20.4 cents' worth of inputs will be needed from firms in all sectors outside of the region. And for each dollar's worth of final demand in the region for sector 2 output, this figure is 15 cents.

If we have estimates of household inputs, household consumption, and income earned *in the region*, the model can be closed with respect to households, allowing calculation of regional total output multipliers. If we assume that the household input coefficients in the region are the same as those for the nation as a whole and that these represent labor supplied by workers living in the region, then $a_{31}^r = 0.30$, $a_{32}^r = 0.25$, and $a_{33}^r = 0.05$. Also, if we assume that sectors 1 and 2 supply 80 percent and 60 percent, respectively, of consumer needs (the same percentages as they supply of the needs for production), then $a_{13}^r = (0.8)(0.05) = 0.04$ and $a_{23}^r = (0.6)(0.40) = 0.24$ so

$$\bar{\mathbf{A}}^r = \begin{bmatrix} .12 & .20 & .04 \\ .12 & .03 & .24 \\ .30 & .25 & .05 \end{bmatrix} \quad \text{and} \quad \bar{\mathbf{L}}^r = (\mathbf{I} - \bar{\mathbf{A}}^r)^{-1} = \begin{bmatrix} 1.217 & 0.282 & 0.123 \\ 0.263 & 1.164 & 0.305 \\ 0.453 & 0.395 & 1.172 \end{bmatrix}$$

Therefore, the regional *total* output multipliers, as in (6.9), are $\bar{m}(o)_1^r = 1.933$ and $\bar{m}(o)_2^r = 1.841$.

With information on regional labor inputs (in monetary terms) and household consumption coefficients, various income multipliers could be found for the region. Value-added multipliers could also be found in exactly parallel ways. No new principles are involved in assessing multiplier effects with a single-region table instead of a national table. However, with many-region input–output models, a wider variety of multipliers is possible. We examine these in the interregional and multiregional cases in turn.

6.3.2 Interregional Input–Output Multipliers

With interregional and multiregional input–output models output, various multiplier effects can be calculated (a) for a single region (region r), (b) for each of the other regions, (c) for the "rest of the economy" (aggregated over *all* regions outside of r), and (d) for the total, many-region (national) economy.

We illustrate the possibilities using a set of hypothetical data for a two-region model. Consider the following coefficients matrices for an interregional model with (the same) three sectors in each region

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^{rr} & \mathbf{A}^{rs} \\ \mathbf{A}^{sr} & \mathbf{A}^{ss} \end{bmatrix} = \begin{bmatrix} .150 & .250 & .050 & .021 & .094 & .017 \\ .200 & .050 & .400 & .167 & .125 & .133 \\ .300 & .250 & .050 & .050 & .050 & .000 \\ .075 & .050 & .060 & .167 & .313 & .067 \\ .050 & .013 & .025 & .125 & .125 & .047 \\ .025 & .100 & .100 & .250 & .250 & .133 \end{bmatrix} \tag{6.22}$$

and

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{bmatrix} = \begin{bmatrix} 1.462 & .506 & .332 & .259 & .382 & .147 \\ .721 & 1.514 & .761 & .558 & .629 & .324 \\ .678 & .578 & 1.378 & .318 & .390 & .147 \\ .318 & .253 & .251 & 1.428 & .649 & .190 \\ .177 & .123 & .124 & .268 & 1.315 & .114 \\ .346 & .365 & .365 & .598 & .695 & 1.300 \end{bmatrix} \tag{6.23}$$

Recall from Chapter 3 that we use subscript numbers for elements (submatrices) of a partitioned interregional $\mathbf{L}$ matrix because $\mathbf{L}^{rr}$ and $\mathbf{L}^{ss}$ are used for $(\mathbf{I} - \mathbf{A}^{rr})^{-1}$ and $(\mathbf{I} - \mathbf{A}^{ss})^{-1}$, respectively.

Intraregional Effects For exogenous changes in final demands for region r goods (the first three elements in a six-element $\mathbf{f}$ vector), the elements in the 3×3 submatrix $\mathbf{L}_{11}$ represent impacts on the outputs of sectors in region r. Here

$$\mathbf{L}_{11} = \begin{bmatrix} 1.462 & .506 & .332 \\ .721 & 1.514 & .761 \\ .678 & .578 & 1.378 \end{bmatrix} \tag{6.24}$$

Simple intraregional output multipliers for region r are found as the column sums of $\mathbf{L}_{11}$;

$$\mathbf{m}(o)^{rr} = \mathbf{i}'[\mathbf{L}_{11}] = \begin{bmatrix} 2.861 & 2.598 & 2.471 \end{bmatrix} \tag{6.25}$$

Similarly, for region s,

$$\mathbf{m}(o)^{ss} = \mathbf{i}'[\mathbf{L}_{22}] = \begin{bmatrix} 2.294 & 2.659 & 1.604 \end{bmatrix} \tag{6.26}$$

If we had household input coefficients in monetary terms for regions r ($a^{rr}_{n+1,j}$) and s ($a^{ss}_{n+1,j}$), we could find simple intraregional household income multipliers and type I income multipliers. Note that finding total intraregional output multipliers, household income multipliers, or type II income multipliers requires that we have labor input coefficients (in monetary terms) and household consumption coefficients for four different matrices. Initially, the input coefficients matrix for region r – $\mathbf{A}^{rr}$ in (6.22) – must be closed with respect to households. This then adds a row to $\mathbf{A}^{rs}$ and a column to $\mathbf{A}^{sr}$. The former represents inputs of labor from region r to sector 1, 2, and 3 production in region s (for example, commuters). The latter represents purchases of outputs of sectors 1, 2, and 3 in region s by consumers located in region r (imports of consumer goods). For complete consistency, in order to capture income-generating effects throughout the entire (here, two-region) system, the input coefficients matrix for region s – $\mathbf{A}^{ss}$ in (6.22) – should also be closed with respect to households. This then additionally requires a new row in $\mathbf{A}^{sr}$ and a new column in $\mathbf{A}^{rs}$. These new coefficients represent inputs of labor from region s to production in r and purchases by consumers in s of goods made in r, respectively. Thus the $\bar{\mathbf{A}}$ and $\bar{\mathbf{L}}$ matrices, for our numerical example, would grow from 6×6 to 8×8.

Given this $\bar{\mathbf{L}}$ matrix, total intraregional output multipliers, household income multipliers, and type II income multipliers for region r would be found using the elements from the upper left submatrix – now 4×4 – in $\bar{\mathbf{L}}$. Similarly, using intraregional physical labor input coefficients or value-added coefficients for both regions, total intraregional employment or value-added multipliers and type II multipliers could be found.

Interregional Effects The essence of an interregional (or multiregional) input–output model is that it includes impacts in one region that are caused by changes in another region; these are often termed the interregional spillover effects. In our example, these are reflected in the $\mathbf{L}_{12}$ and $\mathbf{L}_{21}$ matrices; here

$$\mathbf{L}_{21} = \begin{bmatrix} .318 & .253 & .251 \\ .177 & .123 & .124 \\ .346 & .365 & .365 \end{bmatrix} \tag{6.27}$$

Consider, $(l_{21})_{23} = 0.124$; this indicates that for each dollar's worth of final demand for the output of sector 3 in region r, 12.4 cents' worth of output from sector 2 in region s is required as input.

Thus, in an interregional input–output model, we can calculate simple interregional multipliers, $m(o)^{sr}_{\cdot j}$ – the total value of output from all sectors in region s used to satisfy a dollar's worth of final demand for sector j in region r. Here,

$$\mathbf{m}(o)^{sr} = \mathbf{i}'[\mathbf{L}_{21}] = \begin{bmatrix} 0.841 & 0.741 & 0.740 \end{bmatrix} \tag{6.28}$$

These are output impacts that are transmitted across regional boundaries – here from r (where the exogenous change occurs) to s (where production occurs). As the reader can perhaps imagine by now, we have the same set of possibilities for measuring various interregional income effects, interregional employment effects, and total interregional effects using the same kinds of calculations as for intraregional effects, now using $\mathbf{L}_{21}$ (and $\bar{\mathbf{L}}_{21}$ if the regions were closed with respect to households). Interregional effects whose origins are in new final demand in region s would be calculated using the elements of $\mathbf{L}_{12}$ (or $\bar{\mathbf{L}}_{12}$). Here

$$\mathbf{m}(o)^{rs} = \mathbf{i}'[\mathbf{L}_{12}] = \begin{bmatrix} 1.135 & 1.401 & 0.618 \end{bmatrix} \tag{6.29}$$

National Effects Assuming, once again, that there are exogenous increases in final demands for region r goods and hence in outputs of region r sectors, we can denote as national effects the sums of columns in both $\mathbf{L}_{11}$ and $\mathbf{L}_{21}$. (These could logically also be termed *total* effects, but we have used *total*, as contrasted with simple, for effects that are calculated from a matrix that has households endogenous.) Arranged as row vectors,

$$\begin{aligned} \mathbf{m}(o)^{r} &= \mathbf{i}'\begin{bmatrix} \mathbf{L}_{11} \\ \mathbf{L}_{21} \end{bmatrix} = \begin{bmatrix} 3.702 & 3.339 & 3.211 \end{bmatrix} \\ \mathbf{m}(o)^{s} &= \mathbf{i}'\begin{bmatrix} \mathbf{L}_{12} \\ \mathbf{L}_{22} \end{bmatrix} = \begin{bmatrix} 3.429 & 4.060 & 2.222 \end{bmatrix} \end{aligned} \tag{6.30}$$

For the two-region interregional system, let $\mathbf{m}(o) = \begin{bmatrix} \mathbf{m}(o)^{r} & \mathbf{m}(o)^{s} \end{bmatrix}$. Here

$$\mathbf{m}(o) = \mathbf{i}'\mathbf{L} = \begin{bmatrix} 3.702 & 3.339 & 3.211 & 3.429 & 4.060 & 2.222 \end{bmatrix} \tag{6.31}$$

A policy implication from these figures is that a dollar's worth of government spending on the output of sector 2 in region s would have the greatest impact throughout the two-region economy, as measured by total output (direct plus indirect) required from all sectors in both regions. Similarly, if the government is interested in acquiring goods from sector 1 or sector 3, the greatest *national* (both regions) economic impact will occur if the purchases are made from firms in region r.

Again, using information on labor inputs or value added in each region, simple and type I income, employment and value-added effects could be calculated at the national (all regions) level. Similarly, for a system in which all regions have been closed with respect to households, total national output, income, employment and value-added effects and type II multipliers can be found.

Sectoral Effects As a final kind of multiplier, we can find the impact on sector i throughout the entire country, because of a dollar's worth of final demand for sector j in either region. (Since this crosses regional boundaries, it is also a kind of "national" effect.) Denote this simple output multiplier as $m(o)_{ij}^{\cdot r}$ and $m(o)_{ij}^{\cdot s}$. For our example,

$$\begin{aligned} m(o)_{13}^{\cdot r} &= (l_{11})_{13} + (l_{21})_{13} = 0.332 + 0.251 = 0.583 \\ m(o)_{21}^{\cdot s} &= (l_{22})_{21} + (l_{12})_{21} = 0.268 + 0.558 = 0.826 \end{aligned}$$

and so on. With additional region-specific information (labor input or value-added coefficients) we could find various simple or type I effects; with elements from $\bar{\mathbf{L}}$, we would find total multipliers and type II effects. (These kinds of sectoral effects are only meaningful when each region contains the same sectors.)

More Than Two Regions With models of more than two regions, there are no new principles involved, although the possibilities increase. For example, with three regions, one can trace interregional effects in now six different ways: (1) exogenous changes in region 1 affecting outputs in region 2 and/or region 3, (2) exogenous changes in 2 affecting outputs in 1 and/or 3, and (3) exogenous changes in 3 affecting outputs in 1 and/or 2.

6.3.3 Multiregional Input–Output Multipliers

All of the multipliers found in the interregional input–output model have their counterparts in the multiregional model. This is to be expected, since the multiregional model is an attempt to capture all of the connections in the interregional model using a simpler set of data. Each of the components in the interregional case – for example, $\mathbf{A}^{rr}$ and $\mathbf{A}^{rs}$ – has its counterpart estimate – $\hat{\mathbf{c}}^{rr}\mathbf{A}^{r}$ and $\hat{\mathbf{c}}^{rs}\mathbf{A}^{s}$ – in the multiregional case. A thorough exploration of multipliers in the multiregional input–output model can be found in DiPasquale and Polenske (1980).

The final form of the multiregional model was

$$\mathbf{x} = (\mathbf{I} - \mathbf{CA})^{-1}\mathbf{Cf} \tag{6.32}$$

Here $\mathbf{A} = \begin{bmatrix} \mathbf{A}^{r} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{s} \end{bmatrix}$ is a block diagonal matrix whose submatrices represent regional technical (not regional input) coefficients and $\mathbf{C} = \begin{bmatrix} \hat{\mathbf{c}}^{rr} & \hat{\mathbf{c}}^{rs} \\ \hat{\mathbf{c}}^{sr} & \hat{\mathbf{c}}^{ss} \end{bmatrix}$, where the components of the submatrices in $\mathbf{C}$ represent flows between regions in the form of proportions of a commodity in a region that come from within the region and from each of the other regions.

The important point to be recalled is that in the interregional model the exogenous sectors represent final demands, wherever located, for goods made by producers in a particular region. In the multiregional model, the $\mathbf{f}$'s represent demands exercised by exogenous sectors located in a given region for goods, wherever produced. For a two-region multiregional model, it is the $\hat{\mathbf{c}}^{rr}$ and $\hat{\mathbf{c}}^{sr}$ matrices that spatially distribute the final demand in region r between producers in r and producers in s.

For example, assume that there are two sectors in each of the two regions and that we want to assess the impact throughout the two-region system of an increase of \$100 in final demand for good 1 by households located in region r, so $\mathbf{f}^r = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$ and $\mathbf{f}^s = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$;

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}^r \\ \mathbf{f}^s \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let

$$\hat{\mathbf{c}}^{rr} = \begin{bmatrix} 0.7 & 0 \\ 0 & 0.4 \end{bmatrix}, \hat{\mathbf{c}}^{rs} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.3 \end{bmatrix}, \hat{\mathbf{c}}^{sr} = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.6 \end{bmatrix}, \hat{\mathbf{c}}^{ss} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.7 \end{bmatrix}$$

Then

$$\mathbf{C} = \begin{bmatrix} \hat{\mathbf{c}}^{rr} & \hat{\mathbf{c}}^{rs} \\ \hat{\mathbf{c}}^{sr} & \hat{\mathbf{c}}^{ss} \end{bmatrix} = \begin{bmatrix} 0.7 & 0 & 0.2 & 0 \\ 0 & 0.4 & 0 & 0.3 \\ 0.3 & 0 & 0.8 & 0 \\ 0 & 0.6 & 0 & 0.7 \end{bmatrix}$$

and the $\mathbf{Cf}$ term that postmultiplies $(\mathbf{I} - \mathbf{CA})^{-1}$ in (6.32) is

$$\mathbf{Cf} = \begin{bmatrix} 0.7 & 0 & 0.2 & 0 \\ 0 & 0.4 & 0 & 0.3 \\ 0.3 & 0 & 0.8 & 0 \\ 0 & 0.6 & 0 & 0.7 \end{bmatrix} \begin{bmatrix} 100 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 70 \\ 0 \\ 30 \\ 0 \end{bmatrix}$$

The impact of the new \$100 is not felt exclusively in region r, rather only \$70 (70 percent) is presented as new demand for good 1 made in region r, and \$30 (30 percent) turns out to be new demand for good 1 in region s.

The $\mathbf{C}$ matrix distributes the final demands in the multiregional model across supplying regions in accordance with the percentages embodied in the components of $\mathbf{C}$. Premultiplication of $\mathbf{Cf}$ by $(\mathbf{I} - \mathbf{CA})^{-1}$ then converts these distributed final demands into necessary outputs from each sector in each region in the usual way. Thus the matrix from which the various multipliers are derived in the multiregional model is $(\mathbf{I} - \mathbf{CA})^{-1}\mathbf{C}$.

In the numerical illustration in section 3.4.4, with two regions of three sectors each, we found

$$
(\mathbf{I}-\mathbf{CA})^{-1}\mathbf{C}=\begin{bmatrix}
1.127 & .447 & .300 & .478 & .418 & .153 \\
.628 & 1.317 & .606 & .552 & 1.115 & .323 \\
.512 & .526 & 1.101 & .335 & .470 & .247 \\
 & & & & & \\
.625 & .369 & .250 & 1.224 & .456 & .216 \\
.238 & .385 & .205 & .278 & .650 & .167 \\
.472 & .445 & .589 & .594 & .529 & 1.232
\end{bmatrix}
$$

in (3.31). This matrix plays the same role for multiplier analysis in the multiregional model that $\mathbf{L}=(\mathbf{I}-\mathbf{A})^{-1}=\begin{bmatrix}\mathbf{L}_{11} & \mathbf{L}_{12} \\ \mathbf{L}_{21} & \mathbf{L}_{22}\end{bmatrix}$ in (6.23) did for the interregional case. We examine some of these possibilities; the parallels with the interregional case should be clear, so the illustrations need not be exhaustive. To emphasize the parallel, we define

$$
\boldsymbol{\mathcal{L}}=(\mathbf{I}-\mathbf{CA})^{-1}\mathbf{C}=\begin{bmatrix}\boldsymbol{\mathcal{L}}_{11} & \boldsymbol{\mathcal{L}}_{12} \\ \boldsymbol{\mathcal{L}}_{21} & \boldsymbol{\mathcal{L}}_{22}\end{bmatrix}
$$

Intraregional Effects Column sums of elements in $\boldsymbol{\mathcal{L}}_{11}$ and $\boldsymbol{\mathcal{L}}_{22}$ are simple intraregional output multipliers. These multipliers correspond to (6.25) and (6.26), above; here

$$
\begin{aligned}
\mathbf{m}(o)^{rr} &= \mathbf{i}'\boldsymbol{\mathcal{L}}_{11}=\begin{bmatrix}2.267 & 2.290 & 2.007\end{bmatrix} \\
\mathbf{m}(o)^{ss} &= \mathbf{i}'\boldsymbol{\mathcal{L}}_{22}=\begin{bmatrix}2.096 & 1.635 & 1.615\end{bmatrix}
\end{aligned}
\tag{6.33}
$$

As before, income, employment or value-added multipliers could be found if we had the requisite additional data. Closing the multiregional model with respect to households, in order to be able to calculate total and type II multipliers, requires the addition of regional labor input coefficient rows and household consumption coefficient columns to each of the regional input matrices in $\mathbf{A}$, and it requires estimates of $c^{rr}_{n+1,n+1}$, $c^{rs}_{n+1,n+1}$, and so on – these are the proportions of household demands for labor services that are expected to be supplied from within and from outside of each region. These coefficients would be added to the lower right of each diagonal matrix $\hat{\mathbf{c}}^{rr}$, $\hat{\mathbf{c}}^{rs}$, etc. Given $(\mathbf{I}-\bar{\mathbf{C}}\bar{\mathbf{A}})^{-1}\bar{\mathbf{C}}$, using overbars to indicate a model in which households have been made endogenous, we could find these various intraregional multipliers in the usual way, from the upper left and lower right submatrices. Also, with information on value added in each sector in each region, value-added multipliers could be found as in the interregional case.

Interregional Effects As in the interregional model, these effects are derived from $\mathcal{L}_{12}$ and $\mathcal{L}_{21}$. Here, corresponding to (6.28) and (6.29), we have

$$\begin{aligned} \mathbf{m}(o)^{sr} &= \mathbf{i}'[\mathcal{L}_{21}] = \begin{bmatrix} 1.335 & 1.199 & 1.044 \end{bmatrix} \\ \mathbf{m}(o)^{rs} &= \mathbf{i}'[\mathcal{L}_{12}] = \begin{bmatrix} 1.365 & 2.003 & 0.723 \end{bmatrix} \end{aligned} \tag{6.34}$$

National Effects Corresponding to (6.30), we have the following simple output multipliers that reflect production in all sectors in all (here, the two) regions to support a dollar's worth of new final demand for a particular good. Here

$$\begin{aligned} \mathbf{m}(o)^{r} &= \mathbf{i}'\begin{bmatrix} \mathcal{L}_{11} \\ \mathcal{L}_{21} \end{bmatrix} = \begin{bmatrix} 3.602 & 3.489 & 3.051 \end{bmatrix} \\ \mathbf{m}(o)^{s} &= \mathbf{i}'\begin{bmatrix} \mathcal{L}_{12} \\ \mathcal{L}_{22} \end{bmatrix} = \begin{bmatrix} 3.461 & 3.638 & 2.338 \end{bmatrix} \end{aligned} \tag{6.35}$$

Thus, a new dollar's worth of demand from households located in r for good 2 generates a total of \$3.49 new output throughout the entire multiregional system. Arranged in a single row vector, and parallel to (6.31), we have

$$\mathbf{m}(o) = \mathbf{i}'\mathcal{L} = \begin{bmatrix} 3.602 & 3.489 & 3.501 & 3.461 & 3.638 & 2.338 \end{bmatrix} \tag{6.36}$$

and similar kinds of policy implications can be drawn from these figures. For example, assume that the government could stimulate consumer demand in a particular region for a particular product (e.g., through tax credits, as for insulation and storm windows in cold regions). The greatest overall (national) effect, as measured by these simple national output multipliers, would come from consumer demand in region s for good 2.

Sectoral Effects Finally, as with the interregional model, we can assess the impact on sector i throughout the economy of one dollar's worth of new final demand in region r for good j. For example, $m(o)^{\cdot r}_{13} = (\ell_{11})_{13} + (\ell_{21})_{13} = 0.300 + 0.250 = 0.550$, $m(o)^{\cdot s}_{21} = (\ell_{22})_{21} + (\ell_{12})_{21} = 0.278 + 0.552 = 0.830$, and so on.

Final Demand for Goods Made in a Particular Region If one is using the version of the multiregional input–output model in which impacts of new region-specific final demands are being assessed (as in the example of a foreign airline's new order for Boeing jetliners made in the state of Washington), where

$$\mathbf{x} = (\mathbf{I} - \mathbf{CA})^{-1}\mathbf{f}^*$$

as in (3.32) in Chapter 3, then all of the multiplier calculations outlined above would be found from the elements in $(\mathbf{I} - \mathbf{CA})^{-1}$ rather than $(\mathbf{I} - \mathbf{CA})^{-1}\mathbf{C}$. The $(\mathbf{I} - \mathbf{CA})^{-1}$ matrix for this numerical example was given in (3.33) in that chapter. The interested reader may wish to find the various multipliers, as in (6.33) through (6.35).

More Than Two Regions As before, with models of more than two regions, there are no new principles involved, although the possibilities for multiplier calculations increase. For example, with three regions, there are three possible settings in which to calculate various intraregional multiplier effects and six in which to calculate interregional effects.

In section 3.4.6 we introduced a three-sector, three-region aggregation of the Chinese 2000 multiregional model. The $\boldsymbol{\mathcal{L}} = (\mathbf{I} - \mathbf{CA})^{-1}\mathbf{C}$ matrix for that model is repeated below, in Table 6.5. (This was Table 3.9 in Chapter 3.) The regional aggregations used in this table result in very large geographic aggregates, and the relative uniformity of the simple output multipliers across regions, as indicated in the tables to follow, reflects this. Simple intra- and interregional output multipliers for this Chinese model are presented in Table 6.6. In addition, simple national (all-region) multipliers are shown.

For example, a ¥1 change in final demand in the North for manufacturing and construction (sector 2) requires ¥0.41 from all sectors in the South and ¥0.04 from the Rest of China. In view of the sectoral breakdown used in this model it is not surprising that manufacturing and construction (sector 2) has the largest simple output multiplier in each region and in the nation as a whole, or that the services sector has the second-largest multipliers with natural resources a rather distant third.

In terms of regional dependencies, we see that the South is much more dependent on the North than on the Rest of China for the inputs that would be needed to satisfy one unit of final demand in each of the sectors in the South – from the sums of the three elements in the North row for the South, 0.4683, vs. the sums of the three elements in the Rest of China row, 0.1467. Similar aggregate measures can be derived for the other regions.

Sector-specific simple output multipliers, $m(o)_{ij}^{\cdot r}$, are shown in Table 6.7. There is a great deal of uniformity across regions. For example, ¥1 of new demand for manufacturing and construction output by households located in the North, South or Rest of China regions generates a national impact in terms of ¥ worth of new output in sector 1 of 0.3321, 0.3249, or 0.3413 in the three regions. Similarly, ¥1 worth of new final demand for services generates a need for inputs of ¥0.5801, 0.6157 or 0.5033 worth of new manufacturing and construction output in the three regions. The figures are generally similar in other rows of Table 6.7. Again, this is primarily because of the very large sizes of the three regions in this model illustration.

Hioki (2005) presents an empirical analysis for the Chinese economy, using the same Chinese MRIO data but at greater levels of disaggregation. This is an analysis of the magnitude of interregional spread or "trickle down" effects, especially from eastern Chinese (coastal) regions to the less developed western (inland) regions. The study calculated intraregional and interregional simple output multipliers for an eight-region, 17-sector version of the CMRIO model. Illustrative of the kinds of conclusions drawn in this study is the observation that around 20 percent of the total output in the Central region is induced by final demands of the coastal regions (p. 170). This suggests that the government's strategy, begun during the 1980s, favoring development of the coastal

Table 6.5 Leontief Inverse Matrix, $\mathcal{L}$, for the Chinese Multiregional Economy, 2000

	North			South			Rest of China		
	Nat. Res.	Manuf. & Const.	Services	Nat. Res.	Manuf. & Const.	Services	Nat. Res.	Manuf. & Const.	Services
North									
Natural Resources	1.1631	0.2561	0.0965	0.0227	0.0582	0.0268	0.0064	0.0161	0.0085
Manuf. & Const.	0.3008	1.7275	0.4080	0.0537	0.1596	0.0849	0.0191	0.0529	0.0314
Services	0.0840	0.1686	1.1794	0.0115	0.0306	0.0202	0.0035	0.0093	0.0054
South									
Natural Resources	0.0325	0.0681	0.0321	1.1919	0.2504	0.1114	0.0245	0.0459	0.0232
Manuf. & Const.	0.1194	0.2943	0.1588	0.3258	1.9193	0.5036	0.0742	0.2010	0.1187
Services	0.0193	0.0447	0.0284	0.0848	0.1920	1.1965	0.0142	0.0375	0.0252
ROC									
Natural Resources	0.0034	0.0079	0.0039	0.0062	0.0164	0.0082	1.1958	0.2793	0.1061
Manuf. & Const.	0.0098	0.0245	0.0133	0.0176	0.0478	0.0272	0.2068	1.5681	0.3532
Services	0.0021	0.0051	0.0030	0.0045	0.0114	0.0075	0.0730	0.1916	1.1716

Table 6.6 Simple Intra- and Interregional Output Multipliers for the Chinese Multiregional Input–Output System, 2000

	Region and Sector Experiencing a One-Unit Change in Final Demand								
	North			South			Rest of China		
	1	2	3	1	2	3	1	2	3
Total Output to Satisfy the Final Demand Change									
North	1.5479	2.1522	1.6840	0.0879	0.2485	0.1319	0.0289	0.0783	0.0454
South	0.1711	0.4071	0.2193	1.6024	2.3616	1.8115	0.1128	0.2844	0.1670
RoC	0.0154	0.0375	0.0202	0.0283	0.0756	0.0428	1.4755	2.0389	1.6309
Nation	1.7344	2.5967	1.9234	1.7187	2.6856	1.9862	1.6173	2.4016	1.8433

Table 6.7 Sector-Specific Simple Output Multipliers for the Chinese Multiregional Input–Output System, 2000

	Sector and Region Experiencing a One-Unit Change in Final Demand								
	Natural Resources			Manufacturing and Construction			Services		
	North	South	RoC	North	South	RoC	North	South	RoC
1	1.1990	1.2208	1.2267	0.3321	0.3249	0.3413	0.1325	0.1464	0.1378
2	0.4300	0.3970	0.3000	2.0462	2.1267	1.8220	0.5801	0.6157	0.5033
3	0.1054	0.1008	0.0906	0.2184	0.2340	0.2384	1.2108	1.2241	1.2022

regions (which it was thought would then lead to spillovers inland) has "actually started to work to a certain extent" (p. 171).[9]

6.4 Miyazawa Multipliers

The important work of Miyazawa (1976) on endogenizing households in an input–output model generates various multiplier matrices.[10] A comprehensive overview of the explicit demographic-economic interactions in the Miyazawa structure and its applications can be found in the collection of papers in Hewings *et al.* (1999). In this section we depart from some of the notation used elsewhere in this book, in order to be consistent with that used by Miyazawa, since virtually all subsequent discussion and application of the Miyazawa framework has continued to use his notation. Specifically, this means that we will now define $\mathbf{B} = (\mathbf{I} - \mathbf{A})^{-1}$ (instead of $\mathbf{L}$, since Miyazawa uses $\mathbf{L}$ for another purpose, as we will see below).

6.4.1 Disaggregated Household Income Groups

We assume that households can be separated into q distinct income-bracket groups and that payments by producers to wage earners in each of those groups can be identified. Let $\underset{(q\times n)}{\mathbf{V}} = [v_{gj}]$, where v_{gj} represents income paid to a wage earner in income bracket g $(g = 1, \ldots, q)$ per dollar's worth of output of sector j. This is a generalization (to q rows) of the single row of household input coefficients or labor input coefficients in Chapter 2, $\mathbf{h}_R = [a_{n+1,1}, \ldots, a_{n+1,n}]$. Similarly, let $\underset{(n\times q)}{\mathbf{C}} = [c_{ih}]$, where c_{ih} is the amount of sector i's product consumed per dollar of income of households in income group h $(h = 1, \ldots, q)$; this is a generalization (to q columns) of the single

[9] We examine some of the details of construction of this multiregional model in section 8.7.

[10] The definitive work is Miyazawa (1976), although there were several articles preceding that monograph. Most of these were in the *Hitotsubashi Journal of Economics* in the 1960s and early 1970s and were not widely known outside of Japan. More recent work by Sonis and Hewings (1993, 1995) on extended multiregional Miyazawa multipliers can also be found in that journal, as well as elsewhere (e.g., Sonis and Hewings, 2000).

column of household consumption coefficients in Chapter 2, $\mathbf{h}_C = \begin{bmatrix} a_{1,n+1} \\ \vdots \\ a_{n,n+1} \end{bmatrix}$, and yet another use for $\mathbf{C}$ in input–output discussions. So the augmented matrix of coefficients is $\bar{\mathbf{A}} = \begin{bmatrix} \underset{(n\times n)}{\mathbf{A}} & \underset{(n\times q)}{\mathbf{C}} \\ \underset{(q\times n)}{\mathbf{V}} & \underset{(q\times q)}{\mathbf{0}} \end{bmatrix}$, and the expanded input–output system is

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{V} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} + \begin{bmatrix} \mathbf{f}^* \\ \mathbf{g} \end{bmatrix} \tag{6.37}$$

where $\underset{(q\times 1)}{\mathbf{y}}$ is a vector of total income for each of the income groups, $\underset{(n\times 1)}{\mathbf{f}^*}$ is a vector of final demands excluding household consumption (now endogenized) and $\underset{(q\times 1)}{\mathbf{g}}$ is a vector of exogenous income (if any) for the income groups.

Assume that $\underset{(q\times 1)}{\mathbf{g}} = \mathbf{0}$; then the two matrix equations in the system in (6.37) are

$$\mathbf{x} = \mathbf{Ax} + \mathbf{Cy} + \mathbf{f}^* \text{ and } \mathbf{y} = \mathbf{Vx} \tag{6.38}$$

From (6.37),

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{I} - \mathbf{A} & -\mathbf{C} \\ -\mathbf{V} & \mathbf{I} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{f}^* \\ \mathbf{0} \end{bmatrix} \tag{6.39}$$

Using results on inverses of partitioned matrices (Appendix A) it is not difficult to show that the elements of the partitioned inverse in (6.39) can be expressed as

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{B}[\mathbf{I} + \mathbf{C}(\mathbf{I} - \mathbf{VBC})^{-1}\mathbf{VB} & \mathbf{BC}(\mathbf{I} - \mathbf{VBC})^{-1} \\ (\mathbf{I} - \mathbf{VBC})^{-1}\mathbf{VB} & (\mathbf{I} - \mathbf{VBC})^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{f}^* \\ \mathbf{0} \end{bmatrix} \tag{6.40}$$

where, as noted, $\mathbf{B} = (\mathbf{I} - \mathbf{A})^{-1}$.

This can be simplified if, following Miyazawa, we define $\mathbf{VBC} = \mathbf{L}$ and $\mathbf{K} = (\mathbf{I} - \mathbf{L})^{-1} = (\mathbf{I} - \mathbf{VBC})^{-1}$, so that

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \underset{(n\times n)}{\mathbf{B}(\mathbf{I} + \mathbf{CKVB})} & \underset{(n\times q)}{\mathbf{BCK}} \\ \underset{(q\times n)}{\mathbf{KVB}} & \underset{(q\times q)}{\mathbf{K}} \end{bmatrix} \begin{bmatrix} \mathbf{f}^* \\ \mathbf{0} \end{bmatrix} \tag{6.41}$$

Miyazawa defines $\mathbf{L}$ as the matrix of “inter-income-group coefficients” and $\mathbf{K}$ as the “interrelational income multiplier” matrix. A typical element of $\mathbf{L}$ is $l_{gh} = v_{gi}b_{ij}c_{jh}$; this shows the direct increase in the income of group g resulting from expenditure of an additional unit of income by group h. Reading from right to left, household demand (expenditure) of c_{jh} by group h for the output of sector j requires $b_{ij}c_{jh}$ in output from sector i and this, in turn, means income payments from sector i in the amount of $v_{gi}b_{ij}c_{jh}$ to households in group g. Similarly, each element in $\mathbf{K} = (\mathbf{I} - \mathbf{L})^{-1}$ indicates the total increase (direct, indirect and induced) in the income of one group that results from expenditure of an additional unit of income by another group. (An illustration of this approach can be found in the matrix of interrelational income multipliers, $\mathbf{K}$, for 11 income groups in the USA for 1987 that is shown in Rose and Li, 1999.)

From (6.41),

$$\mathbf{x} = \mathbf{B}(\mathbf{I} + \mathbf{CKVB})\mathbf{f}^* \tag{6.42}$$

and

$$\mathbf{y} = \mathbf{KVBf}^* \tag{6.43}$$

In (6.42), the effect of final demands on outputs is seen to be the product of two distinct matrices. The first is the Leontief inverse of the open model, $\mathbf{B}$. The second is $(\mathbf{I} + \mathbf{CKVB})$; this augments the final demand stimulus, $\mathbf{If}^*$, by $\mathbf{CKVBf}^*$, which endogenizes the total income spending effect. Again, starting at the right, $\mathbf{Bf}^*$ generates the initial output (without household spending), $\mathbf{VBf}^*$ indicates the resultant initial income payments to each group, $\mathbf{KVBf}^*$ multiplies that into total income received in each group – this is exactly what is described by the result in (6.43) – and, finally, $\mathbf{CKVBf}^*$ translates that received income into consumption (demand) by each group on each sector's output. Miyazawa denotes $\mathbf{KVB}$ the "multi-sector income multiplier" matrix (or the "matrix multiplier of income formation"), indicating the direct, indirect and induced incomes for each income group generated by the initial final demand.

6.4.2 Miyazawa's Derivation

Miyazawa first derives the results on the interrelational multiplier matrix without reference to partitioned matrices [in Miyazawa, 1976, Chapter 1, sections II(2)–III(1); the partitioned inverse structure appears later in Chapter 1, section III(3)]. He makes extensive use of partitioned matrices later in the book – especially in Part 2 on internal and external matrix multipliers. This is a direction that has been explored and expanded considerably in much of the work of Sonis, Hewings and others (summarized in Sonis and Hewings, 1999, which also contains an extensive set of references to their work). A second direction of research that extends the input–output framework to incorporate interactions between economic and demographic components is associated with the many publications of Batey, Madden and others (summarized in Batey and Madden, 1999, again with many references).

We present Miyazawa's initial approach here primarily for completeness, and because the results are often discussed (briefly) in this form in the literature. He begins with

$$\mathbf{x} = \mathbf{Ax} + \mathbf{CVx} + \mathbf{f}^*$$

from (6.38). From this,

$$\mathbf{x} = (\mathbf{I} - \mathbf{A} - \mathbf{CV})^{-1}\mathbf{f}^* \tag{6.44}$$

and with $\mathbf{B} = (\mathbf{I} - \mathbf{A})^{-1}$, straightforward matrix algebra gives

$$(\mathbf{I} - \mathbf{A} - \mathbf{CV}) = (\mathbf{B}^{-1} - \mathbf{CV})\mathbf{BB}^{-1} = (\mathbf{I} - \mathbf{CVB})\mathbf{B}^{-1}$$

Substituting into (6.44),

$$\mathbf{x} = [(\mathbf{I} - \mathbf{CVB})\mathbf{B}^{-1}]^{-1}\mathbf{f}^*$$

and, from the rule for inverses of products,

$$\mathbf{x} = \mathbf{B}(\mathbf{I} - \mathbf{CVB})^{-1}\mathbf{f}^* \tag{6.45}$$

In this form, we find the original Leontief inverse, $\mathbf{B}$, postmultiplied by $(\mathbf{I} - \mathbf{CVB})^{-1}$, which Miyazawa termed the "subjoined inverse matrix."

A further variation is possible and is sometimes used. Starting with (6.45) and, as earlier, with $\mathbf{VBC} = \mathbf{L}$ and $\mathbf{K} = (\mathbf{I} - \mathbf{L})^{-1}$, then

$$\mathbf{K}(\mathbf{I} - \mathbf{VBC}) = \mathbf{I}$$

Premultiply both sides by $\mathbf{C}$ and postmultiply both sides by $\mathbf{VB}$,

$$\mathbf{CK}(\mathbf{I} - \mathbf{VBC})\mathbf{VB} = \mathbf{CVB} \text{ or } \mathbf{CK}(\mathbf{VB} - \mathbf{VBCVB}) = \mathbf{CVB}$$

Factor out $\mathbf{VB}$ to the left and then subtract both sides from $\mathbf{I}$, giving

$$\mathbf{I} - \mathbf{CKVB}(\mathbf{I} - \mathbf{CVB}) = \mathbf{I} - \mathbf{CVB} \text{ or } \mathbf{I} = \mathbf{CKVB}(\mathbf{I} - \mathbf{CVB}) + \mathbf{I} - \mathbf{CVB}$$

Regrouping terms

$$\mathbf{I} = (\mathbf{I} + \mathbf{CKVB})(\mathbf{I} - \mathbf{CVB})$$

and so, from the fundamental definition of an inverse,

$$(\mathbf{I} - \mathbf{CVB})^{-1} = (\mathbf{I} + \mathbf{CKVB})$$

Putting this result into (6.45) gives

$$\mathbf{x} = \mathbf{B}(\mathbf{I} + \mathbf{CKVB})\mathbf{f}^* \tag{6.46}$$

as in (6.42).

Miyazawa suggests that if labor input coefficients, in $\mathbf{V}$, and household consumption coefficients, in $\mathbf{C}$, are less stable than interindustry coefficients (in $\mathbf{A}$ and consequently in $\mathbf{B}$), there is an advantage to using the format in (6.46) instead of (6.45). Namely, a revised subjoined inverse, $(\mathbf{I} - \mathbf{CVB})^{-1}$, whose order is n, can be found by using $\mathbf{K}$, whose order is q "… [which] in most cases is very much smaller than n …" (Miyazawa, 1976, p. 7). However, inverting large matrices is no longer the concern that it was in the 1970s.

From (6.46), household income, $\mathbf{y} = \mathbf{Vx}$, is seen to be

$$\mathbf{y} = \mathbf{VB}(\mathbf{I} + \mathbf{CKVB})\mathbf{f}^* = (\mathbf{I} + \mathbf{VBCK})\mathbf{VBf}^* = (\mathbf{I} + \mathbf{LK})\mathbf{VBf}^*$$

But since $\mathbf{K} = (\mathbf{I} - \mathbf{L})^{-1}$, $(\mathbf{I} - \mathbf{L})\mathbf{K} = \mathbf{I}$, $\mathbf{LK} = \mathbf{K} - \mathbf{I}$, so $(\mathbf{I} + \mathbf{LK}) = \mathbf{K}$, and

$$\mathbf{y} = \mathbf{KVBf}^* \tag{6.47}$$

as in (6.43).

6.4.3 Numerical Example

We expand the numerical example from Chapter 2, assuming a three-sector economy with households divided into two income groups. Let the augmented coefficients matrix be

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{V} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0.15 & 0.25 & 0.05 & 0.1 & 0.05 \\ 0.2 & 0.05 & 0.4 & 0.2 & 0.1 \\ 0.3 & 0.25 & 0.05 & 0.01 & 0.1 \\ 0.05 & 0.1 & 0.08 & 0 & 0 \\ 0.12 & 0.05 & 0.1 & 0 & 0 \end{bmatrix}$$

In particular, labor income coefficients for the two household groups are given in the two rows of $\mathbf{V} = \begin{bmatrix} 0.05 & 0.1 & 0.08 \\ 0.12 & 0.05 & 0.1 \end{bmatrix}$, and consumption coefficients for those same two groups are given in the two columns of $\mathbf{C} = \begin{bmatrix} 0.1 & 0.05 \\ 0.2 & 0.1 \\ 0.01 & 0.1 \end{bmatrix}$.

Given $\mathbf{V}$, $\mathbf{C}$, and $\mathbf{B} = (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.3651 & .4253 & .2509 \\ .5273 & 1.3481 & .5954 \\ .5698 & .4890 & 1.2885 \end{bmatrix}$, the relevant Miyazawa matrices are easily found to be

$$\mathbf{VBC} = \begin{bmatrix} .0574 & .0454 \\ .0601 & .0480 \end{bmatrix} \quad \text{and} \quad \mathbf{K} = (\mathbf{I} - \mathbf{VBC})^{-1} = \begin{bmatrix} 1.0642 & .0507 \\ .0671 & 1.0536 \end{bmatrix}$$

For example, in this illustration, a direct increase of \$1 in income to households in group 1 leads to a 6.7 cent (k_{21}) increase in income payments to households in group 2. Similarly,

$$\mathbf{KVB} = \begin{bmatrix} .1898 & .2162 & .1960 \\ .2716 & .1894 & .2106 \end{bmatrix}$$

In this case, for example, an additional unit of final demand for the goods of sector 1 generates 27.16 cents in new income for group 2. Furthermore,

$$\mathbf{B}(\mathbf{I} - \mathbf{CVB})^{-1} = \begin{bmatrix} 1.4445 & .4994 & .3234 \\ .6496 & 1.4609 & .7062 \\ .6577 & .5644 & 1.3648 \end{bmatrix} \quad \text{and} \quad \mathbf{BCK} = \begin{bmatrix} .2476 & .1545 \\ .3642 & .2492 \\ .1923 & .2258 \end{bmatrix}$$

(The reader can make appropriate interpretations of the elements in each of these matrices.)

In this case, the Leontief inverse for the augmented system can easily be found directly; it is[11]

$$(\mathbf{I}-\bar{\mathbf{A}})^{-1}=\bar{\mathbf{B}}=\begin{bmatrix}\bar{\mathbf{B}}_{11} & \bar{\mathbf{B}}_{12}\\ \bar{\mathbf{B}}_{21} & \bar{\mathbf{B}}_{22}\end{bmatrix}=\begin{bmatrix}1.4445 & .4994 & .3234 & .2476 & .1545\\ .6496 & 1.4609 & .7062 & .3642 & .2492\\ .6577 & .5644 & 1.3648 & .1923 & .2258\\ \\ .1898 & .2162 & .1960 & 1.0642 & .0507\\ .2716 & .1894 & .2106 & .0671 & 1.0536\end{bmatrix}$$

and the correspondences with elements in $\bar{\mathbf{B}}$ are exactly as expected, namely $\mathbf{K}=\bar{\mathbf{B}}_{22}$, $\mathbf{KVB}=\bar{\mathbf{B}}_{21}$, $\mathbf{BCK}=\bar{\mathbf{B}}_{12}$ and $\mathbf{B}(\mathbf{I}-\mathbf{CVB})^{-1}=\bar{\mathbf{B}}_{11}$.

6.4.4 Adding a Spatial Dimension

We saw in Chapter 3 that interregional or multiregional input–output models were conveniently represented in partitioned matrix form. To incorporate the Miyazawa structure into an IRIO- or MRIO-style model, assume that we have p regions $(k, l = 1, \ldots, p)$ with n sectors $(i, j = 1, \ldots, n)$ each and that we have identified q household income groups $(g, h = 1, \ldots, q)$ in each region. Then the augmented $\mathbf{A}$ matrix would be

$$\bar{\mathbf{A}}=\begin{bmatrix}\underset{(np\times np)}{\mathbf{A}} & \underset{(np\times pq)}{\mathbf{C}}\\ \\ \underset{(pq\times np)}{\mathbf{V}} & \underset{(pq\times pq)}{\mathbf{0}}\end{bmatrix}$$

where

$$\underset{(np\times np)}{\mathbf{A}}=\begin{bmatrix}\underset{(n\times n)}{\mathbf{A}^{11}} & \cdots & \underset{(n\times n)}{\mathbf{A}^{1p}}\\ \vdots & \ddots & \vdots\\ \underset{(n\times n)}{\mathbf{A}^{p1}} & \cdots & \underset{(n\times n)}{\mathbf{A}^{pp}}\end{bmatrix}=\left[a_{ij}^{kl}\right],\quad \underset{(np\times qp)}{\mathbf{C}}=\begin{bmatrix}\underset{(n\times q)}{\mathbf{C}^{11}} & \cdots & \underset{(n\times q)}{\mathbf{C}^{p1}}\\ \vdots & \ddots & \vdots\\ \underset{(n\times q)}{\mathbf{C}^{p1}} & \cdots & \underset{(n\times q)}{\mathbf{C}^{pp}}\end{bmatrix}=\left[c_{ih}^{kl}\right],$$

and

$$\underset{(pq\times np)}{\mathbf{V}}=\begin{bmatrix}\underset{(q\times n)}{\mathbf{V}^{11}} & \cdots & \underset{(q\times n)}{\mathbf{V}^{1p}}\\ \vdots & \ddots & \vdots\\ \underset{(q\times n)}{\mathbf{V}^{p1}} & \cdots & \underset{(q\times n)}{\mathbf{V}^{pp}}\end{bmatrix}=\left[v_{gj}^{kl}\right].$$

Notice that consumption coefficients require knowledge of the spending habits of consumers in each income group in each region on goods from each sector in each region. Similarly, the labor input coefficients require knowledge on payments to laborers in each income group in each region by each sector in each region.

[11] Again, we use $\bar{\mathbf{B}}$ rather than $\bar{\mathbf{L}}$ to be consistent with the Miyazawa literature.

Table 6.8 Interrelational Interregional Income Multipliers

	Region of Income Origin				
Region of Income Receipt	1	2	3	4	Row Total
1	1.23	0.12	0.16	0.07	1.57
2	0.11	1.28	0.13	0.05	1.57
3	0.11	0.03	1.06	0.01	1.14
4	0.44	0.56	0.50	1.77	3.28
Column Total	1.81	1.99	1.85	1.90	

Source: Hewings, Okuyama and Sonis, 2001, Table 9.

The elements in the partitioned inverse in (6.41) will have the same dimensions as $\bar{\mathbf{A}}$, namely

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \underset{(np \times np)}{\mathbf{B}(\mathbf{I} + \mathbf{CKVB})} & \underset{(np \times pq)}{\mathbf{BCK}} \\ \underset{(pq \times np)}{\mathbf{KVB}} & \underset{(pq \times pq)}{\mathbf{K}} \end{bmatrix} \begin{bmatrix} \mathbf{f}^* \\ \mathbf{0} \end{bmatrix}$$

Clearly, this is potentially very demanding of data. However, an illustrative application can be found in Hewings, Okuyama and Sonis (2001) for a 53-sector, four-region model (Chicago and three surrounding suburbs), without division into income groups – that is, $n = 53$, $p = 4$, and $q = 1$. In this case the income formation impacts are across regions rather than income groups. In particular, $\mathbf{K}$ is a 4×4 matrix; it is shown in Table 6.8.[12]

Reading down column 1 for illustration, we find that from an increase of \$1 in income in Region 1, an additional \$0.23 is generated in Region 1, \$0.11 in Regions 2 and 3, and \$0.44 in Region 4. Column sums have an interpretation similar to the more usual output multipliers; they indicate the new income generated throughout the four-region system (Chicago metropolitan area) of an additional \$1 in income in the region at the top of the column. Row sums are a measure of additional income in each region at the left as a result of a \$1 income increase in each region. (As with row sums of the usual Leontief inverse, these are generally less useful results than the column sums.) Often, results in empirically derived interrelational multiplier matrices are normalized in some way to account, for example, for differences in sizes of the regions being studied. A complete interregional Miyazawa analysis would require that we distinguish several income brackets in each region (that is, $q > 1$) and then create consumption coefficients and labor input coefficients for each of those brackets (in each region).

[12] For additional data and details on this application, see Hewings and Parr (2007).

6.5 Gross and Net Multipliers in Input–Output Models

6.5.1 Introduction

Leontief's earliest formulations (for the USA in 1919, 1929, and 1939) were in terms of "net" accounts. The fundamental balance equations had no z_{ii} or a_{ii} terms; in the empirical tables the on-diagonal elements were zero.

> [The interindustry transactions table] would naturally have many empty squares. Those lying along the main diagonal are necessarily left open because our accounting principle does not allow for registration of any transaction within the same firm …" (Leontief, 1951, p. 13)

> The output of an industry … is defined with exclusion of the products consumed by the same industry in which they have been produced. Thus $a_{11} = a_{22} = \cdots = a_{ii} = \cdots = a_{mm} = 0$ by definition. (Leontief, 1951, p. 189)

The 1947 US input–output tables discussed and published in Evans and Hoffenberg (1952) include on-diagonal transactions, coefficients, and inverse elements; in that sense these tables are "gross." They point out that the inverse figures can be adjusted to exclude intra-sector transactions but they do not suggest that as a preferable alternative.[13] In Leontief *et al.* (1953, Chapter 2 by Leontief) the equations in the text are gross but the tables and the equations in the Mathematical Note to Chapter 2 are net. In virtually all later publications (for example, Leontief, 1966, Chapters 2 and 7) on-diagonal elements are included.[14] (For a thoughtful discussion of net and gross input–output accounts, see Jensen, 1978.) This net/gross distinction led to the concept of input–output "net" multipliers, which we explore below.

6.5.2 Multipliers in the Net Input–Output Model

We consider only square systems. Generating a net model simply means that the main diagonals of $\mathbf{Z}$ and $\mathbf{A}$ contain only zeros, and that the gross output vector is reduced by the amount of each sector's intraindustry transactions. As usual, denote by $\hat{\mathbf{Z}}$ the diagonal matrix containing the elements z_{ii}. Then let $\mathbf{Z}_{net} = \mathbf{Z} - \hat{\mathbf{Z}}$, and $\hat{\mathbf{x}}_{net} = \hat{\mathbf{x}} - \hat{\mathbf{Z}}$; this latter is a diagonal matrix of sectoral outputs in the *net* system from which on-diagonal (intrasectoral) transactions have been removed.[15] As usual, input coefficients are found for the net system as

$$\mathbf{A}_{net} = \mathbf{Z}_{net}(\hat{\mathbf{x}}_{net})^{-1} = (\mathbf{Z} - \hat{\mathbf{Z}})(\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1}$$

and

$$(\mathbf{I} - \mathbf{A}_{net}) = \mathbf{I} - (\mathbf{Z} - \hat{\mathbf{Z}})(\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1}$$

[13] In contrast, Georgescu-Roegen (1971) argues that diagonal elements in an input–output model ("internal flows") must be suppressed.

[14] Early input–output tables in the UK (for example, for 1954 and 1963) were presented in "net" form (UK, Central Statistical Office, 1961 and 1970). Fifteen-sector versions of these tables appear in Allen and Lecomber (1975) and Barker (1975).

[15] Alternative notation uses $\check{\mathbf{Z}}$ instead of $\mathbf{Z}_{net}$, and similarly for $\mathbf{A}_{net}$ and $\mathbf{x}_{net}$. We avoid that convention because it becomes cumbersome when the vector $\mathbf{x}_{net}$ needs a hat to indicate the associated diagonal matrix – and a "$\wedge$" on top of a "$\vee$" is just too much.

We now examine an alternative expression for the right-hand side. [This demonstration appears to have originated in Weber, 1998 (in German). It is apparently not widely known, at least outside the German-speaking world.] Using the observation that $(\hat{\mathbf{x}} - \hat{\mathbf{Z}})(\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1} = \mathbf{I}$, it can be shown that[16]

$$(\mathbf{I} - \mathbf{A}_{net}) = [(\mathbf{I} - \mathbf{A})\hat{\mathbf{x}}](\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1}$$

Taking the inverse of both sides,

$$\mathbf{L}_{net} = (\mathbf{I} - \mathbf{A}_{net})^{-1} = \{[(\mathbf{I} - \mathbf{A})\hat{\mathbf{x}}](\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1}\}^{-1}$$

and using the matrix algebra rule for inverses of products (for appropriately sized matrices) that $(\mathbf{MNP})^{-1} = \mathbf{P}^{-1}\mathbf{N}^{-1}\mathbf{M}^{-1}$,

$$\mathbf{L}_{net} = (\hat{\mathbf{x}} - \hat{\mathbf{Z}})\hat{\mathbf{x}}^{-1}(\mathbf{I} - \mathbf{A})^{-1} = \hat{\mathbf{x}}_{net}\hat{\mathbf{x}}^{-1}\mathbf{L} \tag{6.48}$$

from which

$$(\hat{\mathbf{x}}_{net})^{-1}\mathbf{L}_{net} = \hat{\mathbf{x}}^{-1}\mathbf{L} \tag{6.49}$$

[Notice from (6.48) that $\mathbf{L}_{net} = (\hat{\mathbf{x}} - \hat{\mathbf{Z}})\hat{\mathbf{x}}^{-1}\mathbf{L} = (\mathbf{I} - \hat{\mathbf{A}})\mathbf{L}$, where $\hat{\mathbf{A}} = \hat{\mathbf{Z}}\hat{\mathbf{x}}^{-1}$.][17]

Consider household income multipliers for the two systems. Given a vector of total household income by sector, $\mathbf{z}_h = [z_{n+1,1}, \ldots, z_{n+1,n}]$, then $\mathbf{h} = \mathbf{z}_h\hat{\mathbf{x}}^{-1}$ and $\mathbf{h}_{net} = \mathbf{z}_h(\hat{\mathbf{x}}_{net})^{-1}$ are the vectors of earnings coefficients in the gross and net systems, respectively. From (6.49),

$$\mathbf{z}_h(\hat{\mathbf{x}}_{net})^{-1}\mathbf{L}_{net} = \mathbf{z}_h\hat{\mathbf{x}}^{-1}\mathbf{L}$$

or

$$\mathbf{h}_{net}\mathbf{L}_{net} = \mathbf{hL}$$

Thus, the income multipliers in the two systems are equal, and therefore for studies in which these kinds of multiplier results are of interest, it makes no difference which model is used.

This result is equally valid for most other multipliers – value-added, household income, pollution-generation, energy use, etc. – associated with productive activity (Table 6.4). The only exception is for output multipliers – $\mathbf{m}(o) = \mathbf{i}'\mathbf{L}$ and $\mathbf{m}(o)_{net} = \mathbf{i}'\mathbf{L}_{net}$; they will not be equal,[18] since from (6.48) $\mathbf{L}_{net} = \hat{\mathbf{x}}_{net}\hat{\mathbf{x}}^{-1}\mathbf{L}$. However, the transformation from one to the other is straightforward, namely

$$\mathbf{m}(o)_{net} = \mathbf{i}'\mathbf{L}_{net} = \mathbf{i}'\hat{\mathbf{x}}_{net}\hat{\mathbf{x}}^{-1}\mathbf{L}$$

[16] This particular expression for the identity matrix may seem unmotivated, but it cleverly allows for a significant rewriting of the expression for $(\mathbf{I} - \mathbf{A}_{net})$. For the interested reader, the derivation is: $(\mathbf{I} - \mathbf{A}_{net}) = (\hat{\mathbf{x}} - \hat{\mathbf{Z}})(\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1} - (\mathbf{Z} - \hat{\mathbf{Z}})(\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1} = [(\hat{\mathbf{x}} - \hat{\mathbf{Z}}) - (\mathbf{Z} - \hat{\mathbf{Z}})](\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1} = (\hat{\mathbf{x}} - \mathbf{Z})(\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1} = [(\mathbf{I} - \mathbf{Z}\hat{\mathbf{x}}^{-1})\hat{\mathbf{x}}](\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1} = [(\mathbf{I} - \mathbf{A})\hat{\mathbf{x}}](\hat{\mathbf{x}} - \hat{\mathbf{Z}})^{-1}$.

[17] This fact was noted by Evans and Hoffenberg (1952, p. 140) who used a verbal argument and not a matrix algebra demonstration.

[18] Except for the trivial and uninteresting case when $\mathbf{x} = \mathbf{x}_{net}$.

or

$$\mathbf{m}(o) = \mathbf{i}'\mathbf{L} = \mathbf{i}'\hat{\mathbf{x}}(\hat{\mathbf{x}}_{net})^{-1}\mathbf{L}_{net}$$

(Recall that order of multiplication of diagonal matrices makes no difference.)

Numerical Example[19] Let $\mathbf{Z} = \begin{bmatrix} 150 & 500 & 50 \\ 200 & 100 & 400 \\ 300 & 500 & 50 \end{bmatrix}$ so $\mathbf{Z}_{net} = \mathbf{Z} - \hat{\mathbf{Z}} = \begin{bmatrix} 0 & 500 & 50 \\ 200 & 0 & 400 \\ 300 & 500 & 0 \end{bmatrix}$. If $\mathbf{x} = \begin{bmatrix} 1000 \\ 2000 \\ 1000 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} .15 & .25 & .05 \\ .2 & .05 & .4 \\ .3 & .25 & .05 \end{bmatrix}$; $\mathbf{x}_{net} = \begin{bmatrix} 850 \\ 1900 \\ 950 \end{bmatrix}$, $\mathbf{A}_{net} = \mathbf{Z}_{net}(\hat{\mathbf{x}}_{net})^{-1} = \begin{bmatrix} 0 & .2632 & .0526 \\ .2353 & 0 & .4211 \\ .3529 & .2632 & 0 \end{bmatrix}$.
Then $\mathbf{L} = \begin{bmatrix} 1.3651 & .4253 & .2509 \\ .5273 & 1.3481 & .5954 \\ .5698 & .4890 & 1.2885 \end{bmatrix}$ and $\mathbf{L}_{net} = (\mathbf{I} - \mathbf{A}_{net})^{-1} = \begin{bmatrix} 1.1603 & .3615 & .2133 \\ .5010 & 1.2807 & .5656 \\ .5414 & .4646 & 1.2241 \end{bmatrix}$.
In this case,

$$\mathbf{m}(o) = \mathbf{i}'\mathbf{L} = \begin{bmatrix} 2.4623 & 2.2624 & 2.1348 \end{bmatrix}$$

$$\mathbf{m}(o)_{net} = \mathbf{i}'\mathbf{L}_{net} = \begin{bmatrix} 2.2026 & 2.1067 & 2.0030 \end{bmatrix}$$

Here $\hat{\mathbf{x}}(\hat{\mathbf{x}}_{net})^{-1} = \begin{bmatrix} 1.1765 & 0 & 0 \\ 0 & 1.0526 & 0 \\ 0 & 0 & 1.0526 \end{bmatrix}$ and so $\mathbf{m}(o) = \mathbf{i}'\hat{\mathbf{x}}(\hat{\mathbf{x}}_{net})^{-1}\mathbf{L}_{net} =$

$$\begin{bmatrix} 1.1765 & 1.0526 & 1.0526 \end{bmatrix} \begin{bmatrix} 1.1603 & .3615 & .2133 \\ .5010 & 1.2807 & .5656 \\ .5414 & .4646 & 1.2241 \end{bmatrix} = \begin{bmatrix} 2.4623 & 2.2624 & 2.1348 \end{bmatrix}$$

as expected.

Finally, let $\mathbf{z}_h = [100,\ 120, 80]$ (household income payments); then

$$\mathbf{h} = \begin{bmatrix} 0\,.10 & 0.06 & 0.08 \end{bmatrix} \text{ and } \mathbf{h}_{net} = \begin{bmatrix} 0\,.1176 & 0.0632 & 0.0842 \end{bmatrix}$$

from which

$$\mathbf{hL} = \mathbf{h}_{net}\mathbf{L}_{net} = \begin{bmatrix} 0.2137 & 0.1625 & 0.1639 \end{bmatrix}$$

again as expected.

6.5.3 Additional Multiplier Variants

(Indirect Effects)/(Direct Effects) A number of analysts have taken the view that multipliers should not include the initial stimulus, as they do when the basic

[19] We use the 3 × 3 example from earlier but now disregard the fact that sector 3 is households and simply treat this as a general three-sector model illustration.

definition is "total effects"/"direct effects." For example, for output multipliers this means the \$1 of new final-demand for sector j which turns into \$1 of new sector j output. The usual resolution is simply to subtract 1 from each of the elements in $\mathbf{m}(o)$. This is equivalent to replacing $\mathbf{L}$ by $(\mathbf{L}-\mathbf{I})$ in the formula for $\mathbf{m}(o)$, since $\mathbf{i}'(\mathbf{L}-\mathbf{I}) = \mathbf{i}'\mathbf{L} - \mathbf{i}'\mathbf{I} = \mathbf{m}(o) - \mathbf{i}'$. (For example, see Oosterhaven, Piek and Stelder, 1986.)[20] Of course this will not change the *rankings* of the sectors, but it certainly has implications for other kinds of calculations in which the multipliers are used.

The same adjustment [subtracting 1 or using $(\mathbf{L}-\mathbf{I})$] is appropriate for any Type I or Type II multiplier (Table 6.3). As an example, when $\mathbf{r} = \mathbf{h}$, the Type I multiplier, $\mathbf{m}(h) = \mathbf{hL}$ would be converted to $\mathbf{h}(\mathbf{L}-\mathbf{I})\hat{\mathbf{h}}^{-1} = \mathbf{hL}\hat{\mathbf{h}}^{-1} - \mathbf{hI}\hat{\mathbf{h}}^{-1} = \mathbf{m}(h) - \mathbf{i}'$.

"Growth Equalized" Multipliers Policy makers may wish to know the impact on a particular sector of a general expansion in final demand in all sectors (for example, to help identify "bottlenecks") or of changing patterns of final demand. One approach involves what have been called "growth-equalized" multipliers. (See, for example, Gray *et al.*, 1979, and Gowdy, 1991, for these and many additional multipliers.) The motivation is clear: "… size variation among economic sectors prevents meaningful comparisons of multipliers … to add \$1 of output to some sectors represents a much larger rate of growth than it would for other sectors" (Gray *et al.*, 1979, pp. 68, 72, respectively).

Consider output multipliers; again, the principles are the same for all the other possible multipliers. The idea begins with the multiplier *matrix* $\mathbf{M}(o) = \mathbf{L}$. Row sums, $\mathbf{M}(o)\mathbf{i} = \mathbf{Li}$, indicate output effects in each sector when final demand for each sector increases by \$1.00. This is generally considered an unlikely scenario; an obvious variation is to posit an unequal increase in final demand across sectors. For example, instead of $\mathbf{Li}$ one could use $\mathbf{L}\langle \mathbf{f}\langle \mathbf{i}'\mathbf{f}\rangle^{-1}\rangle\mathbf{i}$, where $\langle \mathbf{f}\langle \mathbf{i}'\mathbf{f}\rangle^{-1}\rangle$ is a diagonal matrix showing each sector's final demand as a *proportion* of total final demand, $f_j/\sum_j f_j$; that is, a measure of relative sector size (or importance). (Base-year output proportions, $x_j/\sum_j x_j$, could also be used.) Element (i,j) in the matrix $\mathbf{L}\langle \mathbf{f}\langle \mathbf{i}'\mathbf{f}\rangle^{-1}\rangle$ shows the effect on sector i output of a $\$(f_j/\sum_j f_j)$ increase in j's final demand. Then $\mathbf{L}\langle \mathbf{f}\langle \mathbf{i}'\mathbf{f}\rangle^{-1}\rangle\mathbf{i}$ shows the multiplier effect on each sector's output of a \$1 final-demand increase distributed across sectors according to their proportion of total final demand.

Another possibility is to use equal percentage, not absolute, demand increases across sectors. This is the "growth equalization." For example, elements of the column vector $[\mathbf{M}(o)](0.01)\mathbf{f} = (0.01)\mathbf{Lf}$ indicate output effects in each sector when final demand for each sector increases by one percent, and $(0.01)\mathbf{i}'\mathbf{Lf} = (0.01)[\mathbf{m}(o)]\mathbf{f}$ indicates the economy-wide total output generated. We illustrate with the same three-sector figures.

[20] Since $(\mathbf{L}-\mathbf{I}) = \mathbf{L}(\mathbf{I}-\mathbf{L}^{-1}) = \mathbf{LA}$ or $(\mathbf{L}-\mathbf{I}) = (\mathbf{I}-\mathbf{L}^{-1})\mathbf{L} = \mathbf{AL}$, these modified multipliers could also be found as $\mathbf{i}'\mathbf{AL}$ or $\mathbf{i}'\mathbf{LA}$ (see de Mesnard, 2002, or Dietzenbacher, 2005).

For the example,

$$\mathbf{f} = \begin{bmatrix} 300 \\ 1300 \\ 150 \end{bmatrix} \text{ and } \langle \mathbf{f} \langle \mathbf{i}'\mathbf{f} \rangle^{-1} \rangle = \left[f_j / \sum_j f_j \right] = \begin{bmatrix} 0.1714 & 0 & 0 \\ 0 & 0.7429 & 0 \\ 0 & 0 & 0.0857 \end{bmatrix}$$

In this case,

$$\mathbf{L}\langle \mathbf{f} \langle \mathbf{i}'\mathbf{f} \rangle^{-1} \rangle = \begin{bmatrix} 0.2340 & 0.3159 & 0.0215 \\ 0.0904 & 1.0015 & 0.0510 \\ 0.0977 & 0.3633 & 0.1104 \end{bmatrix} \text{ and } [\mathbf{L}\langle \mathbf{f} \langle \mathbf{i}'\mathbf{f} \rangle^{-1} \rangle]\mathbf{i} = \begin{bmatrix} 0.5714 \\ 1.1429 \\ 0.5714 \end{bmatrix}$$

Using a one percent increase for the growth equalization illustration,

$$\mathbf{L}\langle (0.01)\mathbf{f} \rangle = \begin{bmatrix} 4.0953 & 5.5284 & 0.3764 \\ 1.5820 & 17.5250 & 0.8930 \\ 1.7095 & 6.3576 & 1.9328 \end{bmatrix}$$

and

$$\mathbf{i}'\mathbf{L}\langle (0.01)\mathbf{f} \rangle = [\, 7.3868 \;\; 29.4110 \;\; 3.2022 \,]$$

Recall that for this example the simple output multipliers were

$$\mathbf{m}(o) = \mathbf{i}'\mathbf{L} = \left[2.4623 \;\; 2.2624 \;\; 2.1348 \right]$$

and we see that the relative importance of the sectors is altered (now it is final demand for sector 2 that is the most stimulative; previously – in $\mathbf{m}(o)$ – it was sector 1).

Another Kind of Net Multiplier Standard input–output multipliers (Tables 6.3 and 6.4) are designed to be used with (multiplied by) final demand. Oosterhaven and Stelder (2002a, 2002b) have observed that in the real world, "practitioners" sometimes (perhaps often) use them incorrectly, to multiply total sectoral output (or value added or employment). So they propose *net* multipliers (the terminology could be confusing; these are not multipliers in a net model, as in section 6.5.2). Essentially, they simply convert a standard multiplier so that it can be used in conjunction with total outputs. For example, their Type I *net* output multipliers are $\mathbf{i}'\mathbf{L}\hat{\mathbf{f}}_c$, where $\mathbf{f}_c = [f_j/x_j]$; in their terms, f_j/x_j is the fraction of j's output that may "rightfully be considered exogenous" (Oosterhaven and Stelder, 2002a, p. 536). Specifically, they "decompose" $\mathbf{i}'\mathbf{L}\mathbf{f}$ as follows:

$$\mathbf{i}'\mathbf{L}\mathbf{f} = \mathbf{m}(o)\mathbf{f} = \mathbf{m}(o)\hat{\mathbf{f}}\mathbf{i} = \mathbf{m}(o)\hat{\mathbf{f}}\hat{\mathbf{x}}^{-1}\hat{\mathbf{x}}\mathbf{i} = \mathbf{m}(o)\hat{\mathbf{f}}_c\mathbf{x} = \mathbf{i}'\mathbf{L}\hat{\mathbf{f}}_c\mathbf{x}$$

The *net* multiplier *matrix* is thus $\mathbf{L}\hat{\mathbf{f}}_c$ and the associated *vector* of economy-wide multipliers is $\mathbf{i}'\mathbf{L}\hat{\mathbf{f}}_c = \mathbf{m}(o)\hat{\mathbf{f}}_c$. Other multipliers can be similarly modified.

This work generated considerable discussion and a lengthy and elaborate exchange (de Mesnard, 2002, 2007a, 2007b; Dietzenbacher, 2005, Oosterhaven, 2007), with a variety of interpretations and alternative terminology. In the end, "net contribution" or "net backward linkage" indicators were suggested as a more appropriate label than "multiplier." We will return to an aspect of this in Chapter 12 on linkage measures in input–output models.

6.6 Multipliers and Elasticities

6.6.1 Output Elasticity

Another approach to compensating for differences in industry size is one step further from simply considering percentage increases in final demand (as above, in growth equalized multipliers). The idea is to measure both the stimulus *and* its effect in percentage terms – in this case the percentage change in total output (or income or employment, etc.) due to a percentage change in a given industry's final demand. (See, for example, Mattas and Shrestha, 1991 or Ciobanu, Mattas and Psaltopoulos, 2004.) These (percentage change)/(percentage change) measures are "elasticities" in economics terms.

In particular, consider a one percent change in f_j only, so $(\Delta\mathbf{f})' = [0, \ldots, (0.01)f_j, \ldots, 0]$. Then $\Delta\mathbf{x} = \mathbf{L}\Delta\mathbf{f} =, \begin{bmatrix} l_{1j} \\ \vdots \\ l_{nj} \end{bmatrix}(0.01)f_j$. The economy-wide output change is $\mathbf{i}'\Delta\mathbf{x} = \mathbf{i}'\begin{bmatrix} l_{1j} \\ \vdots \\ l_{nj} \end{bmatrix}(0.01)f_j = \mathbf{m}(o)_j(0.01)f_j$. This percentage change in total output (across all industries) that is generated by $(0.01)f_j$ has been labeled the *output elasticity* of industry j (oe_j) and is defined as

$$oe_j = 100 \times (\mathbf{i}'\Delta\mathbf{x}/\mathbf{i}'\mathbf{x}) = 100 \times \mathbf{m}(o)_j[(0.01)f_j/\mathbf{i}'\mathbf{x}] = \mathbf{m}(o)_j[f_j/\mathbf{i}'\mathbf{x}]$$

(It would be more precise to call this an *output-to-final demand* elasticity, to distinguish it from other elasticities, below.)

Modification of any of the other multipliers in section 6.2.2 – through multiplication by $[f_j/\mathbf{i}'\mathbf{x}]$ – produces exactly parallel results, giving income, employment, etc., elasticities to final demand. Note that these are very similar to the "growth-equalized" multipliers above; in that case, the modification was produced by $\left[f_j \Big/ \sum_j f_j\right]$ while here it is $\left[f_j \Big/ \sum_j x_j\right]$.

6.6.2 Output-to-Output Multipliers and Elasticities

Direct Effects Starting with $z_{ij} = a_{ij}x_j$, consider the direct effect of an exogenous change in industry j's output (Δx_j) – $\Delta x_j \to \Delta z_{ij} = a_{ij}\Delta x_j$. This Δz_{ij} represents new i output *directly* required by j, so $\Delta x_i = \Delta z_{ij}$, and thus $\Delta x_i = a_{ij}\Delta x_j$ or $\Delta x_i/\Delta x_j = a_{ij}$. Now consider a one percent increase in j's output $\Delta x_j = (0.01)x_j$; this means

$\Delta x_i = (0.01)a_{ij}x_j$. So the (i, j)th element of the matrix $(0.01)\mathbf{A}\hat{\mathbf{x}}$ measures the direct effect of j's one percent increase in output on industry i. Expressed as a percentage of i's output, we have $100(\Delta x_i/x_i) = 100(0.01)a_{ij}x_j/x_i = a_{ij}x_j/x_i$. And in matrix form, this is the (i, j)th element of the matrix $\hat{\mathbf{x}}^{-1}\mathbf{A}\hat{\mathbf{x}}$, showing the *direct* effect on industry i's output (percentage change) resulting from a one percent change in industry j's output. This is a *direct output-to-output* elasticity. We will meet the matrix $\hat{\mathbf{x}}^{-1}\mathbf{A}\hat{\mathbf{x}}$ again in Chapter 12, where we explore supply-side input–output models.

Total Effects Elements of the Leontief inverse matrix translate final demand changes into *total* output changes – $\Delta x_i = l_{ij}\Delta f_j$ and $l_{ij} = \Delta x_i/\Delta f_j$. These encompass direct and indirect effects, and they are at the heart of the multipliers explored in previous sections in this chapter. Again, it would be slightly cumbersome but completely accurate to call l_{ij} an *output-to-final-demand multiplier*. Consider l_{jj}, the on-diagonal element in the jth column of $\mathbf{L}$ – $l_{jj} = \Delta x_j/\Delta f_j$ or $\Delta x_j = l_{jj}\Delta f_j$. Define l^*_{ij} as l_{ij}/l_{jj}; then

$$l^*_{ij} = l_{ij}/l_{jj} = [\Delta x_i/\Delta f_j]/[\Delta x_j/\Delta f_j] = \Delta x_i/\Delta x_j$$

or $\Delta x_i = l^*_{ij}\Delta x_j$. Thus, l^*_{ij} could be (and has been) viewed as a *total output-to-output* multiplier.

The matrix of these multipliers, $\mathbf{L}^* = [l^*_{ij}]$, is created by dividing each element in a column of $\mathbf{L}$ by the on-diagonal element for that column – $\mathbf{L}^* = \mathbf{L}(\hat{\mathbf{L}})^{-1}$(as usual, $\hat{\mathbf{L}}$ is a diagonal matrix created from the on-diagonal elements in $\mathbf{L}$). Then each of the elements in column j of $\mathbf{L}^*$ indicates the amount of change in industry i output (the row label) that would be required if the *output* of industry j were increased by one dollar.[21]

Suppose, then, that industry j is projected to increase its output to some new amount, $\bar{x}_j$. Postmultiplication of $\mathbf{L}^*$ by a vector, $\bar{\mathbf{x}}$, with $\bar{x}_j$ as its jth element and zeros elsewhere, will generate a vector of total new outputs, $\mathbf{x}^*$, necessary from each industry in the economy because of the exogenously determined output in industry j. That is,

$$\mathbf{x}^* = \mathbf{L}^*\bar{\mathbf{x}} \tag{6.50}$$

We return to this matrix in Chapter 13 in the context of "mixed" input–output models in which final demands (for some industries) and gross outputs (for the other industries) are specified exogenously.

Moving to elasticity terms, the (i, j)th element of $(0.01)\mathbf{L}\hat{\mathbf{x}}$ gives the (total) new output in industry i caused by a one-percent output increase in industry j. So, exactly parallel to the direct elasticity case, above, the (i, j)th element of $\hat{\mathbf{x}}^{-1}\mathbf{L}\hat{\mathbf{x}}$ gives the percent increase in industry i total output due to an initial exogenous one percent increase in industry j output – the "direct and indirect output elasticity of industry i with respect to the output

[21] This is equivalent to the "total flow" approach of Szyrmer (for example, Szyrmer, 1992). He makes a case for the unsuitability of the usual output multipliers (from the standard demand-driven input–output model) for a wide variety of real-world impact studies. Some analysts argue that the *initial* exogenous one-dollar stimulus should be removed from the "total effect" calculation. As was seen above (section 6.5.3), this can be accomplished by replacing $\mathbf{L}$ by $(\mathbf{L} - \mathbf{I})$. The interested reader should see de Mesnard (2002) and Dietzenbacher (2005) for details.

in industry j" (Dietzenbacher, 2005, p. 426). We will also meet this matrix, $\hat{\mathbf{x}}^{-1}\mathbf{L}\hat{\mathbf{x}}$, again in Chapter 12 in the discussion of supply-side input–output models.

6.7 Multiplier Decompositions

A number of approaches have been suggested for analyzing the economic "structure" that is portrayed in input–output data. Multiplier decompositions are a prominent part of this research, and we explore two of these in this section.[22]

6.7.1 Fundamentals

We start with the fundamental input–output accounting relationship

$$\underset{(n\times 1)}{\mathbf{x}} = \underset{(n\times n)}{\mathbf{A}}\ \underset{(n\times 1)}{\mathbf{x}} + \underset{(n\times 1)}{\mathbf{f}} \tag{6.51}$$

from which $\mathbf{x} = (\mathbf{I}-\mathbf{A})^{-1}\mathbf{f} = \mathbf{Lf}$. We now introduce some algebra that initially appears unmotivated but it will soon be clear what is accomplished. Given some $\underset{(n\times n)}{\tilde{\mathbf{A}}}$, adding and subtracting $\tilde{\mathbf{A}}\mathbf{x}$ to (6.51) and rearranging produces

$$\mathbf{x} = \mathbf{Ax} - \tilde{\mathbf{A}}\mathbf{x} + \tilde{\mathbf{A}}\mathbf{x} + \mathbf{f} \Rightarrow (\mathbf{I} - \tilde{\mathbf{A}})\mathbf{x} = (\mathbf{A} - \tilde{\mathbf{A}})\mathbf{x} + \mathbf{f} \tag{6.52}$$

and, solving[23] for $\mathbf{x}$,

$$\mathbf{x} = (\mathbf{I} - \tilde{\mathbf{A}})^{-1}(\mathbf{A} - \tilde{\mathbf{A}})\mathbf{x} + (\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f}$$

Let $\mathbf{A}^* = (\mathbf{I} - \tilde{\mathbf{A}})^{-1}(\mathbf{A} - \tilde{\mathbf{A}})$; then this is

$$\mathbf{x} = \mathbf{A}^*\mathbf{x} + (\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f} \tag{6.53}$$

Next, premultiply both sides of (6.53) by $\mathbf{A}^*$

$$\mathbf{A}^*\mathbf{x} = (\mathbf{A}^*)^2\mathbf{x} + \mathbf{A}^*(\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f} \tag{6.54}$$

and substitute this for $\mathbf{A}^*\mathbf{x}$ in the right-hand side of (6.53)

$$\mathbf{x} = (\mathbf{A}^*)^2\mathbf{x} + \mathbf{A}^*(\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f} + (\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f} = (\mathbf{A}^*)^2\mathbf{x} + (\mathbf{I} + \mathbf{A}^*)(\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f} \tag{6.55}$$

Again, solving for $\mathbf{x}$,

$$\mathbf{x} = \underbrace{[\mathbf{I} - (\mathbf{A}^*)^2]^{-1}}_{\mathbf{M}_3}\underbrace{(\mathbf{I} + \mathbf{A}^*)}_{\mathbf{M}_2}\underbrace{(\mathbf{I} - \tilde{\mathbf{A}})^{-1}}_{\mathbf{M}_1}\mathbf{f} \tag{6.56}$$

In this way the usual Leontief inverse (multiplier) matrix, $(\mathbf{I} - \mathbf{A})^{-1}$, has been decomposed into the product of three matrices.

[22] For an overview of these and several others, see Sonis and Hewings (1988) or additional references noted in section 14.2, below.
[23] Here and throughout we assume nonsingularity of the matrices whose inverses are shown.

This algebra can be continued. Premultiply both sides of (6.55) by $\mathbf{A}^*$,

$$\mathbf{A}^*\mathbf{x} = (\mathbf{A}^*)^3\mathbf{x} + [\mathbf{A}^* + (\mathbf{A}^*)^2](\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f} \tag{6.57}$$

and, again, substitute for $\mathbf{A}^*\mathbf{x}$ in the right-hand side of (6.53)

$$\mathbf{x} = (\mathbf{A}^*)^3\mathbf{x} + [\mathbf{I} + \mathbf{A}^* + (\mathbf{A}^*)^2](\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f} \tag{6.58}$$

Solving for $\mathbf{x}$, we now find

$$\mathbf{x} = \underbrace{[\mathbf{I} - (\mathbf{A}^*)^3]^{-1}}_{\mathbf{M}_3}\underbrace{[\mathbf{I} + \mathbf{A}^* + (\mathbf{A}^*)^2]}_{\mathbf{M}_2}\underbrace{(\mathbf{I} - \tilde{\mathbf{A}})^{-1}}_{\mathbf{M}_1}\mathbf{f} \tag{6.59}$$

[Compare with the results in (6.56).]

In the context of social accounting matrices (Chapter 11), where much of the fundamental work on multiplier decompositions originated, $\mathbf{M}_1$ is said to capture a "transfer" effect, $\mathbf{M}_2$ embodies "open-loop" effects and $\mathbf{M}_3$ contains "closed-loop" effects. (For example, see Pyatt and Round, 1979.) The logic of these labels will be clear in the interregional context, below.

These iterations can continue any number of times. After k steps, the parallel to (6.58) is

$$\mathbf{x} = (\mathbf{A}^*)^k\mathbf{x} + [\mathbf{I} + \mathbf{A}^* + (\mathbf{A}^*)^2 + \cdots + (\mathbf{A}^*)^{k-1}](\mathbf{I} - \tilde{\mathbf{A}})^{-1}\mathbf{f} \tag{6.60}$$

and the parallel to (6.59) is

$$\mathbf{x} = \underbrace{[\mathbf{I} - (\mathbf{A}^*)^k]^{-1}}_{\mathbf{M}_3}\underbrace{[\mathbf{I} + \mathbf{A}^* + (\mathbf{A}^*)^2 + \cdots + (\mathbf{A}^*)^{k-1}]}_{\mathbf{M}_2}\underbrace{(\mathbf{I} - \tilde{\mathbf{A}})^{-1}}_{\mathbf{M}_1}\mathbf{f} \tag{6.61}$$

6.7.2 Decompositions in an Interregional Context

For a two-region interregional model (section 3.3) the input–output accounting relationship $\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{f}$ becomes

$$\begin{bmatrix} \mathbf{x}^r \\ \mathbf{x}^s \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{rr} & \mathbf{A}^{rs} \\ \mathbf{A}^{sr} & \mathbf{A}^{ss} \end{bmatrix}\begin{bmatrix} \mathbf{x}^r \\ \mathbf{x}^s \end{bmatrix} + \begin{bmatrix} \mathbf{f}^r \\ \mathbf{f}^s \end{bmatrix}$$

With a view toward decompositions, we can isolate the intraregional and interregional elements in $\mathbf{A}$; let

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^{rr} & \mathbf{A}^{rs} \\ \mathbf{A}^{sr} & \mathbf{A}^{ss} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^{rr} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{ss} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{A}^{rs} \\ \mathbf{A}^{sr} & \mathbf{0} \end{bmatrix}$$

Define $\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A}^{rr} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{ss} \end{bmatrix}$ from which $(\mathbf{I} - \tilde{\mathbf{A}}) = \begin{bmatrix} \mathbf{I} - \mathbf{A}^{rr} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} - \mathbf{A}^{ss} \end{bmatrix}$. Then, using the decomposition in (6.56), for example,

$$\mathbf{M}_1 = (\mathbf{I} - \tilde{\mathbf{A}})^{-1} = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^{rr})^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I} - \mathbf{A}^{ss})^{-1} \end{bmatrix}$$

(from the rule that the inverse for a block-diagonal matrix is made up of the inverses of the matrices on the main diagonal). Also,

$$\begin{aligned} \mathbf{A}^* &= (\mathbf{I} - \tilde{\mathbf{A}})^{-1}(\mathbf{A} - \tilde{\mathbf{A}}) \\ &= \begin{bmatrix} (\mathbf{I} - \mathbf{A}^{rr})^{-1} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I} - \mathbf{A}^{ss})^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{A}^{rs} \\ \mathbf{A}^{sr} & \mathbf{0} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} & (\mathbf{I} - \mathbf{A}^{rr})^{-1}\mathbf{A}^{rs} \\ (\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr} & \mathbf{0} \end{bmatrix} \end{aligned}$$

and so, again from (6.56),

$$\mathbf{M}_2 = \mathbf{I} + \mathbf{A}^* = \begin{bmatrix} \mathbf{I} & (\mathbf{I} - \mathbf{A}^{rr})^{-1}\mathbf{A}^{rs} \\ (\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr} & \mathbf{I} \end{bmatrix}$$

Finally, from straightforward matrix multiplication,

$$(\mathbf{A}^*)^2 = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^{rr})^{-1}\mathbf{A}^{rs}(\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr} & \mathbf{0} \\ \mathbf{0} & (\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr}(\mathbf{I} - \mathbf{A}^{rr})^{-1}\mathbf{A}^{rs} \end{bmatrix}$$

and so

$$\mathbf{M}_3 = [\mathbf{I} - (\mathbf{A}^*)^2]^{-1} = \begin{bmatrix} [\mathbf{I} - (\mathbf{I} - \mathbf{A}^{rr})^{-1}\mathbf{A}^{rs}(\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr}]^{-1} & \mathbf{0} \\ \mathbf{0} & [\mathbf{I} - (\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr}(\mathbf{I} - \mathbf{A}^{rr})^{-1}\mathbf{A}^{rs}]^{-1} \end{bmatrix}$$

(again from the rule for the inverse of a block-diagonal matrix).

In terms of intra- and interregional effects, the matrices in $\mathbf{M}_1$ are seen to capture *intraregional* (Leontief inverse or "transfer") effects, those in $\mathbf{M}_2$ contain *interregional spillover* ("open-loop") effects, and the matrices in $\mathbf{M}_3$ record *interregional feedback* ("closed-loop") effects (Round, 1985, 2001; Dietzenbacher, 2002).[24] As usual, define

$$\mathbf{L}^{rr} = (\mathbf{I} - \mathbf{A}^{rr})^{-1} \quad \text{and} \quad \mathbf{L}^{ss} = (\mathbf{I} - \mathbf{A}^{ss})^{-1}$$

[24] There have been other definitions of these various effects in the input–output literature, beginning perhaps with Miller (1966, 1969) but also including, among others, Yamada and Ihara (1969), Round (1985, 2001), or Sonis and Hewings (2001).

These are the intraregional effects in each region ($\mathbf{M}_1$). The two spillover matrices in $\mathbf{M}_2$ may be represented as

$$\mathbf{S}^{rs} = \mathbf{L}^{rr}\mathbf{A}^{rs} \text{ and } \mathbf{S}^{sr} = \mathbf{L}^{ss}\mathbf{A}^{sr}$$

and the two feedback matrices in $\mathbf{M}_3$ can be defined as

$$\mathbf{F}^{rr} = [\mathbf{I} - \mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{L}^{ss}\mathbf{A}^{sr}]^{-1} \text{ and } \mathbf{F}^{ss} = [\mathbf{I} - \mathbf{L}^{ss}\mathbf{A}^{sr}\mathbf{L}^{rr}\mathbf{A}^{rs}]^{-1}$$

or

$$\mathbf{F}^{rr} = [\mathbf{I} - \mathbf{S}^{rs}\mathbf{S}^{sr}]^{-1} \text{ and } \mathbf{F}^{ss} = [\mathbf{I} - \mathbf{S}^{sr}\mathbf{S}^{rs}]^{-1}$$

Therefore, in the two-region interregional context, $\mathbf{x} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{f}$ becomes

$$\begin{bmatrix} \mathbf{x}^r \\ \mathbf{x}^s \end{bmatrix} = \begin{bmatrix} \mathbf{F}^{rr} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}^{ss} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{S}^{rs} \\ \mathbf{S}^{sr} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{L}^{rr} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}^{ss} \end{bmatrix} \begin{bmatrix} \mathbf{f}^r \\ \mathbf{f}^s \end{bmatrix} \tag{6.62}$$

or, carrying out the multiplications,

$$\begin{bmatrix} \mathbf{x}^r \\ \mathbf{x}^s \end{bmatrix} = \begin{bmatrix} \mathbf{F}^{rr}\mathbf{L}^{rr} & \mathbf{F}^{rr}\mathbf{S}^{rs}\mathbf{L}^{ss} \\ \mathbf{F}^{ss}\mathbf{S}^{sr}\mathbf{L}^{rr} & \mathbf{F}^{ss}\mathbf{L}^{ss} \end{bmatrix} \begin{bmatrix} \mathbf{f}^r \\ \mathbf{f}^s \end{bmatrix} \tag{6.63}$$

6.7.3 Stone's Additive Decomposition

An alternative decomposition isolates *net* effects. Starting with the multiplicative result in (6.56) [or (6.59), or (6.61)], namely $\mathbf{x} = \mathbf{M}\mathbf{f}$, where $\mathbf{M} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1$, Stone (1985) proposed the additive form

$$\mathbf{M} = \mathbf{I} + \underbrace{(\mathbf{M}_1 - \mathbf{I})}_{\tilde{\mathbf{M}}_1} + \underbrace{(\mathbf{M}_2 - \mathbf{I})\mathbf{M}_1}_{\tilde{\mathbf{M}}_2} + \underbrace{(\mathbf{M}_3 - \mathbf{I})\mathbf{M}_2\mathbf{M}_1}_{\tilde{\mathbf{M}}_3}$$

(This is easily seen to be true by simply carrying out the algebra on the right-hand side.) Therefore,

$$\mathbf{x} = \mathbf{M}\mathbf{f} = \mathbf{I}\mathbf{f} + \underbrace{(\mathbf{M}_1 - \mathbf{I})}_{\tilde{\mathbf{M}}_1}\mathbf{f} + \underbrace{(\mathbf{M}_2 - \mathbf{I})\mathbf{M}_1}_{\tilde{\mathbf{M}}_2}\mathbf{f} + \underbrace{(\mathbf{M}_3 - \mathbf{I})\mathbf{M}_2\mathbf{M}_1}_{\tilde{\mathbf{M}}_3}\mathbf{f} \tag{6.64}$$

To paraphrase Stone (p. 162) – in the context of an interregional model – we start with a matrix of initial injections, $\mathbf{I}\mathbf{f}$. The second term ($\tilde{\mathbf{M}}_1\mathbf{f}$) adds on the *net* intraregional effects captured in $\mathbf{M}_1$. Next (in $\tilde{\mathbf{M}}_2\mathbf{f}$) we add in the net interregional spillover effects in $\mathbf{M}_2$. Finally, the fourth term ($\tilde{\mathbf{M}}_3\mathbf{f}$) captures the net interregional feedback effects in

$\mathbf{M}_3$. In the two-region example, these are

$$\tilde{\mathbf{M}}_1 = \mathbf{M}_1 - \mathbf{I} = \begin{bmatrix} \mathbf{L}^{rr} - \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}^{ss} - \mathbf{I} \end{bmatrix}$$

$$\tilde{\mathbf{M}}_2 = (\mathbf{M}_2 - \mathbf{I})\mathbf{M}_1 = \begin{bmatrix} \mathbf{0} & \mathbf{S}^{rs} \\ \mathbf{S}^{sr} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{L}^{rr} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}^{ss} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{S}^{rs}\mathbf{L}^{ss} \\ \mathbf{S}^{sr}\mathbf{L}^{rr} & \mathbf{0} \end{bmatrix}$$

$$\tilde{\mathbf{M}}_3 = (\mathbf{M}_3 - \mathbf{I})\mathbf{M}_2\mathbf{M}_1 = \begin{bmatrix} \mathbf{F}^{rr}\mathbf{L}^{rr} - \mathbf{L}^{rr} & \mathbf{F}^{rr}\mathbf{S}^{rs}\mathbf{L}^{ss} - \mathbf{S}^{rs}\mathbf{L}^{ss} \\ \mathbf{F}^{ss}\mathbf{S}^{sr}\mathbf{L}^{rr} - \mathbf{S}^{sr}\mathbf{L}^{rr} & \mathbf{F}^{ss}\mathbf{L}^{ss} - \mathbf{L}^{ss} \end{bmatrix}$$

While these appear (and are) increasingly complex, they serve to disentangle the complex net of intraregional, spillover, and feedback effects.

6.7.4 A Note on Interregional Feedbacks

Interregional feedback effects in a two-region input–output model were explored in section 3.3.2. They were defined early on (Miller 1966, 1969) for the specific scenario of a change in final demand in region r only – so $\Delta\mathbf{f}^r \neq \mathbf{0}$ and $\Delta\mathbf{f}^s = \mathbf{0}$. Then a measure of the interregional feedback effect is found as the difference between the output change in region r that would be generated by the complete two-region model and the output change in region r that would be calculated from a single-region model. These outputs are

$$\Delta\mathbf{x}_T^r = [(\mathbf{I} - \mathbf{A}^{rr}) - \mathbf{A}^{rs}\mathbf{L}^{ss}\mathbf{A}^{sr}]^{-1}\Delta\mathbf{f}^r \text{ and } \Delta\mathbf{x}_S^r = (\mathbf{I} - \mathbf{A}^{rr})^{-1}\Delta\mathbf{f}^r$$

(with subscripts indicating "two-region" and "single-region" models, respectively). Consider the inverse matrix in $\Delta\mathbf{x}_T^r$, $[(\mathbf{I} - \mathbf{A}^{rr}) - \mathbf{A}^{rs}(\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr}]^{-1}$.

1. Factoring out $(\mathbf{I} - \mathbf{A}^{rr})$ gives

$$\{(\mathbf{I} - \mathbf{A}^{rr})[\mathbf{I} - (\mathbf{I} - \mathbf{A}^{rr})^{-1}\mathbf{A}^{rs}(\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr}]\}^{-1}$$

2. Using the rule that $(\mathbf{MN})^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}$, we have

$$[\mathbf{I} - (\mathbf{I} - \mathbf{A}^{rr})^{-1}\mathbf{A}^{rs}(\mathbf{I} - \mathbf{A}^{ss})^{-1}\mathbf{A}^{sr}]^{-1}(\mathbf{I} - \mathbf{A}^{rr})^{-1}$$

Using $\mathbf{L}^{rr} = (\mathbf{I} - \mathbf{A}^{rr})^{-1}$ and $\mathbf{L}^{ss} = (\mathbf{I} - \mathbf{A}^{ss})^{-1}$, we have

$$\Delta\mathbf{x}_T^r = [\mathbf{I} - \mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{L}^{ss}\mathbf{A}^{sr}]^{-1}\mathbf{L}^{rr}\Delta\mathbf{f}^r \text{ and } \Delta\mathbf{x}_S^r = \mathbf{L}^{rr}\Delta\mathbf{f}^r$$

Finally, using $\mathbf{F}^{rr} = [\mathbf{I} - \mathbf{L}^{rr}\mathbf{A}^{rs}\mathbf{L}^{ss}\mathbf{A}^{sr}]^{-1}$ from $\mathbf{M}_3$, above,

$$\Delta\mathbf{x}_T^r - \Delta\mathbf{x}_S^r = \mathbf{F}^{rr}\mathbf{L}^{rr}\Delta\mathbf{f}^r - \mathbf{L}^{rr}\Delta\mathbf{f}^r = (\mathbf{F}^{rr}\mathbf{L}^{rr} - \mathbf{L}^{rr})\Delta\mathbf{f}^r = (\mathbf{F}^{rr} - \mathbf{I})\mathbf{L}^{rr}\Delta\mathbf{f}$$

The $\mathbf{F}^{rr}\mathbf{L}^{rr}$ term is exactly the upper left element in the multiplier matrix from the multiplicative decomposition in (6.63), and the $(\mathbf{F}^{rr} - \mathbf{I})\mathbf{L}^{rr}$ term (for the difference in gross outputs in the two models) is exactly the upper left element in $\tilde{\mathbf{M}}_3$ from the additive decomposition of net effects.

6.7.5 *Numerical Illustration*

We reconsider the two-region example from Chapter 3, in light of these decomposition possibilities. In that example we had

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}^{rr} & \mathbf{Z}^{rs} \\ \mathbf{Z}^{sr} & \mathbf{Z}^{ss} \end{bmatrix} = \begin{bmatrix} 150 & 500 & 50 & 25 & 75 \\ 200 & 100 & 400 & 200 & 100 \\ 300 & 500 & 50 & 60 & 40 \\ 75 & 100 & 60 & 200 & 250 \\ 50 & 25 & 25 & 150 & 100 \end{bmatrix}$$

and

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}^r \\ \mathbf{x}^s \end{bmatrix} = \begin{bmatrix} 1000 \\ 2000 \\ 1000 \\ 1200 \\ 800 \end{bmatrix}$$

with associated direct and total requirements matrices of

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}^{rr} & \mathbf{A}^{rs} \\ \mathbf{A}^{sr} & \mathbf{A}^{ss} \end{bmatrix} = \begin{bmatrix} 0.1500 & 0.2500 & 0.0500 & 0.0208 & 0.0938 \\ 0.2000 & 0.0500 & 0.4000 & 0.1667 & 0.1250 \\ 0.3000 & 0.2500 & 0.0500 & 0.0500 & 0.0500 \\ 0.0750 & 0.0500 & 0.0600 & 0.1667 & 0.3125 \\ 0.0500 & 0.0125 & 0.0250 & 0.1250 & 0.1250 \end{bmatrix}$$

and

$$\mathbf{L} = \begin{bmatrix} 1.4234 & 0.4652 & 0.2909 & 0.1917 & 0.3041 \\ 0.6346 & 1.4237 & 0.6707 & 0.4092 & 0.4558 \\ 0.6383 & 0.5369 & 1.3363 & 0.2501 & 0.3108 \\ 0.2672 & 0.2000 & 0.1973 & 1.3406 & 0.5473 \\ 0.1468 & 0.0908 & 0.0926 & 0.2155 & 1.2538 \end{bmatrix}$$

In addition,[25]

$$\mathbf{L}^{rr} = (\mathbf{I} - \mathbf{A}^{rr})^{-1} = \begin{bmatrix} 1.3651 & 0.4253 & 0.2509 \\ 0.5273 & 1.3481 & 0.5954 \\ 0.5698 & 0.4890 & 1.2885 \end{bmatrix}$$

and

$$\mathbf{L}^{ss} = (\mathbf{I} - \mathbf{A}^{ss})^{-1} = \begin{bmatrix} 1.2679 & 0.4528 \\ 0.1811 & 1.2075 \end{bmatrix}$$

From these we can generate the additional components needed for these decompositions, namely

$$\mathbf{S}^{rs} = \mathbf{L}^{rr}\mathbf{A}^{rs} = \begin{bmatrix} 0.1119 & 0.1937 \\ 0.2654 & 0.2477 \\ 0.1578 & 0.1790 \end{bmatrix} \text{ and } \mathbf{S}^{sr} = \mathbf{L}^{ss}\mathbf{A}^{sr} = \begin{bmatrix} 0.1177 & 0.0691 & 0.0874 \\ 0.0740 & 0.0242 & 0.0411 \end{bmatrix}$$

$$\mathbf{F}^{rr} = [\mathbf{I} - \mathbf{S}^{rs}\mathbf{S}^{sr}]^{-1} = \begin{bmatrix} 1.0296 & 0.0134 & 0.0191 \\ 0.0535 & 1.0262 & 0.0359 \\ 0.0343 & 0.0164 & 1.0228 \end{bmatrix}$$

and

$$\mathbf{F}^{ss} = [\mathbf{I} - \mathbf{S}^{sr}\mathbf{S}^{rs}]^{-1} = \begin{bmatrix} 1.0488 & 0.0599 \\ 0.0228 & 1.0297 \end{bmatrix}$$

The **M** matrices for the multiplicative decomposition are easily found to be

$$\mathbf{M}_1 = \begin{bmatrix} 1.3651 & 0.4253 & 0.2509 & 0 & 0 \\ 0.5273 & 1.3481 & 0.5954 & 0 & 0 \\ 0.5698 & 0.4890 & 1.2885 & 0 & 0 \\ 0 & 0 & 0 & 1.2679 & 0.4528 \\ 0 & 0 & 0 & 0.1811 & 1.2075 \end{bmatrix}$$

for *intraregional transfer* effects, as is expected, since only $\mathbf{L}^{rr}$ and $\mathbf{L}^{ss}$ appear in this matrix. Next

$$\mathbf{M}_2 = \begin{bmatrix} 1 & 0 & 0 & 0.1119 & 0.1937 \\ 0 & 1 & 0 & 0.2654 & 0.2477 \\ 0 & 0 & 1 & 0.1578 & 0.1790 \\ 0.1177 & 0.0691 & 0.0874 & 1 & 0 \\ 0.0740 & 0.0242 & 0.0411 & 0 & 1 \end{bmatrix}$$

[25] Remember that $\mathbf{L}^{rr}$ does not designate the 3×3 submatrix in the upper left of $\mathbf{L}$, and similarly $\mathbf{L}^{ss}$ is not the 2×2 submatrix in the lower right of $\mathbf{L}$.

contains *interregional spillover* ("open-loop") effects only, transmitted from r to s (upper right) and from s to r (lower left). Finally

$$\mathbf{M}_3 = \begin{bmatrix} 1.0296 & 0.0134 & 0.0191 & 0 & 0 \\ 0.0535 & 1.0262 & 0.0359 & 0 & 0 \\ 0.0343 & 0.0164 & 1.0228 & 0 & 0 \\ 0 & 0 & 0 & 1.0488 & 0.0599 \\ 0 & 0 & 0 & 0.0228 & 1.0297 \end{bmatrix}$$

identifies *interregional feedback* ("closed-loop") effects.

We first use the multiplicative decomposition to find $\mathbf{x}^{new} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{f}^{new}$ for our example (Chapter 3) with $(\mathbf{f}^{new})' = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 \end{bmatrix}$. This will generate $\mathbf{x}^{new} = \begin{bmatrix} 142.34 \\ 63.46 \\ 63.83 \\ 26.72 \\ 14.68 \end{bmatrix}$, as we found in that chapter. Now, however, the effects can be disentangled. Specifically,

1. First, $\mathbf{M}_1\mathbf{f}^{new} = \begin{bmatrix} 136.51 \\ 52.73 \\ 56.98 \\ 0 \\ 0 \end{bmatrix}$ indicates the initial impact in region r, the origin of the final demand change.
2. Next, $\mathbf{M}_2\mathbf{M}_1\mathbf{f}^{new} = \begin{bmatrix} 136.51 \\ 52.73 \\ 56.98 \\ 24.69 \\ 13.71 \end{bmatrix}$ adds to (1) the increases in the two sectors of region s because of the spillovers from r. Note that outputs in r are unchanged from (1), since this calculation is concerned with spillovers only. Clearly the difference between the results in (2) and (1), $\mathbf{M}_2\mathbf{M}_1\mathbf{f}^{new} - \mathbf{M}_1\mathbf{f}^{new}$, will be the vector of changes in s only.
3. Finally, $\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{f}^{new} = \begin{bmatrix} 142.34 \\ 63.46 \\ 63.83 \\ 26.72 \\ 14.68 \end{bmatrix} = \mathbf{L}\mathbf{f}^{new}$ then adds in the feedback effects in the two regions – in r where the stimulus originated and in s because of the stimulus from the spillovers. In this case, the difference between the results in (3) and (2),

$\begin{bmatrix} 5.83 \\ 10.73 \\ 6.84 \\ 2.03 \\ 0.97 \end{bmatrix}$, nets out the feedback effects by themselves. The first three elements, $\begin{bmatrix} 5.83 \\ 10.73 \\ 6.84 \end{bmatrix}$, are exactly the interregional feedback amounts that we found for region r in Chapter 3.

Consider now the components of the additive decomposition

$$\mathbf{x}^{new} = \mathbf{M}\mathbf{f}^{new} = \mathbf{I}\mathbf{f}^{new} + \underbrace{(\mathbf{M}_1 - \mathbf{I})}_{\tilde{\mathbf{M}}_1}\mathbf{f}^{new} + \underbrace{(\mathbf{M}_2 - \mathbf{I})\mathbf{M}_1}_{\tilde{\mathbf{M}}_2}\mathbf{f}^{new} + \underbrace{(\mathbf{M}_3 - \mathbf{I})\mathbf{M}_2\mathbf{M}_1}_{\tilde{\mathbf{M}}_3}\mathbf{f}^{new}$$

These provide the net effects. For this example, these multiplier matrices are

$$\tilde{\mathbf{M}}_1 = \begin{bmatrix} 0.3651 & 0.4253 & 0.2509 & 0 & 0 \\ 0.5273 & 0.3481 & 0.5954 & 0 & 0 \\ 0.5698 & 0.4890 & 0.2885 & 0 & 0 \\ 0 & 0 & 0 & 0.2679 & 0.4528 \\ 0 & 0 & 0 & 0.1811 & 0.2075 \end{bmatrix}$$

$$\tilde{\mathbf{M}}_2 = \begin{bmatrix} 0 & 0 & 0 & 0.1769 & 0.2845 \\ 0 & 0 & 0 & 0.3814 & 0.4193 \\ 0 & 0 & 0 & 0.2325 & 0.2876 \\ 0.2469 & 0.1859 & 0.1833 & 0 & 0 \\ 0.1371 & 0.0841 & 0.0858 & 0 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{M}}_3 = \begin{bmatrix} 0.0583 & 0.0400 & 0.0400 & 0.0148 & 0.0195 \\ 0.1073 & 0.0756 & 0.0753 & 0.0278 & 0.0365 \\ 0.0684 & 0.0478 & 0.0477 & 0.0176 & 0.0232 \\ 0.0203 & 0.0141 & 0.0141 & 0.0727 & 0.0944 \\ 0.0097 & 0.0067 & 0.0067 & 0.0343 & 0.0463 \end{bmatrix}$$

The pieces of the decomposition in (6.64) are:

1. $\mathbf{I}\mathbf{f}^{new} = \begin{bmatrix} 100 \\ 0 \\ 0 \\ \\ 0 \\ 0 \end{bmatrix}$ is just the initial "shock."
2. Then $\tilde{\mathbf{M}}_1\mathbf{f}^{new} = \begin{bmatrix} 36.51 \\ 52.73 \\ 56.98 \\ \\ 0 \\ 0 \end{bmatrix}$ accounts for the indirect effects in r; the sum of (1) and (2) is just $\mathbf{M}_1\mathbf{f}^{new}$, by definition.
3. Next, $\tilde{\mathbf{M}}_2\mathbf{f}^{new} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ 24.69 \\ 13.71 \end{bmatrix}$ captures the spillovers; this is $\mathbf{M}_2\mathbf{M}_1\mathbf{f}^{new} - \mathbf{M}_1\mathbf{f}^{new}$, also by definition.
4. Finally, $\tilde{\mathbf{M}}_3\mathbf{f}^{new} = \begin{bmatrix} 5.83 \\ 10.73 \\ 6.84 \\ \\ 2.03 \\ 0.97 \end{bmatrix}$ isolates the contribution from the interregional feedbacks; by definition this is $\mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\mathbf{f}^{new} - \mathbf{M}_2\mathbf{M}_1\mathbf{f}^{new}$.

The matrix components of these decompositions, $\mathbf{M}$ and $\tilde{\mathbf{M}}$, are multiplier matrices, and so various multipliers can be calculated in the same way as was done earlier in this chapter for $\mathbf{L}$ – for example, simple column sums, or weighted sums if employment, value added or other economic impacts are of interest.

An empirical example applying these kinds of decompositions can be found in Zhang and Zhao (2005). They present a detailed set of decompositions of initial, spillover, and feedback effects derived from the 17-sector version of the 2000 Chinese multiregional (CMRIO) model that has been aggregated spatially into two mega-regions – Coastal and Non-coastal regions.

6.8 Summary

In this chapter we have introduced the reader to a wide variety of multipliers that are frequently calculated and used in real-world applications of the input–output framework. While the array may seem bewildering at first glance, it is, in fact, incomplete. For example, instead of using household input coefficients, as in (6.11), to generate a household income multiplier, one can weight the elements of a column of **L** by the parallel concept of "government input" coefficients, representing dollar's worth of government payments by a sector per dollar's worth of that sector's output. These would be the elements needed in the added row of an **A** matrix that was being closed with respect to government operations, not households. In this way, we would generate government multipliers. And similarly, other multipliers associated with exogenous sectors can be calculated – for example, foreign trade multipliers.

The use of the input–output framework for impact analysis, due to changing final demands, using multipliers, constitutes one of the most frequent uses of the model. In subsequent chapters we will explore extensions to deal specifically with energy (Chapter 9) and environmental problems (Chapter 10), and alternative uses of the model, in which the data are transformed into alternative summary measures of economic activity such as decomposition of changes over time and linkage analysis, in which the relative "importance" of sectors is assessed.

We explored the added richness of the Miyazawa formulation of a "closed" model in which various income-consumption-output impacts can be isolated. And we also examined some of the many variations on early multiplier formulations – for example, when the approach is changed from (direct + indirect effects)/(direct effects) to (indirect effects)/(direct effects) – which essentially means subtracting one from a traditional multiplier. We also examined the conversion of (multiplier) effects into elasticity terms, giving percentage changes due to a one percent increase in an industry's final demand or output. Finally, we examined two approaches to the decomposition of multiplier effects; these provide mechanisms that explicitly identify the routes of transmission of the initial exogenous stimulus. (Additional approaches to disentangling economic structure are explored briefly in Chapter 14.) We illustrated these in the spatial case, with interregional spillovers and feedbacks. The approach is equally insightful for extended input–output models, as illustrated by the Miyazawa structure. This is a feature of many studies employing social accounting matrices (SAMs) and will be discussed further in Chapter 11.

Appendix 6.1 The Equivalence of Total Household Income Multipliers and the Elements in the Bottom Row of $(\mathbf{I} - \bar{\mathbf{A}})^{-1}$

Consider the general representation of our 3×3 model closed with respect to households (sector 3), and its inverse, similarly partitioned.

$$(\mathbf{I} - \bar{\mathbf{A}}) = \begin{bmatrix} (1 - a_{11}) & -a_{12} & -a_{13} \\ -a_{21} & (1 - a_{22}) & -a_{23} \\ -a_{31} & -a_{32} & (1 - a_{33}) \end{bmatrix} = \begin{bmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{bmatrix}$$

$$(\mathbf{I}-\bar{\mathbf{A}})^{-1}=\bar{\mathbf{L}}=\begin{bmatrix}\bar{l}_{11} & \bar{l}_{12} & \bar{l}_{13}\\ \bar{l}_{21} & \bar{l}_{22} & \bar{l}_{23}\\ \\ \bar{l}_{31} & \bar{l}_{32} & \bar{l}_{33}\end{bmatrix}=\begin{bmatrix}\mathbf{S} & \mathbf{T}\\ \\ \mathbf{U} & \mathbf{V}\end{bmatrix}$$

From results on inverses of partitioned matrices in Appendix A, particularly (2) in (A.4), $\mathbf{GS}+\mathbf{HU}=\mathbf{0}$. Here, since $\mathbf{H}=1-a_{33}$, we can write $\mathbf{U}=a_{33}\mathbf{U}-\mathbf{GS}$, or

$$\begin{bmatrix}\bar{l}_{31} & \bar{l}_{32}\end{bmatrix}=a_{33}\begin{bmatrix}\bar{l}_{31} & \bar{l}_{32}\end{bmatrix}+\begin{bmatrix}a_{31} & a_{32}\end{bmatrix}\begin{bmatrix}\bar{l}_{11} & \bar{l}_{12}\\ \bar{l}_{21} & \bar{l}_{22}\end{bmatrix}$$

Written out and rearranged, this is

$$\bar{l}_{31}=a_{31}\bar{l}_{11}+a_{32}\bar{l}_{21}+a_{33}\bar{l}_{31}$$
$$\bar{l}_{32}=a_{31}\bar{l}_{12}+a_{32}\bar{l}_{22}+a_{33}\bar{l}_{32}$$

The three terms on the right-hand sides are exactly the terms in (6.12) – $\bar{m}(h)_j=\sum_{i=1}^{n+1}a_{n+1,i}\bar{l}_{ij}$ – for $j=1$ and $j=2$, where the $(n+1)=3$ and $i=3$ terms are those in the household row (or column). Thus, $\bar{m}(h)_1=\bar{l}_{31}$ and $\bar{m}(h)_2=\bar{l}_{32}$, and this is always true, for any $\bar{m}(h)_j$, for a model of any size with households endogenous. This is (6.13), namely $\bar{m}(h)_j=\bar{l}_{n+1,j}$.

Appendix 6.2 Relationship Between Type I and Type II Income Multipliers

To examine the value of the ratio between type II and type I income multipliers, we again use results on the inverse of a partitioned matrix. To begin we note, for any sector j, that both multipliers – in (6.14) and (6.15) – have the same denominator, $a_{n+1,j}$, and thus the ratio of the two multipliers for sector j is

$$R_j=\frac{m(h)_j^{II}}{m(h)_j^{I}}=\frac{\bar{l}_{n+1,j}}{\sum_{i=1}^{n}a_{n+1,i}l_{ij}} \tag{A6.2.1}$$

In matrix terms, with $\bar{\mathbf{L}}=\begin{bmatrix}\bar{\mathbf{L}}_{11} & \bar{\mathbf{L}}_{12}\\ \\ \bar{\mathbf{L}}_{21} & \bar{\mathbf{L}}_{22}\end{bmatrix}$, the numerator of the ratio in (A6.2.1) is the jth element of $\bar{\mathbf{L}}_{21}$ and the denominator is the corresponding element of $\mathbf{h}'_c\mathbf{L}$. Thus the n-element row vector of these ratios is

$$\mathbf{R}=[R_1,\ldots,R_n]=\bar{\mathbf{L}}_{21}[\langle\mathbf{h}'_c\mathbf{L}\rangle]^{-1} \tag{A6.2.2}$$

The reader should be clear that this matrix operation divides each $\bar{l}_{n+1,1},\ldots,\bar{l}_{n+1,n}$ by the corresponding $\sum_{i=1}^{n}a_{n+1,i}l_{ij}$. (Recall also that the notation $\langle\mathbf{x}\rangle$ is used instead of $\hat{\mathbf{x}}$

when the vector being diagonalized is represented by a matrix expression containing several elements, so that the hat does not fit easily.)

Again using results from Appendix A on the inverse of a partitioned matrix (specifically A.5), and with $(\mathbf{I}-\bar{\mathbf{A}}) = \begin{bmatrix} (1-a_{11}) & -a_{12} & -a_{13} \\ -a_{21} & (1-a_{22}) & -a_{23} \\ -a_{31} & -a_{32} & (1-a_{33}) \end{bmatrix} = \begin{bmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{bmatrix}$, we see that the components in (A6.2.2) are

$$\bar{\mathbf{L}}_{21} = -\bar{\mathbf{L}}_{22}(\mathbf{G}\mathbf{E}^{-1}) = -\bar{\mathbf{L}}_{22}(\mathbf{G}\mathbf{L}) \text{ and } \mathbf{h}'_c\mathbf{L} = -\mathbf{G}\mathbf{L}$$

Thus $\mathbf{R} = -\underset{(1\times 1)}{\bar{\mathbf{L}}_{22}}\,(\underset{(1\times n)}{\mathbf{G}\mathbf{L}})[\underset{(n\times n)}{\langle -\mathbf{G}\mathbf{L}\rangle}]^{-1} = \underset{(1\times 1)}{\bar{\mathbf{L}}_{22}}\,\underset{(1\times n)}{[1,\ldots,1]} = \bar{\mathbf{L}}_{22}\mathbf{i}'$; that is, the ratios are all the same and are equal to the element in the lower-right of the closed model inverse.

For the numerical example in section 6.2.2, we found that the ratio of these multipliers, which we designated k, was 1.29. Recall the inverse for our small example, in (6.6), namely $\bar{\mathbf{L}} = \begin{bmatrix} 1.365 & 0.425 & 0.251 \\ 0.527 & 1.348 & 0.595 \\ 0.570 & 0.489 & 1.289 \end{bmatrix}$, where, in particular (to two decimals), $\bar{\mathbf{L}}_{22} = 1.29$. (Differences are due to rounding and the detailed precision of the inversion process.)

This constancy of the ratios of the two types of multipliers was apparently first demonstrated by Sandoval (1967), in an article in which he showed that the ratio is equal to $\left|(\mathbf{I}-\bar{\mathbf{A}})\right| / |(\mathbf{I}-\mathbf{A})|$, the ratio of the determinants of the Leontief matrices (not inverses) of the closed and open models. [The reader familiar with determinants can easily verify this for the numerical example in this chapter – $|(\mathbf{I}-\mathbf{A})| = 0.7575$, $\left|(\mathbf{I}-\bar{\mathbf{A}})\right| = 0.587875$ and (to two decimal places) $|(\mathbf{I}-\mathbf{A})|/|(\mathbf{I}-\bar{\mathbf{A}})| = 1.29$.] In producing his result, Sandoval did not use results from the inverses of partitioned matrices but rather from the general definitions of inverses in terms of determinants and cofactors. (Other discussions of these topics can be found in Bradley and Gander, 1969, Katz, 1980, and ten Raa and Chakraborty, 1983.)

Problems

6.1 Rank sectors in terms of their importance as measured by output multipliers in each of the economies represented by the data in problems 2.1, 2.2, and 2.4–2.9 (include problem 2.10 if you did it.)

6.2 Consider one (or more) of the problems in Chapter 2. Using output multipliers, from problem 6.1, in conjunction with the new final demands in the problem in Chapter 2, derive the total value of output (across all sectors) associated with the new final demands. Compare your results with the total output obtained by summing the elements in the gross output vector which you found as the solution to the problem in Chapter 2. [In matrix notation, this is comparing $\mathbf{m}(o)\Delta\mathbf{f}$ with $\mathbf{i}'\Delta\mathbf{x} = \mathbf{i}'\mathbf{L}\Delta\mathbf{f}$; we know that they must be equal, since output multipliers are the column sums of the Leontief inverse – $\mathbf{m}(o) = \mathbf{i}'\mathbf{L}$.]

6.3 Using the data in problem 2.3, find output multipliers and also both type I and type II income multipliers for the two sectors. Check that the ratio of the type II to the type I income multiplier is the same for both sectors.

6.4 You have assembled the following facts about the two sectors that make up the economy of a small country that you want to study (data pertain to the most recent quarter). Total interindustry inputs were \$50 and \$100, respectively, for Sectors 1 and 2. Sector 1's sales to final demand were \$60 and Sector 1's total output was \$100. Sector 2's sales to Sector 1 were \$30 and this represented 10 percent of Sector 2's total output. After national elections are held, it may turn out that different government policy will be forthcoming during the first quarter of the coming year.

a. In which of the two sectors does an increase of \$100 in government purchases have the larger effect?
b. How much larger is it than if the \$100 were spent on purchases of the other sector?

6.5 Consider an input output economy defined by $\mathbf{Z} = \begin{bmatrix} 140 & 350 \\ 800 & 50 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1000 \\ 1000 \end{bmatrix}$.

a. In the situation depicted in that question, if you were asked to design an advertising campaign to stimulate export sales of one of the goods produced in the country, would you concentrate your efforts on the product of sector 1 or of sector 2 or on some combination of the two? Why?
b. If labor input coefficients for the two sectors in the region were found to be $a_{31} = 0.1$ and $a_{32} = 0.18$, how might your answer to part (a) of this question be changed, if at all?

6.6 Using the elements in the full two-region interregional Leontief inverse from problem 3.2, find:

a. Simple intraregional output multipliers for sectors 1 and 2 [the vectors $\mathbf{m}(o)^{rr}$ and $\mathbf{m}(o)^{ss}$, as in (6.25) and (6.26)];
b. Simple national (total) output multipliers for sectors 1 and 2 (vectors $\mathbf{m}(o)^{r}$ and $\mathbf{m}(o)^{s}$, as was done in (6.30) in the text];
c. Sector-specific simple national output multipliers for sectors 1 and 2 in regions r and s. (This means finding the four multipliers in $\mathbf{m}(o)^{\cdot r} = [m(o)^{\cdot r}_{11}\ m(o)^{\cdot r}_{21}\ m(o)^{\cdot r}_{12}\ m(o)^{\cdot r}_{22}]$ and $\mathbf{m}(o)^{\cdot s}$, defined similarly.)

6.7 On the basis of the results in problem 6.6, above:

a. For which sector's output does new final demand produce the largest total intraregional output stimulus in region r? In region s?
b. For which sector in which region does an increase in final demand have the largest national (two-region) impact?
c. To increase the output of sector 1 nationally (i.e., in both regions), would it be better to institute policies that would increase household demand in region r or in region s?
d. Answer question (c) if the objective is now to increase sector 2 output nationally.

6.8 Answer problems 6.6 and 6.7, above, for the multiregional case, using the elements in $(\mathbf{I}-\mathbf{CA})^{-1}\mathbf{C}$ from problem 3.3.

6.9 The government in problem 3.4 is interested in starting an overseas advertising and promotion campaign in an attempt to increase export sales of the products of the country. There is specialization of production in the regions of the country; in particular, the products are shown in the table below:

	Region A	Region B	Region C
Manufacturing	Scissors	Cloth	Pottery
Agriculture	Oranges	Walnuts	None

For which product (or products) would increased export sales cause the greatest stimulation of the national economy?

6.10 If you have software (or patience), find $\left|(\mathbf{I}-\bar{\mathbf{A}})\right| / \left|(\mathbf{I}-\mathbf{A})\right|$ for our numerical example in which $\mathbf{A} = \begin{bmatrix} .15 & .25 \\ .20 & .05 \end{bmatrix}$ and $\bar{\mathbf{A}} = \begin{bmatrix} .15 & .25 & .05 \\ .20 & .05 & .40 \\ .30 & .25 & .05 \end{bmatrix}$, demonstrating that it is equal to $(1/g) = 1.29$, as in Appendix 6.2.

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7 Nonsurvey and Partial-Survey Methods: Fundamentals

7.1 Introduction

The heart of any input–output analysis is the table of input–output coefficients describing the relationships between inputs and outputs for a particular economy. To produce a table based on a survey of establishments in the economy is an expensive and time-consuming task, not only at a national level, but also for regions (states, counties, metropolitan areas, etc.). In this chapter we examine some approaches that attempt to adapt older tables to reflect more recent economic conditions or to borrow information in a table for one economy to use for a different economy. In a very general way, these may be thought of as modifications of tables over time or across space, respectively.

7.2 The Question of Stability of Input–Output Data

One of the most serious concerns of those who use input–output models in applied work is that the table of technical coefficients available to them for the economy that they are studying will generally reflect data from a much earlier year. For example, a survey-based or so-called benchmark US input–output table based upon 2002 transactions was not generally available until 2007. These time lags reflect the fact that when establishments in different industries are surveyed for information regarding their purchases of inputs and their sales of output, it takes a great deal of time to obtain the data, organize the information, and reconcile inconsistencies – for example, reported purchases of sector i goods by sector j establishments may differ from reported sales by sector i to sector j establishments. (We will return to this reconciliation problem in section 8.9.) This is a general and continuing problem with survey-based tables.

It is clear that techniques of production will and do change over time, for a variety of reasons. Among others:

1. There is technological change itself, whereby new techniques of production are introduced in a sector (e.g., replacement of some human labor with robots in automobile production).
2. If there is a large increase in demand for the products of a particular sector, output will increase (subject, of course, to capacity constraints), and the producer may

experience economies of scale. For example, if the scale of operation of a firm was very small at the time it was surveyed, relatively large material inputs per dollar of its output might be recorded. Later, after the level of production is increased, economies of scale might be reflected in lower amounts of at least some inputs per dollar of output. (In terms of the usual production function geometry, as in Figure 2.1(b), such scale economies mean that each isoquant represents a higher level of output than under the original conditions of production.)

3. New products are invented (e.g., plastics) which means both that (a) there may be an entirely new sector – row and column – in a sufficiently disaggregated table or at least the product mix will change in an existing sector if the new product is classified there, and (b) it may be used to replace an older product as an input to production in other sectors (e.g., plastic bottles rather than glass for soft drinks).
4. Relative prices change, and this may cause substitution among inputs in a production process (e.g., a switch from oil to natural gas as an energy source after a sharp increase in oil prices).
5. The more aggregated the input–output table, the greater the number of distinct products that are encompassed under one sectoral classification. To recall an extreme example from Chapter 3, if the food and kindred products sector produces mostly tomato soup in one year, there will be a need for tin cans in which to package the output. If, in a later year, the output of the food and kindred products sector is primarily chocolate bars, paper will be required for wrapping the product, not tin. Thus the relative proportions of products that are mixed together in a sector will influence the aggregate production recipe (column of input coefficients) for that sector.
6. Changes from domestically produced to imported inputs – or from imported to domestically produced – will alter the economic interrelationships between sectors in the domestic economy. This is particularly noticeable in interregional and multiregional input–output models.

For reasons such as these, an economy's technical coefficients matrix will change over time. Attempts to quantify these changes are often termed studies of *structural change*. Many of the earliest studies were primarily concerned with *measurement* of this change, and we examine several of these in this section. A second avenue of inquiry has concentrated on the *decomposition* of changes into two or more components of the overall change. We explore a number of these studies later, in section 13.1.

7.2.1 Stability of National Coefficients

Leontief (1951, 1953) was the first to use a national input–output model to study structural change, specifically for the US economy over the period 1919–1939. Structural change in his view is a change in the technical coefficient matrix of the system. Leontief also introduced the idea of substituting one or more (ultimately, all) columns of old input coefficients into a new technical coefficients matrix. Kanemitsu and Ohnishi (1989) used a similar partial substitution method to study technological change in the Japanese economy for the 1970–1980 period.

In the 1953 publication, Leontief examined the overall effects of structural change by forcing (1) the 1919 US economy to satisfy 1929 final demands (and comparing the result with actual 1929 outputs) and (2) the 1929 US economy to satisfy 1939 final demands (and comparing this result with actual 1939 outputs). This has become a standard approach to measuring the overall effects of technical change. An early application is in Rasmussen (1957, esp. Chapter 9), where changes were measured for the Danish economy over the 1947–1949 period.

Following this approach, Carter (1970) analyzed the changes in the US economy in some detail as they were reflected in the 1939, 1947, and 1958 US input–output data. With, say, a 50-sector classification, each year's table of technical coefficients would contain some 2500 a_{ij} coefficients, or there would be 2500 elements in each of the Leontief inverse matrices. It is not immediately obvious how best to compare three sets of 2500 coefficients in order to judge how "different" they are. In general, then, summary measures of comparison become necessary. We briefly explore two kinds of comparisons; one uses a_{ij} coefficients directly, and the other is based on the Leontief inverse.

Comparisons of Direct-Input Coefficients If one constructs two-dimensional plots, in which the horizontal axis is used to measure the size of particular coefficients in the earlier year (t_0) and the vertical axis measures the size of coefficients in the later year (t_1), where the scales along the two axes are the same, then a particular a_{ij} coefficient will have as its horizontal/vertical coordinates the value of that coefficient at time t_0 and at time t_1 – $a_{ij}(t_0)$ and $a_{ij}(t_1)$. For an n-sector economy, there will be n^2 points in such a figure.

If all coefficients remained unchanged over the period, then all the points would fall along a 45-degree line. On the other hand, for coefficients that have increased over time the points will fall above the 45-degree line. Similarly, if coefficients have decreased over time, the points will tend to fall below the 45-degree line. Carter examined figures of this sort for given sets of sectors as inputs (that is, the a_{ij} for specific i's) and found, for example, that input coefficients for the "general inputs" sectors (energy, transportation, trade, communications, and other services) tended to increase over time, while those for materials inputs did not. Industry-specific analyses showed, for example, that coefficients measuring iron and steel inputs to productive sectors (a_{ij}, where $i =$ iron and steel) clustered generally below the 45-degree line, when $t_0 = 1947$ and $t_1 = 1958$; similarly, those for aluminum inputs (a_{ij}, where $i =$ aluminum) tended to cluster above the 45-degree line, for the same time period. This clearly reflects decreased use of iron and steel and increased use of aluminum as inputs to productive processes over the 1947–1958 period.

Comparisons of Leontief Inverse Matrices One way to quantify in an aggregate way the effects of input–output coefficient change over time is to compare the total output vector that would be needed for a given set of final demands, using the Leontief inverses from various technical coefficients matrices. For example, Carter used actual US final demand in 1961, $\mathbf{f}(1961)$, in conjunction with $\mathbf{L}(1939) = [\mathbf{I} - \mathbf{A}(1939)]^{-1}$,

$\mathbf{L}(1947)$ and $\mathbf{L}(1958)$ to calculate $\mathbf{x}(1961/1939)$, $\mathbf{x}(1961/1947)$ and $\mathbf{x}(1961/1958)$. For example,

$$\mathbf{x}(1961/1939) = \mathbf{L}(1939)\mathbf{f}(1961)$$

Here $\mathbf{x}(1961/1939)$ represents the gross output that would be needed from each sector of the economy to satisfy 1961 final demands if the structure of production were that of 1939. (In all cases, these were technical coefficients matrices that excluded households.) Representative results (Carter, 1970, Table 4.1, pp. 35–36) were as follows for total intermediate output – total output, $\mathbf{x}(1961/19xy)$, less final demand, $\mathbf{f}(1961)$ – to satisfy known 1961 final demands (in millions of 1947 dollars and for $xy = 39$, 47, or 58):

- Using 1939 coefficients – 324,288
- Using 1947 coefficients – 336,296
- Using 1958 coefficients – 336,941
- Actual 1961 output – 334,160

The implications are that, over time, intermediate input requirements are relatively stable. Carter suggests that the small increase in total intermediate input represents a slight increase in specialization within sectors and a relative decrease in the use of labor and capital in later years. Overall, while there were noteworthy changes in specific sectors, it appeared from this study that in most sectors structural change was very gradual. This, of course, supports the contention that input–output coefficient tables may remain useful for a number of years, even though the year in which they were constructed may appear to make them out of date.

A sampling of later studies following this same general approach includes:

- Vaccara (1970). The issue was US structural change over 1947, 1958, and 1961 using the 1947 and 1958 US input–output models, in this case focusing on both gross and intermediate output, the latter to remove the possibly dominating influence of sales to final demand.
- Bezdek (1978) looked at the same question of structural change and extended Vaccara's analysis to 1963 and 1966, using data based on somewhat different conventions (for example, regarding transfers).[1]
- Bezdek and Dunham (1978) also employed this line of inquiry. They used an aggregation of 80-order data sets (for 1947, 1958, and 1963) to 11 "functional industries" and made comparisons of their results on intermediate output change over 1947–1963 with the similar work (using other aggregations) by Carter (1970) for the USA. They also compared their 1958–1963 results with those reported by Stäglin and Wessels (1972) in a study with a similar purpose for (what was then) West Germany over 1958–1962.

[1] There is a good deal of other work, not all of it published, by Vaccara and/or Bezdek and others who were at one time associated with the US input–output projects in the Office of Business Economics (OBE) or, more recently, the Bureau of Economic Analysis (BEA) of the US Department of Commerce.

In many of these studies that used data from several consecutive time periods, the objective was often to try to determine whether trends observed in earlier periods appeared to continue to later periods. To the extent that regularities could be uncovered, the hope was that they might suggest approaches that could be used to update or project interindustry data in the absence of complete surveys. In general, that goal proved elusive; as observed in one study, the changes seemed to be "highly erratic, uneven and unpatterned." (Bezdek, 1978, p. 224).

Blair and Wyckoff (1989) examined changes in the US economy over 1963–1980. They considered not only the endpoint years (the 1963 and 1980 tables, the latter an update of 1977) but also data from the intervening 1967, 1972, and 1977 survey-based input–output tables. To assess the effects of changes in final demand, they held production technology in its 1980 form and forced that structure to satisfy, in turn, the final demands for 1972, 1977, 1980, and 1984. In addition, they also fixed a vector of final demands (for 1984) and used it with the varying technical coefficients matrices for 1972, 1977, and 1980. From these experiments, they concluded that the two methods for assessing structural changes overall produce roughly similar results.

Other Summary Measures Column sums of **A** matrices (with, say, households exogenous) show how a given sector depends on other sectors for inputs. If $\sum_i a_{ij}(t_0) = 0.32$ and $\sum_i a_{ij}(t_1) = 0.54$, we would conclude that sector j became more dependent upon other sectors in the economy in the period from t_0 to t_1 and also that sector j depended less on primary inputs – labor, capital, imports. These represent kinds of sectoral "linkage" in an economy, as do column sums of Leontief inverse matrices (output multipliers, Chapter 6). These and other linkage concepts will be taken up in Chapter 12. The point here is simply to note that they provide alternative kinds of summary measures by which to examine coefficients over time.

Data for the US Economy Appendix B contains a representative set of historical input–output data for the US economy aggregated to seven sectors. Other data for the US and additional economies with more sectoral detail are on the website at www.cambridge.org/millerandblair.

7.2.2 Constant versus Current Prices

In studies such as Carter's that attempt to identify structural (technological) change it is appropriate to express the input–output relationships in constant dollars. Suppose that $z_{ij}(t_0) = \$40$, $x_j(t_0) = \$1000$, $z_{ij}(t_1) = \$160$, and $x_j(t_1) = \$2000$. Recall (Chapter 2) that a transaction in value terms, z_{ij}, is a physical flow from i to j, s_{ij}, multiplied by the price of input i, p_i. Then, in terms of current values (at time t_0 and at time t_1), $a_{ij}(t_0) = 0.04$ and $a_{ij}(t_1) = 0.08$. This doubling of the input coefficient from sector i to sector j might be interpreted as a reflection of technological change – a doubling of the importance of good i in industry j's production process. However, if the price of input i increased over the period, then the difference between $a_{ij}(t_0)$ and $a_{ij}(t_1)$ would be at least partly due to this price change, and to the extent that this was the case it would not reflect any changed technological relationships at all. To cite an extreme case, if the price of good i had doubled and if the same physical flow was used in time t_1, then the $z_{ij}(t_1) = s_{ij}(t_1)p_i(t_1) = \$160$ reflects entirely a change in the price of i. If this were reduced to the price level at t_0 – if $p_i(t_1)$ were divided by 2 – then in constant (t_0-level) not current (t_1-level) dollars $z_{ij}(t_1)$ is just \$80; thus, expressed in constant dollars, $a_{ij}(t_1) = \$80/\$2000 = 0.04$, and we would conclude that there has been no structural change at all in the way input i is used in production by sector j. This is why constant-dollar comparisons are generally used in studies that attempt to identify structural change in an economy.

However, in addressing the question of coefficient stability over time (which is, ultimately, the question of whether or not "old" tables can be used reasonably in "new" times), current values are appropriate. There are two reasons for this. In the first place, when input prices increase, the price of the output produced from them will tend to increase also. Recall that the denominator of a_{ij} is x_j, which is a physical output, s_j, multiplied by the price of j, p_j. In the example above, if good i were the only input to sector j whose price had increased, it is not likely that the price of j would have doubled also, but it might well have increased slightly in the period from t_0 to t_1. However, if prices of all (or most) inputs to j had increased over the period, then the price of j is almost certain to have gone up also, so there will be some compensating movement in the numerators and the denominators of the a_{ij}. Thus, coefficients using current prices are likely to exhibit more stability, since price changes will be reflected in both numerators and denominators. This has been noted repeatedly; early studies include Tilanus and Rey (1964) at a national level and Conway (1980) at a regional level. More recently, in a very large study Shishido *et al.* (2000) use 45 individual coefficient tables for 20 countries and one region in China (there were tables for several different years for many of the countries) to examine coefficient change as an economy develops.

Secondly, due to the necessity of dealing with aggregated classifications, sectors contain a wide variety of individual products. Suppose that products a and b are classified as belonging to sector i (for example, heating oil and natural gas in the energy sector). If the price of one of these products, say a, rises relative to the other, then in establishments in sector j where substitution is possible between a and b, there will tend to be replacement of the higher-priced input, a, by the lower-priced one, b. This substitution, in turn, will tend to stabilize the value of the transaction z_{ij}, when that value is measured in current dollars, even though the physical composition of the transaction may be quite different at t_1 from what it was at t_0. (For example, if the price of oil rises relative to that of natural gas, a transaction from the energy sector to sector j may contain relatively more natural gas than oil in t_1 as compared with t_0.)

7.2.3 *Stability of Regional Coefficients*

In Chapter 3 we saw that a regional technical coefficient, a_{ij}^{r}, can be broken down to the sum of the regional input coefficient, a_{ij}^{rr}, and the coefficient representing the amount of good i produced in other regions that is used per dollar's worth of output of sector j in region r, $a_{ij}^{\tilde{r}r}$ (where $\tilde{r}$ indicates regions other than r); $a_{ij}^{rr} = a_{ij}^{r} - a_{ij}^{\tilde{r}r}$. (For studies that concentrate on a specific region, where it is not necessary to use a superscript to designate the particular region, the simpler notation $r_{ij} = a_{ij} - m_{ij}$ is often used for regional input coefficients, technical coefficients, and "import" coefficients.)[2] Both the technical coefficients and the import coefficients, which represent trade patterns, are likely to be subject to variations over time. This has led to the speculation that regional coefficients are likely to be more unstable than technical coefficients, since they are made up of two unstable components – technical coefficients and import coefficients. For example, suppose $a_{ij}(t_0) = 0.1$, $a_{ij}(t_1) = 0.2$, $m_{ij}(t_0) = 0.05$, and $m_{ij}(t_1) = 0.1$. Then $r_{ij}(t_0) = 0.05$, $r_{ij}(t_1) = 0.1$, and the percentage increases in a_{ij}, m_{ij} and r_{ij} from t_0 to t_1 are all 100. On the other hand, if $m_{ij}(t_1) = 0.08$, then $r_{ij}(t_0) = 0.05$, $r_{ij}(t_1) = 0.12$, and the percentage increases are 100, 60, and 140, for a_{ij}, m_{ij}, and r_{ij}, respectively. Thus, in this case, the regional input coefficient is more unstable than either the technical coefficient or the import coefficient, even though the latter two moved in the same direction over t_0 to t_1.

An early study of coefficient stability at the regional level can be found in Beyers (1972), who used three survey-based input–output tables for the state of Washington, for 1963, 1967, and 1972 (Bourque and Weeks, 1969; Beyers *et al.*, 1970; and Bourque and Conway, 1977, respectively). Results of an examination of the regional input coefficients for the 1963 and 1967 Washington survey-based tables, in current dollars, are not conclusive (Beyers, 1972, Table 4, p. 372). For example, examination of the 888 coefficients for which a_{ij} experienced a change over the 1963–1967 period revealed that in 21.3 percent of the cases there was no change in m_{ij}; the change in r_{ij} was the same as in a_{ij}. In 16.2 percent of the cases, a_{ij} and m_{ij} moved in the same direction and there was no change in r_{ij}; in these cases the presence of both a_{ij} and m_{ij} in the definition of r_{ij} was "compensating." In 10.4 percent of the cases, a_{ij} and m_{ij} moved in opposite directions and hence led to a more unstable r_{ij}. However, in the remaining 52.1 percent of the cases, the effects were ambiguous – either a_{ij}, m_{ij}, and r_{ij} all moved in the same direction or a_{ij} and m_{ij} moved in the opposite direction (both of these kinds of movements may or may not lead to more instability in r_{ij} than in either a_{ij} or m_{ij}). For example, if $a_{ij}(t_0) = 0.2$, $a_{ij}(t_1) = 0.19$, $m_{ij}(t_0) = 0.05$, $m_{ij}(t_1) = 0.01$, then $r_{ij}(t_0) = 0.15$ and $r_{ij}(t_1) = 0.18$. While both a_{ij} and m_{ij} have decreased over time, r_{ij} has increased, and the percentage change in r_{ij} (in absolute terms) is larger than the percentage change in a_{ij} – a 20 percent increase versus a 5 percent decrease, respectively.

[2] In interregional and multiregional models we generally distinguish between inputs that come from other regions in the national economy and those that are imported from outside the nation. In the general discussion of this chapter, "import" means "not produced in the region."

Examination of the Leontief inverses for the regional input coefficients and regional technical coefficients for the two years showed “the regional [input coefficients] matrix appears somewhat less stable than the [regional] technical requirements matrix” (Beyers, 1972, p. 372). However, the amount of change was relatively unimportant for overall impact analysis. For example, in an analysis of the Leontief–Carter type, total 1967 output calculated by using the 1963 coefficients matrix and the 1967 final demand was found to be only 2.3 percent larger than total actual 1967 output; intermediate output was 10.5 percent larger. However, the usual caveat applies; namely, some individual sectoral outputs were badly estimated using the 1963 matrix (the worst being overestimated by 77 percent). Further analyses of the Washington survey-based data (Conway, 1977, 1980) arrive at similar conclusions.

Another early study using survey-based state-level data is to be found in Emerson (1976), based on tables for Kansas for 1965 and 1970, including full import and export matrices. The results are, like those for Washington, not terribly conclusive. Although there were some changes in the import coefficients, and consequently in the Kansas regional input coefficients, the problem was judged to be “not acute but . . . of sufficient importance to warrant concern” (Emerson, 1976, p. 275). Also, Baster (1980) supplied some evidence on relative stability of trade coefficients in a study for the Strathclyde region in Scotland. At the level of the individual firm or establishment, 79 percent of the coefficients showing imports from the rest of Scotland were constant over the 1974–1976 period, and an additional 13.5 percent of the coefficients varied by no more than 10 percent over the period. At the sectoral level (that is, aggregating establishments), over 90 percent of the import coefficients were stable.

7.2.4 Summary

There is no question but that coefficients change over time, at both national and at regional levels. It is also apparent that for aggregate kinds of measures, such as total economy-wide output associated with a specific vector of final demand, the error introduced by using an “old” table may not be large. On the other hand, there are other much simpler methods for forecasting total output that are not much worse. As an example, Conway (1975) estimates total Washington 1967 output, $\mathbf{i}'\mathbf{x}^W(1967)$, using known total final demands for 1963 and 1967 – $\mathbf{i}'\mathbf{f}^W(1963)$ and $\mathbf{i}'\mathbf{f}^W(1967)$ – and total 1963 output, $\mathbf{i}'\mathbf{x}^W(1963)$. His estimate is simply

$$\mathbf{i}'x^W(1967) = [\mathbf{i}'\mathbf{x}^W(1963)]\left[\frac{\mathbf{i}'\mathbf{f}^W(1967)}{\mathbf{i}'\mathbf{f}^W(1963)}\right]$$

This is known as a “final-demand blowup” approach; in the Washington case it led to an overestimate of 3.1 percent (Conway, 1975, p. 67), as opposed to the input–output-generated error of 2.3 percent noted above (Beyers, 1972, p. 368). That is, there are much simpler ways to be not much worse off, at this very aggregate level. Of course, the main point of the input–output model is precisely that it generates results at the sectoral level, and for this kind of detail out-of-date tables can produce considerable error. For this reason, there is ongoing concern with improving techniques for updating or projecting input–output data. We examine some of these below and in the following chapter.

7.3 Updating and Projecting Coefficients: Trends, Marginal Coefficients, and Best Practice Methods

7.3.1 Trends and Extrapolation

Early in the history of input–output models it was thought that analysis of the trends in input–output coefficients might be a tempting approach to the problem of estimating probable changes in input–output coefficients over time. Given two or more coefficient matrices defined for an economy over the same set of sectors, linear (or nonlinear) trends could be established for each particular coefficient, and then extrapolations could be made to the year in question (with negative coefficients set equal to zero). For example, if a particular a_{ij} at time t_0 equals 0.2 and if the coefficient for the same i and j is 0.15 three years later (t_0+ 3), then a linear trend extrapolation would suggest that at t_0+ 6, a_{ij} would be equal to 0.10. This is of course a very elementary kind of "analysis." Two early studies found, not surprisingly, that such extrapolations generated worse results than simply using the most recent coefficients table; see Tilanus (1966) for the Netherlands and Barker (Allen and Gossling, 1975, Ch. 2) for UK tables. This approach is no longer given much attention.

7.3.2 Marginal Input Coefficients

Suppose that one is forecasting into the future from the current year, t, to some future year, $t + s$. Given $\mathbf{A}(t)$ and a forecast of $\mathbf{f}(t + s)$, one would then estimate $\mathbf{x}(t + s)$ as

$$\mathbf{x}(t + s) = \mathbf{L}(t)\mathbf{f}(t + s) \tag{7.1}$$

where $\mathbf{L}(t) = [\mathbf{I} - \mathbf{A}(t)]^{-1}$. Suppose that, in addition to the current-year data, there is a set of input–output data for a previous year, $t - r$. Then one could generate a set of marginal input coefficients, a^*_{ij}, defined as

$$a^*_{ij}(t) = \frac{z_{ij}(t) - z_{ij}(t - r)}{x_j(t) - x_j(t - r)} = \frac{\Delta z_{ij}}{\Delta x_j}$$

These coefficients relate the *change* (from year $t - r$ to year t) in the amount of input i purchased by industry j to the *change* (over the same period) in the total amount of j produced. To the extent that the average and marginal coefficients differ, the latter may reflect scale effects. The argument can be made that the marginal coefficient better reflects the inputs from i to j that would be used when the output of sector j changes, due to new (forecast) final demands.

For example, let $z_{ij}(t-r) = \$500$, $z_{ij}(t) = \$560$, $x_j(t-r) = \$5000$, and $x_j(t) = \$6000$, so that $a_{ij}(t) = \$560/\$6000 = 0.0933$ and $a^*_{ij}(t) = \$60/\$1000 = 0.06$. Putting ourselves back to year $t - r$, $a_{ij}(t - r) = \$500/\$5000 = 0.1$. If at time $t - r$ we had "forecast" $x_j(t)$ to be \$6000, our estimate of $z_{ij}(t)$, based on the usual *average* input coefficient, would have been $a_{ij}(t-r)x_j(t) = (0.1)(\$6000) = \$600$. However, if we had had a marginal coefficient at $t-r$, $a^*_{ij}(t-r)$, we could have made a forecast of $z_{ij}(t)$ as $z_{ij}(t) = z_{ij}(t-r) + \Delta z_{ij} = z_{ij}(t-r) + a^*_{ij}(t-r)\Delta x_j = \$500 + a^*_{ij}(t-r)(\$1000)$. In particular, if our estimate of $a^*_{ij}(t-r)$ had been 0.06 [which is our $a^*_{ij}(t)$], our estimate of $z_{ij}(t)$would have been perfect, at \$560. This is the basic idea behind the use of marginal coefficients for forecasting. The alternative to estimating directly the level of new output, at time $t+s$, as in (7.1), is to forecast the change in output, using marginal coefficients, and add it to the current level; that is

$$\mathbf{x}(t+s) = \mathbf{x}(t) + \Delta\mathbf{x} = \mathbf{L}(t)\mathbf{f}(t) + \mathbf{L}^*(t)\Delta\mathbf{f} \tag{7.2}$$

where $\Delta\mathbf{f} = \mathbf{f}(t+s) - \mathbf{f}(t)$, $\mathbf{L}^*(t) = [\mathbf{I} - \mathbf{A}^*(t)]^{-1}$ and $\mathbf{A}^*(t)$ is the matrix of marginal input coefficients. Since the elements in $\mathbf{x}(t+s)$ are found using a combination of current average coefficients, $\mathbf{A}(t)$, and marginal coefficients, $\mathbf{A}^*(t)$, this is in effect a way of introducing changing coefficients over time into the analysis.

Although the idea of using marginal coefficients to reflect changes in input–output structure has a certain logical appeal, early experiments by Tilanus (1967) on a series of Dutch national input–output tables for 13 consecutive years (1948 through 1960) were not encouraging. For $r = 5$ (that is, calculating marginal coefficients over the previous five-year period) and $s = 1, 2, 3, 4^1/_2$ and $6^1/_2$ (years of projection), using marginal coefficients in this way gave results that were not as good as when the most recent table of average coefficients was used – the approach in (7.1) turned out to be better than that in (7.2).

7.3.3 "Best Practice" Firms

An alternative approach for projecting the technology in an input–output table in the future is the "best practice" firm idea pioneered by Miernyk (for example, in Miernyk, 1965). In constructing a table for short-term forecasting into the future – say, three to six years – Miernyk suggested that one not gather current information from *all* firms in each sector, or even from some random sample of firms. Rather, one should obtain data only from the "best practice" firms in a sector – those that are technologically most advanced at present. Such firms can be defined as those for which the ratios of employment or wage payments to total gross output are relatively low ("low labor intensity") or those with relatively high ratios of profits to total gross output. Firms could be identified as belonging to the best practice group if they satisfied any one or only if they satisfied several of these (or similar) criteria simultaneously.

The logic is that these firms, which are somewhat unusual currently (in the sense of being "better than average" for their sector), probably represent the technology that will be generally in use in the future – the best of today will be the average of the future. There are many obvious objections to this idea – why should "best" today be "average" in five years for *all* sectors? Is this approach valid for three years, or five years, or seven years in the future? And so on. But in its favor is the fact that it is a workable, feasible way of constructing technical coefficients matrices that are more likely to represent the future structure of production than would a table that was constructed to represent the average structure in each sector today.

7.4 Updating and Projecting Coefficients: The RAS Approach and Hybrid Methods

7.4.1 The RAS Technique

Early work at updating input–output information, done under Stone's direction, is reported in Stone (1961); Stone and Brown (1962); Cambridge University, Department of Applied Economics (1963); and Bacharach (1970). Because this technique requires less information than is usually obtained in a survey of the sort that underlies survey-based input–output tables, it is often referred to as a partial-survey, or a nonsurvey method. It is now recognized that full surveys are generally impractical and that a "hybrid" approach is called for, in which some kinds of superior information (from small, focused surveys, expert opinion, etc.) are incorporated into an otherwise "nonsurvey" procedure.[3] In this section we examine the widely-used "RAS" procedure (also known as a "biproportional" matrix balancing technique); the origin of the name will become clear in what follows. There have been numerous variations – attempts at refinement and improvement of this procedure – and research continues to be active.[4] Later we will see how additional information can be incorporated into the basic RAS procedure, producing an example of a hybrid technique.

To begin, assume that we have an input–output direct input coefficients table for an n-sector economy for a given year in the past (in what follows, we will designate this as year "0") and that we would like to update those coefficients to a more recent year (for example, the current year, which we will designate year "1"). Using obvious notation, we have $\mathbf{A}(0)$ and we want $\mathbf{A}(1)$, the n^2 coefficients for the n sectors in the economy for the more recent or current year.[5]

The RAS technique generates an estimate of these coefficients from $3n$ pieces of information for the year of interest (year 1). These are: (1) total gross outputs, x_j (which are also needed with survey-based transactions information); (2) total interindustry (intermediate) sales, by sector – for sector i this is $\sum\limits_{j=1}^{n} z_{ij}$, which is the same as total output of sector i less sector i's sales to final demand (since $x_i = \sum\limits_{j=1}^{n} z_{ij} + f_i$) and (3) total interindustry purchases, by sector – for sector j this is $\sum\limits_{i=1}^{n} z_{ij}$, which is the same

[3] See Lahr (1993) for a thorough discussion and an extensive set of references. As noted by Richardson (1985, p. 624): "If survey-based models are too expensive, conversion of national coefficients too mechanical, and short cuts too unreliable, the hybrid approaches are the wave of the future."

[4] An excellent overview of RAS and similar matrix adjustment techniques is to be found in several of the chapters in Allen and Gossling (1975), which also contains a good list of early references. See also Polenske (1997) for a thorough critical review. An important newer reference is the June, 2004, issue of *Economic Systems Research*, a special issue on "Biproportional Techniques in Input–Output Analysis," edited by Lahr and de Mesnard. See especially the lead article by the editors (Lahr and de Mesnard, 2004).

[5] The RAS approach is usually presented in the context of updating *coefficients*, and we maintain that viewpoint in this section. As we will later see, one can equally well use RAS to update *transactions* and then derive updated coefficients from those updated transactions.

as $x_j - v_j$ (total output of sector j less total purchases by j from the payments sector – labor inputs to sector j, imported inputs to sector j, taxes paid for government services, interest paid on capital loans, rental payments for land, etc.)

It has become conventional in the RAS literature to define $u_i = \sum_{j=1}^{n} z_{ij}$ and $v_j = \sum_{i=1}^{n} z_{ij}$; as vectors, these are $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$. Since these need to be known for year 1, they will be designated $\mathbf{u}(1)$ and $\mathbf{v}(1)$. (In using $\mathbf{u}$ and $\mathbf{v}$, we are following established convention in the literature on nonsurvey techniques. Throughout this text we have also, following convention, used $\mathbf{v}'$ for the value-added (row) vector. Also, in Chapter 5 we followed another convention in using $\mathbf{U}$ and $\mathbf{V}$ for the Use and Make *matrices*, respectively, in a commodity-by-industry input–output accounting framework. The context should always make clear what is intended.)

Thus, the problem that the RAS procedure addresses is: given an $n \times n$ matrix $\mathbf{A}(0)$ and given three n-element vectors for a more recent year – $\mathbf{x}(1)$, $\mathbf{u}(1)$, and $\mathbf{v}(1)$ – estimate $\mathbf{A}(1)$. We denote this estimate as $\tilde{\mathbf{A}}(1)$. If we are dealing with, say, a 25-sector economy, we are estimating the 625 coefficients in $\tilde{\mathbf{A}}(1)$ from 75 pieces of information. These are: (1) the 25 row sums of the unknown transactions matrix, $\mathbf{Z}(1)$, namely $\mathbf{u}(1) = \mathbf{Z}(1)\mathbf{i}$; (2) the 25 column sums of the same matrix, $\mathbf{v}(1)' = \mathbf{i}'\mathbf{Z}(1)$ [or $\mathbf{v}(1) = \mathbf{Z}(1)'\mathbf{i}$]; and (3) the 25 year-1 gross outputs, $\mathbf{x}(1)$, which are needed to convert an estimate of a $z_{ij}(1)$ into an estimate of a technical coefficient $a_{ij}(1)$.

We develop the procedure for the general 3×3 case, and then present a 3×3 numerical example. The potential usefulness of the technique is in real-world applications, where the number of sectors is much larger than three and hence the difference between n^2 and $3n$ is large. For example, with an 80-sector table, $n^2 = 6400$, whereas $3n = 240$. For the general 3×3 case, we assume that base-year coefficients are known,

$$\mathbf{A}(0) = \begin{bmatrix} a_{11}(0) & a_{12}(0) & a_{13}(0) \\ a_{21}(0) & a_{22}(0) & a_{23}(0) \\ a_{31}(0) & a_{32}(0) & a_{33}(0) \end{bmatrix} \tag{7.3}$$

and for the "target" year we have

$$\mathbf{x}(1) = \begin{bmatrix} x_1(1) \\ x_2(1) \\ x_3(1) \end{bmatrix}, \ \mathbf{u}(1) = \begin{bmatrix} u_1(1) \\ u_2(1) \\ u_3(1) \end{bmatrix}, \ \mathbf{v}(1) = \begin{bmatrix} v_1(1) \\ v_2(1) \\ v_3(1) \end{bmatrix} \tag{7.4}$$

Initially, assume $\mathbf{A}(0) = \mathbf{A}(1)$, namely that the technical coefficients have remained stable over time. To test the credibility of this hypothesis, we investigate whether or not it is consistent with year-1 information on intermediate sales and purchases. These are row and column sums of the transactions matrix, so it is necessary to convert coefficients into transactions – here this means that our *initial* estimate of the target transactions matrix is $\mathbf{Z}^0 = \mathbf{A}(0)\hat{\mathbf{x}}(1)$.[6] Here this is

$$\mathbf{Z}^0 = \mathbf{A}(0)\hat{\mathbf{x}}(1) = \begin{bmatrix} a_{11}(0) & a_{12}(0) & a_{13}(0) \\ a_{21}(0) & a_{22}(0) & a_{23}(0) \\ a_{31}(0) & a_{32}(0) & a_{33}(0) \end{bmatrix} \begin{bmatrix} x_1(1) & 0 & 0 \\ 0 & x_2(1) & 0 \\ 0 & 0 & x_3(1) \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}(0)x_1(1) & a_{12}(0)x_2(1) & a_{13}(0)x_3(1) \\ a_{21}(0)x_1(1) & a_{22}(0)x_2(1) & a_{23}(0)x_3(1) \\ a_{31}(0)x_1(1) & a_{32}(0)x_2(1) & a_{33}(0)x_3(1) \end{bmatrix} \tag{7.5}$$

The issue is whether (or how well) the row sums and the column sums of the matrix in (7.5) correspond to our information about the target year economy – $\mathbf{u}(1)$ and $\mathbf{v}(1)$. Starting with row sums, we need to compare $\mathbf{u}^0 = \mathbf{Z}^0\mathbf{i} = [\mathbf{A}(0)\hat{\mathbf{x}}(1)]\mathbf{i}$ with $\mathbf{u}(1)$.[7]

If $\mathbf{u}^0 = \mathbf{u}(1)$, $\mathbf{Z}^0$ has the correct row sums. It then remains to be seen whether the column sums of $\mathbf{Z}^0$ match the known interindustry purchases given in $\mathbf{v}(1)$. If $\mathbf{i}'\mathbf{Z}^0 = \mathbf{v}(1)$, our work is finished, since the old technical coefficient matrix, $\mathbf{A}(0)$, in conjunction with the new gross outputs, $\mathbf{x}(1)$, generates the proper target year interindustry sales and purchases. Since the $\mathbf{u}(1)$ and $\mathbf{v}(1)$ are row and column sums of the (unknown) $\mathbf{Z}(1)$ matrix, they are sometimes referred to as "marginals" or "row and column margins" of $\mathbf{Z}(1)$.

It is much more likely that the no-change hypothesis fails – that $\mathbf{u}^0 \neq \mathbf{u}(1)$ and/or $\mathbf{v}^0 \neq \mathbf{v}(1)$. Specifically, suppose that row sums of the matrix in (7.5) are unsatisfactory;

$$\begin{aligned} a_{11}(0)x_1(1) + a_{12}(0)x_2(1) + a_{13}(0)x_3(1) &= u_1^0 \neq u_1(1) \\ a_{21}(0)x_1(1) + a_{22}(0)x_2(1) + a_{23}(0)x_3(1) &= u_2^0 \neq u_2(1) \\ a_{31}(0)x_1(1) + a_{32}(0)x_2(1) + a_{33}(0)x_3(1) &= u_3^0 \neq u_3(1) \end{aligned} \tag{7.6}$$

If a particular $u_i^0 > u_i(1)$, the elements in row i – $a_{i1}(0)$, $a_{i2}(0)$, $a_{i3}(0)$, in the example – are larger than they should be, since the $x_1(1)$, $x_2(1)$, and $x_3(1)$ contain "updated" (target year) information. [Similarly, if $u_k^0 < u_k(1)$, the elements of row k in $\mathbf{A}(0)$ are smaller than they should be.]

Let $u_i(1)/u_i^0 = r_i^1$ (the first of what will be a series of adjustment terms); when $u_i^0 > u_i(1)$, $r_i^1 < 1$. Let $i = 1$ for illustration. If each element in row 1 of $\mathbf{A}(0)$ is multiplied by r_i^1, each of those elements will be reduced. In particular, this operation generates a new set of coefficients in that row which, when multiplied by the $\mathbf{x}(1)$, will sum to $u_1(1)$ exactly, which is what we want.[8] Letting $r_1^1 a_{11}(0) = a_{11}^1$, $r_1^1 a_{12}(0) = a_{12}^1$,

[6] We use the notation $\mathbf{Z}^0$ because this represents an estimate of $\mathbf{Z}(1)$ based on no change in $\mathbf{A}(0)$. Subsequent estimates of the true $\mathbf{Z}(1)$ will be denoted $\mathbf{Z}^1, \mathbf{Z}^2, \ldots, \mathbf{Z}^k$.

[7] Similarly, $\mathbf{u}^0 = \mathbf{Z}^0\mathbf{i}$ will be the first of a series of estimates of the true $\mathbf{u}(1)$, again based on the no-change hypothesis.

[8] From (7.6) we have $a_{11}(0)x_1(1) + a_{12}(0)x_2(1) + a_{13}(0)x_3(1) = u_1^0$ where $u_1^0 > u_1(1)$. Letting $r_1^1 = u_1(1)/u_1^0$, and multiplying through by r_1^1, we have $r_1^1 a_{11}(0)x_1(1) + r_1^1 a_{12}(0)x_2(1) + r_1^1 a_{13}(0)x_3(1) = r_1^1 u_1^0 = \left(\frac{u_1(1)}{u_1^0}\right) u_1^0 = u_1(1)$.

and $r_1^1 a_{13}(0) = a_{13}^1$, row 1 of $\mathbf{A}(0)$ has been altered to produce a new set of coefficients that constitute our first estimate of a better set of values, in the sense that they satisfy the target year information in $u_1(1)$ exactly.

Similarly, if $u_2^0 < u_2(1)$, we form $r_2^1 = u_2(1)/u_2^0 > 1$. Multiplying the elements in row 2 of $\mathbf{A}(0)$ by r_2^1 has the effect of *increasing* each of them sufficiently so that the new second row sum will equal the known $u_2(1)$. (The demonstration follows exactly the same argument as in footnote 8.) Letting $r_2^1 a_{21}(0) = a_{21}^1, r_2^1 a_{22}(0) = a_{22}^1$ and $r_2^1 a_{23}(0) = a_{23}^1$, we find a modified second row of $\mathbf{A}(0)$, where in this example all elements in this row have been increased. These are our first estimates of a better set of values for row 2 of $\mathbf{A}(0)$. Similarly, for row 3, since $u_3^0 \neq u_3(1)$ in (7.6), we use $r_3^1 = u_3(1)/u_3^0$ to multiply each coefficient in the third row of $\mathbf{A}(0)$ – reducing them if $u_3(1) < u_3^0$ and expanding if $u_3(1) > u_3^0$ – producing the known target year row sum $u_3(1)$.

This is the logic of the row adjustments. Algebraically, we want to multiply row 1 of $\mathbf{A}(0)$ by r_1^1, row 2 of $\mathbf{A}(0)$ by r_2^1 and row 3 of $\mathbf{A}(0)$ by r_3^1, and this is accomplished using a diagonal matrix made up of the r^1. (As we have seen earlier in this book, premultiplication of any matrix, $\mathbf{M}$, by a diagonal matrix, $\mathbf{D} = [d_i]$, has the effect of multiplying row i of $\mathbf{M}$ by the element d_i.) Thus a first estimate of a target-year $\mathbf{A}$ matrix, denoted $\mathbf{A}^1$, is given by

$$\mathbf{A}^1 = \begin{bmatrix} r_1^1 & 0 & 0 \\ 0 & r_2^1 & 0 \\ 0 & 0 & r_3^1 \end{bmatrix} \mathbf{A}(0) \tag{7.7}$$

The superscripts (at present, 1) in the description of the RAS technique refer to the "step" in the procedure; $\mathbf{A}^1$ is our first estimate, which means our estimate after the first step of the procedure; $\mathbf{A}^2$ will be our second estimate (and not "$\mathbf{A}$ squared"), and so on. This may appear cumbersome at first, but it turns out to be useful notation, as we will see. Letting $\mathbf{r}^1 = [r_1^1, r_2^1, r_3^1]$,

$$\hat{\mathbf{r}}^1 = \begin{bmatrix} r_1^1 & 0 & 0 \\ 0 & r_2^1 & 0 \\ 0 & 0 & r_3^1 \end{bmatrix}$$

the result in (7.7) can be expressed as

$$\mathbf{A}^1 = \hat{\mathbf{r}}^1 \mathbf{A}(0) \tag{7.8}$$

The composition of $\hat{\mathbf{r}}^1$ can easily be described using the "hat" notation once again to convert a vector into a diagonal matrix and recalling that the inverse of a diagonal matrix is another diagonal matrix whose elements are the reciprocals of those in the original matrix. Therefore

$$\hat{\mathbf{r}}^1 = [\hat{\mathbf{u}}(1)](\hat{\mathbf{u}}^0)^{-1} \tag{7.9}$$

Following this first adjustment of $\mathbf{A}(0)$ we have a better estimate of $\mathbf{Z}(1)$, namely $\mathbf{Z}^1 = \mathbf{A}^1\hat{\mathbf{x}}(1) = \hat{\mathbf{r}}^1\mathbf{A}(0)\hat{\mathbf{x}}(1)$, with a set of row sums, $\mathbf{u}^1$, that correspond exactly to $\mathbf{u}(1)$. From (7.9) and $\mathbf{A}(0)\hat{\mathbf{x}}(1) = \hat{\mathbf{u}}^0$, we know that

$$\mathbf{u}^1 = \mathbf{Z}^1\mathbf{i} = [\hat{\mathbf{r}}^1\mathbf{A}(0)\hat{\mathbf{x}}(1)]\mathbf{i} = \{[\hat{\mathbf{u}}(1)](\hat{\mathbf{u}}^0)^{-1}\hat{\mathbf{u}}^0\}\mathbf{i} = \mathbf{u}(1) \tag{7.10}$$

(It was to ensure this equality that the modification of $\mathbf{A}(0)$ to $\mathbf{A}^1$ was made; this was illustrated in the previous footnote.)

The next issue, then, is whether or not the *column* sum information for the target year is captured in the improved matrix, $\mathbf{A}^1$. For that question we need to compare $\mathbf{v}(1)$ and $(\mathbf{Z}^1)'\mathbf{i} = \mathbf{v}^1 = \begin{bmatrix} v_1^1 & v_2^1 & v_3^1 \end{bmatrix}'$, the new column sums. These are

$$\begin{aligned} a_{11}^1x_1(1) + a_{21}^1x_1(1) + a_{31}^1x_1(1) &= (a_{11}^1 + a_{21}^1 + a_{31}^1)x_1(1) = v_1^1 \\ a_{12}^1x_2(1) + a_{22}^1x_2(1) + a_{32}^1x_2(1) &= (a_{12}^1 + a_{22}^1 + a_{32}^1)x_2(1) = v_2^1 \\ a_{13}^1x_3(1) + a_{23}^1x_3(1) + a_{33}^1x_3(1) &= (a_{13}^1 + a_{23}^1 + a_{33}^1)x_3(1) = v_3^1 \end{aligned} \tag{7.11}$$

If $v_1^1 = v_1(1)$, $v_2^1 = v_2(1)$, and $v_3^1 = v_3(1)$, then $\mathbf{A}^1 = \tilde{\mathbf{A}}(1)$, since it generates row and column sums that correspond to the observed $\mathbf{u}(1)$ and $\mathbf{v}(1)$.

In most cases, however, $\mathbf{v}^1 \neq \mathbf{v}(1)$, and so it is now necessary to modify the elements in $\mathbf{A}^1$ column by column. For example, if $v_1^1 > v_1(1)$ – the first $\mathbf{A}^1$ column sum in (7.11) is larger than it should be – let $v_1(1)/v_1^1 = s_1^1$ and multiply through the first equation in (7.11).[9] The superscript on s_1^1 indicates that this is our *first* modification of coefficients in order to meet column sum information. The modified coefficients in column 1 are then $s_1^1a_{11}^1$, $s_1^1a_{21}^1$, and $s_1^1a_{31}^1$; we denote these a_{11}^2, a_{21}^2, and a_{31}^2. The superscript 2 on the coefficients denotes that this is our *second* modification of elements from the original $\mathbf{A}(0)$ matrix.

Similarly, let $s_2^1 = v_2(1)/v_2^1$ and $s_3^1 = v_3(1)/v_3^1$. If a particular $v_j(1) > v_j^1$, the associated $s_j^1 > 1$ and the elements in the jth column of $\mathbf{A}^1$ are all increased when multiplied by s_j^1. On the other hand, if $v_k(1) < v_k^1$, then $s_k^1 < 1$, and each element in the kth column of $\mathbf{A}^1$ is reduced when it is multiplied by s_k^1. When a particular $v_m(1) = v_m^1$, the corresponding $s_m^1 = 1$, and the elements in column m of $\mathbf{A}^1$ will not be changed.

Algebraically, we now want to multiply column 1 of $\mathbf{A}^1$ by s_1^1, column 2 by s_2^1, and column 3 by s_3^1. Postmultiplication of $\mathbf{M}$ by a diagonal matrix has the effect of multiplying column j of $\mathbf{M}$ by the element d_j, so we form a second estimate, $\mathbf{A}^2$, as

$$\mathbf{A}^2 = \mathbf{A}^1 \begin{bmatrix} s_1^1 & 0 & 0 \\ 0 & s_2^1 & 0 \\ 0 & 0 & s_3^1 \end{bmatrix} \tag{7.12}$$

Letting $\mathbf{s}^1 = [s_1^1, s_2^1, s_3^1]$, this is

$$\mathbf{A}^2 = \mathbf{A}^1\hat{\mathbf{s}}^1 \tag{7.13}$$

[9] This gives $s_1^1(a_{11}^1 + a_{21}^1 + a_{31}^1)x_1(1) = s_1^1v_1^1 = [v_1(1)/v_1^1]v_1^1 = v_1(1)$, which is what we want.

Given $\mathbf{v}(1)$ and $\mathbf{v}^1$, we see that

$$\hat{\mathbf{s}}^1 = [\hat{\mathbf{v}}(1)](\hat{\mathbf{v}}^1)^{-1} \tag{7.14}$$

[Compare $\hat{\mathbf{r}}^1$ in (7.9).] With this set of adjustments, we know that the column sums are correct; $\mathbf{Z}^2 = \mathbf{A}^2[\hat{\mathbf{x}}(1)]$, and

$$(\mathbf{Z}^2)'\mathbf{i} = [\mathbf{A}^2\hat{\mathbf{x}}(1)]'\mathbf{i} = \mathbf{v}(1) \tag{7.15}$$

precisely, since it was to ensure this equality that the change of $\mathbf{A}^1$ to $\mathbf{A}^2$ was made.

Note, from (7.8) and (7.13),

$$\mathbf{A}^2 = \hat{\mathbf{r}}^1\mathbf{A}(0)\hat{\mathbf{s}}^1 \tag{7.16}$$

Ignoring superscripts, hats, lower-case letters, and the (0), denoting base-year information, we have "**RAS**" on the right-hand side of (7.16). This is the origin of the name of the technique. The point here is that the **R** is seen to refer to a diagonal matrix of elements modifying rows, the **A** to the coefficient matrix being modified, and the **S** to a diagonal matrix of column modifiers.

While $\mathbf{A}^2$ in (7.13) now contains elements that, in conjunction with $\mathbf{x}(1)$, satisfy the $\mathbf{v}(1)$ margins [as in (7.15)], it will generally be the case that in modifying $\mathbf{A}^1$ to $\mathbf{A}^2$ we will have disturbed the row sum property of $\mathbf{A}^1$, given in (7.10). [Except in the case where $\hat{\mathbf{s}}^1 = \mathbf{I}$, meaning that $\mathbf{A}^1$ also satisfies *all* of the column margins exactly, and then $\mathbf{A}^1$ is our desired $\tilde{\mathbf{A}}(1)$.] Therefore, we must now test $\mathbf{A}^2$ for row sum conformability, in the same way that we tested $\mathbf{A}(0)$, originally, and the reader can see where this is going. Each subsequent row modification will generally upset the previous column modification, and vice versa – a column modification will upset the previous row modification. We explore one more iteration – a row and then a column modification.

Thus, we now find $\mathbf{Z}^2\mathbf{i}$; that is

$$\begin{bmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 \\ a_{21}^2 & a_{22}^2 & a_{23}^2 \\ a_{31}^2 & a_{32}^2 & a_{33}^2 \end{bmatrix} \begin{bmatrix} x_1(1) & 0 & 0 \\ 0 & x_2(1) & 0 \\ 0 & 0 & x_3(1) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} u_1^2 \\ u_2^2 \\ u_3^2 \end{bmatrix} \tag{7.17}$$

and let $\mathbf{u}^2 = \begin{bmatrix} u_1^2 \\ u_2^2 \\ u_3^2 \end{bmatrix}$. (The superscript on **u** indicates our *second* set of row sum estimates.) If, as is likely, $\mathbf{u}^2 \neq \mathbf{u}(1)$, we repeat the steps used in forming the diagonal row-modifying matrix – $r_1^2 = u_1(1)/u_1^2$, $r_2^2 = u_2(1)/u_2^2$ and $r_3^2 = u_3(1)/u_3^2$ – and define

$$\hat{\mathbf{r}}^2 = \begin{bmatrix} r_1^2 & 0 & 0 \\ 0 & r_2^2 & 0 \\ 0 & 0 & r_3^2 \end{bmatrix} = [\hat{\mathbf{u}}(1)](\hat{\mathbf{u}}^2)^{-1} \tag{7.18}$$

[Compare $\hat{\mathbf{r}}^1$ in (7.9).] Note that the numerators in the r_i ratios are always the same, namely $u_i(1)$ – the number that we want. The denominators change, since they represent the "latest" estimates – here u_i^2 instead of u_i^1.

The entire procedure now follows the pattern that we have already established. If $\hat{\mathbf{r}}^2 = \mathbf{I}$, then $\mathbf{A}^2$ contains elements that satisfy both column and row margins, and we use it as $\tilde{\mathbf{A}}(1)$. If not – if $\mathbf{u}^2 \neq \mathbf{u}(1)$ – then we generate a further estimate of $\mathbf{A}(0)$ as

$$\mathbf{A}^3 = \hat{\mathbf{r}}^2\mathbf{A}^2 \tag{7.19}$$

The construction of $\hat{\mathbf{r}}^2$ assures that the row margins are now met.

The issue then (again) is whether the column sum properties of $\mathbf{A}^3$ satisfy the known target-year information in $\mathbf{v}(1)$. Thus v_1^2, v_2^2, and v_3^2 are generated, as in (7.11), with a_{ij}^3 here replacing a_{ij}^1 in that equation. Let $\mathbf{v}^2 = \begin{bmatrix} v_1^2 \\ v_2^2 \\ v_3^2 \end{bmatrix}$; if $\mathbf{v}^2 = \mathbf{v}(1)$, then we have in $\mathbf{A}^3$ a matrix that satisfies both row and column margins, and we use it for $\tilde{\mathbf{A}}(1)$. If $\mathbf{v}^2 \neq \mathbf{v}(1)$, we form

$$\hat{\mathbf{s}}^2 = [\hat{\mathbf{v}}(1)](\hat{\mathbf{v}}^2)^{-1} \tag{7.20}$$

exactly as in (7.14), but using the elements in $\mathbf{v}^2$ rather than those in $\mathbf{v}^1$. Then our next estimate of $\mathbf{A}(0)$ is given by

$$\mathbf{A}^4 = \mathbf{A}^3\hat{\mathbf{s}}^2 \tag{7.21}$$

Note, from (7.16) and (7.19), that

$$\mathbf{A}^3 = [\hat{\mathbf{r}}^2\hat{\mathbf{r}}^1]\mathbf{A}(0)[\hat{\mathbf{s}}^1] \tag{7.22}$$

and from (7.21)

$$\mathbf{A}^4 = [\hat{\mathbf{r}}^2\hat{\mathbf{r}}^1]\mathbf{A}(0)[\hat{\mathbf{s}}^1\hat{\mathbf{s}}^2] \tag{7.23}$$

Clearly, $\hat{\mathbf{r}}^1$, $\hat{\mathbf{r}}^2$, $\hat{\mathbf{s}}^1$, and $\hat{\mathbf{s}}^2$ are all diagonal matrices (3×3 in this example), so, for example,

$$[\hat{\mathbf{r}}^2\hat{\mathbf{r}}^1] = \begin{bmatrix} r_1^2r_1^1 & 0 & 0 \\ 0 & r_2^2r_2^1 & 0 \\ 0 & 0 & r_3^2r_3^1 \end{bmatrix}$$

And similarly for $[\hat{\mathbf{s}}^1\hat{\mathbf{s}}^2]$. By repetition of these procedures, we would find

$$\begin{aligned} &\mathbf{A}^5 = [\hat{\mathbf{r}}^3\hat{\mathbf{r}}^2\hat{\mathbf{r}}^1]\mathbf{A}(0)[\hat{\mathbf{s}}^1\hat{\mathbf{s}}^2] \\ &\mathbf{A}^6 = [\hat{\mathbf{r}}^3\hat{\mathbf{r}}^2\hat{\mathbf{r}}^1]\mathbf{A}(0)[\hat{\mathbf{s}}^1\hat{\mathbf{s}}^2\hat{\mathbf{s}}^3] \\ &\vdots \\ &\mathbf{A}^{2n} = [\hat{\mathbf{r}}^n\cdots\hat{\mathbf{r}}^1]\mathbf{A}(0)[\hat{\mathbf{s}}^1\cdots\hat{\mathbf{s}}^n] \end{aligned} \tag{7.24}$$

Letting $\hat{\mathbf{r}} = [\hat{\mathbf{r}}^n \cdots \hat{\mathbf{r}}^1]$ and $\hat{\mathbf{s}} = [\hat{\mathbf{s}}^1 \cdots \hat{\mathbf{s}}^n]$, and, again, ignoring hats, lower-case letters and the (0), the right-hand side of (7.24) is "**RAS**." As mentioned earlier, this is the origin of the name of the procedure.

One may reasonably ask: how many alterations using row and column balancing factors *will* be needed until the adjusted matrix satisfies the row and column marginal totals for year 1? And, for that matter, do we know that, eventually, they *will* be satisfied, or may the sequence of adjustments make things continually worse instead of better? In general, it has been found that the RAS procedure in fact does converge. That is, after row adjustment $\hat{\mathbf{r}}^{k+1}$ we are closer to $\mathbf{u}(1)$ than we were after the previous adjustment, $\hat{\mathbf{r}}^k$, and after column adjustment $\hat{\mathbf{s}}^{k+1}$ we are closer to $\mathbf{v}(1)$ than we were after $\hat{\mathbf{s}}^k$.[10] The number of adjustments needed depends at least in part on how close one wants the row and column margins of the adjusted matrix to be to the known target-year values $\mathbf{u}(1)$ and $\mathbf{v}(1)$. One criterion is to continue the matrix adjustments until all elements in both $[|\mathbf{u}(1) - \mathbf{u}^k|]$ and $[|\mathbf{v}(1) - \mathbf{v}^k|]$ are no more than ε, where ε is some small positive number, say 0.001. This means that each u_i^k is within 0.001 of the desired $u_i(1)$, and also that each v_j^k is within 0.001 of its associated $v_j(1)$.

For cases in which one is interested in assessing impacts on an economy of some *future* event, a *projection* of an existing technical coefficients matrix is called for. One approach is again to use the RAS procedure, where now the values in the $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{x}$ vectors must be forecast into the future year τ; these estimates $\mathbf{u}(\tau)$, $\mathbf{v}(\tau)$, and $\mathbf{x}(\tau)$ will then be used along with the current or most recent base matrix, $\mathbf{A}(0)$.

7.4.2 Example of the RAS Procedure

We illustrate the mathematics with a 3×3 example. Let

$$\mathbf{A}(0) = \begin{bmatrix} .120 & .100 & .049 \\ .210 & .247 & .265 \\ .026 & .249 & .145 \end{bmatrix} \tag{7.25}$$

The information necessary for a full survey-based coefficients table for the target year, $\mathbf{A}(1)$, would be interindustry flows, $\mathbf{Z}(1)$, and total outputs, $\mathbf{x}(1)$. Suppose, in fact, that we have

$$\mathbf{Z}(1) = \begin{bmatrix} 98 & 72 & 75 \\ 65 & 8 & 63 \\ 88 & 27 & 44 \end{bmatrix} \tag{7.26}$$

and

$$\mathbf{x}(1) = \begin{bmatrix} 421 \\ 284 \\ 283 \end{bmatrix} \tag{7.27}$$

[10] These technical matters, dealing with properties of the RAS technique, including convergence, are beyond the scope of this book.

Consequently,

$$\mathbf{u}(1) = \begin{bmatrix}245 & 136 & 159\end{bmatrix}' \tag{7.28}$$

and

$$\mathbf{v}(1) = \begin{bmatrix}251 & 107 & 182\end{bmatrix}' \tag{7.29}$$

and

$$\mathbf{A}(1) = [\mathbf{Z}(1)][\hat{\mathbf{x}}(1)]^{-1} = \begin{bmatrix} .2328 & .2535 & .2650 \\ .1544 & .0282 & .2226 \\ .2090 & .0951 & .1555 \end{bmatrix} \tag{7.30}$$

The point of partial-survey techniques is to develop reasonable estimates of the elements in $\mathbf{A}(1)$ in the absence of this kind of information on the full set of transactions in $\mathbf{Z}(1)$. To use the RAS approach, we need only the marginal information in $\mathbf{u}(1)$ and $\mathbf{v}(1)$, along with $\mathbf{x}(1)$ – as in (7.27), (7.28), and (7.29) – and the original or base year coefficients matrix, $\mathbf{A}(0)$, as in (7.25).

Beginning with the conjecture that the coefficients have not changed, we first examine the row sums of $\mathbf{A}(0)\hat{\mathbf{x}}(1)$, as in (7.5), in light of $\mathbf{u}(1)$. Here

$$\mathbf{Z}^1 = \mathbf{A}(0)\hat{\mathbf{x}}(1) = \begin{bmatrix} 50.520 & 28.400 & 13.867 \\ 88.410 & 70.148 & 74.995 \\ 10.946 & 70.716 & 41.035 \end{bmatrix}$$

and

$$\mathbf{u}^1 = \mathbf{Z}^1\mathbf{i} = \begin{bmatrix}92.787 & 233.553 & 122.697\end{bmatrix}'$$

Clearly, this is nowhere near to $\mathbf{u}(1)$ in (7.28) and adjustment is needed. To begin, then, $r_1^1 = u_1(1)/u_1^1 = 245/92.787 = 2.6405$, $r_2^1 = 0.5823$ and $r_3^1 = 1.2959$. Forming $\hat{\mathbf{r}}^1$ as in (7.9), we have

$$\hat{\mathbf{r}}^1 = [\hat{\mathbf{u}}(1)](\hat{\mathbf{u}}^1)^{-1} = \begin{bmatrix} 2.6405 & 0 & 0 \\ 0 & 0.5823 & 0 \\ 0 & 0 & 1.2959 \end{bmatrix}$$

and our first adjusted matrix, $\mathbf{A}^1$, is

$$\mathbf{A}^1 = \hat{\mathbf{r}}^1\mathbf{A}(0) = \begin{bmatrix} .3169 & .2640 & .1294 \\ .1223 & .1438 & .1543 \\ .0337 & .3227 & .1879 \end{bmatrix} \tag{7.31}$$

The elements in $\hat{\mathbf{r}}^1$ assure that the row sums of $\mathbf{A}^1\hat{\mathbf{x}}(1)$ will equal $\mathbf{u}(1)$, as in (7.10). Checking the column sums of $\mathbf{A}^1\hat{\mathbf{x}}(1)$ against $\mathbf{v}(1)$, we have

$$\mathbf{v}^1 = [\mathbf{A}^1\hat{\mathbf{x}}(1)]'\mathbf{i} = \begin{bmatrix}199.06 & 207.48 & 133.46\end{bmatrix}'$$

and this is wide of the mark, since

$$\mathbf{v}(1) = \begin{bmatrix} 251 & 107 & 182 \end{bmatrix}$$

Then, as in (7.14),

$$\hat{\mathbf{s}}^1 = [\hat{\mathbf{v}}(1)](\hat{\mathbf{v}}^1)^{-1} = \begin{bmatrix} 1.2609 & 0 & 0 \\ 0 & 0.5157 & 0 \\ 0 & 0 & 1.3637 \end{bmatrix}$$

and, following (7.13),

$$\mathbf{A}^2 = \mathbf{A}^1\hat{\mathbf{s}}^1 = \begin{bmatrix} .3995 & .1219 & .1764 \\ .1542 & .0661 & .2104 \\ .0425 & .1664 & .2562 \end{bmatrix}$$

In this example, we arbitrarily set $\varepsilon = 0.005$, meaning that the alternating row and column adjustments would continue through the kth adjustment, when $|u_i(1) - u_i^k| \leq 0.005$ and $|v_j(1) - v_j^k| \leq 0.005$ for $i, j = 1, 2, 3$. For this example, $k = 12$ (six row adjustments and six column adjustments were needed). The final matrix, $\mathbf{A}^{12}$, is

$$\tilde{\mathbf{A}}(1) = \mathbf{A}^{12} = \begin{bmatrix} .3924 & .1219 & .1596 \\ .1509 & .0661 & .1897 \\ .0592 & .1887 & .2938 \end{bmatrix} \tag{7.32}$$

Rather than print all the present matrices, $\mathbf{A}^1$ through $\mathbf{A}^{11}$, Table 7.1 gives the successive values of two representative coefficients, a_{11} and a_{23}, beginning with the original $\mathbf{A}(0)$ matrix and continuing through each RAS iteration. In Table 7.2 we record the three elements in $[\mathbf{u}(1) - \mathbf{u}^k]$ and the three elements in $[\mathbf{v}(1) - \mathbf{v}^k]$ (transposed to make them row vectors, for ease of presentation), for $k = 0$ through 13. The $k = 0$ line shows the row and column differences using $\mathbf{A}(0)\hat{\mathbf{x}}(1)$ – that is, assuming $\mathbf{A}(0) = \mathbf{A}(1)$. As expected, at $k = 1$ the row margins, in $\mathbf{u}(1)$, are satisfied exactly – all zero elements in $[\mathbf{u}(1) - \mathbf{u}^1]$ – but the column margins, in $\mathbf{v}(1)$, are not. Then, step 2 adjusts for these column constraints – generating zeros in $[\mathbf{v}(1) - \mathbf{v}^2]$ – but throwing the row sums out of balance with $\mathbf{u}(1)$. Therefore, for odd values of k, the $\mathbf{u}$ differences are all zero; for even values of k, the $\mathbf{v}$ differences are all zero. At $k = 13$ (that is, following $k = 12$), all differences are less than 0.005 in absolute value (for the first time), and hence the RAS adjustment is terminated. Finally, in Table 7.3 we present the elements of each of the matrices, $\hat{\mathbf{r}}^1$ through $\hat{\mathbf{r}}^7$ and $\hat{\mathbf{s}}^1$ through $\hat{\mathbf{s}}^7$, as in $\mathbf{A}^{2n}$ in (7.24).

It is of interest to compare our RAS-generated target-year matrix, $\tilde{\mathbf{A}}(1)$ with $\mathbf{A}(1)$ in (7.30), which we would have available to us if the entire set of interindustry transactions in $\mathbf{Z}(1)$ had been known.

$$\tilde{\mathbf{A}}(1) = \begin{bmatrix} .3924 & .1219 & .1596 \\ .1509 & .0661 & .1897 \\ .0529 & .1887 & .2938 \end{bmatrix} \text{ and } \mathbf{A}(1) = \begin{bmatrix} .2328 & .2535 & .2650 \\ .1544 & .0282 & .2226 \\ .2090 & .0951 & .1555 \end{bmatrix}$$

Table 7.1 Values of a_{11} and a_{23} at Each Step in the RAS Adjustment Procedure

k	a_{11}	a_{23}
0	.120	.265
1	.3169	.1543
2	.3995	.2104
3	.3812	.1966
4	.3957	.1913
5	.3902	.1912
6	.3931	.1900
7	.3920	.1900
8	.3926	.1898
9	.3923	.1898
10	.3925	.1897
11	.3924	.1897
12	.3924	.1897

Table 7.2 Differences from Row and Column Margins at Each Step in the RAS Adjustment Procedure

k	$[\mathbf{u}(1) - \mathbf{u}^k]'$			$[\mathbf{v}(1) - \mathbf{v}^k]'$		
0	152.2130	−97.5530	36.3030	101.1240	−62.2640	52.1030
1	0	0	0	51.9376	−100.4759	48.5383
2	−11.8055	−9.5328	21.3383	0	0	0
3	0	0	0	9.2120	−4.1679	−5.0441
4	−3.4458	−.0723	3.5181	0	0	0
5	0	0	0	1.8586	−.6862	−1.1724
6	−.7098	−.0024	.7122	0	0	0
7	0	0	0	.3798	−.1394	−.2404
8	−.1452	−.0007	.1459	0	0	0
9	0	0	0	.0778	−.0286	−.0492
10	−.0297	−.0002	.0299	0	0	0
11	0	0	0	.0159	−.0059	−.0101
12	−.0061	0	.0061	0	0	0
13	0	0	0	.0033	−.0012	−.0021

Even casual inspection shows that there are significant differences in most of the elements in these two matrices.

Define an error matrix, $\mathbf{E}(\mathbf{A})$, as $\mathbf{E}(\mathbf{A}) = \tilde{\mathbf{A}}(1) - \mathbf{A}(1)$. Here

$$\mathbf{E}(\mathbf{A}) = \begin{bmatrix} .1596 & -.1316 & -.1054 \\ -.0035 & .0379 & -.0329 \\ -.1561 & .0936 & .1383 \end{bmatrix}$$

Table 7.3 Elements in the Diagonal Matrices $\hat{\mathbf{r}}^k$ and $\hat{\mathbf{s}}^k$, for $k = 1, \ldots, 7$

k	$\hat{\mathbf{r}}^k$			$\hat{\mathbf{s}}^k$		
1	2.6405	.5823	1.2959	1.2609	.5157	1.3637
2	.9540	.9345	1.1550	1.0381	.9625	.9730
3	.9861	.9995	1.0226	1.0075	.9936	.9936
4	.9971	1.0000	1.0045	1.0015	.9987	.9987
5	.9994	1.0000	1.0009	1.0003	.9997	.9997
6	.9999	1.0000	1.0002	1.0001	.9999	.9999
7	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Notice that column sums of $\mathbf{E}(\mathbf{A})$ are zero; this reflects the fact that column sums of $\tilde{\mathbf{A}}(1)$ and $\mathbf{A}(1)$ are equal, except for rounding in this small example.[11]

An alternative way to express the errors in each of the coefficients is to convert the elements in $\mathbf{E}(\mathbf{A})$ to percentages. Define $\mathbf{P}(\mathbf{A}) = [p(a)_{ij}]$ where

$$p(a)_{ij} = [|\tilde{a}_{ij} - a_{ij}(1)|/a_{ij}(1)] \times 100 = [|e(a)_{ij}|/a_{ij}(1)] \times 100$$

These are the absolute values of the errors as a percentage of the corresponding true coefficients in $\mathbf{A}(1)$. For this example,

$$\mathbf{P}(\mathbf{A}) = \begin{bmatrix} 68.6 & 51.9 & 39.8 \\ 2.3 & 134.4 & 14.8 \\ 74.7 & 98.4 & 88.9 \end{bmatrix}$$

Viewed in this way, also, it is clear that some of the RAS-estimated coefficients are wildly different from their survey counterparts. Six of the nine RAS-generated coefficients are in error by more than 50 percent – not a very successful estimate.

There are many measures available for quantifying the "difference" between two matrices. We illustrate several of them. *The mean absolute deviation* (MAD) simply averages the elements in $\mathbf{E}(\mathbf{A})$, ignoring sign:

$$\text{MAD} = (1/n^2) \sum_{i=1}^{n} \sum_{j=1}^{n} |e(a)_{ij}|$$

In our example, MAD = (1/9)(0.8589) = 0.0954. This represents the average amount (whether positive or negative) by which an estimated coefficient differs from the true coefficient. The *mean absolute percentage error* (MAPE) performs the same averaging

[11] The RAS marginal constraints assure that $\mathbf{i}'\tilde{\mathbf{Z}}(1) = \mathbf{i}'\mathbf{Z}(1)$. Since $\tilde{\mathbf{Z}}(1) = \tilde{\mathbf{A}}(1)\hat{\mathbf{x}}(1)$ and $\mathbf{Z}(1) = \mathbf{A}(1)\hat{\mathbf{x}}(1)$, $\mathbf{i}'\tilde{\mathbf{A}}(1)\hat{\mathbf{x}}(1) = \mathbf{i}'\mathbf{A}(1)\hat{\mathbf{x}}(1)$ and so (postmultiplying by $[\hat{\mathbf{x}}(1)]^{-1}$), $\mathbf{i}'\tilde{\mathbf{A}}(1) = \mathbf{i}'\mathbf{A}(1)$.

on the elements in $\mathbf{P}(\mathbf{A})$, namely

$$\text{MAPE} = (1/n^2)\sum_{i=1}^{n}\sum_{j=1}^{n} p(a)_{ij}$$

For this example, MAPE = (1/9)(575.38) = 63.76, which means that, on average, each coefficient will be either 63.8 percent larger or smaller than its true value; that is, it will be "in error" by 63.8 percent. [If the *direction* of error is thought to be important, then we could generate the elements in the $\mathbf{P}(\mathbf{A})$ matrix, retaining the signs. However, in that case, it is not very meaningful to find an average over all elements, since positive and negative errors would offset each other.] By these measures (and others, which we need not explore here), the matrix produced by the RAS procedure in this small example does not appear to be a particularly good reflection of $\mathbf{A}(1)$. At least this is the implication of these measures that examine the element-by-element accuracy of $\tilde{\mathbf{A}}(1)$ as compared with $\mathbf{A}(1)$. In larger examples, more representative of real-world input–output tables, there are more elements available for adjustment in any row or column and, in that sense, there is more flexibility in producing an estimate of the target-year matrix.

Another point of view is that while this individual cell accuracy (sometimes called *partitive* accuracy) may be important for some kinds of problems, the ultimate test of a set of input–output coefficients is how well they perform in practice (also sometimes known as *holistic* accuracy).[12] That is, perhaps we should be more concerned with the relative accuracy in the Leontief inverse matrices associated with $\tilde{\mathbf{A}}(1)$ and $\mathbf{A}(1)$. Here

$$\mathbf{L}(1) = \begin{bmatrix} 1.5651 & .4684 & .6146 \\ .3463 & 1.1599 & .4144 \\ .4264 & .2465 & 1.3829 \end{bmatrix} \tag{7.33}$$

and

$$\tilde{\mathbf{L}}(1) = [(\mathbf{I} - \tilde{\mathbf{A}}(1)]^{-1} = \begin{bmatrix} 1.7703 & .3298 & .4888 \\ .3310 & 1.1940 & .3955 \\ .2210 & .3438 & 1.5583 \end{bmatrix} \tag{7.34}$$

The associated error matrices are

$$\mathbf{E}(\mathbf{L}) = \begin{bmatrix} .2052 & -.1386 & -.1258 \\ -.0153 & .0341 & -.0189 \\ -.2054 & .0973 & .1754 \end{bmatrix}$$

and

$$\mathbf{P}(\mathbf{L}) = \begin{bmatrix} 13.1 & 29.6 & 20.5 \\ 4.4 & 2.9 & 4.6 \\ 48.2 & 39.5 & 12.7 \end{bmatrix}$$

12 These terms are from Jensen (for example, Jensen, 1980).

For this small example, percentage errors in $\mathbf{P}(\mathbf{L})$, associated with the Leontief inverse matrices, are generally considerably smaller than those in $\mathbf{P}(\mathbf{A})$.

Alternatively, consider the output multipliers associated with $\mathbf{L}(1)$ and $\tilde{\mathbf{L}}(1)$, $\mathbf{m}(o) = \begin{bmatrix} 2.3378 & 1.8748 & 2.4119 \end{bmatrix}$ and $\tilde{\mathbf{m}}(o) = \begin{bmatrix} 2.3223 & 1.8676 & 2.4426 \end{bmatrix}$. The vector of percentage errors, expressing each $[m(o)_j - \tilde{m}(o)_j]$ as a percentage of $m(o)_j$, is $\mathbf{p}(\mathbf{m}) = \begin{bmatrix} 0.66 & 0.38 & -1.27 \end{bmatrix}$. This indicates much closer correspondence between the estimated and true multipliers than might be expected from **E(A)** and **P(A)**, and even from **E(L)** and **P(L)**.

The power series expressions for L and $\tilde{\mathbf{L}}$ are helpful here, namely

$$\mathbf{L} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots \text{ and } \tilde{\mathbf{L}} = \mathbf{I} + \tilde{\mathbf{A}} + \tilde{\mathbf{A}}^2 + \tilde{\mathbf{A}}^3 + \cdots$$

From these, the (row) vector of multiplier differences can be expressed as

$$\mathbf{i}'\mathbf{L} - \mathbf{i}'\tilde{\mathbf{L}} = \mathbf{i}'(\mathbf{L} - \tilde{\mathbf{L}}) = \mathbf{i}'(\mathbf{I} - \mathbf{I}) + \mathbf{i}'(\mathbf{A} - \tilde{\mathbf{A}}) + \mathbf{i}'(\mathbf{A}^2 - \tilde{\mathbf{A}}^2) + \mathbf{i}'(\mathbf{A}^3 - \tilde{\mathbf{A}}^3) + \cdots$$

Clearly $\mathbf{i}'(\mathbf{I} - \mathbf{I}) = \mathbf{0}$, and also $\mathbf{i}'(\mathbf{A} - \tilde{\mathbf{A}}) = \mathbf{i}'\mathbf{E}(\mathbf{A}) = \mathbf{0}$, as noted above. Therefore,

$$\mathbf{i}'\mathbf{L} - \mathbf{i}'\tilde{\mathbf{L}} = \mathbf{0} + \mathbf{0} + \mathbf{i}'(\mathbf{A}^2 - \tilde{\mathbf{A}}^2) + \mathbf{i}'(\mathbf{A}^3 - \tilde{\mathbf{A}}^3) + \cdots$$

We see that the first two terms in the expression for the output multiplier differences are zero. (In the example in Table 2.5 we saw that these two terms in the power series accounted for between 85 and 92 percent of the total output effect.)

Comparison of multipliers is a test of the model in use, with specific final-demand vectors – $[1,\ 0, 0]'$, $[0,\ 1, 0]'$, and $[0,\ 0,\ 1]'$, respectively. We can also compare the performance using any arbitrarily chosen $\mathbf{f}(1)$ vector. For example, let $\mathbf{f}(1) = \begin{bmatrix} 800 \\ 700 \\ 300 \end{bmatrix}$; then from the Leontief inverses in (7.33) and (7.34),

$$\mathbf{x}(1) = \begin{bmatrix} 1764.20 \\ 1213.29 \\ 928.54 \end{bmatrix} \text{ and } \tilde{\mathbf{x}}(1) = \begin{bmatrix} 1793.74 \\ 1219.25 \\ 884.95 \end{bmatrix}$$

Again, expressing the differences as a percentage of $x_i(1)$,

$$\mathbf{p}(\Delta\mathbf{x}) = \begin{bmatrix} 1.67 \\ 0.49 \\ -4.69 \end{bmatrix}$$

The effect on the gross output of sector 3 is underestimated by almost five percent while the other two outputs are much more accurately estimated. Of course, results of this kind depend on the arbitrary $\mathbf{f}(1)$vector used for the illustration.

Conclusions suggested by this example are: (1) the RAS procedure may generate a technical coefficients matrix that does not look very much like an associated full-survey matrix, but (2) an **A** matrix estimated by RAS may perform relatively well in practice, that is, when converted to its associated Leontief inverse, in terms of the sectoral gross outputs that it produces in conjunction with a given $\mathbf{f}(1)$ vector. We will examine another holistic measure of performance in Chapter 8, when we explore differences in output multipliers in a regional input–output model.

7.4.3 Updating Coefficients vs. Transactions

Early discussions of the technique assumed that one begins with a base year **A**; this is explicit in the "**RAS**" name. It appears that Deming and Stephan (1940) first used the biproportional adjustment technique that later became known as RAS. Leontief (1941) suggested a similar pair of influences (on rows and on columns) to account jointly for coefficient change. Stone and his colleagues at Cambridge apparently were unaware of this work when they proposed it in 1962 (Bacharach, 1970, p. 4; see also Lahr and de Mesnard, 2004). The Cambridge work emphasized operations on a base-year *coefficient* matrix, even though Bacharach (1970, p. 20) suggests that the ultimate interest was in a target-year *transactions* matrix.

In fact, this biproportional matrix balancing approach can be equally well applied directly to a base-year transactions matrix, $\mathbf{Z}(0)$, in conjunction with the required marginal information, $\mathbf{x}(1)$, $\mathbf{u}(1)$, and $\mathbf{v}(1)$. In this case, there is no need to convert the coefficients at each step, $\mathbf{A}^k$, to transactions, $\mathbf{Z}^k$, in order to check the degree of conformity with $\mathbf{u}(1)$ and $\mathbf{v}(1)$. There seems to have been some uncertainty in the literature on whether or not the end results of the two exercises – directly altering **A** vs. directly altering **Z** – are the same.[13]

In the former case (updating **A**), denote $\tilde{\mathbf{A}}^A(1) = \hat{\mathbf{r}}^A\mathbf{A}(0)\hat{\mathbf{s}}^A$, leading to $\tilde{\mathbf{Z}}^A(1) = \tilde{\mathbf{A}}^A(1)\hat{\mathbf{x}}(1)$ and in the latter case (updating **Z**), let $\tilde{\mathbf{Z}}^Z(1) = \hat{\mathbf{r}}^Z\mathbf{Z}(0)\hat{\mathbf{s}}^Z$, leading to $\tilde{\mathbf{A}}^Z(1) = \tilde{\mathbf{Z}}^Z(1)\hat{\mathbf{x}}(1)^{-1}$. The question is whether or not $\tilde{\mathbf{A}}^A(1) = \tilde{\mathbf{A}}^Z(1)$ or $\tilde{\mathbf{Z}}^A(1) = \tilde{\mathbf{Z}}^Z(1)$ (where superscripts indicate which matrix was used in the updating procedure). The answer is that it makes no difference which kind of matrix is used as the base for the updating procedure (coefficients or transactions); the results from the two approaches are the same. (See Dietzenbacher and Miller, 2009, for a proof).

Numerical Illustration This is the data set for the closed model in Chapter 2. Call this year 0 data:

$$\mathbf{Z}(0) = \begin{bmatrix} 150 & 500 & 50 \\ 200 & 100 & 400 \\ 300 & 500 & 50 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 1000 \\ 2000 \\ 1000 \end{bmatrix}, \quad \mathbf{A}(0) = \begin{bmatrix} .15 & .25 & .05 \\ .20 & .05 & .40 \\ .30 & .25 & .05 \end{bmatrix}$$

[13] See for example, the sequence of opinions expressed in Okuyama *et al.* (2002), Jackson and Murray (2004), and Oosterhaven (2005).

Assume that we have the following year 1 information (necessary for RAS):

$$\mathbf{x}(1) = \begin{bmatrix} 1200 \\ 2500 \\ 1400 \end{bmatrix}, \quad \mathbf{u}(1) = \begin{bmatrix} 780 \\ 810 \\ 1050 \end{bmatrix}, \quad \mathbf{v}(1) = \begin{bmatrix} 740 \\ 1270 \\ 630 \end{bmatrix}$$

Coefficient updating. Start with $\mathbf{A}(0)$. In this case we find

$$\tilde{\mathbf{A}}^A(1) = \hat{\mathbf{r}}^A \mathbf{A}(0)\hat{\mathbf{s}}^A = \begin{bmatrix} 0.1370 & 0.2205 & 0.0460 \\ 0.1752 & 0.0423 & 0.3529 \\ 0.3046 & 0.2452 & 0.0511 \end{bmatrix}$$

For ease of presentation, we have rounded all coefficients to four digits and all transactions to whole numbers. In this case, the associated transactions matrix is

$$\tilde{\mathbf{Z}}(1)^A = \tilde{\mathbf{A}}^A \hat{\mathbf{x}}(1) = \begin{bmatrix} 164 & 551 & 64 \\ 210 & 106 & 494 \\ 365 & 613 & 72 \end{bmatrix}$$

Transaction updating. Start with $\mathbf{Z}(0)$. RAS provides the update

$$\tilde{\mathbf{Z}}^Z(1) = \hat{\mathbf{r}}^Z \mathbf{Z}(0)\hat{\mathbf{s}}^Z = \begin{bmatrix} 164 & 551 & 64 \\ 210 & 106 & 494 \\ 365 & 613 & 72 \end{bmatrix}$$

illustrating that $\tilde{\mathbf{Z}}^A(1) = \tilde{\mathbf{Z}}^Z(1)$. Also, from this,

$$\tilde{\mathbf{A}}^Z(1) = \tilde{\mathbf{Z}}^Z(1)[\hat{\mathbf{x}}(1)]^{-1} = \begin{bmatrix} 0.1370 & 0.2205 & 0.0460 \\ 0.1752 & 0.0423 & 0.3529 \\ 0.3046 & 0.2452 & 0.0511 \end{bmatrix}$$

and $\tilde{\mathbf{A}}^A(1) = \tilde{\mathbf{A}}^Z(1)$. This does not prove but illustrates what is a general result.

7.4.4 An Economic Interpretation of the RAS Procedure

In the preceding sections, we have illustrated the mathematics of the RAS procedure for sequentially adjusting rows and columns of a given coefficient matrix, $\mathbf{A}(0)$, in order to generate an estimate of a more recent matrix, $\mathbf{A}(1)$, where only $\mathbf{x}(1)$, $\mathbf{u}(1)$, and $\mathbf{v}(1)$ are assumed known for the target year, 1. When the adjustment process is terminated – because the row and column margins are within the prespecified level of error, ε, from the elements in $\mathbf{u}(1)$ and $\mathbf{v}(1)$ – we have

$$\mathbf{A}(1) = \hat{\mathbf{r}}A(0)\hat{\mathbf{s}} \tag{7.35}$$

As we have seen, each element r_i in $\hat{\mathbf{r}}$ multiplies each element in row i of $\mathbf{A}(0)$ and each element s_j of $\hat{\mathbf{s}}$ multiplies each element in column j of $\mathbf{A}(0)$ – for $i, j = 1, \ldots, n$.

In this "updating" procedure, one might well ask why this kind of uniform proportional change should be expected for the elements in rows or columns of $\mathbf{A}(0)$.

In the early development of the RAS procedure, Stone (1961) described the uniform changes along any row and down any column in **A** as reflecting what he termed the economic phenomena of *substitution effects* and *fabrication effects*, respectively. The former refers to the emergence of substitutes as production inputs; that is, the substitution of one input for another – for example, the use (throughout industrial processes) of plastic products in place of metal ones. The implication is that all a_{ij} in the plastics row (i) would increase (for example, be multiplied by 1.4) and all a_{kj} in the metals row (k) would decrease (for example, be multiplied by 0.82). The term fabrication effect refers to the altered proportion of value-added items in a sector's total purchases. For example, over time, the product of a particular sector may come to depend more on high-technology capital equipment and/or skilled labor. Thus, a dollar's worth of the product embodies proportionately less of interindustrial inputs and proportionately more of value-added inputs, and the a_{ij} in the column representing the industry in question would decrease (for example, be multiplied by 0.79).

To the extent that technological change in the style of production may be reflected in such substitution and fabrication effects, the RAS procedure has a logical economic basis. However, many researchers discount this oversimplified view of the way in which such change is distributed throughout an economy. Instead, they view RAS as a purely mathematical procedure. It can be shown that the RAS technique in fact emerges as the solution to a constrained optimization problem in which, subject to the row and column margins given in $\mathbf{u}(1)$ and $\mathbf{v}(1)$, we want to generate a new coefficient matrix, $\mathbf{A}(1)$, that "differs" as little as possible from our previous observation, $\mathbf{A}(0)$. The underlying logic is simply that, in the absence of any new information, we would assume that $\mathbf{A}(0)$ is still the best representation of interindustrial relationships. However, given some updated information – in $\mathbf{x}(1)$, $\mathbf{u}(1)$, and $\mathbf{v}(1)$ – a modified matrix, $\mathbf{A}(1)$, will usually be called for.

Two properties of the RAS procedure bear noting. Signs are preserved in the sense that no $a_{ij}(0) > 0$ will ever be changed to a negative-valued coefficient. As the fundamental definitions of $\hat{\mathbf{r}}$ and $\hat{\mathbf{s}}$ make clear, all the r_i and s_j modifiers of $\mathbf{A}(0)$ are non-negative. Thus, no matter how much a particular $a_{ij}(0)$ is modified, it will remain non-negative. Secondly, any $a_{ij}(0)$ that equals zero will remain zero throughout the RAS procedure, since all that happens to it is that it is multiplied by non-negative numbers. Suppose that sector i represents potatoes and sector j is automobiles; if $a_{ij}(0) = 0$, this represents the (believable) fact that potatoes were not purchased as direct inputs to automobile manufacturing in year 0. The RAS technique assures us that in the updated matrix $a_{ij}(1)$ will still be zero. This feature is a mixed blessing. In some cases, such as potatoes and automobiles, it is probably good that a zero-valued coefficient is preserved; potatoes were not used as a direct input to automobiles in year 0 and most probably not in year 1, either. On the other hand, if sector k is plastics and sector j is automobiles, it may be (if year 0 was long enough ago) that $a_{kj}(0) = 0$, but we know that for our more recent year 1, $a_{kj}(1) \neq 0$. Nevertheless, the RAS procedure by itself will predict $a_{kj}(1) = 0$.

7.4.5 Incorporating Additional Exogenous Information in an RAS Calculation

The RAS technique, as discussed above, assumes only target-year information on $\mathbf{x}$, $\mathbf{u}$ and $\mathbf{v}$. Often one may have particular information about specific transactions or specific coefficients. If a particular $z_{ij}(1)$is exogenously known, then since $x_j(1)$ is also known, so is $a_{ij}(1)$. Such information may come from a survey of an "important" industry in the economy, from an independent forecast of a particular sector's sales to one or more sectors, from expert opinions about production practices in a particular sector, and so on.

Suppose that a particular $z_{ij}(1)$ is known. Then one can subtract $z_{ij}(1)$ from both $u_i(1)$ and $v_j(1)$; this is equivalent to inserting a zero in the i, jth cell of $\mathbf{Z}(0)$ and hence of $\mathbf{A}(0)$. Continuing with our general 3×3 example, suppose $z_{31}(1)$ is known. Since $x_1(1)$ is also known, $a_{31}(1)$ is known as well.

Define $\bar{\mathbf{A}}(0)$ to be the same as $\mathbf{A}(0)$ except that $a_{31}(0)$ has been replaced with a zero. Define a 3×3 matrix $\mathbf{K}$ as

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{31}(1) & 0 & 0 \end{bmatrix}$$

This is just the null matrix with k_{31} replaced by the known target-year coefficient, $a_{31}(1)$. Then $\mathbf{A}(0) = \bar{\mathbf{A}}(0) + \mathbf{K}$. Denote by $\bar{\mathbf{u}}(1)$ and $\bar{\mathbf{v}}(1)$ the vectors that remain after $z_{31}(1)$ is subtracted from $u_3(1)$ and $v_1(1)$. These become the relevant new margins, and the RAS procedure is utilized, as usual, but with $\bar{\mathbf{A}}(0)$ as the base-year matrix, to be modified according to the (altered) row and column sum information for the target year, $\bar{\mathbf{u}}(1)$ and $\bar{\mathbf{v}}(1)$. The RAS technique will leave the new zero element, $a_{31}(0)$, unchanged. When the approximating technique is completed we construct our estimate of $\mathbf{A}(1)$ as[14]

$$\tilde{\mathbf{A}}(1)_{31} = \mathbf{K} + \hat{\mathbf{r}}\bar{\mathbf{A}}(0)\hat{\mathbf{s}} \tag{7.36}$$

Clearly, in an economy represented by a larger number of sectors, we may have estimates of several $z_{ij}(1)$ and hence of several of the target-year coefficients, $a_{ij}(1)$. In fact, if there is a "key" sector that is known to play a particularly important role in the economy, an entire column (intermediate inputs to the key sector) and/or an entire row (intermediate sales by the key sector) may be known or somehow independently determined. And indeed there may be more than one key sector. In all of these cases, there is no difference in the approach outlined. Of course, the matrix $\mathbf{K}$ will contain more nonzero (known) elements, the matrix $\bar{\mathbf{A}}(0)$ will contain more zeros, and the adjustments to $\mathbf{u}(1)$ and $\mathbf{v}(1)$ – to generate $\bar{\mathbf{u}}(1)$ and $\bar{\mathbf{v}}(1)$ – will be more extensive.[15]

[14] We use the "31" subscript to indicate which element was replaced by its true value. This does not generalize easily to cases in which more than one element is replaced by exogenous information, but it serves adequately for present purposes.

[15] See section 7.4.8, below, on the role of zeros in creating infeasible problems – where RAS fails to generate a solution.

7.4.6 Modified Example: One Coefficient Known in Advance

Here is an illustration. Suppose a_{31} is known in advance for the example in section 7.4.2; from (7.30), $a_{31}(1) = 0.209$, and therefore

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ .209 & 0 & 0 \end{bmatrix}$$

and so

$$\bar{\mathbf{A}}(0) = \begin{bmatrix} .120 & .100 & .049 \\ .210 & .247 & .265 \\ 0 & .249 & .145 \end{bmatrix}$$

This is $\mathbf{A}(0)$ in (7.25) with $a_{31}(1)$ replaced by 0.

To employ the RAS procedure on $\bar{\mathbf{A}}(0)$ we find $\bar{\mathbf{u}}(1)$ and $\bar{\mathbf{v}}(1)$. The (known) interindustry flow in the target year, from sector 3 to sector 1, is $z_{31}(1) = a_{31}(1)x_1(1) = (0.209)(421) = 87.989$; this therefore must be netted out of both $u_3(1)$ and $v_1(1)$, leading to $\bar{\mathbf{u}}(1) = \begin{bmatrix} 245 & 136 & 71.011 \end{bmatrix}'$ and $\bar{\mathbf{v}}(1) = \begin{bmatrix} 163.011 & 107 & 182 \end{bmatrix}'$. Following (7.36) we find

$$\tilde{\mathbf{A}}(1)_{31} = \begin{bmatrix} .2909 & .1892 & .2431 \\ .0963 & .0884 & .2486 \\ .2090 & .0992 & .1514 \end{bmatrix} \tag{7.37}$$

Recall, from (7.30), that

$$\mathbf{A}(1) = \begin{bmatrix} .2328 & .2535 & .2650 \\ .1544 & .0282 & .2226 \\ .2090 & .0951 & .1555 \end{bmatrix}$$

and the error matrix for this estimate, $\mathbf{E}(\mathbf{A}) = \tilde{\mathbf{A}}(1)_{31} - \mathbf{A}(1)$, is

$$\mathbf{E}(\mathbf{A}) = \begin{bmatrix} -.0581 & -.0643 & -.0219 \\ .0581 & .0602 & .0260 \\ 0 & .0041 & -.0041 \end{bmatrix}$$

In this case, the reader can easily find that MAD = (1/9)(0.2968) = 0.0330 and MAPE = 36.5; in the original example, without any prior information on coefficient values, we found MAD = 0.0954 and MAPE = 63.8. By these measures, then, the RAS estimate in (7.37), which includes exogenous information on $a_{31}(1)$ in the target year, is more accurate than was $\tilde{\mathbf{A}}(1)$ in (7.32).

It turns out, however, that assessment of the performance of modified RAS estimates depends very much on the measure used to measure the differences between matrices, specifically $\tilde{\mathbf{A}}(1) - \mathbf{A}(1)$ (no exogenous information) and $\tilde{\mathbf{A}}(1)_{ij} - \mathbf{A}(1)$ [substitution of the true $a_{ij}(1)$]. Table 7.4 illustrates this sensitivity for the numerical example begun

Table 7.4 MAD and MAPE when One Coefficient is Known in Advance in an RAS Estimate

Element known in advance	MAD ($\times$100)	MAPE
None	9.55	63.8
a_{11}	5.52	31.6
a_{12}	7.24	36.6
a_{13}	8.53	62.1
a_{21}	9.49	63.0
a_{22}	8.80	48.6
a_{23}	9.45	60.8
a_{31}	3.30	36.5
a_{32}	9.17	**69.4**
a_{33}	7.48	47.7

in section 7.4.2 and continued above. This table presents the MAD (multiplied by 100 for easier reading) and MAPE measures associated with each of the $\tilde{\mathbf{A}}(1)_{ij}$ matrices generated using prior information on a single $a_{ij}(1)$ cell in $\mathbf{A}(1)$. In this small example, there is improvement (over the no-prior-information case) as measured by MAD, but using the MAPE measure we find that correct prior information on a_{32} (in bold type) makes the overall estimate worse. (This sensitivity to alternative "metrics" for comparing closeness of matrices is discussed, with numerical examples, in de Mesnard and Miller, 2006.)

This result (worse results with better information) has been discussed before in the literature, although the importance of the measure of distance between matrices was not emphasized. In an early example that was frequently cited, Miernyk (1977) presented this counterintuitive result, using "mean percentage difference" as the measure of distance between the predicted and true target-year matrix. The idea was later taken up by Miller and Blair (1985) in the first edition of this text, where a further example appeared to illustrate the same point. In fact, both of these results have been shown to be flawed – there were errors with the RAS procedures (improper computer programs, stopping criteria that were too loose, etc.).[16] Nonetheless, later experiments with data sets that are much larger and more reflective of real-world applications have identified examples in which additional (correct) information generates poorer RAS estimates, under several fairly common distance measures. (Examples can be found in Szyrmer, 1989, and Lahr, 2001.) Nonetheless, the overwhelming majority of the evidence suggests the contrary. As a general rule, introduction of accurate exogenous information in RAS improves the resulting estimates. This is what hybrid models are designed to do.

[16] These are taken up in detail in de Mesnard and Miller (2006).

7.4.7 Hybrid Models: RAS with Additional Information

In the decades since RAS was first proposed, there have been many applications at both national and regional levels. These have led to numerous variations, modifications, and extensions of the technique. An examination of the tables of contents or annual indexes of many journals in the field – especially *Economic Systems Research* and *Journal of Regional Science* – will reveal a large number of articles with "RAS," "partial-survey methods," "nonsurvey methods," "biproportional methods" or "hybrid models" in the title. Among the modifications are methods labeled "TRAS" (identified by its originators as a "three-stage RAS" or as a "two-stage RAS algorithm"; see Gilchrist and St. Louis, 1999, p. 186 and Gilchrist and St. Louis, 2004, p. 150, respectively), "GRAS" (for "generalized" RAS, for matrices that include negative numbers; see Junius and Oosterhaven, 2003) or "ERAS" (for "extended" RAS; see Israilevich, 1986).

Indeed, the preponderance of tables that are currently (beginning of the twenty-first century) being produced employ the "hybrid" notion of combining some kind of balancing of tables (usually using RAS or a variant) after "superior" information has been introduced, in the style of the example in sections 7.4.4 and 7.4.5. For example, the Bureau of Economic Analysis at the US Department of Commerce uses an adjusted RAS procedure to generate annual input–output tables for non-benchmark-table years in the USA.[17] In Europe, Eurostat is the agency that oversees collection and compilation of input–output data for the European Union member countries. Tables for non-benchmark years are produced using the Eurostat method, a modified and expanded RAS approach. (See Eurostat, 2002, esp. Chapter 14.)

A major trick in these kinds of applications is establishing which sectors (columns, rows or even individual cells) are most "important" to the economy, since these are the elements for which superior information would be preferred. In Chapter 12 we examine some of the approaches to identifying "important" sectors in an economy on the basis of their input–output data. As noted, this kind of exploration also identifies important (sets of) coefficients for which one would ideally like to have superior data to combine with RAS or some similar procedure for the remaining cells. There is an immense literature on this subject, and we will explore some of it in Chapter 12. Some of the approaches are essentially *mathematical* in nature – for example, those that are concerned with the influence of errors in one or more elements in a matrix on the resulting elements in the associated inverse matrix – and others are more *economic* in nature, in which attempts are made to identify important, or "key," sectors in an economy. In actuality this distinction tends to blur, since influential elements often belong to what turn out to be important sectors.

[17] This is described in Planting and Guo, 2004. The authors speak of "… [the] new automated updating and balancing method …" (p. 157).

7.4.8 The Constrained Optimization Context

The notion of the “difference” between two matrices is a subtle one; there are many alternative measures. The RAS procedure can be shown to minimize

$$D[\mathbf{A}(0) : \tilde{\mathbf{A}}(1)] = \sum_i \sum_i \left\{ \tilde{a}_{ij}(1) \ln \left[\frac{\tilde{a}_{ij}(1)}{a_{ij}(0)} \right] \right\}$$

subject to the constraints on row and column sums given by $\mathbf{u}(1) = [(\tilde{\mathbf{A}})(1)\hat{\mathbf{x}}(1)]\mathbf{i}$ and $\mathbf{v}(1) = \mathbf{i}'[(\tilde{\mathbf{A}})(1)\hat{\mathbf{x}}(1)]$. (This is explored in Appendix 7.2 to this chapter.) The objective function, $D[\mathbf{A}(0) : \tilde{\mathbf{A}}(1)]$, has an interpretation as the “information” measure of distance between $\mathbf{A}(0)$ and $\tilde{\mathbf{A}}(1)$. In a sense, it generates the $\tilde{\mathbf{A}}(1)$ which, given $\mathbf{A}(0)$ and the information in $\mathbf{x}(1)$, $\mathbf{u}(1)$, and $\mathbf{v}(1)$, generates the least “surprise.”

Many other potentially attractive measures have been proposed to represent the difference (or distance) between the estimated matrix and the base-year matrix. These become an objective function in an associated constrained optimization problem. The constraints continue to be the row and column margins, as in RAS. However, $\tilde{a}_{ij} \geq 0$ for all i and j must be added as an additional n^2 constraints because, unlike the RAS procedure, non-negativity of the solutions to these programming problems cannot be assured. Sometimes, also, bounds have been set on the sizes of relative change allowed for the elements. For example, $(0.5)a_{ij}(0) \leq \tilde{a}_{ij}(1) \leq (1.5)a_{ij}(0)$ would assure that each original coefficient did not increase or decrease by more than 50 percent.

Some of the objectives that have been proposed in this input–output updating context are:[18]

- Total absolute deviation: $\sum_i \sum_j \left| a_{ij}(0) - \tilde{a}_{ij}(1) \right|$. Divided by n^2, this is known as the mean absolute deviation (MAD). This and the following two objectives can be converted to a linear form, thus creating a linear program which is easily solved. (Jackson and Murray, 2004.)
- Weighted absolute deviation: $\sum_i \sum_j a_{ij}(0) \left| a_{ij}(0) - \tilde{a}_{ij}(1) \right|$. (Lahr, 2001.)
- Relative deviation: $\sum_i \sum_j \frac{\left| a_{ij}(0) - \tilde{a}_{ij}(1) \right|}{a_{ij}(0)}$. (Matuszewski, Pitts and Sawyer, 1964.) Multiplied by 100 and divided by n^2, this is known as the mean absolute percentage error (MAPE).
- Squared (or quadratic) deviation: $\sum_i \sum_j [a_{ij}(0) - \tilde{a}_{ij}(1)]^2$. (Almon, 1968.) This and the next two objectives require solution of a nonlinear program, which may be problematic.
- Weighted squared deviation: $\sum_i \sum_j a_{ij}(0)[a_{ij}(0) - \tilde{a}_{ij}(1)]^2$. [Canning and Wang, 2005, use a weighted quadratic penalty function in a program designed to estimate the $z_{ij}^{\cdot r}$ and z_i^{rs} components of a multiregional input–output model (Chapter 3).]

[18] Constraints always include non-negativity of the $\tilde{a}_{ij}(1)$, along with the row and column margins, namely $\sum_j \tilde{a}_{ij}(1)x_j = u_i$ and $\sum_i \tilde{a}_{ij}(1)x_j = v_j$, respectively.

- Relative squared deviation: $\sum_i \sum_j \frac{[a_{ij}(0) - \tilde{a}_{ij}(1)]^2}{a_{ij}(0)}$. (Friedlander, 1961.) This is Pearson's Chi-square measure, used early by Deming and Stephan (1940).
- Sign-preserving absolute differences: $\sum_i \sum_j \left| a_{ij}(0) - y_{ij}a_{ij}(0) \right|$, where $y_{ij}a_{ij}(0) = \tilde{a}_{ij}(1)$. (Junius and Oosterhaven, 2003.)[19]

The nonlinear alternatives require solution of possibly large and complex nonlinear programs, with their attendant difficulties, including computational issues (despite powerful computer programs and software), local rather than global optima, etc. Early overviews of some of these alternative minimization objectives can be found in Lecomber (Allen and Gossling, 1975, Ch. 1) and Hewings and Janson (1980, Appendix). Many recent proposals and extensive discussions are contained in Lahr and de Mesnard (2004), de Mesnard (2004) and Jackson and Murray (2004). In particular, Jackson and Murray present extensive results for applications of a total of 10 model formulations (including those listed above) to the problem of estimating the 1972 23-sector US industry-by-industry data from a 1967 matrix and 1972 margins. They found that, generally, RAS produced the best results. Canning and Wang (2005) contains a discussion of the advantages of a mathematical programming approach to constrained matrix-balancing problems and reviews some of the important contributions in the literature.

7.4.9 Infeasible Problems

In general, the RAS procedure converges to within acceptable tolerance in a reasonable number of iterations – often less than 50. However, examples of nonconvergence have appeared in the literature. The usual explanation is that the matrix being adjusted is too sparse – contains too many zeros. A very disaggregated transactions matrix (hundreds of sectors) or interregional trade-flow matrices would have more zeros than, say, a highly aggregated national table.[20] Intuitively, the problem with zeros is that the entire burden of change is forced onto the remaining, nonzero elements, and they may be inadequate to the task (depending in large part on the locations of the zeros relative to the nonzeros).

Here is a very simple illustration of the issue:[21] Let

$$\mathbf{Z}(0) = \begin{bmatrix} 5 & 0 \\ 4 & 3 \end{bmatrix}, \ \mathbf{u}(1) = \begin{bmatrix} 10 \\ 2 \end{bmatrix}, \text{ and } \mathbf{v}(1) = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

[19] In this case, the constraints are non-negativity of the y_{ij} and margin constraints of $\sum_j y_{ij}a_{ij}(0)x_j = u_i$ and $\sum_i y_{ij}a_{ij}(0)x_j = v_j$. Linearization is possible, as in the first three cases.

[20] For example, nonconvergence occurred while working with inter-state trade tables in developing the US 1967 multiregional model (see Möhr, Crown and Polenske, 1987).

[21] From de Mesnard (2003).

The difficulty with the problem is clear when we look at the required new margins in relation to the structure that $\tilde{\mathbf{Z}}(1)$ must have, namely

$$\begin{array}{cc} \begin{bmatrix} \tilde{z}_{11}(1) & 0 \\ \tilde{z}_{21}(1) & \tilde{z}_{22}(1) \end{bmatrix} & \begin{bmatrix} 10 \\ 2 \end{bmatrix} \\ \begin{bmatrix} 7 & \quad\quad 5 \end{bmatrix} & \end{array}$$

To satisfy $u_1(1) = 10$, it is clear that $\tilde{z}_{11}(1) = 10$, since $\tilde{z}_{12}(1) = 0$ because zeros are perpetuated in RAS. Clearly, if $u_1(1) = 10$ then $\tilde{z}_{21}(1)$ would need to be -3 in order to satisfy $v_1(1) = 7$, but this is impossible since RAS does not generate negative elements from those that are positive. One straightforward way out of the problem is to assign a small positive number to zero-valued cells in the base matrix.[22] In this small illustration, changing $z_{12}(0)$ from zero to, say, 0.5, introduces exactly the flexibility that is needed, and as a consequence RAS will produce (rounded)[23]

$$\tilde{\mathbf{Z}}(1) = \begin{bmatrix} 6.5911 & 3.4089 \\ 0.4099 & 1.5901 \end{bmatrix}$$

An argument made in defense of this approach is that the original zero-valued elements could be the result of rounding; that is, these elements were actually very small flows that fell below the "reduce to zero" threshold in recording the data. On the other hand, some zeros represent true technological facts – as above in the illustration of a zero flow from potatoes to automobiles, which should be maintained in the target matrix. Moreover, in a large problem it may not be necessary to change all zeros into small positives, and then the issue is to decide which zeros should be altered. One approach uses a linear programming problem to select subsets of elements for augmentation (from zero to positive numbers); see Möhr, Crown and Polenske, 1987 for a discussion and illustration of this approach.

7.5 Summary

In this chapter we have examined approaches to estimating tables of input–output coefficients when a full matrix of interindustry transactions is not available. No nonsurvey or partial-survey technique can be expected to generate a table that is a perfect copy of what could be obtained if a complete survey were undertaken. On the other hand, errors and compromises of many sorts enter into the production of even the best survey-based table, so it can be argued that even a survey-based table is not a completely accurate snapshot of an economy. The updating problem has given rise to a number of approaches, usually including an RAS adjustment at some point, often combined with either survey data or expert opinion on certain key elements – sometimes individual coefficients, sometimes entire rows or columns. This hybrid strategy is an attempt to capture the best of several approaches – selective survey information, expert opinion, and the attractive mathematical features of the RAS technique.

22 Apparently this was first done by Hewings, 1969, in his dissertation. (Cited in de Mesnard, 2003.)
23 After nine iterations, using $|u_i(1) - u_i^k| \le 0.001$ and $|v_i(1) - v_i^k| \le 0.001$ for all i as the stopping criterion.

Appendix 7.1 RAS as a Solution to the Constrained Minimum Information Distance Problem

The problem is to choose the elements of $\tilde{\mathbf{A}}(1)$ so as to minimize the information measure of distance between $\mathbf{A}(0)$ and $\tilde{\mathbf{A}}(1)$, namely

$$D[\mathbf{A}(0) : \tilde{\mathbf{A}}(1)] = \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{a}_{ij}(1) \ln \left[\frac{\tilde{a}_{ij}(1)}{a_{ij}(0)}\right] \tag{A7.1.1}$$

subject to

$$\sum_{j=1}^{n} \tilde{a}_{ij}(1)x_j(1) = u_i(1) \; (i = 1, \ldots, n) \tag{A7.1.2}$$

$$\sum_{i=1}^{n} \tilde{a}_{ij}(1)x_j(1) = v_j(1) \; (j = 1, \ldots, n) \tag{A7.1.3}$$

Notice that the expression in (A7.1.1) is only defined for $a_{ij}(0) \neq 0$. The associated Lagrangian expression is

$$\begin{aligned} L = &\sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{a}_{ij}(1) \ln \left[\frac{\tilde{a}_{ij}(1)}{a_{ij}(0)}\right] \\ &- \sum_{i=1}^{n} \lambda_i \left[\sum_{j=1}^{n} \tilde{a}_{ij}(1)x_j(1) - u_i(1)\right] - \sum_{j=1}^{n} \mu_j \left[\sum_{i=1}^{n} \tilde{a}_{ij}(1)x_j(1) - v_j(1)\right] \end{aligned} \tag{A7.1.4}$$

and the appropriate first-partial derivatives are

$$\partial L/\partial \tilde{a}_{ij}(1) = 1 + \ln \tilde{a}_{ij}(1) - \ln a_{ij}(0) - \lambda_i x_j(1) - \mu_j x_j(1) \tag{A7.1.5}$$

Setting $\partial L/\partial \tilde{a}_{ij}(1) = 0$ yields

$$\ln \tilde{a}_{ij}(1) = \ln a_{ij}(0) - 1 + \lambda_i x_j(1) + \mu_j x_j(1)$$

and, taking antilogarithms,

$$\tilde{a}_{ij}(1) = a_{ij}(0)\mathrm{e}^{[-1+\lambda_i x_j(1)+\mu_j x_j(1)]}$$

or, rearranging,

$$\tilde{a}_{ij}(1) = \mathrm{e}^{[\lambda_i x_j(1)-1/2]} a_{ij}(0) \mathrm{e}^{[\mu_j x_j(1)-1/2]} \tag{A7.1.6}$$

Let

$$r_i = \mathrm{e}^{[\lambda_i x_j(1)-1/2]} \tag{A7.1.7}$$

which is a function of λ_i only (that is, a row constraint), and let

$$s_j = \mathrm{e}^{[\mu_j x_j(1)-1/2]} \tag{A7.1.8}$$

which is a function of μ_j only (that is, a column constraint). Then the right-hand side of (A7.1.6) can be shown as

$$\tilde{a}_{ij}(1) = r_i a_{ij}(0) s_j \tag{A7.1.9}$$

The new coefficient, $\tilde{a}_{ij}(1)$, is derived as the old coefficient, $a_{ij}(0)$, modified by a row-constraint term, r_i, and a column-constraint term, s_j.

The constraints of the problem, (A7.1.2) and (A7.1.3), are reproduced in the remaining first-order conditions, as usual, when we set $\partial L/\partial \lambda_i = 0$ $(i = 1, \ldots, n)$ and $\partial L/\partial \mu_j = 0$ $(j = 1, \ldots, n)$. Inserting (A7.1.9) into these two constraints gives

$$r_i = u_i(1) \Big/ \sum_{j=1}^{n} a_{ij}(0) s_j x_j(1)$$

and

$$s_j = v_j(1) \Big/ \sum_{j=1}^{n} r_i a_{ij}(0) x_j(1)$$

The values of r_i and s_j are found through iterative solution of these two equations. This is what the RAS procedure accomplishes. (See Macgill, 1977 or Bacharach, 1970 for details.)

The matrix equivalent of (A7.1.9) is

$$\tilde{\mathbf{A}}(1) = \hat{\mathbf{r}} \mathbf{A}(0) \hat{\mathbf{s}} \tag{A7.1.10}$$

as in (7.35) in the text, where

$$\hat{\mathbf{r}} = \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & & 0 \\ \vdots & & & \vdots \\ 0 & & & r_n \end{bmatrix} \text{ and } \hat{\mathbf{s}} = \begin{bmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & & 0 \\ \vdots & & & \vdots \\ 0 & & & s_n \end{bmatrix}$$

Examining second-partial derivatives, we find

$$\partial^2 L/\partial \tilde{a}_{ij}(1)^2 = 1/\tilde{a}_{ij}(1) \tag{A7.1.11}$$

This is strictly positive for all $\tilde{a}_{ij}(1) > 0$. From (A7.1.9), this means for all $a_{ij}(0) > 0$, since $r_i > 0$ and $s_j > 0$ [(A7.1.7) and (A7.1.8)]. Thus the RAS solution minimizes $D[\mathbf{A}(0) : \tilde{\mathbf{A}}(1)]$ in (A7.1.1).

Problems

7.1 Consider the following US input–output tables for 1997[24], 2003, and 2005 (in $ millons).

Produce industry-by-industry transactions tables using the assumption of industry-based technology for these three years. Suppose historical price indices for these tables

[24] The tables for 1997 differ from those provided in Appendix B in that they reflect data assembled "before redefinitions," as discussed in Chapter 4.

US Use 1997	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Imports*
1 Agriculture	74,938	15	1,121	150,341	2,752	13,400	11	(23,123)
2 Mining	370	19,461	4,281	112,513	53,778	5,189	30	(64,216)
3 Construction	1,122	29	832	7,499	11,758	50,631	27	–
4 Manufacturing	49,806	19,275	178,903	1,362,660	169,915	418,412	1,914	(765,454)
5 Trade, Transport & Utilities	21,650	11,125	76,056	380,272	199,004	224,271	612	6,337
6 Services	32,941	45,234	107,723	483,686	545,779	1,592,426	3,801	(16,942)
7 Other	63	781	422	33,905	19,771	26,730	–	(126,350)
US Make 1997	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Industry Output*
1 Agriculture	284,511	–	65	356	455	1,152	–	286,539
2 Mining	–	158,239	109	9,752	295	258	–	168,653
3 Construction	–	–	670,210	–	–	–	–	670,210
4 Manufacturing	–	727	1,258	3,703,275	39,720	36,034	3,669	3,784,683
5 Trade, Transport & Utilities	556	381	21,393	15,239	2,201,532	141,674	–	2,380,776
6 Services	–	410	54,850	1,306	109,292	6,444,098	1,821	6,611,778
7 Other	–	–	6,206	–	–	7,010	947,023	960,238
Commodity Output	285,067	159,757	754,091	3,729,928	2,351,295	6,630,226	952,513	14,862,876
US Use 2003	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Imports*
1 Agriculture	61,946	1	1,270	147,559	231	18,453	2,093	(26,769)
2 Mining	441	33,299	6,927	174,235	89,246	1,058	11,507	(125,508)
3 Construction	942	47	1,278	8,128	10,047	65,053	48,460	–
4 Manufacturing	47,511	22,931	265,115	1,249,629	132,673	516,730	226,689	(1,075,128)
5 Trade, Transport & Utilities	24,325	13,211	100,510	382,630	190,185	297,537	123,523	8,065
6 Services	25,765	42,276	147,876	509,084	490,982	2,587,543	442,674	(44,060)
7 Other	239	1,349	2,039	48,835	35,110	83,322	36,277	(177,578)

Table (*cont.*)

		1	2	3	4	5	6	7	
US Make 2003		*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Output*
1	Agriculture	273, 244	–	–	67	–	1, 748	–	275, 058
2	Mining	–	232, 387	–	10, 843	–	–	–	243, 231
3	Construction	–	–	1, 063, 285	–	–	–	–	1, 063, 285
4	Manufacturing	–	–	–	3, 856, 583	–	30, 555	3, 278	3, 890, 416
5	Trade, Transport & Utilities	–	570	–	–	2, 855, 126	41	957	2, 856, 693
6	Services	–	475	–	–	133	9, 136, 001	3, 278	9, 139, 886
7	Other	3, 359	896	–	3, 936	104, 957	323, 996	1, 827, 119	2, 264, 263
Commodity Output		276, 602	234, 328	1, 063, 285	3, 871, 429	2, 960, 216	9, 492, 341	1, 834, 631	19, 732, 832
US Use 2005		*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Imports*
1	Agriculture	71, 682	1	1, 969	174, 897	335	18, 047	1, 671	(31, 248)
2	Mining	524	57, 042	8, 045	297, 601	123, 095	1, 290	16, 570	(226, 059)
3	Construction	1, 597	74	1, 329	7, 886	12, 449	74, 678	54, 282	–
4	Manufacturing	61, 461	34, 860	339, 047	1, 452, 738	183, 135	589, 452	255, 456	(1, 372, 424)
5	Trade, Transport & Utilities	26, 501	17, 197	136, 193	460, 348	244, 153	362, 324	127, 266	6, 790
6	Services	27, 274	52, 297	165, 179	543, 690	610, 978	3, 017, 728	529, 779	(50, 588)
7	Other	240	1, 323	2, 021	61, 316	44, 561	90, 071	39, 656	(208, 971)
US Make 2005		*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Output*
1	Agriculture	310, 868	–	–	65	–	1, 821	–	312, 754
2	Mining	–	373, 811	–	22, 752	–	–	–	396, 563
3	Construction	–	–	1, 302, 388	–	–	–	–	1, 302, 388
4	Manufacturing	–	–	–	4, 454, 957	–	26, 106	4, 467	4, 485, 529
5	Trade, Transport & Utilities	–	808	–	–	3, 354, 043	47	1, 046	3, 355, 944
6	Services	–	556	–	–	152	10, 473, 161	3, 771	10, 477, 640
7	Other	4, 657	1, 410	–	4, 111	115, 428	339, 582	2, 061, 136	2, 526, 325
Commodity Output		315, 525	376, 586	1, 302, 388	4, 481, 885	3, 469, 622	10, 840, 717	2, 070, 419	22, 857, 143

are given in the following table (price indices in percent relative to some arbitrary earlier year):

	1997	2003	2005
Agriculture	100	113.5	122.7
Mining	96.6	131.3	201
Construction	181.6	188.9	209.9
Manufactuirng	133.7	150.8	156.9
Trade, Transport & Utilities	200.4	205.7	217.1
Services	129.3	151.6	219.8
Other	140	144.7	161.4

Produce a set of constant price input–output tables for the same years using 2005 as the base year for prices.

7.2 For the constant price tables constructed in problem 7.1, suppose we measure year-to-year change as the average of the absolute value of differences between the column sums of A for the same industry sectors in two different years. Which three sectors exhibited the most change from 1997 to 2005? How does that compare with the three most changed sectors measured in nominal dollars rather than constant dollars? Why are they different?

7.3 Using the current price tables constructed in problem 7.1, compute the marginal input coefficients between the years 1997 and 2005.

7.4 Consider the following interindustry transactions and total outputs two-sector input–output economy for the year 2000:

2000	*A*	*B*	Total Output
A	1	2	10
B	3	4	10
VA	6	4	

Suppose estimates are generated for the year 2010 for the vectors of total final demand, total value-added, and total output in the following table.

2010	Final Demand	Value Added	Total Output
A	12	10	25
B	6	8	20

Using the 2000 table as a base and using the 2010 projections for final demand, value-added and total output, compute an estimate of the 2010 technical coefficients table using the RAS technique.

7.5 Using the 1997 input–output table expressed in 1997 dollars constructed in problem 7.1 and the vectors of intermediate inputs, intermediate outputs, and total outputs from the corresponding input–output table for 2005, compute an RAS estimate of the 2005 table using the 1997 table as a base. Compute the mean absolute percentage error (MAPE) of the RAS-estimated table for 2005 compared with the "real" 2005 table.

7.6 Suppose we have a baseline transactions matrix defined as $\mathbf{Z}(0) = \begin{bmatrix} 100 & 55 & 25 \\ 50 & 75 & 45 \\ 25 & 10 & 110 \end{bmatrix}$.

We are provided with estimates of intermediate inputs and outputs, $\mathbf{v}(1) = \begin{bmatrix} 265 \\ 225 \\ 325 \end{bmatrix}$ and $\mathbf{u}(1) = \begin{bmatrix} 325 \\ 235 \\ 255 \end{bmatrix}$, respectively.

a. Compute an estimate of the transactions table for the next year, $\tilde{\mathbf{Z}}^z(1)$ using $\mathbf{Z}(0)$, $\mathbf{v}(1)$ and $\mathbf{u}(1)$, using the RAS technique.

b. Suppose we know the vector of total outputs, $\mathbf{x}(1) = \begin{bmatrix} 750 \\ 500 \\ 1000 \end{bmatrix}$, corresponding to $\mathbf{Z}(0)$, and we also have an estimate of total outputs for next year, $\mathbf{x}(1) = \begin{bmatrix} 1000 \\ 750 \\ 1500 \end{bmatrix}$.
Compute $\mathbf{A}(0)$ and use it along with $\mathbf{v}(0)$ and $\mathbf{u}(0)$ to generate an estimate of the technical coefficients matrix for next year $\tilde{\mathbf{A}}^A(1)$. Finally, compute $\tilde{\mathbf{A}}^z(1) = \tilde{\mathbf{Z}}^z(1)\hat{\mathbf{x}}(1)^{-1}$. Is $\tilde{\mathbf{A}}^A(1) = \tilde{\mathbf{A}}^z(1)$? Why or why not?

7.7 For the economy in problem 7.6, suppose we acquire a survey-based table of technical coefficients next year of $\mathbf{A}(1) = \begin{bmatrix} .2 & .1 & .033 \\ .035 & .167 & .05 \\ .03 & .033 & .133 \end{bmatrix}$. At the beginning of the survey we know only $a(1)_{32} = .033$ and we use that along with $\mathbf{A}(0)$, $\mathbf{v}(0)$, and $\mathbf{u}(0)$ to generate an intermediate estimate of the entire matrix of coefficients, $\tilde{\mathbf{A}}^*(1)$. If we measure difference between two matrices as MAPE, which estimate of $\mathbf{A}(1)$ is better – $\tilde{\mathbf{A}}(1)$ or $\tilde{\mathbf{A}}^*(1)$? Suppose early in the survey period we determine $a(1)_{11} = .2$ instead of knowing $a(1)_{32}$. Which estimate of $\mathbf{A}(1)$ is better – $\tilde{\mathbf{A}}(1)$ or $\tilde{\mathbf{A}}^*(1)$? How does this case differ from the case where $a(1)_{32}$ is known?

7.8 Consider the transactions matrix $\mathbf{Z}(0) = \begin{bmatrix} 100 & 55 & 25 \\ 0 & 75 & 25 \\ 25 & 10 & 110 \end{bmatrix}$ and projected vectors of intermediate inputs and outputs, $\mathbf{v}(1) = \begin{bmatrix} 125 \\ 140 \\ 160 \end{bmatrix}$ and $\mathbf{u}(1) = \begin{bmatrix} 180 \\ 100 \\ 145 \end{bmatrix}$, respectively. Compute the RAS estimate, $\tilde{\mathbf{Z}}(1)$. Suppose we learn that $v_1(0) = 100$ instead of 125. Is it possible to compute $\tilde{Z}(1)$ via the RAS technique? Why or why not?

7.9 For the US input–output tables for 1997 and 2005 (from problem 7.1, expressed in current year dollars rather than constant year dollars), compute the RAS estimate $\tilde{\mathbf{A}}(2005)$ using $\mathbf{A}(1997)$, $\mathbf{v}(2005)$, and $\mathbf{u}(2005)$. Compute the MAPE for $\tilde{\mathbf{A}}(2005)$ compared with $\mathbf{A}(2005)$. How does that error compare with the MAPE for $\tilde{\mathbf{L}}(2005) = [\mathbf{I} - \tilde{\mathbf{A}}(2005)]^{-1}$ when compared with $\mathbf{L}(2005)$?

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8 Nonsurvey and Partial-Survey Methods: Extensions

8.1 Introduction

Regional input–output tables share with their national counterparts the problem of becoming outdated simply because of the passage of time. But smaller geographic scale introduces other problems. For example, if the only automobile assembly plant in Michigan closes and its replacement opens (as the only such plant) in Tennessee, a "national" table would still reflect automobile assembly (although perhaps under more modern methods in the new plant) whereas that activity would disappear entirely from a Michigan table and appear as a completely new activity in a Tennessee table. In addition, states or counties or even smaller economic areas may have fewer resources available for the kinds of data collection needed for survey-based input–output tables – although since the economy is smaller (in terms of number of square miles covered, numbers of active plants, etc.) the effort involved in surveying may be less. In addition, when one is concerned with models in which two or more regions are connected (or a single region and the rest of the country) shipments out of and into the regions assume a much more important role – the former providing inputs to production and the latter representing markets for outputs. Consequently, considerable effort has been devoted to estimation of interregional flows of goods in an effort to construct approximations to and estimates of the a_{ij}^{rs} or $c_{i\cdot}^{rs}$ coefficients of the IRIO or MRIO models (Chapter 3).

As noted in Chapter 3, some of the earliest attempts at estimating interindustry relationships at a regional level employed national input coefficients along with estimates of regional supply percentages showing, for each supplying sector, the proportion of total regional requirements of that good that could be expected to originate within the region. One procedure for obtaining this estimate for sector i was to find the ratio of total regional output, less exports, of sector i, to the total output, less exports, plus imports, of sector i. As in Chapter 3, for a particular region r,

$$p_i^r = \frac{x_i^r - e_i^r}{x_i^r - e_i^r + m_i^r}$$

Thus, when none of good i was imported, $p_i^r = 1$, and the assumption is that all of the region's needs for i can be supplied internally. The regional input coefficient matrix is then estimated as

$$\mathbf{A}^{rr} = \hat{\mathbf{p}}\mathbf{A}^{n}$$

where $\mathbf{p} = [p_i^r]$ and $\mathbf{A}^n$ is the national technical coefficients matrix. As we saw in Chapter 3, this represents a uniform alteration of each of the coefficients in row i of $\mathbf{A}^n$ by p_i^r.

As we saw in section 3.2, a regional input coefficient, a_{ij}^{rr}, is defined as the difference between a regional technical coefficient, a_{ij}^{r}, and a regional import coefficient, a_{ij}^{sr}, where s indicates "outside of r." (When it is clear what particular region is intended, the simpler notation $r_{ij} = a_{ij} - m_{ij}$ is used.) If we have available a complete set of intra- and interregional data (as is needed in constructing an interregional input–output model, for example), then we observe the a_{ij}^{rr}'s (and a_{ij}^{sr}'s) directly. However, if we are trying to estimate a_{ij}^{rr} from national data, the estimation problem can be posed in the following way: (1) estimate a regional technical coefficient, a_{ij}^{r}, from the corresponding national coefficient, a_{ij}^{n}, and then (2) estimate the regional input coefficient, a_{ij}^{rr}, as some proportion of the regional technical coefficient; that is, $a_{ij}^{rr} = p_{ij}^r a_{ij}^r$ (where $0 \le p_{ij}^r \le 1$). Instead of estimating, a_{ij}^{r} and a_{ij}^{sr} we estimate a_{ij}^{r} and p_{ij}^{r}. The two steps in this procedure for estimating a_{ij}^{rr} from a_{ij}^{n} would therefore be: (1) find $\alpha_{ij}^r \ge 0$ such that

$$a_{ij}^{r} = (\alpha_{ij}^{r})(a_{ij}^{n}) \tag{8.1}$$

and (2) find β_{ij}^r $(0 \le \beta_{ij}^r \le 1)$ such that

$$a_{ij}^{rr} = (\beta_{ij}^{r})(a_{ij}^{r}) \tag{8.2}$$

[Of course, if we indeed can find α_{ij}^r and β_{ij}^r for every i and j, this is equivalent to finding $a_{ij}^{rr} = (\gamma_{ij}^r)(a_{ij}^n)$ where $\gamma_{ij}^r = (\alpha_{ij}^r)(\beta_{ij}^r)$].

The basic point is that in general there is not enough regional information to find the α_{ij}^r and β_{ij}^r. For example, in the simple procedure described at the beginning of this section, we see that (1) a_{ij}^r was assumed equal to a_{ij}^n; in terms of (8.1), $\alpha_{ij}^r = 1$ for all i and j (region r and national production recipes are identical) and (2) each regional purchaser, j, of input i was assumed to buy the same proportion of those inputs from within the region; in terms of (8.2), $\beta_{ij}^r = p_i^r$ for all i.

In the absence of specific survey information, it is customary, at least initially, to invoke assumption (1). This overlooks probable regional differences in product mixes within a sector (as discussed in Chapter 3), especially at anything but the finest level of disaggregation; it also ignores relative sizes and ages of firms within a particular regional sector (with differing efficiencies, for example), differences in quality of capital stocks, etc. The prevalent view in the mid-1980s was

> … in the absence of any information about many of these characteristics, one is left with very few options but to adopt a very conservative strategy, namely, one in which a minimum of speculation is applied to the modification process. (Hewings, 1985, p. 47.)

We will now explore a number of nonsurvey techniques for regionalization of national coefficients – through adjustments based entirely on published information on regional employment, income, or output, by industry – and see where they fit in the general scheme given by (8.1) and (8.2). Later we will examine more recent and more comprehensive "regionalization" approaches; but since, historically, the techniques discussed in section 8.2 have been used in a great many regional studies, it is imperative that we understand them.

8.2 Location Quotients and Related Techniques

8.2.1 Simple Location Quotients

Let x_i^r and x^r denote gross output of sector i in region r and total output of all sectors in region r, respectively, and let x_i^n and x^n denote these totals at the national level. Then the simple location quotient for sector i in region r is defined as

$$LQ_i^r = \left(\frac{x_i^r/x^r}{x_i^n/x^n}\right) \tag{8.3}$$

(often in the literature these are denoted by SLQ_i). In cases where regional output data are not consistently available, or where analysts feel it is appropriate, other measures of regional and national economic activity are often used – including employment (probably the most popular), personal income earned, value added, and so on, by sector.

The interpretation of this measure is straightforward. The numerator in (8.3) indicates the proportion of region r's total output that is contributed by sector i. The denominator represents the proportion of total national output that is contributed by sector i, nationally. If $LQ_i^r = (0.034)/(0.017) = 2$, sector i's output represents 3.4 percent of all regional gross output while, at the national level, sector i's output represents only 1.7 percent of the total national output. In a case like this – in fact, whenever $LQ_i^r > 1$ – sector i is more localized, or concentrated, in the region than in the nation as a whole. Conversely, if $LQ_i^r = (0.015)/(0.045) = 0.33$, we understand that while sector i's output is 4.5 percent of the total national gross output, it represents only 1.5 percent of the gross output in the region. In this situation, sector i is less localized, or less concentrated, in region r than in the nation as a whole.

Note that simple algebra generates an alternative expression, namely

$$LQ_i^r = \left(\frac{x_i^r/x_i^n}{x^r/x^n}\right)$$

This tells a somewhat different "story." The numerator measures the proportion of total national output of commodity i that is produced in region r. The denominator is the proportion of total national output of all commodities that is produced in region r. But the interpretation is much the same; $LQ_i^r > 1$ indicates a commodity whose production is relatively localized in region r.

The simple location quotient has been viewed as a measure of the ability of regional industry i to supply the demands placed upon it by other industries (and by final demand) in that region, in the following way. If industry i is less concentrated in the region than in the nation ($LQ_i^r < 1$), it is seen as less capable of satisfying regional demand for its output, and its regional direct input coefficients, $a_{ij}^{rr}(j=1,\ldots,n)$ are created by reducing the national coefficients, a_{ij}^{n}, by multiplying them by LQ_i^r. However, if industry i is more highly concentrated in the region than in the nation ($LQ_i^r > 1$), then it is assumed that the national input coefficients from industry i, a_{ij}^n ($j = 1, \ldots, n$), apply to the region, and the regional "surplus" produced by i will be exported to the rest of the nation. Thus, for each row i of an estimated regional table,

$$a_{ij}^{rr} = \left\{ \begin{matrix} (LQ_i^r)a_{ij}^n & \text{if } LQ_i^r < 1 \\ a_{ij}^n & \text{if } LQ_i^r \geq 1 \end{matrix} \right\} \tag{8.4}$$

[If a national sector is not present in the region ($LQ^r = 0$), that row and column are simply deleted from $\mathbf{A}^n$.]

In terms of the general scheme in (8.1) and (8.2), we see that this procedure is equivalent to (1) assuming $\alpha_{ij}^r = 1$ for all i and j and (2) letting $\beta_{ij}^r = LQ_i^r$ when $LQ_i^r < 1$ and $\beta_{ij}^r = 1$ when $LQ_i^r \geq 1$. Note that there is a distinct asymmetry in this approach. When a sector is import-oriented ($LQ_i^r < 1$), the modification of the national coefficient varies with the strength of the import orientation – $a_{ij}^{rr} = (LQ_i^r)a_{ij}^n$. When a sector is export-oriented ($LQ_i^r > 1$), the strength of that orientation is not reflected in the modification – $a_{ij}^{rr} = (1)a_{ij}^n$.

A complication arises if the estimates of regional industry output that are obtained using LQ coefficients exceed actual output for some industries. In this event, coefficients developed by this method have often been "balanced" to ensure that they do not overestimate the regional output of each sector. The notion of a balancing method is simply that if estimated coefficients generate a regional output for sector i ($\tilde{x}_i$) that is too large (meaning $\tilde{x}_i^r > x_i^r$), then the row-i estimates, a_{ij}^{rr} (for all j), should be uniformly reduced – multiplied by $(x_i^r/\tilde{x}_i^r)$.

For example, calculate estimated sector i output on the basis of actual regional industry outputs (these are necessary data for this correction) and the LQ-estimated regional input coefficients (and regional final-demand purchase coefficients). For sector i, this is

$$\tilde{x}_i^r = \sum_j a_{ij}^{rr} x_j^r + \sum_f c_{if}^{rr} f_f^r \tag{8.5}$$

where

$\tilde{x}_i^r$ = estimated regional output of sector i,
f_f^r = total regional final demand of final-demand sector f, and
c_{if}^{rr} = estimated regional final-demand purchase coefficient of regional final-demand sector f from industry i.

The c_{if}^{rr} elements reflect purchases of regionally produced output i by regional final-demand sector f. Typically, the regional final-demand sectors will be personal consumption expenditures, investment, state and local government, as well as both foreign and rest-of-the-country exports (a part of which will be federal government purchases, except for those purchases made by federal installations located in the region). These estimates are found in much the same manner as were the a_{ij}^{rr}; that is, using national data and the region-specific location quotients. In particular,

$$c_{if}^{rr} = \begin{Bmatrix} (LQ_i^r)c_{if}^n & \text{if } LQ_i^r < 1 \\ c_{if}^n & \text{if } LQ_i^r \geq 1 \end{Bmatrix} \tag{8.6}$$

where

$$c_{if}^n = f_{if}/f_f,$$

f_{if} = national sales of industry i to final-demand sector f, and
f_f = total national purchases of final-demand sector f.

Thus, when $LQ_i^r \geq 1$, it is assumed that purchases of good i by final-demand sector f are the same proportion of total sector f purchases in the region as in the nation. For example, if purchases of electricity (sector i) by consumers (final-demand sector f) constitute 3 percent of total consumer expenditures nationally ($c_{if}^n = 0.03$), and if $LQ_i^r \geq 1$, then it is assumed that 3 percent of the total expenditures by consumers in region r will be on electricity produced in region r; $c_{if}^{rr} = 0.03$. When $LQ_i^r < 1$, then the national proportion is modified downward. If $LQ_i^r = 0.67$, then it would be assumed that only 2 percent of the total expenditures by consumers in region r will be on electricity produced in region r; $c_{if}^{rr} = 0.02$.

The next step in the balancing procedure is to calculate the ratio of estimated to actual regional output; denote this by Z_i^r. Then

$$Z_i^r = x_i^r/\tilde{x}_i^r \tag{8.7}$$

Each row of estimated regional input coefficients for which Z_i^r is less than one is adjusted downward. That is, adjusted ("balanced") regional input coefficients are estimated as

$$\bar{a}_{ij}^{rr} = \begin{Bmatrix} Z_i^r a_{ij}^{rr} & \text{if } Z_i^r < 1 \\ a_{ij}^{rr} & \text{if } Z_i^r \geq 1 \end{Bmatrix} \tag{8.8}$$

As noted above, in this LQ and other quotient approaches, $\alpha_{ij}^r = 1$ is assumed. The observed national technology is uniform across regions; regional input coefficients vary only because of varying regional capacities to satisfy own-region demand. For some kinds of production, this is quite reasonable; for others it is not. Coca Cola made in Boston probably has the same production "recipe" as Coca Cola made in San Francisco (even though local ability to supply any given input may vary). However, an "airplane" made in Seattle (for example, a Boeing commercial airliner with two jet engines) is quite a different product from an airplane made in Wichita (for example, a Cessna private aircraft with one propeller engine). So in a model with a highly aggregated "aircraft" sector, there is clearly non-uniformity in production recipes for "aircraft" across states in the USA, and the $\alpha_{ij}^r = 1$ assumption is invalid. This is the product-mix issue, and the level of aggregation is decisive. In a model with a sector labeled "Commercial aircraft, 2 jet engines" it is apparent that wherever produced, two jet engines will be used per aircraft. Similarly, for the "Private aircraft, one propeller engine" sector, one propeller engine will be required per aircraft. At that level of disaggregation, the assumption of constant (national) technology across regions ($\alpha_{ij}^{r} = 1$) may be reasonable.

Another complaint made about this approach (and many of its variants, to be examined below) is that it underestimates regional trade since it ignores cross-hauling – the situation in which a region exports and imports the same goods. Cross-hauling is a generally observed phenomenon, but it is also difficult to capture in an estimation technique. To take a very simple illustration, at a level of aggregation that includes a sector labeled "agriculture," a specific region (say Washington State) exports peaches (to California, for example) and imports avocados (from California); both are products of the "agriculture" sector. Using an *LQ* approach, a specific sector in a specific region must be either a net exporter or a net importer of any particular good. When $LQ_i^r > 1$, industry i is seen as producing more than its share of the national output of i, and region r is assumed to be a net exporter of the "excess" output of i. Conversely, if $LQ_j^r < 1$, the region is less than self-sufficient in good j and will therefore be a net importer of that good. (When $LQ_k^r = 1$, the region would neither import nor export good k.) This quirk of the location quotient approach thus leads to a tendency for underestimation of interregional trade (agricultural products cannot be shipped from Washington to California and also from California to Washington) and thus for overestimation of intraregional economic activity, and therefore it also tends to generate regional multipliers that are too large.[1] Later in this section we will examine an approach that attempts to overcome this problem.

There are several variants of the simple location quotient approach, all of which are used in the same general way in adjusting national to regional coefficients. We examine some of these in what follows. Since the *LQ* approach in (8.4) will never increase a national coefficient (they are either left unchanged or made smaller), this procedure is also called *reducing* the national coefficients table, and hence these are sometimes referred to as *reduction* techniques.

This $a_{ij}^{rr} \leq a_{ij}^{n}$ characteristic of the *LQ* approach has also been called into question (see, for example, McCann and Dewhurst, 1998). A producer in sector j might use relatively fewer imported inputs than is reflected in the national coefficients for sector j, and thus at least some regionally supplied inputs *could* be larger, per unit of output j in that region than in the nation as a whole. And in general, if the national coefficient is an average of observed regional coefficients, then some coefficients in some regions should be expected to be above average while others in other regions would necessarily be below average. One of the variants to be examined below (section 8.2.4) allows for $a_{ij}^{rr} > a_{ij}^{n}$.

[1] Robison and Miller (1991) calculate the amount of overestimation of intraregional trade in a model for a small multicounty area in Idaho.

8.2.2 Purchases-Only Location Quotients

The purchases-only location quotient (*PLQ*) for sector i in region r relates regional to national ability to supply sector i inputs, but only to those sectors that use i as an input. That is,

$$PLQ_i^r = \left(\frac{x_i^r / x^{*r}}{x_i^n / x^{*n}} \right) \tag{8.9}$$

where x_i^r and x_i^n are regional and national output of good i, as before, and where x^{*r} and x^{*n} are total regional and national output of only those sectors that use i as an input. The idea here is simply that if input i is not used by sector k, then the size of sector k's output is not relevant in determining whether or not the region can supply all of its needs for input i. [For example, whether or not region r can supply all of its needs for potatoes (sector i) is probably not affected by the amount of automobiles produced (sector k) in region r, since potatoes are not a direct input to automobile manufacturing.] PLQ_i^r is used in the same way as LQ_i^r to uniformly adjust the elements in row i of a national coefficients table, as in (8.4).

8.2.3 Cross-Industry Quotients

Another variant is the cross-industry quotient (*CIQ*). This allows for differing modifiers within a given row of the national matrix; that is, it allows for differing cell-by-cell adjustments within $\mathbf{A}^n$ rather than uniform adjustments along each row. What is now of interest is the relative importance of both selling sector i and buying sector j in the region and in the nation. Specifically,

$$CIQ_{ij}^r = \left(\frac{x_i^r / x_i^n}{x_j^r / x_j^n} \right) \tag{8.10}$$

Then

$$a_{ij}^{rr} = \begin{Bmatrix} (CIQ_{ij}^r) a_{ij}^n & \text{if } CIQ_{ij}^r < 1 \\ a_{ij}^n & \text{if } CIQ_{ij}^r \geq 1 \end{Bmatrix} \tag{8.11}$$

The idea is that if the output of regional sector i relative to the national output of i is larger than the output of regional sector j relative to the national output of sector j ($CIQ_{ij}^r > 1$), then all of j's needs of input i can be supplied from within the region. Similarly, if sector i at the regional level is relatively smaller than sector j at the regional level ($CIQ_{ij}^r < 1$), then it is assumed that some of j's needs for i inputs will have to be imported. Note that $CIQ_{ij}^r = LQ_i^r / LQ_j^r$. Note also that $CIQ_{ii}^r = 1$ (along the main diagonal, when $i = j$), and hence this technique would make no adjustments to on-diagonal coefficients. This has been called into question, and often the diagonal elements are adjusted using their associated LQ_i's in place of CIQ_{ii}^r (Smith and Morrison, 1974; Flegg, Webber and Elliott, 1995). More completely, then,

$$a_{ij}^{rr} = \begin{Bmatrix} (CIQ_{ij}^r)a_{ij}^n & \text{if } CIQ_{ij}^r < 1 \\ a_{ij}^n & \text{if } CIQ_{ij}^r \geq 1 \end{Bmatrix} \quad \text{for } i \neq j$$

$$a_{ij}^{rr} = \begin{Bmatrix} (LQ_i^r)a_{ij}^n & \text{if } LQ_i^r < 1 \\ a_{ij}^n & \text{if } LQ_i^r \geq 1 \end{Bmatrix} \quad \text{for } i = j$$

8.2.4 The Semilogarithmic Quotient and its Variants, FLQ and AFLQ

Rewrite LQ_i in (8.3) as $LQ_i^r = (x_i^r/x_i^n) \div (x_r/x_n)$. This clearly distinguishes the measure of the relative size of the regional (selling) sector (x_i^r/x_i^n) and the relative size of the region (x_r/x_n), but the sizes of buying sectors are ignored. The cross-industry quotient includes relative sizes of both selling (x_i^r/x_i^n) and buying (x_j^r/x_j^n) sectors but contains no x_r/x_n term. In the 1970s, Round conjectured that an appropriate approach should include all three measures. He proposed, among others, a "semilogarithmic quotient (SLQ)" which he defined (Round, 1978a, p. 182) as

$$SLQ_{ij}^r = LQ_i^r / \log_2(1 + LQ_j^r)$$

suggesting that it "... was devised simply to account for all three ratios in a way which maintains the basic properties of both the LQ and CIQ methods."[2] Notice that $\log_2(1+LQ_j^r) = 1$ when $LQ_j^r = 1$ and so in that case $SLQ_{ij}^r = LQ_i^r$; for $LQ_j^r > 1, \log_2(1+LQ_j^r) > 1$ and the adjustment means that $SLQ_{ij}^r < LQ_i^r$ and the reverse is the case when $LQ_j^r < 1$. Rewriting SLQ_{ij}^r we have

$$SLQ_{ij}^r = [(x_i^r/x_i^n) \div (x^r/x^n)] / \log_2\{1 + [(x_j^r/x_j^n) \div (x^r/x^n)]\}$$

and we see that along with relative sizes of both industries, i and j, this includes the regional size component in both numerator and denominator but not in such a way that the terms cancel out.

Perhaps surprisingly, applications using this *SLQ* generally failed to demonstrate any particular improvement over simpler measures like *LQ* and *CIQ*.[3] This spurred attempts to include these three factors in a measure that might perform better. One approach was developed in several articles by Flegg and others – hence the acronym *FLQ*. (See, for example, Flegg, Webber and Elliott, 1995; Flegg and Webber, 1997, 2000, and references cited in those articles.) This measure is generated by modifying the CIQ_{ij}^r to incorporate an additional measure of the relative size of the region; namely,

$$FLQ_{ij}^r = (\lambda) CIQ_{ij}^r$$

2 "The semilogarithmic form is arbitrary, but is among the simplest functions which maintains basic properties of the [quotient] values without further parameterization." (Round, 1978a, p. 182, note 4.) The first mention of this quotient seems to be in Smith and Morrison (1974, p. 43); they indicate that it was suggested in a personal communication from Round dated 1971. See also Flegg, Webber and Elliott (1995).

3 For example, in Smith and Morrison (1974) and Harrigan, McGilvray and McNicoll (1981).

where $\lambda = \{\log_2[1 + (x_E^r/x_E^n)]\}^\delta, 0 \le \delta < 1$.[4] Then

$$a_{ij}^{rr} = \begin{Bmatrix} (FLQ_{ij}^r)a_{ij}^n & \text{if } FLQ_{ij}^r < 1 \\ a_{ij}^n & \text{if } FLQ_{ij}^r \ge 1 \end{Bmatrix}$$

Flegg *et al.*, as well as many other regional analysts, use employment rather than output as the relevant measures of regional and national activity; these are x_E^r and x_E^n for the region and the nation, respectively, so x_E^r/x_E^n provides an alternative to the output ratio (x^r/x^n) as a measure of relative regional size. They also use employment as the measure of sector i and j activity (output). The general idea is to reduce national coefficients less for larger regions – on the belief that larger regions import (relatively) less than smaller ones.[5] The problem, however, is that the analyst must specify a value of δ in advance (β in the earlier formulation in footnote 5), and it is not at all clear what this value (or range of values) should be. Empirical work has suggested that $\delta = 0.3$ seems to work well in a variety of situations (see the articles by Flegg and associates cited above). The approach has been shown to be an improvement in at least one study that compared LQ, CIQ and (the earlier version of) FLQ for a region in Finland for which there were also survey-based coefficients to serve as a standard against which to measure the estimates (Tohmo, 2004). (Problem 8.4 asks the reader to examine the behavior of λ for various values of x_E^r/x_E^n and δ to see how this adjustment might work.)

An additional variant of the FLQ is designed to reflect regional specialization (Flegg and Webber, 2000). This was developed in response to the observation (McCann and Dewhurst, 1998) that such specialization might lead to increased intraregional purchases (by the specialized industry) and hence to intraregional input coefficients that were larger than their national counterparts. As noted earlier, national coefficients can never be increased by any of the quotient techniques examined thus far. In this case, the proposed augmentation of the FLQ (termed $AFLQ$) is

$$AFLQ_{ij}^r = \begin{Bmatrix} [\log_2(1 + LQ_j^r)]FLQ_{ij}^r & \text{if } LQ_j^r > 1 \\ FLQ_{ij}^r & \text{if } LQ_j^r \le 1 \end{Bmatrix}$$

and so

$$a_{ij}^{rr} = \begin{Bmatrix} (AFLQ_{ij}^r)a_{ij}^n & \text{if } LQ_j^r > 1 \\ (FLQ_{ij}^r)a_{ij}^n & \text{if } LQ_j^r \le 1 \end{Bmatrix}$$

This adjustment term, $[\log_2(1 + LQ_j^r)]$, is the modifier used for Round's SLQ_{ij}^r, only now it appears as a multiplier and not a divisor. Now FLQ is increased in those cases (only) in which sector j is relatively specialized in region r (when $LQ_j^r > 1$, so $[\log_2(1+ LQ_j^r)] > 1$). For example, as LQ_j^r increases from 1 to 5, $\log_2(1 + LQ_j^r)$ goes from 1 to 2.585. (There are some issues regarding the possibility of a national coefficient being

[4] Flegg and Webber (1997) use λ^* in their formulation because they used λ in an earlier and less successful version of their formula – $FLQ_{ij}^r = (\lambda^\beta)CIQ_{ij}^r$ where $\lambda = (x_E^r/x_E^n)/\{\log_2[1 + (x_E^r/x_E^n)]\}$.

[5] This logic has been questioned. See Brand (1997), McCann and Dewhurst (1998) and replies from Flegg and Webber (1997 and 2000, respectively).

increased to more than 1.0.) The argument is that a large industry (j) in a particular region may attract in-movement to the region of firms in other sectors that supply j; hence j's *intra*regional input purchases may be larger than the national coefficient would suggest. However, limited empirical evidence suggests that not much is gained in performance over *FLQ* by this augmentation (Flegg and Webber, 2000).

8.2.5 Supply–Demand Pool Approaches

The supply–demand pool (*SDP*) technique estimates regional from national coefficients in much the same way as the procedure that was used to balance the regional coefficients estimated by the simple location quotient technique. National technical coefficients are taken as the first approximation to regional coefficients. Regional output by sector is then found, as above, by multiplying each of these coefficients by the appropriate actual regional output of that sector (and similarly for final-demand sectors, but using the *national* final-demand input proportions, c_{if}^n) and summing:

$$\tilde{x}_i^r = \sum_j a_{ij}^n x_j^r + \sum_f c_{if}^n f_f^r \tag{8.12}$$

Then the regional commodity balance, b_i^r, is calculated for industry i as $b_i^r = x_i^r - \tilde{x}_i^r$.

If this balance is positive (or zero), using national coefficients as estimates of regional coefficients does not generate an overestimate of regional production and so $a_{ij}^{rr} = a_{ij}^n$ and $c_{if}^{rr} = c_{if}^n$ are acceptable estimates. However, if the balance is negative, national coefficients are too large, in the sense that they generate unrealistically high regional outputs, by sector, so $a_{ij}^{rr} = a_{ij}^n (x_i^r / \tilde{x}_i^r)$ and $c_{if}^{rr} = c_{if}^n (x_i^r / \tilde{x}_i^r)$ – the national coefficients are reduced by the amount necessary to make the regional balance for that sector exactly zero. Summarizing,

$$a_{ij}^{rr} = \begin{Bmatrix} (x_i^r / \tilde{x}_i^r) a_{ij}^n & \text{if } b_i^r < 0 \\ a_{ij}^n & \text{if } b_i^r \geq 0 \end{Bmatrix} \tag{8.13}$$

In terms of the general approaches in (8.1) and (8.2), we see that the supply-demand pool technique assumes that $\alpha_{ij}^r = 1$, as do all of the quotient techniques mentioned above. Further, $\beta_{ij}^r = x_i^r / \tilde{x}_i^r$ when $x_i^r - \tilde{x}_i^r < 0$ and $\beta_{ij}^r = 1$ when $x_i^r - \tilde{x}_i^r \geq 0$. As with the *LQ*-based techniques, only reductions of national coefficients are possible and cross-hauling is not captured.

8.2.6 Fabrication Effects

Round (1972, 1978a, 1983) has suggested an adjustment to account for differing regional "fabrication" effects that reflect differing value-added/output ratios for a specific sector across regions. Define the regional fabrication effect for sector j in region r as

$$\rho_j^r = \frac{1 - (w_j^r / x_j^r)}{1 - (w_j^n / x_j^n)} \tag{8.14}$$

In the numerator, w_j^r is total value-added payments by sector j in region r and x_j^r is, as usual, gross output of sector j in r. Thus (w_j^r/x_j^r) is the proportion of the total output of sector j in region r accounted for by value-added elements, and $1-(w_j^r/x_j^r)$ is the proportion of total output that is due to interindustry inputs from the processing sectors (including imports). Roughly, then, the numerator represents the relative dependence of sector j in region r on inputs from itself and all other sectors. For example, if $w_j^r = \$400$ and $x_j^r = \$1000$, then $1-(w_j^r/x_j^r) = 0.6$; 60 percent of the value of sector j's total output is derived from inputs from producing sectors. The denominator in (8.14) is this same measure of industrial dependence for sector j nationally. Suppose $w_j^n = \$300{,}000$ and $x_j^n = \$1{,}000{,}000$, so that the denominator in (8.14) is 0.7; at the national level sector j is relatively more dependent on industrial inputs and relatively less dependent on value-added inputs. For this example, $\rho_j^r = 0.6/0.7 = 0.857$.

Round suggests that ρ_j^r be used as α_{ij}^r in (8.1), so that the estimate of a_{ij}^r (for $i = 1, \ldots, n$) is found as

$$a_{ij}^r = (\rho_j^r)(a_{ij}^n)$$

This is a column modification, as opposed to the row modifications of the quotient-like techniques – the entire jth column of $\mathbf{A}^n$ is multiplied by ρ_j^r to generate an estimate of the jth column of $\mathbf{A}^r$. The idea is that since interindustrial inputs are relatively less important to industry j's production in region r than at the national level, national input coefficients for sector j should be scaled down. Similarly, if $\rho_k^r > 1$, then all of the elements in the kth column of $\mathbf{A}^n$ would be scaled upward, to generate the estimates of $a_{ik}^r (i = 1, \ldots, n)$. Unlike most *LQ*-based techniques, national coefficients can be increased with this approach.[6] This a_{ij}^r can then be further adjusted to create an estimate of a_{ij}^{rr} via a quotient-like modification.

8.2.7 Regional Purchase Coefficients

Work at the Regional Science Research Institute (as discussed, for example, in Stevens and Trainer, 1976, 1980 and in Stevens *et al.*, 1983) concentrated on estimation of what are essentially the regional supply proportions, p_i^r, that were mentioned in section 8.1 (and earlier in Chapter 3). These were termed *regional purchase coefficients* (RPCs) in the RSRI work; they operate uniformly across rows, as do *LQ*-based methods. In terms of (8.1), $\alpha_{ij}^r = 1$ and, in (8.2), $\beta_{ij}^r = p_i^r \; (= RPC_i^r)$.

The regional purchase coefficient for a sector is defined as the proportion of regional demand for that sector's output that is fulfilled from regional production. Formally, for region r and good i,

$$\mathrm{RPC}_i^r = z_i^{rr}/(z_i^{rr} + z_i^{sr})$$

[6] This "fabrication" adjustment is similar in spirit to the column adjustments (the s's) in the RAS updating procedure, which multiply all elements in the kth column of the coefficient matrix by s_k. This is what Stone termed the "fabrication effect" – the possibility that there is a change in the proportion of value-added inputs in a sector's output over time.

where, as in Chapter 3, z_i^{rr} accounts for shipments of good i from producers in r to all buyers in r and z_i^{sr} represents imports of i from outside r to buyers in r.[7] Dividing numerator and denominator by z_i^{rr},

$$\mathrm{RPC}_i^r = 1/[1 + 1/(z_i^{rr}/z_i^{sr})]$$

Effort was concentrated on estimating the magnitude of the *relative shipments* term, z_i^{rr}/z_i^{sr}. Assuming that *relative* terms designate ratios of values in region r to national values, relative shipments are estimated as a function of relative delivered costs (made up of relative unit production costs and relative unit shipment costs). These, in turn, depend on relative wages, relative output levels, and average shipping distances from producers within and outside region r. Various relationships between RPC_i^r and proxies for these relative terms have been proposed and fitted by regression techniques to data that are available in US published sources such as *County Business Patterns*, *Census of Transportation*, and *Census of Manufactures*, as well as a national input–output technical coefficients table. Comparisons with LQ-based approaches suggest the superiority of this method (Stevens, Treyz and Lahr, 1989). An alternative approach to estimation of RPCs (at the county level in the USA) is suggested by Lindall, Olson and Alward (2006) in the context of a gravity model for estimating intercounty commodity flows.[8]

8.2.8 "Community" Input–Output Models

In section 3.6 we cited Robison and Miller (1988, 1991) and Robison (1997) as examples of "regional" input–output modeling at a very small spatial scale:

> Our greatest departure from the traditional I-O approach stems from a fundamental redefinition of region. The traditional I-O approach models uniform regions, e.g., counties and multicounty areas, states, and so on. In contrast, we build models for punctiform regions, i.e., models constructed for individual cities, towns, and hamlets. (Robison, 1997, p. 326.)

Robison and Miller (1991) introduce ideas from Central Place Theory in modeling such small area economies – here a small Idaho timber (logging/sawmills) economy, the "rural West-Central Idaho Highlands Highway 55 economy" (six communities, five containing sawmills, with a combined population around 20,000). The authors suggest that principles from that theory can help to guide construction of such "intercommunity" input–output models. For example, they consider an intra- and intercommunity coefficients matrix of the sort $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \cdots & \mathbf{A}_{1m} \\ \vdots & \ddots & \vdots \\ \mathbf{A}_{m1} & \cdots & \mathbf{A}_{mm} \end{bmatrix}$, where communities $1, \ldots, m$

[7] In the context of the multiregional input–output model (Chapter 3), these coefficients are the c_i^{rr} – for example, as in (3.27). However, in the MRIO model, they are used to modify a *regional* matrix, $\mathbf{A}^r$, that is not assumed simply to be the same as $\mathbf{A}^n$, the national table. In terms of equations (8.1) and (8.2), in the MRIO model $a_{ij}^r \neq 1$, at least for some i, j and r, and $\beta_{ij}^r = c_i^{rr}$ for all i.

[8] Since these authors use a gravity model formulation, there is further discussion of their work below in section 8.6.1.

are arranged from upper left to lower right in descending hierarchical (rank) order. Then strict hierarchical trade (meaning that goods flow primarily from higher- to lower-order places) would be reflected in an **A** matrix that is (close to) upper-triangular (zeros below the main diagonal).[9] In such multiregional economies cross-hauling is much less likely to be present, and hence estimation techniques that fail to account for cross-hauling (such as *LQ* and *SDP* approaches, as noted above) may be acceptable regionalization approaches.

Robison (1997) discusses hybrid procedures to estimate trade among such places in a set of hierarchically structured areas (the upper-right off-diagonal elements in **A**) as well as other special features of small-area rural economies such as their extreme openness and the importance of transboundary income and expenditure flows. In this case the application was for a rural two-county region in central Idaho (total population less than 12,000) which was disaggregated into seven community-centered sub-county regions.

County-level data from IMPLAN ("impact analysis for planning") form a basis for much of the estimation, but further disaggregation to subcounty community level regions is required. For some data, surveys were used; for others, published sources contribute information (for example, business listings in local telephone directories); in still others, *SDP* and/or *LQ* estimates were generated. Other applications of these ideas can be found in Hamilton *et al.* (1994).

8.2.9 Summary

It is worth noting that these approaches (or variants) are frequently used in applied regional analysis. Even the straightforward location quotients of section 8.2.1 are often employed. For example, in the USA, multipliers for any selected single- or multi-county region can be purchased from the Bureau of Economic Analysis of the US Department of Commerce through their *Regional Impact Modeling System* (RIMS II).[10] This system uses location quotients to derive estimates of intraregional input coefficients. As noted earlier, these intraregional coefficients will tend to be overestimated; in fact, Robison and Miller (1988) advise caution in using either RIMS or IMPLAN estimates in small area studies – specifically:

> We argue that pool and quotient techniques, used in nonsurvey models such as IMPLAN and RIMSII, should not be applied to a single county situation, or to any aggregation of counties that is not, in some sense, a functional economic area [p. 1523] . . . Because of the likelihood of cross-hauling when state and functional economic boundaries diverge, we suspect that many of these models of states [e.g., RIMSII] possess overstated multipliers . . . There could be large errors in reported RIMSII multipliers for states [p. 1529].

[9] The authors recognize that there can be shipments up the hierarchy: "What about trade in the opposite direction, from lower to higher-order places? Examples from rural regions would include agricultural and other raw materials shipped to higher-order places for processing. Rural economies are simple and normally raw materials trade will have to be obtained from observation rather than technique" (Robison, 1997, pp. 335–336).

[10] Currently these are based on 2004 national input–output accounts and 2004 regional economic accounts. See US Department of Commerce (1997); also www.bea.gov.

As another example, the bulk of the discussion in Gerking *et al.* (2001) deals with imaginative ways of filling in for "suppressed" data at the county level (due to disclosure concerns), in order to estimate industry-specific employment at the county level. These employment data are used to calculate county-level location quotients which are then used in the usual way to regionalize (down to the county level) a national direct coefficients table. The authors argue that estimating at the lowest possible level of sectoral aggregation has the effect of minimizing the consequences of the no cross-hauling feature of location quotient reduction techniques. The approach is illustrated with an economic impact analysis of an energy project for a county in Wyoming.

It is generally recognized that the reduction techniques discussed above are less than totally successful. Yet the need for input–output data at a regional level continues to increase and has stimulated much discussion and many approaches. Often, these are *hybrid* techniques which include use of the RAS procedure (originally devised for updating national input–output information) along with additional information. We turn to some of these developments next, looking first at the use of RAS for regionalization of a national input–output table.

8.3 RAS in a Regional Setting

As we saw in section 7.4, the RAS technique generates a coefficient matrix for a particular year, $\mathbf{A}(1)$, given observations on total outputs, total interindustry sales, and total interindustry purchases for that year – $\mathbf{x}(1)$, $\mathbf{u}(1)$, and $\mathbf{v}(1)$, and using as a starting point an earlier coefficient matrix, $\mathbf{A}(0)$. While it is inherently a mathematical technique, we have also seen that the economic notions of uniform substitution and fabrication effects are compatible with the procedure. Since coefficient tables for regional input–output models are essential for regional analysis, one way to have a wider variety of tables available for various regions of a nation is to apply the same RAS principles, where we utilize a (relatively up-to-date) *national* input–output table, $\mathbf{A}^n$, and current marginal information about *regional* economic activity – $\mathbf{x}^r, \mathbf{u}^r$, and $\mathbf{v}^r$. Or, for that matter, instead of $\mathbf{A}^n$, one may have a current input–output table for some *other* region in the country, s, and then use the known $\mathbf{A}^s$ as the matrix to be adjusted to satisfy the observed marginal information for region r. Thus, instead of using the RAS procedure to adjust coefficient matrices across time (the updating problem), it has also been used to adjust coefficient matrices across space (the regionalization problem). To the extent that a national table, $\mathbf{A}^n$, reflects an average of input–output relationships in various regions of the nation, the minimization of "information distance" or "surprise" that is inherent in the RAS technique may also be appropriate at the regional level. Or if there is an input–output coefficient table for a region, s, that is thought to be economically similar to the region in question, r, then this same "minimal surprise" characteristic of the RAS procedure is possibly an attractive one.

On the (different) problem of updating an existing regional table via RAS-like techniques, see, among others, the early work of McMenamin and Haring (1974) (and also the Giarratani, 1975, comment on this work)[11] and Malizia and Bond (1974). Many studies have compared results from the RAS approach with one or more of the reduction techniques in section 8.2 for deriving a regional from a national input–output table. We illustrate this kind of comparison in the following section. More recently, analysts have combined both kinds of techniques into hybrid approaches. We examine a few examples in section 8.5. Later, in section 8.7, we consider the additional problem of estimating interregional flows in order to create a model for two or more connected regions.

8.4 Numerical Illustration

In Table 8.1 we present illustrative results from application of some of these techniques to estimate matrices for Region 1 (North China) from the three-region, three-sector data set for China for 2000. (These were used for illustration in Chapter 3, section 3.4.6.) More detailed results – for example, the complete 3×3 coefficients matrices and their associated Leontief inverses, are shown in Web Appendix 8W.1 for the interested reader. This is done in part because, as always when comparing matrices, a good deal of individual detail is inevitably lost when summary measures are used. Coefficients and inverse matrices were estimated using *LQ*, *CIQ*, *FLQ*, *AFLQ*, *RPC*, and *RAS* techniques, first on an unadjusted national table, $\mathbf{A}^n$, created by spatial aggregation of the data in Chapter 3, and then on a regional technical coefficients table, $\mathbf{A}^r$, created using Round's fabrication effect adjustment, where $\mathbf{A}^r = \mathbf{A}^n \hat{\rho}^r$.

Differences between survey-based total intraregional intermediate inputs and those in each of the estimated matrices (column sums of $\mathbf{A}^{rr}$ and each estimate, $\tilde{\mathbf{A}}^{rr}$) are one way to condense n^2 pieces of information (here 9) into n. Differences in column sums of each of the Leontief inverses, $\mathbf{L}^{rr}$ and each of the $\tilde{\mathbf{L}}^{rr}$ (intraregional output multipliers) are another (and more frequently used) summary measure. Both of these mask individual cell differences in the process of summation down the columns. We also include one additional measure, the mean absolute percentage error (MAPE), already used in section 7.4.2. This is the average of the percentage differences in corresponding cells of $\mathbf{A}^{rr}$ and $\tilde{\mathbf{A}}^{rr}$ or of $\mathbf{L}^{rr}$ and $\tilde{\mathbf{L}}^{rr}$ (irrespective of whether positive or negative), so it too masks a wide variety of individual differences.

Notice that *RAS* always estimates total intraregional intermediate inputs correctly. In this small set of examples *RPC* was the best of the quotient techniques and *RAS* performed best overall, on the basis of either intermediate inputs or multipliers. The fabrication adjustment suggested by Round appeared to be a significant help for *RPC* only, and only when assessed on the basis of the average percentage difference. It made no difference for *RAS* because the initial matrices (with and without the fabrication adjustment) were very close. These results pertain only to this one small illustration.

[11] A particular feature of the McMenamin–Haring approach is that it employs the RAS technique on an entire transactions table, including the sales to final-demand sectors and the purchases from value-added sectors. That is, $\mathbf{u}^r$ and $\mathbf{v}^r$ are not needed; only $\mathbf{x}^r$ is used. This relaxes the data requirements but imposes the biproportionality assumption on not only the interindustry transactions but also on final-demand and value-added data. This is the basic point raised by Giarratani (1975).

Table 8.1 Total Intraregional Intermediate Inputs and Intraregional Output Multipliers for Region 1 (North China) Calculated from Several Regionalization Techniques

	Total Intraregional Intermediate Inputs			Percentage Differences[a]			Average Percentage Difference[b]	MAPE[c]
Survey	0.2891	0.5781	0.3466					
Using $\mathbf{A}^n$								
LQ	0.3166	0.6774	0.4019	9.54	17.19	15.96	14.23	12.41
CIQ	0.3169	0.6717	0.4022	9.64	16.20	16.03	13.96	12.54
FLQ	0.2541	0.5189	0.3113	−12.10	−10.23	−10.18	−10.84	13.04
AFLQ	0.2541	0.5290	0.3113	−12.10	−8.50	−10.18	−10.26	12.49
RPC	0.2827	0.5850	0.3495	−2.19	1.19	0.84	1.41[d]	7.17
RAS	0.2891	0.5781	0.3466	0	0	0	0	6.94
Using Round's $\mathbf{A}^r = \mathbf{A}^n \hat{\rho}^r$								
LQ	0.3222	0.6720	0.3943	11.45	16.24	13.75	13.82	12.17
CIQ	0.3225	0.6663	0.3945	11.56	15.27	13.82	13.55	12.31
FLQ	0.2585	0.5148	0.3054	−10.57	−10.95	−11.90	−11.14	13.33
AFLQ	0.2585	0.5247	0.3054	−10.57	−9.23	−11.90	−10.57	12.79
RPC	0.2877	0.5803	0.3429	−0.48	0.38	−1.08	0.65[d]	7.11
RAS	0.2891	0.5781	0.3466	0	0	0	0	6.94
	Intraregional Output Multipliers			**Percentage Differences[e]**			**Average Percentage Difference**	**MAPE[f]**
Survey	1.5311	2.1115	1.6620					
Using $\mathbf{A}^n$								
LQ	1.6765	2.5684	1.9201	9.50	21.63	15.53	15.55	25.06
CIQ	1.6734	2.5480	1.9148	9.29	20.67	15.21	15.06	23.74
FLQ	1.4309	1.9294	1.5515	−6.55	−8.63	−6.65	−7.28	15.93
AFLQ	1.4353	1.9590	1.5578	−6.26	−7.22	−6.27	−6.58	14.81
RPC	1.5108	2.1318	1.6700	−1.33	0.96	0.48	0.92[d]	3.50
RAS	1.5219	2.1145	1.6618	−0.60	0.14	−0.01	0.25[d]	2.79
Using Round's $\mathbf{A}^r = \mathbf{A}^n \hat{\rho}^r$								
LQ	1.6841	2.5425	1.8933	9.99	20.41	13.92	14.77	23.76
CIQ	1.6810	2.5226	1.8882	9.79	19.47	13.61	14.29	22.46
FLQ	1.4369	1.9172	1.5375	−6.15	−9.20	−7.49	−7.62	16.45
AFLQ	1.4413	1.9463	1.5436	−5.87	−7.82	−7.12	−6.94	15.35
RPC	1.5179	2.1163	1.6524	−0.86	0.23	−0.58	0.56[d]	3.13
RAS	1.5219	2.1145	1.6618	−0.60	0.14	−0.01	0.25[d]	2.79

[a] This is $\{[(\mathbf{i}'\tilde{\mathbf{A}} - \mathbf{i}'\mathbf{A}) \oslash \mathbf{i}'\mathbf{A}] \times 100\}$, where "$\oslash$" indicates element-by-element division.

[b] This is a simple, unweighted average. Various kinds of weightings (e.g., using some measure of the size of each sector) are frequently used.

[c] Calculated as $\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|a_{ij}-\tilde{a}_{ij}|}{a_{ij}}\right) \times 100$.

[d] This is the average of the absolute values of the differences, so that the negatives and positives do not cancel out.

[e] Calculated as $\{[(\mathbf{i}'\tilde{\mathbf{L}} - \mathbf{i}'\mathbf{L}) \oslash \mathbf{i}'\mathbf{L}] \times 100\}$.

[f] Calculated as $(\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{|l_{ij}-\tilde{l}_{ij}|}{l_{ij}}) \times 100$.

Other applications (and other measures of error) could easily generate different outcomes in terms of rankings of the techniques. Problem 8.9 asks the reader to create similar results for either (or both) of the other two regions, the South and the Rest of China, in the Chinese data in Chapter 3. Results for those exercises are shown in the Solutions, for those who want to bypass the work. As can be seen, the sizes of errors vary a great deal across the three region results, and in one case *FLQ* and *AFLQ* perform very badly. Generally, *RAS* is seen to provide the best results.

Over the years there have been many empirical studies, generally much larger than the illustration in Table 8.1, in which various location quotient approaches to regional coefficient estimation, often along with RAS, have been compared, and, not unexpectedly, the results have varied. Examples include (but are not limited to) Czamanski and Malizia (1969), Schaffer and Chu (1969), Hewings (1969, 1971), Round (1972), Morrison and Smith (1974), Smith and Morrison (1974), Eskelinen and Suorsa (1980), Cartwright, Beemiller and Gusteley (1981), Alward and Palmer (1981), Harrigan, McGilvray and McNicoll (1981), Sawyer and Miller (1983), Stevens, Treyz and Lahr (1989), Flegg and Webber (2000), Tohmo (2004) and Riddington, Gibson and Anderson (2006). As always, the results often depend on the statistic(s) used to rate the techniques.

8.5 Exchanging Coefficients Matrices

Early in applied regional input–output work it was thought that an alternative to adapting a national table to reflect the economic characteristics of a particular region might be to adapt an existing table for some other region or, indeed, simply to use a table for one region as representing another region as well. For example, a coefficients table for a particular wheat-growing county in North Dakota might reflect very well the economic interrelations in another wheat-growing county in North Dakota, or probably also in South Dakota or Nebraska. However, less plausible would be the use of a survey-based table for Philadelphia to represent interrelations in the Boston or, less likely, San Francisco economy. How much and what kind of modifications would be necessary are much more complicated questions. In this regard, one can only make very broad and general statements; for example, if in the opinion of experts, two regions are very similar economically, then it is possible that a coefficients table for one of them may prove to be useful for the other also. Or it may be useful with appropriate modification; the problem is always how to decide what needs to be modified and how to go about doing it.

As an example of coefficient exchange at the regional level, Hewings (1977) used a survey-based table for Washington State for 1963 (Bourque and Weeks, 1969) to estimate Kansas interindustry structure in 1965; he also used a survey-based Kansas table for 1965 (Emerson, 1969) to estimate Washington's structure in 1963. After appropriate classification of the two tables into a comparable set of sectors, it was clear from inspection that there were many individual coefficients that were vastly different in the two tables. In a simple coefficient change, estimating Washington output with Kansas technology, as $\mathbf{x}^W = (\mathbf{I}-\mathbf{A}^{KK})^{-1}\mathbf{f}^W$ and similarly, estimating Kansas output with Washington technology, $\mathbf{x}^K = (\mathbf{I}-\mathbf{A}^{WW})^{-1}\mathbf{f}^K$, it was found that aggregate errors (for total output, summed over all sectors) were 4.8 percent (overestimate) for Washington and −12.6 percent (underestimate) for Kansas. However, as usual with aggregate measures of error, individual sector estimates were often very far off; the worst in Washington was overestimated by 336 percent and the worst in Kansas was overestimated by 114 percent. Thus, straightforward coefficient exchange could not be considered a success.

However, using the RAS procedure in conjunction with Kansas survey-based information on total intermediate outputs, total intermediate inputs, and total output, by sector, produced far superior results. That is, the Washington table (instead of a national table) was "balanced" by the RAS technique to conform to the observed Kansas marginal information. With the modification, total estimated Kansas output was underestimated by only 0.008 percent, and the largest error for an individual sector's output was only 0.195 percent.

To emphasize the relative importance of the marginal information in the RAS procedure, Hewings also "balanced" an artificial coefficient matrix made up of random numbers (but with column sums less than one). That is, the "base" matrix was a totally artificial one, which did not correspond to any national or regional table. Using a randomly generated new final-demand vector, he compared "true" gross outputs (using the actual Kansas table) with the RAS-adjusted Washington table and the RAS-adjusted random table. In these two cases, the total Kansas output, summed over all sectors, was overestimated by only 0.028 percent and underestimated by 0.192 percent, respectively. The worst errors in individual sector outputs were 3.7 percent (Washington table) and 5.6 percent (random table). The main lesson from this experiment appears to be that information on region-specific sectoral total intermediate outputs, $\mathbf{u}$, and inputs, $\mathbf{v}$, along with sectoral gross outputs, $\mathbf{x}$, are of dominant importance (as opposed to the base matrix) in an RAS adjustment procedure. (For a comment on the Hewings study and a reply, see Thumann, 1978 and Hewings and Janson, 1980. Also, see Szyrmer, 1989, for a discussion of experiments that indicate the importance of correct target-year marginal information in an RAS procedure.)

8.6 Estimating Interregional Flows

Earlier in this chapter, we examined some techniques that have been proposed and used to estimate regional input coefficients from existing regional or national tables. If two or more regions are to be connected in the model, then interregional coefficients are also needed. In Chapter 3 we saw what data are necessary in both the interregional and multiregional cases, and an example was provided from multiregional data for China in 2000 (in section 3.3.5).

Because of the extremely detailed data that are necessary for a full interregional model and because the US multiregional model was itself an extremely ambitious and time-consuming project, there are not many existing tables of interregional commodity flows or their associated coefficients that can be used as "base" tables to be updated, projected, or exchanged. Rather, a number of proposals have been explored for estimating these flows between sectors and regions. The techniques are sometimes relatively advanced, and a thorough survey is beyond the scope of this book. We indicate only some of the broad ideas that have been used.

8.6.1 Gravity Model Formulations

Many versions of gravity model formulations have been proposed and explored for estimating commodity flows between regions. The basic idea is that the flow of good i from region r to region s can be looked upon as a function of (1) some measure of the total output of i in $r, x_i^{r\cdot}$, (2) some measure of the total purchases of i in $s, x_i^{\cdot s}$, and (3) the distance (as a measure of "impedance") between the two regions, d^{rs}. One straightforward function, taking inspiration from Newton's observations on gravity (and hence the name for this class of models), would involve the product of the two "masses" ($x_i^{r\cdot}$ and $x_i^{\cdot s}$) divided by the square of the distance. A bit more generally,

$$z_i^{rs} = \frac{(c_i^r x_i^{r\cdot})(d_i^s x_i^{\cdot s})}{(d^{rs})^{e_i}} = (k_i^{rs}) \frac{x_i^{r\cdot} x_i^{\cdot s}}{(d^{rs})^{e_i}} \tag{8.15}$$

where c_i^r, d_i^s (alternatively, k_i^{rs}) and e_i are parameters to be estimated. (In the strictest Newtonian form, $e_i = 2$.)

As noted in Chapter 3, the gravity approach was suggested initially in an input–output context in Leontief and Strout (1963); it was also explored in Theil (1967). Leontief and Strout suggested the relatively simplified form

$$z_i^{rs} = \frac{x_i^{r\cdot} x_i^{\cdot s}}{x_i^{\cdot\cdot}} Q_i^{rs} \tag{8.16}$$

where $x_i^{r\cdot}$ is labeled the "supply pool" of good i in region r, $x_i^{\cdot s}$ is labeled the "demand pool" of good i in region s, $x_i^{\cdot\cdot}$ is the total production of commodity i in the system and Q_i^{rs} is a parameter. The authors write:

> The multiplicative form in which the total output of good i in the exporting and its total input in the importing regions enter into [(8.15)] permits us to characterize it as a special type of Gravity or Potential Model. It implies that there can be no flow from region r to region s if either one of those two magnitudes is equal to zero. The introduction of the aggregate output of good i into the denominator implies that, if the aggregate output [$x_i^{\cdot\cdot}$], as well as output [$x_i^{r\cdot}$] in region r and total input [$x_i^{\cdot s}$] in region s, double, the flow of that good from region r to region s will double too. [Leontief (1966) p. 226. The authors use g and h in place of r and s.]

Notice that the denominator in this formulation [(8.16)] is aspatial; that is, its magnitude is unrelated to any measure of "distance" between r and s. Rather, it provides the flexibility necessary so that if, for good i, the supply pool in r, the demand pool in s and total output all increase by p percent, then z_i^{rs} increases by that same percent (assuming $Q_i^{rs} > 0$). So the Q_i^{rs} term has something of the look of $\frac{k_i^{rs}}{(d^{rs})^{e_i}}$ from (8.15).

An important feature of this kind of formulation is that cross-hauling is allowed; that is, good i can be shipped simultaneously from r to s and from s to r. Specifically, if $x_i^{r\cdot}$, $x_i^{\cdot r}$, $x_i^{s\cdot}$, and $x_i^{\cdot s}$ are all nonzero, and if $Q_i^{rs} > 0$ and $Q_i^{sr} > 0$, then both $z_i^{rs} > 0$ and $z_i^{sr} > 0$.

The most optimistic scenario is that values of $x_i^{r\cdot}$, $x_i^{\cdot s}$, $x_i^{\cdot\cdot}$, and z_i^{rs} are known from some base period or for some subset of transportation data. In that case, one can evaluate the parameter Q_i^{rs} from those data, as

$$Q_i^{rs} = \frac{\bar{z}_i^{rs}\bar{x}_i^{\cdot\cdot}}{\bar{x}_i^{r\cdot}\bar{x}_i^{\cdot s}}$$

where overbars indicate known values. Leontief and Strout also discuss a number of alternative ways of estimating the Q_i^{rs} in cases where there is no base-case information.

Polenske (1970a) tested the Leontief–Strout gravity approach, using Japanese interregional flow data. She also compared the gravity formulation with the Chenery–Moses MRIO model (section 3.4) and one other alternative, known as a "row-coefficient" version of the MRIO model. The gravity and MRIO estimates were about equally good and far better than those obtained from the row-coefficient model (Polenske, 1970b). Estimates based on gravity models have also appeared in Uribe, de Leeuw and Theil (1966) and Gordon (1976) among others. Lindall, Olson, and Alward (2006) use a gravity formulation to estimate gross trade flows for some 509 commodities and 3140 counties in the USA. One outcome is that their results allow for estimation of a set of regional purchase coefficients (RPCs) for each county, using their results for each county's commodity i trade with itself divided by total county demand for i.

The gravity approach was embedded in a general entropy-maximizing framework in a number of papers by Wilson. An overview is provided by Wilson (1970, especially Chapter 3).[12] Connections with information theory have been suggested, and this has been thoroughly explored by Batten (1982, 1983) and applied in Snickars (1979). Batten's empirical studies combine iterative (RAS-like) methods with a maximum entropy formulation and, if required, additional variations ("minimum information gain" procedures). (See Batten, 1983, especially Chapter 5 and Appendix E.) Batten and Boyce (1986) review gravity-based and other spatial interaction models.

8.6.2 Two-Region Interregional Models

A number of estimation methods for interregional models are simplifications or variants of the quotient techniques discussed above. Essentially, they use some measure of a region's import or export orientation with respect to each good; and if region r is found to be an exporter of good i, then it is assumed that all the requirements for i in region r will be met by local production and hence there will be no imports of i to region r (no cross-hauling). One important feature in a two-region interregional model is that one region's (domestic) exports of a particular good are the other region's (domestic)

[12] A compact discussion of MRIO, gravity, and entropy-maximizing models can also be found in Toyomane (1988). He also develops and applies two alternative multinomial logit models of trade coefficients to an Indonesian example. Amano and Fujita (1970) combine MRIO and econometric models to allow both input coefficients and trade coefficients to change over time. Details of these models are beyond the scope of this book.

imports. From (8.4), since

$$a_{ij}^{rr} = \begin{Bmatrix} (LQ_i^r)a_{ij}^n & \text{if } LQ_i^r < 1 \\ a_{ij}^n & \text{if } LQ_i^r \geq 1 \end{Bmatrix}$$

then, in a two-region interregional model (with regions r and s),

$$a_{ij}^{sr} = \begin{Bmatrix} (1 - LQ_i^r)a_{ij}^n & \text{if } LQ_i^r < 1 \\ 0 & \text{if } LQ_i^r \geq 1 \end{Bmatrix}$$

For example, if $LQ_i^r = 0.65$, then the assumption is that 35 percent of the needs of input i by sectors in region r will be met by imports from region s.

A simple procedure of this sort was used in early studies by Nevin, Roe and Round (1966) for a two-region model in the United Kingdom and by Vanwynsberghe (1976) for a three-region Belgian model. Examination of a wide variety of nonsurvey techniques in an (especially two-region) interregional setting is contained in a series of papers by Round (1972, 1978a, 1978b, 1979, and 1983), to which the interested reader is referred. An alternative approach used in several Swedish regional studies is outlined in Andersson (1975) and modifications are suggested in Bigsten (1981). As we will see below, there have been attempts to modify the two-region approach for cases in which more than two regions are present.

8.6.3 Two-Region Logic with more than Two Regions

The logic of the "balancing" inherent in two-region models – where one region's (domestic) exports of i are the other region's (domestic) imports of i – appears to have first been extended to more than two regions by Hulu and Hewings (1993; five regions); later examples include Hewings, Okuyama and Sonis (2001; four regions) and Bonet (2005; seven regions). The essential idea is to use location quotients, a sequence of two-region models, and an RAS balancing approach. A three-region setting is adequate to illustrate the process.

1. Consider a two-region context where regions 2 and 3 have been aggregated; let r = region 1 and $\tilde{r}$ = the rest of the economy (the remaining two regions). Use location quotients for r to estimate $\mathbf{A}^{rr} = [a_{ij}^{rr}]$ in the usual way from a known national coefficient matrix, $\mathbf{A}^n$; $a_{ij}^{rr} = \begin{Bmatrix} (LQ_i^r)a_{ij}^n & \text{if } LQ_i^r < 1 \\ a_{ij}^n & \text{if } LQ_i^r \geq 1 \end{Bmatrix}$. Then import coefficients from the rest of the economy to r, $\mathbf{A}^{\tilde{r}r} = [a_{ij}^{\tilde{r}r}]$, are found as $a_{ij}^{\tilde{r}r} = a_{ij}^n - a_{ij}^{rr}$.

Similarly, find $\mathbf{A}^{\tilde{r}\tilde{r}} = [a_{ij}^{\tilde{r}\tilde{r}}]$ using location quotients for the aggregate "rest of the economy" region (regions 2 and 3 in this case). Finally, imports from region 1 to the rest of the economy, $\mathbf{A}^{r\tilde{r}} = [a_{ij}^{r\tilde{r}}]$, are found as $a_{ij}^{r\tilde{r}} = a_{ij}^{n} - a_{ij}^{\tilde{r}\tilde{r}}$. The result is $\begin{bmatrix} \mathbf{A}^{11} & \mathbf{A}^{1\tilde{1}} \\ \mathbf{A}^{\tilde{1}1} & \mathbf{A}^{\tilde{1}\tilde{1}} \end{bmatrix}$.

2. Repeat this procedure for each of the other possible two-region partitions ($r = 2, \tilde{r} = 1, 3$ and $r = 3, \tilde{r} = 1, 2$), giving $\begin{bmatrix} \mathbf{A}^{22} & \mathbf{A}^{2\tilde{2}} \\ \mathbf{A}^{\tilde{2}2} & \mathbf{A}^{\tilde{2}\tilde{2}} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{A}^{33} & \mathbf{A}^{3\tilde{3}} \\ \mathbf{A}^{\tilde{3}3} & \mathbf{A}^{\tilde{3}\tilde{3}} \end{bmatrix}$.
   This information can be arranged as in the table below. Missing, of course, are the interregional coefficients (shaded areas).

$\mathbf{A}^{11}$			$\mathbf{A}^{1\tilde{1}}$
	$\mathbf{A}^{22}$		$\mathbf{A}^{2\tilde{2}}$
		$\mathbf{A}^{33}$	$\mathbf{A}^{3\tilde{3}}$
$\mathbf{A}^{\tilde{1}1}$	$\mathbf{A}^{\tilde{2}2}$	$\mathbf{A}^{\tilde{3}3}$	

3. Convert coefficients to flows. For example, using known outputs $\mathbf{x}^1$ and $\mathbf{x}^{\tilde{1}}$, find $\begin{bmatrix} \mathbf{A}^{11} & \mathbf{A}^{1\tilde{1}} \\ \mathbf{A}^{\tilde{1}1} & \mathbf{A}^{\tilde{1}\tilde{1}} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}^1 & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{x}}^{\tilde{1}} \end{bmatrix} = \begin{bmatrix} \mathbf{Z}^{11} & \mathbf{Z}^{1\tilde{1}} \\ \mathbf{Z}^{\tilde{1}1} & \mathbf{Z}^{\tilde{1}\tilde{1}} \end{bmatrix}$. Similar calculations can be made for $r = 2$ and 3, producing

$\mathbf{Z}^{11}$			$\mathbf{Z}^{1\tilde{1}}$
	$\mathbf{Z}^{22}$		$\mathbf{Z}^{2\tilde{2}}$
		$\mathbf{Z}^{33}$	$\mathbf{Z}^{3\tilde{3}}$
$\mathbf{Z}^{\tilde{1}1}$	$\mathbf{Z}^{\tilde{2}2}$	$\mathbf{Z}^{\tilde{3}3}$	

   (The three $\mathbf{Z}^{\tilde{r}\tilde{r}}$ matrices that are generated in these calculations are ignored.)

4. The (shaded) off-diagonal flow matrices remain to be estimated. If these empty cells are filled with initial estimates, an RAS procedure can be applied. The extremely simplifying assumption is made that imports to any particular (on-diagonal) region come equally from all other (here both) regions; e.g., $\mathbf{Z}^{21} = \mathbf{Z}^{31} = (1/2)\mathbf{Z}^{\tilde{1}1}$. Thus all cells now contain initial estimates.

$\mathbf{Z}^{11}$	$\mathbf{Z}^{12}$	$\mathbf{Z}^{13}$	$\mathbf{Z}^{1\tilde{1}}$
$\mathbf{Z}^{21}$	$\mathbf{Z}^{22}$	$\mathbf{Z}^{23}$	$\mathbf{Z}^{2\tilde{2}}$
$\mathbf{Z}^{31}$	$\mathbf{Z}^{32}$	$\mathbf{Z}^{33}$	$\mathbf{Z}^{3\tilde{3}}$
$\mathbf{Z}^{\tilde{1}1}$	$\mathbf{Z}^{\tilde{2}2}$	$\mathbf{Z}^{\tilde{3}3}$	

5. Eliminate the on-diagonal matrices. Given the row and column margins, which account only for interregional flows (the shaded portions), use RAS to create a balanced table. (The table conforms to the column sums by the way in which it was constructed, but not to the row sums.)

$\mathbf{0}$	$\mathbf{Z}^{12}$	$\mathbf{Z}^{13}$	$\mathbf{Z}^{1\tilde{1}}$
$\mathbf{Z}^{21}$	$\mathbf{0}$	$\mathbf{Z}^{23}$	$\mathbf{Z}^{2\tilde{2}}$
$\mathbf{Z}^{31}$	$\mathbf{Z}^{32}$	$\mathbf{0}$	$\mathbf{Z}^{3\tilde{3}}$
$\mathbf{Z}^{\tilde{1}1}$	$\mathbf{Z}^{\tilde{2}2}$	$\mathbf{Z}^{\tilde{3}3}$	

If the presence of null matrices creates convergence problems (as it did in the articles cited), reintroduce the on-diagonal matrices, alter the margins accordingly, and reapply RAS.

8.6.4 Estimating Commodity Inflows to a Substate Region

Liu and Vilain (2004) start with known commodity flow data for US states from the 1993 US Commodity Flow Survey (US Department of Commerce, 1993) and national commodity-by-industry input–output data. They derive commodity inflows to a substate region using features of a supply-side, commodity-by-industry model and secondary data on the region's industrial structure. Their two-step procedure first scales each commodity's national "output coefficients" (using the terminology of the supply-side input–output model which is discussed in Chapter 12) to a state level. These coefficients are commodity sales to industries as a proportion of the selling sector's output, rather than commodity purchases as a proportion of the buying sector's output; they represent the distribution across buyers instead of across sellers in the usual input–output model. The second step then scales the state-level coefficients to a regional (substate) level. In both cases, the scaling is done using location quotients, but on these output coefficients, rather than on input coefficients as in the approaches of section 8.2.

Given a national Use matrix, $\mathbf{U}^N$, and total commodity outputs, $\mathbf{q}^N$, find $\mathbf{B}^N = (\hat{\mathbf{q}}^N)^{-1}\mathbf{U}^N$, where b_{ij}^N indicates the share of total commodity i output that is sold to industry j.[13] Convert these national output shares to the state-level shares using state-specific location quotients. [The authors use industry earnings as the basis for their location quotients but recognize that other alternatives (e.g., employment) are possible.] That is, create $\mathbf{B}^S = \mathbf{B}^N \langle \mathbf{lq}^S \rangle$ where $\mathbf{lq}^S = [LQ_i^S]$ and LQ_i^S is the earnings-based location quotient for sector i in the state.[14] In contrast to the way in which location quotients are used for regionalizing national input coefficients (sections 8.2.1–8.2.4), if a given $LQ_i^S > 1$ then the associated b_{ij}^N (for all j) are increased. Next, normalize the elements in each row of $\mathbf{B}^S$ by dividing by the row sum (so that all row sums in the normalized matrix will equal 1) – $\tilde{\mathbf{B}}^S = \mathbf{B}^S \langle \mathbf{B}^S \mathbf{i} \rangle^{-1}$, so $\tilde{b}_{ij}^S = b_{ij}^S / \sum_j b_{ij}^S$. Each $\tilde{b}_{ij}^S$ is an estimate of the proportion of commodity i shipped into the state that will be used by industry j in the state.[15] Let $\mathbf{m}^S$ be a vector of inflows to the state of the m commodities (known from the Commodity Flow Survey). Then the matrix $\boldsymbol{\rho}^S = \hat{\mathbf{m}}^S \tilde{\mathbf{B}}^S$ apportions the inflows of the m commodities among the n industries (including households); ρ_{ij}^S is the amount of commodity i flowing to industry j in the state. (In the notation of Chapter 3, this is an estimate of $z_{ij}^{\cdot S}$, an element in the MRIO model.)

The next step moves to the regional (substate) level. Estimate another matrix of location quotients, $\mathbf{LQ}^R$, this time for the region, measuring the relative representation of each industry in the region. Then define $\boldsymbol{\rho}^R = \boldsymbol{\rho}^S \langle \mathbf{lq}^R \rangle = \hat{\mathbf{m}}^S \tilde{\mathbf{B}}^S \langle \mathbf{lq}^R \rangle$; ρ_{ij}^R is an approximation of the amount of commodity i shipped to the state that is used by industry j in the region in question.[16] In terms of transportation planning, row sums of $\boldsymbol{\rho}^R$ may also be of interest; they are estimates of the total amount of each commodity that is shipped to the region – $\boldsymbol{\varphi}^R = \boldsymbol{\rho}^R \mathbf{i} = [\phi_i^R]$, where ϕ_i^R is the total regional inflow of commodity i.

The authors apply the method to commodity inflow to seven states and compare their results with known inflows (from the 1993 Commodity Flow Survey; that is, they assume the "regions" are in fact states in order to have data with which to compare their estimates). For six of the seven states, mean absolute percentage errors (MAPEs) were between 16 and 30 (with large variation among commodities) and for one state, the MAPE was 71. This method is compared with results from the Jackson *et al.* approach, discussed immediately below.

[13] We use upper-case superscripts "N," "S," and "R" to denote nation, (subnational) state and (substate) region, respectively, since lower-case superscripts are generally reserved for individual and distinct regions (e.g., "r" and "s").

[14] This notation is a bit unconventional. The vector of location quotients for the state is denoted by lower-case bold letters, $\mathbf{lq}^S$ (our convention for vectors throughout this book), but its elements are upper-case, LQ_i^S, to conform with the usual convention for representing location quotients, as in section 8.2.1, even though the usual notation for elements of the $\mathbf{lq}^S$ vector would be lq_i^S.

[15] The normalization is done so that 100 percent of the inflow of each commodity will be used up by purchasing sectors in the state. The authors work with a closed model, so household consumption of imported commodities is accounted for.

[16] Notice that these are akin to the *regional sales coefficients* of Oosterhaven and his colleagues in the Netherlands (section 8.7.2).

8.6.5 Additional Studies

Commodity Flows among US States

Interregional Social Accounts Model (ISAM). In two articles (Jackson *et al.*, 2006; Schwarm, Jackson and Okuyama, 2006) single-state SAMs are constructed using data derived from IMPLAN. (In the second article, the acronym *ISAG* is used, for *Interregional Social Accounts Generator*.) The authors estimate interregional commodity-by-industry flows connecting the states in an attempt to improve on the commodity flow survey (CFS) data from the Bureau of Transportation Statistics. The system consists of 51 regions (states plus DC), 54 industry/commodity sectors, 4 factors of production, and 18 institutions. The primary effort is to derive an estimating equation to distribute known regional domestic exports (from the single-region SAMs) from each region to each other domestic region in the model. (Intraregional flows are generated in the construction of each of the single-region SAMs, and they are assumed to be correct.) The authors assume that distributions of exports from one region to all others are fixed, while export levels vary with regional production.

The preferred estimating equation is a function of transportation costs (interregional distances) and region-specific commodity demand. It has the form

$$z_i^{rs} = \frac{(w_i^s)^{\alpha_i} \exp(-\beta_i d^{rs})}{\sum_s (w_i^s)^{\alpha_i} \exp(-\beta_i d^{rs})} z_i^{r\cdot}$$

where w_i^s is a measure of region s's demand for imports of commodity i and d^{rs} is some measure of the distance between r and s, and where α and β are elasticities on commodity demand and distance, respectively. These elasticities are estimated in an optimization model in which a measure of total absolute deviation between estimated flows and their associated observed benchmark flows is minimized. (The many details, including how a set of benchmark figures is generated, are relatively complex. The interested reader is referred to Jackson *et al.*, 2006.) At various points, biproportional (RAS) adjustments are required to ensure consistency with known national figures.

National Interstate Economic Model (NIEMO). An ambitious project to revitalize the US MRIO model is underway at the University of Southern California in its Center for Risk and Economic Analysis of Terrorism Events (CREATE). This effort updates the outdated US MRIO models for 1963 and 1977 (Chapter 3) in a framework of 47 sectors and 52 regions (50 states, the District of Columbia and the Rest of the World) and uses the model in many empirical applications. There are numerous publications, beginning around 2005, that discuss the derivation of the model and various extensions and applications; these include Park *et al.* (2004) and Richardson, Gordon and Moore (2007, especially the chapters by Park *et al.* and Richardson *et al.*).

The basic model-building idea is to integrate data from 2001 IMPLAN state-level input–output models (for intrastate coefficients) with commodity flow data from the US Department of Transportation's 1997 Commodity Flow Survey (for interregional coefficients) in an MRIO framework. There are many issues with the data sets alone – for example, reconciling the 509 IMPLAN sectors with the 43 sectors in the CFS data and dealing with the absence of interstate trade in services and many other empty cells in the CFS data. Instead of a gravity-model approach (as in the work of Jackson *et al.*, above), NIEMO uses a doubly constrained Fratar model (a biproportional matrix balancing technique – similar to RAS – from the transportation engineering literature) to generate interregional coefficients.

This work has been extended in many directions: (a) to a supply-driven model, for example, to quantify effects of terrorist attacks on ports (import disruptions), (b) to a price-sensitive supply-side model, incorporating exogenous price elasticities of demand and (c) to a flexible model, in which input and output coefficients matrices are altered in an RAS procedure as a result of natural disaster or terrorist attack. Among the several applications, in addition, are assessments of the sectoral/spatial impact of an outbreak of mad cow disease, Mexico–US border closure and attacks on theme parks.

An Optimization Model for Interregional Flows In Canning and Wang (2005), the authors formulate a quadratic programming problem to estimate interregional, interindustry transaction flows in a national system of regions. They choose a mathematical programming approach because of the flexibility that such a format provides for incorporating constraints (adding-up constraints, upper and/or lower limits on values of individual variables, etc.). While the ultimate goal is to estimate the elements of a several-region interregional input–output (IRIO) model, practical considerations mandated that a simpler multiregional (MRIO) model be used. (These models are explored in Chapter 3.) Variables to be estimated are regional inputs (ignoring regional origin), $z_{ij}^{\cdot r}$, and interregional flows (ignoring sectoral destination), z_i^{rs}. The approach requires a national input–output table and regional data on gross outputs (x_i^r), value added (v_i^r), final demand ($f_i^{\cdot r}$), exports to foreign destinations (e_i^r) and imports from abroad (m_i^r).

Specifically, consistency conditions for variables z_i^{rs} and $z_{ij}^{\cdot r}$ in the MRIO model require:

1. For each commodity i and region r, total output is completely distributed to users (intermediate and final) in all regions plus overseas

$$\sum_{s=1}^{p} z_i^{rs} + e_i^r = x_i^r$$

2. For each i and r, the value of gross output is attributable to intermediate inputs (regardless of their origin) plus primary inputs (value added)

$$\sum_{j=1}^{n} z_{ji}^{\cdot r} + v_i^r = x_i^r$$

3. Total requirements (intermediate plus final) for i in r are completely met by shipments from all regions (including r) plus imports from overseas

$$\sum_{j=1}^{n} z_{ij}^{\cdot r} + f_i^r = \sum_{s=1}^{p} z_i^{sr} + m_i^r$$

4. Intermediate purchases of commodity i by sector j in region r, when summed over all regions, must equal the national (superscript N) transaction amount

$$\sum_{r=1}^{p} z_{ij}^{\cdot r} = z_{ij}^N$$

In addition, constraints from the national accounts specify that regional output, value added, final demand, foreign exports and imports, for each commodity i, when summed over all regions, are equal to their associated national totals. That is, $\sum_{r=1}^{p} x_i^r = x_i^N, \sum_{r=1}^{p} v_i^r = v_i^N, \sum_{r=1}^{p} f_i^r = f_i^N, \sum_{r=1}^{p} e_i^r = e_i^N$, and $\sum_{r=1}^{p} m_i^r = m_i^N$. These linear equations can be incorporated easily as constraints into a mathematical programming format.

Subject to these (or similar) constraints, the authors suggest an objective function in which deviation from prespecified "estimates" of the unknowns is minimized. In their formulation, this takes the form of a weighted quadratic function. There are many options for how this is specified, and the details are beyond the scope of this book.

The authors present one application to a 4-region, 10-sector data set.[17] The results for this one application indicated relative success with respect to estimates of the interregional flows (z_i^{rs}), with mean average percentage errors (MAPEs) in the 4–7 percent range, while MAPEs for the regional inputs ($z_{ij}^{\cdot r}$) were less impressive (in the 15–20 percent range). Notice that a two-region model (region r and the rest of the nation) could be cast in this format, taking advantage of the adding-up constraints above.

8.7 Hybrid Methods

In this section we summarize a few of the (many) approaches that have been used by researchers in many parts of the world to derive regional input–output data. This represents only a small sample of real-world studies, virtually all of which use a hybrid approach with a combination of "superior data" or partial surveys and RAS or other techniques. As will be seen, these methods often embed the (intra)regional table estimation problem in a larger several-region system. Because of their tendency to (at least originally) focus on a regional table, we include them here rather than below, in section 8.8, on estimating interregional flows. However, the division is rather arbitrary, since some of the approaches in the later section also generate estimates of intraregional data.

[17] The "regions" were large – Japan, the USA, the EU, and the rest of the world.

8.7.1 Generation of Regional Input–Output Tables (GRIT)

A great deal of work has been done by Jensen and West and their colleagues in Australia on procedures for deriving input–output tables for various regions of that country, starting with a national table, employing allocation and quotient methods and paying attention to "superior data" and expert opinion when and as available. They have named this the GRIT technique. (See, for example, Jensen, Mandeville and Karunaratne, 1979 or West, 1990.) It has a long history, beginning in the late 1970s (those results are now known as "GRIT I"). Modifications led to "GRIT II" in the 1980s and then a version for estimating two or more regional tables and merging them into an interregional table ("GRIT III").[18] It is generally described as consisting of five steps (see Hewings and Jensen, 1986, for example):

1. Identify and adjust a "parent" table. Generally this will be a national table from time $(t-1)$ for the country in which the region of interest is located. This may be a transactions table or a coefficients table. Assume that the table incorporates competitive imports, so that the coefficients are true national technical coefficients, $\mathbf{A}^{\cdot n}(t-1)$. This will generally need to be updated from time $(t-1)$ to time t using RAS or some alternative technique – $\mathbf{A}^{\cdot n}(t-1) \rightarrow \mathbf{A}^{\cdot n}(t)$.[19]
2. Use some allocation or quotient method to convert national to regional coefficients; $a_{ij}^{\cdot r}(t) = r_{ij}^{r} a_{ij}^{\cdot n}(t)$ and then adjust for regional imports (e.g., using regional purchase coefficients) to produce an initial estimate of intraregional input coefficients, $a_{ij}^{rr}(t) = \rho_i^r a_{ij}^{\cdot r}(t)$.
3. Insert superior data from surveys, expert opinion, etc.
4. Define the appropriate regional sectors, usually through (weighted) aggregation of the national sectors. Insert additional superior data again, after the aggregation, in those cases where such information is known only at this more aggregated level. This might be done especially for "critical" (e.g., "inverse-important") cells, however determined (see section 12.3.3). The results are a prototype regional transactions table, $\mathbf{Z}^{rr}(p) = [z(p)_{ij}^{rr}]$, with an associated coefficients matrix $\mathbf{A}^{rr}(p)$ and Leontief inverse, $\mathbf{L}^{rr}(p) = [\mathbf{I} - \mathbf{A}^{rr}(p)]^{-1}$.
5. Using superior data and opinion once again – for example, by comparing multipliers derived from $\mathbf{L}^{rr}(p)$ in step (4) with those for "similar" regions – derive final versions of $\mathbf{Z}^{rr}$, $\mathbf{A}^{rr}$, and $\mathbf{L}^{rr}$.

Over the recent past, increasing emphasis has been placed on obtaining superior data from the outset, including extensive searches of published data (public and private sources) and special requests to national, state and local government agencies, followed by surveys.

[18] A description of much of this history can be found in West, Morison and Jensen (1982).
[19] If the initial tables are *national* transactions (exports excluded) then the import element for each sector (column) must be allocated up that column to the individual entries.

Table 8.2 Components in the DEBRIOT Approach

Intra- and Interregional Transactions	To Region r	To Region s	Regional Sales to Domestic Markets
From Region r	$\mathbf{Z}^{rr}, \mathbf{F}^{rr}$	$\mathbf{Z}^{rs}, \mathbf{F}^{rs}$	$\mathbf{Z}^{rn}, \mathbf{F}^{rn}$
From Region s	$\mathbf{Z}^{sr}, \mathbf{F}^{sr}$	$\mathbf{Z}^{ss}, \mathbf{F}^{ss}$	$\mathbf{Z}^{sn}, \mathbf{F}^{sn}$
Regional Use of Domestic Products	$\mathbf{Z}^{nr}, \mathbf{F}^{nr}$	$\mathbf{Z}^{ns}, \mathbf{F}^{ns}$	

8.7.2 Double-Entry Bi-Regional Input–Output Tables (DEBRIOT)

Researchers in the Netherlands have developed an extensive set of regional (and interregional) input–output tables for that country. [See Oosterhaven, 1981, for work up until the 1980s, Boomsma and Oosterhaven, 1992, for a description of the DEBRIOT approach and Eding *et al.*, 1999, for the procedure when one starts with regional Make (supply) and Use tables.] Most of these are of the two-region sort – the region of interest (r) and the rest of the country (s). The primary object is to estimate an intraregional transactions matrix, $\mathbf{Z}^{rr}$ (the elements in the light gray area in Table 8.2). Toward that end, the procedure requires estimates of regional sales to domestic markets and regional use of domestic products, the elements in the dark gray areas. As a result of the two-region nature of the accounts, the approach also generates $\mathbf{Z}^{rs}, \mathbf{Z}^{sr}$, and $\mathbf{Z}^{ss}$ (the matrices in the medium gray areas).[20]

The approach is based on observation that firms in the Netherlands are generally better informed about the spatial destination of their sales than they are about the spatial origin of their purchases. Thus attention is directed not primarily to purchase data (as is the case with regional purchase coefficients) but to information on the sectoral and spatial destination of sales. Also, there is an almost total absence of quotient methods, and hence the inherent upward bias associated with the no cross-hauling feature of those methods may be mitigated.

These major components of DEBRIOT are:

1. $\mathbf{Z}^{nr} = \left[z_{ij}^{nr}\right]$, the regional *domestic use matrix* for region r. Here the superscript n indicates the nation, i.e., $r + s$.[21] So z_{ij}^{nr} is the use by sector j in region r of i goods produced domestically, in either r or s. Estimate the regional *technology* matrix (transactions) by applying national technology coefficients, $(z_{ij}^{\cdot n}/x_j^n)$, assumed

[20] In all cases, $\mathbf{F}$ (a matrix) is used to allow for disaggregation of final demand, including possibly households distinguished by income brackets, etc. In the simplest of models, we would have $\mathbf{f}$ (a vector). In this brief summary we use $\mathbf{Z}$ (transactions) matrices. In a commodity-by-industry accounting setting, one would deal with $\mathbf{U}$ (Use) matrices.

[21] Recall that notation such as $\mathbf{Z}^{rs}$ describes transactions between sectors in two spatially distinct regions, r and s. Here $\mathbf{Z}^{nr}$ describes purchases by sectors in r from the national pool of domestic outputs, some of which come from sectors in r and some from sectors in s.

known, to regional total use, also known: $z_{ij}^{\cdot r} = (z_{ij}^{\cdot n}/x_j^n)x_j^r$.[22] Next, it is assumed that each $z_{ij}^{\cdot r}$ can be broken down into its domestic and foreign components: $z_{ij}^{\cdot r} = z_{ij}^{nr} + m_{ij}^r$. This may be done by using the national import coefficients, m_{ij}^n, to reduce $z_{ij}^{\cdot r}$ by the *national* proportion of imports of i to total use of i

$$z_{ij}^{nr} = z_{ij}^{\cdot r} - (m_{ij}^n/z_{ij}^{\cdot n})z_{ij}^{\cdot r} = [1 - (m_{ij}^n/z_{ij}^{\cdot n})]z_{ij}^{\cdot r}$$

Construct $\mathbf{Z}^{ns} = \left[z_{ij}^{ns}\right]$ similarly from information on rest-of-nation output, x_j^s, and construct $\mathbf{F}^{nr}$ and $\mathbf{F}^{ns}$ similarly.

2. $\mathbf{Z}^{rn} = \left[z_{ij}^{rn}\right]$, the regional *domestic sales matrix* for region r. Survey to find an *overall regional domestic export coefficient* for sector i in region r, $t_i^{rs} = (z_{i\cdot}^{rs} + f_{i\cdot}^{rs})/(x_i^r - e_i^r)$. The denominator is total domestic sales of i made in r, the numerator is the total amount of i made in r that went to s, and so this ratio is the *proportion* of r's total domestic sales of i that went to s. Similarly, $(1 - t_i^{rs}) = t_i^{rr}$ is the *proportion* that remained in r.[23] Using t_i^{rs}, estimate *nonsurvey region r domestic sales coefficients* as a weighted average of the demand structure of the rest of the country and the region of interest:

$$s_{ij}^{rn} = t_i^{rs}[z_{ij}^{ns}/(z_{i\cdot}^{ns} + f_{i\cdot}^{ns})] + (1 - t_i^{rs})[z_{ij}^{nr}/(z_{i\cdot}^{nr} + f_{i\cdot}^{nr})]$$

The denominator in the first bracketed expression is the total amount of i from all domestic sources $(r + s)$ that is demanded in region s, and so the expression in brackets is the proportion of that total used by sector j in s. Multiplication by t_i^{rs} (the proportion of r's domestic sales of i that went to s) generates an estimate of the proportion of domestically supplied i from r used by j in s. The second bracketed term on the right represents the proportion of the total amount of i from all domestic sources used by sector j in r. Thus the sum on the right-hand side represents the proportion of domestically supplied i from r used by j in the nation. Then the regional sales to the domestic market are estimated as $z_{ij}^{rn} = s_{ij}^{rn}(x_i^r - e_i^r)$. Construct $\mathbf{Z}^{sn}, \mathbf{F}^{rn},$ and $\mathbf{F}^{sn}$ similarly.

3. Construction of $\mathbf{Z}^{rr}, \mathbf{Z}^{rs}$, and $\mathbf{Z}^{sr}$. Note, initially, that

$$z_{ij}^{rr}(\max) = \min(z_{ij}^{nr}, z_{ij}^{rn})$$

from which

$$z_{ij}^{rs}(\min) = z_{ij}^{rn} - z_{ij}^{rr}(\max)$$

and

$$z_{ij}^{sr}(\min) = z_{ij}^{nr} - z_{ij}^{rr}(\max)$$

22 Boomsma and Oosterhaven use $z_{ij}^{\cdot r} = [z_{ij}^{\cdot n}/(x_j^n - v_j^n)](x_j^r - v_j^r)$ to account for Round's fabrication effect (see above), but that detail need not concern us at this point.

23 These have been called *regional sales coefficients* (*RSC*), in contrast to *regional purchase coefficients* (*RPC*) that were discussed in Chapter 3 and earlier in this chapter.

Survey "important" cells (again, however defined) to find cell-specific domestic export coefficients t_{ij}^{rs}. Then $z_{ij}^{rs} = t_{ij}^{rs} z_{ij}^{rn}$, from which, by subtraction, $z_{ij}^{rr} = z_{ij}^{rn} - z_{ij}^{rs} = (1 - t_{ij}^{rs}) z_{ij}^{rn}$. For all other cells in $\mathbf{Z}^{rr}$, z_{ij}^{rr}(max) is decreased until it reaches a level that is consistent with the overall domestic export coefficient t_i^{rs}, from which z_{ij}^{rs} and z_{ij}^{sr} can then also be found.[24]

4. Finally, $\mathbf{Z}^{ss} = \mathbf{Z}^{nn} - \mathbf{Z}^{rr} - \mathbf{Z}^{rs} - \mathbf{Z}^{sr}$.

8.7.3 The Multiregional Input–Output Model for China, 2000 (CMRIO)

Early work on national input–output tables in China apparently began in the 1960s. Starting in 1987, the National Bureau of Statistics (NBS) produced survey-based tables every five years (1987, 1992, 1997, etc.), and regions (provinces) construct their own regional input–output tables (except Tibet and Hainan), with the same sector classifications and for the same years as the national tables. (See Chen, Guo and Yang, 2005. Also see Polenske and Chen, 1991, for a history of Chinese input–output work up to that time.)

There also was some early work on connected-regional models, with three regions and ten sectors. A much more ambitious multiregional model for China was produced for the year 2000 (CMRIO), with eight regions (provinces) and four levels of aggregation – three, eight, 17, and 30 sectors. [The main references are Institute of Developing Economies–Japan External Trade Organization (IDE-JETRO), 2003 and Okamoto and Ihara, 2005.[25] The IDE-JETRO publication contains results for three, eight, and 17 sectors; the 30-sector data are on an accompanying CD-ROM disc.] Details concerning the construction of this ambitious data set are contained in Okamoto and Zhang (2003), who note that regional economic disparity "… has become the main topic for Central government of China" (p. 9), and this underscored the need for a multiregional input–output approach. Similar observations are made in Okamoto and Ihara (p. 201)[26]:

> Recent studies on the regional development of China have shown that regional disparity has become a significant problem and this has led many policy makers and researchers to pay attention to the issue of how we might develop the underdeveloped regions of the nation. It should be noted, however, that most of the approaches to date have focused on the situations in specific regions, rather than considering interregional interdependency. Therefore, in order to add something substantial to these previous studies, we felt the need to consider the interregional feedback effects and/or spatial interactions quantitatively. This was the main reason why we compiled a full-scale interregional input-output model for China as a useful analytical tool for considering spatial economy.

[24] Details abound, and can be found in Boomsma and Oosterhaven, 1992.

[25] IDE in Tokyo, Japan, was founded in 1960 as an organization under the jurisdiction of the then Ministry of International Trade and Industry (MITI; now the Ministry of Economy, Trade and Industry) to act as a social science institute for basic and comprehensive research activities in the areas of economics, politics, and social issues in developing countries and regions. In July 1998 it merged with the Japan External Trade Organization (JETRO) and became IDE-JETRO. It is now a major source of input–output data assembly and analysis.

[26] The authors use "interregional" in a general sense. The implementation is a hybrid approach but essentially in the "multiregional" style (estimation of interregional transactions z_i^{rs}, not z_{ij}^{rs}).

The compilation consisted of three broad phases of essentially two steps each:

1. Collection and estimation of exogenous data. Provincial input–output data were collected (these are unpublished data that cannot be accessed by foreigners), with attempts to check for consistency with national data. Estimates were required for provincial value added, final demand, and foreign trade. The final result of this phase is a collection of regional input coefficients for each region.
2. Estimation of interregional commodity flows. Survey data were collected from over 500 "important" enterprises. For other commodities, estimates were generated using a Leontief–Strout gravity model approach (section 8.6.1), complemented by superior data, where available. This phase generates sets of interregional trade coefficients.
3. Compilation of the multiregional model. Here the results of the two earlier phases were joined together and the (inevitable) discrepancies reconciled.

The various chapters in Okamoto and Ihara (2005) explore a number of applications of the CMRIO model with the goal of analyzing such important regional economic phenomena as interregional multipliers, feedbacks and spillovers and spatial linkages.

8.8 International Input–Output Models

8.8.1 Introduction

The notion of extending the several-*region* input–output model framework to several *countries* apparently first appeared in Wonnacott (1961), who created a connected Canada–USA two-country model. In what follows we explore several more elaborate applications of this idea – involving more than two countries – including examples for Asia, the European Community, and other many-country models. The model structures follow exactly the logical lines of the interregional or multiregional cases. In some instances, data collection is made easier because of the "national" nature of the "regions." For example, while "export" and "import" figures are often sketchy or nonexistent for regions, they are often available, in various forms, for a nation's external trade.[27]

8.8.2 Asian International Input–Output Tables

The idea of modeling the input–output connections among Asian nations became attractive to scholars in that area because of the emerging interdependence of many Asian economies. Initial work was carried out by researchers at the Institute of Developing Economies (IDE) in Japan. Their first attempt at an "international" input–output table began in 1965; it covered six "mega-regions" (North America, Europe, Oceania, Latin America, Asia, and Japan).[28] In various publications, these and subsequent tables have

[27] There can be many compatibility issues with regard to a country's export and import data – for example, distinguishing competitive vs. noncompetitive imports, valuation of imports at *ex customs* prices and exports at producers' prices.

[28] This can be viewed as an early example of a "global" or "world" model. See section 8.8.5, below.

also been labeled "multinational" and "multilateral." These are more appropriate labels, since the work builds on an MRIO (Chenery–Moses) framework.

A comprehensive history of the IDE work on international input–output tables can be found in Institute of Developing Economies (2006a), especially Part 1: "Compilation of the Asian international input–output table." (This material is also covered in the "Introduction" in Furukawa, 1986.) The historical overview encompasses three phases.

The first phase (1973–1977) launched comprehensive development of an international input–output structure for East and Southeast Asian countries – three national tables (Indonesia, Singapore, and Thailand) and three "bilateral" tables (Korea–Japan, USA–Japan, and Philippines–Japan) were produced.

The second phase (1978–1982) encompassed construction of a 1975 multilateral table among ASEAN (Association of Southeast Asian Nations) countries, Japan, Korea, and the USA. This included estimation of national tables for 1975 where necessary (and updates for Malaysia, Philippines, Singapore, and the USA and construction of bilateral tables for Indonesia–Japan, Thailand–Japan, and Korea–Japan). Finally, these were linked together as a single international (multilateral) table for 1975. This work was completed in 1983.

In the third phase (1988–present) an international table was created for 1985, now including China and Taiwan. Since then IDE has created multilateral tables every five years – thus far for 1990, 1995, and 2000 – with 10 countries and 7-, 24-, and 76-sector levels of aggregation.[29]

As might be imagined, the data compilation problems are enormous. For example, the 10 different national tables exhibit a number of differences. These tables must all be made "consistent" in order to be included (as on-diagonal blocks) in the overall multinational table.[30] In estimating the international transactions, export vectors and import matrices are created in a very detailed set of procedures (and then converted to producers' prices). Import statistics are relied on more heavily than export, because import data are more carefully collected for customs duties in each country. Details can be found, for example, in IDE-JETRO (2006a), Part 1, III "Linking of the tables."

This work provides an extremely rich data set for empirical studies that make use of analytical methods that depend on input–output data sets. Among these are linkage analysis (both sectoral and spatial) and other techniques designed to assess relative importance of sectors (or regions). These topics are covered in some detail in Chapter 12. Representative examples include Sano and Osada (1998) [sector linkages]; Meng *et al.* (2006) [linkages for 1985, 1990, 1995, and 2000 for sectors within each country and across countries] and Kuwamori and Meng (2006) [sectoral linkages over time within individual countries, evolution over time of total intermediate inputs (sectoral input

[29] The countries are: China, Indonesia, Japan, Korea, Malaysia, Philippines, Singapore, Taiwan, Thailand, and the USA. The data appear in Institute of Developing Economies (2006b).

[30] From IDE-JETRO (2006a, p. 15): "… one of the most complicated, nerve-racking tasks of compilation is the adjustment of national tables to conform to a common format."

structures) and total final demands (sectoral demand structures), linkages between countries, the impact of Beijing Olympic Games-related investments on regional economic growth in China]. Using the 1990 and 2000 Asian input–output data sets, Kuwamori (2007) examines the relative importance of each of the 10 countries on each of the others, as well as specific industries in those countries, via the "hypothetical extraction" process (also Chapter 12). The emerging influence of the Chinese economy is made clear through comparison of some of the results from the 1990 and 2000 data.

8.8.3 "Hybrid" Many-Region Models for the EC

The formation of the European Community (EC) in 1971 [preceded by the European Economic Community (EEC) from 1958] generated an interest in and need for consistent economic data on each member country's economic activities, not only internal transactions but also intercountry connections. Van der Linden and Oosterhaven (1995) address the need for a consistent set of intra- and intercountry input–output tables for the EC in order to address a variety of important policy issues, such as interregional and intercountry income spillovers.

Presently Eurostat (the statistical office of the EC) produces consolidated tables for the EC as a whole, including what amounts to $\mathbf{Z}^{rr}$ and $\mathbf{x}^r$ for each of the member states.[31] Additional information, including $\mathbf{Z}^{\cdot r}$ (where the "." indicates shipments from all other EC countries) and other import data, are also available. From these, the authors create a kind of many-region (or many-nation) model for the EC that lies between the IRIO and MRIO styles.[32] The presence of $\mathbf{Z}^{rr}$ and $\mathbf{x}^r$ permits calculation of intracountry input coefficient matrices of the IRIO type, namely $\mathbf{A}^{rr} = \mathbf{Z}^{rr}(\hat{\mathbf{x}}^r)^{-1}$. From data on imports, the authors estimate $c_i^{sr} = m_i^{sr}/m_i^{\cdot r}$; these are used across rows of $\mathbf{Z}^{\cdot r}$ to approximate $\mathbf{Z}^{sr}(s \neq r)$ in standard MRIO fashion, namely, $\mathbf{Z}^{sr} = \hat{\mathbf{c}}^{sr}\mathbf{Z}^{\cdot r}$. There are problems associated with accounting for services and with the kinds of prices in which the data are available (e.g., producers' vs. CIF or ex-customs prices). Also, there are discrepancies with import and export data that are created by the use of the MRIO approach; for example, initial estimates of country r's total exports of commodity i to other EC countries generally differ from the figure found by summing imports of i from r to each of the countries. The authors use an RAS balancing approach to deal with these issues.

An illustration of the kinds of questions that can be addressed with these intercountry EC models is provided by Hoen (2002), who uses these EC tables as the starting point for his input–output analysis of the economic effects of European integration. For example, he examines various multipliers and spillovers, and he presents a decomposition of value-added growth, among others, all based on the input–output data. For these purposes, however, he requires a set of data in constant prices, not current (as are generated in the van der Linden and Oosterhaven work). To achieve this Hoen employs an RAS approach and compares his results with those from the usual "double deflation"

[31] See Eurostat, 2002, for a thorough discussion. Some years do not include tables for all member states. See van der Linden and Oosterhaven, 1995, or Hoen, 2002, for details.

[32] This work builds on Schilderinck, 1984, where the first attempt at a consistent set of connected-country tables for the EC was presented.

Table 8.3 Structure of the TIIO Model

	ASEAN5	C1	···	C7	J1	···	J8	EA	USA
ASEAN5									
C1									
⋮									
C7									
J1									
⋮									
J8									
EA									
USA									

procedure. In fact, he suggests that double deflation can be viewed as a special case of a more general RAS approach.

8.8.4 China–Japan "Transnational Interregional" Input–Output (TIIO) Model, 2000

This is another ambitious IDE-JETRO undertaking. It is a ten-sector model that combines a "multinational" character – China, Japan, ASEAN5 (Indonesia, Malaysia, the Philippines, Singapore, and Thailand), East Asia (Korea and Taiwan) and the USA – with *regional* disaggregations of China into seven regions and Japan into eight regions. Thus there are 18 geographic areas; some are true sub-national regions (the 15 in China and Japan), one is a nation (USA) and two are multinational areas (ASEAN5, East Asia). (Primary references are Inomata and Kuwamori, 2007, and Development Studies Center, IDE-JETRO, 2007.) This is known as the "Transnational Interregional Input–Output (TIIO)" model.

The tables are compiled from the 2000 Asian data (section 8.8.3) and the interregional tables for China (section 8.7.3) and Japan. As might be expected, there were many issues regarding data compatibility, and many assumptions were required to translate national trade data to the regional level (in the cases of China and Japan). The overall structure of this ambitious project is indicated in Table 8.3. Each cell contains a 10×10 transactions (or coefficients) matrix. We explore details of estimations in the lighter shaded area – namely Chinese exports to Japan at the regional level in both countries. Estimation of elements in the darker shaded area follows the same approach.

Chinese Exports to Japan for Intermediate Demand We use Japan (J) as an illustration of the external country – the lighter shaded portion of Table 8.3. Procedures for ASEAN5, East Asia, and the USA do not involve regional breakdowns for the receiving (importing) area. The object here is to include region and sector specificity for the Chinese origins and Japanese destinations of Chinese exports to Japan, as indicated in Table 8.4, where $\mathbf{Z}^{rs} = [z_{ij}^{rs}]$ for $i, j = 1, \ldots, 10$.

Table 8.4 China-to-Japan Intermediate Transactions in TIIO

		Japanese Region				
		J1	⋯	Js	⋯	J8
Chinese Region	C1	$\mathbf{Z}^{11}$	⋯	$\mathbf{Z}^{1s}$	⋯	$\mathbf{Z}^{18}$
	⋮	⋮		⋮		⋮
	Cr	$\mathbf{Z}^{r1}$	⋯	$\mathbf{Z}^{rs}$	⋯	$\mathbf{Z}^{r8}$
	⋮	⋮		⋮		⋮
	C7	$\mathbf{Z}^{71}$	⋯	$\mathbf{Z}^{7s}$	⋯	$\mathbf{Z}^{78}$

The following are known from trade data ($\breve{m}$ indicates Japanese import data, $\frown{e}$ indicates Chinese export data):

z_{ij}^{CJ} = total Chinese exports of good i to Japanese sector j (= total Japanese imports by sector j of good i from China),
$\breve{m}_{i.}^{Cs}$ = Japanese region s imports of i from China,
$\breve{m}_{i.}^{CJ} = \sum_{s=1}^{8} \breve{m}_{i.}^{Cs}$ = total Japanese imports of i from China,
$[(\breve{m}_{i.}^{Cs}/\breve{m}_{i.}^{CJ}) \times 100]$ = percentage of Japanese imports of i from China that goes to region s in Japan (comparable to the $c_{i.}^{rs}$ data in MRIO models),
$\overset{\frown}{e}_{i.}^{rJ}$ = Chinese region r exports of i to Japan,
$\overset{\frown}{e}_{i.}^{CJ} = \sum_{r=1}^{7} \overset{\frown}{e}_{i.}^{rJ}$ = total Chinese exports of i to Japan,
$[(\overset{\frown}{e}_{i.}^{rJ}/\overset{\frown}{e}_{i.}^{CJ}) \times 100]$ = percentage of Chinese exports of i to Japan that comes from region r in China.

Assumptions:

1. Each sector j in region s in Japan gets $[(\breve{m}_{i.}^{Cs}/\breve{m}_{i.}^{CJ}) \times 100]$ percent of its i from China. (This is the standard MRIO model assumption.) That is,

$$\tilde{z}_{ij}^{Cs} = (\breve{m}_{i.}^{Cs}/\breve{m}_{i.}^{CJ}) z_{ij}^{CJ}$$

Suppose $[(\breve{m}_{i.}^{Cs}/\breve{m}_{i.}^{CJ}) \times 100] = 12$ (meaning that 12 percent of the input of i for *each* sector in region s in Japan comes from China); then if $z_{ij}^{CJ} = 2000$, $\tilde{z}_{ij}^{Cs} = 240$.

2. Each region r in China contributes $[(\overset{\frown}{e}_{i.}^{rJ}/\overset{\frown}{e}_{i.}^{CJ}) \times 100]$ percent of China's exports of i to Japan. Then

$$\tilde{\tilde{z}}_{ij}^{rs} = (\overset{\frown}{e}_{i.}^{rJ}/\overset{\frown}{e}_{i.}^{CJ})\tilde{z}_{ij}^{Cs} = (\overset{\frown}{e}_{i.}^{rJ}/\overset{\frown}{e}_{i.}^{CJ})(\breve{m}_{i.}^{Cs}/\breve{m}_{i.}^{CJ}) z_{ij}^{CJ}$$

Suppose $[(\overset{\frown}{e}_{i.}^{rJ}/\overset{\frown}{e}_{i.}^{CJ}) \times 100] = 10$; this means that 10 percent of Chinese exports of i to Japan come from region r in China. Then $\tilde{\tilde{z}}_{ij}^{rs} = 24$.

The authors recognize that this is admittedly a strong assumption (Development Studies Center, 2007, p. 67):

> In general, it seems unlikely to assume that the proportion of inputs of [sector 1] in region 1 of China to the inputs as a whole in the industry related to daily lives in Hokkaido [Japan region 1] is identical to the proportion of the inputs in the industry related to daily lives in the Kanto Region [Japan Region 8]. Even so, since there is no information available which proves that this is "not true", estimation has been made under the assumption [that the data on each Chinese region's input supply proportions] are applicable to all [eight Japanese regions].

This produces the estimate of one element, $\tilde{z}_{ij}^{rs}$, of the 100 in $\mathbf{Z}^{rs}$. Similar calculations are needed for each additional element in $\mathbf{Z}^{rs}$ and for each of the remaining 55 matrices (each 10×10) in Table 8.4.

Applications This data set has prompted a large number of studies of international linkages, including feedbacks and spillovers (Chapter 3), for individual countries as a whole (all industries) as well as for individual sectors. Some of these results appeared in Inomata and Sato (2007) and a large collection of such studies can be found in Inomata and Kuwamori (2007).[33]

8.8.5 Leontief's World Model

Another example illustrating expansion from a "multiregional" to "multinational" input–output perspective is to be found in Leontief's world model[34]. This huge project from the early 1970s was sponsored by the United Nations as part of its search for "… possible alternative policies to promote development while at the same time preserving and improving the environment" (United Nations, 1973, p. 2).

The final version consisted of 15 regions (four advanced industrial countries, four centrally planned economies and two groups of developing countries (three resource-rich and four resource-poor), each with 48 sectors, including eight exhaustible resources as well as eight types of major pollutants and five types of abatement activities (since the motivation was one of environmental impacts). Data were assembled for the base year of 1970, and projections were made to 1980, 1990, and 2000. National input–output tables formed the basis of the intraregional data sets; various accounting practices and sectoring schemes created many consistency issues.

For the interregional data, Leontief created "world trade pools" to model trading relationships for each traded good. For each good and each region there are two sets of parameters: import coefficients and export shares. For good i and region r, an element of the latter specifies the proportion of the total amount of world exports of good i that is provided by region r to the world pool of good i, $e_i^r = x_i^{r\cdot} / \sum_{q=1}^{p} x_i^{q\cdot}$. The import

[33] A paper by Oosterhaven and Stelder (2007) contains an extensive and informative comparison of results from four "hybrid" models with the IDE-JETRO Asian 2000 table. The four alternative approaches reflect the differing kinds of national import and export data that may be available in real-world situations – such as with or without separate import matrices (in *ex customs* prices), with or without export matrices (in producers' prices), and using RAS procedures at various stages in the process.

[34] This is outlined in Leontief, 1974 (his Nobel Memorial Lecture). The evolution of the model is presented in Fontana, 2004; see also Duchin, 2004, for more background.

shares indicate the volume of competitive imports as a fraction of domestic production of the same good, $m_i^r = x_i^{\cdot r}/x_i^r$. These parameters are estimated, based on observed data (the x's) and expert opinion.

> The world pool idea avoided the need for building an input–output international trade model, with country-to-country flows for each commodity . . . [W]ith Leontief's idea of world pools . . . nothing is required to be known about the bilateral relations between regions. (Fontana, 2004, p. 34.)

In effect, this approach carries the simplification of the IRIO model one step beyond the MRIO formulation: z_{ij}^{rs} (IRIO) $\rightarrow$ $z_{i\cdot}^{rs}$ (MRIO) $\rightarrow$ $z_{i\cdot}^{r\cdot}$ and $z_{i\cdot}^{\cdot s}$ (World Model).

Results from this project were published in Leontief, Carter and Petri (1977). Other applications include Leontief, Mariscal and Sohn, 1983b, Leontief *et al.*, 1983a and Leontief and Duchin, 1983. However, as Duchin (2004) noted, the model generated little in the way of long-term interest.

> Among economists, even those in the IO community have paid relatively little attention to the World Model . . . [T]he descendant of Leontief's World Model was last used for research completed in the early 1990s (Duchin and Lange, 1994), and the team that did the analysis has dispersed (p. 59).

In Duchin and Lange (1994) the world model framework was employed to examine alternative environmental futures for the planet. It consisted of 16 world regions and about 50 sectors. A sense of the broad-brush approach necessary in such an ambitious world model is given in Appendix 8.1, where 189 countries are grouped into the 16 geographical classifications used in the model.[35]

Duchin (2005) presents a generalization of the World Model in a linear programming format that is designed to be particularly applicable for analyzing scenarios dealing with environmental impact and sustainable development. As Fontana (2004, p. 37) notes "The Leontief world model was a stepping stone for explorers of the long-term future of the world economy."

8.9 The Reconciliation Issue

In section 7.2 we noted that problems can arise in constructing survey-based interindustry transactions tables when the row total for a sector differs from the column total for that same sector. This happens also in hybrid approaches to both updating and regionalization. Since one common approach to reconciliation uses an RAS approach, this discussion was postponed until we had introduced the RAS technique in its more usual updating or regionalization role.

Some input–output tables (especially at the regional level) have been constructed exclusively on the basis of information on purchases by sectors in the economy. A sample of establishments in each sector are asked to identify the magnitudes of their inputs, by sector and by region – or at least whether the input came from inside the region in question or was imported from outside that region. This is sometimes known as the "purchases only" or "columns only" approach, since the transactions table (and

[35] The complete list of countries and their geographic assignments to world regions can be found in Appendix C, "World Model Geographic Classification," in Duchin and Lange (1994).

hence the direct-input coefficients matrix) is compiled column by column. It depends on information from establishments regarding the distribution of their costs. (This was used in constructing the 496-sector Philadelphia table for 1959; see Isard and Langford, 1971.) Similarly, a "sales only" or "rows only" procedure depends entirely upon information on the magnitudes of sales from a particular sector to all other regional sectors, and to final-demand purchasers. This relies on information from establishments regarding the distribution of their products. [For a study that used this approach, see Hansen and Tiebout, 1963, and recall that DEBRIOT (section 8.7.2) emphasizes sales over purchases information.]

Usually, there will be some (but not complete) information on purchases and some (but also not complete) information on sales – for example, from a questionnaire in which firms are asked for data on both sales and purchases. Thus, for many cells there may be two estimates of the z_{ij}^r transaction. If one has independent estimates of regional total gross outputs, x_j^r, from published sources, this of course means that there will be two estimates of the regional direct-input coefficient. The issue then is one of reconciling the two estimates. (Early examples of empirical studies using both row and column information include Bourque *et al.*, 1967; Beyers *et al.*, 1970; Bourque and Conway, 1977; Miernyk *et al.*, 1967 and Miernyk *et al.*, 1970.)

Often, the reconciliation is made entirely on the basis of the judgment of the researchers, reflecting their knowledge of the regional economy and comparisons with national coefficients; Bourque *et al.* (1967) provides one such example. Building on the general discussion in Miernyk *et al.* (1970), in which an attempt was made to estimate the relative accuracy (reliability) of various pieces of information, Jensen and McGaurr (1976) propose a two-stage procedure. Let the two transactions estimates for the i, jth cell be r_{ij} and c_{ij}, from the "rows-only" and "columns-only" information. On the basis of knowledge of sampling procedures and other features of the data and of probable sources of error, a pair of what Jensen and McGaurr termed reliability weights are chosen for the two estimates. Let k_{ij} denote this weight for the rows-only estimate ($k_{ij} \geq 0$), then $(1 - k_{ij})$ will be the weight for the columns-only estimate. Then, a first approximation to the reconciled transactions estimate for the i, jth cell is found as the simple weighted sum $z_{ij}^1 = k_{ij} r_{ij} + (1 - k_{ij}) c_{ij}$. The superscript 1 represents the fact that this is a first estimate. For example, if one believed that a rows-only estimate r_{ij} was "almost" completely accurate, its k_{ij} might be set at 0.9; if, in the judgment of the researchers, the row and column estimates for a particular cell were equally likely to be correct, k_{ij} would be 0.5 for that cell, and so on.

In addition to the total output vector, $\mathbf{x}^r$, suppose that independent estimates have been made of the magnitudes of final-demand purchases from each sector, so the final-demand column vector is known, and also assume that there are estimates of all value-added payments by each sector (including imports), so the value-added row vector, $\mathbf{va}^r$, is also known.[36] Then the total value of interindustry transactions is given by $T^r = \mathbf{i}'(\mathbf{x}^r - \mathbf{f}^r) = \sum_i (x_i^r - f_i^r)$, or, equivalently, by $T^r = (\mathbf{x}^r - \mathbf{va}^r)\mathbf{i} = \sum_j (x_j^r - va_j^r)$. It is then

[36] Here we resort to using va_j^r for value added in sector j in region r because v_j^r will be needed for total intermediate inputs in the RAS balancing technique that follows.

necessary to check the total of the estimated transactions, $Z^1 = \sum_i \sum_j z_{ij}^1$, against this (independently estimated) total figure, T^r. If, as is very likely, these are not equal, each z_{ij}^1 is scaled upward or downward through multiplication by T^r/Z^1. This produces a second set of estimates of reconciled transactions, $z_{ij}^2 = z_{ij}^1[T^r/Z^1]$. These estimates are consistent in the aggregate, in that $\sum_i \sum_j z_{ij}^2 = T^r$. This concludes stage one.

While the transactions z_{ij}^2 have now been adjusted so that they sum to the proper aggregate flow, they must also be consistent with the *individual* row and column sums. This is where RAS comes in. Since x_i^r and f_i^r have been independently estimated, then total intermediate output for each sector, u_i^r, is found as $u_i^r = x_i^r - f_i^r$. Similarly, given estimates of va_j^r, then total intermediate input for each sector, v_j^r, is found as $v_j^r = x_j^r - va_j^r$. The issue then is whether $\sum_j z_{ij}^2 = u_i^r$, for each sector $(i = 1, \ldots, n)$ and also whether $\sum_i z_{ij}^2 = v_j^r$, for each sector $(j = 1, \ldots, n)$. In general, not all of these constraining equations will be met, and so the estimates in z_{ij}^2 must be further adjusted to conform to the marginal information for each row and each column. This is exactly the kind of problem for which the RAS technique is suited, and it is the procedure that is suggested by Jensen and McGaurr. This is stage two of the adjustment. The result will be a third and final set of transactions estimates z_{ij}^3. Given the estimates of x_i^r the direct input coefficients can then be estimated.[37]

This approach has been discussed because it represents one formalized way of attempting to incorporate subjective judgments (via the reliability weights) and also a certain amount of objective structure (via the RAS adjustments) in the reconciliation procedure. Many researchers have incorporated alternative approaches to reconciling conflicting estimates. An example can be found in step 3 of the DEBRIOT procedure (section 8.7.2), above, and there is considerably more detail in Boomsma and Oosterhaven (1992).

An entirely different approach was suggested by Gerking, in the context of a stochastic view of input–output models (Gerking, 1976a, b). He proposes that coefficients can be estimated and that the reconciliation problem can be addressed using regression techniques (Gerking, 1976c, 1979b). This generated a good deal of critical comment and response in the literature. (For example, Brown and Giarratani, 1979; Miernyk, 1976, 1979; Gerking, 1979a, c.) The reconciliation issue is far from settled; the range of possibilities from wholly subjective to entirely mathematical is very wide indeed.

8.10 Summary

In this chapter we have looked at some options that are available for estimating a table of regional input–output coefficients when a full matrix of regional transactions is not

[37] If independent estimates of f_j^r and va_j^r are not available, then one can employ the same procedure as outlined above on an expanded transactions table; this would include estimates from firms on not only their interindustry transactions but also on sales to final-demand sectors and purchases from value-added sectors. In this case the first reconciliation would scale all transactions so that their total was $\sum_i x_i^r (= \mathbf{i}'\mathbf{x})$, and the second reconciliation would compare row and column sums against each x_i^r and x_j^r. This is, in fact, the procedure used by Jensen and McGaurr (1976, 1977) in their discussion and in their empirical work.

available. In the regional case, location-quotient like procedures and regional purchase coefficient information take advantage of comparative economic data on the region vs. the nation of which it is a part. In addition, the RAS procedure is as applicable to the regionalization problem as it was to the updating problem of the last chapter. And, again, it is usual to find a combination of partial-survey information, expert opinion and RAS (or RAS-like) techniques blended into a hybrid approach that generates superior results. We also explored some real-world applications of these techniques in both sub- and super-national studies.

Appendix 8.1 Geographical Classifications in the World Input–Output Model

The 16 world regions covering 189 countries in the World Input–Output Model are (Duchin and Lange, 1994):

High-income North America (5)
Newly industrializing Latin America (5)
Low-income Latin America (40)
High-income Western Europe (23)
Medium-income Western Europe (8)
Eastern Europe (7)
Former Soviet Union
Centrally planned Asia (3)
Japan
Newly industrializing Asia (7)
Low-income Asia (16)
Major oil producers (15)
Northern Africa and other Middle East (16)
Sub-Saharan Africa (34)
Southern Africa (5)
Oceania (3)

Problems

8.1 The economy of the Land of Lilliput is described by the following input–output table:

	Interindustry Transactions		Total Outputs
	A	*B*	
A	1	6	20
B	4	2	15

Land of Brobdingnag is described by another input–output table:

	Interindustry Transactions		Total Outputs
	A	*B*	
A	7	4	35
B	1	5	15

The economy of the distant land of the Houyhnhnms is described by yet another input–output table:

	Interindustry Transactions		Total
	A	*B*	Outputs
A	20	30.67	100
B	2.86	38.33	15

a. Compute the vectors of value-added, intermediate inputs, final-demand, and intermediate outputs for each economy.
b. A Lilliputian economist is interested in examining the structure of the Brobdingnagian economy. Likewise, a Brobdingnagian economist is interested in examining the structure of the Lilliputian economy. However, each economist only has available to him the value-added, final-demand, and total-output vectors for the foreign economy. Each economist knows the RAS modification procedure and uses it with the technical coefficients matrix of his own economy serving as the base **A** matrix. Which of the two economists calculates a better estimate of the foreign economy's technical coefficients matrix in terms of mean absolute deviation (all elements of **A**)?
c. An economist in the distant land of the Houyhnhnms learned of the two other economies from a world traveler. He becomes interested in the economic structures of these foreign lands but is only able to obtain the final-demand, value-added, and total-output vectors for each economy from the world traveler. The economist uses RAS with his own country's **A** matrix as a base to estimate the interindustry structure of the two distant lands. Which economy does he estimate more accurately in terms of a mean absolute deviation? Do you notice anything peculiar about the comparative structures of the Lilliputian, Brobdingnagian, and Houyhnhnm economies?
d. The Land of Lilliput plans to build a new power plant which will require the following value of output (in millions of dollars) from each of the economy's industries (directly, so it can be thought of as a final demand presented to the Lilliputian economy) of $\mathbf{f} = [100 \quad 150]'$. How accurate, measured as an average mean absolute deviation, is the Houyhnhnms' estimate of the total industrial activity (output) in the Lilliputian economy required to construct this power plant?

8.2 Suppose the economies given in problem 8.1 are really three-sector economies where the economy of the Land of Lilliput is described by the following input–output table:

	Interindustry Transactions			
	A	*B*	*C*	Total Outputs
A	1	6	6	20
B	4	2	1	15
C	4	1	1	12

The economy of the neighboring land of Brobdingnag is described by another input–output table:

	Interindustry Transactions			
	A	*B*	*C*	Total Outputs
A	7	4	8	35
B	1	5	1	15
C	6	2	7	30

The economy of the distant land of Houyhnhnms is described by yet another input–output table:

	Interindustry Transactions			
	A	*B*	*C*	Total Outputs
A	5.5	33	33	1,101
B	22	11	5.5	82.5
C	22	5.5	5.5	66

Solve parts a, b, and c of problem 8.1 for these new economies.

8.3 Consider the following input–output table for Region 1:

	A	*B*	Total Outputs
A	1	2	10
B	3	4	10

We are interested in determining the impact of a particular final demand in another region (Region 2). Suppose we have the following data concerning Region 2.

	Value Added	Final Demand	Total Outputs
A	10	11	15
B	13	12	20

Suppose that the cost of computing an RAS estimate of Region 2's input–output table using Region 1's **A** matrix as a base table is given by nc_1, where n is the number of RAS iterations. One iteration is defined by one row and one column adjustment, that is, $\mathbf{A}' = \mathbf{RAS}$ (a row adjustment alone as the last iteration would also be counted as an iteration). We ultimately wish to compute the impact of a new final demand in Region 2. This impact (the total outputs required to support the new final demand) can be computed exactly or by using the round-by-round approximation of the inverse. We know that: (1) The cost of computing the inverse exactly on a computer is c_1 and the cost of using this inverse in impact analysis is c_2 (let us assume that $c_2, = 10c_1$, that is, the cost of computing the inverse is ten times the cost of using it in impact analysis). (2) The cost of a round-by-round approximation of impact analysis is mc_1, where m is the order of the round-by-round approximation, that is, $\mathbf{f} + \mathbf{Af} + \mathbf{A}^2\mathbf{f} + \cdots + \mathbf{A}^m\mathbf{f}$.

a. Assuming that a fourth-order round-by-round approximation is sufficiently accurate ($m = 4$), which method of impact analysis should we use to minimize cost – (1) or (2)?
b. What is the total cost of performing impact analysis, including the cost of the RAS approximation (tolerance of 0.01) and of the impact analysis scheme you chose in a?
c. If the budget for the entire impact-analysis calculation is $7c_1$, what level of tolerance can you afford: 0.01, 0.001, 0.0001, 0.00001, or 0.000001?

8.4 Examine the behavior of the adjustment term that converts location-quotient approach *FLQ* to *FLQA*, $\lambda = \{\log_2[1 + (x^r/x^n)]\}^\delta$ for values of $x^r/x^n = .01, .1, .25, .5, .75$, and 1 cross tabulated with values of $\delta = 0, .1, .3, .5$, and 1.

8.5 The matrix of technical coefficients for a national economy, $\mathbf{A}^N$, and the vector of total outputs, $\mathbf{x}^N$, as well as the corresponding values for a target region, $\mathbf{A}^R$ and $\mathbf{x}^R$, are

$$\mathbf{A}^N = \begin{bmatrix} .1830 & .0668 & .0087 \\ .1377 & .3070 & .0707 \\ .1603 & .2409 & .2999 \end{bmatrix} \quad \mathbf{x}^N = \begin{bmatrix} 518,288.6 \\ 4,953,700.6 \\ 14,260,843.0 \end{bmatrix}$$

$$\mathbf{A}^R = \begin{bmatrix} .1092 & .0324 & .0036 \\ .0899 & .0849 & .0412 \\ .1603 & .1170 & .2349 \end{bmatrix} \quad \mathbf{x}^R = \begin{bmatrix} 8,262.7 \\ 95,450.8 \\ 170,690.3 \end{bmatrix}$$

Compute the matrix of simple location quotients (SLQ) and the estimate of the matrix of regional technical coefficients using the SLQ.

8.6 For the national and regional data specified in problem 8.5, compute the matrix of Cross-Industry Quotients (CLQ) and the estimate of the matrix of regional technical coefficients using the CLQ.

8.7 Consider once again the national and regional data specified in problem 8.5. Estimate the matrix of regional technical coefficients using the RAS technique.

8.8 Compare the estimated regional matrices of technical coefficients computed in problems 8.5, 8.6, and 8.7. In terms of mean absolute deviation from the actual regional technical coefficients, which technique provides the most accurate estimate?

8.9 Using the three-sector, three-region Chinese MRIO data for 2000 from Table 3.7, create estimates of the intraregional input coefficients and their associated Leontief inverses for regions 2 (South China) and 3 (Rest of China), using the same reduction techniques and measures of difference that appear in Table 8.1 for region 1 (North China).

8.10 The following are the 1997 matrix of technical coefficients and vector of total outputs for the State of Washington as well as the 2003 matrix of technical coefficients for the United States, where the sectors are defined as (1) agriculture, (2) mining, (3) construction, (4) manufacturing, (5) trade, transportation and utilities, (6) services, and (7) other:

$$\mathbf{A}^W = \begin{bmatrix} .1154 & .0012 & .0082 & .0353 & .0019 & .0033 & .0016 \\ .0008 & .0160 & .0057 & .0014 & .0022 & .0002 & .0001 \\ .0072 & .0084 & .0066 & .0043 & .0074 & .0196 & .0133 \\ .0868 & .0287 & .0958 & .0766 & .0289 & .0244 & .0205 \\ .0625 & .0278 & .0540 & .0525 & .0616 & .0317 & .0480 \\ .0964 & .1207 & .0704 & .0596 & .1637 & .1991 & .2224 \\ .0020 & .0031 & .0056 & .0019 & .0045 & .0051 & .0066 \end{bmatrix} \quad \mathbf{x}^W = \begin{bmatrix} 7{,}681.0 \\ 581.7 \\ 17{,}967.1 \\ 77{,}483.7 \\ 56{,}967.2 \\ 109{,}557.6 \\ 4{,}165.5 \end{bmatrix}$$

$$\mathbf{A}^{US} = \begin{bmatrix} .2225 & .0000 & .0012 & .0375 & .0001 & .0020 & .0010 \\ .0021 & .1360 & .0072 & .0453 & .0311 & .0003 & .0053 \\ .0034 & .0002 & .0012 & .0021 & .0035 & .0071 & .0214 \\ .1724 & .0945 & .2488 & .3204 & .0468 & .0572 & .1004 \\ .0853 & .0527 & .0912 & .0950 & .0643 & .0314 & .0526 \\ .0902 & .1676 & .1339 & .1261 & .1655 & .2725 & .1882 \\ .0101 & .0140 & .0103 & .0214 & .0206 & .0200 & .0247 \end{bmatrix}$$

Using the RAS technique, estimate the Washington State table using the US matrix of technical coefficients as a starting point. Compute the mean absolute deviation of the estimated state technical coefficients matrix from the actual state matrix.

8.11 Suppose in problem 8.10 that we do not know all of the technical coefficients for the Washington State economy, $\mathbf{A}^W$, but we do know several, namely a_{11}^W, a_{62}^W, and a_{65}^W. Using RAS as an estimating procedure, how do we incorporate knowing these coefficients only into the process of estimating the balance of the Washington State technical coefficients using $\mathbf{A}^{US}$ as the initial estimate using the total outputs and intermediate inputs and outputs that you found in problem 8.10? Compute a revised estimate of the Washington State economy. How does it compare with the original estimate you found in problem 8.10?

8.12 Suppose in problem 8.11 you are able to determine from exogenous sources some alternative technical coefficients, namely a_{67}^W, a_{42}^W, and a_{54}^W. Compute a revised estimate of the Washington State matrix of technical coefficients using these known coefficients. Compute another estimate using both these and the previously identified known coefficients (from problem 8.11). How does this yet again revised estimate of the Washington State matrix of technical coefficients compare with the estimates you found in problems 8.10 and 8.11?

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9 Energy Input–Output Analysis

9.1 Introduction

Leontief's original input–output framework (Leontief, 1936) conceived of industry production functions, which he frequently referred to as production "recipes," as measured in physical units, such as specifying the technical coefficients in tons of coal or bushels of wheat, as inputs, required per dollar's worth of an industry's output or per ton of steel output. However, the data collection requirements and a number of other constraints rendered implementation of the framework in physical units too unwieldy, certainly at the time and even today to a lesser extent. Hence, the basic methodology for input–output analysis evolved, in both theory and application, through measuring all quantities in value terms with implicit fixed prices, as detailed in earlier chapters. Even late in his life, however, Professor Leontief continued to explore ways in which the framework could be implemented more widely in physical units rather than value terms (Leontief, 1989). A number of researchers, such as Duchin (1992)[1], carried on with that work and continued to develop it.

The contributions of many researchers have extended the input–output framework incrementally in the direction of employing physical units and, in the process, have helped lay the groundwork for new research areas such as industrial ecology and ecological economics, which are topics addressed in more detail in Chapter 10. In addition, there have been substantial developments in related areas where public policy concerns have encouraged such development and data have been collected to help implement the framework. Among these topics, as we will see in this chapter, input–output analysis provides a useful framework for tracing energy use and other related characteristics such as environmental pollution or flows of physical materials associated with interindustry activity.

The generalization of input–output analysis techniques to a much broader conceptual level, such as accounting for social indicators in so-called social accounting matrices and other related constructs that capture many different socioeconomic characteristics of an economy associated with interindustry activity (Chapter 11), began with simpler

[1] Other references to this work include Duchin and Lange (1987, 1994), Duchin and Hertwich (2003), and Weisz and Duchin (2006).

attempts to link input–output models and other national income accounting techniques with many measurable quantities associated with interindustry activity, such as energy use, environmental pollution, and employment. These generalized models are introduced in their most rudimentary form in Chapter 10 as a logical extension to the basic formulation that evolved to handle energy and environmental factors, but the more current, elaborate and general extensions and connections to national economic accounting systems are explored in Chapter 11.

9.1.1 Early Approaches to Energy Input–Output Analysis

In the late 1960s and 1970s, the United States economy was growing increasingly dependent upon foreign sources of oil and was forced to cope with supply shortages following embargoes imposed in the early 1970s by principally Arab countries organized into a cartel known as the Organization of Petroleum Exporting Countries (OPEC). At the same time there was also growing public concern over the environmental impacts associated with increasing energy use, especially air pollution associated with the burning of coal. Since energy was a critical factor of production for many industries in many regions of the country, researchers and government policymakers began to focus on the role of energy in the economy. In particular, input–output models focused on energy use were developed extensively during the oil crises in the early 1970s and there has been a resurgence in their use in recent years to analyze the relationship between energy use and climate change. Cumberland (1966), Strout (1967), Ayres and Kneese (1969), Bullard and Herendeen (1975b), Griffin (1976), Blair (1979 and 1980) and many others chronicle the early developments. Since then, considerable attention in the literature has been focused on extending the Leontief input–output framework to more explicitly account for energy and related environmental activities. Many of these applications are noted throughout this chapter and in Chapter 10.

The simplest and most straightforward of the energy extensions to the Leontief framework is to explicitly account for energy use by simply adding a set of linear energy coefficients that define energy use per dollar's worth of output of industrial sectors. This approach, developed and widely used in the early 1970s, has a number of methodological and practical limitations but continues to be used frequently today largely because it is often difficult to obtain additional data necessary to address the key weaknesses of ensuring internal consistency in accounting for energy supply and use throughout the economy. The strengths and weaknesses of these early approaches to energy input–output analysis are discussed in Appendix 9.1. In much of this chapter we develop a so-called "hybrid units" approach, initially put forward by Bullard and Herendeen (1975b), that addresses the principal weaknesses in the simplest approaches to energy input–output analysis.

9.1.2 Contemporary Energy Input–Output Analysis

The hybrid units formulation of energy input–output analysis defines energy coefficients that inherently conform to a set of "energy conservation conditions." These conditions turn out to be equivalent analytically to ensuring the internal consistency of accounting for physical energy flows in the economy. The alternative and, as noted earlier due to available data limitations, more commonly applied formulations outlined in Appendix 9.1 conform to these conditions only when interindustry prices of energy are uniform across all consuming sectors.

The condition of uniform interindustry energy prices may be present in some applications such as, for example, in some regional or developing economies, but such conditions are not common in many other situations, especially in large modern economies. Bullard and Herendeen's early work was developed further by others in the 1980s and 1990s, especially with respect to characterizing the role of structural change in the economy and its implications for energy and environmental emissions, e.g., Blair (1980), Wang and Chuang (1987), Blair and Wyckoff (1989), OTA (1990), Rose and Chen (1991), Han and Lakshmanan (1994), Casler (2001) and Dietzenbacher and Sage (2006) are also described later in this chapter.

9.2 Overview Concepts of Energy Input–Output Analysis

First we begin with how the basic input–output framework has been extended to account for interindustry energy flows, applications of which, as noted earlier, were particularly extensive in the late 1970s and early 1980s in the wake of the Arab oil embargoes and their effects on the US economy. In Chapter 10 we add other extensions such as accounting for pollution elimination and generation or recycling of materials. The mathematical structure of all these extensions almost mirrors the classical Leontief model that we have discussed in earlier chapters. However, when we seek to ensure consistency between, for example, measured levels of energy consumption (in physical units) and economic activity (usually measured in monetary units), we must add to the basic analytical framework.

In general energy input–output typically determines the total amount of energy required to deliver a product to final demand, both directly as the energy consumed by an industry's production process and indirectly as the energy embodied in that industry's inputs. In engineering parlance, calculating this total energy requirement is the result of what is often called a *process analysis*: a target product is identified either as a good or service, then a list can be compiled of the goods and services directly required to deliver the product. These inputs to the target production process include fuels (direct energy) and other nonenergy goods and services. The nonenergy inputs are then analyzed to determine the inputs to their production processes, which again include some fuels and nonenergy goods and services.

This process analysis traces inputs back to primary resources; the first round of energy inputs is the *direct energy requirement*; subsequent rounds of energy inputs comprise the *indirect energy requirement*. The sum of direct and all indirect energy requirements comprise the *total energy requirement*. For example, the energy used in assembling automobiles would be a direct energy requirement, while the energy *embodied* in (used in providing) the materials employed at the assembly plant (tires, engines, etc.), would comprise an indirect energy requirement.

Special complications arise when substantial portions of the inputs to a production process are imported and such situations have been important in historical applications of these methods. In addition, some sectors produce energy as a by-product or co-product – the same concept of secondary production used in earlier chapters. By-product examples include electricity production as a by-product of oil refinery operations or methane production as a by-product of landfill operations. An example of a co-product would be co-generation of electricity and steam, where the electricity is supplied to the utility grid or perhaps used locally in a manufacturing enterprise and the steam is used as an industrial process heat source. Accommodating these situations will also require adding to the basic framework. Figure 9.1 is a schematic of energy flows and use in the US economy in 2002. The input–output framework is well suited to analyzing in a comprehensive way these energy flows as they relate to interindustry activity.

In the energy input–output framework, computing the total energy requirement of industries, sometimes called the *energy intensity*, is analogous to computing the total dollar requirement or Leontief inverse of the traditional input–output model. In energy input–output analysis, however, we are most often concerned with energy measured in physical units – for example, British thermal units (BTUs) or quadrillions of BTUs

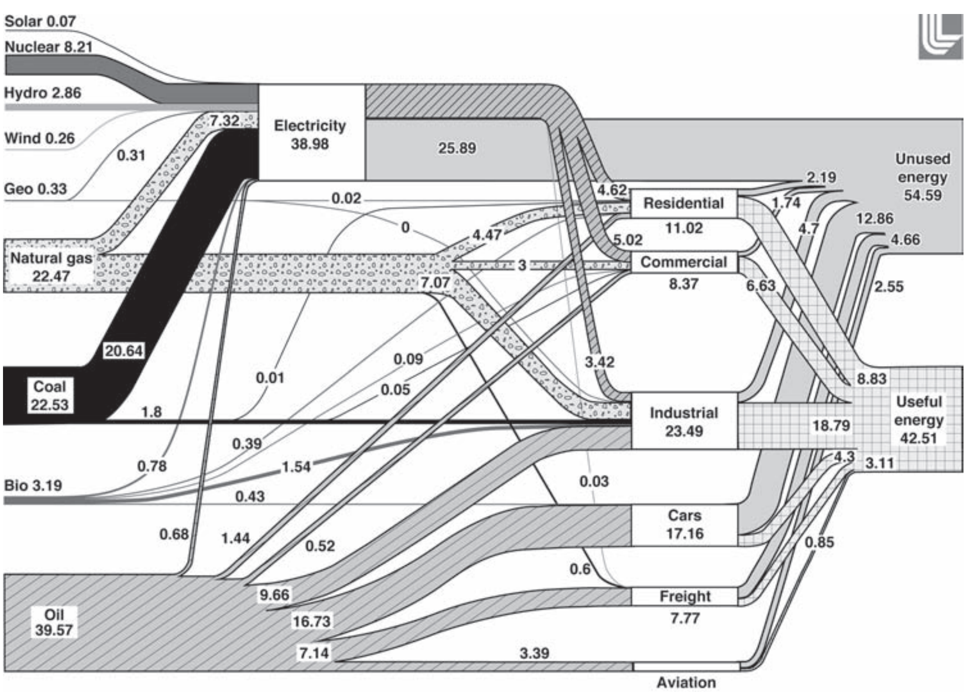

Figure 9.1 US Energy Use for 2006 (Quadrillions of BTUs)
Source: Lawrence Livermore Laboratory, 2007.

(Quads), as in Figure 9.1, barrels of oil, or tons of coal, rather than in dollars or value terms. As may be expected, one way to obtain these quantities in physical units is to first compute the total dollar requirement by conventional input–output analysis, and then convert these values to BTUs or some other appropriate physical units by means of prices relating dollar outputs to energy outputs. We will eventually see, however, that such a procedure (which is essentially the commonly used procedure outlined in Appendix 9.1) introduces inconsistencies in the resultant accounting of energy consumption, necessitating adjustments in the procedure in some circumstances to ensure reasonable results.

To illustrate the potential problem just outlined, in computing the energy intensity of a product (as defined above), we will distinguish between *primary* energy sectors (e.g., crude oil, coal mining, or solar energy) and *secondary* energy sectors (e.g., refined petroleum or electricity). Secondary energy sectors receive primary energy as an input and convert it into secondary energy forms. Hence, if we compute both the total amount of primary energy required to produce an industry's output and the total amount of secondary energy required to produce that same output, they must be equal, net of any energy lost in converting energy from primary to secondary energy forms, such as in producing electric power from coal. Different technologies, of course, have different energy conversion efficiencies and some energy sources, such as nuclear power or solar energy, have other complicating characteristics. In general, however, our energy input–output formulation should include the condition that the total primary energy intensity of a product should equal the total secondary energy intensity of the product plus any amount of energy lost in energy conversion or used for some other purpose.[2] We refer to this condition as an *energy conservation condition*. This condition will be a fundamental determinant in assessing whether or not a particular energy input–output model formulation accurately depicts the energy flows in the economy.

9.2.1 The Basic Formulation

We begin with the most contemporary framework of energy input–output where we construct a transactions table in so-called "hybrid units." That is, we trace energy flows in the economy in BTUs (or some other convenient energy units) and nonenergy flows in value terms such as dollars.[3] We will see later in Appendix 9.1 that such a formulation is generally superior to alternative formulations widely applied in the literature, albeit in some cases less easy to implement practically because of availability of data. We will explore the circumstances under which use of this framework is important as well as when alternatives are appropriate or acceptable.

In energy input–output we seek an analogous set of matrices to **Z**, **A**, and **L**, that is, an *energy* transactions or flows matrix (this time measured in physical inputs of *energy*, e.g., BTUs), a direct *energy* requirements matrix and finally a total *energy* requirements matrix. With only a minor change in the way we represent interindustry

[2] For example, some commodities may be used both as an energy source and as a raw material, such as petroleum.
[3] This "hybrid" formulation was suggested by Bullard and Herendeen (1975a); it is discussed in Blair (1979), Griffin (1976), Casler and Wilbur (1984) and others.

transactions in the basic input–output framework of Chapter 2, we can construct these energy input–output matrices.

We begin with a traditional input–output accounting identity, $\mathbf{Zi}+\mathbf{f}=\mathbf{x}$, where $\mathbf{Z}$ is the matrix of interindustry transactions, $\mathbf{f}$ is the vector of total final demands and $\mathbf{x}$ is the vector of total outputs, all measured in value terms, e.g., dollars. We are interested in measuring energy flows in physical units, so presume we have an analogous identity given by $\mathbf{Ei}+\mathbf{q}=\mathbf{g}$, where $\mathbf{E}$ is the matrix of energy flows from energy-producing sectors to all sectors as consumers of energy, $\mathbf{q}$ is the vector of energy deliveries to final demand[4] and $\mathbf{g}$ is the vector of total energy consumption, all once again measured in physical units. Note that if there are n sectors in the economy, m of which are energy sectors, then $\mathbf{Z}$ will be of dimension $n \times n$, but $\mathbf{E}$ will be of dimension $m \times n$. Similarly, while $\mathbf{f}$ and $\mathbf{x}$ are of dimension $n \times 1$, $\mathbf{q}$ and $\mathbf{g}$ will be of dimension $m \times 1$.

If, as before, $\mathbf{A}$ is the matrix of technical coefficients then $\mathbf{Z}=\mathbf{A}\hat{\mathbf{x}}$ and it follows that $\mathbf{L}=(\mathbf{I}-\mathbf{A})^{-1}$, the familiar Leontief inverse, so that total requirements can be expressed as $\mathbf{x}=\mathbf{Lf}$. We would like to have a matrix analogous to the $\mathbf{L}$ that yields total energy requirements in the equation $\mathbf{g}=\boldsymbol{\alpha}\mathbf{f}$ where $\boldsymbol{\alpha}$ is that $m \times n$ matrix.

9.2.2 The Total Energy Requirements Matrix

If we presume for a moment that the matrix $\boldsymbol{\alpha}$ already exists, we can define a set of energy conservation conditions in an energy input–output model generally that specify the relationship between primary and secondary energy sectors. These conditions were first articulated by Herendeen (1974) and adapted somewhat here as the following:

$$\alpha_{kj}x_j = \sum_{i=1}^{n} \alpha_{ki}z_{ij} + g_{kj} \tag{9.1}$$

where α_{kj} is the total amount of energy of type k required to produce a dollar's worth of sector i's output; x_j is the total dollar output of sector j; and z_{ij} is the dollar value of sector i's product consumed by sector j. The term g_{kj} is the total energy output of an energy sector and we define all elements of the $m \times n$ matrix $\mathbf{G}=[g_{kj}]$ as g_k for elements where energy sector k and industry sector j refer to the same industrial sector and 0 otherwise. Most elements of $\mathbf{G}$ are zero, except for those that correspond to total energy output for the energy sector designated by the row index. For accounting convenience, if the energy sectors $k=1,\ldots,m$ are placed first in the index of industry sectors $j=1,\ldots,n$, i.e., the series $k=1,\ldots,m$ and $j=1,\ldots,m$ both refer to the same collection of industry sectors, then the nonzero entries will appear along the principal diagonal[5] of $\mathbf{G}$ or, equivalently, the locations of nonzero elements in $\mathbf{G}$ are located where $k=j$.

[4] Note that $\mathbf{q}$ here and for most of this chapter should not be confused with its use to designate the vector of total commodity outputs in the commodity-by-industry accounting framework; it should always be clear in the context of the discussion which use of $\mathbf{q}$ applies.

[5] We define the principal diagonal of a nonsquare matrix $\mathbf{A}$ to be the a_{ii} elements.

Conceptually, the energy conservation conditions for all economic sectors j ($j = 1, \ldots, n$) can be described as the energy embodied in any sector output x_j is equal to the amount of energy embodied in all that sector's inputs, z_{ij} for $i = 1, \ldots, n$, plus any primary energy input, g_{kj}, which is nonzero only for primary energy sectors. Translated into matrix terms, (9.1) becomes[6]

$$\boldsymbol{\alpha}\hat{\mathbf{x}} = \boldsymbol{\alpha}\mathbf{Z} + \mathbf{G} \tag{9.2}$$

We illustrate this for the case of three economic sectors ($i, j = 1, 2, 3$), where the first two of the industry sectors are also designated by energy sectors ($k = 1, 2$). Equation (9.2) is then expressed as

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \end{bmatrix} \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \end{bmatrix} \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix} + \begin{bmatrix} g_{11} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that in this example there is only one nonzero element in $\mathbf{G}$, which indicates that there is only one *primary* energy sector. Expanding this equation yields

$$\begin{bmatrix} \alpha_{11}x_1 & \alpha_{12}x_2 & \alpha_{13}x_3 \\ \alpha_{21}x_1 & \alpha_{22}x_2 & \alpha_{23}x_3 \end{bmatrix} =$$
$$\begin{bmatrix} \alpha_{11}z_{11} + \alpha_{12}z_{21} + \alpha_{13}z_{31} & \alpha_{11}z_{12} + \alpha_{12}z_{22} + \alpha_{13}z_{32} & \alpha_{11}z_{13} + \alpha_{12}z_{23} + \alpha_{13}z_{33} \\ \alpha_{21}z_{11} + \alpha_{22}z_{21} + \alpha_{23}z_{31} & \alpha_{21}z_{12} + \alpha_{22}z_{22} + \alpha_{23}z_{32} & \alpha_{21}z_{13} + \alpha_{22}z_{23} + \alpha_{23}z_{33} \end{bmatrix}$$
$$+ \begin{bmatrix} g_{11} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Each term of this matrix is defined generally by (9.1); for example, the upper-left term is $\alpha_{11}x_1 = (\alpha_{11}z_{11} + \alpha_{12}z_{21} + \alpha_{13}z_{31}) + g_{11}$, which is identical to (9.1) for $k = 1, j = 1$ and $i = 1$, 2, and 3. Since $\mathbf{Z} = \mathbf{A}\hat{\mathbf{x}}$, we can write directly from (9.2) $\boldsymbol{\alpha}\hat{\mathbf{x}} = \boldsymbol{\alpha}\mathbf{A}\hat{\mathbf{x}} + \mathbf{G}$. Rearranging terms, we can derive $\boldsymbol{\alpha}(\mathbf{I} - \mathbf{A})\hat{\mathbf{x}} = \mathbf{G}$ and $\boldsymbol{\alpha}(\mathbf{I} - \mathbf{A}) = \mathbf{G}\mathbf{x}^{-1}$ or

$$\boldsymbol{\alpha} = \mathbf{G}\hat{\mathbf{x}}^{-1}(\mathbf{I} - \mathbf{A})^{-1} \tag{9.3}$$

which defines the matrix of total energy requirements.

We have not yet shown the conditions under which a matrix of total energy coefficients satisfies the energy conservation conditions defined by (9.1), but this will become clear when we define interindustry transactions in so-called "hybrid units" in the following. Expressing transactions in hybrid units is accomplished by taking the original interindustry transactions matrix, $\mathbf{Z}$, and replacing the energy rows with the corresponding rows in the energy flows matrix, $\mathbf{E}$. We define a new transactions matrix,

[6] In defining energy conservation, the energy inputs depicted in (9.1) as g_{kj} are the only inputs exogenous to the economy, i.e., primary inputs; all other inputs are embodied in consuming sector j's inputs $i = 1, \ldots, n$. This is illustrated later in Example 9.1.

$\mathbf{Z}^*$, for which the energy rows are measured in energy units and the nonenergy rows are measured in dollars, as usual. We must, of course, define corresponding vectors of total output, $\mathbf{x}^*$, and of final demand, $\mathbf{f}^*$, for which the energy and nonenergy sector quantities are similarly measured in energy units and dollars, respectively. In terms of our earlier notation these quantities are defined as the following:

$$\mathbf{Z}^* = [z_{ij}^*] = \begin{cases} z_{ij} \text{ where } i \text{ is a nonenergy sector} \\ e_{kj} \text{ where } k \text{ is an energy sector} \end{cases}$$

$\mathbf{Z}^*$ is of dimension $n \times n$;

$$\mathbf{f}^* = [f_i^*] = \begin{cases} f_i \text{ where } i \text{ is a nonenergy sector} \\ q_k \text{ where } k \text{ is an energy sector} \end{cases}$$

$\mathbf{f}^*$ is of dimension $n \times 1$;

$$\mathbf{x}^* = [x_i^*] = \begin{cases} x_i \text{ where } i \text{ is a nonenergy sector} \\ g_k \text{ where } k \text{ is an energy sector} \end{cases}$$

$\mathbf{x}^*$ is of dimension $n \times 1$. Finally we define $\mathbf{g}^*$ to be

$$\mathbf{g}^* = [g_i^*] = \begin{cases} 0 \text{ where } i \text{ is a nonenergy sector} \\ g_k \text{ where } k \text{ is an energy sector} \end{cases}$$

where $\mathbf{g}^*$ is of dimension $n \times 1$.

The corresponding matrices, $\mathbf{A}^* = \mathbf{Z}^*(\hat{\mathbf{x}}^*)^{-1}$ and $\mathbf{L}^* = (\mathbf{I} - \mathbf{A}^*)^{-1}$, follow directly from these definitions. However, some of the characteristics of these matrices differ from the traditional Leontief model. For example, the column sums of $\mathbf{A}^*$ are not necessarily less than unity as in the traditional model and are, in fact, meaningless since the units are not consistent – it is not meaningful to add BTUs per dollar of output with dollar's worth of input per dollar of output.

Note also that the units of the direct requirements matrix, $\mathbf{A}^*$, and the total requirements matrix, $\mathbf{L}^*$, reflect hybrid units as well. Consider, for example, the two-sector case where the first sector is an energy sector and the second is a nonenergy sector. The units of such a model formulated in hybrid units, for each element in the matrix, are expressed as $\mathbf{Z}^* = \begin{bmatrix} BTU & BTU \\ \$ & \$ \end{bmatrix}$, $\mathbf{f}^* = \begin{bmatrix} BTU \\ \$ \end{bmatrix}$, $\mathbf{x}^* = \begin{bmatrix} BTU \\ \$ \end{bmatrix}$ and $\mathbf{g}^* = \begin{bmatrix} BTU \\ 0 \end{bmatrix}$. Hence, we obtain

$$\mathbf{A}^* = \mathbf{Z}^*(\hat{\mathbf{x}}^*)^{-1} = \begin{bmatrix} BTU/BTU & BTU/\$ \\ \$/BTU & \$/\$ \end{bmatrix} \tag{9.4}$$

The matrix $\mathbf{L}^*$ will have the same units as $\mathbf{A}^*$ except, of course, that they are in terms of the requirement (BTUs or dollars) per unit (BTU or dollar) of final demand (i.e., total requirement) instead of per unit of total output (direct requirement).

Table 9.1 Energy and Dollar Flows: Example 9.1

	Widgets	Energy	Final Demand	Total Output
Value Transactions in Million of Dollars				
Widgets	10	20	70	100
Energy	30	40	50	120
Energy Transactions in Quadrillions of BTUs				
Energy	60	80	100	240

To obtain the matrices we referred to earlier as the *direct energy requirements matrix* and *total energy requirements matrix* we need only extract the energy rows from $\mathbf{A}^*$ and $\mathbf{L}^*$, respectively. A convenient tool for isolating the energy rows is to construct the matrix product $\mathbf{G}(\hat{\mathbf{x}}^*)^{-1}$, analogous to the matrix product defined earlier as $\mathbf{G}\hat{\mathbf{x}}^{-1}$ but with a special property. Recall that nonzero elements of $\mathbf{G}$ are the elements of $\mathbf{g}$ that are energy sectors. Since the nonzero elements of $\mathbf{g}$ (and of $\mathbf{g}^*$ for that matter) are identical to the corresponding values in $\mathbf{x}^*$ (recall also the definition of x_i^*), the result of this product is a matrix of ones and zeros, where the ones denote the locations of energy sectors. If we postmultiply this vector by $\mathbf{L}^*$, the result includes only the rows of the total energy coefficients, that is, the energy rows of $\mathbf{L}^*$. Similarly, we can premultiply $\mathbf{A}^*$ by this matrix to retrieve *only* the direct energy coefficients from $\mathbf{A}^*$, that is, the energy rows of $\mathbf{A}^*$. Hence, we define the direct and total energy coefficients matrices (which, of course, in the two-sector illustration are actually row vectors since there is only one energy sector) to be $\boldsymbol{\delta}$ and $\boldsymbol{\alpha}$, respectively:

$$\boldsymbol{\delta} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}\mathbf{A}^* \tag{9.5}$$

$$\boldsymbol{\alpha} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}\mathbf{L}^* \tag{9.6}$$

Example 9.1: Two-Sector Illustration of Hybrid Units Input–Output Analysis We consider a two-sector example that will illustrate the essential properties of this "hybrid units" formulation of the energy input–output problem. Table 9.1 includes a table of dollar interindustry transactions and a related table showing energy flows in quadrillions of BTUs corresponding to the dollar transactions from the energy sector to the other sectors, including deliveries to final demand and total output.

From the conventions just described for the hybrid units formulation we can define $\mathbf{Z}^* = \begin{bmatrix} 10 & 20 \\ 60 & 80 \end{bmatrix}$ and $\mathbf{x}^* = \begin{bmatrix} 100 \\ 240 \end{bmatrix}$ and then derive $\mathbf{A}^* = \mathbf{Z}^*(\hat{\mathbf{x}}^*)^{-1} = \begin{bmatrix} 0.100 & 0.083 \\ 0.600 & 0.333 \end{bmatrix}$ and $\mathbf{L}^* = \begin{bmatrix} 1.212 & 1.515 \\ 1.091 & 1.636 \end{bmatrix}$. From (9.4) and (9.5) we compute the direct and total energy requirements matrices (for this two-sector example these matrices are actually row vectors):

$$\boldsymbol{\delta} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}\mathbf{A}^* = \begin{bmatrix} 0 & 240 \end{bmatrix} \begin{bmatrix} 1/100 & 0 \\ 0 & 1/240 \end{bmatrix} \begin{bmatrix} 0.100 & 0.083 \\ 0.600 & 0.333 \end{bmatrix} = \begin{bmatrix} 0.600 & 0.333 \end{bmatrix}$$

or

$$\boldsymbol{\delta} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0.100 & 0.083 \\ 0.600 & 0.333 \end{bmatrix} = \begin{bmatrix} 0.600 & 0.333 \end{bmatrix}$$

and

$$\boldsymbol{\alpha} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}\mathbf{L}^* = \begin{bmatrix} 0 & 240 \end{bmatrix} \begin{bmatrix} 1/100 & 0 \\ 0 & 1/240 \end{bmatrix} \begin{bmatrix} 1.212 & 1.515 \\ 1.091 & 1.636 \end{bmatrix} = \begin{bmatrix} 1.091 & 1.636 \end{bmatrix}.$$

Note that in using the energy input–output model in impact analysis – that is, analogous to $\mathbf{x} = \mathbf{Lf}$ in the traditional Leontief model – the final demand presented to the total requirements matrix must be in hybrid units, that is, $\mathbf{g} = \boldsymbol{\alpha}\mathbf{f}^*$. We can verify for this example, since $\mathbf{f}^* = \begin{bmatrix} 70 \\ 100 \end{bmatrix}$, that $\mathbf{g} = \boldsymbol{\alpha}\mathbf{f}^* = \begin{bmatrix} 1.091 & 1.636 \end{bmatrix} \begin{bmatrix} 70 \\ 100 \end{bmatrix} = 240$.

Example 9.2: Generalization to Several Energy Types In the initial energy input–output formulation, we defined the vector $\mathbf{g}$ to be of length m (the number of energy sectors) denoting the total energy output (in BTUs) of energy sectors. In developing the hybrid units notation further, we defined the vector $\mathbf{g}^*$ to be of length n (the total number of industry sectors, including energy sectors) where the elements representing energy sectors (m of the n elements) denote total energy output (in BTUs) of those sectors; the remaining elements were defined to be zero.

Consider a four-sector economy, in which three sectors are energy sectors, namely crude oil, refined petroleum, and electric power. The fourth sector, automobiles, is the only nonenergy sector. Note that the only primary energy sector in this economy is crude oil since the refined petroleum, and electric power sectors both convert oil into secondary energy products. The dollar transactions for the economy are given in Table 9.2; the energy flows in the economy (measured in 10^{15} BTUs) are as given in Table 9.3.

For the hybrid units energy input–output formulation using the data in Tables 9.2 and 9.3,

$$\mathbf{Z}^* = \begin{bmatrix} 0 & 20 & 20 & 0 \\ 1 & 3 & 0 & 1 \\ 2.5 & 1.25 & 1.25 & 2.5 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{f}^* = \begin{bmatrix} 0 \\ 15 \\ 12.5 \\ 20 \end{bmatrix}, \mathbf{x}^* = \begin{bmatrix} 40 \\ 20 \\ 20 \\ 20 \end{bmatrix}$$

from which we can derive

$$\mathbf{A}^* = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0.0250 & 0.1500 & 0 & 0.0500 \\ 0.0625 & 0.0625 & 0.0625 & 0.0125 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Table 9.2 Interindustry Economic Transactions: Example 9.2 (millions of dollars)

	Crude Oil	Refined Petroleum	Electric Power	Autos	Final Demand	Total Output
Crude Oil	0	5	5	0	0	10
Refined Petroleum	2.5	2.5	0	2.5	12.5	20
Electric Power	2.5	1.25	1.25	2.5	12.5	20
Autos	0	0	0	0	20	20

Table 9.3 Energy Flows: Example 9.2 (10^{15} BTUs)

	Crude Oil	Refined Petroleum	Electric Power	Autos	Final Demand	Total Output
Crude Oil	0	20	20	0	0	40
Refined Petroleum	1	3	0	1	15	20
Electric Power	2.5	1.25	1.25	2.5	12.5	20

and

$$\mathbf{L}^* = \begin{bmatrix} 1.109 & 1.391 & 1.183 & 0.217 \\ 0.033 & 1.217 & 0.035 & 0.065 \\ 0.076 & 0.174 & 1.148 & 0.152 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In deriving the matrix of total energy coefficients, $\boldsymbol{\alpha}$, which in this case is of dimension 3×4, we first compute the matrix $\mathbf{G}$, as defined earlier, describing the total energy consumption of each type as $\mathbf{G} = \begin{bmatrix} x_1^* & 0 & 0 & 0 \\ 0 & x_2^* & 0 & 0 \\ 0 & 0 & x_3^* & 0 \end{bmatrix} = \begin{bmatrix} 40 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & 20 & 0 \end{bmatrix}$. Recall that $\mathbf{G}$ is created by taking the values of total energy production for each of the energy sectors (40, 20 and 20, respectively, for crude oil, refined petroleum, and electricity) and defining them as the g_{ii}^*, elements of $\mathbf{G}$. Given $\mathbf{G}$ and $\mathbf{x}^*$, we can obtain

$$\mathbf{G}(\hat{\mathbf{x}}^*)^{-1} = \begin{bmatrix} 40 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & 20 & 0 \end{bmatrix} \begin{bmatrix} 1/40 & 0 & 0 & 0 \\ 0 & 1/20 & 0 & 0 \\ 0 & 0 & 1/20 & 0 \\ 0 & 0 & 0 & 1/20 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Hence, it follows directly that

$$\boldsymbol{\alpha} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}(\mathbf{I} - \mathbf{A}^*)^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1.109 & 1.391 & 1.183 & 0.217 \\ 0.033 & 1.217 & 0.035 & 0.065 \\ 0.076 & 0.174 & 1.148 & 0.152 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which, as noted earlier, simply retrieves the first three (energy) rows of $\mathbf{L}^*$ or

$$\boldsymbol{\alpha} = \begin{bmatrix} 1.109 & 1.391 & 1.183 & 0.217 \\ 0.033 & 1.217 & 0.035 & 0.065 \\ 0.076 & 0.174 & 1.148 & 0.152 \end{bmatrix}$$

Note the special properties of $\boldsymbol{\alpha}$. Adding the second and third rows yields the first row, except for the first column, which corresponds to the economy's primary energy sector, where the difference between the sum of the second and third elements and the first element is unity. In the following we show that these two results define precisely the conditions of energy conservation in the economy.

9.2.3 The Hybrid Units Formulation and Energy Conservation Conditions

With (9.1) we formally defined the energy conservation conditions for an energy input–output model as

$$\alpha_{kj} x_j = \sum_{i=1}^{n} \alpha_{ki} z_{ij} + g_{kj} \tag{9.7}$$

where, as before, α_{kj} is the amount of energy required to produce a dollar's worth of sector i's output; x_j is the total dollar output of sector j; and z_{ij} is the dollar value of sector i's product consumed by sector j. For this discussion, we restrict g_{kj} to be the total energy output of only primary energy sectors.[7] That is, the energy embodied in any sector output x_j equals the amount of energy embodied in all that sector's inputs $z_{ij} (i = 1, \ldots, n)$ plus the primary energy input, g_{kj}, which is nonzero only for primary energy sectors. As noted earlier, translated into matrix terms the energy conservation conditions can be expressed as

$$\boldsymbol{\alpha}\hat{\mathbf{x}} = \boldsymbol{\alpha}\mathbf{Z} + \mathbf{G} \tag{9.8}$$

In the hybrid units formulation, $\mathbf{x}$, $\mathbf{A}$, and $\mathbf{g}$ in (9.7) are replaced by corresponding values of $\mathbf{x}^*$ and $\mathbf{A}^*$ and $\mathbf{G}$ or $\boldsymbol{\alpha}\hat{\mathbf{x}}^* = \boldsymbol{\alpha}\mathbf{Z}^* + \mathbf{G}$. Expressing the input–output transactions in hybrid units includes specifying the energy transactions in physical units so that the energy conservation conditions can be expressed as a set of physical relationships (independent of the prices of energy), which we illustrate in Example 9.2 (revisited) below. In Appendix 9.1 we show more generally that the hybrid units model satisfies

[7] In defining energy conservation, the energy inputs depicted in (9.1) as g_{kj} are the only inputs exogenous to the economy, i.e., primary inputs; all other inputs are embodied in consuming sector j's inputs $i = 1, 2, \ldots, n$.

the conditions of energy conservation generally while *not* specifying these relationships in hybrid units will satisfy the conditions of energy conservation in only limited circumstances.

Example 9.2: Generalization to Several Energy Types (Revisited) Recall the total energy requirements matrix from Example 9.2,

$$\boldsymbol{\alpha} = \begin{bmatrix} 1.109 & 1.391 & 1.183 & 0.217 \\ 0.033 & 1.217 & 0.035 & 0.065 \\ 0.076 & 0.174 & 1.148 & 0.152 \end{bmatrix}$$

The first energy sector, oil, is the *primary* energy sector while the remaining energy sectors, refined petroleum and electricity, are *secondary* energy sectors. Consider the last column of $\boldsymbol{\alpha}$, that is, the automobile sector (the only nonenergy sector in this example). The term $\alpha_{14} = 0.217$ is the total primary energy intensity of producing automobiles in the economy. That is, it takes 0.217×10^9 BTUs of crude oil to produce (including both direct and indirect energy requirements) one dollar's worth of output in the automobile sector. Similarly, $\alpha_{24} = 0.065$ and $\alpha_{34} = 0.152$ are the secondary energy intensities of automobile production, namely, it takes 0.065 and 0.152 billion BTUs of refined petroleum and electricity, respectively, to produce a dollar's worth of automobiles. However, since both refined petroleum and electricity ultimately come from crude oil in this economy (since crude oil is the only primary energy sector), the energy-conservation condition requires that the sum of secondary energy intensities for automobile production equals the primary energy intensity (minus any losses in energy conversion from crude oil to electricity or refined petroleum, which we ignore for the time being). That is, for this example, $\alpha_{24} + \alpha_{34} = \alpha_{14} = 0.217$. This condition, of course, should hold for all sectors in the economy except for the primary energy sectors, which extract their energy from outside the economy (primary resources).

In other words, if we sum the secondary energy rows (rows 2 and 3 in our example) – the total secondary energy intensity – the result should be the same as the primary energy intensity (row 1 in the example). If there were more primary energy sectors, the total primary energy intensity would be the sum of the primary energy rows. For column 1 (crude oil), however, total secondary energy intensity is $\alpha_{21} + \alpha_{31} = 0.109$ while $\alpha_{11} = 1.109$. The difference can be interpreted as the amount of crude oil received from outside the economy per unit of output of crude oil, namely *all* of it, since it is a primary resource.

9.3 Further Methodological Considerations

We now examine a number of additional methodological considerations that become important in the application of energy input–output analysis.

Table 9.4 Interindustry Transactions in Hybrid Units: Example 9.3

	Coal	Electric Power	Autos	Final Demand	Total Output
Coal (Quadrillion BTU)	0	300	0	0	300
Electric Power (Quadrillion BTU)	20	20	20	60	120
Automobiles (million dollars)	0	0	0	100	100

9.3.1 Adjusting for Energy Conversion Efficiencies

In the version of the energy input–output model discussed thus far that deals with secondary energy production, we ignored the effect of energy conversion efficiencies. For example, in converting coal to electricity, on average across the economy only about one-third of the energy content of coal burned in coal-fired power plants gets distributed as electricity. The rest is dissipated as waste heat. Earlier, we assumed that the energy produced by secondary sources must equal the sum of the amounts of primary sources consumed in producing that secondary energy. It is a quite straightforward extension to modify our hybrid units model to account for conversion efficiencies.

This adjustment must first recognize that for secondary energy sources, $g_k^* \neq x_k^*$, where the value of g_k^* refers to the total energy *input* of type k to the production process and x_k^* refers to the total type k energy *output*. The ratio, g_k^* to x_k^*, is the energy conversion efficiency, which means if the amount of output, x_k^*, is known we multiply by the reciprocal of the conversion efficiency, x_k^* / g_k^*, in order to determine the amount of primary energy required as input. We illustrate the process of adjusting for energy conversion efficiencies in Example 9.3.

Example 9.3: Adjusting for Energy Conversion Efficiencies Consider a three-sector economy with one primary energy sector (coal) and one secondary energy sector (electricity) as depicted in Table 9.4.

The corresponding total-requirements matrix for this example is

$$\mathbf{L}^* = \begin{bmatrix} 1.25 & 3.75 & .75 \\ 0.1 & 1.5 & 0.3 \\ 0 & 0 & 1 \end{bmatrix}$$

In this example the coal sector delivers all of its output to electricity and that output is 300×10^{15} BTUs, that is, $g_1^* = 300$, but the total output of electricity is $x_2^* = 120$; hence the implied conversion efficiency of producing electricity from coal is $x_2^*/g_1^* = 0.4$. Therefore we write $\mathbf{G}(\hat{\mathbf{x}}^*)^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 300/120 & 0 \end{bmatrix}$ and $\boldsymbol{\alpha} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}\mathbf{L}^* = \begin{bmatrix} 1.25 & 3.75 & .75 \\ 0.25 & 3.75 & .75 \end{bmatrix}$. Each element in $\boldsymbol{\alpha}$ reflects the energy conversion efficiency implied in $\mathbf{G}(\hat{\mathbf{x}}^*)^{-1}$. [Remember that nonzero terms in $\mathbf{G}(\hat{\mathbf{x}}^*)^{-1}$ are defined as the reciprocal of efficiency.] For example, the total amount of coal required to deliver the amounts of electricity given in the second row of $\mathbf{L}^*$ is 2.5 times the amount given in that row, since the energy conversion efficiency is 0.4. Hence, the second row in $\boldsymbol{\alpha}$ is 2.5 times the second row in $\mathbf{L}^*$.

9.3.2 Accounting for Imports

As described in Chapter 4, many input–output studies include transferred or competitive imports as part of the total output, that is, imported goods competing with domestically produced goods. To correctly compute the *domestic* total energy intensity via our total energy requirements matrix, the matrix must be adjusted to reflect only domestic output. Reducing total outputs, $\mathbf{x}^*$, by transferred imports, $\mathbf{x}^*_{\mathbf{I}}$, we write $\mathbf{D} = \mathbf{E}(\hat{\mathbf{x}}^* - \hat{\mathbf{x}}^*_{\mathbf{I}})^{-1}$ and $\mathbf{A}^* = \mathbf{Z}^*(\hat{\mathbf{x}}^* - \hat{\mathbf{x}}^*_{\mathbf{I}})^{-1}$. Recall that $\boldsymbol{\alpha} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}\mathbf{L}^*$ so using the adjusted total outputs vector, we can derive $\boldsymbol{\alpha} = \mathbf{G}(\hat{\mathbf{x}}^* - \hat{\mathbf{x}}^*_{\mathbf{I}})^{-1}[\mathbf{I} - \mathbf{Z}^*(\hat{\mathbf{x}}^* - \hat{\mathbf{x}}^*_{\mathbf{I}})^{-1}]^{-1}$. Rearranging and collecting terms, $\boldsymbol{\alpha} = \mathbf{G}[\hat{\mathbf{x}}^* - \hat{\mathbf{x}}^*_{\mathbf{I}} - \mathbf{Z}^*]^{-1}$. For the case where $\mathbf{x}^*_{\mathbf{I}} = \mathbf{0}$, $\boldsymbol{\alpha} = \mathbf{G}(\hat{\mathbf{x}}^* - \mathbf{Z}^*)^{-1}$. The reader can show that this is equivalent to our previous definition, $\boldsymbol{\alpha} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}\mathbf{L}^*$. In studies where the domestic production of energy is the central focus, the adjustment just described would be important. Herendeen (1974) deals with this adjustment in detail.

9.3.3 Commodity-by-Industry Energy Models

In Chapter 5 we introduced the commodity-by-industry accounting identity, $\mathbf{x} = \mathbf{Vi}$. This equation defined the vector of total *industry* outputs as the row sums of the *Make* matrix, **V.** Another commodity-by-industry accounting identity was $\mathbf{q} = \mathbf{Ui} + \mathbf{e}$. This equation defined $\mathbf{q}$, the vector of total *commodity* outputs (not to be confused with the use of $\mathbf{q}$ to denote energy final demand throughout much of this chapter), as the sum of the row sums of the *Use* matrix of commodity inputs, $\mathbf{U}$, and the vector of final demands for commodities, $\mathbf{e}$.

In Chapter 5 we examined the two major assumptions for deriving a total requirements matrix, either an industry-based or commodity-based definition of technologies in the input–output economy. For this discussion we assume the former, but similar results could easily be derived for the latter. Again from Chapter 5, with an industry-based technology, we have $\mathbf{U} = \mathbf{B}\hat{\mathbf{x}}$ and $\mathbf{V} = \mathbf{D}\hat{\mathbf{q}}$ from which we can easily derive $\mathbf{q} = \mathbf{B}\hat{\mathbf{x}}\mathbf{i} + \mathbf{e} = \mathbf{Bx} + \mathbf{e}$, and hence $\mathbf{q} = \mathbf{BVi} + \mathbf{e}$. Finally, substituting $\mathbf{D}\hat{\mathbf{q}}$ for $\mathbf{V}$ gives $\mathbf{q} = \mathbf{BD}\hat{\mathbf{q}}\mathbf{i} + \mathbf{e} = \mathbf{BDq} + \mathbf{e}$ or $\mathbf{q} = (\mathbf{I} - \mathbf{BD})^{-1}\mathbf{e}$, where $(\mathbf{I} - \mathbf{BD})^{-1}$ is the matrix of commodity-by-commodity total requirements.

We now return to the energy-balance equation stated earlier for the traditional Leontief model [shown earlier as (9.8)]: $\boldsymbol{\alpha}\hat{\mathbf{x}} = \boldsymbol{\alpha}\mathbf{Z} + \mathbf{G}$. The corresponding equation for the commodity-by-commodity model is $\boldsymbol{\alpha}\hat{\mathbf{q}} = \boldsymbol{\alpha}(\mathbf{BD}\hat{\mathbf{q}}) + \mathbf{G}$, where $\boldsymbol{\alpha}$, as before, is the matrix of total energy intensities; however, this time we define them as commodity energy intensities rather than industry energy intensities ($\mathbf{G}$ must also be defined in terms of commodities). Rewriting, by ordinary matrix algebra, we have $\boldsymbol{\alpha}(\mathbf{I} - \mathbf{BD})\hat{\mathbf{q}} = \mathbf{G}$, so $\boldsymbol{\alpha} = \mathbf{G}\hat{\mathbf{q}}^{-1}(\mathbf{I} - \mathbf{BD})^{-1}$.

As before, where we computed $\mathbf{G}\hat{\mathbf{x}}^{-1}$ to identify the energy rows of $\mathbf{L}^*$, $\mathbf{G}\hat{\mathbf{q}}^{-1}$ accomplishes the same for $(\mathbf{I}-\mathbf{BD})^{-1}$ if $\mathbf{q}$ is measured in "hybrid units," as before – BTUs for energy sectors and dollars for nonenergy sectors.

As discussed in Chapter 5, in some industries a significant fraction of total output (in value terms) can be attributed to secondary production. In such situations the use of commodity-by-industry accounts is important. Moreover, the secondary production of energy by nonenergy sectors – for example, industrial cogeneration of electricity – can be easily accommodated in this framework.

9.4 Applications

We now consider a number of applications of the energy input–output formulation to several contemporary problems. The intent is not to examine these applications in detail, but only to illustrate the kinds of questions that have been considered in the energy input–output framework.

9.4.1 Net Energy Analysis

Many researchers define net energy analysis as a comparison of the energy produced by a process (or a series of processes) to the energy required to create and sustain that process (Figure 9.2). As noted earlier in this chapter, we restrict the discussion of net energy analysis to this purpose as opposed to its occasional use in promoting an "energy theory of value" (discussed further below). In net energy analysis, the amount of energy to be processed for a given energy production system is the direct energy requirement, as we defined it for the energy input–output model. Similarly, the sum of the direct energy required to create and operate the process and the energy embodied in the industry's inputs is interpreted as the total energy requirement. Alternative approaches to net energy analysis, including the input–output approach summarized below, are considered in detail in Spreng (1988).

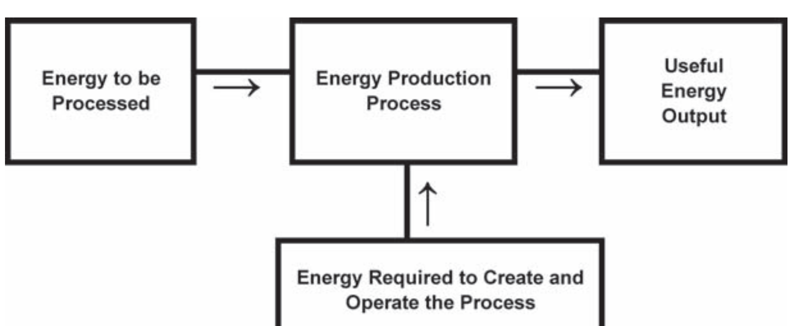

Figure 9.2 Net Energy Analysis

At the same time as the energy input–output framework evolved in the early 1970s, a number of researchers were investigating some of the same issues from the perspective of thermodynamic efficiency from engineering and chemistry, as in Berry (1972). Berry, in collaboration with others (Berry, Fels and Makino, 1974) examined the "energy cost" of automobiles. Berry, Salmon and Heal (1978) subsequently and more formally explored the relationship between economic efficiency and thermodynamic efficiency. Exploring the relationship between economics and thermodynamics led some researchers who are proponents of net energy analysis to develop a controversial "energy theory of value" (Hannon, 1973 or Gilliland, 1975) and perhaps even more controversially an "entropy theory of value" (Georgescu-Roegen, 1971), which invokes the Second Law of Thermodynamics ("useful energy gets dissipated") to assert that an economy faces limits to growth.

Proponents of an embodied energy theory of value (Odum, 1971, or Costanza and Herendeen, 1984) assert that the value of goods and services in an economy is assumed to be related to the direct and indirect energy embodied in them. Most energy analysts today, however, reject the energy theory of value for many of the same reasons economists rejected the labor theory of value advanced by the Physiocrats (see Appendix C). Nonetheless, many researchers find considerable merit in the orderly way of tracing energy flows offered in net energy analysis as a complement to more standard economic analysis frameworks. Examples of moving the framework in this direction can be found in Proops (1977), Bullard, Penner and Pilati (1978), Blair (1979, 1980), Treloar (1997) and Cleveland (1999) and in the efforts to analyze structural change in an economy discussed later in this chapter and in Chapter 13.

Example 9.4: Net Energy Analysis Consider a highly aggregated hybrid units input–output model of the US economy for 1963 depicted in Table 9.5. In this table there are five energy sectors: (1) coal, (2) crude oil and gas, (3) refined petroleum, (4) electricity, and (5) natural gas utilities. The primary energy in the economy consists of crude oil and gas, coal and the electricity produced from nuclear and hydroelectric plants. The nuclear and hydroelectric amounts are relatively small, so that for convenience, this energy is often represented in terms of its fossil fuel equivalent by dividing it by the efficiency of converting fossil fuels to electricity.[8]

We can write an expression for the total *primary energy intensity*[9] of an industry as $\bar{\boldsymbol{\alpha}} = [\bar{\alpha}_j]$ where $\bar{\alpha}_j = \alpha_{1j} + \alpha_{2j} + (\beta/\eta)\,\alpha_{4j}$. Since we define primary energy intensity only in terms of primary energy sectors, for the example economy (Table 9.5) refined petroleum and natural gas utilities are secondary energy sectors. Hence, only coal, crude oil and gas and a portion of electric power sectors appear in the equation, i.e., terms from rows 1, 2, and 4 of the total energy requirements matrix. The term β is the

8 The US input–output table for 1967 was the first US table for which a full survey-based set of corresponding energy transactions measured in physical units was generated to implement energy input–output models of the sort developed in this chapter.

9 This concept is developed in detail in Bullard and Herendeen (1975a) and in many other articles by these authors.

Table 9.5 Input–Output Transactions for the US Economy in Hybrid Units (1967)*

		Coal Mining	Oil & Nat. Gas	Ref. Petrol.	Elec. Utilities	Gas Utilities	Chem.	Agric.	Mining & Manuf.	Transp. & Comm.	Rest of Economy	Final Demand	Total Output
1.	Coal Mining	96			7,750	14	551	71	4,702			2,740	15,924
2.	Oil & Nat. Gas.		1,113	23,326		17,737	148					499	42,823
3.	Ref. Petroleum	32	43	1,624	906	14	741	847	4,030	3,691	2,037	14,037	28,002
4.	Elec. Utilities	16	43	56	445		381	71	1,343	75	509	1,181	4,120
5.	Gas Utilities		86	896	3,148	977	868	212	1,343	151	1,528	4,948	14,157
6.	Chemicals	48	171	616	41		4,025	2,540	10,075	75	1,018	2,672	21,281
7.	Agriculture						763	19,898	36,270	75	3,055	10,498	70,559
8.	Mining & Manuf.	350	1,328	868	943	283	3,008	6,562	255,235	4,897	49,902	348,295	671,671
9.	Transp. & Comm.	32	171	1,344	610	42	635	1,552	16,120	6,102	17,313	31,412	75,333
10.	Rest of Economy	366	4,197	2,968	3,650	849	1,271	11,007	82,616	13,108	99,294	289,873	509,199

*Transactions are in millions of dollars for nonenergy sectors and in Quads (10^{15} BTU) for energy sectors.

fraction of electricity produced directly from hydroelectric and nuclear sources; η is the conversion efficiency of producing electricity from fossil fuels. Recall that α_{kj} is the total energy intensity of energy type k per dollar's worth of final demand of industry j.

The $\mathbf{A}^*$ and $\mathbf{L}^*$ matrices for the US transactions shown in Table 9.5 are given in Tables 9.6 and 9.7, respectively. The total primary energy intensities, $\bar{\boldsymbol{\alpha}}$, are given in Table 9.8. Note that the matrices $\mathbf{A}^*$ and $\mathbf{L}^*$ are given in hybrid units; hence, as discussed earlier, the column sums of $\mathbf{A}^*$ are not necessarily less than unity. Recall from before that the energy rows of $\mathbf{L}^*$ comprise the total energy requirements, $\boldsymbol{\alpha}$. For purposes of our illustration we assume that $\beta = 0.1$, and $\eta = 0.33$.

Suppose that we are considering two alternative designs for an electric power plant, both of which are rated at 1,000 megawatts of electric power output and will operate approximately 7,000 hours per year for 30 years. This means that each power plant will produce 21×10^{10} kilowatt-hours (kwh) of electrical energy (or 0.7×10^{15} BTUs) over its lifetime. The lists of materials required for construction, operation, and maintenance (excluding fuel) for the two plants are given in Table 9.9. We will interpret these expenditures, for impact analysis purposes, as new final demands presented to the US economy.

We refer to the vectors of expenditures for the two power plants in Table 9.9 as $\mathbf{r}^{\mathbf{I}}$ and $\mathbf{r}^{\mathbf{II}}$, respectively. The aggregate primary energy intensities for the two plants are found in 10^{12} BTUs by $\bar{\boldsymbol{\alpha}}\mathbf{r}^{\mathbf{I}} = 683$ and $\bar{\boldsymbol{\alpha}}\mathbf{r}^{\mathbf{II}} = 962$. As a measure of overall "energy efficiency" of technologies, the energy ratio (*ER*) is defined to be the ratio of total energy output of the power plant over its lifetime to the total primary energy intensity. For our example, the energy ratios of the two plants are 1.025 and 0.728, respectively; from a net energy standpoint, power plant design I is more efficient than design II. If the total production is the same for both power plants, then the primary energy intensity gives the same "ranking" as the *ER*.

9.4.2 Energy Cost of Goods and Services

We can use the energy input–output framework to estimate the total energy cost of final demand expenditures such as, for example, the total fuel (direct) and indirect energy consumed in acquiring and using a family automobile. Bullard and Herendeen (1975a) show that only about 30 percent of the total energy consumption attributable to automobile usage is gasoline; the rest includes the "energy cost of energy" (e.g., refinery losses), the direct and indirect energy associated with manufacture of the auto, parts, maintenance, road construction and so on. We can, of course, use this calculation to compare the energy efficiency of alternative modes of transportation, for example, automobile use versus urban mass transit. This problem has been examined in detail by Hannon and Puelo (1974).

Similarly, we might wish to examine the energy intensity of family expenditures as a function of income (see Herendeen, 1974). The US Bureau of Labor Statistics routinely compiles personal consumption expenditure data that can be used in energy input–output analysis. Perhaps the most interesting result of this work is that *direct* energy consumption (e.g., gasoline) appears to level off with increasing income. The result,

Table 9.6 Technical Coefficients: Example 9.4

A*	Coal Mining	Oil & Nat. Gas	Ref. Petrol.	Elec. Utilities	Gas Utilities	Chem.	Agric.	Mining & Manuf.	Transp. & Comm.	Rest of Economy
1. Coal Mining	0.006	0.000	0.000	1.881	0.001	0.026	0.001	0.007	0.000	0.000
2. Oil & Nat. Gas.	0.000	0.026	0.833	0.000	1.253	0.007	0.000	0.000	0.000	0.000
3. Ref. Petroleum	0.002	0.001	0.058	0.220	0.001	0.035	0.012	0.006	0.049	0.004
4. Elec. Utilities	0.001	0.001	0.002	0.108	0.000	0.018	0.001	0.002	0.001	0.001
5. Gas Utilities	0.000	0.002	0.032	0.764	0.069	0.041	0.003	0.002	0.002	0.003
6. Chemicals	0.003	0.004	0.022	0.010	0.000	0.189	0.036	0.015	0.001	0.002
7. Agriculture	0.000	0.000	0.000	0.000	0.000	0.036	0.282	0.054	0.001	0.006
8. Mining & Manuf.	0.022	0.031	0.031	0.229	0.020	0.141	0.093	0.380	0.065	0.098
9. Transp. & Comm.	0.002	0.004	0.048	0.148	0.003	0.030	0.022	0.024	0.081	0.034
10. Rest of Economy	0.023	0.098	0.106	0.886	0.060	0.060	0.156	0.123	0.174	0.195

Table 9.7 Leontief Inverse: Example 9.4

$\mathbf{L}^*$	Coal Mining	Oil & Nat. Gas	Ref. Petrol.	Elec. Utilities	Gas Utilities	Chem.	Agric.	Mining & Manuf.	Transp. & Comm.	Rest of Economy
1. Coal Mining	1.009	0.004	0.012	2.151	0.007	0.086	0.013	0.023	0.006	0.006
2. Oil & Nat. Gas.	0.005	1.036	0.977	1.489	1.397	0.166	0.043	0.033	0.063	0.019
3. Ref. Petroleum	0.003	0.004	1.072	0.305	0.007	0.061	0.028	0.020	0.061	0.011
4. Elec. Utilities	0.001	0.002	0.005	1.132	0.003	0.027	0.004	0.005	0.002	0.002
5. Gas Utilities	0.002	0.005	0.047	0.954	1.081	0.081	0.014	0.012	0.008	0.008
6. Chemicals	0.005	0.007	0.039	0.064	0.012	1.248	0.071	0.039	0.008	0.009
7. Agriculture	0.004	0.008	0.018	0.092	0.015	0.092	1.422	0.133	0.017	0.028
8. Mining & Manuf.	0.046	0.080	0.177	0.966	0.160	0.372	0.298	1.704	0.174	0.221
9. Transp. & Comm.	0.006	0.013	0.079	0.310	0.026	0.070	0.060	0.063	1.108	0.056
10. Rest of Economy	0.041	0.146	0.320	1.837	0.288	0.249	0.354	0.317	0.289	1.301

Table 9.8 Total Primary Energy Intensities: Example 9.4

Industry Sector	Primary Energy Intensity
1. Coal Mining	1.016
2. Crude Oil	1.039
3. Refined Petroleum	0.980
4. Electric Utilities	3.985
5. Gas Utilities	1.405
6. Chemical Products	0.263
7. Agriculture	0.058
8. Mining and Manufacturing	0.057
9. Transportation & Communication	0.069
10. Rest of the Economy	0.028

Table 9.9 Power Plant Inputs: Example 9.4

Industry Sector	Power Plant I	Power Plant II
1. Coal Mining	0	0
2. Crude Oil	0	0
3. Refined Petroleum	0	0
4. Electric Utilities	100	200
5. Gas Utilities	100	0
6. Chemical Products	100	100
7. Agriculture	0	0
8. Mining and Manufacturing	1,000	1,000
9. Transportation & Communication	500	1,000
10. Rest of the Economy	1,000	500

when personal consumption expenditures are translated to indirect energy consumption via the energy input–output model, is that *total* energy consumption attributable to family expenditure does not level off with increasing income. Hence, Herendeen argues, estimates of impacts of energy shortfalls on consumers based on direct energy consumption alone could be quite misleading.

This concept of examining the energy impacts of changes in final demand can, of course, also be used to investigate changes in final demand other than that part which comprises personal consumption. Bezdek and Hannon (1974), for example, examined the impacts of various federal "public works" programs, Blair (1979) examined the regional impact of constructing new electric-power-generating facilities, Battjes, Noorman and Biesiot (1998) explore the energy intensity of imported goods, and Bezdek and Wendling (2005a, 2005b) examine the implications of fuel economy standards on automobile fuel consumption.

Efforts to improve the sustainable use of resources in an economy led researchers to examine the life cycle costs of products, including perhaps especially energy costs. Input–output analysis is, of course, an especially suitable tool to examine the total resource costs of industrial production and use and is used frequently in this type of analysis, such as in Hendrickson, Lave and Matthews (2006), and in the field of industrial ecology, such as in Duchin (1992). This subject is examined more generally and in more detail in Chapter 10.

9.4.3 Impacts of New Energy Technologies

Just (1974) used an input–output model to examine the impact of new energy technologies such as coal gasification or combined gas-and-steam-cycle electric power generation on the US economy. His approach was to estimate the column of technical coefficients, $\mathbf{A}_N$, that would describe the new technology. If $\mathbf{A}_j$ is the technology that might be replaced by $\mathbf{A}_N$, then the new column $\mathbf{A}_j^{new}$, reflecting incorporation of the new technology, would be $\mathbf{A}_j^{new} = g\mathbf{A}_N + (1-g)\mathbf{A}_j$, where g is the fraction of the total production of sector j for which replacement is expected. Gowdy and Miller (1968) employ a similar technique to examine energy efficiency technology in the US and Japan. Other examples of using input–output analysis to examine the impacts of new energy technologies include Herendeen and Plant (1981) and Blair (1979) and Casler and Hannon (1989).

9.4.4 An Energy Tax

Bullard and Herendeen (1975a) examined the impact of a tax on energy use (per BTU). They assumed that all of the tax would be passed on directly to the consumer. The results indicate that the tax would be distributed in such a way as to substantially increase the prices of energy-intensive products. The impacts of such a tax have been estimated for the US economy based on the 367-sector input–output model developed by Herendeen (1974). More contemporary models employ combined input–output and econometric techniques, discussed later in this chapter and in Chapter 14.

9.4.5 Energy and Structural Change

A fascinating time in US economic history was the period from the late 1970s through the end of the twentieth century, when the US economy began fundamental structural change from an economy dominated by manufacturing and related activity to one increasingly dominated by services. The degree to which the accompanying change in energy use in the economy is viewed as either cause or effect (most likely both) of the broader structural change in the economy is a matter of some debate and explains at least in part why energy use patterns associated with economic structural change have been widely studied since the 1980s and, as one might expect, input–output analysis was a common tool in many such analyses. Such work includes Park (1982), Proops (1984, 1988), OTA (1988, 1990), Blair and Wyckoff (1989), Rose and Chen (1991), all of which draw on the much earlier work of Strout (1967), Carter (1970) and Reardon (1976) and others. Many studies attribute the changes in the patterns of US energy consumption in the 1970s and 1980s to a combination of forces: economic growth, adoption of more energy-efficient technology, changes in the economic mix of goods and services away from energy-intensive industry such as steel production and towards high value-added manufacturing and services, and finally the interaction of these forces with one another; see Kelly, Blair and Gibbons (1989).

Analyzing energy use and structural change in an economy are really a subset of a broader collection of work commonly referred to as Structural Decomposition Analysis (SDA), which generally studies economic changes by fashioning a set of comparative static adjustments of key data in input–output tables, usually over a period of years. SDA in its broader context is discussed in detail in Chapter 13. SDA has been employed widely to explore the relationships between energy use and interindustry activity, including the work cited above, but some researchers, such as Rose and Chen (1991), Ang (1995), Lin and Polenske (1995), Mukhopadhyay and Chakraborty (1999), Wilting, Biesiot and Moll (1999), Jacobsen (2000), Casler (2001) and Kagawa and Inamura (2001, 2004) have assembled more elaborate breakdowns of the different sources of energy use changes. As an illustration for purposes of this chapter we describe a relatively simple process employed in OTA (1990) or Casler and Blair (1997), although Dietzenbacher and Sage (2006) outlines a number of limitations to using energy input–output models expressed in hybrid units in SDA.

Consider matrices of total energy coefficients for two different years, 1972 and 1985, as $\boldsymbol{\alpha}^{72}$ and $\boldsymbol{\alpha}^{85}$, respectively. Hence we can define $\mathbf{g}^{72} = \boldsymbol{\alpha}^{72}\mathbf{f}^{72}$ and $\mathbf{g}^{85} = \boldsymbol{\alpha}^{85}\mathbf{f}^{85}$, where $\mathbf{f}$ and $\mathbf{g}$ are the final demand and total energy consumption vectors for the corresponding years indicated by the superscripts. Recall that $\boldsymbol{\alpha}$ is $m \times n$, $\mathbf{f}$ is $n \times 1$ and $\mathbf{g}$ is $m \times 1$, where m is the number of energy sectors and n is the total number of industry sectors. If we consider 1972 the base year, then we seek to measure the sources of the changes of energy consumption resulting from changes in the production recipe (as reflected in the matrix of technical coefficients, $\mathbf{A}$) and from changes in final consumption, $\mathbf{f}$, between 1972 and 1985. So, relative to the base year the consumption for any other year, e.g., 1985, can be written as $\mathbf{g}^{85} = (\boldsymbol{\alpha}^{72}+\Delta\boldsymbol{\alpha})(\mathbf{f}^{72}+\Delta\mathbf{f})$ or, separating terms, $\mathbf{g}^{85} = \boldsymbol{\alpha}^{72}\mathbf{f}^{72} + \boldsymbol{\alpha}^{72}\Delta\mathbf{f} + \Delta\boldsymbol{\alpha}\mathbf{f}^{72} + \Delta\boldsymbol{\alpha}\Delta\mathbf{f}$ where $\Delta\boldsymbol{\alpha}$ is the matrix of changes in technical coefficients and $\Delta\mathbf{f}$ is the vector of changes in final demand between 1972 and 1985. Using this expression for $\mathbf{g}^{72}$ above, we can define the difference in energy consumption between 1985 and 1972 using the expression above for $\mathbf{g}^{85}$ as

$$\mathbf{g}^{85} - \mathbf{g}^{72} = \boldsymbol{\alpha}^{72}\mathbf{f}^{72} + \boldsymbol{\alpha}^{72}\Delta\mathbf{f} + \Delta\alpha\mathbf{f}^{72} + \Delta\boldsymbol{\alpha}\Delta\mathbf{f} - \boldsymbol{\alpha}^{72}\mathbf{f}^{72}$$

Combining terms, we have $\mathbf{g}^{85} - \mathbf{g}^{72} = \boldsymbol{\alpha}^{72}\Delta\mathbf{f} + \Delta\alpha\mathbf{f}^{72} + \Delta\boldsymbol{\alpha}\Delta\mathbf{f}$ and if we define these changes as $\Delta\boldsymbol{\alpha} = \boldsymbol{\alpha}^{85} - \boldsymbol{\alpha}^{72}$ and $\Delta\mathbf{f} = \mathbf{f}^{85} - \mathbf{f}^{72}$ then we have

$$\mathbf{g}^{85} - \mathbf{g}^{72} = \boldsymbol{\alpha}^{72}(\mathbf{f}^{85} - \mathbf{f}^{72}) + (\boldsymbol{\alpha}^{85} - \boldsymbol{\alpha}^{72})\mathbf{f}^{72} + (\boldsymbol{\alpha}^{85} - \boldsymbol{\alpha}^{72})(\mathbf{f}^{85} - \mathbf{f}^{72})$$

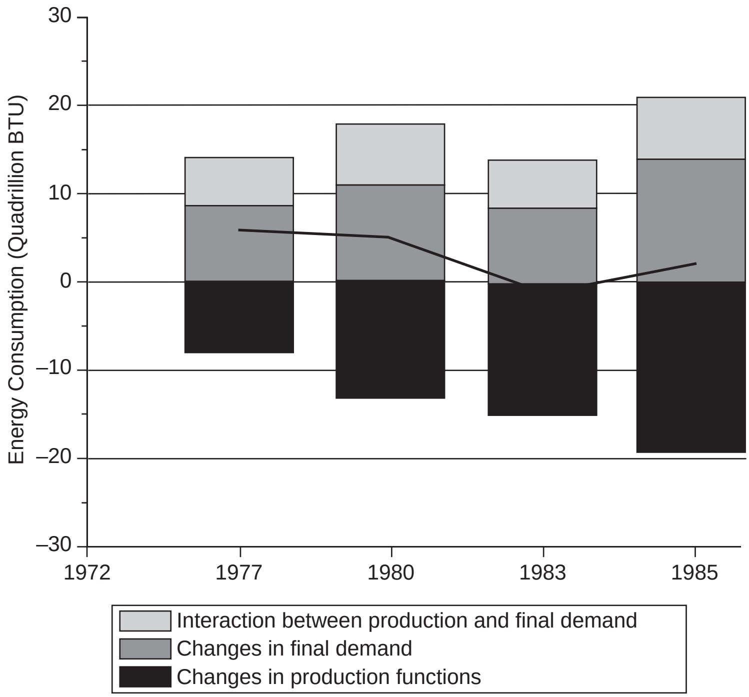

Figure 9.3 Changes in US Energy Consumption: 1972–1985

where $\boldsymbol{\alpha}^{72}(\mathbf{f}^{85} - \mathbf{f}^{72})$ is the effect caused by changing final demand; $(\boldsymbol{\alpha}^{85} - \boldsymbol{\alpha}^{72})\mathbf{f}^{72}$ is the effect caused by changes in production functions; and $(\boldsymbol{\alpha}^{85} - \boldsymbol{\alpha}^{72})(\mathbf{f}^{85} - \mathbf{f}^{72})$ the effect of interaction of final demand and production function changes.

Some simplified results of OTA (1990) are summarized in Figure 9.3. This figure shows that while total consumption of energy changed relatively little in the US over the period 1972–1985, the modest net change disguises, on one hand, substantial economic growth and associated increases in energy use offset by, on the other hand, large improvements in energy efficiency (nearly two-thirds of the total effect on reducing energy consumption) and a shift from more to less energy-intensive industry in the nation's economy overall.

9.4.6 Energy Input–Output and Econometrics

The relative analytical simplicity of input–output analysis can be both a strength and a limitation in its application to public policy problems. On the one hand, the conceptually simplifying assumptions of fixed input requirements and constant relative prices, both implicit in the classical input–output framework, render implementation easier, although satisfying the data requirements can still be quite difficult. On the other hand, these assumptions limit the ability of the framework to deal very effectively with some fundamental features of a modern economic system, such as prices or elasticities.

Nonetheless, many modern economic models that can accommodate such features still have their roots in an input–output framework.

In addition, some of these models seek to make input–output itself more flexible by formally defining input–output coefficients as a function of relative prices of aggregate prices of capital, labor, energy and materials, and in more detailed models of more specific industry output prices. Among the most prominent of these models was applied to long-term energy policy evaluation in Hudson and Jorgenson (1974) and considerable subsequent work by those authors and other associates in both energy and environmental policy applications. The model was also commercially developed and applied for a number of years as the Data Resources Inc. Long-Term Inter-industry Model (LTIM).

In the following we briefly characterize the key features of the Hudson–Jorgenson (HJ) model to illustrate one of the extensions to the basic input–output framework that has developed extensively over the last several decades. Additional discussion of the development of the broader class of econometric/input–output models is included in Chapter 14.

In the simplest terms, the HJ model is constructed as the interaction of an econometric aggregate macroeconomic growth model and an interindustry model with price sensitive input–output technical coefficients. The macroeconomic growth model is beyond the scope of this text, but is used to provide aggregated input prices and final demands that are used by the interindustry model to compute primary input demands to the economy, as depicted in Figure 9.4.

The HJ model, in its first release (Hudson and Jorgenson, 1974), employed a nine-sector input–output framework that included four highly aggregated nonenergy sectors and the five energy sectors common to the aggregation scheme adopted by the Bureau

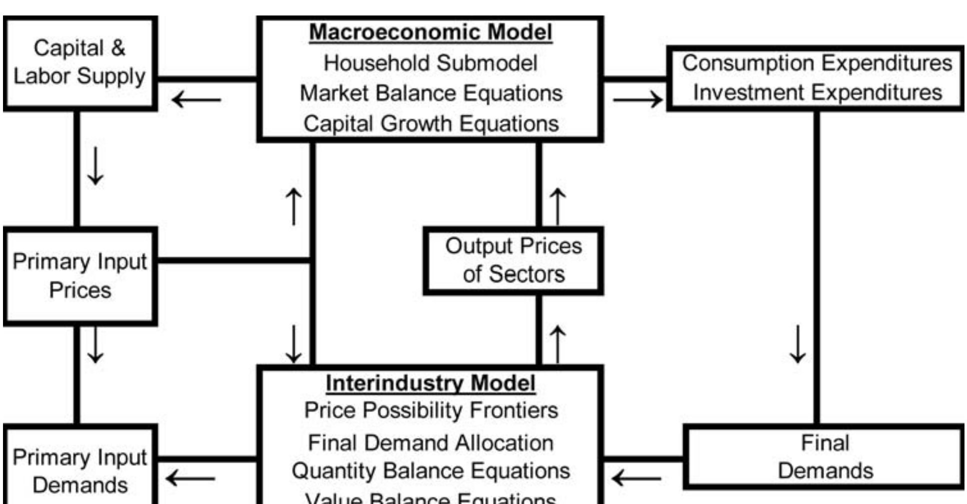

Figure 9.4 Hudson–Jorgenson Model

of Economic Analysis: (1) agriculture, nonfuel mining and construction, (2) manufacturing (excluding petroleum refining), (3) transportation, (4) communications, trade, services, (5) coal mining, (6) crude petroleum and natural gas, (7) petroleum refining, (8) electric utilities and (9) gas utilities. In order to facilitate connection with other model components, the framework also breaks out value-added inputs into three components: (1) imports, (2) capital services and (3) labor services. Similarly, total final demand is disaggregated into (1) personal consumption expenditures, (2) gross domestic private investment, (3) government purchases of goods and services, and (4) exports.

The HJ model operates by first employing the macroeconomic growth model (or macro-growth model, for short) noted earlier to produce levels of final demand for all industries. For the five energy sectors, the prices and levels of imports are taken to be exogenous, although this feature was modified in later versions of the model, such as Jorgenson and Wilcoxen (1990, 1993), Jorgenson, Slesnick and Wilcoxen (1992), or Jorgenson and Stiroh (2000). For the four nonenergy sectors, the prices of imports are taken to be exogenous (again, in the original version of the model), but the quantities of imports and of labor and capital services are subsequently determined endogenously by the interindustry model as modified below.

The prices of capital and labor services are also produced by the macro-growth model, while the quantities of exports and government purchases of all industry sectors are taken to be exogenous along with the allocation of total investment among all industries (although level of total investment itself is generated by the macro-growth model). Perhaps the unique modification to the classical Leontief framework in the HJ model is that of incorporating producer behavior into the interindustry model by specifying production functions (columns of the technical coefficient matrix) in terms of the relative prices of factor inputs (industry output prices and prices of capital and labor services).[10]

The framework still requires the row and column interindustry identities to hold. That is, the row sums of interindustry transactions plus deliveries to final demand equals total output and the column sums of the *value* of interindustry transactions plus value-added services (in this case, capital and labor services) plus the value of imports also equals total output. The row sum version is $x_i = \sum_{j=1}^{9} z_{ij} + f_i$ for $i = 1, \ldots, 9$ and the column sum version is $p_i x_i = \sum_{j=1}^{9} p_j z_{ji} + p_k x_{ki} + p_l x_{li} + p_{ri} x_{ri}$ for $i = 1, \ldots, 9$, where

x_{ki} = quantity of capital services used by industry i,

x_{li} = quantity of labor services used by industry i,

x_{ri} = quantity of competitive imports of the output of industry i,

p_k = aggregate price of capital services,

10 Another approach to models where production functions respond to prices is Liew (1980) and subsequent work, which is discussed in Chapter 14.

p_l = aggregate price of labor services and

p_{ri} = price of competitive imports to industry i.

In the HJ formulation prices are explicitly identified in the value-added identity since prices become variables. The prices of capital and labor services actually can differ among industries (suggesting that p_k and p_l should each have an additional i subscript), but they are computed as the product of aggregate prices from the macro-growth model and the ratios of service prices to each industry that are determined exogenously, so they are actually completely specified by the macro-growth model and just scaled by exogenously specified ratios. Hence, for simplicity, we show them here as the aggregate prices.

The HJ model uses a collection of so-called price possibility frontiers, which econometrically specify a level of output of an industry as a function of the relative prices of industry inputs and the prices of capital, labor, and imports. In the HJ formulation, for each of the nine industry sectors, three relationships are specified:

1. The price of a sector's output as a function of the prices of four aggregate inputs – capital, labor, energy, and materials.
2. The price of aggregate energy input in each sector as a function of the prices of the five types of energy specified in the model – coal, crude oil and natural gas, refined petroleum, electricity and gas delivered by gas utilities.
3. The price of aggregate nonenergy inputs to each sector as a function of the prices of the five types of nonenergy inputs to each sector – agriculture, manufacturing, transportation, communications, and competitive imports.

The formal specification of the price possibility frontiers is beyond the scope of this text, but is covered in detail in Christensen, Jorgenson and Lau (1971, 1973), where it is postulated that a useful local second-order approximation of any price possibility frontier can be specified as a function that is quadratic in the logarithms of the prices of the inputs to that sector. These so-called transcendental logarithmic price possibility frontiers, or translog price possibility frontiers for short, take, as an example, the following form:

$$\ln a_i + \ln p_i = \alpha_0^i + \alpha_K^i \ln p_k + \alpha_L^i \ln p_L + \alpha_E^i \ln p_E + \alpha_M^i \ln p_M + \frac{1}{2}\left[\beta_{KK}^I (\ln p_K)^2 + \beta_{KL}^I \ln p_K \ln p_L + \cdots\right]$$

where in this case a_i is the relative share of input i, p_i is price of commodity i and p_K, p_L, p_E, and p_M are, respectively, the aggregate prices of capital, labor, energy, and materials. The translog production functions make it possible to include price and in this instance especially the price of energy, as a variable in this modified interindustry model. The result is an ability to project energy supply and demand, energy price and cost and energy imports and exports under a much richer variety of economic conditions than with the traditional static input–output framework.

The HJ model was used to explore the potential economic implications of an energy tax, which at the time was defined as a so-called "BTU tax," since the rate would be assessed based on the heating value of all energy fuels or, more specifically, "a uniform rate of tax levied on the energy content of all fuels used outside the energy generation sector." (Hudson and Jorgenson, 1974.) At the time this was one of the policy options being considered by Congress as a means of seeking to achieve "energy independence" in the United States. In the HJ model the tax is introduced as a price markup on all energy sources and new economic and energy use projections are calculated. The authors conclude that major reductions in energy use (in the early 1970s) were achievable in the US economy without major economic cost.

9.4.7 *Other Applications*

The energy input–output model has been used to examine a wide variety of other problems in addition to those just outlined. These include a detailed characterization of the US import–export balance (Bullard and Herendeen, 1975b), analysis of the costs versus benefits of alternative energy conservation programs (Henry, 1977), energy consumption analysis (Bullard and Herendeen, 1975a), regional energy trade balance relationships (Bourque, 1981) and others related to the energy implications of specific policy initiatives, including Almon *et al*. (1974), Bullard, Penner and Pilati (1978), Polenske (1976), Proops (1977, 1984, 1988) and Bezdek and Wendling (2005a, b).

Since the early 1990s, applications of input–output analysis to energy issues reported in the literature have been dominated by three areas: (1) more detailed analysis of energy and material flows in industrial complexes, such as in Albino, Dietzenbacher and Kühtz (2003) or Giljum and Hubacek (2004), (2) analyzing the relationship between energy use and environmental issues in areas such as global climate change and sustainable development, such as in Lenzen, Pade and Munksgaard (2004), Kratena and Schleicher (1999), Zhang and Folmer (1998) or the extensions to the Hudson–Jorgenson model noted earlier (Jorgenson and Wilcoxen, 1990, 1993, or Jorgenson and Stiroh, 2000) (some of these energy-environment extensions are examined in more detail in Chapter 10) and (3) analyzing changes in economic structure related to changing patterns of energy use in economies, as in Kagawa and Inamura (2004) and others mentioned earlier in this chapter. The subject of structural decomposition is explored in more detail and more generally as well in Chapter 13.

9.5 Summary

In this chapter we have presented an energy input–output model by constructing matrices of direct and total energy coefficients in so-called "hybrid units." As noted earlier, alternative approaches are discussed in Appendix 9.1. These approaches were widely used in the late 1960s and early 1970s and continue to be used commonly today for a variety of practical reasons. However, these models produce energy coefficients dependent upon the level of final demand that was presented to the energy input–output model, which we show in Appendix 9.1 to be a fundamental flaw, although the seriousness of

Table 9.10 Summary of Energy Input–Output Relationships: Initial Formulation

	Economic ($n \times n$)	Energy ($m \times n$)
Transactions	$\mathbf{Z}$	$\mathbf{Z}^*, \mathbf{E}$
	$\mathbf{Zi} + \mathbf{f} = \mathbf{x}$	$\mathbf{Z}^*\mathbf{i} + \mathbf{g}^* = \mathbf{x}^*$
		$\mathbf{Ei} + \mathbf{q} = \mathbf{g}$
Direct Requirements	$\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$	$\mathbf{A}^* = \mathbf{Z}^*(\hat{\mathbf{x}}^*)^{-1}; \boldsymbol{\delta} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}\mathbf{A}^*$
	$\mathbf{Ax} + \mathbf{f} = \mathbf{x}$	$\mathbf{A}^*\mathbf{x}^* + \mathbf{f}^* = \mathbf{x}^*; \boldsymbol{\delta}\mathbf{x}^* + \mathbf{q} = \mathbf{g}$
Total Requirements	$\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$	$\mathbf{L}^* = (\mathbf{I} - \mathbf{A}^*)^{-1}; \boldsymbol{\alpha} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}\mathbf{L}^*$
	$\mathbf{x} = \mathbf{Lf}$	$\mathbf{x}^* = \mathbf{L}^*\mathbf{f}^*; \mathbf{g} = \boldsymbol{\alpha}\mathbf{f}^*$

the flaw depends upon the circumstances. The hybrid units approach does not suffer from this limitation. We show that the hybrid units approach yields energy coefficients that conform to a fundamental definition of energy conservation conditions; the alternative formulations are shown in Appendix 9.1 to conform only to these conditions when interindustry prices of energy are uniform across all consuming sectors. Table 9.10 summarizes the energy input–output relationships developed in this chapter.

An analogous table is presented in Appendix 9.1 (Table A9.1.4) showing the relationships among the original Leontief model, the hybrid units as developed here, and alternative energy input–output formulations reviewed in the appendix.

The energy input–output extensions to the classical Leontief model are widely applied in the literature and some of those extensions and applications are summarized. Finally, more advanced extensions are described – such as coupling an interindustry model with econometric models of demand behavior and macroeconomic growth to provide more flexibility for dealing with public policy issues. Such models lay the groundwork for even more flexible general equilibrium models outlined in Chapter 14.

Appendix 9.1 Earlier Formulation of Energy Input–Output Models

A9.1.1 Introduction

In this appendix we present an alternative formulation of the energy input–output model. While still widely applied in the literature, this approach suffers from limitations that in some cases should preclude its use. This formulation was initially adopted by Strout (1967) and Bullard and Herendeen (1975b). In other cases, as we will show, however, the model can be acceptable or even equivalent to the formulation presented in section 9.2.

First, recall the $m \times n$ matrix of energy flows, $\mathbf{E}$, which was defined in the text of this chapter and used in the basic accounting relationship

$$\mathbf{Ei} + \mathbf{q} = \mathbf{g} \tag{A9.1.1}$$

A traditional approach to energy input–output analysis is to define a matrix of *direct energy coefficients*, $\mathbf{D} = [d_{kj}]$ where $d_{kj} = e_{kj}/x_j$, that is, the amount of energy type k (in BTUs or some other convenient energy units for $k = 1, \ldots, m$) required directly to produce a dollar's worth of each producing sector's output ($j = 1, \ldots, n$). Expressed in matrix terms this is $\mathbf{D} = \mathbf{E}\hat{\mathbf{x}}^{-1}$. This is, of course, directly analogous to the direct input coefficients, $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$, except that $\mathbf{D}$ will in general not be square since $m < n$.

For purposes that will become clear later, defining a direct energy coefficient is equivalent to first defining a matrix, $\mathbf{P}$, of *implied energy prices*, with elements defined as $p_{kj} = z_{kj}/e_{kj}$ ($k = 1, \ldots, m$; $j = 1, \ldots, n$), defined only for $e_{kj} \neq 0$. The units of p_{kj} are then dollars paid per unit of energy of type k delivered to consuming sector j. These prices are "implied" since the prices calculated in this way generally do not necessarily correspond to the price actually paid for energy, but their significance, nonetheless, will become clear shortly. For now, implied prices can be used to derive the direct energy coefficients as $d_{kj} = \frac{a_{kj}}{p_{kj}}$. This is equivalent to our previous definition of $\mathbf{D}$, since $d_{kj} = \frac{a_{kj}}{p_{kj}} = \left(\frac{z_{kj}}{x_j}\right)\left(\frac{e_{kj}}{z_{kj}}\right) = \frac{e_{kj}}{x_j}$ or, in matrix terms, $\mathbf{D} = \mathbf{E}\hat{\mathbf{x}}^{-1}$. It follows directly that $\mathbf{E} = \mathbf{D}\hat{\mathbf{x}}$ and from the original energy transactions balance equation $\mathbf{Ei} + \mathbf{q} = \mathbf{g}$, we obtain $\mathbf{D}\hat{\mathbf{x}}\mathbf{i} + \mathbf{q} = \mathbf{g}$ but, since $\hat{\mathbf{x}}\mathbf{i} = \mathbf{x}$, $\mathbf{Dx} + \mathbf{q} = \mathbf{g}$, which, as noted earlier, is directly analogous to $\mathbf{Ax} + \mathbf{f} = \mathbf{x}$ of the traditional Leontief model. The traditional method continues to develop a matrix of total energy coefficients first by substituting $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$ to obtain

$$\mathbf{Dx} = \mathbf{D}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{f} \tag{A9.1.2}$$

The matrix $\mathbf{D}(\mathbf{I}-\mathbf{A})^{-1}$ is defined as the matrix of total interindustry energy coefficients.

In order to account for the energy consumed directly by final demand, the second term in the energy transactions balance equation (A9.1.1), we return to the notion of implied energy prices, this time for the energy that is delivered to final demand (as done in earlier interindustry transactions when the p were defined – recall that direct energy coefficients were defined only for interindustry energy transactions). Now we have $\mathbf{p_f} = [p_{kf}]$ where

$$p_{kf} = f_k/q_k \tag{A9.1.3}$$

Here f_k is the final demand in dollars for the output of energy sector k and p_{kf} is the corresponding implied energy price in units of dollars of final demand per unit of energy type k (for $q_k \neq 0$; for $q_k = 0$ we will define $p_{kf} = 0$). This relationship allows us to express final demand and the corresponding energy requirements in a manner similar to that for interindustry energy requirements associated with interindustry transactions, by rewriting (A9.1.3) as $q_k = (1/p_{kf})f_k$ or in matrix terms as $\mathbf{q} = \tilde{\mathbf{Q}}\mathbf{f}$, where $\tilde{\mathbf{Q}} = [\tilde{q}_k]$ is an $m \times n$ matrix of implied inverse energy prices for final demand whose elements are defined as

$$\tilde{q}_k = \begin{cases} 1/p_{kf}, & \text{when energy sector } k \text{ and industry sector } j \\ & \text{describe the same industrial sector} \\ 0, & \text{otherwise} \end{cases}$$

There will, of course, be at most m nonzero elements in $\tilde{\mathbf{Q}}$ since there are only m elements in $\mathbf{q}$. By constructing $\tilde{\mathbf{Q}}$ of dimension $m \times n$, we can combine it with the interindustry energy coefficients to produce a matrix of total (interindustry plus final-demand) energy coefficients, to obtain $\mathbf{g} = \mathbf{Dx} + \mathbf{q}$ or, substituting $(\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$ for $\mathbf{x}$, we have $\mathbf{g} = \mathbf{D}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{f} + \tilde{\mathbf{Q}}\mathbf{f}$ and collecting terms we have

$$\mathbf{g} = [\mathbf{D}(\mathbf{I} - \mathbf{A})^{-1} + \tilde{\mathbf{Q}}]\,\mathbf{f} \tag{A9.1.4}$$

The bracketed quantity, which we denote by $\boldsymbol{\varepsilon}$, is a matrix of *total energy coefficients* analogous to $\boldsymbol{\alpha}$ defined in the course of developing the energy conservation conditions, which expresses the total amount of energy (BTUs) required of each energy type, $\mathbf{g}$, both directly and indirectly, as a function of final demand $\mathbf{f}$.

Variations of this approach abound in the literature, sometimes ignoring the energy consumed directly in final demand, sometimes assuming uniform energy prices across all consuming sectors, but almost always defining a set of direct energy coefficients in this manner and thereby ignoring or assuming away the technical energy conservation relationships between primary and secondary energy sectors.

A9.1.2 Illustration of the Implications of the Traditional Approach

The following example illustrates the inconsistencies introduced by using variations of the traditional approach just outlined.

Example 9.5: Energy Input–Output Alternative Formulation Consider a simple three-sector input–output economy where two of the sectors are energy sectors, coal and electricity. Assume that transactions (in millions of dollars) observed for a given year are as shown in Table A9.1.1. Here $\mathbf{Z}$, $\mathbf{f}$ and $\mathbf{x}$ are $\mathbf{Z} = \begin{bmatrix} 0 & 40 & 0 \\ 10 & 10 & 10 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbf{f} = \begin{bmatrix} 0 \\ 30 \\ 10 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 40 \\ 60 \\ 100 \end{bmatrix}$. Suppose that the corresponding energy flows of this economy, expressed in quadrillions of BTUs, are given by Table A9.1.2. Hence, using notation introduced earlier, $\mathbf{E} = \begin{bmatrix} 0 & 120 & 0 \\ 20 & 20 & 20 \end{bmatrix}$, $\mathbf{q} = \begin{bmatrix} 0 \\ 60 \end{bmatrix}$ and $\mathbf{g} = \begin{bmatrix} 120 \\ 120 \end{bmatrix}$.

Note some of the special characteristics of this energy economy. First, the coal sector delivers all of its product to the electricity sector, another energy sector. Hence, as discussed earlier, the coal sector is known as a *primary* energy sector and electricity is a *secondary* energy sector. Note also that the total amount of coal used is the same as the amount of electricity consumed in the economy, which seems reasonable since the electricity sector received all of its primary energy from coal (excluding conversion efficiencies, for the moment). Another important peculiarity of this example is the matrix of *implied energy prices*, $\mathbf{P} = [z_{kj}/e_{kj}] = \begin{bmatrix} 0 & 40/120 & 0 \\ 10/20 & 10/20 & 10/20 \end{bmatrix} = \begin{bmatrix} 0 & 0.333 & 0 \\ 0.5 & 0.5 & 0.5 \end{bmatrix}$ and $\mathbf{p_f} = [f_k/q_k] = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$. Note that the price of electricity

Table A9.1.1 Dollar Transactions for Example 9.5 (millions of dollars)

	Coal	Electric Power	Autos	Final Demand	Total Output
Coal	0	40	0	0	40
Electric Power	10	10	10	30	60
Automobiles	0	0	0	100	100

Table A9.1.2 Energy Flows for Example 9.5 (10^{15} BTUs)

	Coal	Electric Power	Autos	Final Energy Demand	Total Energy Output
Coal	0	120	0	0	120
Electric Power	20	20	20	60	120

is the same across all consuming sectors, including final demand (0.5). Hence, by the earlier development, $\tilde{\mathbf{Q}}$, the matrix of inverse energy prices for final demand, and $\mathbf{D}$, the direct energy coefficients matrix are $\tilde{\mathbf{Q}} = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 0 & 120 & 0 \\ 20 & 20 & 20 \end{bmatrix} \begin{bmatrix} 1/40 & 0 & 0 \\ 0 & 1/60 & 0 \\ 0 & 0 & 1/100 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0.5 & 0.333 & 0.2 \end{bmatrix}$. Finally, we can compute $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 0 & 0.667 & 0 \\ 0.25 & 0.167 & 0.1 \\ 0 & 0 & 0 \end{bmatrix}$ and $(\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.25 & 1.00 & 0.10 \\ 0.38 & 1.50 & 0.15 \\ 0 & 0 & 1.00 \end{bmatrix}$. Knowing $\mathbf{D}$, $(\mathbf{I} - \mathbf{A})^{-1}$ and $\tilde{\mathbf{Q}}$ and using (A9.1.4) we can find the total energy requirements, $\boldsymbol{\varepsilon} = \mathbf{D}(\mathbf{I} - \mathbf{A})^{-1} + \tilde{\mathbf{Q}} = \begin{bmatrix} 0.75 & 3 & 0.3 \\ 0.75 & 3 & 0.3 \end{bmatrix}$.

It should not be surprising, at least for this example, that the rows of $\boldsymbol{\varepsilon}$ are identical, because of the peculiarities of this energy economy noted earlier. Suppose, however, we change the example only slightly to remove the uniformity of energy prices across consuming sectors.

Example 9.6: Energy Input–Output Example (Revised) In modifying Example 9.5, only slightly, we redefine only **E** and **Q** as shown in Table A9.1.3, i.e., increasing the amount of electricity consumed by the autos sector from 20 to 30 quads and reducing

Table A9.1.3 Energy Flows for Example 1 Revised (10^{15} BTUs)

	Coal	Electric Power	Autos	Final Energy Demand	Total Energy Output
Coal	0	120	0	0	120
Electric Power	20	20	**30**	**50**	120

the amount of electricity consumed by final demand from 60 to 50 quads (denoted in bold face in the table).

Note that we do not change the total energy consumption of 120 quads in Table A9.1.2 and we do not change the economic transactions measured in dollars, $\mathbf{Z}$. However, since some of the energy transactions measured in quads change, the corresponding relative interindustry and final-demand energy prices change as well. The new implied energy prices for interindustry and final-demand sales along with the matrix of implied inverse energy prices to final demand are then, respectively, given by $\mathbf{P} = \begin{bmatrix} 0 & 0.333 & 0 \\ 0.5 & 0.5 & 0.333 \end{bmatrix}$, $\mathbf{p_f} = \begin{bmatrix} 0 \\ 0.599 \end{bmatrix}$ and $\tilde{\mathbf{Q}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1.67 & 0 \end{bmatrix}$. Note that the prices are no longer uniform. Finally $\boldsymbol{\varepsilon}$, the matrix of total energy coefficients, becomes $\boldsymbol{\varepsilon} = \begin{bmatrix} 0.75 & 3 & 0.3 \\ 0.75 & 2.667 & 0.4 \end{bmatrix}$. Looking at the elements in the third column, this new total energy requirements matrix specifies that one dollar's worth of automobiles requires 0.4×10^{15} BTUs of electricity to produce that output, but only 0.3×10^{15} BTUs of coal. This violates the energy conservation condition for this example, since the electricity-producing sector received all its primary energy from coal (electricity is a pass-through sector for coal). In other words, by design for this example the two total energy requirements matrix rows should be the same.

It should be readily apparent that application of this energy input–output formulation simply yields the output of the traditional Leontief model multiplied by a set of conversion factors – the implied energy prices. Such formulations are frequently applied in the literature, but in the following we show more generally that this formulation provides internally consistent results only when these energy prices are the same across *all* consuming sectors (including final demand) for each energy type or when a new final demand presented to the economy is very close to that from which the input–output model was originally derived. Only under such circumstances will the model always faithfully reproduce the original data. Griffin (1976) shows that the condition of uniform prices across all energy-consuming sectors does not hold at all historically for the US economy. Similar results are illustrated in Weisz and Duchin (2006). Possible cases where it could be more acceptable are discussed later. For reference, Table A9.1.4 summarizes the alternative energy formulation just described compared with the analogous "hybrid units" formulation developed in this chapter and so-called physical input–output models.

Table A9.1.4 Summary of Energy Input–Output Relationships

	Economic Model	Hybrid Units Energy Model Method II	Alternative Energy Model Method I
Transactions	$\mathbf{Z}$ $\mathbf{Zi}+\mathbf{f}=\mathbf{x}$	$\mathbf{Z}^*, \mathbf{E}$ $\mathbf{Z}^*\mathbf{i}+\mathbf{f}^*=\mathbf{x}^*$ $\mathbf{Ei}+\mathbf{q}=\mathbf{g}$	$\mathbf{E}$ $\mathbf{g}=\mathbf{Ei}+\mathbf{q}$
Direct Requirements	$\mathbf{A}=\mathbf{Z}\hat{\mathbf{x}}^{-1}$ $\mathbf{Ax}+\mathbf{f}=\mathbf{x}$	$\mathbf{A}^*=\mathbf{Z}^*\hat{\mathbf{x}}^{-1}; \boldsymbol{\delta}=\mathbf{G}(\hat{\mathbf{x}}^*)^{-1}\mathbf{A}^*$ $\mathbf{A}^*\mathbf{x}+\mathbf{f}^*=\mathbf{x}^*; \boldsymbol{\delta}\mathbf{x}^*+\mathbf{q}=\mathbf{g}$	$\mathbf{D}=\mathbf{E}\hat{\mathbf{x}}^{-1}$ $\mathbf{Dx}+\mathbf{q}=\mathbf{g}$
Total Requirements	$(\mathbf{I}-\mathbf{A})^{-1}$ $\mathbf{x}=(\mathbf{I}-\mathbf{A})^{-1}\mathbf{f}$	$(\mathbf{I}-\mathbf{A}^*)^{-1}; \boldsymbol{\alpha}=\mathbf{G}(\hat{\mathbf{x}}^*)^{-1}(\mathbf{I}-\mathbf{A}^*)^{-1}$ $\mathbf{x}^*=(\mathbf{I}-\mathbf{A}^*)^{-1}\mathbf{f}^*;\ \mathbf{g}=\boldsymbol{\alpha}\mathbf{f}^*$	$\boldsymbol{\varepsilon}=\mathbf{D}(\mathbf{I}-\mathbf{A})^{-1}+\tilde{\mathbf{Q}}$ $\mathbf{g}=\boldsymbol{\varepsilon}\mathbf{f}$

We now explore further the conditions of energy conservation and the conditions under which the alternative model can be applied, first through an example and then more generally. First, however, for reference, in Table A9.1.4 we summarize the relationships developed so far for the traditional Leontief model, the hybrid units interindustry model defined in the text of this chapter, which we will refer to as Method II and, finally, the alternative energy model just defined, which we will refer to as Method I.

Extensions of Example 9.1 Recall the two-sector economy given in Example 9.1 of the text of this chapter where we constructed the following hybrid units energy input–output relationships using Method II: $\mathbf{Z}^* = \begin{bmatrix} 10 & 20 \\ 60 & 80 \end{bmatrix}$ and $\mathbf{x}^* = \begin{bmatrix} 100 \\ 240 \end{bmatrix}$. The direct and total energy requirements matrices (Method II) for this example are $\mathbf{A}^* = \mathbf{Z}^*(\hat{\mathbf{x}}^*)^{-1} = \begin{bmatrix} 0.100 & 0.083 \\ 0.600 & 0.333 \end{bmatrix}$ and $\mathbf{L}^* = \begin{bmatrix} 1.212 & 1.515 \\ 1.091 & 1.636 \end{bmatrix}$ so that $\boldsymbol{\delta} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}\mathbf{A}^* = \begin{bmatrix} 0.600 & 0.333 \end{bmatrix}$ and $\boldsymbol{\alpha} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}\mathbf{L}^* = \begin{bmatrix} 1.091 & 1.636 \end{bmatrix}$. The analogous information for the alternative energy input–output formulation, using Method I, is given by the energy transactions $\mathbf{E} = \begin{bmatrix} 60 & 80 \end{bmatrix}$ and $\mathbf{q} = [100]$ and the interindustry dollar transactions and total outputs by $\mathbf{Z} = \begin{bmatrix} 10 & 20 \\ 30 & 40 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 100 \\ 200 \end{bmatrix}$. Hence, the direct and total energy requirements matrices are $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 0.100 & 0.167 \\ 0.300 & 0.333 \end{bmatrix}$ and $(\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} 1.212 & 0.303 \\ 0.546 & 1.636 \end{bmatrix}$. Using $\mathbf{q}$, we have $\tilde{\mathbf{Q}} = \begin{bmatrix} 0 & 100/50 \end{bmatrix} = \begin{bmatrix} 0 & 2 \end{bmatrix}$ and $\mathbf{D} = \mathbf{E}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 60 & 80 \end{bmatrix} \begin{bmatrix} 1/100 & 0 \\ 0 & 1/120 \end{bmatrix} = \begin{bmatrix} 0.600 & 0.667 \end{bmatrix}$.

It follows directly from (A9.1.4) that $\boldsymbol{\varepsilon} = \mathbf{E}\hat{\mathbf{x}}^{-1}(\mathbf{I}-\mathbf{A})^{-1} + \tilde{\mathbf{Q}} = \mathbf{D}(\mathbf{I}-\mathbf{A})^{-1} + \tilde{\mathbf{Q}}$, which for the example is $\boldsymbol{\varepsilon} = \begin{bmatrix} 0.600 & 0.667 \end{bmatrix} \begin{bmatrix} 1.212 & 0.303 \\ 0.546 & 1.636 \end{bmatrix} + \begin{bmatrix} 0 & 2 \end{bmatrix} =$

$\begin{bmatrix} 1.091 & 3.273 \end{bmatrix}$. Note that $\boldsymbol{\varepsilon}$ is identical to $\boldsymbol{\alpha}$, except that the elements involving energy consumption in $\boldsymbol{\alpha}$ are simply multiplied by the relevant energy price. This is reasonable because $\boldsymbol{\varepsilon}$ is used in conjunction with $\mathbf{f}$ and not with $\mathbf{f}^*$. That is, $\mathbf{f}^* = \begin{bmatrix} 70 \\ 100 \end{bmatrix}$ with an energy price of 2 (10^{15} BTUs/\$$10^6$) is equivalent to $\mathbf{f} = \begin{bmatrix} 70 \\ 50 \end{bmatrix}$, so, we have $\boldsymbol{\varepsilon}\mathbf{f} = \begin{bmatrix} 1.091 & 3.272 \end{bmatrix}\begin{bmatrix} 70 \\ 50 \end{bmatrix} = 240$ and $\boldsymbol{\alpha}\mathbf{f}^* = \begin{bmatrix} 1.091 & 1.636 \end{bmatrix}\begin{bmatrix} 70 \\ 100 \end{bmatrix} = 240$. The first expression, $\boldsymbol{\varepsilon}\mathbf{f}$, generates the total energy requirement (240×10^{15} BTUs) needed to support final demand $\mathbf{f}$. The second expression, $\boldsymbol{\alpha}\mathbf{f}^*$, yields the same result but in terms of supporting the equivalent final demand, $\mathbf{f}^*$, measured in hybrid units.

The result should not be surprising at all, since under conditions of uniform interindustry energy prices, the computation of $\boldsymbol{\varepsilon}$ is simply a price adjustment of $\boldsymbol{\alpha}$. To reflect this in our notation, we define a two-element vector $\mathbf{r} = [1\ 2]$ where the first element is the value that converts the nonenergy units of the original model to the nonenergy units of the hybrid units model. Clearly these units are the same, so the value of this element is always unity. The second element is the interindustry inverse energy price.

Given this vector $\mathbf{r}$, we can easily write $\mathbf{f}^* = \hat{\mathbf{r}}\mathbf{f}$. For the example this is $\mathbf{f}^* = \hat{\mathbf{r}}\mathbf{f} = \begin{bmatrix} 70 \\ 100 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 70 \\ 50 \end{bmatrix}$. Also, $\mathbf{x}^* = \hat{\mathbf{r}}\mathbf{x}$ or $\mathbf{x} = \hat{\mathbf{r}}^{-1}\mathbf{x}^*$. For the example this is $\mathbf{x}^* = \hat{\mathbf{r}}\mathbf{x} = \begin{bmatrix} 100 \\ 240 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 100 \\ 120 \end{bmatrix}$. The implications of this are as follows. For the case of uniform interindustry energy prices, there is no need to account for energy in BTUs at all, since this is equivalent to deriving outputs in dollars and converting to BTUs by simply multiplying by the energy price. However, as we found before, if prices are not uniform for all consumers (both interindustry and final-demand consumers), such procedures are inappropriate.

It is important to note that the above result, i.e., that a vector $\mathbf{r}$ exists such that $\mathbf{x}^* = \hat{\mathbf{r}}\mathbf{x}$ and $\mathbf{f}^* = \hat{\mathbf{r}}\mathbf{f}$, will in general be true *only* under conditions of uniform energy prices, which we will illustrate in the following. Recall that in the case of using the alternative formulation in Example 9.1, when this condition was not met, the model gave inappropriate results. We can test to see if the hybrid units model fares better when we relax the condition of uniform energy prices by considering, once again, the two-sector model of Example 9.1 with new energy flows and corresponding energy prices.

Note that the dollar quantities, $\mathbf{Z}$, $\mathbf{f}$, $\mathbf{x}$, $\mathbf{A}$ and $(\mathbf{I} - \mathbf{A})^{-1}$ do not change at all from the earlier case. However, the hybrid units quantities change since the energy transactions have changed by reducing the amount of energy delivered to final demand by 20 quadrillion BTUs and increasing the amount of energy consumed by the energy sector itself by the same amount, thus keeping total energy output the same. With a change in energy flows but no change in the corresponding dollar transactions, the energy prices change and are no longer uniform for all consumers, as shown in Table A9.1.6.

Table A9.1.5 Energy and Dollar Flows for Example 9.1 (Revised)

	Widgets	Energy	Final Demand	Total Output
	Value Transactions in Millions of Dollars			
Widgets	10	20	70	100
Energy	30	40	50	120
	Energy Transactions in Quadrillions of BTUs			
Energy	60	**100**	**80**	240

Table A9.1.6 Implied Energy Prices for Example 9.1 (Revised)

10^{15} BTUs/ $\$10^6$	Widgets	Energy	Final Demand	Total Output
Energy	2	2.5	1.6	2

Table A9.1.7 Results for Example 9.1 (Revised)

Method I: Alternative Formulation	Method II: Hybrid units Model
$\mathbf{Z} = \begin{bmatrix} 10 & 20 \\ 30 & 40 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 100 \\ 120 \end{bmatrix}$	$\mathbf{Z}^* = \begin{bmatrix} 10 & 20 \\ 60 & 100 \end{bmatrix} \mathbf{x}^* = \begin{bmatrix} 100 \\ 240 \end{bmatrix}$
$\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 0.100 & 0.167 \\ 0.300 & 0.333 \end{bmatrix}$	$\mathbf{A}^* = \mathbf{Z}^*(\hat{\mathbf{x}}^*)^{-1} = \begin{bmatrix} 0.100 & 0.083 \\ 0.600 & 0.417 \end{bmatrix}$
$\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.212 & 0.303 \\ 0.546 & 1.636 \end{bmatrix}$	$\mathbf{L}^* = (\mathbf{I} - \mathbf{A}^*)^{-1} = \begin{bmatrix} 1.228 & 0.175 \\ 1.263 & 1.895 \end{bmatrix}$

As before, from the conventions of the alternative formulation (Method I) and of the hybrid units formulation (Method II) we can derive the results given in Table A9.1.7.

We can now calculate the total energy coefficients by the two methods.

Method 1

$$\mathbf{D} = \mathbf{E}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 60 & 100 \end{bmatrix} \begin{bmatrix} 1/100 & 0 \\ 0 & 1/120 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.833 \end{bmatrix}$$

$$\boldsymbol{\varepsilon} = \mathbf{D}(\mathbf{I} - \mathbf{A})^{-1} + \tilde{\mathbf{Q}} = \begin{bmatrix} 0.6 & 0.833 \end{bmatrix} \begin{bmatrix} 1.212 & 0.303 \\ 0.546 & 1.636 \end{bmatrix} + \begin{bmatrix} 0 & 8/5 \end{bmatrix} = \begin{bmatrix} 1.182 & 3.145 \end{bmatrix}$$

From this we can verify that, since $\mathbf{f} = \begin{bmatrix} 70 \\ 50 \end{bmatrix}$, $\boldsymbol{\varepsilon}\mathbf{f} = \begin{bmatrix} 1.182 & 3.145 \end{bmatrix} \begin{bmatrix} 70 \\ 50 \end{bmatrix} = 240.$

Method 2

$$\mathbf{G}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 0 & 240 \end{bmatrix} \begin{bmatrix} 1/100 & 0 \\ 0 & 1/240 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$\boldsymbol{\alpha} = \mathbf{G}(\hat{\mathbf{x}}^*)^{-1}(\mathbf{I} - \mathbf{A}^*)^{-1} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1.228 & 0.175 \\ 1.263 & 1.895 \end{bmatrix} = \begin{bmatrix} 1.263 & 1.895 \end{bmatrix}$$

From this we can verify that, since $\mathbf{f}^* = \begin{bmatrix} 70 \\ 80 \end{bmatrix}$, $\boldsymbol{\alpha}\mathbf{f}^* = \begin{bmatrix} 1.263 & 1.895 \end{bmatrix} \begin{bmatrix} 70 \\ 80 \end{bmatrix} = 240.$

Both methods thus yield the same total energy requirements for the basic data from which the models were originally formulated. However, this is not generally true. Consider two cases of *new* final-demand vectors for which we wish to compute the total energy requirement by both Methods I and II.

Case 1. Consider two final demand vectors, $\mathbf{f}$ and $\mathbf{f}^*$, which describe the same final demand since the energy price to final demand is 8/5, so that $\mathbf{f} = \begin{bmatrix} 100 \\ 333.1 \end{bmatrix}$ and $\mathbf{f}^* = \begin{bmatrix} 100 \\ 533 \end{bmatrix}$. That is, the relationship between f_2^* and f_2 is $f_2^* = f_2(8/5) = (333.1)(8/5) = 533$. Computing the total energy requirement by the two methods:

Method I	Method II
$\boldsymbol{\varepsilon}\mathbf{f} = \begin{bmatrix} 1.182 & 3.145 \end{bmatrix} \begin{bmatrix} 100 \\ 333.1 \end{bmatrix} = 1,166$	$\boldsymbol{\alpha}\mathbf{f}^* = \begin{bmatrix} 1.263 & 1.895 \end{bmatrix} \begin{bmatrix} 100 \\ 533 \end{bmatrix} = 1,136$

Case 2. Consider another equivalent pair of final demands, defined as $\mathbf{f} = \begin{bmatrix} 1,000 \\ 10 \end{bmatrix}$ and $\mathbf{f}^* = \begin{bmatrix} 1,000 \\ 16 \end{bmatrix}$, for which the total energy requirement by the two methods are:

Method I	Method II
$\boldsymbol{\varepsilon}\mathbf{f} = \begin{bmatrix} 1.182 & 3.145 \end{bmatrix} \begin{bmatrix} 1,000 \\ 10 \end{bmatrix} = 1,031.90$	$\boldsymbol{\alpha}\mathbf{f}^* = \begin{bmatrix} 1.263 & 1.895 \end{bmatrix} \begin{bmatrix} 1,000 \\ 16 \end{bmatrix} = 1,293.32$

Note that in Case 1, using Method I results in a higher total energy requirement than using Method II and a lower amount in Case 2. In the following we will show that Method II always computes the total energy requirement correctly. We can then conclude that in Cases 1 and 2, Method I overestimates and underestimates, respectively, the total energy requirement.

A9.1.3 General Limitations of the Alternative Formulation

We return briefly to the alternative formulation of total energy coefficients derived earlier and defined in (A9.1.4): $\mathbf{g} = [\mathbf{D}(\mathbf{I}-\mathbf{A})^{-1} + \tilde{\mathbf{Q}}]\mathbf{f}$. With an arbitrary final demand, denoted as $\mathbf{f}^{new}$, and the corresponding total energy requirement as $\mathbf{g}^{new}$, then we define the total output vector used in defining the total energy coefficients as $\mathbf{x}^{old}$. $\mathbf{D}$ is computed as $\mathbf{D} = \mathbf{E}(\hat{\mathbf{x}}^{old})^{-1}$. Combining the expressions for $\mathbf{g}^{new}$ and $\mathbf{D}$ we obtain

$$\mathbf{g}^{new} = \mathbf{E}(\hat{\mathbf{x}}^{old})^{-1}(\mathbf{I}-\mathbf{A})^{-1}\mathbf{f}^{new} + \tilde{\mathbf{Q}}\mathbf{f}^{new} = \mathbf{E}(\hat{\mathbf{x}}^{old})^{-1}\mathbf{x}^{new} + \tilde{\mathbf{Q}}\mathbf{f}^{new} \qquad \text{(A9.1.5)}$$

If $\mathbf{x}^{old} = \mathbf{x}^{new}$, then the product $(\hat{\mathbf{x}}^{old})^{-1}\mathbf{x}^{new}$ will be a column vector of ones. In addition, by definition, $\mathbf{q} = \tilde{\mathbf{Q}}\mathbf{f}^{new}$, and, hence, (A9.1.5) becomes $\mathbf{g}^{new} = \mathbf{Ei} + \mathbf{q}$, which is (A9.1.1) from which the total energy coefficients were originally derived. If $\mathbf{x}^{old} \neq \mathbf{x}^{new}$, however, which is the case for most applications, the model does not reduce to (A9.1.1) and does not accurately reflect the energy flows generated by a new final demand.

We can conclude that while Method II (the hybrid units formulation) correctly computes in all cases the total energy requirement for any arbitrary vector of final demands consistent with our energy conservation condition, Method I yields correct results only for the base case of final demands from which the model was originally derived, or, as it turns out, if the new final-demand vector is a linear combination of the reference case of final demands (the same scalar multiplied by every element of the final-demand vector, which might be interpreted as uniform economic growth). Hence, in general, if the necessary data are available, the only defense for using Method I in practice is when impact analysis involves new final demands that are not substantially different from the basic data from which the model was derived or when there are uniform interindustry energy prices throughout the economy.[11]

Problems

9.1 Consider the following three-sector input–output economy; two sectors are energy sectors (oil is the primary energy sector and refined petroleum is the secondary energy sector):

Interindustry Transactions ($\$10^6$)	Oil	Refined Petroleum	Manuf.	Final Demand	Total Output
Oil	0	20	0	0	20
Refined Petroleum	2	2	2	24	30
Manufacturing	0	0	0	20	20

The energy sector transactions are also measured in quadrillions of BTUs in the following table:

[11] Herendeen (1974) suggested an ad hoc modification procedure for enforcing consistency.

Energy Sector Transactions (10^{15} BTUs)	Oil	Refined Petroleum	Manuf.	Final Demand	Total Output
Oil	0	20	0	0	20
Refined Petroleum	1	1	1	17	20

Given this information, do the following:

a. Compute (1) the matrix of implied inverse energy prices, (2) the direct energy requirements matrix, and (3) the total energy requirements matrix (including an accounting for energy consumed by final demand) using the method developed in Appendix 9.1. Do you notice anything peculiar about the total energy requirements matrix?

b. Reformulate this problem as a hybrid units input–output model; that is, recompute the technical coefficients and Leontief inverse using value terms for nonenergy sectors and energy units (BTUs) for energy sectors. Does this model conform to the conditions of energy conservation?

9.2 Consider the following input–output transactions table in value terms (millions of dollars) for two industries – A and B:

	A	*B*	Total Output
A	2	4	100
B	6	8	100

Suppose we have a direct energy requirements matrix for this economy that is given by:

$$\mathbf{D} = \begin{bmatrix} .2 & .3 \\ .1 & .4 \end{bmatrix} \begin{array}{l} 10^{15} \text{ BTUs of oil per million dollars of output} \\ 10^{15} \text{ BTUs of coal per million dollars of output} \end{array}$$

a. Compute the total energy requirements matrix (neglecting energy consumption by final demand).

b. Suppose that the final demands for industries A and B are projected to be \$200 million and \$100 million respectively for the next year. What is the net increase in energy (both oil and gas) required to support this new final demand (neglect energy consumed directly by final demand, since you do not have the information to do this calculation anyway)? What fraction of this net increase is a direct energy requirement and what fraction is indirect (total minus direct)?

c. Suppose an energy conservation measure in industry B caused the direct energy requirement of that industry for coal to be reduced from 0.4 to 0.3 (10^{15} BTUs of coal per dollar of output of industry B). How does this change the direct and total energy requirements needed to support the new final demand given in b?

9.3 Consider the following input–output table ($\$10^6$)

	Transactions			
	Autos	Oil	Electricity	Total Output
Autos	2	6	1	10
Oil	0	0	20	20
Electricity	3	2	1	30

Assume that the implied inverse energy price matrix for this economy is given by the following (in dollars per billion BTU)

	Autos	Oil	Electricity	Final Demand
Oil	0	0	0.4082	0
Electricity	0.3333	0.2857	0.5	1.2912

a. Compute the current energy flows matrix, that is, the distribution of each energy type among the industries in the economy measured in BTUs.
b. Compute the direct energy coefficients matrix.
c. If a final demand vector of \$2 million worth of autos and 18 quadrillion BTUs of electricity is presented to this economy, what would be the total amount of energy (of each type) required to support this final demand?
d. Compute the total energy requirement using the alternative method of Appendix 9.1.

9.4 Recall that the conditions for energy conservation in an input–output model can be expressed as $\boldsymbol{\alpha}\hat{\mathbf{x}} = \boldsymbol{\alpha}\mathbf{Z} + \mathbf{G}$ where $\boldsymbol{\alpha}$ is the matrix of total energy coefficients, $\mathbf{Z}$ is the matrix of interindustry transactions, $\mathbf{x}$ is the vector of total outputs, and $\mathbf{G}$ is the matrix of primary energy outputs.

a. Show that the hybrid units formulation of the energy input–output model – that is, where $\mathbf{x}$ is replaced by $\mathbf{x}^*$ and $\mathbf{Z}$ is replaced by $\mathbf{Z}^*$ – satisfies these conditions in general.
b. Given the following two tables of total energy coefficients, explain which of them satisfies the conditions of energy conservation and why. Use the convention that crude oil is a primary energy sector and refined petroleum and electricity are secondary energy sectors.

Case 1	Crude Oil	Refined Petroleum	Electricity	Autos
Crude Oil	0	.6	.5	.3
Refined Petroleum	0	.4	.5	.2
Electricity	0	.2	0	.1

Case 2	Crude Oil	Refined Petroleum	Electricity	Autos
Crude Oil	0	.6	.5	.3
Refined Petroleum	0	.4	.2	.1
Electricity	0	.2	0	.1

9.5 An energy input–output model is defined (in $\$10^6$ units) by $\mathbf{Z} = \begin{bmatrix} 0 & 10 & 0 \\ 5 & 5 & 5 \\ 0 & 0 & 0 \end{bmatrix}$, $\mathbf{f} = \begin{bmatrix} 0 \\ 25 \\ 20 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 10 \\ 40 \\ 20 \end{bmatrix}$. Industries I and II are energy industries with patterns of output allocation expressed in energy terms (10^{15} BTUs) by $\mathbf{E} = \begin{bmatrix} 0 & 40 & 0 \\ 5 & 5 & 15 \end{bmatrix}$ and $\mathbf{g} = \begin{bmatrix} 0 \\ 15 \end{bmatrix}$.

a. Compute $\boldsymbol{\varepsilon}$, the total energy requirements matrix (via the traditional method outlined in Appendix 9.1).

b. Now compute $\boldsymbol{\alpha}$, the hybrid units total energy requirements matrix.

9.6 Consider the following hybrid units transactions matrix and vector of total outputs, i.e., the first three rows of the energy sectors (oil, coal, and electricity) are measured in millions of BTU and the last row, manufacturing, is measured in millions of dollars:

$$\mathbf{Z}^* = \begin{bmatrix} 0 & 0 & 40 & 0 \\ 0 & 0 & 60 & 0 \\ 2 & 3 & 12 & 48 \\ 15 & 20 & 30 & 40 \end{bmatrix} \text{ and } \mathbf{x}^* = \begin{bmatrix} 40 \\ 60 \\ 100 \\ 200 \end{bmatrix}.$$

Suppose final demand for manufactured goods increased by \$200 billion. What would be the increase in total primary energy used to satisfy this growth in final demand?

9.7 For the economy specified in problem 9.6, two alternative technologies are proposed for generating electric power, which involve alternative new specifications for the technical coefficients matrix depicting different "recipes" for electric power production in the economy, $\mathbf{A}^{*(I)}$ and $\mathbf{A}^{*(II)}$. The original electric power generation column of the technical coefficients matrix is given by $\mathbf{A}^*$. Suppose the two alternative changed columns of the technical coefficients matrix corresponding to the alternative technologies are given by $\mathbf{A}_{.3}^{*(I)} = \begin{bmatrix} .2 \\ .7 \\ .1 \\ .4 \end{bmatrix}$ and $\mathbf{A}_{.3}^{*(II)} = \begin{bmatrix} .5 \\ .4 \\ .12 \\ .4 \end{bmatrix}$ and a change in final demand of $\Delta\mathbf{f}^* = \begin{bmatrix} 0 \\ 0 \\ 20 \\ 30 \end{bmatrix}$. Which economy [matrix incorporating the specifications $\mathbf{A}^*$, $\mathbf{A}^{*(I)}$ or $\mathbf{A}^{*(II)}$] reflects the most energy-intensive manufacturing, i.e., which one of the two new technologies consumes the least primary energy per unit of final demand of manufacturing and how much less primary energy does that technology consume than the other to support final demand $\Delta\mathbf{f}^*$?

9.8 For the original energy-economy defined in problem 9.6 ($\mathbf{A}^*$), suppose an energy-conserving manufacturing process is developed that can be depicted as a new column of the technical coefficients matrix for manufacturing, given by $\mathbf{A}_{.4}^{*(new)} = \begin{bmatrix} 0 \\ 0 \\ .12 \\ .20 \end{bmatrix}$. If this new process were adopted, how much primary energy would be saved in the economy, both directly in terms of fuel used directly in manufacturing, and indirectly in the energy embodied in the inputs to manufacturing?

9.9 Suppose the original energy economy used in problem 9.6 is faced with an oil supply shortage of a ten percent reduction in total input of oil available in the economy. What would be the corresponding reduction in GDP? To do this calculation you will need to know the energy prices to final demand, which are given by $\mathbf{p}_f = [p_{kf}] = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$. If new electric power generation technology I from problem 9.7 and the energy conserving manufacturing process from problem 9.8 were both incorporated into the economy, what would be the change in GDP with the same oil shortage?

9.10 Below are nine-sector 1963 and 1980 input–output tables for the United States expressed in hybrid units (quadrillions of BTUs for energy sectors and millions of dollars for non-energy sectors). The first five sectors are energy sectors: (1) coal, (2) oil, (3) refined petroleum products, (4) electricity, and (5) natural gas. The remaining four sectors are non-energy sectors: (6) natural resources, (7) manufacturing, (8) transportation, and (9) services. Using the approach derived in section 9.4.5, determine the amounts of the change in total energy use of each energy type between 1963 and 1980 and the components of that change that are attributable to change in production functions, to change in final demand, and to the interaction between the changes in production functions and final demand.

1980	1	2	3	4	5	6	7	8	9	Total Output
1	0.0012	0.0000	0.0007	1.5464	0.0000	0.0000	0.0002	0.0000	0.0000	18, 597
2	0.0001	0.0319	0.8960	0.0001	0.8707	0.0000	0.0001	0.0000	0.0000	36, 842
3	0.0063	0.0024	0.0612	0.3344	0.0008	0.0005	0.0002	0.0023	0.0002	31, 215
4	0.0026	0.0021	0.0035	0.0822	0.0020	0.0000	0.0001	0.0000	0.0001	7, 827
5	0.0006	0.0461	0.0301	0.4856	0.0720	0.0001	0.0003	0.0000	0.0001	19, 244
6	0.2092	1.4027	0.5040	7.8254	0.4350	0.0896	0.0628	0.0355	0.0289	6, 194, 571
7	2.6323	0.8480	2.4090	3.5155	0.1804	0.2672	0.3780	0.0493	0.0626	18, 081, 173
8	0.1773	0.0806	2.1831	4.8195	0.0794	0.0199	0.0251	0.1289	0.0141	2, 240, 904
9	1.8576	2.6159	2.7945	8.5173	1.2302	0.1831	0.1238	0.1224	0.2027	23, 803, 723
1963										
1	0.0019	0.0000	0.0008	1.7415	0.0010	0.0000	0.0004	0.0001	0.0000	12, 476
2	0.0000	0.0423	0.7996	0.0007	0.9308	0.0000	0.0003	0.0000	0.0000	30, 384
3	0.0015	0.0011	0.0600	0.1973	0.0031	0.0004	0.0003	0.0021	0.0002	19, 878
4	0.0015	0.0007	0.0018	0.0963	0.0002	0.0000	0.0001	0.0000	0.0000	3, 128
5	0.0001	0.0035	0.0330	0.7046	0.0919	0.0000	0.0003	0.0001	0.0001	13, 194
6	0.0456	0.4582	0.5926	7.9623	0.6565	0.1111	0.0835	0.0415	0.0426	4, 865, 092
7	0.8684	0.4081	1.1700	1.0933	0.0937	0.2340	0.4035	0.0498	0.0496	11, 333, 710
8	0.1105	0.0655	1.1964	4.5632	0.3965	0.0231	0.0256	0.0863	0.0121	1, 131, 226
9	0.4794	2.2388	1.9461	8.0643	1.1016	0.1121	0.0881	0.1203	0.1721	10, 588, 385

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10 Environmental Input–Output Analysis

10.1 Introduction

Since the late 1960s the input–output framework has been extended by many researchers to account for environmental pollution generation and abatement associated with interindustry activity. Leontief (1970) himself provided one of the key methodological extensions that has since been applied widely and extended further. In this chapter we will examine several of the most prominent environmental input–output formulations and discuss many of the features, advantages, and limitations of each. Much like the discussion in Chapter 9 – modification of the traditional Leontief model to deal with energy flows – in the environmental extensions we must include some additional conditions in order to enforce consistency among interindustry production, pollution generation, and pollution abatement activities.

10.2 Basic Considerations

A principal problem to be resolved in environmental models is the appropriate unit of measurement of environmental (or ecological) quantities – for example, in monetary or physical units. In the alternatives we consider here, we will see formulations using each approach. We will examine three basic categories of environmental input–output models:

1. *Generalized Input–Output Models*. These are formed by augmenting the technical coefficients matrix with additional rows and/or columns to reflect pollution generation and abatement activities. We explore two variations on such models – one aimed at analysis of impacts and another aimed at planning applications.
2. *Economic–Ecologic Models*. These models result from extending the interindustry framework to include additional "ecosystem" sectors, where flows will be recorded between economic and ecosystem sectors along the lines of an interregional input–output model.
3. *Commodity-by-Industry Models*. Such models express environmental factors as "commodities" in a commodity-by-industry input–output table, as described in Chapters 4 and 5.

10.3 Generalized Input–Output Analysis: Basic Framework

A very common public policy analysis problem is to analyze the implications of a new spending program (usually government, but certainly not exclusively so) on an economy, not just the traditional impact analysis developed in earlier chapters, or the special case of energy consumption examined in the previous chapter, but a more comprehensive examination of a wide variety of factors associated with that spending program, such as impacts on employment, pollution, or capital expenditures. In this section we develop a general framework for tracing these impacts associated with interindustry production generated in response to a spending program interpreted as a new vector of final demands presented to the economy.

10.3.1 Accounting for Pollution Impacts

A very straightforward approach to accounting for pollution generation associated with interindustry activity is to first assume a matrix of pollution output or direct impact coefficients, $\mathbf{D}^p = [d_{kj}^p]$, each element of which is the amount of pollutant type k, e.g., sulfur dioxide, generated per dollar's worth of industry j's output. Hence, the level of pollution associated with a given vector of total outputs can be expressed as

$$\mathbf{x}^{p*} = \mathbf{D}^p\mathbf{x} \tag{10.1}$$

where $\mathbf{x}^{p*}$ is the vector of pollution levels. Hence, by adding the traditional Leontief model, $\mathbf{x} = \mathbf{Lf}$ where $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$, we can compute $\mathbf{x}^{p*}$ as a function of final demand, that is, the total pollution of each type generated by the economy directly and indirectly in supporting that final demand:

$$\mathbf{x}^{p*} = [\mathbf{D}^p\mathbf{L}]\mathbf{f} \tag{10.2}$$

We can view the bracketed quantity as a matrix of total environmental impact coefficients; that is, an element of this matrix is the total pollution impact generated per dollar's worth of final demand presented to the economy.

10.3.2 Generalized Impacts

Even though we are concerned primarily with environmental extensions to input–output analysis in this chapter, we could easily replace the pollution coefficients matrix with a corresponding matrix for virtually any factor associated with interindustry activity that we assume varies linearly with output, for example, employment or energy consumption (as we did in Chapter 9 in the case of energy in the alternate formulation developed in Appendix 9.1). The use of employment coefficients is essentially equivalent to the notion of employment multipliers introduced in Chapter 6, and extensions that incorporate more detailed disaggregation of final-demand and value-added sectors, and especially how these sectors interact with one another in the economy in so-called social accounting matrices, are the subject of Chapter 11. In this chapter, however,

Table 10.1 Input–Output Transactions (millions of dollars)

	Industry *A*	Industry *B*	Final Demand	Total Output
Industry *A*	3	2	5	10
Industry *B*	1	7	2	10

Table 10.2 Direct Impact Coefficients

Direct Impact Coefficients	Industry *A*	Industry *B*	Direct Impact per $ (millions) of Output
Energy			
Oil	0.2	0.3	Billion BTUs
Coal	0.1	0.4	
Pollution			
Sulfur Dioxide	0.5	1.1	Thousand lbs.
Hydrocarbons	0.7	0.7	
Employment			
Employment	0.1	0.2	Person-years

we restrict the generalized framework to energy use, environmental pollution, and employment as illustrative of the more general case. We begin with an example.

Example 10.1: Generalized Input–Output Analysis Consider the two-sector input–output table of transactions shown in Table 10.1. The corresponding technical coefficient and Leontief inverse matrices are $\mathbf{A} = \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.7 \end{bmatrix}$ and $\mathbf{L} = \begin{bmatrix} 1.58 & 1.05 \\ 0.53 & 3.68 \end{bmatrix}$. We now define three direct-impact coefficient matrices relating energy requirements, pollution generation, and employment to total output[1] (see Table 10.2).

The corresponding direct impact coefficient matrices can be specified for energy, pollution, and employment by

$$\mathbf{D}^e = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.4 \end{bmatrix}, \quad \mathbf{D}^p = \begin{bmatrix} 0.5 & 1.1 \\ 0.7 & 0.7 \end{bmatrix}, \quad \text{and} \quad \mathbf{D}^l = [0.1 \quad 0.2]$$

For convenience, we can easily *concatenate* these matrices (that is, stack these matrices $\mathbf{D}^e$, $\mathbf{D}^p$, and $\mathbf{D}^l$ on top of one another as sub-matrices in a single matrix), to yield a

[1] Recall from Chapter 9 that developing such coefficients for energy, in particular, is in effect using the methodology of Appendix 9.1, which must be applied carefully in order to avoid inconsistent results.

direct-impact coefficient matrix **D**:

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}^e \\ \\ \mathbf{D}^p \\ \\ \mathbf{D}^l \end{bmatrix} = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.4 \\ & \\ 0.5 & 1.1 \\ 0.7 & 0.7 \\ & \\ 0.1 & 0.2 \end{bmatrix}$$

We can similarly define a vector of total impacts, $\mathbf{x}^*$, by concatenating $\mathbf{x}^{e*} = \mathbf{D}^e\mathbf{x}$, $\mathbf{x}^{p*} = \mathbf{D}^p\mathbf{x}$, and $\mathbf{x}^{l*} = \mathbf{D}^l\mathbf{x}$ to yield $\mathbf{x}^* = \begin{bmatrix} \mathbf{x}^{e*} \\ \mathbf{x}^{p*} \\ \mathbf{x}^{l*} \end{bmatrix}$. Hence, we can write $\mathbf{x}^* = \mathbf{D}\mathbf{x}$. For accounting convenience, we may wish to record $\mathbf{x}^*$ along with the corresponding final demands associated with the generation of a particular vector of total impacts, $\mathbf{x}^*$. We can accomplish this easily by defining a new vector of total impacts and concatenating the vector of final demands with $\mathbf{x}^*$; we define this expanded vector of total impacts as $\tilde{\mathbf{x}}$, so that $\tilde{\mathbf{x}} = \begin{bmatrix} \mathbf{x}^* \\ \mathbf{f} \end{bmatrix}$. We can similarly expand the matrix of direct-impact coefficients by concatenating **D** with $(\mathbf{I} - \mathbf{A})$; we define this new expanded direct-impact coefficients matrix to be $\mathbf{G} = \begin{bmatrix} \mathbf{D} \\ (\mathbf{I} - \mathbf{A}) \end{bmatrix}$.

In our example, for \$10 million worth of total production of each industry, *A* and *B*, a total of 5 million BTUs each of oil and coal is required to support that production. Similarly, 16,000 and 14,000 pounds, respectively, of SO_2 and hydrocarbons are generated by this production; 3,000 person-years of employment are also associated with the level of industrial production. Hence, we can write $\tilde{\mathbf{x}} = \mathbf{G}\mathbf{x}$, which, for our example, is

$$\tilde{\mathbf{x}} = \mathbf{G}\mathbf{x} = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.4 \\ & \\ 0.5 & 1.1 \\ 0.7 & 0.7 \\ & \\ 0.1 & 0.2 \\ & \\ 0.7 & -0.2 \\ -0.1 & 0.3 \end{bmatrix} \begin{bmatrix} 10 \\ \\ 10 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ \\ 16 \\ 14 \\ \\ 3 \\ \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}^* \\ \\ \mathbf{f} \end{bmatrix}$$

This formulation is particularly well suited to using input–output with mathematical programming models as in Thoss (1976) and Blair (1979), one example of which we develop later in this chapter.

As an alternative to the preceding formulation, we may wish to express total impacts as a function of final demands. For example, we may wish to employ the coefficients in impact analysis in much the same way we traditionally use the Leontief inverse. That is, we may find the total impacts in terms of energy, pollution generation, and employment associated with some given level of final demand. This formulation was originally applied by Just (1974) and Folk and Hannon (1974) to examine the impacts of new energy technologies. Other more recent applications are summarized in Forssell and Polenske (1998), including, in particular, Qayum (1994), Schäfer and Stahmer (1989) and Lange (1998).

In this case we can also write our earlier expression for total impacts, $\mathbf{x}^* = \mathbf{D}\mathbf{x}$, equivalently as $\mathbf{x}^* = [\mathbf{DL}]\mathbf{f}$, where, as before for pollution coefficients alone, the bracketed quantity is the matrix of total-impact coefficients. Let us denote the bracketed quantity by $\mathbf{D}^*$ so that, for our example, we can find the levels of energy needs, pollution generation, and employment associated with the basic data:

$$\tilde{\mathbf{x}}^* = \mathbf{D}^*\mathbf{f} = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.4 \\ \\ 0.5 & 1.1 \\ 0.7 & 0.7 \\ \\ 0.1 & 0.2 \end{bmatrix} \begin{bmatrix} 1.58 & 1.05 \\ 0.53 & 3.68 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} \begin{bmatrix} 0.47 & 1.32 \\ 0.37 & 1.58 \\ \\ 1.37 & 4.58 \\ 1.47 & 3.32 \\ \\ 0.26 & 0.84 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ \\ 16 \\ 14 \\ \\ 3 \end{bmatrix}$$

Note that here we compute $\mathbf{x}^*$ as a function of $\mathbf{f}$ while in $\mathbf{x}^* = \mathbf{D}\mathbf{x}$ we computed $\mathbf{x}^*$ as a function of $\mathbf{x}$. Finally, for convenience, we may wish to include $\mathbf{x}$ itself in our vector of total impacts. This can be accomplished easily by concatenating $\mathbf{x}$ with the vector of total impacts in the same manner we concatenated $\mathbf{x}^*$ with $\mathbf{f}$ in constructing $\tilde{\mathbf{x}}$; we define a new expanded vector of total impacts to be $\bar{\mathbf{x}}$:

$$\bar{\mathbf{x}} = \begin{bmatrix} \mathbf{x}^* \\ \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ \\ 16 \\ 14 \\ \\ 3 \\ \\ 10 \\ 10 \end{bmatrix}$$

We can similarly expand the total-impacts coefficients matrix by concatenating the Leontief inverse with the total-impact coefficients; we call the new expanded total-impacts coefficients matrix, $\mathbf{H}$, so that $\mathbf{H} = \begin{bmatrix} \mathbf{D}^* \\ \mathbf{L} \end{bmatrix}$. Hence, for the example we have the following:

$$\bar{\mathbf{x}} = \begin{bmatrix} \mathbf{x}^* \\ \mathbf{x} \end{bmatrix} = \mathbf{Hf} = \begin{bmatrix} 0.47 & 1.32 \\ 0.37 & 1.58 \\ 1.37 & 4.58 \\ 1.47 & 3.32 \\ 0.26 & 0.84 \\ 1.58 & 1.05 \\ 0.53 & 3.68 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 16 \\ 14 \\ 3 \\ 10 \\ 10 \end{bmatrix}$$

Note that $\tilde{\mathbf{x}}$ and $\bar{\mathbf{x}}$ are equivalent descriptions of the same situation, since $\mathbf{x}$ and $\mathbf{f}$ uniquely define one another in a Leontief model – for every given $\mathbf{f}$, there is one and only one $\mathbf{x}$, and vice versa. Note also that we can create a matrix of impacts generated by each industry separately by diagonalizing $\mathbf{f}$ to yield $\mathbf{H}\hat{\mathbf{f}}$. For the last example, $\mathbf{H}\hat{\mathbf{f}} = \begin{bmatrix} 2.37 & 2.63 \\ 1.84 & 3.16 \\ 6.84 & 9.16 \\ 7.37 & 6.63 \\ 1.32 & 1.68 \\ 7.89 & 2.11 \\ 2.63 & 7.37 \end{bmatrix}$. Hence, for example, of the 5×10^9 BTUs of oil consumed by the economy in the course of satisfying final demand $\mathbf{f}' = \begin{bmatrix} 5 & 2 \end{bmatrix}$, 2.37×10^9 BTUs are attributed to industry A; 2.63×10^9 BTUs are attributed to industry B. (These sums across the rows of these values do not equal $\bar{\mathbf{x}}$ due to rounding.) This is analogous to retrieving the matrix of transactions, $\mathbf{Z}$, from $\mathbf{Z} = \mathbf{A}\hat{\mathbf{x}}$.

10.3.3 Summary: Generalized Input–Output Formulations

Recall that the generalized input–output model becomes possible with a set of direct impact coefficients, $\mathbf{D} = [d_{kj}]$, each element of which is the amount of an impact variable k, for example, pollution or energy, generated per dollar's worth of industry j's output. Using $\mathbf{D}$ we can pose the generalized input–output model in what we will refer to as either its *impact analysis* or *planning* forms, which we define as the following:

Case I: Impact Analysis Form

$$\bar{\mathbf{x}} = \mathbf{Hf} \text{ where } \mathbf{H} = \begin{bmatrix} \mathbf{D}^* \\ \mathbf{L} \end{bmatrix} \text{ and } \bar{\mathbf{x}} = \begin{bmatrix} \mathbf{x}^* \\ \mathbf{x} \end{bmatrix} \text{ and } \mathbf{D}^* = \mathbf{DL}$$

Case II: Planning Form

$$\tilde{\mathbf{x}} = \mathbf{Gx} \text{ where } \mathbf{G} = \begin{bmatrix} \mathbf{D} \\ (\mathbf{I} - \mathbf{A}) \end{bmatrix} \text{ and } \tilde{\mathbf{x}} = \begin{bmatrix} \mathbf{x}^* \\ \mathbf{f} \end{bmatrix}$$

The impact analysis form is the version most traditionally considered in input–output applications where the question is what industry outputs and factors associated with interindustry activity, such as energy use, environmental pollution levels, and employment, result from a given schedule of final demands presented to the economy, as in Johnson and Bennett (1981), Hannon, Costanza and Herendeen (1983) and many others. However, the planning form has advantages in applications where one seeks to optimize an objective other than the objective implicit in a traditional input–output model.[2] In the following we explore some examples of how to extend this framework to planning applications.

10.4 Generalized Input–Output Analysis: Extensions of the Planning Approach

In Chapters 1 and 2 we developed the basic input–output framework as the solution of a system of n linear equations in n unknowns, which is certainly one of the most attractive features of the framework – its very straightforward and unique solution. At various points in this text we relax fundamental assumptions in the framework in order to adapt it to specific situations, such as allowing technical coefficients to vary as a function of relative prices in the case of econometric extensions to the basic model, adding capital coefficients in dynamic input–output models, or adding trade coefficients in multi- or interregional models, as examples. Similarly, in using input–output in planning applications, where one seeks to optimize (maximize or minimize) some objective function related to interindustry activity, it is useful to begin by thinking of input–output as a very simple linear programming problem.

10.4.1 Linear Programming: A Brief Introduction by Means of the Leontief Model

Recall from Chapter 2 (Figure A2.2.1), the basic two-sector formulation of the Leontief model; we rearrange and collect terms to yield

$$(1 - a_{11})x_1 - a_{12} = f_1$$
$$-a_{21}x_1 + (1 - a_{22}) = f_2$$

This is depicted graphically for the general case in Figure 10.1 – the solution is the intersection of the two lines. Suppose we relax the equality in this expression to an *inequality*, interpreting the change as requiring that the value of total outputs less the

[2] We will see later that the implicit objective function in an input–output model is to maximize the sum of all final demands or, equivalently, to minimize the sum of all value-added inputs.

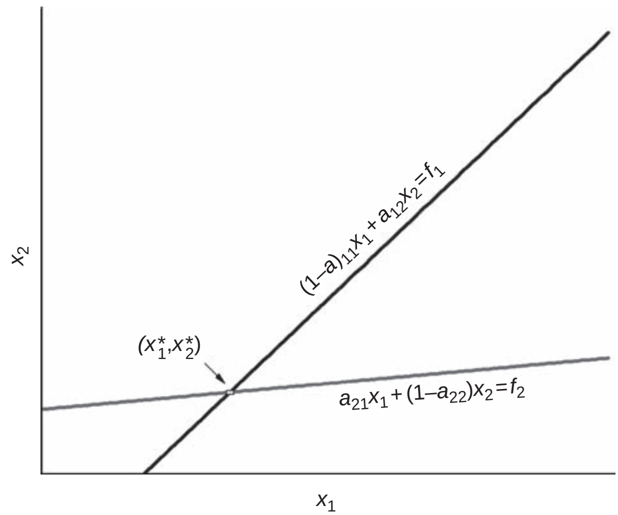

Figure 10.1 Two-Sector Leontief Model

value of all interindustry outputs yield *at least* the value of deliveries to final demand:

$$(1-a_{11})x_1 - a_{12}x_2 \geq f_1$$
$$-a_{21}x_1 + (1-a_{22})x_2 \geq f_2$$

or, equivalently, we introduce a "surplus variable" (a concept we will use later) that nets out any excess between the value of total output and its use to satisfy intermediate inputs plus final demand:

$$(1-a_{11})x_1 - a_{12}x_2 - s_1 = f_1$$
$$-a_{21}x_1 + (1-a_{22})x_2 - s_2 = f_2$$

where s_1 and s_2 are the surplus variables (≥ 0).

Consider an example where $\mathbf{A} = \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.7 \end{bmatrix}$ and $\mathbf{f} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$. We have $\mathbf{Ax} + \mathbf{f} \geq \mathbf{x}$ or $(\mathbf{I}-\mathbf{A})\mathbf{x} \geq \mathbf{f}$, which for the example is $\begin{bmatrix} 0.7 & -0.2 \\ -0.1 & 0.3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 5 \\ 5 \end{bmatrix}$ or, equivalently, with the surplus variables to account for the inequalities:

$$0.7x_1 - 0.2x_2 - s_1 = 5$$
$$-0.1x_1 + 0.3x_2 - s_2 = 5$$

These equations are shown graphically in Figure 10.2. The shaded area of the graph shows all possible non-negative solutions where final demand is at least satisfied by some combination of production of x_1 and x_2 as well as $s_1 \geq 0$ and $s_2 \geq 0$. If $s_1 = s_2 = 0$

then there is no surplus production and the two equations can be solved for the two unknowns, x_1 and x_2, which is $\mathbf{x}^* = (x_1^*, x_2^*) = (13.15, 21.05)$, the intersection of the two equations on the graph and, of course, equivalent to the input–output solution shown in Figure 10.1.

Let us explore more what this solution means in terms of the production possibilities represented by the shaded region on the graph. Recall from Chapter 2 that the value-added coefficients for our two-sector economy are defined by $v_k = 1 - (a_{1k} + a_{2k})$, for $k = 1, 2$. Hence, we have $(\mathbf{I} - \mathbf{A})\mathbf{x} \geq \mathbf{f}$ or $\begin{bmatrix} 0.7 & -0.2 \\ -0.1 & 0.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 5 \\ 5 \end{bmatrix}$ and the total value added in the economy, which here we will call q, can be found by summing the value added for all sectors, in this case two sectors, $q = v_1x_1 + v_2x_2$ or, for this example: $q = .6x_1 + .1x_2$. Figure 10.3 shows the previous figure but with lines representing different values of q. We can ask what is the minimum value of q, which we will call q^*, that satisfies deliveries to final demand or, equivalently, what values of x_1 and x_2 minimize q while satisfying final demand? The figure indicates that the value would be $q^* = 10$, which in this example we can verify by recalling from Chapter 2 that in a Leontief economy the GNP is the sum of all value added or the sum of all final demands; for the example: $q^* = f_1 + f_2 = 5 + 5 \equiv v_1 + v_2 = 7.895 + 2.105 = 10$. This is the solution to a simple linear programming (LP) problem. The shaded area is called the *feasible region* or the set of all possible solutions satisfying the inequality constraints specified in $(\mathbf{I} - \mathbf{A})\mathbf{x} \geq \mathbf{f}$. The *objective function,* which we seek to minimize in this LP problem, is the expression for total value added (GNP) defined above.

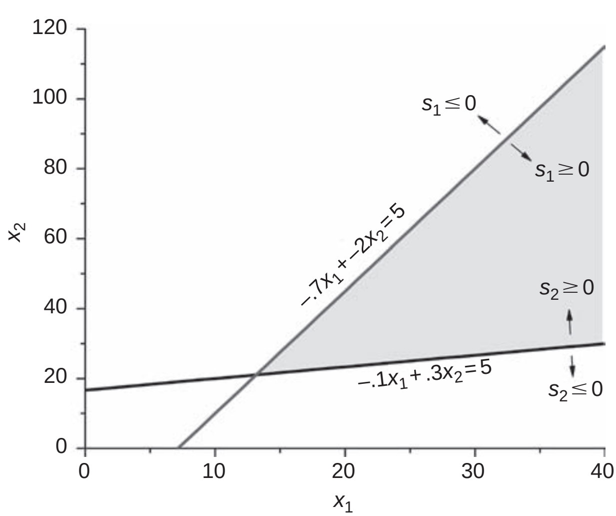

Figure 10.2 Input–Output and Linear Programming: Example 10.1

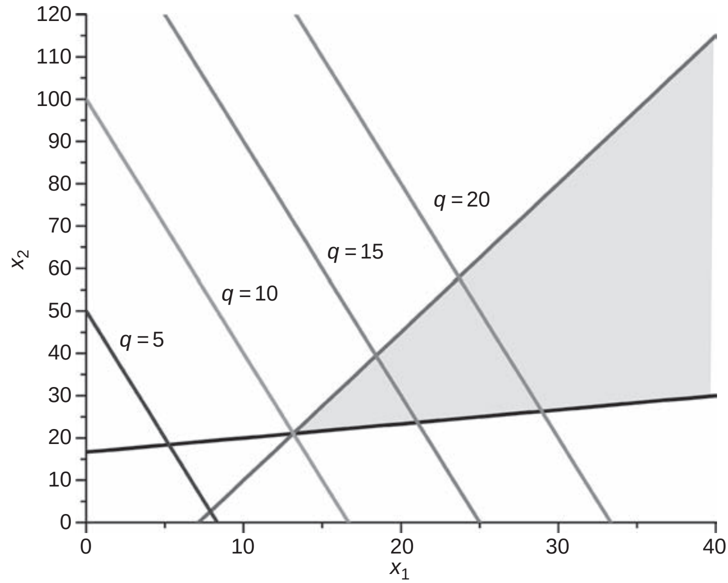

Figure 10.3 Different Values of GNP for Example 10.1

The following notation is often used in specifying an LP problem:

$$\begin{aligned}&\text{Min} \quad q = v_1x_1 + v_2x_2\\&\text{subject to:}\\&(1 - a_{11})x_1 - a_{12}x_2 \geq f_1\\&-a_{21}x_1 + (1 - a_{22}x_2) \geq f_2\end{aligned}$$

One can imagine that for problems any larger than two variables, the solution to an LP problem becomes much more complex than the graphical representation given in Figure 10.3.

The general LP case of m variables and n equations is mathematically described as minimizing (or maximizing) a linear function over a convex polyhedron and is a very well-developed tool in the field of operations research (see, for example, Miller, 2000). More extensive economic interpretations of the Leontief model as a linear programming problem are included in Dorfman, Samuelson and Solow (1958) and Intriligator (1971). An important advantage of posing the input–output framework in this way, at least for the purposes of this chapter, is that we can consider alternative objective functions and/or additional constraints as part of a planning problem.

As an example, the shaded region depicted in both Figures 10.2 and 10.3 defines all production possibilities for x_1 and x_2 that satisfy the structural conditions of the Leontief economy and it turns out that minimizing total value added as the objective subject to these conditions is equivalent to the original input–output problem, but we may be interested in finding combinations of $\mathbf{x}$ that satisfy these structural conditions but minimize pollution emissions or energy consumption or perhaps any other factor that is assumed to vary with industry output.

To illustrate this let us return to Example 10.1, but add generalized impact coefficients as developed in section 10.3. Clever readers will notice that the example Leontief economy used to illustrate the LP formulation is the same as that used in developing the generalized input–output framework earlier in this chapter. Recall that the generalized input–output formulation could be presented either in its *impact analysis* form or its *planning* form. For now we use the planning form: $\tilde{\mathbf{x}} = \mathbf{G}\mathbf{x} = \begin{bmatrix} \mathbf{D} \\ (\mathbf{I} - \mathbf{A}) \end{bmatrix} \mathbf{x}$, where $\mathbf{D}$ is the matrix of direct impact coefficients relating factors such as energy use, pollution emissions, and employment to industry output. For our example, we have

$$\tilde{\mathbf{x}} = \mathbf{G}\mathbf{x} = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.4 \\ 0.5 & 1.1 \\ 0.7 & 0.7 \\ 0.1 & 0.2 \\ 0.7 & -0.2 \\ -0.1 & 0.3 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 16 \\ 14 \\ 3 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}^* \\ \mathbf{f} \end{bmatrix}$$

where $\mathbf{x}^*$ represents the levels of total impact of energy use, pollution, and employment associated with output $\mathbf{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$ and, of course, final demand $\mathbf{f} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$. We use $\mathbf{G}$ to represent the more generalized structural relationships governing not only the Leontief production possibilities but also the levels of energy consumed, pollution discharged, and employment generated that are associated with those production possibilities. Hence, in its simplest form we could rewrite the LP problem as the following:

$$\text{Min} \;\; q = \mathbf{v}'\mathbf{x}$$
$$\text{subject to: } \mathbf{G}\mathbf{x} \geq \tilde{\mathbf{x}}$$

but, more generally, the inequalities for impact coefficients, $\mathbf{D}$, could go in either direction ($\leq$ or $\geq$). In some environmental applications, for example, one would likely be working with a constraint on total emissions for various types of environmental pollutants associated with industrial activity.[3]

Since the input–output model has a unique solution for a given level of final demand, and those conditions are part of the constraint equations, $\mathbf{G}\mathbf{x} \geq \tilde{\mathbf{x}}$, then either of two

[3] However, recall that for a linear inequality if one multiplies both sides by a negative number, that operation changes the direction of the inequality, but does not change the nature of the constraint. Hence, without loss of generality, we can actually use the notation $\mathbf{G}\mathbf{x} \geq \tilde{\mathbf{x}}$ to describe all constraints.

situations apply: 1. the additional constraints beyond the Leontief conditions included in the constraint equations (e.g., energy, environmental, and employment equations) over-constrain the problem, i.e., present conflicting constraints, so that there is no feasible region and, hence, no possible LP solution or 2. the additional conditions are not binding constraints, i.e., they are fully satisfied by the Leontief conditions. In our example so far, of course, the latter situation applies and the LP solution is identical to the impact analysis solution determined earlier. In the former case, however, when there is no feasible region, we will need to resort to other approaches in order to find a solution. We explore such an approach later in this chapter.

10.4.2 Multiple Objectives

The LP formulation to the generalized input–output planning problem also gives us the flexibility to include alternative or even multiple objective functions in approaching planning problems. For example, one might be interested in minimizing the value-added cost to meeting a target final demand, minimizing pollution emissions and energy conservation all as goals. Decision making with multiple objectives is another well-developed area in operations research with many approaches available. Surveys of such approaches are found in Cohen (1978), Cochrane and Zeleny (1973), Nijkamp and Rietveld (1976), Trzaskalik and Michnik (2002), and Tanino, Tanaka and Inuiguchi (2003). In this text we consider one commonly applied extension to LP to accommodate multiple objectives known as linear goal programming (GP) that can be used very straightforwardly to extend the Leontief framework to deal with environmental issues.

10.4.3 Conflicting Objectives and Linear Goal Programming

Consider a very simple LP problem given by

$$\begin{aligned}
&\text{Max}\ \ q = x_1 + 2x_2 \\
&\text{subject to:} \\
&x_1 + x_2 \le 8 \\
&x_1 \le 7 \\
&x_2 \le 4.5
\end{aligned}$$

which is shown graphically in Figure 10.4. The reader can verify that the optimal solution to this LP problem is $\mathbf{x}^* = (x_1^*, x_2^*) = (3.5, 4.5)$ and $q^* = 12.5$.

Suppose that there are really two objectives to this linear programming problem. The first is implicit in a traditional linear programming problem – the optimal solution must lie within the feasible region (shaded region in Figure 10.4) defined by the inequality equations. The second objective is to maximize the objective function q subject to having already satisfied the first objective. Conceptually, the key to converting this LP problem into a goal programming (GP) problem[4] is to consider the implicit objective

[4] Goal programming was first suggested by Charnes and Cooper (1961). Useful characterizations of the approach and further refinements are included in Lane (1970), Lee (1971, 1972, 1973), Ignizio (1976).

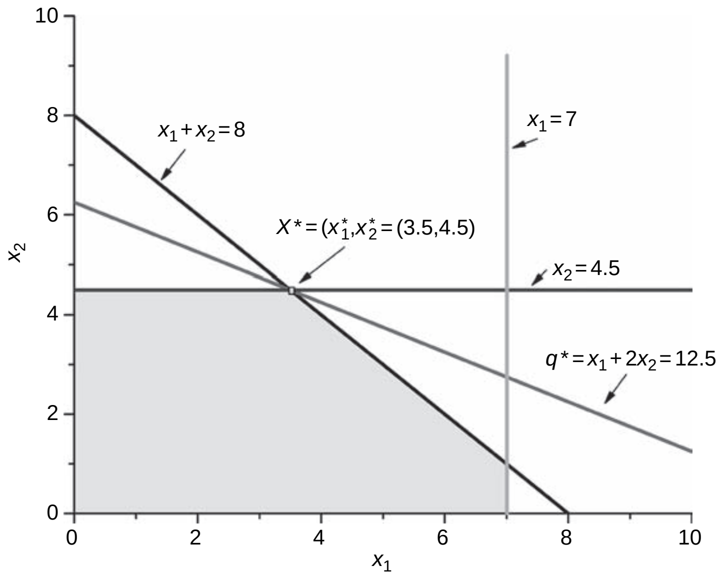

Figure 10.4 Linear Programming Solution

as explicit. To do this, we refine the notion of *surplus* variables introduced earlier in developing the linear programming problem to define so-called *deviational* variables d_1, d_2, and d_3 that measure "deviation" from the right-hand side values of the constraint equations and another, d_4, that measures deviation from an established goal for the explicit objective function, q. In all cases we assume that $d_k \geq 0$.

We can think of the deviational variables as either exhibiting *overachievement* or *underachievement* of the established goal. For overachievement of goal k we call the level of overachievement a *positive deviation*, which is indicated by a non-zero value for a *positive deviational variable,* d_k^+ (like the *surplus* variable used earlier). Likewise, for underachievement of the established goal, we call the level of underachievement a *negative deviation* from the goal indicated by a nonzero value for a *negative deviational variable*, d_k^- (sometimes referred to as a *slack* variable). Note that if $d_k^+ > 0$ then $d_k^- = 0$ and vice versa, i.e., at most one of the paired positive and negative deviational variables, d_k^- and d_k^+, can be greater than 0; if both are 0 then, of course, the goal is achieved exactly.

In GP, to account for the relative order of priority on the objectives, we assign the objectives to *pre-emptive priority* classes, each of which is denoted by P_l, for $l = 1, \ldots, L$ where L is the total number of priority classes, which for our example reduces to P_1 and P_2 associated with first satisfying all three constraints and then with maximizing q, respectively. For our example to be consistent with the implicit priority order in the LP problem we assign the objective functions derived from the LP constraint equations as within P_1 and the explicit objective function, q, in the LP problem as within P_2. We write this as

$$\text{Min } \; P_1(d_1^+ + d_2^+ + d_3^+) + P_2(d_4^-) \tag{10.3}$$

subject to :

$$x_1 + x_2 + d_1^- - d_1^+ = 8 \tag{10.4}$$

$$x_1 + d_2^- - d_2^+ = 7 \tag{10.5}$$

$$x_2 + d_3^- - d_3^+ = 4.5 \tag{10.6}$$

$$x_1 + 2x_2 + d_4^- - d_4^+ = 20 \tag{10.7}$$

Note that, in general, minimizing positive deviational variables is equivalent to satisfying a $\leq$ constraint or minimizing an objective function and minimizing negative deviational variables is equivalent to satisfying a $\geq$ constraint or maximizing an objective function. Hence, seeking to drive d_k^+ , for $k = 1, 2$ and 3 to zero is equivalent to satisfying the constraints originally specified in the LP problem and seeking to drive d_4^- to zero is equivalent to maximizing the objective function specified in the LP problem. In effect, in GP we no longer really distinguish between what in the LP formulation were objective and constraint equations[5] and, rather, consider them all goal equations that are optimized in a specified preemptive order of priority.[6] For the example all the former objective and constraint equations are posed as GP goal equations denoted as (10.4), (10.5), (10.6), and (10.7). Also, in this example, the right-hand side to the last goal equation, i.e., (10.7), is an arbitrarily large value (we chose 20 for this example) as a goal, since minimizing negative deviation of the objective function from an arbitrarily high goal is the same as maximizing the objective function.

We solve the example graphically in Figures 10.5–10.8. Conceptually we can think of the solution as successively narrowing the feasible region of solutions until the feasible region is reduced to a single point – the optimal solution:

- Objective 1: The first objective is to minimize d_1^+. We are able to minimize d_1^+ in (10.4) to zero, which reduces the solution region from the starting point of all non-negative values of x_1 and x_2 to those points beneath the line defined by $x_1 + x_2 = 8$ in Figure 10.5 (the shaded region).
- Objective 2: The second objective is to minimize d_2^+. We are able to minimize d_2^+ in (10.5) to zero, which reduces the solution region from the shaded portion in Figure 10.5 (all points beneath the line defined by $x_1+x_2 = 8$ in that figure) to the intersection of that region and the region comprised of points to left the line defined by $x_1 = 7$, shown as the dark shaded region in Figure 10.6.

[5] Constraints in LP are generally specified as inequalities, but by introduction of slack or surplus variables, they can be specified as equations.

[6] This applies to the linear version of GP; the features of alternative GP formulations, such as developed in Lane (1970) or Cohen (1978), address some of the limitations of the linear approach that will be apparent in what follows, e.g., when problems are very tightly constrained.

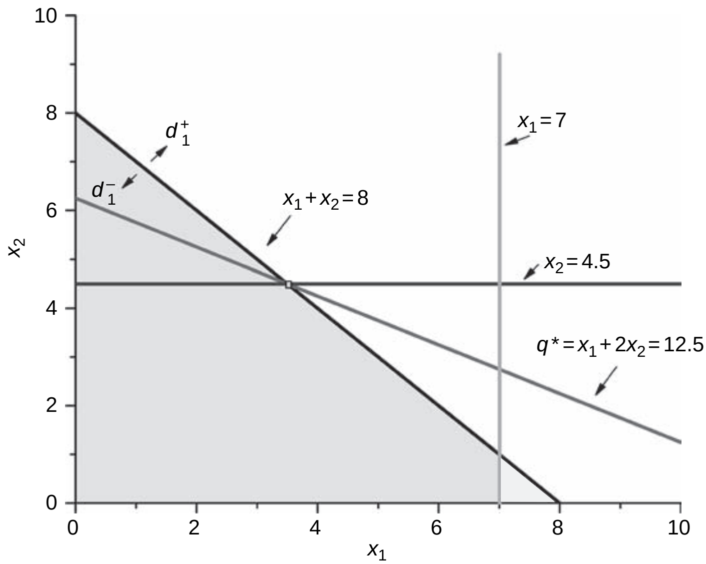

Figure 10.5 Goal Programming Solution: Objective 1

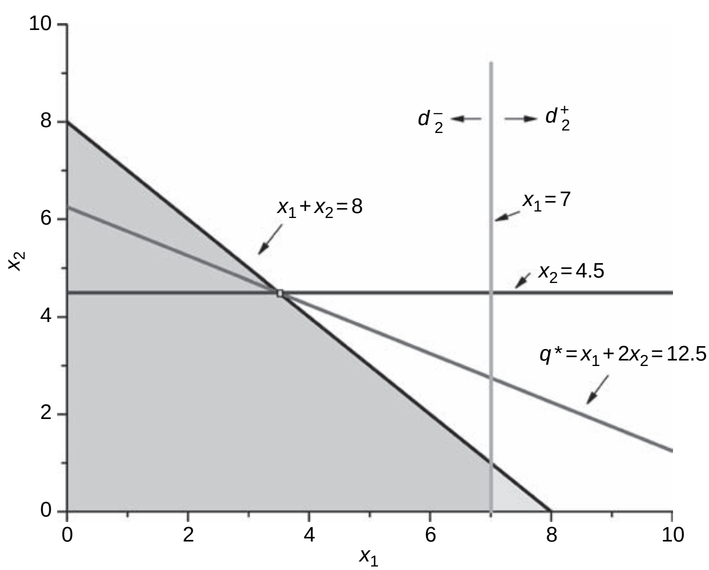

Figure 10.6 Goal Programming Solution: Objective 2

- Objective 3: The third objective is to minimize d_3^+. We are able to minimize d_3^+ in (10.6) to zero, which reduces the solution region from the dark shaded portion in Figure 10.6 (optimizing all objectives so far) to the intersection of that region and the region comprised of points to below the line defined by $x_2 = 4.5$, as shown in Figure 10.7.

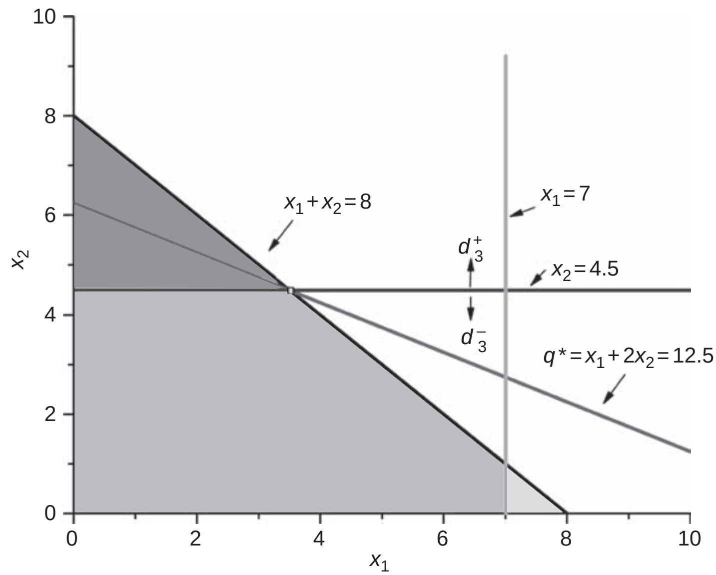

Figure 10.7 Goal Programming Solution: Objective 3

- Objective 4: Finally, the last objective is to minimize d_4^-. We can test successive values of $q = 5, 9, 12.5, 15$, and 20, and if we were able to drive d_4^- to zero then q would be 20. However, at that value we would violate several of the previous, higher priority objectives. The farthest we can get and still satisfy the higher order objectives is at $q = 12.5$ or where $d_4^- = 7.5$, which is at the point, $\mathbf{x}^* = (3.5, 4.5)$, as shown in Figure 10.8. This is, of course equivalent to the LP solution since we expressed the GP pre-emptive priority classes in the same order as that of the implicit and explicit objectives for the LP problem, respectively. Other solutions are possible, however, with alternative specifications the pre-emptive priority levels. We leave it to the reader to experiment with such situations with this example, although we illustrate this flexibility later as GP is applied to the case of generalized input–output planning models.

10.4.4 Additional Observations

Specifying Objectives It is important to note that GP provides considerable flexibility in handling multiple, even conflicting objectives. In LP we are required to begin with a feasible region in order to find our way to an optimal solution. In GP the feasible region is essentially unspecified since any point can be completely specified in terms of the deviational variables and it is the priority order of objectives that determines the solution, because by specifying a priority order a solution can be found even if the objectives are conflicting. This places much more of the planning burden on determining the priority order, which in a policy context is often quite complex, especially with

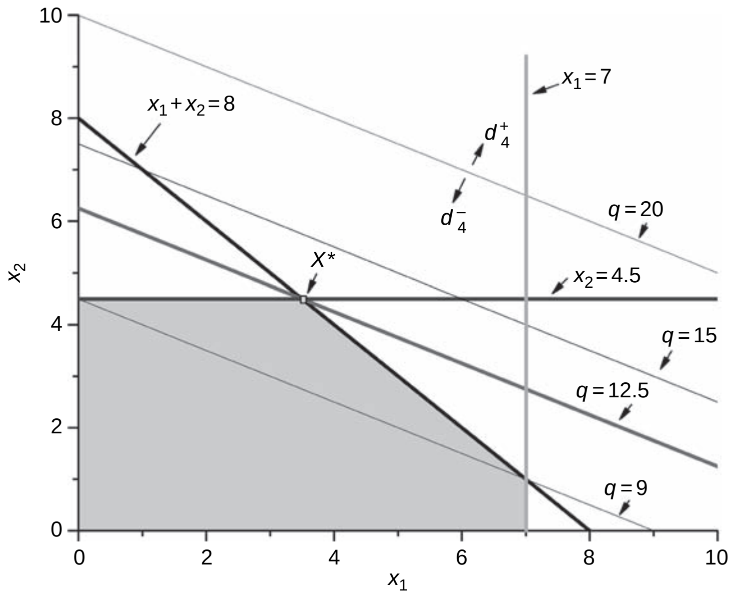

Figure 10.8 Goal Programming Solution: Objective 4

sometimes competing policy objectives, such as economic growth and environmental quality. Blair (1979) employs an approach called *analytic hierarchies* (Saaty, 1980) for this purpose. Others use a wide range of multiobjective decision-making approaches, such as Nijkamp and van Delft (1977) and Cohen (1978).

Tightly Constrained Problems Another commonly cited limitation of linear goal programming (and linear programming for that matter) is that tightly constrained problems are insensitive to how close one is to a given target because solutions, at least as we have developed the methodology so far, are developed by satisfying goals completely in pre-emptive order (ordinal) or, as it is sometimes called, lexicographic order. This means that one has to fully satisfy a higher order goal equation before moving on to the next. This can lead to illogical solutions, especially in tightly constrained problems. For example, if an employment goal has a higher priority than, say, a pollution goal, then the last unit of employment achieved could be at the expense of an enormous amount of pollution. The literature includes many approaches to address this problem, such as Lane (1970).

Solution Methods As with LP, when the number of variables and equations increases beyond two, solution procedures become much more complex. However, there are a variety of solution approaches to GP problems. In GP, as we have just seen with an example, through sequential imposition of constraints we arrive at a solution – GP is sometimes referred to as "weighting within constraints." However, in our graphical solution we did not distinguish between different pre-emptive classes of objectives and objectives within a pre-emptive priority class. In practice, the solution to objectives within a pre-emptive priority class is determined simultaneously rather than sequentially. Linear GP problems (the only sort covered in this text) can be solved via a basic simplex algorithm similar to that commonly used in LP, as in Blair (1979) and Lee (1971, 1972, 1973). Other approaches are explored in Ijiri (1965), Cohen (1978), and Ignizio (1976).

10.4.5 Applications to the Generalized Input–Output Planning Problem

Let us return to our generalized input–output planning example (Example 10.1) to illustrate the GP solution in this context. Recall the constraint equations, which in the GP context are no longer called constraints, but rather goal equations. We refer to the relationships among the goal equations collectively as the set of *system process functions*:

$$\tilde{\mathbf{x}} = \mathbf{G}\mathbf{x} = \begin{bmatrix} \mathbf{D} \\ (\mathbf{I}-\mathbf{A}) \end{bmatrix}\mathbf{x} = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.4 \\ 0.5 & 1.1 \\ 0.7 & 0.7 \\ 0.1 & 0.2 \\ 0.7 & -0.2 \\ -0.1 & 0.3 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 16 \\ 14 \\ 3 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}^* \\ \mathbf{f} \end{bmatrix}$$

where the partitions of the matrix $\mathbf{G}$ are the coefficients for the energy, environment, employment, and economic (Leontief) equations, respectively.

These relationships are depicted in Figure 10.9, which is a trivial example because the right-hand-side quantities were derived from the impact analysis version of the model for the case where the energy, pollution, and employment values were found from $\mathbf{f} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ in the first place. It will be more interesting if we choose different values for $\mathbf{x}^*$ and $\mathbf{f}$. We consider the case where there is growth in Industry B, so that final demand becomes $\mathbf{f} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$ and where we consider the energy and environmental targets to be re-established as $\mathbf{x}^{e*} = \begin{bmatrix} 20 \\ 20 \end{bmatrix}$ and $\mathbf{x}^{p*} = \begin{bmatrix} 16 \\ 20 \end{bmatrix}$, respectively, which we seek not to exceed and the employment target $x^{l*} = 12$, which we seek to not drop below.

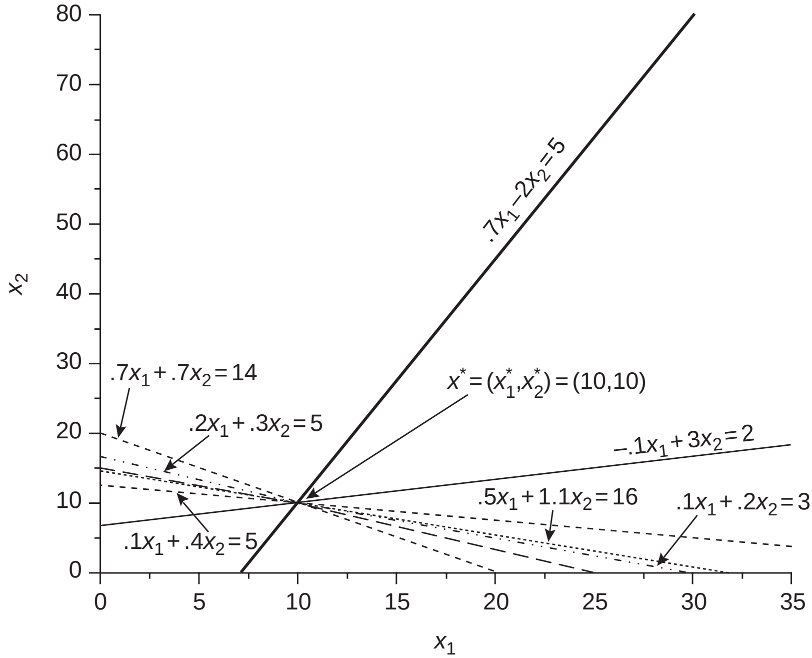

Figure 10.9 Goal Programming Generalized Input–Output Initial Solution

The problem can be restated as

$$\tilde{\mathbf{x}} = \mathbf{G}\mathbf{x} = \begin{bmatrix} \mathbf{D} \\ (\mathbf{I}-\mathbf{A}) \end{bmatrix}\mathbf{x} = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.4 \\ 0.5 & 1.1 \\ 0.7 & 0.7 \\ 0.1 & 0.2 \\ 0.7 & -0.2 \\ -0.1 & 0.3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{matrix} \le \\ \le \\ \le \\ \le \\ \ge \\ \ge \\ \ge \end{matrix} \begin{bmatrix} 20 \\ 20 \\ 16 \\ 20 \\ 12 \\ 5 \\ 10 \end{bmatrix} = \begin{bmatrix} \mathbf{x}^* \\ \mathbf{f}' \end{bmatrix}$$

and translated into a GP format, the goal equations become

$$0.2x_1 + 0.3x_2 \le 20 \rightarrow 0.2x_1 + 0.3x_2 + d_1^- - d_1^+ = 20 \ \text{(minimize } d_1^+) \quad (10.8)$$

$$0.1x_1 + 0.4x_2 \le 20 \rightarrow 0.1x_1 + 0.4x_2 + d_2^- - d_2^+ = 20 \ \text{(minimize } d_2^+) \quad (10.9)$$

$$0.5x_1 + 1.1x_2 \le 16 \rightarrow 0.5x_1 + 1.1x_2 + d_3^- - d_3^+ = 16 \ \text{(minimize } d_3^+) \quad (10.10)$$

$$0.7x_1 + 0.7x_2 \leq 20 \rightarrow 0.7x_1 + 0.7x_2 + d_4^- - d_4^+ = 20 \text{ (minimize } d_4^+) \quad (10.11)$$

$$0.1_1 + 0.2x_2 \geq 12 \rightarrow 0.1x_1 + 0.2x_2 + d_5^- - d_5^+ = 12 \text{ (minimize } d_5^-) \quad (10.12)$$

$$0.7x_1 - 0.2x_2 \geq 5 \rightarrow 0.7x_1 - 0.2x_2 + d_6^- - d_6^+ = 5 \text{ (minimize } d_6^-) \quad (10.13)$$

$$-0.1x_1 + 0.3x_2 \geq 10 \rightarrow -0.1x_1 + 0.3x_2 + d_7^- - d_7^+ = 10 \text{ (minimize } d_7^-) \quad (10.14)$$

The objective function, assuming a specific set of pre-emptive priorities placing (arbitrarily for now), say, employment in the highest class with others following in lower classes, is the following:

$$P_1(d_5^-) + P_2(d_6^- + d_7^-) + P_3(d_1^+ + d_2^+) + P_4(d_3^+ + d_4^+) \quad (10.15)$$

Graphically, we depict this solution to this problem in Figures 10.10–10.15, considering the goal equations in priority order:

- Objective 1: The first objective is to minimize d_5^- in (10.12), which restricts the solution region from the starting point of all non-negative values of x_1 and x_2 to the area above the line associated with the equation $0.1x_1 + 0.2x_2 = 12$. We are able to minimize d_5^- in (10.12) to zero, the result of which is the shaded region in Figure 10.10.
- Objective 2: The next objective is to minimize d_6^- in (10.13), subject to the constraint already imposed by Objective 1, which restricts the solution region to the intersection of the region above the line defined by $0.1x_1 + 0.2x_2 = 12$ (optimizing Objective 1)

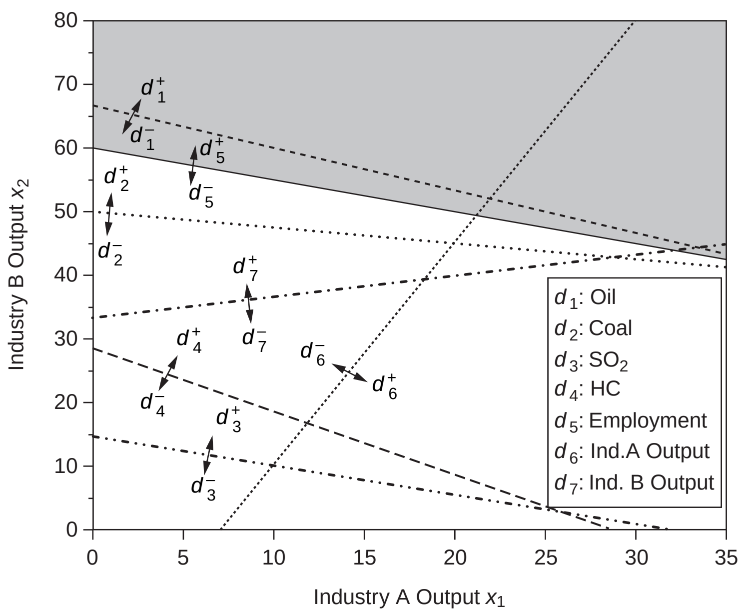

Figure 10.10 Generalized Input–Output Goal Programming: Example 10.1 (Objective 1)

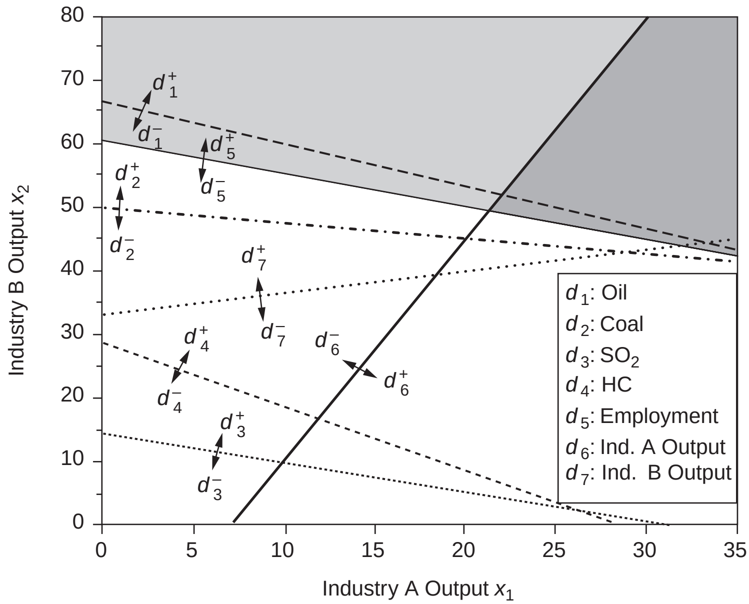

Figure 10.11 Generalized Input–Output Goal Programming: Example 10.1 (Objective 2)

with the region to the right of the line defined by $0.7x_1 - 0.2x_2 = 5$, as shown by the dark shaded region in Figure 10.11. Hence, we are able to minimize d_6^- to zero.

- Objective 3: The next objective is to minimize d_7^- in (10.14), subject to the earlier constraints (optimizing Objectives 1 and 2), which restricts the solution region further by allowing only points above the line defined by $-0.1x_1 + 0.3x_2 = 10$, as shown by the dark shaded region in Figure 10.12.
- Objective 4: The next objective is to minimize d_1^+ in (10.8), subject to the constraints already imposed in satisfying Objectives 1, 2, and 3. This attempts to restrict the solution region to points below the line defined by $0.2x_1 + 0.3x_2 = 20$, which restricts the solution region to the darkly shaded region shown in Figure 10.13. We are able to minimize d_1^+ to zero.
- Objective 5: The next objective is to minimize d_2^+ in (10.9), subject to the previous higher priority constraints imposed so far. We cannot minimize d_2^+ all the way to zero – the closest we can get, without compromising higher priority objectives, is to reduce the solution region to the points along the line defined by $0.1x_1 + 0.2x_2 = 12$ (the same line defining the lower boundary of the solution region satisfying objective 1), as shown in Figure 10.14.
- Objective 6: The next objective is to minimize d_3^+ in (10.10). We cannot reduce d_3^+ to zero – the closest we can get, without compromising earlier constraints, is by restricting the solution region to the point shown as $\mathbf{x}^*$ in Figure 10.15.
- Objective 7: Attempting to move towards any other goal will result in compromising on a higher order goal. Hence we have found the GP solution, which is at the point $\mathbf{x}^* = (x_1^*, x_2^*) = (21.25, 49.375)$.

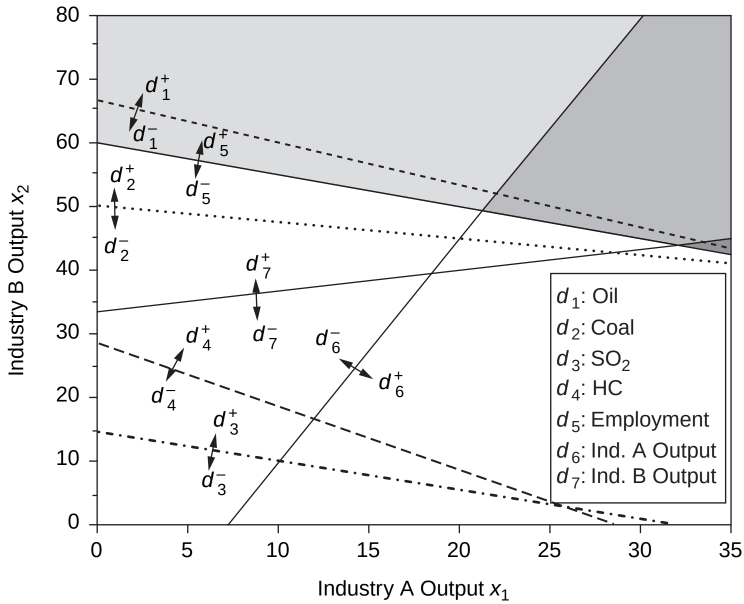

Figure 10.12 Generalized Input–Output Goal Programming: Example 10.1 (Objective 3)

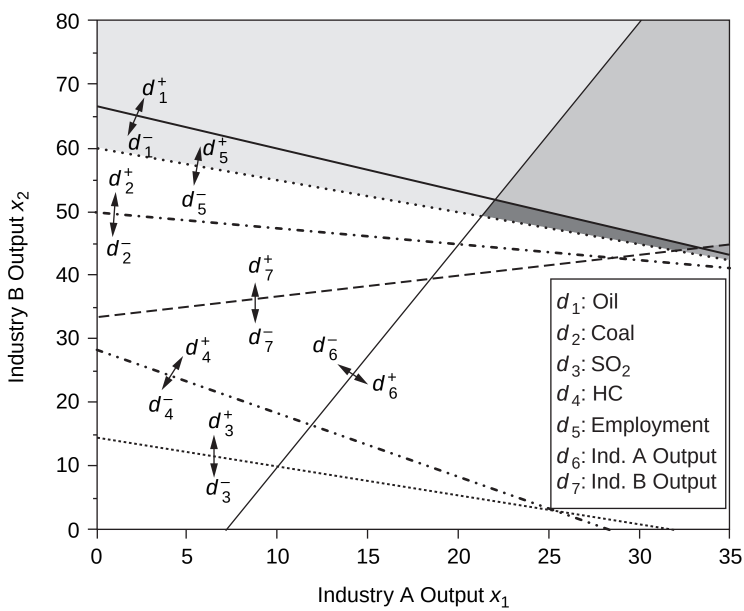

Figure 10.13 Generalized Input–Output Goal Programming: Example 10.1 (Objective 4)

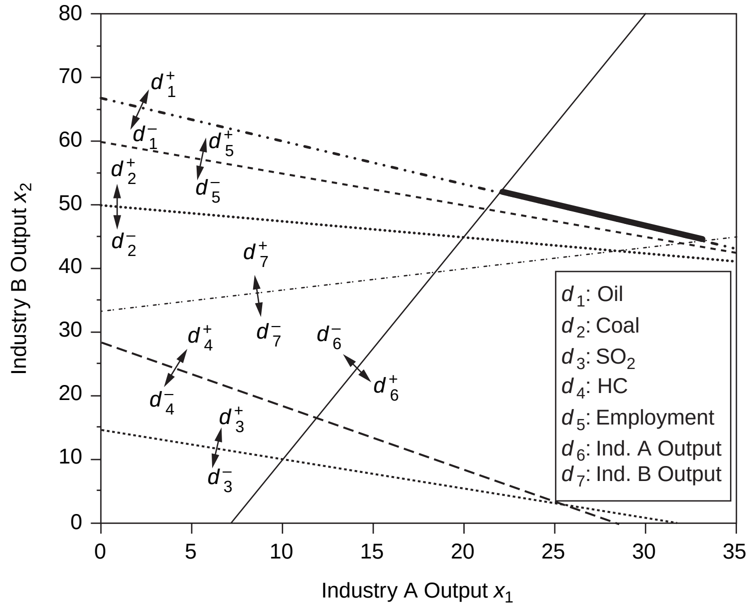

Figure 10.14 Generalized Input–Output Goal Programming: Example 10.1 (Objective 5)

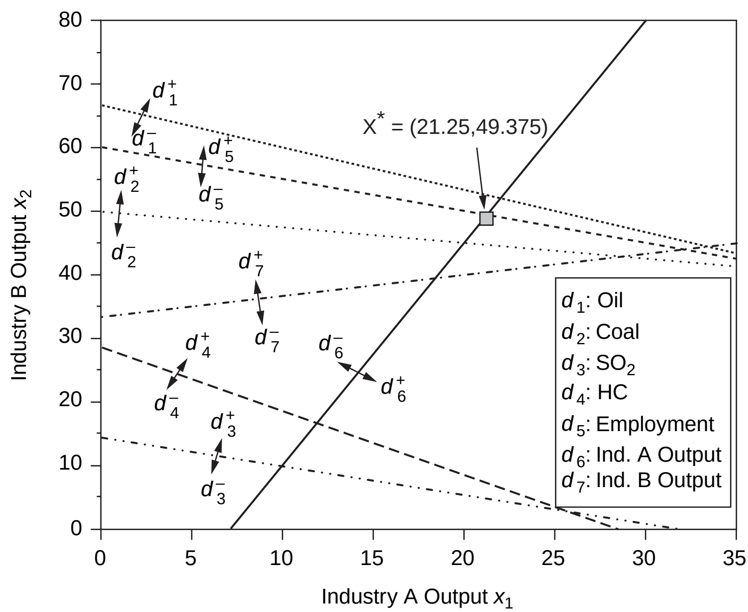

Figure 10.15 Generalized Input–Output Goal Programming: Example 10.1 (Objective 6)

If we change the order of priority in the GP objective function, then, of course, a different solution will result. For example, suppose the highest priority is determined to be minimizing sulfur dioxide emissions (and we arbitrarily relegate maximizing employment to the lowest priority class), then the GP objective function would be $P_1(d_3^+) + P_2(d_6^- + d_7^-) + P_3(d_1^+ + d_2^+) + P_4(d_4^+) + P_5(d_5^-)$. We leave it as an exercise to the reader to show that the solution to the GP problem with the new objective function is the point $\mathbf{x}^* = (x_1^*, x_2^*) = (10, 10)$.

10.4.6 Policy Programming

Blair (1979) combines the impact analysis version of the generalized input–output model along with the planning form and goal programming in an integrated approach called *policy programming*. In this approach, first, a number of alternative future regional energy development scenarios, or *future scenarios*, for short, are defined. These scenarios are all defined to satisfy a set of generalized input–output system process equations, as defined earlier. That is, each future scenario is defined in terms of values of industry output, energy consumption, pollution emissions, and regional employment that comply with the basic Leontief identities and accompanying direct impact coefficients for energy use, environmental pollution emissions, and regional employment. We define a collection of values of these variables that comply with these system process functions as a *consistent* scenario.

In policy programming the method of analytic hierarchies[7] is applied to define *preference scenarios*, which are simply linear combinations of future scenarios. A set of relative weights derived from the method of analytic hierarchies is applied to assemble the linear combination of future scenarios that reflect the relative desirability of the alternative future scenarios. Preference scenarios are assembled for each of a number of defined policy makers or other significant decision-makers in the planning process, e.g., electric utilities, government regulators, or industrial consumers. Since these preference scenarios are linear combinations of consistent scenarios, they are also consistent by the above definition (a theorem from Blair, 1979), and illustrated below.

The variables defined in the system process functions are then divided into clusters, over which different policy makers (or others relevant to the planning problem) have differing degrees of relative influence. The methodology of analytic hierarchies is used again, this time to define weights of relative influence of policy makers over clusters of decision variables. A *composite scenario* is then assembled by bringing together all clusters of variables, with the values derived from the corresponding clusters of variable values drawn from preference scenarios weighted by the relative influence of policy makers over those clusters. This composite scenario, however, will not likely be consistent (by the earlier definition) with the set of system process functions since the relative weights over clusters are derived independently and the composite scenario

[7] The method of analytic hierarchies, often referred to as the Analytic Hierarchy Process, is a theory and method of decision-making based on deriving priorities from a matrix of pairwise comparisons of alternatives; see Saaty (1980).

is not a linear combination of consistent scenarios. We show this by returning to our example.

In the earlier example (Example 10.1) we defined a basic *consistent* future scenario. This scenario can be expressed in either the *impact analysis* and *planning* forms that equivalently describe the same situation as:

Impact Analysis Form

$$\bar{\mathbf{x}} = \begin{bmatrix} \mathbf{x}^* \\ \mathbf{x} \end{bmatrix} = \mathbf{Hf} = \begin{bmatrix} \mathbf{D}^* \\ \mathbf{L} \end{bmatrix} \mathbf{f} = \begin{bmatrix} 0.47 & 1.32 \\ 0.37 & 1.58 \\ 1.37 & 4.58 \\ 1.47 & 3.32 \\ 0.26 & 0.84 \\ 1.58 & 1.05 \\ 0.53 & 3.68 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 16 \\ 14 \\ 3 \\ 10 \\ 10 \end{bmatrix}$$

Planning Form

$$\tilde{\mathbf{x}} = \begin{bmatrix} \mathbf{x}^* \\ \mathbf{f} \end{bmatrix} = \mathbf{Gx} = \begin{bmatrix} \mathbf{D} \\ (\mathbf{I} - \mathbf{A}) \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.4 \\ 0.5 & 1.1 \\ 0.7 & 0.7 \\ 0.1 & 0.2 \\ 0.7 & -0.2 \\ -0.1 & 0.3 \end{bmatrix} \begin{bmatrix} 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 16 \\ 14 \\ 3 \\ 5 \\ 2 \end{bmatrix}$$

We define three new final-demand vectors $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{f}_3$ that correspond to different possible *future scenarios* as $\mathbf{f}_1 = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$, $\mathbf{f}_2 = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$, and $\mathbf{f}_3 = \begin{bmatrix} 4.5 \\ 4.5 \end{bmatrix}$. For each final-demand vector, we can compute the generalized impact of each as $\bar{\mathbf{x}}_i = \mathbf{Hf}_i$ for $i = 1$, 2 and 3. For convenience, if we define $\mathbf{F} = \begin{bmatrix} \mathbf{f}_1 & \mathbf{f}_2 & \mathbf{f}_3 \end{bmatrix}$ as a matrix, the columns of which are the final-demand vectors, then it is easy to define $\bar{\mathbf{X}} = [\bar{\mathbf{x}}_1 \quad \bar{\mathbf{x}}_2 \quad \bar{\mathbf{x}}_3]$ as the matrix of corresponding generalized impact vectors, which can be expressed as

$$\bar{\mathbf{X}} = \mathbf{HF} = \begin{bmatrix} \bar{\mathbf{x}}_1 & \bar{\mathbf{x}}_2 & \bar{\mathbf{x}}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{x}_1^* & \mathbf{x}_2^* & \mathbf{x}_3^* \\ \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 6.8 & 10.1 & 8.1 \\ 7 & 11.8 & 8.8 \\ 21.9 & 34.7 & 26.7 \\ 18.8 & 26.1 & 21.6 \\ 4.1 & 6.4 & 5 \\ 12.6 & 10.5 & 11.8 \\ 14.2 & 26.8 & 18.9 \end{bmatrix}$$

These scenarios (each of which is specified in a column of $\bar{\mathbf{X}}$) are *consistent* as defined earlier, since in each case $\mathbf{x}_i^* = \mathbf{DLf}_i$ for $i = 1, 2,$ and 3. If we define a *composite scenario* as a linear combination of these future scenarios, then it is easy to show that the composite scenario is also consistent. For example, consider the composite scenario $\mathbf{f}_c$ defined as a simple average of the future scenarios, i.e., $\mathbf{f}_c = \sum_{i=1}^{3} \beta_i \mathbf{f}_i$, where $\beta_i = 1/3$ for $i = 1, 2,$ and 3 (note that $\sum_{i=1}^{3} \beta_i = 1$). It is easy to show that $\mathbf{x}_c^* = \mathbf{DLf}_c$ thereby confirming that the composite scenario is consistent as defined above. Blair (1979) shows that this is true for any linear combination of consistent future scenarios (not just a simple average as above). For example, consider the case where $\beta_1 = 0.2$, $\beta_2 = 0.3$, and $\beta_3 = 0.5$ (note again that $\sum_{i=1}^{3} \beta_i = 1$, although that also is not necessarily required for the composite scenario to be consistent):

$$\bar{\mathbf{x}}_c = \mathbf{Hf}_c = \mathbf{H}\left[\beta_1\mathbf{f}_1 + \beta_2\mathbf{f}_2 + \beta_3\mathbf{f}_3\right] = \mathbf{H}\left(.2\begin{bmatrix}6\\3\end{bmatrix} + .3\begin{bmatrix}2\\7\end{bmatrix} + .5\begin{bmatrix}4.5\\4.5\end{bmatrix}\right)$$

$$= \mathbf{H}\begin{bmatrix}4.05\\4.95\end{bmatrix} = \begin{bmatrix}8.4\\9.3\\28.2\\22.4\\5.2\\11.6\\20.4\end{bmatrix}$$

Since $\bar{\mathbf{x}} = \begin{bmatrix}\mathbf{x}^*\\ \mathbf{x}\end{bmatrix}$, we write $\bar{\mathbf{x}}_c = \begin{bmatrix}\mathbf{x}_c^*\\ \mathbf{x}_c\end{bmatrix}$ and if $\mathbf{x}_c^* = \mathbf{Dx}_c$, then the composite scenario is consistent as defined above; this will be true for any linear combination of consistent future scenarios.

Now, suppose we now define variable clusters for these scenarios as energy, pollution, and employment (as in Table 10.2). If we define a different composite scenario as one in which the relative weights for combining the future scenarios are different for each cluster, then it is easy to show that the resulting composite scenario is not necessarily consistent, and in general unlikely. So instead of β_1, β_2 and β_3 we then define these weights for each cluster of variables as β_i^e, β_i^p and β_i^l for scenarios $i = 1, 2,$ and 3, where e, p, and l denote, as before, the energy, pollution, and employment clusters of variables, respectively. Assume for our example these weights (we assume economic output variables for the three scenarios are equally weighted) are given in Table 10.3.

Table 10.3 Policy Programming: Composite Scenario Weights

		Scenario		
		$i = 1$	$i = 2$	$i = 3$
Variable Cluster				
Energy	e	0.1	0.7	0.2
Pollution	p	0.8	0.1	0.1
Employment	l	0.3	0.2	0.5

To account for the variable clusters, we can express each future scenario i as $\bar{\mathbf{x}}_i = \begin{bmatrix} \mathbf{x}_i^e \\ \mathbf{x}_i^p \\ \mathbf{x}_i^l \\ \\ \mathbf{x}_i \end{bmatrix}$ or equivalently as $\tilde{\mathbf{x}}_i = \begin{bmatrix} \mathbf{x}_i^e \\ \mathbf{x}_i^p \\ \mathbf{x}_i^l \\ \\ \mathbf{f}_i \end{bmatrix}$. Hence, we can define a composite scenario, reflecting the different weights for each cluster of variables, as

$$\bar{\mathbf{x}}_c = \begin{bmatrix} \sum_{i=1}^{3} \beta_i^e \mathbf{x}_i^e \\ \sum_{i=1}^{3} \beta_i^p \mathbf{x}_i^p \\ \sum_{i=1}^{3} \beta_i^l \mathbf{x}_i^l \\ \\ \sum_{i=1}^{3} \mathbf{x}_i \end{bmatrix} = \begin{bmatrix} 0.1 \begin{bmatrix} 6.8 \\ 7.0 \end{bmatrix} + 0.7 \begin{bmatrix} 10.1 \\ 11.8 \end{bmatrix} + 0.2 \begin{bmatrix} 8.1 \\ 8.8 \end{bmatrix} \\ 0.8 \begin{bmatrix} 21.9 \\ 18.8 \end{bmatrix} + 0.1 \begin{bmatrix} 34.7 \\ 26.1 \end{bmatrix} + 0.1 \begin{bmatrix} 26.7 \\ 21.6 \end{bmatrix} \\ 0.3\,[4.1] + 0.2\,[6.4] + 0.5\,[5] \\ \\ 1/3 \begin{bmatrix} 12.6 \\ 14.2 \end{bmatrix} + 1/3 \begin{bmatrix} 10.5 \\ 26.8 \end{bmatrix} + 1/3 \begin{bmatrix} 11.8 \\ 18.9 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 9.4 \\ 10.7 \\ 23.7 \\ 19.5 \\ 5.0 \\ \\ 11.6 \\ 20.0 \end{bmatrix}$$

If we now test to see whether or not this new composite scenario is consistent, as defined earlier, we find it is not, i.e., $\mathbf{x}_c^* \neq \mathbf{D}\mathbf{x}_c$, as one might expect:

$$\mathbf{x}_c^* = \begin{bmatrix} 9.4 \\ 10.7 \\ 23.7 \\ 19.5 \\ 5.0 \end{bmatrix} \neq \mathbf{D}\mathbf{x}_c = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.4 \\ 0.5 & 1.1 \\ 0.7 & 0.7 \\ .1 & .2 \end{bmatrix} \begin{bmatrix} 11.6 \\ 20.0 \end{bmatrix} = \begin{bmatrix} 8.3 \\ 9.2 \\ 27.8 \\ 22.1 \\ 5.2 \end{bmatrix}$$

The problem of defining a *consistent composite scenario* can be solved via GP, which can be used to find a consistent scenario that is as close as possible to the composite scenario but that complies fully with the defined system process functions. The literature includes a variety of attempts to extend the input–output framework applied to environmental problems using various kinds of multiobjective decision-making tools in addition to the approach illustrated here (one illustrative example is Hipel, 1992).

10.4.7 Ecological Commodities

In the preceding discussion and in Chapter 9, we defined a set of direct and indirect impacts as factors – such as energy consumption, pollution generation, and employment – that are associated with interindustry activity. In evaluating many environmental issues, we may wish to distinguish between such factors viewed as inputs to an industry production process, for example, energy and employment, and those factors viewed as outputs generated by a production process, for example, pollution.

We might view all these factors as flows into and out of the ecosystem in which the interindustry economic system exists, that is, as ecological input and output commodities. Further, we might restrict our consideration of ecological commodities to nonmarket materials, since we can adequately deal with the marketable commodities through the Leontief model itself (sometimes with modifications, as we found in earlier chapters).

We define a set of ecological commodity inputs – for example, water, land, or air – the magnitudes of which we will capture in a matrix $\mathbf{M} = [m_{kj}]$, an element of which reflects the amount of ecological input of type k used in the production of economic sector j's total output. Similarly, we define a set of ecological commodity outputs – for example, pounds of sulfur dioxide air pollution. The corresponding matrix of ecological commodity output flows is $\mathbf{N} = [n_{kj}]$, an element of which specifies the amount of ecological commodity output k associated with the output of sector j.

Johnson and Bennett (1981) classify ecological commodities according to the sources from which they are extracted and the sinks to which they are eventually discharged. For example, consider the table of economic and ecologic commodity flows in Table 10.4.

For purposes of this illustration, we consider the interindustry transactions to be measured in monetary units, while ecological commodity inputs and outputs are measured in physical units such as acre-feet of water, acres of land, and tons of SO_2 or HC. In Table 10.4 we identify the matrices of ecological commodity inputs and outputs, respectively, i.e., $\mathbf{M}$ and $\mathbf{N}$, as well as the interindustry transactions, $\mathbf{Z}$, vector of total final demands, $\mathbf{f}$, and the vector of total industry outputs, $\mathbf{x}$, corresponding to Table 10.4.

We can now define ecological commodity input and output coefficients in much the same way we defined direct impact coefficients earlier, by first recalling that $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$, which defines the matrix of technical coefficients; hence, similarly we define the matrices of ecological commodity input and output coefficients as $\mathbf{R} = \mathbf{M}\hat{\mathbf{x}}^{-1}$ which

Table 10.4 Economic–Ecologic Commodity Flows

	Interindustry Transactions					Ecological Commodity Outputs	
	Consuming Sectors			Final	Total		
	Agriculture	Mining	Manuf.	Demand	Output	SO_2	HC
Producing Sectors							
Agriculture	1	3	5	3	12	0	1
Mining	0	2	10	0	12	0	2
Manufacturing	0	2	6	16	24	4	3
Ecological Commodity Inputs							
Water	5	4	8				
Land	10	10	1				

defines the matrix of ecological commodity input coefficients, that is, the elements of $\mathbf{R} = [r_{kj}]$ specify the amount of commodity k required per dollar's worth of output of industry j. Also, $\mathbf{Q} = \mathbf{N}'\hat{\mathbf{x}}^{-1}$ defines the ecological commodity output coefficients, that is, $\mathbf{Q} = [q_{kj}]$ specifies the amount of commodity k generated per dollar's worth of output of industry j. Note that $\mathbf{N}'$ is the transpose of the matrix of ecological commodity output flows. For the data given in Table 10.4, we find

$$\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 1 & 3 & 5 \\ 0 & 2 & 10 \\ 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} 1/12 & 0 & 0 \\ 0 & 1/12 & 0 \\ 0 & 0 & 1/24 \end{bmatrix} = \begin{bmatrix} 0.083 & 0.250 & 0.208 \\ 0 & 0.167 & 0.417 \\ 0 & 0.167 & 0.250 \end{bmatrix}$$

$$\mathbf{R} = \mathbf{M}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 5 & 4 & 8 \\ 10 & 10 & 1 \end{bmatrix} \begin{bmatrix} 1/12 & 0 & 0 \\ 0 & 1/12 & 0 \\ 0 & 0 & 1/24 \end{bmatrix} = \begin{bmatrix} 0.417 & 0.333 & 0.333 \\ 0.833 & 0.833 & 0.042 \end{bmatrix}$$

$$\mathbf{Q} = \mathbf{N}'\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 0 & 0 & 4 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1/12 & 0 & 0 \\ 0 & 1/12 & 0 \\ 0 & 0 & 1/12 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0.167 \\ 0.083 & 0.167 & 0.125 \end{bmatrix}$$

Using $\mathbf{R}$ and $\mathbf{Q}$ as computed above, total impact coefficients – in this case, ecological commodity input and output coefficients as a function of final demands – can be written as $\mathbf{R}^* = \mathbf{R}(\mathbf{I}-\mathbf{A})^{-1}$ and $\mathbf{Q}^* = \mathbf{Q}(\mathbf{I}-\mathbf{A})^{-1}$, respectively. For the example,

$$\mathbf{R}^* = \mathbf{R}(\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} 0.417 & 0.333 & 0.333 \\ 0.833 & 0.833 & 0.042 \end{bmatrix} \begin{bmatrix} 1.091 & 0.436 & 0.545 \\ 0 & 1.350 & 0.750 \\ 0 & 0.300 & 1.500 \end{bmatrix}$$

$$= \begin{bmatrix} 0.455 & 0.732 & 0.977 \\ 0.909 & 1.501 & 1.142 \end{bmatrix}$$

$$\mathbf{Q}^* = \mathbf{Q}(\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} 0 & 0 & 0.167 \\ 0.083 & 0.167 & 0.125 \end{bmatrix} \begin{bmatrix} 1.091 & 0.436 & 0.545 \\ 0 & 1.350 & 0.750 \\ 0 & 0.300 & 1.500 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.050 & 0.250 \\ 0.011 & 0.299 & 0.358 \end{bmatrix}$$

The elements in $\mathbf{R}^* = [r^*_{ij}]$ reflect the amount of ecologic input i required directly and indirectly to deliver a dollar's worth of industry j's output to final demand. For example, $r^*_{11} = 0.455$ indicates that 0.455 units of water are required to deliver a dollar's worth of agricultural products to final demand. Similarly, the elements in $\mathbf{Q}^* = [q^*_{ij}]$ reflect the amount of ecologic output i associated with delivering a dollar's worth of industry j's output to final demand directly and indirectly. For example, $q^*_{23} = 0.358$ means that associated with delivering one dollar's worth of manufacturing goods to final demand is production of 0.358 units of hydrocarbon pollutants.

10.5 An Augmented Leontief Model

Another straightforward approach to accounting for pollution generation and abatement in a traditional Leontief model is simply to augment the technical coefficients matrix with a set of pollution generation and/or abatement coefficients. In the case of pollution generation, the coefficients reflect the amount of a particular pollutant generated per dollar's worth of industry output. Similarly, the pollution abatement coefficients reflect inputs to pollution-elimination activities. This procedure was first proposed in Leontief (1970) and has been developed further by Qayum (1991) and Luptacik and Böhm (1994, 1999).

10.5.1 Pollution Generation

Consider the two-sector input–output data presented in Table 10.5 (originally shown in Chapter 2). Suppose that sector 1, in producing the $1,000 output indicated in Table 10.6, generates 50 units of "pollution" or "waste" – for example, emits 50 pounds of solid pollutant into the air.[8] Sector 2, while producing its $2,000 output, may have been observed to have generated 80 pounds of the same solid pollutant. Dividing each of

[8] Of course, there may (and generally will) be several kinds of pollution generation associated with any production process. The basic ideas, however, are adequately illustrated with the example of a single pollutant. Extension to several pollution types is covered in Ayres and Kneese (1969), Gutmanis (1975), and Leontief and Ford (1972).

Table 10.5 Economic–Ecologic Commodity Flows: Matrix Definitions

	Interindustry Transactions					Ecological Commodity Outputs	
	Consuming Sectors			Final	Total		
	Agriculture	Mining	Manuf.	Demand	Output	SO_2	HC
Producing Sectors							
Agriculture							
Mining		**Z**		**f**	**x**	**N**	
Manufacturing							
Ecological Commodity							
Water		**M**					
Land							

Table 10.6 Pollution-Generation Example: Dollar Transactions

		Purchasing Sector		Final	Total
		1	2	Demand	Output
Selling Sector	1	150	500	350	1,000
	2	200	100	1,700	2,000

these by the total output of the sector responsible for their production would give a kind of pollution-generation or waste-generation coefficient – pounds of pollutant generated per dollar's worth of output (as in **Q** in the last section).

Since these pollutants are outputs or by-products of a given production process, they could be interpreted as "negative inputs," in which case we might define them in the **A** matrix in the columns of the producing sectors, 1 or 2, with a minus sign. This is unnecessary, however, if we interpret pollutant generation in terms of the services required to dispose of pollution, for example, waste-disposal services. Hence we measure waste-disposal services in units of pollution disposed of or generated.

Letting p denote pollution generation, $z_{p1} = 50$ pounds says that sector 1 *generated* 50 pounds of pollutant; similarly, $z_{p2} = 80$ indicates that sector 2 *generated* 80 pounds of pollutant. Thus, the pollution-generation coefficients are $z_{p1}/x_1 = a_{p1} = 50/1,000 = 0.05$, and $z_{p2}/x_2 = a_{p2} = 80/2,000 = 0.04$; both are in units of pounds of pollutant per dollar of output.

If the technological relationships implied by these coefficients are assumed to remain as stable as the others in input–output analysis, then the total amount of solid pollutant emitted into the air, x_p, for any given values of $x_1$and x_2, would be given by $x_p = a_{p1}x_1 + a_{p2}x_2$ or, in this example, $x_p = (0.05)x_1 + (0.04)x_2$.

This relationship, defining the total pollution output, can be added directly to the general two-sector model. From before we have

$$\begin{aligned}(1 - a_{11})x_1 - a_{12}x_2 &= f_1 \\ -a_{21}x_1 + (1 - a_{22})x_2 &= f_2\end{aligned} \tag{10.16}$$

Adding a third linear equation involving a third variable, x_p, which does not appear in the previous two equations, is accomplished easily:

$$\begin{aligned}(1 - a_{11})x_1 - a_{12}x_2 + 0x_p &= f_1 \\ -a_{21}x_1 + (1 - a_{22})x_2 + 0x_p &= f_2 \\ -a_{p1}x_1 - a_{p2}x_2 + x_p &= 0\end{aligned}$$

or, in more compact matrix terms, $\begin{bmatrix} (1 - a_{11}) & -a_{12} & 0 \\ -a_{21} & (1 - a_{22}) & 0 \\ -a_{p1} & -a_{p2} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_p \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ 0 \end{bmatrix}$. The original $(\mathbf{I} - \mathbf{A})$ matrix is essentially bordered by a row of (the negatives of) pollution-generation coefficients and a column of zeros, and the $\mathbf{x}$ and $\mathbf{f}$ vectors are appropriately expanded. Denote this expanded coefficient matrix as $(\mathbf{I} - \mathbf{A}_p)$. Note that this is, in practice, similar to closing the basic Leontief model with respect to households, as we did in Chapter 2.

Fundamentally, this simply enables the amount of pollution generated, x_p, to be calculated along with x_1 and x_2 for any given f_1 and f_2. This could also be done in two steps, using the smaller two-equation input–output model, (10.16), and then using the resulting gross outputs, x_1 and x_2, to evaluate x_p via (10.1) – as in section 10.3, with the matrix of direct-impact coefficients.

With the expanded inverse, and the data from the example in section 2.3 (in particular, from Numerical Example: Hypothetical Figures – Approach I), and the hypothesized values for a_{p1} and a_{p2}, we have $\begin{bmatrix} x_1 \\ x_2 \\ x_p \end{bmatrix} = \begin{bmatrix} 0.85 & -0.25 & 0 \\ -0.20 & 0.95 & 0 \\ -0.05 & -0.04 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 600 \\ 1,500 \\ 0 \end{bmatrix}$.

If we calculate the inverse for this numerical example, we find that the elements of the original $(\mathbf{I} - \mathbf{A})^{-1}$ matrix still appear in the upper-left area of the expanded inverse, $(\mathbf{I} - \mathbf{A}_p)^{-1}$. This new inverse is $(\mathbf{I} - \mathbf{A}_p)^{-1} = \left(\dfrac{1}{0.758}\right) \begin{bmatrix} 0.950 & 0.250 & 0 \\ 0.200 & 0.850 & 0 \\ 0.055 & 0.046 & 0.758 \end{bmatrix}$ or

$(\mathbf{I} - \mathbf{A}_p)^{-1} = \begin{bmatrix} 1.254 & 0.330 & 0 \\ 0.264 & 1.122 & 0 \\ 0.073 & 0.061 & 1 \end{bmatrix}$. Hence, $\begin{bmatrix} x_1 \\ x_2 \\ x_p \end{bmatrix} = \begin{bmatrix} 1.254 & 0.330 & 0 \\ 0.264 & 1.122 & 0 \\ 0.073 & 0.061 & 1 \end{bmatrix}$ $\begin{bmatrix} 600 \\ 1,500 \\ 0 \end{bmatrix}$. Thus the pollution generated during production to meet final demands of $f_1 = 600$ and $f_2 = 1,500$ is $x_p = (0.073)(600) + (0.061)(1,500) = 43.80 + 91.50 = 135.50$ pounds. Rounding to whole numbers gives $44 + 92 = 136$ pounds.

Clearly, having $x_1 = 1,247$ and $x_2 = 1,842$ from our earlier calculations, and having the pollution-generation coefficients ($a_{p1} = 0.05,\ a_{p2} = 0.04$), the pollution generated by this production could have been found straightforwardly as $x_p = (0.05)(1,247) + (0.04)(1,842) = 62.35 + 73.68 = 136.03$. Again, rounding to whole numbers gives $62 + 74 = 136$ pounds. [The difference in the results from the two approaches is due to rounding errors that occur in working with decimals in finding the inverse of $(\mathbf{I} - \mathbf{A}_p)$].

Since finding $(\mathbf{I} - \mathbf{A}_p)^{-1}$ involves additional calculations, we would expect the approach using this inverse to have some advantages, and it does. It allows us to impute the total amount of pollution generated back to the final users, whose demands, f_1 and f_2, were responsible for production in the first place. The first two elements in the bottom row of $(\mathbf{I} - \mathbf{A}_p)^{-1}$ do just that; each dollar of final demand for the output of sector 1 causes the generation of 0.073 pounds of pollutant, and each dollar of final demand for the output sector 2 causes 0.061 pounds. Thus, of the 136 pounds produced, f_1 is the cause of 44 pounds and f_2 is the cause of 92 pounds. Note that from the point of view of producers, sector 1's gross output, x_1, generates 62 pounds; sector 2's gross output, x_2, accounts for the remaining 74 pounds. The division of responsibility differs, depending on whether one is viewing the production (supply) side or the final consumption (demand) side. That is, a \$100 reduction of sector 1's gross output means 5 pounds less of solid waste; a \$100 reduction of final demand for sector 1's output means 7.33 pounds less.

For some kinds of environmental policy questions it is useful to be able to assign responsibility not to producers themselves but to the ultimate consumers. The "pollution multipliers" in the bottom row of $(\mathbf{I} - \mathbf{A}_p)^{-1}$ give an indication of the effects on pollution generation that might be expected to accompany, for example, a government tax policy aimed at decreasing final demands by consumers by, say, a selective sales tax.[9]

10.5.2 Pollution Elimination

In a similar fashion, pollution abatement or waste disposal could be introduced into a Leontief framework as one or more columns representing sectors whose function it is to reduce or eliminate various pollutants. Consider only one such sector in a model that includes a pollution-generation row, as above. The coefficients in this column (except

[9] Chatterji (1975) extends the augmented Leontief model to include the concepts of a balanced regional model discussed earlier in Chapter 3.

Table 10.7 Input–Output Transactions: Pollution-Expanded Model Example

	Manuf.	Services	Pollution Abatement	Intermediate Output	Final Demand	Total Output
Manufacturing	15	25	0.6	40.6	59.4	100
Services	20	5	1.2	26.2	73.8	100
Pollution Generation	5	4	0	9	−3	6

for the last one) would represent inputs to the technological process that removes (or disposes of) the pollutant. If the process itself generates additional pollution, this would appear in the form of a coefficient in the pollution-generation row. Let x_p now represent pollution *eliminated*. If all pollution is to be eliminated, the appropriate set of equations would be

$$
\begin{aligned}
(1-a_{11})x_1 - a_{12}x_2 - a_{1p}x_p &= f_1 \\
-a_{21}x_1 + (1-a_{22})x_2 - a_{2p}x_p &= f_2 \\
-a_{p1}x_1 - a_{p2}x_2 + (1-a_{pp})x_p &= 0
\end{aligned}
\tag{10.17}
$$

The coefficients a_{1p} and a_{2p} represent inputs from the other sectors to pollution abatement, and the third equation simply defines the total amount of pollution generated and eliminated as $x_p = a_{p1}x_1 + a_{p2}x_2 + a_{pp}x_p$.

If it is not technologically or economically feasible for *all* waste to be eliminated, let x_p be the amount *eliminated*, only, and let f_p be the amount not eliminated (and hence, in some way, "tolerated" by society, if not exactly "demanded"). The total pollution *generated* is $a_{p1}x_1 + a_{p2}x_2 + a_{pp}x_p$, analogous to the intermediate output in the traditional Leontief model. We must subtract the amount of pollution tolerated (add a negative value, f_p) to yield total pollution eliminated, x_p. Then the third equation in (10.17) would simply be $-a_{p1}x_1 - a_{p2}x_2 + (1-a_{pp})x_p = -f_p$. Hence, the relationships in (10.17) would become

$$
\begin{aligned}
(1-a_{11})x_1 - a_{12}x_2 - a_{1p}x_p &= f_1 \\
-a_{21}x_1 + (1-a_{22})x_2 - a_{2p}x_p &= f_2 \\
-a_{p1}x_1 - a_{p2}x_2 + (1-a_{pp})x_p &= -f_p
\end{aligned}
$$

and the total amount of pollution generated would be $x_p + f_p = a_{p1}x_1 + a_{p2}x_2 + a_{pp}x_p$.

Example 10.2: Pollution-Activity-Augmented Leontief Model Consider the table of transactions, including pollution levels, as given in Table 10.7. The row sums of the interindustry transactions matrix yield intermediate industry output for the economic sectors and total pollution *generated* for the pollution-generation sector. In the final-demand column are final demands for the economic sectors and the amount of pollution tolerated by society, which is entered as a negative value since x_p is a measure of the total amount of pollution *eliminated*; that is, the total amount eliminated, $x_p = 6$ units, should equal the total amount produced, $z_{p1} = 5$ plus $z_{p2} = 4$ or 9 units, less the amount tolerated, which is recorded in the table by the value $-f_p = -3$.

Viewing tolerated pollution as a negative final demand allows us to retain the Leontief identity of intermediate outputs *plus* final demands equals total outputs for pollution sectors as well as economic sectors. Note also that for purposes of this example we presume that the pollution-abatement sector does not generate pollution in the process of eliminating pollution from other sectors and the final demand ($a_{pp} = 0$).

For this example

$$\mathbf{A}_p = \begin{bmatrix} 15 & 25 & 0.6 \\ 25 & 5 & 1.2 \\ 5 & 4 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{100} & 0 & 0 \\ 0 & \frac{1}{100} & 0 \\ 0 & 0 & \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 0.15 & 0.25 & 0.10 \\ 0.25 & 0.05 & 0.20 \\ 0.05 & 0.04 & 0 \end{bmatrix}$$

The last row, the pollution-generation row, indicates that the economic sectors, manufacturing and services, generate 0.05 and 0.04 units of pollution, respectively, per dollar's worth of output. Hence, $(\mathbf{I} - \mathbf{A}_p) = \begin{bmatrix} 0.85 & -0.25 & -0.10 \\ -0.20 & 0.95 & -0.20 \\ -0.05 & -0.04 & 1.00 \end{bmatrix}$ and

$$\mathbf{x}_p = (\mathbf{I} - \mathbf{A}_p)^{-1}\mathbf{f}_p = \begin{bmatrix} 1.630 & 1.806 & 0.195 \\ 0.283 & 1.138 & 0.256 \\ 0.075 & 0.063 & 1.020 \end{bmatrix} \begin{bmatrix} 59.4 \\ 73.8 \\ -3.0 \end{bmatrix} = \begin{bmatrix} 100 \\ 100 \\ 6 \end{bmatrix}.$$

10.5.3 Existence of Non-negative Solutions

In Chapter 2 we presented a set of conditions ensuring the non-negativity of total outputs computed in a Leontief model for a set of given (positive vector of) final demands, the Hawkins–Simon conditions. These conditions will turn out to be much more important in the context of environmental input–output models for assuring the existence of non-negative solutions than they were in the traditional Leontief framework. The corresponding conditions for the Leontief model augmented with pollution-generation and/or pollution-abatement sectors can be derived directly from original Hawkins–Simon conditions.

Recall the simple extended model which includes both generation and elimination of pollution; in matrix terms this is given as

$$\begin{bmatrix} 1 - a_{11} & -a_{12} & -a_{1p} \\ -a_{21} & 1 - a_{22} & -a_{2p} \\ -a_{p1} & -a_{p2} & 1 - a_{pp} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_p \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ -f_p \end{bmatrix}$$

We can rewrite this relationship in terms of its submatrix components as

$$\begin{bmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -a_{1p} \\ -a_{2p} \end{bmatrix} x_p = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$\begin{bmatrix} -a_{p1} & -a_{p2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (1-a_{pp})x_p = -f_p$$

Note that in the first equation we can recognize the 2×2 submatrix $\begin{bmatrix} 1-a_{11} & -a_{12} \\ -a_{21} & 1-a_{22} \end{bmatrix}$ as simply $(\mathbf{I}-\mathbf{A})$, the Leontief matrix of economic sectors unexpanded by the pollution-elimination and -generation sectors. We can assume that $(\mathbf{I}-\mathbf{A})$ alone satisfies the Hawkins–Simon conditions. Rearranging terms in the first equation above of submatrix components we write $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (\mathbf{I}-\mathbf{A})^{-1} \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} a_{1p} \\ a_{2p} \end{bmatrix} x_p \right\}$. We want to show the conditions for ensuring that all elements of the vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ are positive. Since we assume that $(\mathbf{I}-\mathbf{A})$ by itself satisfies the Hawkins–Simon conditions, all the elements of $(\mathbf{I}-\mathbf{A})^{-1}$ are positive. Moreover, $\mathbf{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ is a vector of positive final demands presented to the economy and, since a_{1p} and a_{2p} represent inputs from other sectors to the pollution-abatement sector, they are also non-negative. Hence all elements of the vector $\mathbf{x}$ will be non-negative when x_p is non-negative. The conditions for non-negativity of x_p are found by first rearranging terms to obtain

$$x_p = (1-a_{pp})^{-1} \left\{ \begin{bmatrix} a_{p1} & a_{p2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - f_p \right\}$$

As discussed earlier, the term a_{pp} is simply the technical coefficient describing pollution generation associated with pollution-abatement activities; it is therefore non-negative. Hence the terms $(1-a_{pp})$ and, consequently, $1/(1-a_{pp})$ are non-negative if $a_{pp} < 1$, that is, if the amount of pollution generated by the pollution-abatement sector is less than the amount it eliminates. Finally, therefore, x_p will be non-negative if the expression $\begin{bmatrix} a_{p1} & a_{p2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - f_p$ is non-negative, that is, when $\begin{bmatrix} a_{p1} & a_{p2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > f_p$. As defined earlier, f_p represents the amount of pollution not eliminated, or tolerated, which by definition is non-negative, the coefficients a_{p1} and a_{p2} give the pollution generated per unit of output x_1 and x_2, respectively, which are also positive by definition. This simply implies that x_p will be positive; consequently, the Hawkins–Simon conditions are satisfied for the extended model when the amount of pollution generated in the economy is greater than the amount desired. More generally, this indicates that in polluted areas where pollution generally exceeds the tolerated or desired levels, the augmented model satisfies the Hawkins–Simon conditions. If this is not the case, this augmented model is not necessary.

Example 10.2 (Revisited): Pollution-Activity-Augmented Leontief Model We can expand the Leontief model given in Example 10.2 to obtain

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.254 & 0.330 \\ 0.264 & 1.122 \end{bmatrix} \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} .1 \\ .2 \end{bmatrix} x_p \right\} \text{ and } x_p = (1.0) \left\{ \begin{bmatrix} .05 & .04 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - f_p \right\}$$

Clearly $(\mathbf{I}-\mathbf{A})$, alone, satisfies the Hawkins–Simon conditions (as defined in Chapter 2), namely $|\mathbf{I}-\mathbf{A}| = 0.7575 > 0$, $1-a_{11} = 0.15 > 0$, and $1-a_{22} = 0.05 > 0$. Therefore, all the elements of $(\mathbf{I}-\mathbf{A})^{-1}$ are non-negative, as shown above. Recall that in this example $a_{pp} = 0$ and therefore $(1-a_{pp}) = 1$, which satisfies the Hawkins–Simon conditions. Since $(1-a_{pp}) > 0$, this ensures that the pollution-abatement sector eliminates more pollution than it generates. Finally, since $(1-a_{pp})$ is non-negative, x_p will be non-negative if $\begin{bmatrix} a_{p1} & a_{p2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > f_p$ or, for this case, $\begin{bmatrix} 0.05 & 0.04 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > f_p$. We demonstrate the implications of this condition as follows. First, recall that the expanded Leontief inverse was found to be $(\mathbf{I}-\mathbf{A}_p)^{-1} = \begin{bmatrix} 1.630 & 1.806 & 0.195 \\ 0.283 & 1.138 & 0.256 \\ 0.075 & 0.063 & 1.020 \end{bmatrix}$. Consider two cases, I and II, defined by the following $\mathbf{f}$ vectors: $\mathbf{f}_p^I = \begin{bmatrix} 1.0 \\ 1.0 \\ -0.1 \end{bmatrix}$ and $\mathbf{f}_p^{II} = \begin{bmatrix} 1.0 \\ 1.0 \\ -0.5 \end{bmatrix}$.

Case I depicts final demands of unity for economic sectors and a level of "tolerated" pollution output of 0.1. Case II depicts the same economic sector final demands of unity with an *increased* level of tolerable pollution generation of 0.5 units. The corresponding values of $\mathbf{x}_p^I$ and $\mathbf{x}_p^{II}$ are $\mathbf{x}_p^I = (\mathbf{I}-\mathbf{A}_p)^{-1}\mathbf{f}_p^I = \begin{bmatrix} 1.591 \\ 1.395 \\ 0.035 \end{bmatrix}$ and $\mathbf{x}_p^{II} = (\mathbf{I}-\mathbf{A}_p)^{-1}\mathbf{f}_p^{II} = \begin{bmatrix} 1.513 \\ 1.293 \\ -0.373 \end{bmatrix}$. In Case II, $x_p = -0.373$, the total amount of pollution eliminated, is negative. This is difficult to interpret; that is, it violates the Hawkins–Simon conditions. More specifically, recalling the submatrix equation condition defined earlier, for Case I, $\begin{bmatrix} a_{p1} & a_{p2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > f_p$ or $\begin{bmatrix} 0.05 & 0.04 \end{bmatrix} \begin{bmatrix} 1.591 \\ 1.395 \end{bmatrix} = 0.135 > 0.1$. For Case II we have $\begin{bmatrix} a_{p1} & a_{p2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} < f_p$ or $\begin{bmatrix} 0.05 & 0.04 \end{bmatrix} \begin{bmatrix} 1.513 \\ 1.293 \end{bmatrix} = 0.127 < 0.5$. In Case I the total amount of pollution generated is 0.135 units, which is greater than the $f_p = 0.1$ units tolerated by society. In Case II, however, the total pollution generated is 0.127 units. This is less than the amount tolerated ($f_p = 0.5$), so the Hawkins–Simon conditions are not fulfilled, that is, x_p can be negative. In this case the augmented model is unnecessary. The $x_p = -0.373$ thus represents the fact that the amount of pollution generated in Case II is 0.373 units *less* than the amount tolerated.

The discussion we have presented here has been restricted to a single pollutant. The general framework can be easily extended to several pollutants (see Ayres, 1978).

Table 10.8 Basic Structure of Economic–Ecologic Models

	Industries	Ecologic Processes
Industries	Flows between economic sectors	Flows from industry to the ecosystem
Ecologic Processes	Flows from the ecosystem to industry	Flows within the ecosystem

Additional comments and enhancements to the augmented Leontief model are given in Flick (1974), Steenge (1978), Lee (1982), Chen (1973) and Rhee and Miranowski (1984).

Finally, possibilities of recycling (certainly only feasible for certain kinds of generated pollutants) can be incorporated into the model through changes in the coefficients in the pollution-generation row. For example, sector 2 might generate only one half as much pollutant per dollar's worth of output, due to its ability to use some of the waste as an input to production (for example a_{p2} might be 0.02 instead of 0.04). In addition, entirely new recycling sectors could be introduced. Their output would be the end product of recycling (scrap metal, for example); their inputs would be purchases from other productive sectors and from the waste-generation sector.

10.6 Economic–Ecologic Models

In section 10.2 we introduced the notion of ecological commodities, which we defined as nonmarketable quantities that are either inputs used by or outputs discharged from a production process. Given this definition, we can quite easily extend the notion of commodity-by-industry accounts to accommodate environmental activities in terms of these ecological commodities. Moreover, as an alternative to simply appending environmental intensity rows to the technical coefficients, as we did in the last section when dealing with pollution and its elimination, we can account more specifically for environmental (or ecosystem) flows by creating an "ecosystem submatrix" that is linked to interindustry economic flows matrix in the same manner that regions are interconnected in an interregional input–output model. Such a model is often called a fully integrated model.

10.6.1 Fully Integrated Models

Both Daly (1968) and Isard *et al*. (1972) developed similar procedures along these lines for incorporating environmental activities into an input–output framework. Both approaches employ flow matrices within and between both economic activities and environmental processes. As shown in Table 10.8, transactions can be grouped into four basic submatrices; the diagonal submatrices depict flows within the economy and the ecosystem, and the off-diagonal submatrices depict flows between the economy and the ecosystem and vice versa.

Daly's version employs a highly aggregated industry-by-industry characterization of the economic submatrix (upper-left submatrix of Table 10.8) and a classification of ecosystem processes, including life processes such as plants and animals and nonlife processes such as chemical reactions in the atmosphere. Isard *et al.* refines this basic paradigm by recognizing that secondary production of ecologic outputs – for example, pollution generation – is incompatible with the assumption of one-product industries inherent in traditional Leontief models.

Instead, Isard *et al.* adopts the commodity-by-industry accounting scheme along the lines described earlier in Chapter 5, which permits an accounting of multiple commodities, economic and ecologic, produced by a single industry. The technical coefficients in the model used by Isard *et al.* are estimated directly from technical data, but since this model was never fully implemented, the adequacy of available data for such estimation is very difficult to judge. Richardson (1972), Victor (1972) and Isard *et al.* (1972) discuss the strengths and weaknesses of this approach in more detail. The availability of data for the ecosystem submatrix appears to be the most troublesome point.

10.6.2 Limited Economic–Ecologic Models

Victor (1972) limits the scope of the fully integrated economic–ecologic model of Isard *et al.* to account only for flows of ecological commodities from the environment into the economy and of the waste products from the economy into the environment. Thus, by limiting the scope of the analysis, the data are generally available and the model can be implemented with little difficulty. The basic accounting framework is shown schematically in Table 10.9.

Table 10.9 is the familiar commodity-by-industry format, but augmented with additional rows of ecological inputs ($\mathbf{T}$) and columns of ecological outputs ($\mathbf{R}$). The submatrices are defined as follows:

Economic Subsystem $\mathbf{U} = \left[u_{ij}\right]$ is the economic "Use" matrix; u_{ij} represents the amount of economic commodity i used by industry j. For n industries and m commodities, $\mathbf{U}$ is $m \times n$. $\mathbf{V} = \left[v_{ij}\right]$ is the economic "Make" matrix; v_{ij} represents the amount of economic commodity j produced by industry i; $\mathbf{V}$ is $n \times m$. The vector of economic commodity final demands is $\mathbf{e} = [e_i]$. The vector of economic commodity gross outputs is $\mathbf{q} = [q_i]$, where $\mathbf{q}$ is $m \times 1$. The vector of industry value-added inputs is $\mathbf{v}' = \left[v_j\right]$; v_j represents the total of value-added inputs to industry j and $\mathbf{v}'$ is $1 \times n$. (We noted earlier that $\mathbf{v}$ is traditionally used for value-added elements and $\mathbf{V}$ designates the Make matrix in commodity–industry models. The context in which they are used should eliminate any possible confusion.) Finally, as before the vector of industry total outputs is $\mathbf{x} = \left[x_j\right]$; x_j represents the total output of industry j; $\mathbf{x}$ is $n \times 1$. Note that all of these submatrices of the economic subsystem are defined in the discussion of commodity-by-industry accounts in Chapter 5.

Table 10.9 Limited Commodity-by-Industry Economic–Ecologic Model

	Economic Subsystem			Ecosystem	
	Commodities	Industries	Final Demand	Total Output	Ecologic Commodities
Commodities		$\mathbf{U}$	$\mathbf{e}$	$\mathbf{q}$	$\mathbf{R}$
Industries	$\mathbf{V}$			$\mathbf{x}$	
Value Added		$\mathbf{v}'$	GNP		
Total Output	$\mathbf{q}'$	$\mathbf{x}'$			
Ecologic Commodities		$\mathbf{T}$			

Ecologic Subsystem $\mathbf{R} = [r_{ik}]$ is the matrix of economic commodity by ecologic commodity outputs; r_{ik} is the amount of ecologic commodity k discharged as a result of production of economic commodity i; for l ecologic commodities, $\mathbf{R}$ is $m \times l$. $\mathbf{T} = \left[t_{kj}\right]$ is the matrix of ecologic commodity-by-industry inputs; t_{kj} is the amount of ecologic commodity k used-by-industry j; $\mathbf{T}$ is $l \times n$.

Commodity-by-Industry Formulation As before, with the conventional commodity-by-industry accounts, we can recall from Chapters 4 and 5 that $\mathbf{B} = \mathbf{U}\hat{\mathbf{x}}^{-1}$ where $\mathbf{B} = \left[b_{ij}\right]$ is the matrix of economic commodity-by-industry direct requirements; b_{ij} is the amount of economic commodity i required per dollar's worth of output of industry j; $\mathbf{B}$ is $m \times n$. We also can recall from Chapters 4 and 5 that $\mathbf{C} = \mathbf{V}'\hat{\mathbf{x}}^{-1}$ where $\mathbf{C} = \left[c_{ij}\right]$ is the matrix of industry output proportions; c_{ij} is the fraction of industry j's output that is distributed as commodity i; $\mathbf{C}$ is $m \times n$.

With the accounting system expanded to include ecologic commodities, we can also define $\mathbf{G} = \mathbf{T}\hat{\mathbf{x}}^{-1}$ where $\mathbf{G} = \left[g_{kj}\right]$ is the matrix of ecologic commodity input coefficients; $g_{kj} = t_{kj}/x_j$ is the amount of ecologic commodity k used in the production of a dollar's worth of industry j's output; $\mathbf{G}$ is $l \times n$.

Example 10.3: Limited Economic–Ecologic Models To illustrate Victor's approach, first we review the system of commodity-by-industry accounts given earlier in the example in section 10.3. To this system we append the ecologic commodity accounts, that is, the production of ecologic commodities ($\mathbf{R}$) and the use of ecologic commodities in producing industry output ($\mathbf{T}$) as in Table 10.10.

For purposes of illustration, we restrict the model to an industry-based technology, as described in Chapter 5. Recall that this simply means we assume that an industry consumes economic (and ecologic) commodities in fixed proportions. In this case we need to compute the matrix of commodity input proportions, $\mathbf{D}$, and the matrix of commodity-by-industry direct requirements, namely $\mathbf{B} = \mathbf{U}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 0.111 & 0.091 \\ 0.111 & 0.064 \end{bmatrix}$

Table 10.10 Economic–Ecologic Models: Example 10.3

	Commodities		Industries				Ecological Commodities	
	A	*B*	*A*	*B*	Final Demand	Total Output	SO_2	Water
Commodities			$\mathbf{U}$		$\mathbf{f}$	$\mathbf{q}$	$\mathbf{R}$	
A			10	10	80	100	0	9
B			10	7	83	100	0	0
Industries	$\mathbf{V}$					$\mathbf{x}$		
A	90	0				90		
B	10	100				110		
Value Added			$\mathbf{v}'$		GNP			
			70	93	163			
Total Inputs	$\mathbf{q}'$		$\mathbf{x}'$					
	100	100	90	110		200		
			$\mathbf{T}$					
SO_2			0	0				
Water			10	7				

and $\mathbf{D} = \mathbf{V}\hat{\mathbf{q}}^{-1} = \begin{bmatrix} 0.9 & 0 \\ 0.1 & 1 \end{bmatrix}$. Hence, as before, the industry-by-commodity total requirements matrix is $\mathbf{D}(\mathbf{I} - \mathbf{BD})^{-1} = \begin{bmatrix} 1.022 & 0.099 \\ 0.243 & 1.092 \end{bmatrix}$. The ecologic commodity input coefficient is $\mathbf{G} = \mathbf{T}\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 0 & 0 \\ 0.111 & 0.182 \end{bmatrix}$.

We can find the vector of total ecologic commodity *inputs* for all industries of the economy as the vector of row sums of $\mathbf{T}$. That is, industry *A* consumes 10 units of water in its production process and industry *B* consumes 20 units of water; total water consumption is 30 units. We denote the vector of total ecologic commodity consumption (inputs) by $\bar{\mathbf{t}}$; hence, in matrix terms $\bar{\mathbf{t}} = \mathbf{Ti} = \begin{bmatrix} 0 & 0 \\ 10 & 20 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 30 \end{bmatrix}$.

Since $\mathbf{T} = \mathbf{G}\hat{\mathbf{x}}$ then $\bar{\mathbf{t}} = \mathbf{G}\hat{\mathbf{x}}\mathrm{i}$; but since $\hat{\mathbf{x}}\mathrm{i} = \mathbf{x}, \bar{\mathbf{t}} = \mathbf{Gx}$. Hence, the total ecologic commodity requirement for a given vector of economic commodity final demands, $\mathbf{e}$, is found by $\mathbf{x} = \mathbf{D}(\mathbf{I} - \mathbf{BD})^{-1}\mathbf{e}$ and $\bar{\mathbf{t}} = \mathbf{Gx} = [\mathbf{GD}(\mathbf{I} - \mathbf{BD})^{-1}]\mathbf{e}$. The bracketed quantity is the ecologic input *intensity* (ecologic commodity inputs by economic commodity total requirements matrix).

Note that by using the alternative commodity-by-industry total requirements matrices we derived in Chapter 5 we could easily compute, for example, ecologic inputs and outputs as a function of final demands for industries rather than for commodities, as we have shown here. Hannon, Costanza and Herendeen (1983) have shown circumstances under which the commodity-based and industry-based technology assumptions give equivalent results in specifying ecosystem models.

10.7 Pollution Dispersion

In the environmental extensions to input–output that we have dealt with so far, we have measured pollution in terms of total emissions or discharges of various pollutants. The ultimate effect of pollutant emissions on a region depends not only on the total amount of pollutant generated, but also on the manner in which that pollutant is discharged into the environment. For example, sulfur dioxide produced in electric power plants is emitted and dispersed from tall smokestacks. The concentrations of that pollutant at various points in the region surrounding the plant depend upon a variety of technical, climatic, and geographic factors such as the stack height, wind direction and speed, and local topography.

10.7.1 Gaussian Dispersion Models

A number of researchers – for example, Berlinsky, Carter and First (1973) and Coupé (1977) – have coupled models of pollution dispersion to input–output models. Most of these approaches assume that air pollution from point sources (as opposed to mobile sources, such as automobiles) is dispersed as a "Gaussian plume." In a Gaussian plume the pollutant is assumed to disperse symmetrically about a centerline (the x-axis) of the plume in both the y and z directions (horizontal and vertical, respectively).

The cross section of the plume is an ellipse which indicates that the rate of dispersion is greater in the horizontal direction than in the vertical direction. Pasquill (1962), Gifford (1961), Seinfeld (1975), and others derive a formula describing this dispersion, which gives the pollutant concentration at a "receptor point" (x, y, z) measured downwind from that pollution source. The x-axis measures distance downwind, the y-axis measures horizontal distance from the x-axis, and the z-axis measures vertical distance from the x-axis. The formula for computing pollutant concentrations at ground level ($z = 0$) is

$$C(x, y, 0) = \frac{Q}{2\pi u \sigma_y \sigma_z} \exp\left\{-\frac{1}{2}\left[\left(\frac{y}{\sigma_y}\right)^2 + \left(\frac{H}{\sigma_z}\right)^2\right]\right\}$$

where

$C(x, y, 0)$ = the pollutant concentration in $\mu g/m^3$ (micrograms per cubic meter);
H = height of the pollution source (stack height) in meters;
Q = the pollution emission rate in μg/minute;
σ_y, σ_z = the standard deviation of the horizontal and vertical dispersion distributions, respectively (usually a function of x);
u = the wind speed (meters/minute).

The parameters σ_y and σ_z are functions of the local meteorology and, in particular, the stability of the atmosphere or the air turbulence characteristics. Derivation of specific values of σ_y and σ_z are beyond the scope of this book, but the subject is dealt with in detail in Seinfeld (1975) and elsewhere in the literature, where in particular, much more sophisticated characterizations of pollution dispersion are developed.[10]

10.7.2 Coupling Pollution Dispersion and Input–Output Models

In coupling dispersion models with input–output models, Hordijk (1980) and others distinguish among pollution emissions (at the source), primary concentrations (at specified receptor points) and "cumulated" levels of pollution (accumulated over time at the receptor points). The emission rate is usually assumed to vary linearly with economic output of the industry. Hence, we can determine the location of a pollution source or a number of locations over which total emissions of this type of pollutant in the region are averaged and subsequently dispersed from these locations. We can then select a number of receptor points for which we wish to record pollutant concentrations. This could be a collection of several strategic points or an entire map of all points in the region. We then add the values of pollutant concentration resulting from each pollution source to yield total concentration of that pollutant at that receptor point.

Example 10.4: Coupling Input–Output and Pollution Dispersion Models Consider the region defined in Figure 10.16. Note that there are two pollution sources and four receptor points where we wish to measure pollutant concentration. The following input–output model describes interindustry activity in the region (millions of dollars): $\mathbf{Z} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$. The local government has decided to stimulate industrial activity through fiscal measures which, in terms of the input–output model, translate to increased final demand activity. Suppose the stimulus amounts to $\Delta\mathbf{f} = \begin{bmatrix} \Delta f_1 \\ \Delta f_2 \end{bmatrix}$, where Δf_1 and Δf_2 are \$4 million and \$3 million, respectively.

Assume that the economic activity is related to pollution emission by average values of 30.77 grams per second per million dollars' worth of output from industry A and 27.59 grams per second per million dollars' worth of output from industry B. Further, we assume that $u = 15$ meters/second prevailing from the West, $H = 250$ meters, and $\sigma_y = \sigma_z = ax^b$ where $a = 0.24$ and $b = 0.88$ are empirically derived constants. We are interested in pollutant concentrations at the receptor points (at ground level, that is, where $z = 0$) so we can apply the Gaussian plume equation above. With the increased interindustry activity, pollutant concentrations increase proportionally with the level of increased total output. We can compute the region's new level of total production as the sum of current production, $\mathbf{x}$, and the production prompted by the new final demand,

$$\mathbf{x}^{new} = \mathbf{x} + (\mathbf{I} - \mathbf{A})^{-1}\Delta\mathbf{f} = \begin{bmatrix} 10 \\ 10 \end{bmatrix} + \begin{bmatrix} 1.250 & 0.416 \\ 0.625 & 1.875 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 16.250 \\ 18.125 \end{bmatrix}$$

Since the prevailing wind is from the West, and the pollution sources are east of Receptor Points 3 and 4, the ambient pollutant concentrations at these points are both zero. The

[10] Alternatives to Gaussian dispersion models that are much more sophisticated characterizations of pollution dispersion are described in Seinfeld (1975), Turner (1961), Jacobson (1998), and Borrego and Schayes (2002).

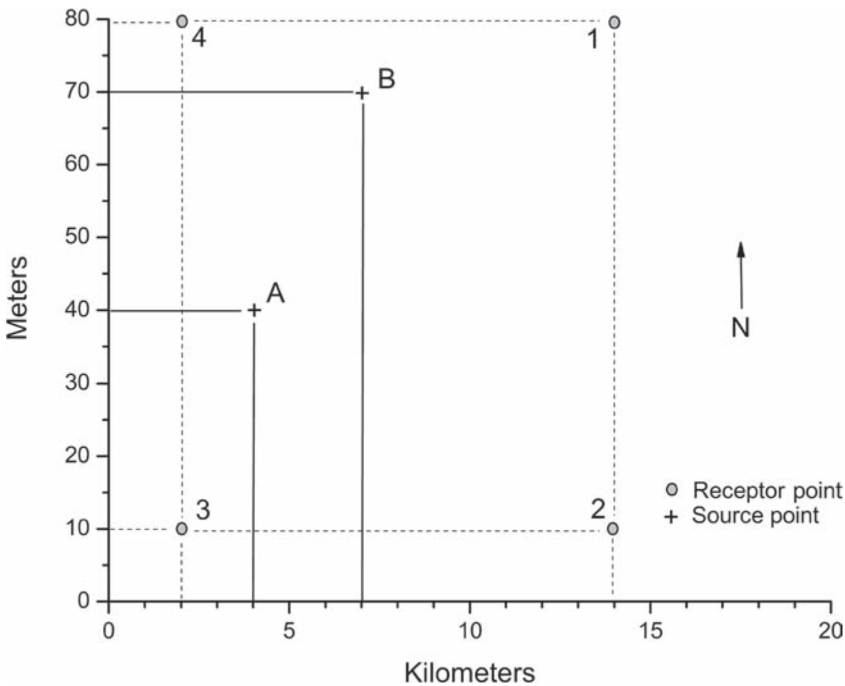

Figure 10.16 Location of Air Pollution Sources and Receptor Points

concentrations at receptor points 1 and 2, however, must be computed by the dispersion formula, so for Receptor Point 1, $C = 4.31 \times 10^{-5} \mu g/m^3$ and for Receptor Point 2, $C = 4.16 \times 10^{-5} \mu g/m^3$.

10.8 Other Applications

Many researchers have employed input–output analysis to examine the implications of major environmental policy initiatives, such as efforts to reduce greenhouse gas emissions from burning fossil fuels, as in Kratena and Schleicher (1999) or Lenzen, Pade and Munksgaard (2004), the economic and environmental implications of recycling, as in Nakamura (1999), ecosystem restoration as in Weisskoff (2000), and economic and environmental issues associated with international trade, as in Reinert and Roland-Holst (2001) or Ahmad and Wyckoff (2003).

Some researchers expand the environmental input–output framework to analyze the social costs of environmental pollution production and elimination, as in Steenge and Voogt (1994) or Steenge (2004), and to adapt national accounting systems to better accommodate environmental costs and benefits of interindustry activity, as in Duchin and Lange (1994), Duchin and Steenge (1999) and United Nations (2000). Much work in the 1970s and 1980s was done specifically in analyzing air pollution issues, as in Berlinksy, Carter and First (1973), and more recently in Lutz (2000).

Much recent work in environmental input–output analysis appears in the literature of ecological economics, which addresses the dynamic and spatial interdependence between human economies and natural ecosystems, such as Duchin and Lange (1994). Finally, input–output analysis is a widely used method for examining the life cycle economic and environmental cost of materials, such as in Lloyd and Lave (2003) or Hendrickson, Lave and Matthews (2006).

10.9 Summary

In this chapter we examined several approaches to accounting for pollution generation and elimination in input–output models. The models ranged from using a matrix of pollution-generation coefficients in conjunction with the Leontief inverse, to coupling input–output relationships with mathematical programming models, to an economic–ecologic model that couples a Leontief model of economic flows with an ecosystem model of environmental commodity flows. Finally we discussed coupling an environmental input–output model to a model of pollution dispersion over a geographic region.

Many input–output models applied to environmental problems appeared in the 1970s, including the Strategic Environmental Assessment System (House, 1977), Miernyk and Sears (1974), Cumberland and Stram (1976), Leontief and Ford (1972), Converse (1971), Page (1973), Stone (1972), and Lowe (1979). More recently applications of input–output models to evaluating the effectiveness of alternative technology options for pollution control have appeared in Ketkar (1999), Rose (1983), and Forssell (1998), which built upon the earlier work of Miernyk (1973), Cumberland (1966), and Giarrantani (1974). Finally, the current frontier of environmental models is similar to that of energy models in the 1970s, where econometric extensions became important (as described in Chapter 9). More recently extensions have appeared to dynamic models, as in Duchin (1990, and 1992), general equilibrium models, as in Conrad and Schmidt (1998) and Zhang (1998), and structural decomposition analysis, as in Wier (1998).

Problems

10.1 Assume that we have the following direct coefficient matrices for energy, air pollution, and employment ($\mathbf{D}^e$, $\mathbf{D}^p$, and $\mathbf{D}^l$, respectively) for two industries, 1 and 2: $\mathbf{D}^e = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.3 \end{bmatrix}$, $\mathbf{D}^p = \begin{bmatrix} 0.2 & 0.5 \\ 0.2 & 0.3 \end{bmatrix}$, and $\mathbf{D}^l = \begin{bmatrix} 0.2 & 0.5 \end{bmatrix}$. Notice that industry 2 is both a high-polluting and high-employment industry. Suppose that the local government has an opportunity to spend a total of \$10 million on a regional development project. Two projects are candidates: (1) Project 1 would spend appropriated dollars in the ratio of 60 percent to industry 1 and 40 percent to industry 2; the minimum size of this project is \$4 million; (2) Project 2 would spend appropriated dollars in the ratio of 30 percent to industry 1 and 70 percent to industry 2; the minimum size of this project is \$2 million. The government can adopt either project or a combination of the two projects (as long as the minimum size of each project is at least maintained and the total budget is not overrun). In other words, we might describe the options available to the government as: $\begin{bmatrix}\beta_a\\ \beta_b\end{bmatrix} = \alpha_1 \begin{bmatrix}0.6\\ 0.4\end{bmatrix} + \alpha_2 \begin{bmatrix}0.3\\ 0.7\end{bmatrix}$ where α_1 and α_2 are budgets allocated to projects 1 and 2, respectively. β_a and β_b are the total final demands presented to the regional economy by the combination of projects for industries A and B, respectively. Suppose that four alternative compositions of these projects are being considered (1) $\begin{cases}\alpha_1 = 4\\ \alpha_2 = 2\end{cases}$, (2) $\begin{cases}\alpha_1 = 5\\ \alpha_2 = 5\end{cases}$, (3) $\begin{cases}\alpha_1 = 10\\ \alpha_2 = 0\end{cases}$ and (4) $\begin{cases}\alpha_1 = 0\\ \alpha_2 = 10\end{cases}$. The following table of constraints describes the local regulation on energy consumption and environmental pollution in the region:

	Maximum Allowable Changes Collectively by All Industries
Oil Consumption (10^{15} BTUs)	3.0
Coal Consumption (10^{15} BTUs)	no limit
SO_2 Emissions (tons)	14.5
NO_x Emissions (tons)	10

Finally, suppose that the regional economy is currently described by the following input–output transactions table (in millions of dollars):

	A	B	Total Output
A	1	3	10
B	5	1	10

a. Which of the proposed combinations of projects (1), (2), (3), and (4) permit the region to operate within the above constraints on energy consumption and air pollution emission and within the established budget constraint?
b. Which of these "legal" projects that you identified in (a) should be adopted to maximize the employment in the region?

10.2 Assume that a regional economy has two primary industries: A and B. In producing these two products it was observed last year that air pollution emissions associated with this industrial activity included 3 pounds of SO_2 and 1 pound of NO_x emitted per dollar's worth of output of industry A, and 5 pounds of SO_2 and 2 pounds of NO_x emitted per dollar's worth of output of industry B. It was also observed that industries A and B consumed 1×10^6 tons and 6×10^6 tons of coal respectively during that year. Industry A also consumed 2×10^6 barrels of oil. Total employment in the region was 100,000 (40 percent of which were employed by industry A and the rest by industry

B). The regional planning agency has constructed the following input–output table of interindustry activity in the region (in \$$10^6$):

	A	*B*	Total Output
A	2	6	10
B	6	12	10

Assume that with growth in the region during the next year the new final-demand vector will be $\begin{bmatrix} 15 & 25 \end{bmatrix}'$. Using what you know about constructing a generalized input–output model, determine the following:

a. the total consumption of each energy type (coal and oil) during the next year;
b. the total pollution emission (of each type) during the next year; and
c. the level of total employment during the next year.

10.3 A regional planning agency initiates a regional development plan. Four projects are being considered that would represent government purchases of regionally produced products, that is, final demands presented to the regional economy (see table).

	Project Expenditure (millions of dollars)			
Regional Industry	Project 1	Project 2	Project 3	Project 4
A	2	4	2	2
B	2	0	0	2
C	2	2	4	3

You are given additional information. The matrix of technical coefficients is $\mathbf{A} = \begin{bmatrix} 0.04 & 0.23 & 0.38 \\ 0.33 & 0.52 & 0.47 \\ 0 & 0 & 0.1 \end{bmatrix}$. The relationships between the following quantities and total output are also known:

	Industry		
	1	2	3
Pollution Emission (grams/\$ output)	4.2	7.0	9.1
Energy Consumption (bbls oil/\$ output)	7.6	2.6	0.5
Employment (workers/\$ output)	7.3	3.3	6.3

a. Which of the four projects contributes most to gross regional output?
b. Which of the projects causes regional consumption of energy to increase the most?
c. Which of the projects contributes most to regional employment?

10.4 Consider an input output economy defined by $\mathbf{Z} = \begin{bmatrix} 140 & 350 \\ 800 & 50 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1000 \\ 1000 \end{bmatrix}$. Suppose this is an economy in deep economic trouble. The federal government has at its disposal policy tools that can be implemented to stimulate demand for goods from one sector or the other. Also suppose that the plants in sector 1 discharge 0.3 lbs of airborne particulate substances for every dollar of output (0.3 lbs/$ output), while sector 2 pollutes at 0.5 lbs/$ output. Finally, let labor input coefficients be 0.005 and 0.07 for sectors 1 and 2, respectively.

a. Would a conflict of interest arise between unions and environmentalists in determining the sector toward which the government should direct its policy effort? (You need not close the matrix with respect to either households or pollution generation to answer this question.)

b. Can you think of a technological reason why or why not a dispute might arise?

10.5 Consider the following interindustry transactions table:

		Purchasing Sector		
		1	2	Total Output
Selling Sector	1	140	350	2000
	2	800	50	1850

The amount of pollution generated by sector 1 is 10 units and by sector 2 is 25 units. Pollution abatement reduced pollution by 5 units in sector 1 and 12 units in sector 2. Total pollution permitted by local regulation is 12 units. Using a pollution-activity-augmented Leontief formulation, what is the level of output for each industry and the total pollution generated if final demands for both sectors increase by 100?

10.6 In problem 8.5 national and regional input–output tables are defined with three sectors (natural resources, manufacturing, and services) with the following matrices of technical coefficients and vectors of total outputs, respectively, $\mathbf{A}^N = \begin{bmatrix} .1830 & .0668 & .0087 \\ .1377 & .3070 & .0707 \\ .1603 & .2409 & .2999 \end{bmatrix}$, $\mathbf{x}^N = \begin{bmatrix} 518,288.6 \\ 4,953,700.6 \\ 14,260,843.0 \end{bmatrix}$, $\mathbf{A}^R = \begin{bmatrix} .1092 & .0324 & .0036 \\ .0899 & .0849 & .0412 \\ .1603 & .1170 & .2349 \end{bmatrix}$ and $\mathbf{x}^R = \begin{bmatrix} 8,262.7 \\ 95,450.8 \\ 170,690.3 \end{bmatrix}$.

The following table of energy use, pollution, and employment coefficients are defined that apply to both the regional and national economies:

	Industry		
	Nat. Res.	Manuf.	Services
Pollution Emission (grams/$ output)	4.2	7	9.1
Energy Consumption (BTUs/$ output)	7.6	2.6	0.5
Employment (person-hrs/$ output)	7.3	3.3	6.3

Suppose a major new public works initiative by the federal government is characterized by the following vector of increases in federal spending:

$\Delta\mathbf{f}' = \begin{bmatrix} 250 & 3,000 & 7,000 \end{bmatrix}$, of which 20 percent will be spent in the region. How do the percentage changes in total impacts on pollution, energy use, employment, and total industrial output of each industry sector for the region compare with those of the nation as a whole?

10.7 For the regional economy described in problem 10.6 prior to the projected final demand, if there were a 10 percent shortfall in the availability of energy, what would be the corresponding impacts on GDP?

10.8 An input–output economy is specified by $\mathbf{A} = \begin{bmatrix} .3 & .1 \\ .2 & .5 \end{bmatrix}$ and $\mathbf{f} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$. Write the linear programming (LP) formulation for finding $\mathbf{x}$, the vector of total outputs. Solve this LP problem graphically. Suppose also that pollution is generated at a rate of 2.5 units per dollar of output of industry 1 and 2 units per dollar of output of industry 2. Replace the objective function for the LP problem above with minimizing pollution emissions. Solve this LP problem graphically and compare the solution with that of the first LP problem.

10.9 For the economy specified in problem 10.8, suppose that employment is generated at a rate of 6 and 3 units per dollar's worth of output for industries 1 and 2, respectively, and that there is a very high priority employment target of 7.5 units for industry 2. Find the vector of total outputs that meets the employment target for industry 2 as the highest priority, then as the next highest priority meets final-demand requirements while minimizing pollution generation to the extent possible and if possible to a total level of 10 units of pollution between the two industries.

10.10 For the 1997 US input–output table provided in Appendix B, suppose the vector of units of carbon dioxide emissions generated per dollar of total output is given by $\mathbf{d} = [2 \quad 3 \quad 4 \quad 7 \quad 10 \quad 5 \quad 4]'$. Assume that the availability of new technology enables the manufacturing sector to reduce the emissions per dollar of output in the year 2005 by 10 percent and the construction sector to reduce emissions by 15 percent. The input–output table for 2005 is also provided in Appendix B. For this case how much do total emissions of carbon dioxide increase or decrease in the United States in 2005 relative to 1997 levels?

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11 Social Accounting Matrices

11.1 Introduction

The System of National Accounts (SNA) described in Chapter 4 was developed from the basic concepts of the circular flow of income and expenditures in an economy. As noted in that chapter, the SNA provides a convenient and essentially standardized framework for compiling and organizing aggregate national statistics that characterize the economic profile of an economy. When the SNA is combined with the input–output accounts, which incorporate the interindustry activity associated with intermediate as well as final production and consumption of goods and services in the economy, the picture of the economy becomes more comprehensive. However, the framework as it has been developed so far in this text provides relatively little insight into the role of people and social institutions in the economy, e.g., labor and households, human capital, and social welfare.

A logical goal then, and the focus of this chapter, is to extend the SNA/IO framework to add a more detailed characterization of the roles of labor, households, and the social institutions of the economy. In particular, we seek to capture in more detail the employment features of the economy, including such factors as income from employment and its disposition, labor costs, and the demographics of the work force that comprise the market for supply and demand of labor. Moreover, in many nations' compilations of national statistics, no framework exists to ensure consistency across statistics from various sources, let alone reconciling them with basic economic accounts. Both goals can be accomplished by means of a so-called Social Accounting Matrix (SAM), development of which is the principal focus of this chapter.

11.2 Social Accounting Matrices: Background

A SAM can be thought of as a generalization of the SNA framework developed in Chapter 4, but we will find that the principal new feature added is to incorporate transactions and transfers between institutions related to distribution of income in the economy.[1] In the construction of SAMs we will find similarities with the energy and environmental input–output extensions developed in Chapters 9 and 10, and especially with the commodity-by-industry or supply and use[2] framework developed in Chapters 4 and 5. In particular, by incorporating SAMs into the supply and use framework, we can develop an extended input–output model that can be employed for analyzing social and economic policy in a more comprehensive way.

A useful conceptual point of departure for the construction of SAMs is to return to the very simple concept of chronicling the circular flow of income and expenditures in the economy (Chapter 4, Figure 4.1). If we change the perspective of viewing the circular flow of income and expenditure to that of transactions between institutions rather than or perhaps in addition to that associated with industry and commodity flows, it is helpful to explicitly identify the places where those transactions occur, namely the markets. That is, consider Figure 11.1, which depicts the flow of income and expenditure, now including product markets where transactions involving consumption of goods and services occur and resource markets where the transactions for value added factors of production occur, such as wages and salaries to employees, entrepreneurship or profits, taxes, and consumption of fixed capital including land.

In developing SAMs, we seek specifically to elaborate on the detailed accounting of what goes on in the product and resource markets, and in particular, delineating the characteristics of the labor force, government policies such as taxation and welfare transfers, and other allocations of income.

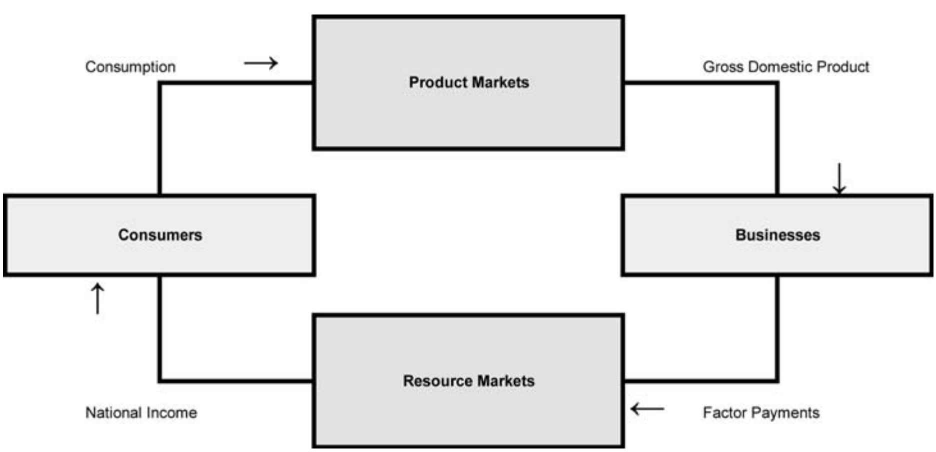

Figure 11.1 Circular Flow of Income, Expenditure, and Market

[1] Extensive discussions of the rationale and development of SAMs are provided in Pyatt (1991a, 1991b, 1994a, 1994b and 1999) or Pyatt and Round (1977, 1985a and 1985b).

[2] Often in discussions about SAMs the terms supply and use replace the term commodity-by-industry, since many additional sectors are incorporated beyond interindustry transactions, although conceptually the terms are identical.

Table 11.1 The Basic National Accounts Balance Statement in Matrix Form

	Prod.	Cons.	Cap.	ROW	Govt.
Production		C	I	X	G
Consumption	Q		D	H	
Capital Accum.		S			
Rest of World	M	O	L		
Govt.		T	B		

11.3 Social Accounting Matrices: Basic Concepts

We begin by recalling the matrix form of the national economic accounts introduced in Chapter 4, shown in Table 11.1, where we reintroduce the variables used in defining the basic national economic accounts using the SNA conventions.

The transactions reflected in the table are the following:

C = total consumption of goods and services in the economy
I = total investment in capital goods
X = total exports of goods and services
G = government expenditures
Q = total income generated in the economy
D = depreciation or consumption of capital goods
H = income generated overseas
S = total private savings
M = total imports of goods and services
O = transfers of money overseas
L = net lending of resources from overseas
T = total direct taxation of consumers
B = total government deficit spending

The reader will also recall from Chapter 4 that the row and column sums of this matrix constitute a set of macroeconomic accounting balance equations, which also corresponded to a set of accounting "T accounts" corresponding to each major set of economic activity:

- Production Account: $Q + M = C + I + X + G$
- Consumption Account: $C + S + O + T = Q + D + H$
- Capital Accumulation Account: $I + D + L + B = S$
- Balance of Payments Account: $X + H = M + O + L$
- Government Account: $G = T + B$

Table 11.2 (Table 4.5 Revisited) The Basic National Accounts Balance Statement in Matrix Form: Example

	Prod.	Cons.	Cap.	ROW	Govt.	Total
Production		475	75	25	25	600
Consumption	550		−19	14		545
Capital Accum.		40				40
Rest of World	50	10	−21			39
Govt.		20	5			25
Total	600	545	40	39	25	

In Chapter 4 the development of the SNA conventions from the basic concept of the circular flow of income and expenditure was illustrated by means of an example shown earlier as Table 4.5 (and graphically in Figure 4.6), which we revisit here as Table 11.2.

The reader will recall from Chapter 4 that these tables resulted from gradually expanding the accounting detail, chronicling the circular flow of income and expenditures between production (defined as the *production account*) and consumption (defined as the *consumption account*) in the economy by sequentially adding new accounts to distinguish major types of economic activity such as savings and investment (the *capital accumulation account*), imports and exports (the *balance of payments account*), and the role of government (the *government account*). Finally, we also expanded the representation of the consumption account to capture the role of individual industries and specific products (goods and services), which also provided the link to input–output analysis.

In the following we will expand the accounting framework further to include a more detailed characterization of the roles of labor and households, which will turn out to be equivalent to defining a SAM. We begin by expanding the consumption account related to labor and households, but we will subsequently expand the role of labor and households in the other major accounts as well. Finally, we can also incorporate non-monetary factors, such as environmental information, in the expanded framework, which we will refer to more generally as extended input–output models (as in Chapter 9 with energy production and use associated with interindustry activity and in Chapter 10 with environmental pollution generation and mitigation).

11.4 The Households Account

Table 11.3 shows the national accounts matrix expanded to distinguish between current consumption by intermediate consumers (industries) and final consumers (households) and to distinguish the role of households as a provider of labor services, a value-added factor of production. This expansion results in an additional row and column, each labeled Households, which we record in a new account defined as the *households account*.

Table 11.3 The Basic National Accounts Balance Statement in Matrix Form Expanded to Include the Households Account

	Prod.	Cons.	Cap.	ROW	Govt.	Households
Production		$\boldsymbol{U}$	I	X	G	$\boldsymbol{F}$
Consumption	Q		D	H		
Capital Accum.						S
Rest of World	M		L			O
Govt.			B			T
Households		$\boldsymbol{V}$				

Since the principal goal is to distinguish between intermediate consumption by businesses and final consumption by households, note that several changes have occurred in the table. First, C – the total consumption of goods and services – has disappeared and been replaced by three new variables along with some changes in several other variables. The three new variables, highlighted in boldface in the table, are the following:

$U =$ total "use" of goods and services by businesses
$F =$ total of final consumption of goods and services by households
$V =$ total of "value-added" inputs consumed by businesses

For the present we assume that all value-added inputs, such as labor and capital, are provided by households. Note also that the values of S, O, and T have moved to the households column, since these transactions are related to household consumption rather than intermediate consumption of goods and services by businesses. That is, total savings (S) refers to the allocation of income to savings by final consumers (principally households), which accumulates for use by businesses (and others); overseas transfers (O) refers to overseas transfers of income by households, and taxes (T) refers to direct taxes paid by households that provide the principal revenue for government. We will incorporate taxes paid by businesses later, which are termed indirect taxes since they are not direct taxes on consumers but, rather, appear as part of the price of goods and services to consumers.

For the revised table of accounts, now including households, the corresponding balance equations are:

- Production Account: $Q + M = U + F + I + X + G$
- Consumption Account: $U + V = Q + D + H$
- Capital Accumulation Account: $I + D + L + B = S$
- Balance of Payments Account: $X + H = M + O + L$
- Government Account: $G = T + B$
- Households Account: $V = F + T + S + O$

The corresponding table for the example is given in Table 11.4.

Table 11.4 The Basic National Accounts Balance Statement in Matrix Form: Example, Expanded to Include the Households Account

	Prod.	Cons.	Cap.	ROW	Govt.	Households	Total
Production		**219**	75	25	25	**256**	600
Consumption	550		−19	14			545
Capital Accum.						**40**	40
Rest of World	50		−21			**10**	39
Govt.			5			**20**	25
Households		**326**					326
Total	600	545	40	39	25	326	

11.5 The Value-Added Account

In earlier chapters we defined value-added inputs to industry production as those comprising primary factors of production, such as capital and labor, which are drawn from outside the interindustry network of producers and consumers of intermediate goods and services. Such primary factors include government services (paid for in taxes, referred to earlier as indirect taxes), consumption of capital goods (depreciation), land (rental payments), and entrepreneurship (normal profits of businesses). We now can introduce to our set of accounts a *value-added account* to capture these inputs. Many of these inputs come from households (as we have assumed so far) who are the principal suppliers of capital and labor, but we need to be able to accommodate other sources as well, such as government and foreign sources.

As a starting point, Table 11.5 includes the matrix form of the national accounts, this time expanded to include the *value-added account*. Note that Table 11.5 includes the new value added row and column as well as a new variable, W, which is defined as the total of value-added payments to households (wages and salaries, interest on capital, etc.) and the corresponding additional balance equation for the value-added account is simply $V = W$, at least so far. In addition, the households account balance equation becomes $W = F + T + S + O$. The corresponding table for the example is given in Table 11.6.

The type of SAM represented by Tables 11.5 and 11.6 is often referred to as a "macro SAM" since it aggregates all of the sectoral details of transactions into a single macroeconomic transaction. In the next section we explore how to expand the SAM to include more detailed sector transactions.

11.6 Interindustry Transactions and the Connection to the Input–Output Framework

In Tables 11.5 and 11.6 the row and column sums from the households and value-added accounts are the same since we have assumed for the present that all value-added inputs are generated by households. In addition, so far, we have also implicitly made a

Table 11.5 The Basic National Accounts Balance Statement in Matrix Form Expanded to Include the Value-Added Account

	Prod.	Cons.	Cap.	ROW	Govt.	Households	Value Added
Production		U	I	X	G	F	
Consumption	Q		D	H			
Capital Accum.						S	
Rest of World	M		L			O	
Govt.			B			T	
Households							W
Value Added		V					

Table 11.6 The Basic National Accounts Balance Statement in Matrix Form: Example, Expanded to Include the Value-Added Account

	Prod.	Cons.	Cap.	ROW	Govt.	Households	Value Added	Total
Production		**219**	75	25	25	**256**		600
Consumption	550		−19	14				545
Capital Accum.						**40**		40
Rest of World	50		−21			**10**		39
Govt.			5			**20**		25
Households							326	326
Value Added		**326**						326
Total	600	545	40	39	25	326	326	

number of somewhat restrictive assumptions, such as that only households pay taxes and generate savings. We will return to the framework later to address these assumptions, but first we expand the consumption account to identify the interindustry transactions involved in intermediate consumption of goods and services (the total value of which is U), as well as the sectoral detail of deliveries to final markets, including the distinction between types of final markets, i.e., personal consumption expenditures (F), net exports (X), government expenditures (G) and investments or capital accumulation (I). These expansions will allow us to recast input–output relationships in a social accounting matrix.

Let us begin with a very simple input–output example and recast it as a SAM. Consider the basic input–output transactions table shown in Table 11.7. (The fact that there

Table 11.7 SAM Framework Example: Input–Output Representation

	Nat. Res.	Manuf.	Services	Households	Total Output
Natural Resources	50	30	0	60	140
Manufacturing	60	40	40	40	180
Services	0	0	0	100	100
Value Added					
Labor	10	70	10		
Capital	20	40	50		
Total Inputs	140	180	100		

Table 11.8 SAM Framework Example Using Social Accounting Conventions

	Expenditures						
	Nat. Res.	Manuf.	Serv.	Labor	Capital	Households	Total Output
Income							
Natural Resources	50	30	0			60	140
Manufacturing	60	40	40			40	180
Services	0	0	0			100	100
Labor	10	70	10				90
Capital	20	40	50				110
Households				90	110		200
Total Inputs	140	180	100	90	110	200	

are no interindustry deliveries of services is arbitrary.) We now recast the input–output table as a SAM using the conventions we have established so far in Table 11.8.

The difference between the basic input–output table representation and the SAM representation seems trivial in this simple example, but the key concept as we consider more complex features, such as distinctions between supply and use (industries and commodities) or adding more detail in value-added or household consumption categories, is that balance equations are maintained via the requirement that row and column sums be equal for all entries. As we add detail for the capital, labor, households, and additional accounts the result will be a much more detailed picture of the economy including not only the input–output table of interindustry income and output, but also the institutional income and expenditures associated with Final Demand and Value Added sectors. As we have seen from the connection to and, indeed, derivation from the SNA, the framework also provides essentially a complete accounting of the circular flow of income and expenditure in an economy.

11.7 Expanding the Social Accounts

Now let us reverse course and begin with the aggregated SNA example developed earlier in the chapter, the final installment of which was presented in Table 11.6. We now add sectoral detail for the supply and use of goods and services as well as categories of value-added income and final consumption. Consider the input–output table in Table 11.9 that is compatible with the SAM depicted in Table 11.6. That is, in Table 11.9 the production and consumption accounts have been added to include sector and commodity detail. This example is, of course, one of many possibilities. The lightly shaded portion of the accounts in the table comprises the Use matrix defined in Chapters 4 and 5 and the darkly shaded portion comprises the Make matrix.

We can further expand the Value Added and Final Demand sectors in Table 11.9 to include more detail in the various types of value-added income and in final commodity consumption patterns. These expanded Value Added and Final Demand accounts are shown in Tables 11.10 and 11.11, respectively.

Finally, we can specify how the sources of valued-added income relate to final demands, completing the circulation of income and expenditure in the economy, as shown in Table 11.12. The corresponding expanded input–output and SAM representations of these accounts including the new table segments from Tables 11.10, 11.11, and 11.12 that completes the circular flow of income and expenditure are shown in Table 11.13. The SAM representation is shown in Table 11.14.

Note that in Table 11.14, as we expand the accounts to include more sectoral detail, we have changed the notation for total income to the economy (the transaction between the consumption row and the production column and until now in this chapter denoted by Q), by the matrix $\mathbf{V}$ to reflect the more familiar input–output notation of the Make matrix. We have also replaced the notation for the sources of value-added income (until now denoted by V) by the matrix $\mathbf{R}$.

11.8 Additional Social Accounting Variables

In the following sections, in order to make the presentation less complex, we simplify the SAM/SNA framework somewhat by returning to an industry-by-industry framework, where interindustry transactions are recorded in a single transactions matrix rather than the complementary pair of Supply and Use matrices of the SNA framework. We also begin once again with macro-SAMs where we return to the use of Q to denote total income to the economy (the transaction between the consumption row and the production column) V to denote total value-added income. However, all the concepts presented apply also in the commodity-by-industry framework as well.

In Tables 11.5 and 11.6 the row and column sums from the households and value added accounts are the same since we have assumed for the present that all value-added inputs are generated by households. In addition, so far, we have also implicitly made a number of other somewhat restrictive assumptions, such as that only households pay

Table 11.9 Input–Output Accounts for Table 11.6 Revisited

	Commodities					Industries					
	Agric.	Energy	Manuf. Prod.	Fin. Serv.	Other Serv.	Nat. Res.	Manuf.	Serv.	Final Demand	Total Output	
Commodities											
Agriculture						**25**	**10**	**15**	83	133	
Energy						**13**	**7**	**9**	58	87	
Manuf. Products						**10**	**20**	**7**	72	109	600
Fin. Services						**10**	**10**	**25**	82	127	
Other Services						**8**	**30**	**20**	86	144	
Value Added						90	174	117	381		
Industries											
Nat. Resources	**88**	**68**	**0**	**0**	**0**					156	
Manufacturing	**45**	**10**	**98**	**10**	**88**					251	600
Services	**0**	**9**	**11**	**117**	**56**					193	
Total Output	133	87	109	127	144	156	251	193			
			600				600				

Table 11.10 Expanded Value-Added Accounts

Value Added Categories	Industries: Natural Resources	Manuf.	Services
Depreciation	−5	−10	−4
Export Income	3	6	5
Taxes	10	45	31
Welfare Transfers	25	33	4
Indirect Taxes	30	55	44
Interest Income	27	45	37
Total Value Added	**90**	**174**	**117**
		381	

Table 11.11 Expanded Final-Demand Accounts

Commodities	Final-Demand Categories: Households	Investment	Govt.	Exports	Total Final Demand	
Agriculture	61	9	3	10	**83**	
Energy	45	6	2	5	**58**	
Manuf. Products	50	12	5	5	**72**	**381**
Fin. Services	50	20	10	2	**82**	
Other Services	50	28	5	3	**86**	

Table 11.12 Sources of Value-Added Income

	Value-Added Sectors: Deprec.	Exp. Inc.	Cons. Taxes	Welfare Transfers	Indirect Taxes	Int. Income	Total	
Final Demand Sectors								
Households			86	51	119		**256**	
Investment	−19				10	84	**75**	**381**
Government						25	**25**	
Exports		14		11			**25**	
Total	**−19**	**14**	**86**	**62**	**129**	**109**		
				381				

taxes and generate savings. We can now begin to relax these assumptions by including some additional variables:

P = government transfers to households, such as welfare transfers
S_G = government savings
S_F = foreign savings
T_B = indirect taxes or taxes paid by businesses
T_I = taxes on imported goods and services

We now introduce in Table 11.15 these variables just defined into our matrix of national accounts as transactions that affect the relevant accounts involved. For example, government revenues now result from not only taxes paid by consumers and deficit spending, but also from indirect taxes paid by businesses as well as taxes on imported goods and services. Similarly, capital is accumulated not only from private savings but now also from foreign and government sources. And finally, household revenues are not only income from remuneration for creation of value-added inputs but also from welfare transfers from government.

The corresponding balance equations become:

- Production Account: $Q + M + T_I = U + F + I + X + G$
- Consumption Account: $U + V + T_B = Q + D + H$
- Capital Accumulation Account: $I + D + L + B = S + S_G + S_F$
- "Rest of World" Account: $X + H + S_F = M + O + L$
- Government Account: $G + P + S_G = T + B + T_B + T_I$
- Household Account: $P + W = F + T + S + O$
- Value Added Account: $V = W$

The table for the example is given in Table 11.16.

11.9 A "Fully Articulated" SAM

Until this point the SAMs we have developed are of a type commonly referred to as a "macro SAM" since the table aggregates many sectoral details of transactions into a single macroeconomic transaction, such as all of what will become the Make and Use matrices into one aggregated number for each. Expanding the macroeconomic transactions shown in Table 11.16 to include the detailed transactions among sectors and institutions results in what is sometimes referred to as a "fully articulated" SAM, although this usually also means much more detail is included in the household and value-added partitions of the matrix assembled to analyze issues related to labor, households consumption and income, social institutions, human capital, and social welfare.

Table 11.13 Expanded Input–Output Accounts

	Commodities					Industries			Final Demand Sectors					
	Agric.	Energy	Manuf. Prod.	Fin. Serv.	Other Serv.	Nat. Res.	Manuf.	Serv.	House holds	Invest.	Govt.	Exports	Total	
Commodities														
Agriculture						25	10	15	61	9	3	10	133	
Energy						13	7	9	45	6	2	5	87	
Manuf. Products						10	20	7	50	12	5	5	109	600
Fin. Services						10	10	25	50	20	10	2	127	
Other Services						8	30	20	50	28	5	3	144	
Industries														
Nat. Resources	88	68	0	0	0								156	
Manufacturing	45	10	98	10	88								251	
Services	0	9	11	117	56								193	600
Value-Added Inputs														
Depreciation						−5	−10	−4					−19	
Export Income						3	6	5					14	
Taxes						10	45	31					86	
Welfare Transfers						25	33	4					62	381
Indirect Taxes						30	55	44					129	
Interest Income						27	45	37					109	
Total	133	87	109	127	144	156	251	193	256	75	25	25		
	600					600			381					

Table 11.14 SAM Representation of Expanded Input–Output Accounts

	Commodities					Industries			Value Added Sectors						Final Demand Sectors					
	Agric.	Energy	Manuf. Prod	Fin. Serv.	Other Serv.	Nat. Res.	Manuf.	Serv.	Deprec.	Exp. Inc.	Cons. Taxes	Welfare Transfers	Indirect Taxes	Int. Income	House-holds	Invest.	Govt.	*Exports*	Total Output	
Commodities																				
Agriculture						25	10	15							61	9	3	10	**133**	
Energy						13	7	9							45	6	2	5	**87**	
Manuf. Products						10	20	7							50	12	5	5	**109**	**600**
Financial Services						10	10	25							50	20	10	2	**127**	
Other Services						8	30	20							50	28	5	3	**144**	
Industries																				
Natural Resources	88	68	0	0	0														**156**	
Manufacturing	45	10	98	10	88														**251**	**600**
Services	0	9	11	117	56														**193**	
Value Added																				
Depreciation						−5	−10	−4											**−19**	
Export Income						3	6	5											**14**	
Consumer Taxes						10	45	31											**86**	**381**
Welfare Transfers						25	33	4											**62**	
Indirect Taxes						30	55	44											**129**	
Interest Income						27	45	37											**109**	
Final Demand Sectors																				
Households											86	51	119						**256**	
Investment									−19				10	84					**75**	**381**
Government														25					**25**	
Exports										14		11							**25**	
Total Income	**133**	**87**	**109**	**127**	**144**	**156**	**251**	**193**	**−19**	**14**	**86**	**62**	**129**	**109**	**256**	**75**	**25**	**25**		
	600					**600**			**381**						**381**					

	Comm.	Ind.	Val. Add.	Fin. Dem.	Total
Commodities		U		F	**g**
Industries	V				**x**
Value Added Sectors		W			GDP
Final Demand Sectors			R		GDP
Total	**g**	**x**	GDP	GDP	

U = Commodities used by Industry (Use matrix)
V = Commodities supplied by Industry (Make matrix)
F = Final Demand of commodities (g is total commodity output)
W = Value Added inputs to Industry (x is total industry output)
R = Sources of Value Added Income
GDP = Sum of all value added inputs or sum of all final demands

Table 11.15 The Basic National Accounts Balance Statement in Matrix Form Expanded to Include Additional Macro Transactions

	Prod.	Cons.	Cap.	ROW	Govt.	Households	Value Added
Production		U	I	X	G	F	
Consumption	Q		D	H			
Capital Accum.				$\boldsymbol{S_F}$	$\boldsymbol{S_G}$	S	
Rest of World	M		L			O	
Government	$\boldsymbol{T_I}$	$\boldsymbol{T_B}$	B			T	
Households					$\boldsymbol{P}$		W
Value Added		V					

Table 11.16 Expanded National Accounts Balance Statement in Matrix Form: Example Expanded to Include Additional Macro Transactions

		1	2	3	4	5	6	7	
		Prod.	Cons.	Capital	Rest of World	Govt.	Households	Value Added	Total
1	Production		219	75	21	17	**268**		600
2	Consumption	550		−19	14				545
3	Capital Accum.				**4**	**6**	30		40
4	Rest of World	48		−21			12		39
5	Government	**2**	**3**	5			20		30
6	Households					**7**		**323**	330
7	Value Added		**323**						323
Total		600	545	40	39	30	330	323	

11.10 SAM Multipliers

In Chapter 6 we explored one of the principal uses of input–output analysis as assessing the effect on an economy of changing elements that are exogenous to the model of economy being studied through a variety of summary measures known as multipliers that can be derived from the elements of the Leontief inverse matrix. That chapter focused on multipliers involving total output, income, employment, and value-added inputs. A key decision in employing multipliers in any given analysis was determining which sectors would be viewed as exogenous to the input–output model and which would be incorporated endogenously into the structure of the model – for example, "closing" the model to the households sector to distinguish so-called Type II from Type I multipliers. As one might expect, similar decisions must be faced in constructing multipliers for SAMs.

Since SAMs are generally designed to try to capture transactions and transfers between all economic agents in a system, it is a somewhat arbitrary decision as to which transactions and transfers are considered to be exogenous for modeling purposes. Round (1988) observes that most often in construction of SAMs used for modeling, and in particular calculation of multipliers, the government, capital, and "rest of world" accounts are considered to be exogenous.

11.10.1 SAM Multipliers: Basic Structure

We begin by defining a "fully articulated" SAM, including essentially all economic transactions and transfers between all economic agents, as the matrix $\bar{\mathbf{Z}}$ similar to the matrix of interindustry transactions in a closed input–output model. The matrix $\bar{\mathbf{Z}}$ in a SAM, like a fully closed Leontief model, is a square matrix for which the row and column sums are identical, which we designate as $\bar{\mathbf{x}}$.

Also in a manner similar to the basic input–output framework, we define part of the economy to be exogenously specified as in "opening" in the input–output model as described in Chapter 2. To do this we first define $\bar{\mathbf{G}} = \begin{bmatrix} \bar{\mathbf{Z}} & \mathbf{F} \\ \mathbf{W} & \mathbf{B} \end{bmatrix}$ where $\mathbf{F}$ is the matrix of exogenous final expenditures (row indices are of industries and column indices are of final expenditure categories), $\mathbf{W}$ is the matrix of exogenous income generated (row indices are of exogenous income categories and column indices are of industries), and $\mathbf{B}$ is the matrix of exogenous income allocations to final expenditures (row indices are exogenous income categories and column indices are of final expenditure categories). However, columns of $\mathbf{F}$ would be only categories of final demand we choose to specify exogenously such as capital expenditures, government expenditures, or exports. Rows of $\mathbf{W}$, as with $\mathbf{F}$, would include only categories we choose to specify exogenously – in this case of value-added categories, such as capital inputs, government subsidies and imports. Our definition of $\mathbf{F}$, $\mathbf{W}$, and $\mathbf{B}$ means we have chosen to treat some categories of final demand and value added endogenously, in which case we include them the matrix partition, $\bar{\mathbf{Z}}$, which will comprise the endogenous portion of the SAM. Hence the row and column indices of $\bar{\mathbf{Z}}$ will be of industries plus any final-demand and value-added categories we have chosen to treat endogenously. It is common to show $\bar{\mathbf{G}}$ normalized by its column sums and defined as the matrix of normalized expenditure shares $\mathbf{G} = \bar{\mathbf{G}}\hat{\mathbf{g}}^{-1}$ where $\mathbf{g} = \bar{\mathbf{G}}\mathbf{i} = \mathbf{i}'\bar{\mathbf{G}}$.

Finally, in order to construct a SAM model we need to distinguish in $\bar{\mathbf{Z}}$ between interindustry transactions and transactions with final-demand and value-added categories. To do this we further partition as $\bar{\mathbf{Z}} = \begin{bmatrix} \mathbf{Z} & \mathbf{0} & \bar{\mathbf{C}} \\ \bar{\mathbf{V}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{Y}} & \bar{\mathbf{H}} \end{bmatrix}$, where $\bar{\mathbf{C}}$ is the matrix of final-demand expenditures we choose to specify as endogenous variables, $\bar{\mathbf{V}}$ is the matrix of value-added inputs we choose to specify endogenously, $\bar{\mathbf{Y}}$ is the matrix of transactions distributing income to value-added categories that we choose to specify endogenously and $\bar{\mathbf{H}}$ is the matrix of transactions distributing institutional and household income to final-demand sectors that we choose to specify endogenously. We now define the matrix of SAM coefficients as $\mathbf{S} = \mathbf{Z}\hat{\bar{\mathbf{x}}}^{-1}$ where the partitions of $\mathbf{S}$ corresponding to the partitions of $\bar{\mathbf{Z}}$ are defined by

$$\mathbf{S} = \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{C} \\ \mathbf{V} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y} & \mathbf{H} \end{bmatrix} \tag{11.1}$$

where $\mathbf{A}$ is the matrix of interindustry technical coefficients, $\mathbf{C}$ is the matrix of endogenous final expenditure coefficients, $\mathbf{V}$ is the matrix of endogenous value-added input shares, $\mathbf{Y}$ is the matrix of endogenous coefficients distributing income to value-added categories and $\mathbf{H}$ is the matrix of endogenous coefficients for distributing institution and household income.

We also define the vector $\bar{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{v} \\ \mathbf{y} \end{bmatrix}$ where $\mathbf{x}$ is the vector of total interindustry sector outputs, $\mathbf{v}$ is the vector of total value-added inputs, and $\mathbf{y}$ is the vector of total household income. We can now specify the basic SAM model as the following:

$$\bar{\mathbf{x}} = \mathbf{S}\bar{\mathbf{x}} + \bar{\mathbf{f}} \tag{11.2}$$

where $\bar{\mathbf{f}} = \begin{bmatrix} \mathbf{f} \\ \mathbf{w} \\ \mathbf{h} \end{bmatrix}$ and $\mathbf{f}$ is the vector of exogenously specified commodity demand, $\mathbf{w}$ is the vector of value-added inputs that are exogenously specified and $\mathbf{h}$ is the vector of household income categories, the levels of which we exogenously specify. Since $\mathbf{S} = \mathbf{Z}\hat{\bar{\mathbf{x}}}^{-1}$ we can rewrite (11.2) as $\bar{\mathbf{x}} = (\mathbf{I} - \mathbf{S})^{-1}\bar{\mathbf{f}}$ and we define $\mathbf{M} = (\mathbf{I} - \mathbf{S})^{-1}$ as the matrix of SAM multipliers. The reader should immediately notice the similarity to $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$ in the Leontief model framework.

With the partitions of $\mathbf{S}$ as defined above and their associated vectors relating to totals for industry output, value-added and household income, we can interpret the corresponding partitions of $\mathbf{M}$; this is the subject of the following sections. In developing the discussions it is important to revisit the analogous section on decomposition of multipliers in section 6.7 and several of the features of inversion of partitioned matrices from Appendix A.

11.10.2 Decomposition of SAM Multipliers[3]

Let us begin with a simplified or reduced version of the SAM model introduced as (11.1) and (11.2), defined as the following:

$$\mathbf{S} = \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{H} & \mathbf{0} \end{bmatrix} \tag{11.3}$$

where $\mathbf{A}$ is the matrix of interindustry technical coefficients, $\mathbf{C}$ is the matrix of endogenous final consumption coefficients, and $\mathbf{H}$ is the matrix of coefficients allocating household income to value-added categories. (For this simplified version we lump all of value added into households.)

We can easily define $\mathbf{S}$ as the sum of two matrices, $\mathbf{Q}$ and $\mathbf{R}$, defined by the following:

$$\mathbf{S} = \mathbf{Q} + \mathbf{R} \quad \mathbf{Q}\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \mathbf{R} = \begin{bmatrix} \mathbf{0} & \mathbf{C} \\ \mathbf{H} & \mathbf{0} \end{bmatrix} \tag{11.4}$$

so that

$$\bar{\mathbf{x}} = \mathbf{S}\bar{\mathbf{x}} + \bar{\mathbf{f}} \tag{11.5}$$

where $\bar{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$ and $\bar{\mathbf{f}} = \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix}$. The vector $\mathbf{x}$ is, once again, the vector of total outputs, $\mathbf{y}$ is vector of total household income, $\mathbf{f}$ is the vector of exogenous final demand and $\mathbf{g}$ is the vector of exogenous household income. Using the definitions in (11.4), we can rewrite (11.5) as:

$$\bar{\mathbf{x}} = \mathbf{Q}\bar{\mathbf{x}} + \mathbf{R}\bar{\mathbf{x}} + \bar{\mathbf{f}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \mathbf{Q}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} + \mathbf{R}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} + \begin{bmatrix} \mathbf{f} \\ \mathbf{g} \end{bmatrix} \tag{11.6}$$

It follows directly that $\bar{\mathbf{x}} - \mathbf{Q}\bar{\mathbf{x}} = \mathbf{R}\bar{\mathbf{x}} + \bar{\mathbf{f}}$ or

$$\bar{\mathbf{x}} = (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{R}\bar{\mathbf{x}} + (\mathbf{I} - \mathbf{Q})^{-1}\bar{\mathbf{f}} \tag{11.7}$$

We define $\mathbf{T} = (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{R}$ so that (11.7) becomes

$$\bar{\mathbf{x}} = \mathbf{T}\bar{\mathbf{x}} + (\mathbf{I} - \mathbf{Q})^{-1}\bar{\mathbf{f}} \tag{11.8}$$

If we multiply through (11.8) by $\mathbf{T}$ we find

$$\mathbf{T}\bar{\mathbf{x}} = \mathbf{T}^2\bar{\mathbf{x}} + \mathbf{T}(\mathbf{I} - \mathbf{Q})^{-1}\bar{\mathbf{f}} = \mathbf{T}(\mathbf{T}\bar{\mathbf{x}}) + \mathbf{T}(\mathbf{I} - \mathbf{Q})^{-1}\bar{\mathbf{f}}$$

[3] Much more detailed discussions of the decomposition of SAM multipliers are included in Pyatt and Round (1979 and 1985b), Round (1985), and Thorbecke (1998), although the discussion here more closely parallels that of Holland and Wyeth (1993) and adopts many of the definitions of that presentation since they relate more directly to the notation used earlier in this chapter.

but it also follows directly from (11.8) that $\mathbf{T}\bar{\mathbf{x}} = \bar{\mathbf{x}} - (\mathbf{I} - \mathbf{Q})^{-1}\bar{\mathbf{f}}$ so that

$$\begin{aligned}\bar{\mathbf{x}} &= \mathbf{T}[\mathbf{T}\bar{\mathbf{x}} + (\mathbf{I} - \mathbf{Q})^{-1}\bar{\mathbf{f}}] + (\mathbf{I} - \mathbf{Q})^{-1}\bar{\mathbf{f}} \\ \bar{\mathbf{x}} &= (\mathbf{I} - \mathbf{T}^2)^{-1}(\mathbf{I} + \mathbf{T})(\mathbf{I} - \mathbf{Q})^{-1}\bar{\mathbf{f}} \end{aligned} \tag{11.9}$$

or, more simply

$$\bar{\mathbf{x}} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\bar{\mathbf{f}} \tag{11.10}$$

where $\mathbf{M}_1 = (\mathbf{I} - \mathbf{Q})^{-1}$, $\mathbf{M}_2 = (\mathbf{I} + \mathbf{T})$, and $\mathbf{M}_3 = (\mathbf{I} - \mathbf{T}^2)^{-1}$. This parallels the discussion of multiplier decomposition in Leontief models in section 6.7. Since we defined $\mathbf{M} = (\mathbf{I} - \mathbf{S})^{-1}$, it is clear that $\mathbf{M} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1$, so that $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{M}_3$ comprise a multiplicative partitioning of $\mathbf{M}$.

We can now use special cases of the partitioned inverse from Appendix A to determine the more detailed structure of $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{M}_3$, in a manner similar to the decomposition of multipliers that is in terms of the original partitions of $\mathbf{S}$, namely $\mathbf{A}$, $\mathbf{C}$, and $\mathbf{H}$. For the case of $\mathbf{M}_1$ it is straightforward from one of the special cases of inversion of partitioned matrices from Appendix A[4] as

$$\mathbf{M}_1 = (\mathbf{I} - \mathbf{Q})^{-1} = \begin{bmatrix} (\mathbf{I} - \mathbf{A}) & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^{-1} = \begin{bmatrix} (\mathbf{I} - \mathbf{A})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{11.11}$$

This matrix defines what are often called "direct effect" multipliers since they include what are easily recognized as Leontief output multipliers, but do not include the multiplier effects associated with other sectors such as value added or households, usually treated as exogenous in input–output models. These multipliers are also sometimes referred to as "intragroup" or "own" multipliers.

For the case of $\mathbf{M}_2$ we again use the same special case of the partitioned inverse to obtain

$$\mathbf{M}_2 = \mathbf{I} + \mathbf{T} = \mathbf{I} + (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{R} = \begin{bmatrix} \mathbf{I} & (\mathbf{I} - \mathbf{A})^{-1}\mathbf{C} \\ \mathbf{H} & \mathbf{I} \end{bmatrix} \tag{11.12}$$

The matrix $\mathbf{M}_2$ is often referred to as the matrix of indirect multipliers, since it records how the effects of exogenous inputs of each type get transmitted to the households sector but not the feedback of those increases (or decreases) in household income subsequently on commodity consumption. These multipliers are sometimes referred to as "extragroup" or "open loop" multipliers, since the feedback loop of the impact on household consumption and value added is not included.

[4] In this case if $\mathbf{M} = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{F} \end{bmatrix}$ then $\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{E}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}^{-1} \end{bmatrix}$ so if $\mathbf{F} = \mathbf{I}$ then $\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{E}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$.

For the case of $\mathbf{M}_3$ we begin with $\mathbf{M}_3 = (\mathbf{I} - \mathbf{T}^2)^{-1} = (\mathbf{I} - [(\mathbf{I} - \mathbf{Q})^{-1}\mathbf{R}]^2)^{-1}$. Using once again the special case of the partitioned inverse we have

$$\mathbf{M}_3 = (\mathbf{I} - \mathbf{T}^2)^{-1} = (\mathbf{I} - [(\mathbf{I} - \mathbf{Q})^{-1}\mathbf{R}]^2)^{-1}$$

or

$$\mathbf{M}_3 = \begin{bmatrix} [\mathbf{I} - (\mathbf{I} - \mathbf{A})^{-1}\mathbf{C}\mathbf{H}]^{-1} & \mathbf{0} \\ \mathbf{0} & [\mathbf{I} - \mathbf{H}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{C}]^{-1} \end{bmatrix} \tag{11.13}$$

The matrix of multipliers $\mathbf{M}_3$ is often referred to as the matrix of "cross" or "closed loop" multipliers, since they capture the feedback effects. For example, for an increase in commodity exports, an exogenous demand, there is an accompanying increase in interindustry production to satisfy that demand as well as an increase in household income, which in turn feeds back to further increase demand for commodities, and so on.

Example 11.1: Reduced Form Case We begin with a basic SAM matrix of transactions (Table 11.17), which we define to be $\mathbf{Z}$, that includes all economic transactions of interest, some subset of which we will consider for modeling purposes to be endogenous to the SAM model and others we will consider to be exogenously specified for the model.[5] For this example, we consider the first six defined sectors, the three industry sectors and three classes of households, as the endogenous sectors, and we consider the capital, government, and "rest of world" sectors to be exogenously specified.

For this case we define $\mathbf{S} = \mathbf{Z}\hat{\bar{\mathbf{x}}}^{-1}$ where $\mathbf{Z}$ is formed from the upper left six rows and columns of Table 11.17 ($\bar{\mathbf{G}}$) and $\bar{\mathbf{x}}$ is formed from the first six elements in the last column ($\mathbf{g}$). Equivalently, $\mathbf{S}$ can be formed from the upper left six rows and columns of matrix of normalized expenditure shares, $\mathbf{G} = \bar{\mathbf{G}}\hat{\mathbf{g}}^{-1}$. Also, to aid in the interpretation of the multipliers, we further subdivide $\mathbf{S}$ into sectors related to interindustry transactions ($\mathbf{A}$), endogenous final consumption ($\mathbf{C}$), and household income ($\mathbf{H}$):

$$\mathbf{S} = \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{H} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} .246 & .003 & .005 & .012 & .009 & .008 \\ .345 & .253 & .215 & .845 & .756 & .691 \\ .049 & .14 & .296 & .121 & .058 & .115 \\ .03 & .042 & .032 & 0 & 0 & 0 \\ .059 & .143 & .134 & 0 & 0 & 0 \\ .123 & .154 & .14 & 0 & 0 & 0 \end{bmatrix}$$

[5] This simplified example is adapted from a more complex development of a SAM for the United States using data for 1982 presented in Holland and Wyeth (1993).

Table 11.17 Reduced Form Fully Articulated SAM: Example 11.1

	Industry			Households			Exogenous			
	Nat. Res.	Manuf.	Services	LIH	MIH	HIH	Capital	Govt.	RoW	Total
Industry										
Natural Resources	50	93	10	5	8	7	0	20	10	203
Manufacturing	70	900	400	350	650	600	40	200	350	3, 560
Services	10	500	550	50	50	100	200	150	250	1, 860
Households										
Low Inc. Households	6	150	60					200	(2)	414
Medium Inc. Households	12	510	250					89	(1)	860
High Inc. Households	25	550	260					35	(2)	868
Exogenous										
Capital	15	200	75	(20)	60	100		(10)	290	710
Government	10	400	125	20	50	25	140	25	(15)	780
Rest of World	5	257	130	9	42	36	330	71	0	880
Total	203	3, 560	1, 860	414	860	868	710	780	880	10, 135

To specify the multipliers recall we separate $\mathbf{S}$ into two additive matrices $\mathbf{Q}$ and $\mathbf{R}$, such that $\mathbf{S} = \mathbf{Q} + \mathbf{R}$ where $\mathbf{Q} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} .246 & .003 & .005 & 0 \\ .345 & .253 & .215 & 0 \\ .049 & .14 & .296 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and

$$\mathbf{R} = \begin{bmatrix} \mathbf{0} & \mathbf{C} \\ \mathbf{H} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & .012 & .009 & .008 \\ 0 & 0 & 0 & .845 & .756 & .691 \\ 0 & 0 & 0 & .121 & .058 & .115 \\ .03 & .042 & .032 & 0 & 0 & 0 \\ .059 & .143 & .134 & 0 & 0 & 0 \\ .123 & .154 & .14 & 0 & 0 & 0 \end{bmatrix}$$

Recall that in deriving the three categories of multipliers, it was useful to define the matrix $\mathbf{T} = (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{R}$ that is frequently used in the expressions defining the multipliers, which for our example is

$$\mathbf{T} = (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{R} = \begin{bmatrix} 0 & 0 & 0 & .023 & .018 & .017 \\ 0 & 0 & 0 & 1.265 & 1.108 & 1.04 \\ 0 & 0 & 0 & .425 & .305 & .372 \\ .03 & .042 & .032 & 0 & 0 & 0 \\ .059 & .143 & .134 & 0 & 0 & 0 \\ .123 & .154 & .14 & 0 & 0 & 0 \end{bmatrix}$$

We now have all the information for this example to specify the three classes of multipliers, $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{M}_3$, as the following:

$$\mathbf{M}_1 = (\mathbf{I} - \mathbf{Q})^{-1} = \begin{bmatrix} (\mathbf{I} - \mathbf{A})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} 1.331 & .007 & .012 & 0 & 0 & 0 \\ .68 & 1.423 & .44 & 0 & 0 & 0 \\ .229 & .284 & 1.508 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Recall that $\mathbf{M}_1$ defines the "direct effect" multipliers, including only the Leontief output multiplier, so, for example, $[\mathbf{M}_1]_{12} = .007$ reflects the dollar's worth of natural resources output generated directly and indirectly to support a dollar's worth of the exogenously specified final demand for manufactured products (the sum of capital and government expenditures on and exports of manufactured products). These multipliers do not include the multiplier effects associated with other sectors such as value added or households, which are usually treated as exogenous in input–output models, but are incorporated into other components of $\mathbf{M}$ as endogenously specified in the SAM:

$$\mathbf{M}_2 = \mathbf{I} + \mathbf{T} = \begin{bmatrix} \mathbf{I} & (\mathbf{I} - \mathbf{A})^{-1}\mathbf{C} \\ \mathbf{H} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & .023 & .018 & .017 \\ 0 & 1 & 0 & 1.265 & 1.108 & 1.04 \\ 0 & 0 & 1 & .425 & .305 & .372 \\ .03 & .042 & .032 & 1 & 0 & 0 \\ .059 & .143 & .134 & 0 & 1 & 0 \\ .123 & .154 & .14 & 0 & 0 & 1 \end{bmatrix}$$

Recall that $\mathbf{M}_2$ defines the indirect multipliers and records how the effects of exogenous inputs of each type get transmitted to the households sector. For example, $[\mathbf{M}_2]_{41} = 0.03$ reflects the dollar's worth low income household income attributable directly and indirectly to a dollar's worth of exogenously specified final demand for natural resources. These multipliers do not include the feedback effects of those increases (or decreases) in household income subsequently on commodity consumption, which are captured in $\mathbf{M}_3$:

$$\mathbf{M}_3 = (\mathbf{I} - \mathbf{T}^2)^{-1} = \begin{bmatrix} [\mathbf{I} - (\mathbf{I} - \mathbf{A})^{-1}\mathbf{CH}]^{-1} & \mathbf{0} \\ \mathbf{0} & [\mathbf{I} - \mathbf{H}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{C}]^{-1} \end{bmatrix}$$

or, for the example,

$$\mathbf{M}_3 = (\mathbf{I} - \mathbf{T}^2)^{-1} = \begin{bmatrix} 1.007 & .012 & .011 & 0 & 0 & 0 \\ .449 & 1.721 & .648 & 0 & 0 & 0 \\ .146 & .23 & 1.207 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.131 & .11 & .109 \\ 0 & 0 & 0 & .464 & 1.389 & .387 \\ 0 & 0 & 0 & .499 & .418 & 1.416 \end{bmatrix}$$

Recall that $\mathbf{M}_3$ defines "cross" multipliers since they capture the feedback effects between households and interindustry transactions. The upper left matrix partition captures the income-induced increase in production from the increased income itself reflected in the lower right partition generated in support of exogenously specified final demand. For example, $[\mathbf{M}_3]_{44} = 1.131$ reflects the dollar's worth of low income household income generated by the increased interindustry consumption attributable to increases in income of all household categories. That is, for any increase in exogenously specified demand there is an accompanying increase in interindustry production to satisfy that demand as well as an increase in household income generated, which in turn feeds back to further increase demand for commodities, and so on.

Finally, the total SAM multiplier matrix is

$$\mathbf{M} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1 = (\mathbf{I} - \mathbf{S})^{-1} = \begin{bmatrix} 1.351 & .027 & .034 & .043 & .035 & .033 \\ 1.916 & 2.628 & 1.741 & 2.463 & 2.113 & 2.039 \\ .627 & .672 & 1.923 & .808 & .626 & .691 \\ .141 & .134 & .136 & 1.131 & .11 & .109 \\ .439 & .47 & .51 & .464 & 1.389 & .387 \\ .55 & .505 & .542 & .499 & .418 & 1.416 \end{bmatrix}$$

which accumulates all the effects of $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{M}_3$.

11.10.3 Multipliers in an Expanded SAM

We now remove the simplification of assuming that value added and households are indistinguishable and return to the expanded case, the summary version of which was presented as (11.1):

$$\mathbf{S} = \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{C} \\ \mathbf{V} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y} & \mathbf{H} \end{bmatrix}$$

As with the reduced form version, we disaggregate $\mathbf{S}$ into two additive matrices, $\mathbf{Q}$ and $\mathbf{R}$ so that $\mathbf{S} = \mathbf{Q} + \mathbf{R}$ where $\mathbf{Q} = \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H} \end{bmatrix}$ and $\mathbf{R} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{C} \\ \mathbf{V} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y} & \mathbf{0} \end{bmatrix}$.

Hence, following the methodology used for the reduced form, $\bar{\mathbf{x}} = \mathbf{S}\bar{\mathbf{x}} + \bar{\mathbf{f}}$ (11.2) where $\bar{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{v} \\ \mathbf{y} \end{bmatrix}$ and $\bar{\mathbf{f}} = \begin{bmatrix} \mathbf{f} \\ \mathbf{w} \\ \mathbf{h} \end{bmatrix}$ and $\mathbf{x}$ is, once again, the vector of total outputs, $\mathbf{v}$ is the vector of total value added, $\mathbf{y}$ is vector of total household income, $\mathbf{f}$ is the vector of exogenous final demand, $\mathbf{w}$ is the vector of exogenous value-added income, and $\mathbf{h}$ is the vector of exogenous household income. As with the reduced form version we can rewrite (11.2) as the following:

$$\begin{aligned} \bar{\mathbf{x}} &= (\mathbf{R} + \mathbf{Q})\bar{\mathbf{x}} + \bar{\mathbf{f}} \\ \bar{\mathbf{x}} &= (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{R}\bar{\mathbf{x}} + (\mathbf{I} - \mathbf{Q})^{-1}\bar{\mathbf{f}} \end{aligned} \tag{11.14}$$

Once again defining $\mathbf{T} = (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{R}$, (11.14) becomes

$$\bar{\mathbf{x}} = \mathbf{T}\bar{\mathbf{x}} + (\mathbf{I} - \mathbf{Q})^{-1}\bar{\mathbf{f}} \tag{11.15}$$

and multiplying through (11.15) by $\mathbf{T}$ we find $\mathbf{T}\bar{\mathbf{x}} = \mathbf{T}^2\bar{\mathbf{x}} + \mathbf{T}(\mathbf{I} - \mathbf{Q})^{-1}\bar{\mathbf{f}}$. It also follows directly from (11.15) that $\mathbf{T}\bar{\mathbf{x}} = \bar{\mathbf{x}} - (\mathbf{I} - \mathbf{Q})^{-1}\bar{\mathbf{f}}$ so that $\bar{\mathbf{x}} = \mathbf{T}[\mathbf{T}\bar{\mathbf{x}} + (\mathbf{I} - \mathbf{Q})^{-1}\bar{\mathbf{f}}] + (\mathbf{I} - \mathbf{Q})^{-1}\bar{\mathbf{f}}$ or

$$\bar{\mathbf{x}} = \mathbf{T}^2\bar{\mathbf{x}} + \mathbf{T}(\mathbf{I} - \mathbf{Q})^{-1}\mathbf{f} + (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{f} \tag{11.16}$$

which to this point is the same as the reduced form model. Since there are three block partitions in this expanded form, it turns out to be helpful in the following to expand the equation further multiplying through yet again by $\mathbf{T}$, substituting the result back into (11.15), and rearranging terms (see Holland and Wyeth, 1993):

$$\begin{aligned} \mathbf{T}\bar{\mathbf{x}} &= \mathbf{T}\left[\mathbf{T}^2\bar{\mathbf{x}} + \mathbf{T}(\mathbf{I} - \mathbf{Q})^{-1}\bar{\mathbf{f}} + (\mathbf{I} - \mathbf{Q})^{-1}\bar{\mathbf{f}}\right] \\ &= \mathbf{T}^3\bar{\mathbf{x}} + \mathbf{T}^2(\mathbf{I} - \mathbf{Q})^{-1}\bar{\mathbf{f}} + \mathbf{T}(\mathbf{I} - \mathbf{Q})^{-1}\bar{\mathbf{f}} \end{aligned} \tag{11.17}$$

Substituting this expression for $\mathbf{T}\bar{\mathbf{x}}$ back into (11.16) and using the subsequent results in (11.15) yields

$$\begin{aligned} \bar{\mathbf{x}} &= \mathbf{T}^3\bar{\mathbf{x}} + \mathbf{T}^2(\mathbf{I} - \mathbf{Q})^{-1}\bar{\mathbf{f}} + \mathbf{T}(\mathbf{I} - \mathbf{Q})^{-1}\bar{\mathbf{f}} + (\mathbf{I} - \mathbf{Q})^{-1}\bar{\mathbf{f}} \\ \bar{\mathbf{x}} &= (\mathbf{I} - \mathbf{T}^3)^{-1}(\mathbf{I} + \mathbf{T} + \mathbf{T}^2)(\mathbf{I} - \mathbf{Q})^{-1}\bar{\mathbf{f}} \end{aligned} \tag{11.18}$$

which we can verify by expanding (11.18). Once again, as with reduced form version, we express this as $\bar{\mathbf{x}} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1\bar{\mathbf{f}}$ but in the expanded case, where $\mathbf{M}_1 = (\mathbf{I} - \mathbf{Q})^{-1}$, $\mathbf{M}_2 = (\mathbf{I} + \mathbf{T} + \mathbf{T}^2)$, and $\mathbf{M}_3 = (\mathbf{I} - \mathbf{T}^3)^{-1}$. We leave it to the reader to verify that expanding the expressions for $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{M}_3$ in terms of $\mathbf{A}$, $\mathbf{C}$, $\mathbf{Y}$, $\mathbf{V}$, and $\mathbf{H}$ become the following:

$$\mathbf{M}_1 = \begin{bmatrix} (\mathbf{I} - \mathbf{A})^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & (\mathbf{I} - \mathbf{H})^{-1} \end{bmatrix} \tag{11.19}$$

In comparing this expression for $\mathbf{M}_1$ with the reduced form case, note that these multipliers still only capture the "own effects" and do not capture the impact of other major sectors. The matrix $\mathbf{M}_2$ is the following:

$$\mathbf{M}_2 = \begin{bmatrix} \mathbf{I} & (\mathbf{I} - \mathbf{A})^{-1}\mathbf{C}(\mathbf{I} - \mathbf{H})^{-1}\mathbf{Y} & (\mathbf{I} - \mathbf{A})^{-1}\mathbf{C} \\ \mathbf{V} & \mathbf{I} & \mathbf{V}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{C} \\ (\mathbf{I} - \mathbf{H})^{-1}\mathbf{Y}\mathbf{V} & (\mathbf{I} - \mathbf{H})^{-1}\mathbf{Y} & \mathbf{I} \end{bmatrix} \tag{11.20}$$

For $\mathbf{M}_2$, the "open loop" multipliers capture, once again, the impacts of an exogenous input on each major sector.

Finally, the matrix $\mathbf{M}_3$ is the following:

$$\mathbf{M}_3 = \begin{bmatrix} [\mathbf{I}-(\mathbf{I}-\mathbf{A})^{-1}\mathbf{C}(\mathbf{I}-\mathbf{H})^{-1}\mathbf{YV}]^{-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & [\mathbf{I}-\mathbf{V}(\mathbf{I}-\mathbf{A})^{-1}\mathbf{C}(\mathbf{I}-\mathbf{H})^{-1}\mathbf{Y}]^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & [\mathbf{I}-(\mathbf{I}-\mathbf{H})^{-1}\mathbf{YV}(\mathbf{I}-\mathbf{A})^{-1}\mathbf{C}]^{-1} \end{bmatrix} \tag{11.21}$$

The matrix $\mathbf{M}_3$ captures the final feedback effect of the subsequent rounds of impact on each sector. As before $\mathbf{M} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1$.

Example 11.2: The Expanded Case Consider the SAM transactions matrix in Table 11.18, which is similar to that used in Example 11.1 except we have now introduced additional value-added and institutional sectors.

For the expanded case, the additive partitioned matrices **Q** and **R,** both of which for this example are of dimension 12 $\times$ 12, corresponding to the number of endogenous sectors, are defined by

$$\mathbf{Q} = \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{H} \end{bmatrix} \quad \text{and} \quad \mathbf{R} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{C} \\ \mathbf{V} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y} & \mathbf{0} \end{bmatrix}$$

where

$$\mathbf{A} = \begin{bmatrix} .275 & .023 & .005 \\ .325 & .304 & .213 \\ .050 & .100 & .361 \end{bmatrix}, \mathbf{Y} = \begin{bmatrix} .87 & 0 & 0 \\ 0 & .149 & 0 \\ 0 & .851 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{V} = \begin{bmatrix} .1 & .325 & .26 \\ .2 & .175 & .1 \\ .025 & .05 & .02 \end{bmatrix},$$

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & .011 & .008 & .006 \\ 0 & 0 & 0 & .874 & .688 & .572 \\ 0 & 0 & 0 & .092 & .081 & .082 \end{bmatrix} \quad \text{and} \quad \mathbf{H} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ .094 & .071 & .088 & 0 & 0 & 0 \\ .438 & .429 & .141 & 0 & 0 & 0 \\ .469 & .5 & .265 & 0 & 0 & 0 \end{bmatrix}$$

Table 11.18 Expanded Form Fully Articulated SAM: Example 11.1

	Industry			Value Added			Households & Institutions						Exogenous			
	Nat. Res.	Manuf.	Serv.	Labor	Capital	Taxes	Wages	Profits	Ent.	LIHH	MIHH	HIHH	Exog. Cap.	Govt.	ROW	Total
Industry																
Natural Resources	55	90	10				0	0	0	5	8	7	−2	8	19	200
Manuf.	65	1,215	425				0	0	0	380	680	625	40	450	120	4,000
Services	10	400	722				0	0	0	40	80	90	313	145	200	2,000
Value Added																
Labor	20	1,300	520													1,840
Capital	40	700	200													940
Taxes	5	200	40													245
Households & Institutions																
Wages				1,600	0	0	0	0	0	0	0	0	0	0	0	1,600
Profits				0	140	0	0	0	0	0	0	0	0	0	0	140
Enterprises				0	800	0	0	0	0	0	0	0	0	50	0	850
Low Inc HH				0	0	0	150	10	75	0	0	0	0	204	−4	435
Med Inc HH				0	0	0	700	60	120	0	0	0	0	113	−5	988
High Inc HH				0	0	0	750	70	225	0	0	0	0	50	−2	1,093
Exogenous																
Exog. Cap.	0	0	0	0	0	0	0	0	380	−20	50	151	0	−200	10	371
Govt.	0	0	0	240	0	245	0	0	50	20	150	200	0	315	−20	1,200
Rest of World	5	95	83	0	0	0	0	0	0	10	20	20	20	65	0	318
Total	200	4,000	2,000	1,840	940	245	1,600	140	850	435	988	1,093	371	1,200	318	16,220

As noted earlier, to facilitate constructing the multipliers succinctly, we have

$$\mathbf{T} = (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{R} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .062 & .048 & .039 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.396 & 1.103 & .924 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .367 & .303 & .277 \\ .1 & .325 & .26 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .2 & .175 & .1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .025 & .05 & .02 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .87 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .149 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .851 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .082 & .086 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .38 & .184 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .408 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The multiplier matrices $\mathbf{M}_1$, $\mathbf{M}_2$, and $\mathbf{M}_3$ can then be calculated as follows:

$$\mathbf{M}_1 = (\mathbf{I} - \mathbf{Q})^{-1} = \begin{bmatrix} 1.403 & .049 & .027 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .723 & 1.534 & .516 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ .223 & .244 & 1.648 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .094 & .071 & .088 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .438 & .429 & .141 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .469 & .5 & .265 & 0 & 0 & 1 \end{bmatrix}$$

Like the earlier example, $\mathbf{M}_1$ is a block diagonal matrix defining the "direct effect" multipliers of each of the sector groups – industry, value added, and household institutions – in response to exogenously specified final demand. For example, $[\mathbf{M}_1]_{11,7} = .438$ reflects the dollar's worth of medium income household income generated directly by a dollar's worth of final consumption payments to wage earners. As before, these multipliers do not include the multiplier effects among the product or service groups so it should not be surprising, for example, that multipliers for the value-added group comprise the identity matrix since demand for value added is not exogenously specified.

$$\mathbf{M}_2 = (\mathbf{I} + \mathbf{T} + \mathbf{T}^2) = \left[\begin{array}{ccc|ccc|cccccc}
1 & 0 & 0 & .039 & .026 & 0 & 0 & 0 & 0 & .062 & .048 & .039 \\
0 & 1 & 0 & .91 & .6 & 0 & 0 & 0 & 0 & 1.396 & 1.103 & .924 \\
0 & 0 & 1 & .258 & .17 & 0 & 0 & 0 & 0 & .367 & .303 & .277 \\
.1 & .325 & .26 & 1 & 0 & 0 & 0 & 0 & 0 & .555 & .442 & .376 \\
.2 & .175 & .1 & 0 & 1 & 0 & 0 & 0 & 0 & .293 & .233 & .197 \\
.025 & .05 & .02 & 0 & 0 & 1 & 0 & 0 & 0 & .079 & .062 & .53 \\
.087 & .283 & .226 & .87 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
.03 & .026 & .015 & 0 & .149 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
.17 & .149 & .085 & 0 & .851 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
.025 & .042 & .03 & .082 & .086 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
.075 & .156 & .117 & .38 & .184 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
.101 & .185 & .136 & .408 & .3 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]$$

The matrix $\mathbf{M}_2$ once again defines the indirect multipliers. That is, for an exogenously specified demand in a sector group – again, industry, value added, or household institutions – these multipliers specify the impact generated in other groups. Hence, the block diagonal elements are all identity matrices since those partitions relate to impacts within the group. For example, if we define the partitions of $\mathbf{M}$ as $\mathbf{M} = \begin{bmatrix} \mathbf{M}_{AA} & \mathbf{M}_{AY} & \mathbf{M}_{AC} \\ \mathbf{M}_{YA} & \mathbf{M}_{YY} & \mathbf{M}_{YC} \\ \mathbf{M}_{CA} & \mathbf{M}_{CY} & \mathbf{M}_{CC} \end{bmatrix}$ then $\mathbf{M}_{CA}$ and $\mathbf{M}_{CY}$ capture the interindustry output and value added output groups generated by a dollar's worth final demand in the households institutions group.

$$\mathbf{M}_3 = (\mathbf{I} - \mathbf{T}^3)^{-1} = \left[\begin{array}{ccc|ccc|cccccc}
1.018 & .034 & .025 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
.417 & 1.792 & .586 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
.118 & .225 & 1.166 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.725 & .478 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & .381 & 1.251 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & .102 & .067 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & .924 & .76 & .646 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & .86 & .069 & .058 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & .492 & .391 & .332 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.139 & .111 & .094 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .524 & 1.417 & .355 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & .621 & .494 & 1.42
\end{array}\right]$$

The matrix $\mathbf{M}_3$ once again defines "cross" multipliers since they capture the feedback effects among sector groups. These multipliers are sometimes called "closed loop" multipliers since the capture effects of demands generated in one group produces an effect in another group that, in turn, generates an effect in the original group. Hence only the block diagonal partitions will be nonzero.

Finally, once again, $\mathbf{M}$ captures the cumulative effect of all three types of multipliers with $\mathbf{M} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1$:

$$\mathbf{M} = \begin{bmatrix} 1.459 & .109 & .087 & .077 & .051 & 0 & .089 & .088 & .045 & .12 & .094 & .079 \\ 2.011 & 2.912 & 1.901 & 1.798 & 1.185 & 0 & 2.068 & 2.045 & 1.034 & 2.741 & 2.174 & 1.834 \\ .588 & .635 & 2.041 & .51 & .336 & 0 & .586 & .581 & .293 & .749 & .607 & .535 \\ .952 & 1.122 & 1.157 & 1.725 & .478 & 0 & .833 & .824 & .417 & 1.098 & .874 & .743 \\ .703 & .595 & .554 & .381 & 1.251 & 0 & .438 & .434 & .219 & .579 & .46 & .39 \\ .149 & .161 & .138 & .102 & .067 & 1 & .117 & .116 & .059 & .155 & .123 & .104 \\ .828 & .976 & 1.006 & 1.5 & .415 & 0 & 1.725 & .717 & .363 & .954 & .76 & .646 \\ .105 & .089 & .083 & .057 & .186 & 0 & .065 & 1.065 & .033 & .086 & .069 & .058 \\ .598 & .506 & .472 & .324 & 1.065 & 0 & .373 & .369 & 1.187 & .492 & 391 & .332 \\ .138 & .143 & .142 & .173 & .146 & 0 & .199 & .176 & .141 & 1.139 & .111 & .094 \\ .492 & .536 & .542 & .726 & .412 & 0 & .835 & .822 & .34 & .524 & 1.417 & .354 \\ .599 & .636 & .638 & .817 & .57 & 0 & .94 & .966 & .5 & .621 & .494 & 1.42 \end{bmatrix}$$

For the example, we leave it as an exercise to the reader to verify that $\mathbf{M} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1 = (\mathbf{I} - \mathbf{S})^{-1}$, but we can interpret the multipliers in this example in a variety of ways. One simple way is to explore the linkage between aggregate levels of interindustry activity, final-demands and value-added sectors. For example, using the definitions of the partitions of $\mathbf{M}$ noted earlier, the average of the column sums of $\mathbf{M}_{AA} = 3.91$ would reflect how connected total industrial output is to aggregate final demand for industrial output (the traditional Leontief multipliers from Chapter 6) compared with the average of the column sums of $\mathbf{M}_{CA} = 2.84$, which would reflect how connected overall household income is to aggregate final demand for industrial output.

11.10.4 Additive Multipliers

In many kinds of analysis involving multipliers, it is convenient to formulate them so that their sum rather than their sequential multiplication yields the total multipliers. These "additive" multipliers were first proposed by Stone (1985) and further developed by Pyatt and Round (1985a). Following the development in section 6.7, Stone formulated the following construction:

$$\mathbf{M} = (\mathbf{I} - \mathbf{S})^{-1} = \mathbf{N}_1 + \mathbf{N}_2 + \mathbf{N}_3$$

where $\mathbf{N}_1$, defined as matrix of direct or "own" multipliers, is the same as $\mathbf{M}_1$, i.e., $\mathbf{N}_1 = \mathbf{M}_1$. The matrix of indirect or "open loop" multipliers, $\mathbf{N}_2$, is defined as $\mathbf{N}_2 = \mathbf{M}_2\mathbf{M}_3\mathbf{M}_1 - \mathbf{M}_3\mathbf{M}_1$, and the matrix of cross or "closed loop" multipliers, $\mathbf{N}_3$, is defined as $\mathbf{N}_3 = \mathbf{M}_3\mathbf{M}_1 - \mathbf{M}_1$. We can verify that $\mathbf{M} = \mathbf{N}_1 + \mathbf{N}_2 + \mathbf{N}_3$ by the following:

$$\begin{aligned}
&\mathbf{M} = \mathbf{N}_1 + \mathbf{N}_2 + \mathbf{N}_3 \\
&\mathbf{M} = \mathbf{M}_1 + [\mathbf{M}_2\mathbf{M}_3\mathbf{M}_1 - \mathbf{M}_3\mathbf{M}_1] + [\mathbf{M}_3\mathbf{M}_1 - \mathbf{M}_1] \\
&\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2\mathbf{M}_3\mathbf{M}_1 - \mathbf{M}_3\mathbf{M}_1 + \mathbf{M}_3\mathbf{M}_1 - \mathbf{M}_1 \\
&\mathbf{M} = \mathbf{M}_2\mathbf{M}_3\mathbf{M}_1
\end{aligned}$$

Since the multiplicative form of the multipliers was derived as $\mathbf{M} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1$, we must prove that $\mathbf{M}_2\mathbf{M}_3 = \mathbf{M}_3\mathbf{M}_2$, which follows directly from the special cases of the partitioned inverse described in Appendix A and noted earlier so that

$$\mathbf{M} = (\mathbf{I} - \mathbf{S})^{-\mathbf{1}} = \mathbf{N}_1 + \mathbf{N}_2 + \mathbf{N}_3 = \mathbf{M}_2\mathbf{M}_3\mathbf{M}_1 = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1$$

For Example 11.1 above, the matrix of total multipliers, $\mathbf{M}$, was computed earlier; the additive multipliers for this example are the following:

$$\mathbf{N}_1 = \mathbf{M}_1 = \begin{bmatrix} 1.331 & .007 & .012 & 0 & 0 & 0 \\ .68 & 1.423 & .44 & 0 & 0 & 0 \\ .229 & .284 & 1.508 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{N}_2 = \mathbf{M}_2\mathbf{M}_3\mathbf{M}_1 - \mathbf{M}_3\mathbf{M}_1 = \begin{bmatrix} 0 & 0 & 0 & .043 & .035 & .033 \\ 0 & 0 & 0 & 2.463 & 2.113 & 2.039 \\ 0 & 0 & 0 & .808 & .626 & .691 \\ .141 & .134 & .136 & 0 & 0 & 0 \\ .439 & .47 & .51 & 0 & 0 & 0 \\ .55 & .505 & .542 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{N}_3 = \mathbf{M}_3\mathbf{M}_1 - \mathbf{M}_1 = \begin{bmatrix} .021 & .02 & .022 & 0 & 0 & 0 \\ 1.236 & 1.214 & 1.301 & 0 & 0 & 0 \\ .398 & .388 & .415 & 0 & 0 & 0 \\ 0 & 0 & 0 & .131 & .11 & .109 \\ 0 & 0 & 0 & .464 & .389 & .387 \\ 0 & 0 & 0 & .499 & .418 & .416 \end{bmatrix}$$

We leave it as an exercise for the reader to verify that the additive multipliers sum to the product of the multiplicative multipliers and are both equal to the total multipliers for the expanded case and Example 11.2.

11.11 The Relationship between Input–Output and SAM Multipliers

Input–output multipliers and SAM multipliers have many similarities and seek to capture the same effects. Their relationship to one another parallels that of the development of Miyazawa's model and multipliers examined in section 6.4. In this section we explicitly illustrate the relationship.

Recall the simple input–output example provided in Table 11.7 and recast as a SAM in Table 11.8. The matrix of input–output technical coefficients and the Leontief inverse are $\mathbf{A} = \begin{bmatrix} .357 & .167 & 0 \\ .429 & .222 & .4 \\ 0 & 0 & 0 \end{bmatrix}$ and $(\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix} 1.815 & .389 & .156 \\ 1 & 1.5 & .6 \\ 0 & 0 & 1 \end{bmatrix}$. The corresponding matrix of total expenditure shares, $\bar{\mathbf{S}}$, and corresponding matrix of SAM coefficients, $\mathbf{S}$, assuming households as the exogenous sector are

$$\bar{\mathbf{S}} = \begin{bmatrix} \mathbf{S} & \mathbf{F} \\ \mathbf{W} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} .357 & .167 & 0 & 0 & 0 & .3 \\ .429 & .222 & .4 & 0 & 0 & .2 \\ 0 & 0 & 0 & 0 & 0 & .5 \\ .071 & .389 & .1 & 0 & 0 & 0 \\ .143 & .222 & .5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

and further, $\mathbf{S}$ is partitioned as $\mathbf{S} = \begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{H} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} .357 & .167 & 0 & 0 & 0 \\ .429 & .222 & .4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ .071 & .389 & .1 & 0 & 0 \\ .143 & .222 & .5 & 0 & 0 \end{bmatrix}$. In this case, the computation of the SAM multipliers is quite simple. The additive partitioning of $\mathbf{S}$ in $\mathbf{Q}$ and $\mathbf{R}$ is

$$\mathbf{Q} = \begin{bmatrix} .357 & .167 & 0 & 0 & 0 \\ .429 & .222 & .4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{R} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ .071 & .389 & .1 & 0 & 0 \\ .143 & .222 & .5 & 0 & 0 \end{bmatrix}$$

so that the corresponding computation of the multipliers becomes

$$\mathbf{M}_1 = (\mathbf{I}-\mathbf{Q})^{-1} = \begin{bmatrix} (\mathbf{I}-\mathbf{A})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} 1.815 & .389 & .156 & 0 & 0 \\ 1 & 1.5 & .6 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M}_2 = \mathbf{I} + \mathbf{T} = \mathbf{I} + (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{R} = \begin{bmatrix} \mathbf{I} & (\mathbf{I} - \mathbf{A})^{-1}\mathbf{C} \\ \mathbf{H} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ .071 & .389 & .1 & 0 & 0 \\ .143 & .222 & .5 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M}_3 = (\mathbf{I} - \mathbf{T}^2)^{-1} = \begin{bmatrix} \mathbf{I} - (\mathbf{I} - \mathbf{A})^{-1}\mathbf{CH} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} - \mathbf{H}(\mathbf{I} - \mathbf{A})^{-1}\mathbf{C} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M} = (\mathbf{I} - \mathbf{S})^{-1} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1 = \begin{bmatrix} 1.815 & .389 & .156 & 0 & 0 \\ 1 & 1.5 & .6 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ .519 & .611 & .344 & 1 & 0 \\ .481 & .389 & .656 & 0 & 1 \end{bmatrix}$$

where the frequently used expression for $\mathbf{T}$ noted earlier is defined by

$$\mathbf{T} = (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{R} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ .071 & .389 & .1 & 0 & 0 \\ .143 & .222 & .5 & 0 & 0 \end{bmatrix}$$

This case may not seem very interesting because there is no endogenous final demand, i.e., $\mathbf{C} = \mathbf{0}$, since the feedback between the final demand and value added does not exist, a fundamental assumption of the input–output model unless we "close" the model to households, which is essentially the same as constructing a SAM with an endogenous final-demand component. Note that since there are no endogenous final demands it should not be surprising that the columns of the lower left partition of $\mathbf{M}$ each sum to unity. If we divide the existing vector of final demand into endogenous and exogenous components, we can generate a more interesting result.

To do this we define a new SAM transactions table, $\bar{\mathbf{Z}}$, given in Table 11.19. Note that the sum of endogenous and exogenous final demands from households is the same as the total household figures in Table 11.18 above.

Table 11.19 SAM Framework Example Using Social Accounting Conventions (revised to include endogenous final demand)

	Expenditures							
	Nat. Res.	Manuf.	Serv.	Labor	Capital	Endog. Households	Exog. Households	Total Output
Income								
Natural Resources	50	30	0			40	20	140
Manufacturing	60	40	40			15	25	180
Services	0	0	0			0	100	100
Labor	10	70	10					90
Capital	20	40	50					110
Endog. Households				45	10			55
Exog. Households				45	100			145
Total Inputs	140	180	100	90	110	55	145	

The corresponding matrix of normalized total expenditure shares is

$$\mathbf{G} = \begin{bmatrix} .357 & .167 & 0 & 0 & 0 & .727 & .138 \\ .429 & .222 & .4 & 0 & 0 & .273 & .172 \\ 0 & 0 & 0 & 0 & 0 & 0 & .69 \\ .071 & .389 & .1 & 0 & 0 & 0 & 0 \\ .143 & .222 & .5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .5 & .091 & 0 & 0 \\ 0 & 0 & 0 & .5 & .909 & 0 & 0 \end{bmatrix}$$

and we can employ the expanded form for computing the SAM multipliers, beginning by specifying the SAM coefficients by the following partitioned matrix:

$$\mathbf{S} = \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{C} \\ \mathbf{V} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y} & \mathbf{H} \end{bmatrix} = \begin{bmatrix} .357 & .167 & 0 & 0 & 0 & .727 \\ .429 & .222 & .4 & 0 & 0 & .273 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ .071 & .389 & .1 & 0 & 0 & 0 \\ .143 & .222 & .5 & 0 & 0 & 0 \\ 0 & 0 & 0 & .5 & .091 & 0 \end{bmatrix}$$

Hence $\mathbf{Q} = \begin{bmatrix} .357 & .167 & 0 & 0 & 0 & 0 \\ .429 & .222 & .4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $\mathbf{R} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & .727 \\ 0 & 0 & 0 & 0 & 0 & .273 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ .071 & .389 & .1 & 0 & 0 & 0 \\ .143 & .222 & .5 & 0 & 0 & 0 \\ 0 & 0 & 0 & .5 & .091 & 0 \end{bmatrix}$

and $\mathbf{T} = (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{R} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1.426 \\ 0 & 0 & 0 & 0 & 0 & 1.136 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ .071 & .389 & .1 & 0 & 0 & 0 \\ .143 & .222 & .5 & 0 & 0 & 0 \\ 0 & 0 & 0 & .5 & .091 & 0 \end{bmatrix}$ so that the corresponding computation of the multipliers becomes as follows:

$$\mathbf{M}_1 = (\mathbf{I}-\mathbf{Q})^{-1} = \begin{bmatrix} 1.815 & .389 & .156 & 0 & 0 & 0 \\ 1 & 1.5 & .6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The Leontief inverse, once again, is the upper left partition of $\mathbf{M}_1$, and in this economy there is no direct multiplier effect on output of value added and households since the only exogenous expenditures are from households on industry output:

$$\mathbf{M}_2 = \mathbf{I}+\mathbf{T}+\mathbf{T}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & .713 & .13 & 1.426 \\ 0 & 1 & 0 & 0 & .568 & .103 & 1.136 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ .071 & .389 & .1 & 1 & 1 & 0 & .544 \\ .143 & .222 & .5 & 0 & 0 & 1 & .456 \\ .049 & .215 & .095 & .5 & .091 & .091 & 1 \end{bmatrix},$$

The indirect multipliers, $\mathbf{M}_2$, once again have identity matrices for the block diagonal partitions.

$$\mathbf{M}_3 = (\mathbf{I}-\mathbf{T}^3)^{-1} = \begin{bmatrix} 1.101 & .446 & .198 & 0 & 0 & 0 \\ .081 & 1.355 & .158 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.396 & .072 & 0 \\ 0 & 0 & 0 & .332 & 1.06 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.456 \end{bmatrix}$$

Finally, the closed loop multipliers, $\mathbf{M}_3$, reflect the feedback loops among the sector groups. In particular, in this example, this shows a substantial impact associated with treating a portion of the final demand from households as an endogenous variable in the SAM model.

Table 11.20 Comparative Input–Output and SAM Multipliers

Multiplier Type	Nat. Res.	Manuf.	Services
Input–Output	2.815	1.889	1.756
SAM (no endog. final demand)	3.815	2.889	2.756
SAM (with endog. final demand)	5.828	5.154	4.296

The matrix of total multipliers is found, once again, as $\mathbf{M} = (\mathbf{I} - \mathbf{S})^{-1} = \mathbf{M}_3\mathbf{M}_2\mathbf{M}_1$ and for the example we have

$$\mathbf{M} = \begin{bmatrix} 2.444 & 1.097 & .637 & 1.038 & .189 & 2.077 \\ 1.502 & 2.064 & .984 & .827 & .15 & 1.655 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ .758 & .881 & .528 & 1.396 & .072 & .792 \\ .683 & .615 & .81 & .332 & 1.06 & .664 \\ .441 & .496 & .338 & .728 & .132 & 1.456 \end{bmatrix}$$

Table 11.20 shows the differences among the average total multipliers for the three classes of multipliers computed for this example.

Note that since the SAM endogenizes transactions not previously included in the input–output interindustry accounts, the SAM multipliers will generally be larger than the input–output multipliers and with endogenized final demand as well the multipliers will be larger still.

11.12 Balancing SAM Accounts

SAMS, by their structural requirements and conventions, e.g., requiring a square transactions matrix with row and column totals equal, are useful for reconciling different sources of data that may be inconsistent. The RAS technique developed in Chapter 7 is often employed to balance a SAM. For example, consider the "unbalanced" SAM in Table 11.21 – it is unbalanced since the row and column totals are not the same. In the following we consider two cases of balancing the SAM in Table 11.21. We employ the RAS technique using the transactions matrix developed in Section 7.4.3.

11.12.1 Example: Balancing a SAM

Suppose we determine that the best estimate of total outputs is the average of the row and column totals, $\bar{\mathbf{x}}' = [780 \quad 670 \quad 40 \quad 60]$, which we then use as the row and column totals for applying the RAS procedure. Since $\bar{\mathbf{Z}} = \begin{bmatrix} 0 & 600 & 65 & 45 \\ 700 & 0 & -25 & 15 \\ 0 & 40 & 0 & 0 \\ 50 & 10 & 0 & 0 \end{bmatrix}$ the

Table 11.21 Unbalanced SAM: Examples 11.12.1 and 11.12.2

	Prod.	Cons.	Capital	ROW	Totals
Producer	0	600	65	45	710
Consumers	700	0	−25	15	690
Capital	0	40	0	0	40
Rest of World	50	10	0	0	60
Totals	750	650	40	60	1500

RAS adjustment result using $\bar{\mathbf{x}}'$ for both the row and column constraints is $\bar{\mathbf{Z}}^{RAS} = \begin{bmatrix} 0 & 630 & 40 & 60 \\ 670 & 0 & 0 & 0 \\ 0 & 40 & 0 & 0 \\ 50 & 10 & 0 & 0 \end{bmatrix}$. This result may seem implausible since consumer transactions to capital and exports are driven to zero, so constraining the RAS procedure to fix additional information, if there is any such information, may be warranted, which we explore in the next example.

11.12.2 Example: Balancing a SAM with Additional Information

Suppose we have high confidence in the estimates of consumer capital investment in Table 11.21, i.e., $g_{23} = -25$. We apply the RAS procedure again, but fixing cell g_{23} to -25 using the procedure outlined in Section 7.4.7 to result in

$$\bar{\mathbf{Z}} = \begin{bmatrix} 0 & 620 & 65 & 45 \\ 680 & 0 & -25 & 15 \\ 0 & 40 & 0 & 0 \\ 50 & 10 & 0 & 0 \end{bmatrix}$$

Notice this time the RAS adjustment yields a much more plausible balanced SAM since the capital and exports entries are not driven to zero.

11.13 Some Applications of SAMs

SAMs are widely applied in the literature, particularly in the area of reconciliation of social accounting data, as originally conceived by Richard Stone and chronicled in his Nobel Memorial Lecture published in Stone (1997) and in Stone (1985), Stone and Croft-Murray (1959) and Stone *et al.* (1962), as well as in the work of many other authors such as Pyatt (1985, 1988, 1991a, 1991b, 1994a, 1994b, 1999), Round (1985) and Keuning (1991).

Applications of SAMs to many other policy problems have appeared as well, such as in regional development policy in Pyatt and Round (1977, 1985b), Pyatt and Thorbecke (1976) and Round (1988, 2003), the implications of market integration across Europe in Round (1991), analysis of labor productivity in European Commission (2003), in analyzing spatial patterns of income and wealth generation in Denmark in Masden and Jensen-Butler (2005), and extensions to social and environmental indicators in Bolivia (Alarcón, van Heemst and de Jong, 2000).

11.14 Summary

In this chapter we have introduced the fundamental assumptions and conventions of a social accounting matrix (SAM) and examined how it relates to the system of national accounts and input–output analysis. We explored the essential additional information provided in SAMs compared with input–output tables in terms of a more detailed accounting of the characteristics of the households and labor, government taxation and welfare transfers, and allocations of income. Finally, we examined the concept of SAM multipliers and how they relate to traditional input–output multipliers.

Problems

11.1 Consider a macro economy represented by Figure 11.2, Construct a "macro-SAM" representation of this economy. What is the missing value for sales of exports, X? Show the SAM in two forms: (a) with the Final Consumers sector included as part of the Consumers sector and (2) with the Final Consumers sector included as a separately defined sector.

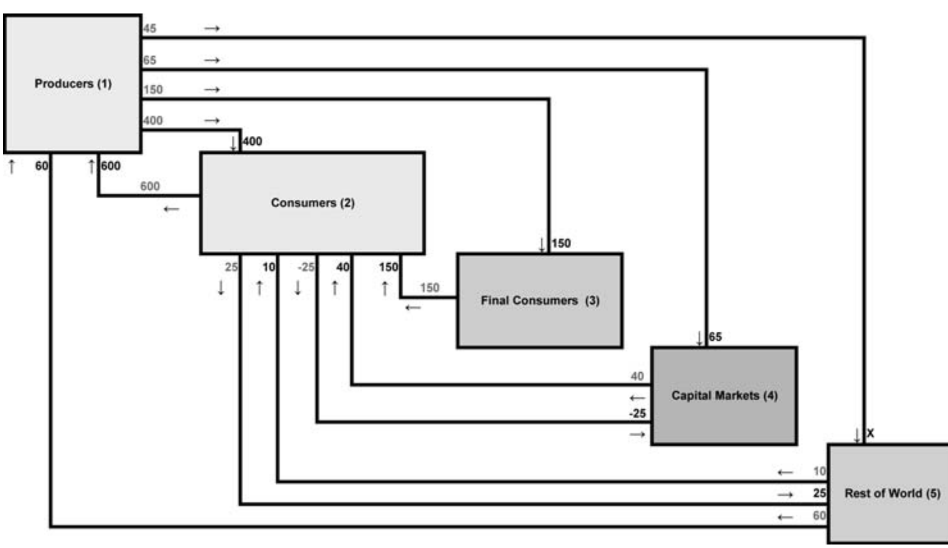

Figure 11.2 Sample Macroeconomy: Problem 11.1

11.2 For the economy depicted in problem 11.1, suppose the following input–output accounts are collected:

	Commodities		Industries		Final Demand	Totals	Grand Total
	Manuf.	Services	Manuf.	Services			
Commodities							
Manuf.			94	96	110	300	660
Services			94	117	148	360	
Industries							
Manuf.	295	0				295	660
Services	5	360				365	
Value Added			106	152	260		
Totals	300	360	295	365			
Grand Total	660		660				

Construct the "fully articulated" SAM, i.e., including the interindustry detail provided by these input–output accounts. Allocate final demand as part of consumer demand and assign commodity imports to v_{ii} for competitive imports to industry i. There is no unique solution.

11.3 For the fully articulated SAM in problem 11.2 expand the SAM to include sectors defined for consumer demand and exports. Again there is no unique solution, but the SAM must be balanced, i.e., row and column sums equal.

11.4 For the SAM developed in problem 11.3:

a. Compute the matrix of total expenditure shares.

b. Assume final-demand and value-added sectors are considered exogenous transactions to this economy. Compute the SAM coefficient matrix.

c. Compute the "direct effect" for this SAM.

11.5 Consider the following SAM for the developing nation of Sri Lanka:[6]

Sri Lanka SAM 1970	Value Added	Institutions	Indirect Taxes	Surplus/ Deficit	Production	Rest of World	Total
Value Added					11473		11473
Institutions	11360	2052	1368			3	14783
Indirect Taxes		389			885	94	1368
Surplus/Deficit		−425				425	0
Production		11312			4660	2113	18085
Rest of World	113	1455			1067		2635
Total	11473	14783	1368	0	18085	2635	

If we consider Surplus/Deficit and Rest of World as external to the SAM, compute the direct, indirect, cross, and total multipliers in the additive form.

6 Adapted from Pyatt and Round (1979), pp. 852–853.

11.6 Consider the unbalanced SAM given in the table below. Independent analysis indicates the total output of each sector; these are given in the additional column specified in the table. Use biproportional scaling to produce a balanced SAM with rows and columns both summing to the independent sector output estimates.

	Prod.	Cons.	Capital	ROW	Totals	Estimated Totals
Producers	0	600	65	45	710	660
Consumers	700	0	−25	15	690	600
Capital	0	40	0	0	40	40
Rest of World	50	10	0	0	60	60
Totals	750	650	40	60	1500	1360

11.7 For the unbalanced SAM given in problem 11.6, if in addition to the estimated totals we become aware that the elements $z_{23} = -25$, $z_{24} = 15$, and $z_{42} = 10$ in the balanced SAM are fixed, use the method of biproportional scaling fixing these selected elements to produce a balanced SAM.

11.8 Consider the following "macro-SAM" for the US economy for 1988[7]:

US SAM 1988	Prod.	Comm.	Labor	Property	Enterprises	Households	Govt.	Capital	Rest of World	Taxes	Errors	Total
Production		4831										4831
Commodities						3235	970	750	431			5386
Labor	2908											2908
Property	1556								117			1673
Enterprises				1589		95	93					1777
Households			2463		1045		556					4064
Government	377		445		138	587		96		18		1661
Capital					594	145			117		−10	846
Rest of World		537		84		2	42					665
Taxes		18										18
Errors & Omissions	−10											−10
Total	4831	5386	2908	1673	1777	4064	1661	846	665	18	−10	

If we consider the first five sectors as the endogenous sectors, compute the direct, indirect, cross, and total multipliers in their multiplicative form.

11.9 For the macro-SAM specified in problem 11.8, compute the direct, indirect, and total multipliers in their additive form. What do you notice about the direct multipliers in the additive form compared with the direct multipliers in their multiplicative form?

11.10 Consider the SAM for the USA (1988) expanded with interindustry detail shown in Table 11.22. If we consider the first nine sectors as the endogenous sectors, compute the total multipliers.

[7] As reported in Reinert and Roland-Holst (1992), pp. 173–187.

Table 11.22 SAM with Expanded Interindustry Detail for United States, 1988[a]

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	
	US SAM 1988 ($ billions)	Agric.	Mining	Const.	Nondur. Manuf	Durable Manuf.	Transp. & Util.	Trade	Finance	Services	Labor	Prop.	Enter-prises	House holds	Govt.	Capital	Rest of World	Tariffs	Errors	TOTAL
1	Agriculture	42	0	2	98	8	0	3	8	7	0	0	0	18	7	1	22	0	0	214
2	Mining	0	10	2	82	8	35	0	0	0	0	0	0	1	0	2	8	0	0	148
3	Construction	2	12	1	7	9	21	6	36	18	0	0	0	0	134	358	0	0	0	602
4	Nondurable Manuf.	30	1	35	370	83	37	24	14	149	0	0	0	453	38	4	93	0	0	1332
5	Durable Manuf.	4	3	175	55	480	19	7	4	81	0	0	0	236	97	296	187	0	0	1643
6	Transport & Utilities	5	1	17	66	65	78	46	31	84	0	0	0	310	34	13	26	0	0	774
7	Trade	8	1	72	57	73	11	14	7	50	0	0	0	529	11	56	43	0	0	932
8	Finance	10	3	10	18	25	14	52	20	79	0	0	0	771	16	22	25	0	0	1065
9	Services	5	1	53	68	74	31	124	93	214	0	0	0	917	632	0	27	0	0	2240
10	Labor	33	18	197	218	430	212	385	217	1198	0	0	0	0	0	0	0	0	0	2908
11	Property	60	56	32	142	69	207	147	511	332	0	0	0	0	0	0	117	0	0	1673
12	Enterprise	0	0	0	0	0	0	0	0	0	0	1589	0	96	92	0	0	0	0	1778
13	Households	0	0	0	0	0	0	0	0	0	2463	0	1046	0	556	0	0	0	0	4064
14	Government	8	12	7	28	18	35	127	113	30	445	0	138	587	0	96	0	16	0	1659
15	Capital	0	0	0	0	0	0	0	0	0	0	0	594	145	0	0	117	0	−10	846
16	Rest of World	8	31	0	115	295	75	0	12	2	0	83	0	2	42	0	0	0	0	665
17	Tariffs	0	0	0	8	8	0	0	0	0	0	0	0	0	0	0	0	0	0	16
18	Errors & Omissions	0	0	−1	−1	−1	−1	−1	−2	−2	0	0	0	0	0	0	0	0	0	−10
	TOTAL	214	148	602	1332	1643	774	932	1065	2240	2908	1673	1778	4064	1659	846	665	16	−10	

[a]As reported in Reinert and Roland-Holst (1992).

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12 Supply-Side Models, Linkages, and Important Coefficients

12.1 Supply Side Input–Output Models

12.1.1 The Early Interpretation

In 1958 Ghosh presented an alternative input–output model based on the same set of base-year data that underpin the demand-driven model in earlier chapters, namely $\mathbf{Z}$, $\mathbf{f}$, and $\mathbf{v}$, from which $\mathbf{x}$ follows as $\mathbf{x} = \mathbf{Zi}+\mathbf{f}$ or as $\mathbf{x}' = \mathbf{i}'\mathbf{Z} + \mathbf{v}'$. In the demand-driven model, direct input coefficients are defined in $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$, leading to $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f} = \mathbf{Lf}$. In this case the Leontief inverse relates sectoral gross outputs to the amount of final product (final demand) – that is, to a unit of product *leaving* the interindustry system at the end of the process. The alternative interpretation that Ghosh suggests relates sectoral gross production to the primary inputs – that is, to a unit of value *entering* the interindustry system at the beginning of the process.

This approach is made operational by essentially "rotating" or transposing our vertical (column) view of the model to a horizontal (row) one. Instead of dividing each *column* of $\mathbf{Z}$ by the gross output of the sector associated with that column, the suggestion is to divide each *row* of $\mathbf{Z}$ by the gross output of the sector associated with that row. We use $\mathbf{B}$ to denote the *direct-output coefficients* matrix that results.[1] For a two-sector example, this means

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} z_{11}/x_1 & z_{12}/x_1 \\ z_{21}/x_2 & z_{22}/x_2 \end{bmatrix} = \begin{bmatrix} 1/x_1 & 0 \\ 0 & 1/x_2 \end{bmatrix} \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} = \hat{\mathbf{x}}^{-1}\mathbf{Z} \tag{12.1}$$

These b_{ij} coefficients represent the distribution of sector i's outputs across sectors j that purchase interindustry inputs from i; these are frequently called *allocation* coefficients, as opposed to *technical* coefficients, a_{ij}. Using

$$\mathbf{x}' = \mathbf{i}'\mathbf{Z} + \mathbf{v}'$$

[1] Early presentations used $\overrightarrow{\mathbf{A}}$ for these coefficients and $\mathbf{A}\!\downarrow$ for the traditional demand-side coefficients, which we have denoted simply by $\mathbf{A}$. This served to make visually explicit the two points of view: $\mathbf{A}\!\downarrow$ resulting from uniform division of all elements in each column of $\mathbf{Z}$ by the associated column output, and $\overrightarrow{\mathbf{A}}$ resulting from division of all elements in each row of $\mathbf{Z}$ by the associated row output.

where $\mathbf{v}' = [v_1, \ldots, v_n]$ – this is (2.29) in Chapter 2 – and

$$\mathbf{Z} = \hat{\mathbf{x}}\mathbf{B} \tag{12.2}$$

from (12.1), we have

$$\mathbf{x}' = \mathbf{i}'\hat{\mathbf{x}}\mathbf{B} + \mathbf{v}' = \mathbf{x}'\mathbf{B} + \mathbf{v}' \tag{12.3}$$

since $\mathbf{i}'\hat{\mathbf{x}} = \mathbf{x}'$. From this,

$$\mathbf{x}' = \mathbf{v}'(\mathbf{I} - \mathbf{B})^{-1} \tag{12.4}$$

Define

$$\mathbf{G} = (\mathbf{I} - \mathbf{B})^{-1} \tag{12.5}$$

with elements g_{ij}. This has been called the *output inverse*, in contrast to the usual Leontief inverse, $\mathbf{L} = [l_{ij}] = (\mathbf{I} - \mathbf{A})^{-1}$ (the *input inverse*). Element g_{ij} has been interpreted as measuring "the total value of production that comes about in sector j per unit of primary input in sector i." (Augustinovics, 1970, p. 252.) Then, (12.4) is

$$\mathbf{x}' = \mathbf{v}'\mathbf{G} \tag{12.6}$$

In terms of *changes* in $\mathbf{v}$, we would find the associated output changes as

$$\Delta\mathbf{x}' = (\Delta\mathbf{v}')\mathbf{G} \tag{12.7}$$

As we have seen earlier with the Leontief price model (section 2.6), we can equally well transpose all elements so that the resulting vector of gross outputs is a column rather than a row. In that case, (12.3) will be

$$\mathbf{x} = \mathbf{B}'\mathbf{x} + \mathbf{v} \tag{12.8}$$

from which

$$\mathbf{x} = (\mathbf{I} - \mathbf{B}')^{-1}\mathbf{v} \tag{12.9}$$

Since[2] $\mathbf{G}' = (\mathbf{I} - \mathbf{B}')^{-1}$, (12.9) is

$$\mathbf{x} = \mathbf{G}'\mathbf{v} \tag{12.10}$$

This is the version of the model that we will use in what follows. However, many analysts use the form in (12.6) and (12.7). Again, in terms of *changes* in $\mathbf{v}$ we would have

$$\Delta\mathbf{x} = \mathbf{G}'(\Delta\mathbf{v}) \tag{12.11}$$

The basic assumption of the supply-side approach is that the output distributions in b_{ij} are stable in an economic system, meaning that if output of sector i is, say, doubled, then the sales from i to each of the sectors that purchase from i will also be doubled. Instead of fixed input coefficients, fixed output coefficients are assumed in the supply-side model.

[2] This follows from matrix algebra results that $(\mathbf{A} \pm \mathbf{B})' = \mathbf{A}' \pm \mathbf{B}'$ and $(\mathbf{A}')^{-1} = (\mathbf{A}^{-1})'$.

For sector j in the n-sector case, from (12.10) we have

$$x_j = v_1 g_{1j} + \cdots + v_i g_{ij} + \cdots + v_n g_{nj} \tag{12.12}$$

Recall the typical equation in the solution to the demand-driven model, from (2.15) in Chapter 2:

$$x_i = l_{i1} f_1 + \cdots + l_{ij} f_j + \cdots + l_{in} f_n$$

The effect on output of sector i, Δx_i, of a \$1.00 change in final demand for sector j goods ($\Delta f_j = 1$), is given by l_{ij}. (Again, for readers who are familiar with differential calculus, $\partial x_i / \partial f_j = l_{ij}$.) Column sums of $\mathbf{L} = [l_{ij}]$ were seen (Chapter 6) to be output multipliers; $\sum_{i=1}^{n} l_{ij}$ denotes the total new output throughout all n sectors of the economy that is associated with a \$1.00 increase in final demand for sector j. Row sums of $\mathbf{L}$ can also be interpreted; $\sum_{j=1}^{n} l_{ij}$ shows the total new sector i intermediate sales to all sectors that would be needed if there were a \$1.00 increase in the final demands for the outputs of *each* of the n sectors in the economy.

From (12.12), the effect on sector j output, Δx_j, of a \$1.00 change in the availability of primary inputs to sector i ($\Delta v_i = 1$) is given by g_{ij}. (In calculus terms, $\partial x_j / \partial v_i = g_{ij}$; note that the order of the subscripts in this partial derivative is the opposite of that for l_{ij}). For example, if $g_{ij} = 0.67$, this has been interpreted to mean that if there is \$1.00 less labor available to sector i as an input to production (due, say, to a strike), then the amount of reduction in sector j output will be \$0.67. The reduction comes about because, in the input–output framework, a decrease in the available labor to sector i means a decrease in sector i output and hence in the outputs of all sectors that depend on sector i's product as an input to their own production processes. This represents the same kind of effect, originating in an exogenous supply change, as is captured in the usual input–output system, which responds to exogenous demand changes.

In this (early) view of the Ghosh model, row and column sums in the output inverse, $\mathbf{G} = (\mathbf{I} - \mathbf{B})^{-1} = [g_{ij}]$ were given interpretations that parallel those in the Leontief quantity model. Row sums, $\sum_{j=1}^{n} g_{ij} = g_{i1} + \cdots + g_{in}$ $(=\partial x_1/\partial v_i + \cdots + \partial x_n/\partial v_i)$, were taken to represent the effect on total output throughout all sectors of the economy that would be associated with a \$1.00 change in primary inputs for sector i. This is the supply-side model's analog to an output (or demand) multiplier – a *column* sum in $\mathbf{L}$. These supply model *row* sums were termed input (or supply) multipliers. Also, column sums, $\sum_{i=1}^{n} g_{ij} = g_{1j} + \cdots + g_{nj}$ $(= \partial x_j/\partial v_1 + \cdots + \partial x_j/\partial v_n)$, were interpreted as the total effect on sector j output if there were a \$1.00 change in the supply of primary factors for *each* of the n sectors in the economy. These *column* sums were the supply-side model's parallel to the *row* sums of $\mathbf{L}$ in the demand model.

Numerical Illustration (Hypothetical Data) Let

$$\mathbf{Z} = \begin{bmatrix} 225 & 600 & 110 \\ 250 & 125 & 425 \\ 325 & 700 & 150 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1200 \\ 2000 \\ 1500 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} 265 \\ 1200 \\ 325 \end{bmatrix}$$

Then

$$\mathbf{B} = \hat{\mathbf{x}}^{-1}\mathbf{Z} = \begin{bmatrix} 1/1200 & 0 & 0 \\ 0 & 1/2000 & 0 \\ 0 & 0 & 1/1500 \end{bmatrix} \begin{bmatrix} 225 & 600 & 110 \\ 250 & 125 & 425 \\ 325 & 700 & 150 \end{bmatrix}$$

$$= \begin{bmatrix} .188 & .5 & .092 \\ .125 & .063 & .213 \\ .217 & .467 & .1 \end{bmatrix}$$

and

$$\mathbf{G} = (\mathbf{I} - \mathbf{B})^{-1} = \begin{bmatrix} 1.484 & .982 & .383 \\ .316 & 1.418 & .367 \\ .521 & .971 & 1.394 \end{bmatrix} \text{ and } \mathbf{G}' = \begin{bmatrix} 1.484 & .316 & .521 \\ .982 & 1.418 & .971 \\ .383 & .367 & 1.394 \end{bmatrix}$$

Thus, for example, if there were \$100 less labor available for sector 1 production and \$300 less for both sector 2 and sector 3 production, we would find, as in (12.11),

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} 1.484 & .316 & .521 \\ .982 & 1.418 & .971 \\ .383 & .367 & 1.394 \end{bmatrix} \begin{bmatrix} -100 \\ -300 \\ -300 \end{bmatrix} = \begin{bmatrix} -399.53 \\ -815.06 \\ -566.47 \end{bmatrix}$$

These figures, $\Delta x_1 = -400$, $\Delta x_2 = -815$ and $\Delta x_3 = -566$, would then be interpreted as the amounts by which the outputs of the three sectors would be reduced, given the decreases in labor inputs to the sectors.

If $\Delta v_1 = 1$ and $\Delta v_2 = \Delta v_3 = 0$,

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} 1.484 \\ .982 \\ .383 \end{bmatrix}$$

These figures represent the total additional outputs possible in each of the three sectors due to the availability of one more unit of primary inputs to sector 1. If $\Delta v_1 = -1$ and $\Delta v_2 = \Delta v_3 = 0$, these numbers will be negative, representing reduced output in the sectors. As suggested above, the sum of the elements in row 1 of $\mathbf{G}$ [column 1 of $\mathbf{G}'$], 2.849, represents the total potential impact throughout the economy of a \$1.00 change in the availability of primary inputs to sector 1. Again, as was noted above, this is parallel to the concept of the output multiplier for sector 1 in the ordinary, demand-driven input–output model. It is, in the context of this supply-side model, an *input* multiplier for sector 1. Similarly this kind of input multiplier for sector 2 is 2.101 and for sector 3 it is 2.886. In this view of the supply-side model, one might use these figures to decide where an additional dollar's worth of provision of primary resources (labor, etc.) would be most beneficial to the total economy, in terms of potential for supporting expanded output. Conversely, these input multipliers can indicate the potential contracting effects of shortages in primary inputs to a particular sector. From this point of view, a reduction by \$1.00 in the availability of a scarce resource could lead to a reduction in economy-wide output of \$2.849, \$2.101, or \$2.886, depending on where the primary input reduction occurs.

Numerical Application (US Data) Giarratani (1978) presents an application of the Ghosh model. He calculated output coefficients, **B**, and the associated output inverse matrix, **G**, using 78-sector 1967 US data. Supply multipliers ranged from a high of 4.01 for iron and ferroalloy-ores mining to 1.09 for medical and educational services and nonprofit organizations. With rankings of sectors such as this, it is possible to determine where primary factor constraints would have the greatest potential for limiting aggregate economic output – for example, a contemplated labor strike in one or more sectors.

Looking down the jth column of **G** allows one to identify supply linkages that have potential for significantly limiting the output of sector j. Among others, Giarratani considered an energy sector, petroleum refining and related industries (sector 31, the only secondary energy sector in the 78-sector 1967 US table). Examination of column 31 in the output inverse identifies the following among the largest coefficients: for sector 8, crude petroleum and natural gas, $g_{8,31} = 0.8605$; for sector 27, chemicals and chemical products, $g_{27,31} = 0.0513$; and for sector 12, maintenance and repair construction, $g_{12,31} = 0.0504$. The suggested interpretation is that interruptions in primary inputs to these sectors have the largest potential for disruptions in refined petroleum output.

Other examples of this kind of empirical analysis using the Ghosh model include Chen and Rose (1986) on the role of bauxite as a critical input in the Taiwanese economy and Davis and Salkin (1984) on the importance of water as an input in a county in California.

12.1.2 Relationships between A *and* B *and between* L *and* G

Given $\mathbf{A} = \mathbf{Z}\hat{\mathbf{x}}^{-1}$ and $\mathbf{B} = \hat{\mathbf{x}}^{-1}\mathbf{Z}$, $\mathbf{Z} = (\hat{\mathbf{x}})\mathbf{B}$; putting this into the definition of **A**,

$$\mathbf{A} = \hat{\mathbf{x}}\mathbf{B}\hat{\mathbf{x}}^{-1} \tag{12.13}$$

(When two matrices, **P** and **Q**, are connected by the relation $\mathbf{P} = \mathbf{M}\mathbf{Q}\mathbf{M}^{-1}$, they are said to be *similar*; this is denoted $\mathbf{P} \sim \mathbf{Q}$. Thus we see that **A** and **B** are similar matrices.) Of course, it also follows straightforwardly that

$$\mathbf{B} = \hat{\mathbf{x}}^{-1}\mathbf{A}\hat{\mathbf{x}} \tag{12.14}$$

Recall in section 6.6.2 on elasticities that element (i, j) in the matrix $\hat{\mathbf{x}}^{-1}\mathbf{A}\hat{\mathbf{x}}$ was shown to capture the *direct* effect on industry i's output (percentage change) resulting from a one percent change in industry j's output. This was termed a *direct output-to-output* elasticity. Hence, from (12.14), these elasticities are precisely the elements in $\mathbf{B} = [b_{ij}]$.[3]

Consider $(\mathbf{I} - \mathbf{A})$. From (12.13), $(\mathbf{I} - \mathbf{A}) = \mathbf{I} - \hat{\mathbf{x}}\mathbf{B}\hat{\mathbf{x}}^{-1}$. Since $\hat{\mathbf{x}}\mathbf{I}\hat{\mathbf{x}}^{-1} = \mathbf{I}$,

$$(\mathbf{I} - \mathbf{A}) = \hat{\mathbf{x}}(\mathbf{I} - \mathbf{B})\hat{\mathbf{x}}^{-1}$$

That is, $(\mathbf{I} - \mathbf{A}) \sim (\mathbf{I} - \mathbf{B})$. Using a basic result on the inverse of a product of matrices – $(\mathbf{PQR})^{-1} = \mathbf{R}^{-1}\mathbf{Q}^{-1}\mathbf{P}^{-1}$ – we find that since $(\mathbf{I} - \mathbf{A})^{-1} = [\hat{\mathbf{x}}(\mathbf{I} - \mathbf{B})\hat{\mathbf{x}}^{-1}]^{-1}$

$$(\mathbf{I} - \mathbf{A})^{-1} = \hat{\mathbf{x}}(\mathbf{I} - \mathbf{B})^{-1}\hat{\mathbf{x}}^{-1} \tag{12.15}$$

or

$$\mathbf{L} = \hat{\mathbf{x}}\mathbf{G}\hat{\mathbf{x}}^{-1} \tag{12.16}$$

Thus $\mathbf{L} \sim \mathbf{G}$. [The interested reader might confirm these similarity relationships in (12.13) and (12.15) for the small numerical illustration above.] The results in (12.16) can equally well be written as

$$\mathbf{G} = \hat{\mathbf{x}}^{-1}\mathbf{L}\hat{\mathbf{x}} \tag{12.17}$$

Again referring to section 6.6.2, we saw that element (i, j) in the matrix $\hat{\mathbf{x}}^{-1}\mathbf{L}\hat{\mathbf{x}}$ gives the percent increase in industry i total output due to an initial exogenous one percent increase in industry j output – the *total output-to-output elasticity* of industry i output with respect to output in industry j. From (12.17) these elasticities are exactly the elements in $\mathbf{G} = [g_{ij}]$.

From these results it is clear that any measures defined for $\mathbf{A}$ – such as output multipliers or backward linkages (section 12.2.1) – can be found from $\mathbf{B}$, provided that $\mathbf{x}$ is also known. Conversely, input multipliers or forward linkages (section 12.2.2) – defined on $\mathbf{B}$ – can be found using $\mathbf{A}$ and $\mathbf{x}$.[4]

12.1.3 Comments on the Early Interpretation

An early application of the Ghosh model is to be found in Augustinovics (1970), where direct-input coefficients ($\mathbf{A}$) and direct-output coefficients ($\mathbf{B}$) are compared for a number of countries and over time. However, reservations to this model began to appear in the early 1980s – for example in Giarratani (1980, 1981). The issue is: essentially what kind of economic behavior is represented by a system with constant supply distribution patterns? Ghosh had in mind the context of a planned economy experiencing severe excess demand, with government-imposed restrictions on supply patterns. This is probably not a very general situation in much of the modern world. However, Giarratani (1981, p. 283) suggested a possibly broader context:

[3] Using this interpretation, de Mesnard (2001) refers to the a_{ij} and b_{ij} coefficients as reflecting the *absolute* and *relative* direct influence of sector j on sector i, respectively.

[4] It is easily shown that $\mathbf{A}$ and $\mathbf{B}$ have the same main diagonal elements; the same is true for $\mathbf{L}$ and $\mathbf{G}$. Using $\hat{\mathbf{M}}$ to denote the diagonal matrix whose elements are the main diagonal of a square matrix $\mathbf{M}$, $\hat{\mathbf{A}} = \hat{\mathbf{x}}\hat{\mathbf{B}}\hat{\mathbf{x}}^{-1} = \hat{\mathbf{B}}$, from (12.13), since order of multiplication of diagonal matrices makes no difference and $\hat{\mathbf{x}}\hat{\mathbf{x}}^{-1} = \mathbf{I}$. Exactly the same line of argument shows that $\hat{\mathbf{L}} = \hat{\mathbf{G}}$.

More interesting perhaps is the prospect that this behavior may be the result of voluntary supply decisions in the same context or, alternatively, given the disruption of some basic commodity. Firms may well attempt to maintain their existing markets … by allocating available product on the basis of deliveries in more normal times. Casual evidence on the U.S. experience would seem to support this hypothesis.

It was in this spirit that the application noted above (Giarratani, 1978) was carried out.

Oosterhaven (1980) raised reservations about the plausibility of the Ghosh model and then in the late 1980s a more vigorous exchange took place, particularly in Oosterhaven (1988, 1989), Gruver (1989) and Rose and Allison (1989). In essence, the problem is that primary input increases in sector j are transmitted forward in the Ghosh model to output increases in all sectors that buy from j, without any corresponding increases in primary input use in those sectors. This is because $\Delta\mathbf{v}$ is viewed as exogenous and (in this example) is fixed at $\Delta\mathbf{v}' = [0,\ldots,0,\Delta v_j,0,\ldots,0]$. This wreaks havoc with the notion of sectoral production functions where material inputs *plus primary inputs* are used in fixed proportions.

12.1.4 Joint Stability

The Issue When the demand-driven input–output model is used in standard fashion for impact analysis – as in $\Delta\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\Delta\mathbf{f}$ – a crucial assumption is that the direct-input coefficients matrix, $\mathbf{A}$, remains constant. As a consequence of the connections between $\mathbf{A}$ and $\mathbf{B}$, or between $\mathbf{L}$ and $\mathbf{G}$, this means that *in general* $\mathbf{B}$ (and therefore $\mathbf{G}$) *cannot* remain constant. This came to be known as the "joint stability" problem.[5] A numerical example illustrates the problem nicely. From the data for the three-sector hypothetical illustration in section 12.1.1, we also find[6]

$$\mathbf{A} = \begin{bmatrix} .188 & .3 & .073 \\ .208 & .063 & .283 \\ .271 & .35 & .1 \end{bmatrix} \text{ and } \mathbf{L} = \begin{bmatrix} 1.484 & .589 & .306 \\ .527 & 1.418 & .489 \\ .651 & .729 & 1.394 \end{bmatrix}$$

It will be useful at this point to use superscripts "0" to represent the base-year data, i.e., the given $\mathbf{A}$, $\mathbf{B}$, $\mathbf{L}$, and $\mathbf{G}$ matrices, as well as the initial output, $\mathbf{x}$, will be denoted $\mathbf{A}^0$, $\mathbf{B}^0$ and so forth. Vectors and matrices that result from some exogenous change will be given superscripts "1". Suppose, for illustration, $\Delta\mathbf{f} = \begin{bmatrix} 100 \\ 40 \\ 30 \end{bmatrix}$; using the demand-driven model – $\Delta\mathbf{x} = \mathbf{L}^0\Delta\mathbf{f}$ – we find $\Delta\mathbf{x}(d) = \begin{bmatrix} 181.166 \\ 124.057 \\ 136.095 \end{bmatrix}$ and $\mathbf{x}^1(d) = \begin{bmatrix} 1381.2 \\ 2124.1 \\ 1636.1 \end{bmatrix}$. [We use (d) to indicate that these are results from the demand-driven model, in which a constant $\mathbf{A}$ matrix is assumed.] From these results, we find the new transactions matrix

[5] See, among others, Dietzenbacher (1989), Miller (1989), Rose and Allison (1989), and Chen and Rose (1991).

[6] These matrices, along with $\mathbf{B}$ and $\mathbf{G}$ in section 12.1.1, illustrate the relationships shown in footnote 2, above.

associated with **A** and the new outputs; namely,

$$\mathbf{Z}^1(d) = \mathbf{A}^0[\hat{\mathbf{x}}^1(d)] = \begin{bmatrix} 258.969 & 637.217 & 119.980 \\ 287.743 & 132.754 & 463.560 \\ 374.066 & 743.420 & 163.610 \end{bmatrix}$$

[The reader can easily check that $\mathbf{Z}^1(d)\mathbf{i} + \mathbf{f}^1 = \mathbf{x}^1(d)$.] The direct-output coefficients matrix associated with these new transactions and new total outputs is found, as in (12.1), as

$$\mathbf{B}^1 = [\hat{\mathbf{x}}^1(d)]^{-1}\mathbf{Z}^1(d) = \begin{bmatrix} .188 & .461 & .087 \\ .136 & .063 & .218 \\ .229 & .454 & .1 \end{bmatrix}$$

Recall from above that

$$\mathbf{B}^0 = \begin{bmatrix} .188 & .5 & .092 \\ .125 & .063 & .213 \\ .217 & .467 & .1 \end{bmatrix}$$

and clearly $\mathbf{B}^1 \neq \mathbf{B}^0$. [One simple measure of the difference is the average of all of the (absolute) percentage differences – $(1/n^2)\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left|b_{ij}^0 - b_{ij}^1\right|/b_{ij}^0\right] \times 100$. Here this is 3.58 percent.] The upshot is that, at least in this example (but actually in general), the assumption of a constant **A** matrix, used in an impact analysis, carries with it the requirement that **B** change as a result of the impact.

An exactly similar problem occurs if one uses the supply-driven model to assess the impact of a change in primary inputs. For example, from the data for this three-sector example, $\mathbf{v}' = \begin{bmatrix} 400 & 575 & 815 \end{bmatrix}$. Suppose that $(\Delta\mathbf{v})' = \begin{bmatrix} 50 & 100 & 20 \end{bmatrix}$; using (12.11) to assess the output effects of this change in primary inputs, we find

$[\Delta\mathbf{x}(s)]' = \begin{bmatrix} 116.221 & 210.325 & 83.720 \end{bmatrix}$ and $[\mathbf{x}^1(s)]' = \begin{bmatrix} 1316.2 & 2210.3 & 1583.7 \end{bmatrix}$ [Now (s) denotes results from the supply-driven model.] Parallel to the demand-driven example, there is now a new transactions matrix,

$$\mathbf{Z}^1(s) = [\hat{\mathbf{x}}^1(s)]\mathbf{B}^0 = \begin{bmatrix} 246.792 & 658.111 & 120.654 \\ 276.291 & 138.145 & 469.694 \\ 343.139 & 739.069 & 158.372 \end{bmatrix}$$

In conjunction with the associated $\mathbf{x}^1(s)$, this $\mathbf{Z}^1(s)$ defines the corresponding direct-input coefficients matrix, $\mathbf{A}^1$, namely

$$\mathbf{A}^1 = \mathbf{Z}^1(s)[\hat{\mathbf{x}}^1(s)]^{-1} = \begin{bmatrix} .188 & .3 & .076 \\ .210 & .063 & .3 \\ .261 & .334 & .1 \end{bmatrix}$$

Originally,

$$\mathbf{A}^0 = \begin{bmatrix} .188 & .3 & .073 \\ .208 & .063 & .283 \\ .271 & .35 & .1 \end{bmatrix}$$

and $\mathbf{A}^1 \neq \mathbf{A}^0$. (In this case, the average absolute difference is 2.06 percent.)[7]

This apparent inconsistency – the fact that the requirement of a constant $\mathbf{A}$ (for demand-driven model impact analysis) implies a non-constant $\mathbf{B}$ in the related supply-driven model or that the constant $\mathbf{B}$ needed for supply-driven model impact analysis carries with it the implication of a non-constant $\mathbf{A}$ in the related demand-driven model – led to several empirical studies on relative joint stability (see, for example, Rose and Allison, 1989, or Chen and Rose, 1991). In general, the conclusion drawn was that instability in actual empirical applications was not a major issue.

Conditions under which both A *and* B *will be Stable* Assume that we have found the new outputs resulting from new final demands using the demand-driven model – $\mathbf{x}^1 = (\mathbf{I} - \mathbf{A}^0)\mathbf{f}^1$, so $\mathbf{A}^1 = \mathbf{A}^0$. From (12.14),

$$\mathbf{B}^1 = (\hat{\mathbf{x}}^1)^{-1}\mathbf{A}^0\hat{\mathbf{x}}^1$$

and substituting $\mathbf{A}^0$ from (12.13)

$$\mathbf{B}^1 = (\hat{\mathbf{x}}^1)^{-1}\hat{\mathbf{x}}^0\mathbf{B}^0(\hat{\mathbf{x}}^0)^{-1}\hat{\mathbf{x}}^1$$

Let $\hat{\mathbf{e}} = \hat{\mathbf{x}}^1(\hat{\mathbf{x}}^0)^{-1}$ where $e_i = x_i^1/x_i^0$ can be thought of as a kind of "growth rate" for sector i (remember that order of multiplication makes no difference when the matrices are diagonal); then

$$\mathbf{B}^1 = \hat{\mathbf{e}}^{-1}\mathbf{B}^0\hat{\mathbf{e}}$$

A similar story holds if the supply-driven model is used, with $\mathbf{B}^1 = \mathbf{B}^0$; namely

$$\mathbf{A}^1 = \hat{\mathbf{e}}\mathbf{A}^{\mathbf{0}}\hat{\mathbf{e}}^{-1}$$

If each sector's output changes at the same rate – $e_i = x_i^1/x_i^0 = \lambda$ for all i – then $\hat{\mathbf{e}} = \lambda\mathbf{I}$ and $\mathbf{B}^1 = [(1/\lambda)\mathbf{I}]\mathbf{B}^0(\lambda\mathbf{I}) = \mathbf{B}^0$. A similar argument shows that $\mathbf{A}^1 = \mathbf{A}^0$ under the same conditions, after an impact analysis with the supply-driven model.[8]

12.1.5 Reinterpretation as a Price Model

In order to overcome the criticisms and implausibilities in the original view of the Ghosh model, Dietzenbacher (1997) proposed an alternative interpretation by suggesting that the model be viewed not as a *quantity* model but as a *price* model (see also extensive discussions on alternative interpretations of the Ghosh model in Oosterhaven, 1996 and de Mesnard, 2007). We illustrate the idea by looking again at results from the numerical example in the previous section. Specifically, for

$$(\mathbf{v}^1)' = (\mathbf{v}^0)' + (\Delta\mathbf{v})' = \begin{bmatrix}400 & 575 & 815\end{bmatrix} + \begin{bmatrix}50 & 100 & 20\end{bmatrix} = \begin{bmatrix}450 & 675 & 835\end{bmatrix}$$

[7] For exactly the same reasons as shown in footnote 3, above, $\hat{\mathbf{A}}^0 = \hat{\mathbf{A}}^1$ and $\hat{\mathbf{B}}^0 = \hat{\mathbf{B}}^1$. These relationships are illustrated by the matrices in this section.

[8] For much more detail on these matters see Dietzenbacher (1989, 1997).

we found, using $\mathbf{x}^1(s) = (\mathbf{G}^0)'\mathbf{v}^1$ [(12.10)],

$$[\mathbf{x}^1(s)]' = \begin{bmatrix} 1316.2 & 2210.3 & 1583.7 \end{bmatrix}$$

Suppose that we view the elements in the supply-driven model not as *quantities* (in which case elements in $\Delta\mathbf{v}$ are interpreted as changes in the *amounts* of primary inputs available to the economy and elements in $\Delta\mathbf{x}$ are interpreted in changes in *quantities* produced) but rather as *values* (in which case elements in $\Delta\mathbf{v}$ reflect changes in the *prices* or *costs* of primary inputs and elements in $\Delta\mathbf{x}$ reflect changes in the *values* of outputs). In the demand-driven model of earlier chapters all prices are assumed fixed in an impact analysis and quantities change as a result of changes in the quantities of final demands. Now we assume that all quantities are fixed and use the Ghosh model to assess the repercussions throughout the economy of changes in primary input prices. In that reinterpretation, we can use the term *Ghosh price model*, which can reasonably be looked upon as a *cost-push input–output model*. Changes in primary input costs are transmitted throughout the economy as they are passed on (completely) by producers in the prices of their products that are purchased by other intermediate users, who in turn increase their prices accordingly, etc.

With this interpretation, we identify the *relative* price changes easily as the ratios of elements in $\mathbf{x}^0$ to those in $\mathbf{x}^1(s)$, since *quantities* are fixed and only *valuations* change. Define $\boldsymbol{\pi}$ as the vector of these price ratios,

$$\boldsymbol{\pi} = (\hat{\mathbf{x}}^0)^{-1}[\mathbf{x}^1(s)] \tag{12.18}$$

where $\pi_j = x_j^1(s)/x_j^0 = p_j^1 q_j^0/p_j^0 q_j^0 = p_j^1/p_j^0$ (where q_j^0 is a physical measure of the output of sector j in the base period). For this three-sector example,

$$\boldsymbol{\pi} = \begin{bmatrix} x_1^1(s)/x_1^0 \\ x_2^1(s)/x_2^0 \\ x_3^1(s)/x_3^0 \end{bmatrix} = \begin{bmatrix} 1316.2/1200 \\ 2210.3/2000 \\ 1583.7/1500 \end{bmatrix} = \begin{bmatrix} 1.0968 \\ 1.1052 \\ 1.0558 \end{bmatrix} \tag{12.19}$$

This indicates that (unit) prices of the products of sectors 1, 2, and 3 would rise by 9.68, 10.52, and 5.58 percent, respectively, in response to primary input cost increases of 12.5 [=(50/400) ×100] percent, 17.39 [= (100/575) ×100] percent and 2.45 [=(20/815) ×100] percent, for the three sectors respectively.

Similarly, when $\Delta v_1 = 1$ and $\Delta v_2 = \Delta v_3 = 0$, we found $\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} 1.484 \\ .982 \\ .383 \end{bmatrix}$ so $\mathbf{x}^1(s) = \begin{bmatrix} 1201.48 \\ 2000.98 \\ 1500.38 \end{bmatrix}$. This can now be interpreted in terms of price ratios for the three sectors of

$$\boldsymbol{\pi} = \begin{bmatrix} 1201.48/1200 \\ 2000.98/2000 \\ 1500.38/1500 \end{bmatrix} = \begin{bmatrix} 1.0012 \\ 1.0005 \\ 1.0003 \end{bmatrix}$$

This says that prices would be expected to increase by 0.12, 0.05, and 0.03 percent in the three sectors, respectively, in the face of a 0.25 percent [=(401/400)×100] increase in the cost of primary inputs to sector 1 only.

Connection to the Leontief Price Model (Algebra) It is straightforward to show that the Ghosh price model and the Leontief price model (section 2.6) generate exactly the same results. The Ghosh price model finds

$$\boldsymbol{\pi} = (\hat{\mathbf{x}}^0)^{-1}[\mathbf{x}^1(s)]$$

Since $[\mathbf{x}^1(s)] = (\mathbf{G}^0)'(\mathbf{v}^1)$[(12.10)],

$$\boldsymbol{\pi} = (\hat{\mathbf{x}}^0)^{-1}(\mathbf{G}^0)'\mathbf{v}^1$$

From $\mathbf{G} = \hat{\mathbf{x}}^{-1}\mathbf{L}\hat{\mathbf{x}}$ in (12.17), $\mathbf{G}' = \hat{\mathbf{x}}\mathbf{L}'\hat{\mathbf{x}}^{-1}$, and so we have

$$\boldsymbol{\pi} = (\hat{\mathbf{x}}^0)^{-1}[\hat{\mathbf{x}}^0(\mathbf{L}^0)'(\hat{\mathbf{x}}^0)^{-1}]\mathbf{v}^1 = (\mathbf{L}^0)'(\hat{\mathbf{x}}^0)^{-1}\mathbf{v}^1$$

Finally, since primary input *coefficients* are found as $v_{ci}^1 = v_i^1/x_i^0$, or $\mathbf{v}_c^1 = (\hat{\mathbf{x}}^0)^{-1}\mathbf{v}_1$,

$$\boldsymbol{\pi} = (\mathbf{L}^0)'(\hat{\mathbf{x}}^0)^{-1}\hat{\mathbf{x}}^0\mathbf{v}_c^1 = (\mathbf{L}^0)'\mathbf{v}_c^1 \tag{12.20}$$

In the Leontief price model of section 2.6, it is also the case that primary input price changes generate relative price changes [as in (2.33), which is repeated below]:

$$\tilde{\mathbf{p}} = [\mathbf{I} - (\mathbf{A}^0)']^{-1}\mathbf{v}_c^1 = (\mathbf{L}^0)'\mathbf{v}_c^1 \tag{12.21}$$

As (12.20) and (12.21) make clear, $\boldsymbol{\pi} = \tilde{\mathbf{p}}$. The Leontief price (cost-push) model (section 2.6) and the Ghosh price (cost-push) model generate the same results; the former directly in terms of the vector of relative price changes, $\tilde{\mathbf{p}}$, and the latter in terms of new outputs, $\mathbf{x}^1(s)$, from which $\boldsymbol{\pi}$ is found as the ratio of new to old output values.

Connection to the Leontief Price Model (Numerical Illustration) Using data from the hypothetical example in section 12.1.1 and 12.1.5, we find the base-year primary input coefficients as

$$\mathbf{v}_c^0 = \begin{bmatrix} 400/1200 \\ 575/2000 \\ 815/1500 \end{bmatrix} = \begin{bmatrix} .3333 \\ .2875 \\ .5433 \end{bmatrix}$$

As expected,

$$\tilde{\mathbf{p}}^0 = (\mathbf{L}^0)'\mathbf{v}_c^0 = \begin{bmatrix} 1.4840 & 0.5266 & 0.6514 \\ 0.5893 & 1.4179 & 0.7287 \\ 0.3064 & 0.4893 & 1.3936 \end{bmatrix}\begin{bmatrix} .3333 \\ .2875 \\ .5433 \end{bmatrix} = \begin{bmatrix} 1.0 \\ 1.0 \\ 1.0 \end{bmatrix}$$

verifies that all prices are one ("per dollar's worth of output") in the base-year Leontief model.

Now consider the primary input price increases from the example above, namely

$$(\mathbf{v}^1)' = (\mathbf{v}^0)' + (\Delta\mathbf{v})' = \begin{bmatrix} 400 & 575 & 815 \end{bmatrix} + \begin{bmatrix} 50 & 100 & 20 \end{bmatrix} = \begin{bmatrix} 450 & 675 & 835 \end{bmatrix}$$

In terms of primary input *coefficients*, we have

$$\mathbf{v}_c^1 = \begin{bmatrix} 450/1200 \\ 675/2000 \\ 835/1500 \end{bmatrix} = \begin{bmatrix} .3750 \\ .3375 \\ .5566 \end{bmatrix}$$

and using (12.21),

$$\tilde{\mathbf{p}}^1 = (\mathbf{L}^0)'\mathbf{v}_c^1 = \begin{bmatrix} 1.4840 & 0.5266 & 0.6514 \\ 0.5893 & 1.4179 & 0.7287 \\ 0.3064 & 0.4893 & 1.3936 \end{bmatrix} \begin{bmatrix} .3750 \\ .3375 \\ .5566 \end{bmatrix} = \begin{bmatrix} 1.0968 \\ 1.1051 \\ 1.0558 \end{bmatrix}$$

As expected, these are precisely the same results as we found above for $\boldsymbol{\pi}$ in (12.19).

Either exercise produces the result that price increases of 9.68, 10.51, and 5.58 percent are to be expected for the output of the three sectors as a result of the primary input cost increases given in $(\Delta\mathbf{v})' = \begin{bmatrix} 50 & 100 & 20 \end{bmatrix}$.

A Ghosh Quantity Model Thus far we have seen a Leontief quantity model, a Leontief price model, and a Ghosh price model. It is logical to expect that a Ghosh quantity model also exists (Dietzenbacher, 1997). From the familiar Leontief quantity model, $\mathbf{x}^1 = \mathbf{L}^0\mathbf{f}^1$, and $\mathbf{L}^0 = \hat{\mathbf{x}}^0\mathbf{G}^0(\hat{\mathbf{x}}^0)^{-1}$ [(12.16)], we have

$$\mathbf{x}^1 = \hat{\mathbf{x}}^0\mathbf{G}^0(\hat{\mathbf{x}}^0)^{-1}\mathbf{f}^1$$

Define the new final-demands as proportions (coefficients) of base-period outputs – $(f_c^1)_i = [f_i^1/x_i^0]$ and $\mathbf{f}_c^1 = (\hat{\mathbf{x}}^0)^{-1}\mathbf{f}^1$ – and premultiply both sides by $(\hat{\mathbf{x}}^0)^{-1}$,

$$\tilde{\mathbf{x}} = (\hat{\mathbf{x}}^0)^{-1}\mathbf{x}^1 = (\hat{\mathbf{x}}^0)^{-1}\hat{\mathbf{x}}^0\mathbf{G}^0(\hat{\mathbf{x}}^0)^{-1}\mathbf{f}^1 = \mathbf{G}^0\mathbf{f}_c^1$$

where $\tilde{x}_i = x_i^1/x_i^0$. In this case changes in final-demand *proportions* (of gross outputs) are translated into relative output measures; that is, an *index* showing new outputs, $\mathbf{x}^1$, as proportions of base-period outputs, $\mathbf{x}^0$.

This is the straightforward algebraic derivation of a Ghosh quantity model. The reader can explore the logic of the "story" behind it. Table 12.1 gathers together some of the relevant information about these four models. The quantity and price models – either Leontief or Ghosh – are often described as "dual" to each other[9], while the Leontief variant of the quantity model has been described as the "mirror image" of the Ghosh quantity model, and similarly for the Leontief and Ghosh price models. (Some of this material appeared earlier in Table 2.13.)

[9] There are some rather detailed mathematical discussions on what constitutes a pair of "dual" models. For our input–output models we simply take the term to mean that one model determines quantities (with prices fixed), the other determines prices (with quantities fixed) and the fundamental structural relationships (in $\mathbf{L}^0$ or in $\mathbf{G}^0$) are at the heart (although transposed) of each model and its dual.

Table 12.1 Overview of the Leontief and Ghosh Quantity and Price Models

Model		Leontief	Ghosh
Price (Cost-push) [Quantities fixed; prices change]	Exogenous Variables	$\mathbf{v}_c^1 = (\hat{\mathbf{x}}^0)^{-1}\mathbf{v}^1 = [v_j^1/x_j^0]$	$\mathbf{v}^1 = [v_j^1]$
	Endogenous Variables	$\tilde{\mathbf{p}}^1 = (\mathbf{L}^0)'\mathbf{v}_c^1$ $[\tilde{p}_i = x_i^1(d)/x_i^0]$	$\mathbf{x}^1(s) = (\mathbf{G}^0)'\mathbf{v}^1$
	Coefficient Stability	$\mathbf{A}^1 \neq \mathbf{A}^0$	$\mathbf{B}^1 = \mathbf{B}^0$
Quantity (Demand-pull) [Prices fixed; quantities change]	Exogenous Variables	$\mathbf{f}^1 = [f_i^1]$	$\mathbf{f}_c^1 = (\hat{\mathbf{x}}^0)^{-1}\mathbf{f}^1 = [f_i^1/x_i^0]$
	Endogenous Variables	$\mathbf{x}^1(d) = \mathbf{L}^0\mathbf{f}^1$	$\tilde{\mathbf{x}} = \mathbf{G}^0\mathbf{f}_c^1$ $[\tilde{x}_i = x_i^1(s)/x_i^0]$
	Coefficient Stability	$\mathbf{A}^1 = \mathbf{A}^0$	$\mathbf{B}^1 \neq \mathbf{B}^0$

12.2 Linkages in Input–Output Models

In the framework of an input–output model, production by a particular sector has two kinds of economic effects on other sectors in the economy. If sector j increases its output, this means there will be increased *demands* from sector j (as a purchaser) on the sectors whose goods are used as inputs to production in j. This is the direction of causation in the usual demand-side model, and the term *backward linkage* is used to indicate this kind of interconnection of a particular sector with those (“upstream”) sectors from which it purchases inputs. On the other hand, increased output in sector j also means that additional amounts of product j are available to be used as inputs to other sectors for their own production – that is, there will be increased *supplies* from sector j (as a seller) for the sectors that use good j in their production. This is the direction of causation in the supply-side model. The term *forward linkage* is used to indicate this kind of interconnection of a particular sector with those (“downstream”) sectors to which it sells its output.

Measures have been proposed to quantify such backward and forward linkages, or economic “connectedness.” Comparisons of the strengths of backward and forward linkages for the sectors in a single economy provide one mechanism for identifying “key” or “leading” sectors in that economy (those sectors that are most connected and therefore, in some sense, most “important”) and for grouping sectors into spatial clusters. And if data are available for more than one time period, the evolution of these interconnections can be studied. Also, examination of these measures for similar sectors in different countries provides one method of making international comparisons of the structure of production.

If the backward linkage of sector i is larger than that of sector j, one might conclude that a dollar's worth of expansion of sector i output would be more beneficial to the economy than would an equal expansion in sector j's output, in terms of the productive activity throughout the economy that would be generated by it. Similarly, if the forward linkage of sector r is larger than that of sector s, it could be said that a dollar's worth of expansion of the output of sector r is more essential to the economy than a similar expansion in the output of sector s, from the point of view of the overall productive activity that it could support.

There have been numerous suggestions for differing definitions and refinements of these linkage and key sector measures and others of economic connectedness. Early work includes Rasmussen (1957),[10] Hirschman (1958), Chenery and Watanabe (1958), Yotopoulos and Nugent (1973), Laumas (1975) and Jones (1976), and there has been, and continues to be, a good deal of discussion [for example, on the "proper" definition, see the debate among several authors in the May 1976 issue of the *Quarterly Journal of Economics*, or the Diamond (1976), Schultz and Schumacher (1976) and Laumas (1976a) exchange in *Kyklos*]. Questions on the exact role of linkage measures and the identification of key sectors in development planning have been raised in McGilvray (1977) and Hewings (1982), among others. Our purpose here is simply to introduce the reader to some of the most prevalent of these measures and, in particular, to indicate how they are derived from information in either the demand-side or the supply-side input–output model.

There also have been numerous suggestions for various ways of combining forward and backward linkage measures (examples can be found in Hübler, 1979; Loviscek, 1982; Meller and Marfán, 1981; Cella, 1984; Clements, 1990 and Adamou and Gowdy, 1990). Generally these combined measures have been superseded by the rankings that emerge from the "hypothetical extraction" approach, to which we turn in section 12.2.5.

12.2.1 Backward Linkage

In its simplest form, a measure of the strength of the backward linkage of sector j – the amount by which sector j production depends on interindustry inputs – is given by the sum of the elements in the jth column of the direct input coefficients matrix, namely $\sum_{i=1}^{n} a_{ij}$. Since the coefficients in $\mathbf{A}$ are measures of direct effects only, this is called the *direct* backward linkage:[11]

$$BL(d)_j = \sum_{i=1}^{n} a_{ij} \tag{12.22}$$

[10] Hirschman (1958) cites an edition of this book (same title) published by Einar Harcks in Copenhagen in 1956. This must be a precursor to the 1957 North-Holland edition (also under the Einar Harcks imprint) which is identified as a "second printing."

[11] It would be more consistent with standard vector-matrix notation to use some lower-case designation such as b_j for sector j's backward linkage (a scalar), but BL_j seems to have become standard.

In terms of transactions (**Z**, not **A**), this is simply the value of total intermediate inputs for sector j $\left(\sum_{i=1}^{n} z_{ij}\right)$ as a proportion of the value of j's total output (x_j). This definition, in transactions terms, was first proposed by Chenery and Watanabe (1958). If we define $\mathbf{b}(d) = [BL(d)_1, \ldots, BL(d)_n]$, then

$$\mathbf{b}(d) = \mathbf{i}'\mathbf{A} \tag{12.23}$$

To capture both direct and indirect linkages in an economy, column sums of the total requirements matrix, $\mathbf{L} = [l_{ij}]$, were proposed as a *total backward* linkage measure (Rasmussen, 1957); these are output multipliers (Chapter 6). For sector j we have

$$BL(t)_j = \sum_{i=1}^{n} l_{ij} \tag{12.24}$$

The corresponding row vector of direct and indirect backward-linkage measures for each sector is

$$\mathbf{b}(t) = \mathbf{i}'\mathbf{L} \tag{12.25}$$

There is some disagreement in the literature on whether the on-diagonal elements in **A** or **L** should be included or netted out of the summations (see, for example, Harrigan and McGilvray, 1988). To the extent that these "internal linkages" constitute part of Hirschman's (1958, p. 100) "… input-provision, derived demand … effects," they are appropriately included. On the other hand, if one is specifically interested in a sector's "backward dependence" on or linkage to the *rest* of the economy, they should be omitted.

Also, various normalizations of these measures have been proposed and used in empirical studies. For example, let

$$\overline{BL}(d)_j = \frac{BL(d)_j}{(1/n)\sum_{j=1}^{n} BL(d)_j} = \frac{\sum_{i=1}^{n} a_{ij}}{(1/n)\sum_{i=1}^{n} a_{ij} \sum_{j=1}^{n} a_{ij}}$$

(where the overbar suggests a normalized measure). In this case, sector j's backward linkage is divided by the (simple) average of all backward linkages. (Various weighted averages have also been suggested.) In (row) vector form, these normalized direct backward linkages are (note that $\mathbf{i}'\mathbf{A}\mathbf{i}$ is a scalar)

$$\bar{\mathbf{b}}(d) = \frac{\mathbf{i}'\mathbf{A}}{(\mathbf{i}'\mathbf{A}\mathbf{i})/n} = \frac{n\mathbf{i}'\mathbf{A}}{\mathbf{i}'\mathbf{A}\mathbf{i}} \tag{12.26}$$

The average value of $\bar{\mathbf{b}}(d)$ is unity – $[\bar{\mathbf{b}}(d)]\mathbf{i}(1/n) = \left[n\mathbf{i}'\mathbf{A}/\mathbf{i}'\mathbf{A}\mathbf{i}\right][\mathbf{i}/n] = 1$ – so that sectors with "above average" (stronger) direct backward linkages have indices that are greater than one and that those with "below average" (weaker) direct backward linkages have indices that are less than one. The same logic generates

$$\bar{\mathbf{b}}(t) = \frac{n\mathbf{i}'\mathbf{L}}{\mathbf{i}'\mathbf{L}\mathbf{i}} \tag{12.27}$$

as a normalized total backward linkage index, also with an average value of unity. (This is the "Index of the Power of Dispersion" suggested by Rasmussen, 1957.)

12.2.2 Forward Linkage

An early measure of *direct forward linkage* was also proposed, based on **A** and **L**, as the row sums **Ai**, along with an associated *total forward linkage* measure, the row sums **Li**.[12] Both of these have been viewed with skepticism, because they are generated by a peculiar stimulus – a simultaneous increase of one unit in the gross outputs of every sector in the case of **Ai** and an increase of one unit in the final demands of every sector in the case of **Li**.[13]

This dissatisfaction led to the suggestion that elements from the Ghosh model would be more appropriate as forward linkage measures (Beyers, 1976; Jones, 1976). The row sums **Bi** were suggested as better measures of *direct forward linkage*. In terms of transactions (**Z**, not **B**), this is simply the value of total intermediate sales by sector $i\left(\sum_{j=1}^{n} z_{ij}\right)$ as a proportion of the value of i's total output (x_i). (This also was first proposed in Chenery and Watanabe, 1958). In addition, row sums of the Ghosh inverse, $\mathbf{G} = [g_{ij}]$, were suggested as a better measure of *total forward linkages*. As with backward linkage measures, inclusion or exclusion of on-diagonal elements is an issue, and normalizations are usual.

Thus, the parallels to (12.22) and (12.24) for direct forward linkages are

$$FL(d)_i = \sum_{j=1}^{n} b_{ij} \tag{12.28}$$

and

$$FL(t)_i = \sum_{j=1}^{n} g_{ij} \tag{12.29}$$

In addition, the same two normalized versions for forward linkages can be found. Matrix expressions for all these results are collected in Table 12.2.

12.2.3 "Net" Backward Linkage

Another linkage measure was proposed by Dietzenbacher (2005) in his interpretation of the content of the Oosterhaven and Stelder net multiplier formulation (section 6.5.3).

12 In normalized form, $n\mathbf{L}\mathbf{i}/\mathbf{i}'\mathbf{L}\mathbf{i}$, this is Rasmussen's (1957) "Index of Sensitivity of Dispersion."

13 Among the first to make an issue of weightings in linkage measures was Laumas (1976b). Others before him (e.g., Hazari, 1970; Diamond, 1974), however, had used sets of weights other than unit vectors.

Table 12.2 Linkage Measures

	BL	FL	$\overline{BL}$	$\overline{FL}$
Direct	$\mathbf{i}'\mathbf{A}$	$\mathbf{Bi}$	$\dfrac{n\mathbf{i}'\mathbf{A}}{\mathbf{i}'\mathbf{Ai}}$	$\dfrac{n\mathbf{Bi}}{\mathbf{i}'\mathbf{Bi}}$
Total	$\mathbf{i}'\mathbf{L}$	$\mathbf{Gi}$	$\dfrac{n\mathbf{i}'\mathbf{L}}{\mathbf{i}'\mathbf{Li}}$	$\dfrac{n\mathbf{Gi}}{\mathbf{i}'\mathbf{Gi}}$

Start with the observation that $\mathbf{L}\hat{\mathbf{f}}$ is a matrix whose i, jth element represents output of i generated by f_j. Row sums of $\mathbf{L}\hat{\mathbf{f}}$ are given by $\mathbf{L}\hat{\mathbf{f}}\mathbf{i} = \mathbf{Lf} = \mathbf{x}$; the ith element of this column vector is simply x_i, the output of i generated by all final demands – the standard interpretation of $\mathbf{x}$. Column sums of $\mathbf{L}\hat{\mathbf{f}}$ are given by $\mathbf{i}'\mathbf{L}\hat{\mathbf{f}}$; the jth element of this row vector is the output needed from *all* sectors to satisfy f_j. The Oosterhaven–Stelder net output multiplier was defined as $\mathbf{i}'\mathbf{L}\hat{\mathbf{f}}_c = \mathbf{i}'\mathbf{L}\hat{\mathbf{f}}\hat{\mathbf{x}}^{-1}$ (a row vector). Replacement of $\hat{\mathbf{x}}$ by $\langle \mathbf{L}\hat{\mathbf{f}}\mathbf{i} \rangle$ leads to

$$\mathbf{i}'\mathbf{L}\hat{\mathbf{f}}_c = \mathbf{i}'\mathbf{L}\hat{\mathbf{f}}\hat{\mathbf{x}}^{-1} = (\mathbf{i}'\mathbf{L}\hat{\mathbf{f}})\langle \mathbf{L}\hat{\mathbf{f}}\mathbf{i} \rangle^{-1}$$

The jth element in this row can be seen to be a ratio, namely

$$(\mathbf{i}'\mathbf{L}\hat{\mathbf{f}}_c)_j = \frac{j\text{th column sum of } \mathbf{L}\hat{\mathbf{f}}}{j\text{th row sum of } \mathbf{L}\hat{\mathbf{f}}}$$

In words: the output generated in all industries by f_j divided by the output generated in j by all final demands. This suggests a kind of "net" backward linkage or net key sector measure. In particular, if $(\mathbf{i}'\mathbf{L}\hat{\mathbf{f}}_c)_j > 1$ then economy-wide output generated by final demand in j is larger than the amount of j's output that is generated by all the other industries' final demands. So industry j can be said to be more important for the others than the others are for industry j, and j would thus be identified as a key sector by this measure.

12.2.4 Classifying Backward and Forward Linkage Results

Studies that attempt to identify key sectors from their backward and forward linkage measures usually calculate both (generally in normalized form) and then select those sectors with a high score on both measures.[14] In normalized form, this means sectors with both backward and forward linkages greater than one.

Often, sectors are distributed over a four-way classification as (1) generally independent of (not strongly connected to) other sectors (both linkage measures less than 1), (2) generally dependent on (connected to) other sectors (both linkage measures greater than 1), (3) dependent on interindustry supply (only backward linkage greater than 1)

[14] There have been suggestions for "combined" measures to capture "total" linkage. For example, Hübler (1979) proposed column sums from $[\mathbf{I} - (0.5)(\mathbf{A} + \mathbf{B}')]^{-1}$ for this purpose. More comprehensive measures of total linkage come from hypothetical extraction approaches (section 12.2.6).

Table 12.3 Classification of Backward and Forward Linkage Results

		Direct or Total Forward Linkage	
		Low (<1)	High (> 1)
Direct or Total Backward Linkage	Low (< 1)	(I) Generally independent	(II) Dependent on interindustry demand
	High (> 1)	(IV) Dependent on interindustry supply	(III) Generally dependent

and (4) dependent on interindustry demand (only forward linkage greater than 1). This can be displayed in a 2×2 table, such as shown in Table 12.3.[15] With data for two or more time periods, a table of this sort for each period will give one indication of the evolution of the economy.[16]

12.2.5 Spatial Linkages

Exactly the same kinds of measures can be applied to multiregional input–output data to assess the types and intensities of spatial interdependence or connectedness. These address the issue of strength of economic connections among regions in an economy and, if data for more than one period are available, how those connections are changing over time – for example, increasing regional self-sufficiency or increasing interregional dependence. These measures can be aggregate – that is, is region r in general dependent on imports or exports (or both) or relatively self-sufficient? Or they can be sector/region specific – assessing the import- or export-dependence of sector i in region r on one sector (or all sectors) in another region (or regions). Recalling that total backward linkage is measured by the output multiplier, it is clear that the interregional multipliers discussed in section 6.3 get at exactly these kinds of questions. (Early presentations of the spatial form of linkage measures are in Miller and Blair, 1988 and Batten and Martellato, 1988.)

In the two-region (nation) context, we have $\mathbf{A} = \begin{bmatrix} \mathbf{A}^{rr} & \mathbf{A}^{rs} \\ \mathbf{A}^{sr} & \mathbf{A}^{ss} \end{bmatrix}$, $\mathbf{L} = \begin{bmatrix} \mathbf{L}^{rr} & \mathbf{L}^{rs} \\ \mathbf{L}^{sr} & \mathbf{L}^{ss} \end{bmatrix}$ and $\mathbf{G} = \begin{bmatrix} \mathbf{G}^{rr} & \mathbf{G}^{rs} \\ \mathbf{G}^{sr} & \mathbf{G}^{ss} \end{bmatrix}$. One straightforward set of spatial linkage measures closely parallels the sectoral linkage cases above. In this two-region case, the direct backward linkage of sector j in region r will have both an intraregional and an interregional

[15] This two-way table arrangement appears to have originated with Chenery and Watanabe (1958).

[16] Further subtleties are possible. Each quadrant can be further subdivided; for example, quadrant III could be divided into four more categories, those above and those below one standard deviation above the mean (of 1). And similarly for the other three numbered quadrants.

component. Specifically,

$$BL(d)_j^r = BL(d)_j^{rr} + BL(d)_j^{sr} = \sum_{i=1}^{n} a_{ij}^{rr} + \sum_{i=1}^{n} a_{ij}^{sr}$$

One measure of the relative strength of intra- vs. interregional (internal vs. external) direct backward dependence is given by the percentages

$$100[BL(d)_j^{rr}/BL(d)_j^r] \quad \text{and} \quad 100[BL(d)_j^{sr}/BL(d)_j^r]$$

or, using an alternative normalization,

$$BL(d)_j^{rr}/x_j^r \quad \text{and} \quad BL(d)_j^{sr}/x_j^r$$

Parallels can be found for total backward linkages, namely

$$BL(t)_j^r = BL(t)_j^{rr} + BL(t)_j^{sr} = \sum_{i=1}^{n} l_{ij}^{rr} + \sum_{i=1}^{n} l_{ij}^{sr}$$

and

$$100[BL(t)_j^{rr}/BL(t)_j^r] \quad \text{and} \quad 100[BL(t)_j^{sr}/BL(t)_j^r]$$
$$BL(t)_j^{rr}/x_j^r \text{ and } \quad BL(t)_j^{sr}/x_j^r$$

In compact matrix form, direct and total intra- and interregional backward linkages for each sector in region r are given by the n elements in the following vectors [the parallels are (12.23) and (12.25)]

$$\mathbf{b}(d)^{rr} = \mathbf{i}'\mathbf{A}^{rr} \quad \text{and} \quad \mathbf{b}(d)^{sr} = \mathbf{i}'\mathbf{A}^{sr}$$
$$\mathbf{b}(t)^{rr} = \mathbf{i}'\mathbf{L}^{rr} \quad \text{and} \quad \mathbf{b}(t)^{sr} = \mathbf{i}'\mathbf{L}^{sr}$$

and

$$\mathbf{b}(d)^r = \mathbf{b}(d)^{rr} + \mathbf{b}(d)^{sr} \quad \text{and} \quad \mathbf{b}(t)^r = \mathbf{b}(t)^{rr} + \mathbf{b}(t)^{sr}$$

Ignoring the sectoral detail, one aggregate measure of a region's direct and total backward linkage to itself and to the other region(s) is found by summing (or averaging) over all sectors. For example,

$$B(d)^{rr} = \mathbf{i}'\mathbf{A}^{rr}\mathbf{i} \quad \text{or} \quad B(d)^{rr} = (1/n)\mathbf{i}'\mathbf{A}^{rr}\mathbf{i}$$

and similarly for $B(d)^{sr}$, $B(t)^{rr}$, and $B(t)^{sr}$. Spatial versions of forward linkages follow the same kind of pattern. These are summarized in Table 12.4.

Examples of applications for single countries can be found in, among others, Blair and Miller (1990) and Shao and Miller (1990) for the US economy, Dietzenbacher (1992) for the Netherlands, Pan and Liu (2005) and Okamoto (2005) for China. The

Table 12.4 Summary of Spatial/Sectoral Linkage Measures (Two-Region Example)

Spatial/Sectoral Linkages			
Backward		Forward	
Direct	Total	Direct	Total
$\mathbf{b}(d)^{rr} = \mathbf{i}'\mathbf{A}^{rr}$, $\mathbf{b}(d)^{sr} = \mathbf{i}'\mathbf{A}^{sr}$	$\mathbf{b}(t)^{rr} = \mathbf{i}'\mathbf{L}^{rr}$, $\mathbf{b}(t)^{sr} = \mathbf{i}'\mathbf{L}^{sr}$	$\mathbf{f}(d)^{rr} = \mathbf{B}^{rr}\mathbf{i}$, $\mathbf{f}(d)^{rs} = \mathbf{B}^{rs}\mathbf{i}$	$\mathbf{f}(t)^{rr} = \mathbf{G}^{rr}\mathbf{i}$, $\mathbf{f}(t)^{rs} = \mathbf{G}^{rs}\mathbf{i}$
Normalizations include division of each direct element by $BL(d)_j^r$ [or each total element by $BL(t)_j^r$] or by x_j; e.g., $\bar{\mathbf{b}}(d)^{rr} = \mathbf{i}'\mathbf{A}^{rr}\langle\mathbf{b}(d)^r\rangle^{-1}$ or $\bar{\mathbf{b}}(d)^{rr} = \mathbf{i}'\mathbf{A}^{rr}(\hat{\mathbf{x}})^{-1}$		Normalizations include division of each direct element by $FL(d)_j^r$ [or each total element by $FL(t)_j^r$] or by x_j; e.g., $\bar{\mathbf{f}}(d)^{rr} = \langle\mathbf{f}(d)^r\rangle^{-1}\mathbf{B}^{rr}\mathbf{i}$ or $\bar{\mathbf{f}}(d)^{rr} = (\hat{\mathbf{x}})^{-1}\mathbf{B}^{rr}\mathbf{i}$	
Spatial Linkages			
Backward		Forward	
Direct	Total	Direct	Total
$B(d)^{rr} = \mathbf{i}'\mathbf{A}^{rr}\mathbf{i}$, $B(d)^{sr} = \mathbf{i}'\mathbf{A}^{sr}\mathbf{i}$	$B(t)^{rr} = \mathbf{i}'\mathbf{L}^{rr}\mathbf{i}$, $B(t)^{sr} = \mathbf{i}'\mathbf{L}^{sr}\mathbf{i}$	$F(d)^{rr} = \mathbf{i}'\mathbf{B}^{rr}\mathbf{i}$, $F(d)^{rs} = \mathbf{i}'\mathbf{B}^{rs}\mathbf{i}$	$F(t)^{rr} = \mathbf{i}'\mathbf{G}^{rr}\mathbf{i}$, $F(t)^{rs} = \mathbf{i}'\mathbf{G}^{rs}\mathbf{i}$
	Normalizations include division of each element by n or by $\mathbf{i}'\mathbf{x}$; e.g., $\bar{B}(d)^{rr} = (1/n)\mathbf{i}'\mathbf{A}^{rr}\mathbf{i}$ or $\bar{B}(d)^{rr} = (1/\mathbf{i}'\mathbf{x})\mathbf{i}'\mathbf{A}^{rr}\mathbf{i}$		

study by Chow, Lee and Ong (2006) for Singapore was based on 144-sector input–output data for 1990, 1995, and 2000. The authors chose to calculate (total) forward linkages using row sums of **A** rather than **B**.

With the emergence of international input–output data sets (for example, for the European Union and for Asia-Pacific economies, as described in section 8.8), these spatial measures are pertinent to questions of international economic connections and dependencies and their evolution over time. Illustrative applications here include Dietzenbacher and van der Linden (1997) for the countries of the European Community (see below) and Wu and Chen (2006) on backward linkages Taiwan ← Japan, Korea←Japan, China←Japan, and also Japan←Taiwan, Japan←Korea, Japan←China, in 1985, 1990, 1995, and 2000.

Alternative definitions for interregional economic connections are grounded in the notions of interregional feedbacks and spillovers (Miller and Blair, 1988; see also Chapters 3 and 6, above). These are closely related to the "hypothetical extraction" method. It provides a general framework for linkage analysis, and we turn to it next.

12.2.6 Hypothetical Extraction

The objective of the hypothetical extraction approach is to quantify how much the total output of an n-sector economy would change (decrease) if a particular sector, say the jth, were removed from that economy. Initially, this was modeled in an input–output context by deleting row and column j from the **A** matrix.[17] Using $\bar{\mathbf{A}}_{(j)}$ for the $(n-1) \times (n-1)$ matrix without sector j and $\bar{\mathbf{f}}_{(j)}$ for the correspondingly reduced final-demand vector, output in the "reduced" economy is found as $\bar{\mathbf{x}}_{(j)} = [\mathbf{I} - \bar{\mathbf{A}}_{(j)}]^{-1}\bar{\mathbf{f}}_{(j)}$.[18] (Instead of physically deleting row and column j in **A** and element j in **f**, they can simply be replaced by zeros.) In the full n-sector model, output is $\mathbf{x} = (\mathbf{I}-\mathbf{A})^{-1}\mathbf{f}$, so $T_j = \mathbf{i}'\mathbf{x} - \mathbf{i}'\bar{\mathbf{x}}_{(j)}$ is one aggregate measure of the economy's loss (decrease in value of gross output) if sector j disappears – as such, it is a measure of the "importance" or *total linkage* of sector j. It has been argued that the first term, $\mathbf{i}'\mathbf{x}$, should not include the (original) output x_j. If x_j is omitted, $(\mathbf{i}'\mathbf{x} - x_j) - \mathbf{i}'\bar{\mathbf{x}}$ would measure j's importance to the remaining sectors in the economy. In either case, normalization through division by total gross output ($\mathbf{i}'\mathbf{x}$) and multiplication by 100 produces an estimate of the percentage decrease in total economic activity; $\bar{T}_j = 100[(\mathbf{i}'\mathbf{x} - \mathbf{i}'\bar{\mathbf{x}}_{(j)})/\mathbf{i}'\mathbf{x}]$. (Web Appendix 12W.1 presents hypothetical extractions in detail in the context of partitioned matrix versions of the Leontief and Ghosh models, along with citations to much more of the relevant literature.)

The hypothetical extraction approach has also been used to measure backward and forward linkage components separately (for example, in Dietzenbacher and van der Linden, 1997). Taking inspiration from the discussion of backward linkage in section 12.2.1 and forward linkage in section 12.2.2, **A** is used for this backward measure and **B** is used for the forward measure.

[17] The original idea seems to have appeared in Paelinck, de Caevel and Degueldre, 1965 (in French) or Strassert, 1968 (in German). The first discussion in English known to us is in Schultz (1976, 1977; the latter paper is a longer version of the former).

[18] We use $x_{(j)}$ to distinguish this linkage measure from x_j which is the jth element in **x**.

Backward Linkage Assume that sector j buys no intermediate inputs from any production sector; that is, remove sector j's backward linkages. This is done by replacing column j in $\mathbf{A}$ by a column of zeros. Denote this new matrix $\bar{\mathbf{A}}_{(cj)}$. (We used $\bar{\mathbf{A}}_{(j)}$ above to denote $\mathbf{A}$ with both row *and* column j deleted; now we need the "c" to indicate that it is *column* j only that is gone.) Then $\bar{\mathbf{x}}_{(cj)} = [\mathbf{I} - \bar{\mathbf{A}}_{(cj)}]^{-1}\mathbf{f}$ and $\mathbf{i}'\mathbf{x} - \mathbf{i}'\bar{\mathbf{x}}_{(cj)}$ is one measure of (aggregate) backward linkage for sector j. If more detail is of interest, each element $x_i - \bar{x}_{(cj)i}$ in $\mathbf{x} - \bar{\mathbf{x}}_{(cj)}$ can be viewed as the backward dependence of sector j on sector i. Normalizations are possible and often used. For example, $[x_i - \bar{x}_{(cj)i}]/x_j$, puts this measure on a per-unit-of-output basis, or $100 \times [x_i - \bar{x}_{(cj)i}]/x_j$ to avoid relatively small numbers.

Forward Linkage A parallel to eliminating column j in $\mathbf{A}$ as a way of identifying backward linkages might appear to be the elimination of row j in $\mathbf{A}$ in order to quantify forward linkages. But the discussion in section 12.2.2 suggests that forward linkages of sector j can be more appropriately identified through elimination of that sector's intermediate sales in the $\mathbf{B}$ matrix. That is, replace *row* j of the output coefficients matrix by a row of zeros. Denote this matrix as $\bar{\mathbf{B}}_{(rj)}$. Then $\mathbf{x}' = \mathbf{v}'(\mathbf{I} - \mathbf{B})^{-1}$ and $\bar{\mathbf{x}}'_{(rj)} = \mathbf{v}'[\mathbf{I} - \bar{\mathbf{B}}_{(rj)}]^{-1}$ indicate pre- and post-extraction outputs, and $\mathbf{x}'\mathbf{i} - [\bar{\mathbf{x}}'_{(rj)}]\mathbf{i}$ is an aggregate measure of sector j's forward linkage. Again, each element in $\mathbf{x}' - \bar{\mathbf{x}}'_{(rj)}$ is an indication of j's dependence on sector i as an intermediate output buyer, and normalizations are usual, as in $[x_i - \bar{x}_{(rj)i}]/x_j$ or $100 \times [x_i - \bar{x}_{(rj)i}]/x_j$.

In Table 12.5 we summarize the main hypothetical extraction results. [We (arbitrarily) use $B(t)_j$ and $F(t)_j$ instead of $BL(t)_j$ and $FL(t)_j$ to indicate results from the hypothetical extraction approach and to distinguish them from the linkage measures in (12.24) and (12.29).] The interested reader can work through the exercise of extending these extraction possibilities to the spatial context in which a *region* is hypothetically extracted from its many-region system in order to assess that region's backward, forward and/or total spatial linkages to the rest of that system. (This would amount to filling in boxes in the style of Table 12.5 for "Region r Backward or Forward Linkage" and "Region r Total Linkage.")

When linkages are being measured in order to make comparisons of the structure of production between countries, the underlying coefficients matrices, whether $\mathbf{A}$ or $\mathbf{B}$, should be derived from *total* interindustry transactions data – that is, a particular z_{ij} should include good i used by sector j, whether good i comes from domestic producers or is imported. This is simply because interest is concentrated on how things are made in various economies, not on where the inputs come from. On the other hand, if linkages are being used to define "key" sectors in a particular economy, then the $\mathbf{A}$ or $\mathbf{B}$ matrices should be derived from a flow matrix that includes only domestically supplied inputs, since it is the impact on the domestic economy that is of interest. In studying the economies of less developed countries, it has been suggested (Bulmer-Thomas, 1982, p. 196) that "linkage analysis for LDCs is probably the most common use to which their input–output tables have been put."

Table 12.5 Hypothetical Extraction Linkages

Sector j Backward or Forward Linkage	
Total Backward	Total Forward
$B(t)_j = \mathbf{i}'\mathbf{x} - \mathbf{i}'\bar{\mathbf{x}}_{(cj)}$ where $\bar{\mathbf{x}}_{(cj)} = [\mathbf{I} - \bar{\mathbf{A}}_{(cj)}]^{-1}\mathbf{f}$	$F(t)_j = \mathbf{x}'\mathbf{i} - [\bar{\mathbf{x}}'_{(rj)}]\mathbf{i}$ where $\bar{\mathbf{x}}'_{(rj)} = \mathbf{v}'[\mathbf{I} - \bar{\mathbf{B}}_{(rj)}]^{-1}$
Normalizations include (1) division of each element by x_j, or (2) division by $\sum_{j=1}^{n} x_j$ to create the percentage decrease in total output, $\bar{B}(t)_j = 100\{[\mathbf{i}'\mathbf{x} - \mathbf{i}'\bar{\mathbf{x}}_{(cj)}]/\mathbf{i}'\mathbf{x}\}$ and $\bar{F}(t)_j = 100\{[\mathbf{x}'\mathbf{i} - \bar{\mathbf{x}}'_{(rj)}\mathbf{i}]/\mathbf{x}'\mathbf{i}\}$ or (3) values relative to the average, $\tilde{B}(t)_j = n\bar{B}(t)_j/\mathbf{i}'\bar{B}(t)_j$ and $\tilde{F}(t)_j = n\bar{F}(t)_j/\mathbf{i}'\bar{F}(t)_j$	
Sector j Total Linkage	
$T_j = \mathbf{i}'\mathbf{x} - \mathbf{i}'\bar{\mathbf{x}}_{(j)}$ or $(\mathbf{i}'\mathbf{x} - x_j) - \mathbf{i}'\bar{\mathbf{x}}_{(j)}$ where $\bar{\mathbf{x}}_{(j)} = [\mathbf{I} - \bar{\mathbf{A}}_{(j)}]^{-1}\mathbf{f}_{(j)}$ Normalize to create percentage decrease in total output, $\bar{T}_j = 100\{[\mathbf{i}'\mathbf{x} - \mathbf{i}'\bar{\mathbf{x}}_{(j)}]/\mathbf{i}'\mathbf{x}\}$ or $\bar{\bar{T}}_j = 100\{[\mathbf{i}'\mathbf{x} - \mathbf{i}'\bar{\mathbf{x}}_{(j)} - x_j]/\mathbf{i}'\mathbf{x}\}$ or to indicate values relative to the average, $\tilde{T}_j = n\bar{T}_j / \sum_{j=1}^{n} \bar{T}_j$ or $\tilde{\bar{\bar{T}}}_j = n\bar{\bar{T}}_j / \sum_{j=1}^{n} \bar{\bar{T}}_j$	

An applied study that uses both backward and forward linkages (as in sections 12.2.1 and 12.2.2) as well as incorporating a spatial dimension (section 12.2.4) along with hypothetical extraction (section 12.2.5) is found in Dietzenbacher and van der Linden (1997). This application is based on 1980 intercountry data for seven countries and 17 sectors of the European Community. It begins with backward and forward sectoral linkages for each country. These are then split into domestic and external linkages (the other six countries). Summations over all sectors give average (backward or forward) linkage of each country to each other. Then hypothetical extraction is applied to each country – for example, removal of sector j in Germany leads to an output reduction in Germany and an output reduction in the other six countries. These are translated into percentages (domestic vs. intercountry), giving a measure of each country's importance in the European Community economic system. Also sums (and averages) over all sectors in each country generate linkages between each pair of countries.

12.2.7 Illustration Using US Data

Results for the US 2003 seven-sector tables (Chapter 2) are collected in Table 12.6. As expected, the elements in $\mathbf{b}(t)$ are the total output multipliers that were found in Chapter 6. Normalized backward linkages (either direct or total) identify the sectors with the three strongest (above average) normalized backward linkages ($\overline{BL} > 1$) as

Table 12.6 Linkage Results, US 2003 Data

Sector	$\mathbf{b}(d)$	$\mathbf{b}(t)$	$\mathbf{f}(d)$	$\mathbf{f}(t)$	$\bar{\mathbf{b}}(d)$	$\bar{\mathbf{b}}(t)$	$\bar{\mathbf{f}}(d)$	$\bar{\mathbf{f}}(t)$
1	0.51	1.92	0.75	2.46	1.26	1.13	1.78	1.42
2	0.37	1.61	0.64	2.11	0.90	0.95	1.51	1.21
3	0.42	1.72	0.13	1.20	1.03	1.02	0.30	0.69
4	0.53	1.93	0.46	1.76	1.30	1.14	1.08	1.01
5	0.30	1.49	0.38	1.63	0.74	0.88	0.90	0.94
6	0.37	1.61	0.45	1.74	0.91	0.95	1.05	1.00
7	0.36	1.60	0.16	1.27	0.88	0.94	0.38	0.73

Table 12.7 Classification of Linkage Results, US 2003 Data

		Direct [$\bar{\mathbf{f}}(d)$] or Total [$\bar{\mathbf{f}}(t)$] Forward Linkage	
		Low (<1)	High (>1)
Direct [$\bar{\mathbf{b}}(d)$] or Total [$\bar{\mathbf{b}}(t)$] Backward Linkage	Low (<1)	5 (Trade, Transp., Utilities), 7 (Other)	2 (Mining), 6 (Services)
	High (>1)	3 (Construction)	1 (Agriculture), 4 (Manufacturing)

Table 12.8 Hypothetical Extraction Results, US 2003 Data

Sector	$\bar{B}(t)_j$	$\bar{F}(t)_j$	$\bar{T}_j$	$\bar{\bar{T}}_j$
1	1.02	1.61	2.12	0.73
2	0.69	1.27	1.84	0.61
3	3.87	1.06	17.78	12.39
4	13.59	11.20	30.02	10.30
5	6.47	8.42	25.51	13.03
6	19.95	24.44	59.69	13.38
7	6.64	2.98	25.85	14.38
Sector	$\tilde{B}(t)_j$	$\tilde{F}(t)_j$	$\tilde{T}_j$	$\tilde{\tilde{T}}_j$
1	0.14	0.22	0.09	0.08
2	0.09	0.17	0.08	0.07
3	0.52	0.15	0.76	1.34
4	1.82	1.54	1.27	1.11
5	0.87	1.16	1.17	1.41
6	2.67	3.36	2.54	1.44
7	0.89	0.41	1.10	1.55

Table 12.9 Classification of Hypothetical Extraction Results, US 2003 Data

		Total Forward Linkage $[\tilde{F}(t)_j]$	
		Low (<1)	High (>1)
Total Backward Linkage $[\tilde{B}(t)_j]$	Low (<1)	1 (Agriculture), 2 (Mining) 3 (Construction), 7 (Other)	5 (Trade, Transp., Utilities)
	High (>1)		4 (Manufacturing), 6 (Services)

(4) Manufacturing, (1) Agriculture, and (3) Construction, in that order. In the case of normalized forward linkages, the three largest (above average) are (1) Agriculture, (2) Mining, and (4) Manufacturing, in that order. These results are arranged in Table 12.7.

Hypothetical extraction results for the seven sectors are shown in Tables 12.8 and 12.9. With so few sectors it is perhaps not surprising that the orderings are similar across these hypothetical extraction measures, although netting out x_j does produce some changes in rankings. Sectors 6 (Services) and 4 (Manufacturing) are identified as the two "most important" to the economy – with $\tilde{B}(t)_j > 1$, $\tilde{F}(t)_j > 1$, $\tilde{T}_j > 1$, and $\tilde{\tilde{T}}_j > 1$. However, as the reader can verify, the four-way classifications differ considerably from those in Table 12.7.

12.3 Identifying Important Coefficients

There is a long history and an enormous amount of published work, both theoretical and empirical, on the impact (transmission, propagation) of errors or changes or uncertainty in basic input–output data on the model outcomes. This has appeared under a variety of titles ("probabilistic" or "stochastic" input–output, "error" analysis and "sensitivity" analysis, and so on). Examples in the "probabilistic" vein go back at least to Quandt (1958, 1959).[19] Approaches that investigate the impacts of discrete changes in one or more model components go back at least to the early 1950s (Dwyer and Waugh, 1953; Evans, 1954). It is beyond the scope of this book to explore all of this literature. (A brief review and a large set of pertinent references can be found in Lahr, 2001.) Instead, we concentrate on the concept of "important coefficients."

Early mathematical work on the notion of "important" coefficients (ICs) in an input–output model explored ways of identifying a_{ij} coefficients that have a particularly strong influence on one or more elements in the model, usually on the associated Leontief inverse matrix and/or on one or more gross outputs – meaning that $\Delta a_{ij} \to$ a "large" Δl_{rs} or that $\Delta a_{ij} \to$ a "large" Δx_r for one or more r and s. (We will explore below what

[19] Also representative of this line of inquiry are Simonovits (1975), Lahiri (1983), West (1986), Jackson and West (1989), Roland-Holst (1989), Kop Jansen (1994), ten Raa (1995, Chapter 14; 2005, Chapter 14), or Dietzenbacher (1995, 2006) and the many additional publications cited in these references.

Table 12.10 Number of Important Transactions in the 2000 China MRIO Model

Criterion	Number of cells	Percentage of total number of cells
$> (\mathbf{i}'\mathbf{Z}\mathbf{i}/n^2)$	4,715	8.19
$> (10) \times (\mathbf{i}'\mathbf{Z}\mathbf{i}/n^2)$	1,042	1.81
$> (100) \times (\mathbf{i}'\mathbf{Z}\mathbf{i}/n^2)$	84	0.15
$> (200) \times (\mathbf{i}'\mathbf{Z}\mathbf{i}/n^2)$	32	0.06

constitutes "large" and what measures of "change" are used in these investigations.) Identification of such coefficients can be helpful in deciding where to expend effort in obtaining superior information for updating or regionalizing a known input–output table using a hybrid model. And *IC*s contribute to some studies of key sectors and of what has come to be known as "fundamental economic structure." Jackson (1991) suggests that a distinction should be made between coefficient error (e.g., estimation error) and coefficient change (e.g., technological change).

In what follows, we examine the mathematical underpinnings of these approaches and then several kinds of studies – primarily influences on inverse elements and on gross outputs. Reviews of much of this work can be found in Xu and Madden (1991), Casler and Hadlock (1997) and Tarancón *et al.* (2008). There are many more published studies than we are able to cite. A large amount of work was done in Germany in the 1970s and 1980s and published in German, making it somewhat less accessible to a segment of the English-speaking audience – for example, Schintke (1979, 1984), Maaß (1980) and numerous references therein.

One very straightforward way to assess "importance" of individual cells in input–output data is simply to compare each transaction (z_{ij}) with the average transaction amount ($\mathbf{i}'\mathbf{Z}\mathbf{i}/n^2$). This is done in Okamoto (2005) for the 2000 China multiregional input–output data (CMRIO) made up of eight regions with 30 sectors each – a total of 57,600 potential elements in $\mathbf{Z}$. Table 12.10 shows results for this particular data set (adapted from Okamoto, 2005, p. 141). A similar approach could also be used on the data in coefficients matrices ($\mathbf{A}$ or $\mathbf{B}$) or total requirements matrices ($\mathbf{L}$ or $\mathbf{G}$).

12.3.1 Mathematical Background

These investigations build on early results in Sherman and Morrison (1949, 1950) and Woodbury (1950) – hereafter SMW – who studied how changes in elements in a (nonsingular) matrix were transmitted to changes in elements in the inverse of that matrix. (Basic results and additional details are presented in Appendix 12.1.) Given a nonsingular matrix, $\mathbf{M}$, and its inverse, $\mathbf{M}^{-1} = [\mu_{ij}]$, assume that one (or more) elements of $\mathbf{M}$ are changed, i.e., $m^*_{ij} = m_{ij} + \Delta m_{ij}$, producing $\mathbf{M}^* = \mathbf{M} + \Delta\mathbf{M}$. SMW show how the elements of $(\mathbf{M}^*)^{-1} = [\mu^*_{ij}]$ can be found by "adjusting" the known elements μ_{ij}. This is addressed by Sherman and Morrison (1950) for the case when only one element is changed, by Sherman and Morrison (1949) for changes in several elements in a given column or row and by Woodbury (1950) for changes in elements in several rows (or columns).[20]

For the simplest situation, when a single element m_{ij} is changed (increased or decreased) by an amount Δm_{ij}, the value of the element in row r and column s of the new inverse is found to be

$$\mu^*_{rs} = \mu_{rs} - \frac{\mu_{ri}\mu_{js}\Delta m_{ij}}{1 + \mu_{ji}\Delta m_{ij}} \tag{12.30}$$

Building on this result, it is possible to trace the influence of a change (or "error") in an element of an $\mathbf{A}$ matrix – and hence in $(\mathbf{I} - \mathbf{A})$ – on the associated Leontief inverse, $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$. In this case, we begin with $\mathbf{A}^* = \mathbf{A} + \Delta\mathbf{A}$. Since our interest is in $\mathbf{L}^* = (\mathbf{I} - \mathbf{A}^*)^{-1}$, the parallel to $\mathbf{M}^* = \mathbf{M} + \Delta\mathbf{M}$ is

$$(\mathbf{I} - \mathbf{A}^*) = [\mathbf{I} - (\mathbf{A} + \Delta\mathbf{A})] = (\mathbf{I} - \mathbf{A}) + (-\Delta\mathbf{A})$$

For individual elements in the new inverse, l^*_{rs}, the result in (12.30) becomes

$$l^*_{rs} = l_{rs} + \frac{l_{ri}l_{js}\Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}} \tag{12.31}$$

[This is (A12.1.4) in Appendix 12.1.] Notice that the signs are reversed from those in (12.30) because of the way in which $\Delta\mathbf{A}$ enters the expression for $(\mathbf{I} - \mathbf{A}^*)$.

12.3.2 Relative Sizes of Elements in the Leontief Inverse

The following observations on Leontief inverse elements are relevant to the problem of identifying important coefficients. As we will see, they help to reduce the number of coefficients that need to be examined when ranking those elements for importance.

Observation 1 From the power series approximation, it is clear that all on-diagonal elements in a Leontief inverse are larger than one. Also, it is virtually always observed in real-world Leontief inverse matrices that $l_{rs} < 1$ $(r \neq s)$ (off-diagonal elements are less than one);[21] thus $l_{ii} > 1 > l_{rs}$ for all $r \neq s$. This will be of use for the results below in (12.32).

[20] For much more detail on all of these results, see Miller (2000, Appendices 5.2 and 6.1).

[21] This is not to say that a counterexample cannot be constructed, but rather that they do not seem to occur in practice. For example

$$\mathbf{A} = \begin{bmatrix} 0.02 & 0.4 & 0.4 \\ 0.3 & 0.05 & 0.3 \\ 0.4 & 0.30 & 0.01 \end{bmatrix} \Rightarrow \mathbf{L} = \begin{bmatrix} 1.7767 & 1.0779 & 1.0445 \\ 0.8711 & 1.6925 & 0.8649 \\ 0.9818 & 0.9484 & 1.6942 \end{bmatrix}.$$

As an illustration, all the US Leontief inverses in Miller and Blair (1985, Appendix B), from 1947 through 1977, at both 23- and seven-sector levels of aggregation, exhibit the properties of Observation 1.

Observation 2 In (12.33) and (12.42), it will be of interest to identify the largest of the ratios $l_{ri}l_{js}/l_{rs}$ for a given i and j. Schnabl (2003, p. 497) reports on results in Maaß (1980, in German)

> Maaß's calculation showed that the maximum [of these ratios] is attained if $r = i$ and $s = j$ because then the main diagonal element of the inverse is involved *twice* and since the main diagonal element is usually the biggest one in a row or column this gives the maximum.

Thus $\underset{r,s=1,\ldots,n}{\text{Max}} \ l_{ri}l_{js}/l_{rs} = l_{ii}l_{jj}/l_{ij}$.

Observation 3 Finally, $\underset{r=1,\ldots,n}{\text{Max}} \ l_{ri}/x_r = l_{ii}/x_i$. It is not at all obvious that this should be the case, since the sizes of sectors (as measured by their gross outputs) can vary greatly in real-world models. Nonetheless, it was observed in some early empirical observations and is proven to always be the case in Tarancón *et al.* (2008)[22]. This is useful for the results in (12.38).

12.3.3 "Inverse-Important" Coefficients

For the remainder of this section, it will be useful to complicate the notation in order to be explicit about the element in **A** that is changed. From (12.31),

$$\Delta l_{rs(ij)} = l^*_{rs(ij)} - l_{rs} = \frac{l_{ri}l_{js}\Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}} = l_{ri}l_{js}k^1_{(ij)} \tag{12.32}$$

where $k^1_{(ij)} = \Delta a_{ij}/(1 - l_{ji}\Delta a_{ij})$, a constant for a given i and j, and $\mathbf{L}^*_{(ij)} = [l^*_{rs(ij)}]$ reminds us that the change is in a_{ij}. From Observation 1, above, Δa_{ij} will exert the largest influence on l_{ij} when $r = i$ and $s = j$, since then both elements multiplying $k^1_{(ij)}$ are larger than one [when l_{ri} is $l_{ii}(>1)$ and l_{js} is $l_{jj}\ (>1)$]. Similarly, next-largest influences will be felt in row i or column j of **L**, since then either $l_{ri} \to l_{ii} > 1$ or $l_{js} \to l_{jj} > 1$. In virtually all other cases (not row i or column j) both elements of the product $l_{ri}l_{js}$ are less than one.

From (12.32), the expression for *relative* changes in Leontief inverse elements is

$$\frac{\Delta l_{rs(ij)}}{l_{rs}} = \frac{l_{ri}l_{js}\Delta a_{ij}}{l_{rs}(1 - l_{ji}\Delta a_{ij})} = \frac{l_{ri}l_{js}}{l_{rs}}k^1_{(ij)} \tag{12.33}$$

This is where Observation 2 becomes relevant. Since $\underset{r,s=1,\ldots,n}{Max} \ l_{ri}l_{js}/l_{rs} = l_{ii}l_{jj}/l_{ij}$, it is clear that, again, Δa_{ij} will create the largest relative change on l_{ij}.

In addition, the elements $\Delta l_{rs(ij)}/l_{rs}$ in *column* i and *row* j of the matrix of relative changes will all be identical. In column i (when $s = i$), $\Delta l_{ri(ij)}/l_{ri} = (l_{ri}l_{ji}/l_{ri})k^1_{(ij)} = l_{ji}k^1_{(ij)}$, and in row j (when $r = j$), $\Delta l_{js(ij)}/l_{js} = (l_{ji}l_{js}/l_{js})k^1_{(ij)} = l_{ji}k^1_{(ij)} = \Delta l_{ri(ij)}/l_{ri}$.

[22] Sekulić (1968) observed this to be true for the Yugoslav economy in the early 1960s. Similar observations are made in Schintke (1979 and elsewhere) based on German data. See also results from US data in Table 12.12, below.

Finally, the *percentage* changes are

$$p_{rs(ij)} = 100\left[\frac{\Delta l_{rs(ij)}}{l_{rs}}\right] = 100\left[\frac{l_{ri}l_{js}\Delta a_{ij}}{1-l_{ji}\Delta a_{ij}}\right]\left[\frac{1}{l_{rs}}\right] = 100\left[\frac{l_{ri}l_{js}}{l_{rs}}k^1_{(ij)}\right] \tag{12.34}$$

Again, $p_{ij(ij)}$ will be the largest percentage change caused by Δa_{ij}.

It has been suggested (for example, Hewings, 1981) that a_{ij} may be viewed as "inverse-important" if, for a specified "threshold" amount of change in an inverse element, β, $p_{rs(ij)} \geq \beta$ for one or more r and s, that is, if

$$p_{rs(ij)} = 100\left[\frac{l_{ri}l_{js}\Delta a_{ij}}{1-l_{ji}\Delta a_{ij}}\right]\left[\frac{1}{l_{rs}}\right] \geq \beta \tag{12.35}$$

Denote the percentage change in a_{ij} by α, so that $\Delta a_{ij} = [\alpha/100]\,a_{ij}$; then we have

$$\left[\frac{l_{ri}l_{js}\alpha a_{ij}}{100-l_{ji}\alpha a_{ij}}\right]\left[\frac{100}{l_{rs}}\right] \geq \beta \tag{12.36}$$

for any l_{rs} and a given α and β. For example, let $\alpha = 20$ and $\beta = 10$. This means that a_{ij} will be considered inverse-important if a 20 percent change in its value generates a 10 percent or larger change in one or more elements in the Leontief inverse. The analyst must specify α and β, on the basis of the particular problem under study.

Given Observations 1 and 2, establishing inverse importance for *each* a_{ij} in an n-sector $\mathbf{A}$ matrix requires only one application of (12.35) [or (12.36)] – for $r = i$ and $s = j$.[23] The virtue of the SMW method is that it finds this information about the inverse by working exclusively with known elements in $\mathbf{L}$ and avoiding direct calculation of the new inverse. This was the whole point of the formulation.

At present, however, finding inverses is not quite the task it was in 1950 when the SMW approach was developed; at least this is true for matrices that are not "too large." Then a straightforward alternative to applying (12.35) or (12.36) is to calculate directly the $\mathbf{L}^*_{(ij)}$ associated with each Δa_{ij} and then find the corresponding matrix of percentage changes, $\mathbf{P}_{(ij)} = [p_{rs(ij)}] = 100\{[\mathbf{L}^*_{(ij)} - \mathbf{L}]\oslash\mathbf{L}\}$, where "$\oslash$" indicates element-by-element division.

This line of work, formulating the notion of inverse-important coefficients, was taken up initially in the early 1980s by Hewings, Jensen, West, and others. Examples are Jensen and West (1980), Hewings (1981), Hewings and Romanos (1981) and Hewings (1984). For hybrid (partial-survey) models, the idea is to identify coefficients (or sectors) for which additional information (survey, expert opinion) would be particularly useful. But of course identifying inverse importance implies that a relevant matrix of coefficients already exists to supply the elements in results like (12.35). For updating,

[23] If one wants not simply to establish inverse-importance, but also *extent* [that is, for a given Δa_{ij}, how many (and which) $p_{rs(ij)}$ exceed the β threshold], then $[n^2 - (2n-1)]$ calculations like those in (12.35) or (12.36) would be needed – the n^2 inverse elements, l_{rs}, for a given a_{ij}, less those in row j and column i that are all identical – and these calculations must be made n times, once for each of the a_{ij}. The "field of influence" approach (section 12.3.6) accomplishes this in one matrix operation.

there is a base matrix to be updated, and the premise is that important coefficients at time "t" will also be important at time "$t + 1$." However, there is no hard evidence to support that argument. In fact, Hewings (1984, p. 325), cautioned that "…only 3 of the cells deemed inverse-important in 1963 [the Washington State 49-sector model] were similarly identified in 1967." For regional models there is often not an "earlier" regional table. In the context of estimating a coefficients table in a regional context, Boomsma and Oosterhaven (1992, p. 276, n. 3) observed:

> Here we have a typical "chicken or egg" problem. Without a regional table one cannot determine the inverse-important cells and without that information one cannot construct a decent regional table. Hence, we suggest use of the national table as second best information on inverse-importance.

12.3.4 Numerical Example

We use the two-sector example closed with respect to households from section 2.5, namely[24]

$$\mathbf{A} = \begin{bmatrix} .15 & .25 & .05 \\ .20 & .05 & .40 \\ .30 & .25 & .05 \end{bmatrix} \text{ and } \mathbf{L} = \begin{bmatrix} 1.3651 & .4253 & .2509 \\ .5273 & 1.3481 & .5954 \\ .5698 & .4890 & 1.2885 \end{bmatrix}$$

Consider $\Delta a_{12} = (0.2)a_{12}$ (that is, $\alpha = 20$); then $\mathbf{A}^* = \begin{bmatrix} .15 & .30 & .05 \\ .20 & .05 & .40 \\ .30 & .25 & .05 \end{bmatrix}$, and we can easily calculate $\mathbf{L}^*_{(12)}$ directly as

$$\mathbf{L}^*_{(12)} = \begin{bmatrix} 1.4021 & .5198 & .2926 \\ .5416 & 1.3846 & .6115 \\ .5853 & .5285 & 1.3060 \end{bmatrix}$$

Then

$$\mathbf{P}_{(12)} = \begin{bmatrix} 2.7080 & 22.2225 & 16.6345 \\ 2.7080 & 2.7080 & 2.7080 \\ 2.7080 & 8.0667 & 1.3521 \end{bmatrix}$$

As expected, with the change Δa_{12}, all elements in column 1 and row 2 are identical.

Further, in this illustration $l_{jj} > 1$ $(j = 1, \ldots, 3)$, $l_{ij} < 1$ $(i = 1, \ldots, 3; i \neq j)$ (Observation 1, above) and indeed the largest change caused by Δa_{12} is in l_{12} – here this is $p_{12(12)} = 22.2$ percent. If we specify $\beta = 10$ as our criterion in (12.36) for inverse-importance – namely when a change of 10 percent or more is experienced by at least one inverse coefficient – then we see that a_{12} would be classified as inverse-important because $\Delta a_{12} = (0.2)a_{12}$ causes both l_{12} and l_{13} to be changed by more than 10 percent. In this small three-sector case, it is relatively easy to modify each element in **A**,

[24] In section 2.5 these matrices contained overbars to indicate a model closed with respect to households and to distinguish them from the earlier open model. At this point the overbars just get in the way of other notation and will be dropped.

in turn, by 20 percent, find the associated Leontief inverse and then the associated $\mathbf{P}$ matrix. (Readers are encouraged to do this, at least for several additional a_{ij}.) If we continue to use $\beta = 10$ in that series of calculations – for $\Delta a_{11} = (0.2)a_{11}, \ldots, \Delta a_{33} = (0.2)a_{33}$ – we will identify a_{21}, a_{23}, a_{31}, and a_{32} also as important.[25] (Higher values of β serve to raise the bar on eligibility for importance. For example, with $\beta = 20$, only a_{12} and a_{23} would be labeled important.)

"Importance" could be identified in many other ways, for example, with respect to changes in output multipliers – as in $100\{[\mathbf{i}'\mathbf{L}^*_{(ij)} - \mathbf{i}'\mathbf{L}] \oslash \mathbf{i}'\mathbf{L}\} = 100\{[\mathbf{i}'\Delta\mathbf{L}_{(ij)}] \oslash \mathbf{i}'\mathbf{L}\}$. If β now refers to percentage change in multipliers, only a_{23} is found to be important with $\beta = 10$; however, at $\beta = 5$ the same five coefficients as above are identified.

As noted, the point of the SMW result is that these percentage changes in inverse coefficients can be found without knowing the new inverse at all. Continuing with $\Delta a_{12} = (0.2)a_{12}$, consider the percentage change in l_{13} [$p_{13(12)}$ in $\mathbf{P}_{(12)}$, above]. Using (12.35) with $i = 1, j = 2,\ r = 1,\ s = 3$, and $\Delta a_{12} = 0.05$, we have

$$p_{13(12)} = \left[\frac{l_{11}l_{23}\Delta a_{12}}{1 - l_{21}\Delta a_{12}}\right]\left[\frac{100}{l_{13}}\right] = \left[\frac{(1.3651)(.5954)(0.05)}{1 - (.5273)(0.05)}\right]\left[\frac{100}{(.2509)}\right] = 16.6359$$

Except for rounding (and the number of significant digits carried in the inversion programs used to find $\mathbf{L}$ and $\mathbf{L}^*_{(12)}$), this corresponds to the $p_{13(12)}$ found above. Any other value in $\mathbf{P}_{(12)}$ could be found in the same way.

The designation of inverse-importance depends crucially on the choice of α and β. In a study of several of the Washington State 49-sector tables, Hewings (1984) used $\alpha = 30$ and $\beta = 20$. Out of $49 \times 49 = 2401$ direct input coefficients, between 24 and 42 (1.0–1.7 percent) were judged inverse-important. In a similar study in Sri Lanka (also Hewings, 1984), 3.5 percent were found important in a 12-sector model (apparently using the same α and β). There were interesting although not surprising variations in a two-region Sri Lanka interregional input–output model between intraregional and interregional coefficients (now with a 24×24 matrix); 3.3 percent of the (possible) 288 intraregional coefficients were important and 0.9 percent of those 288 in the interregional matrices were important. In a similar study (Hewings and Romanos, 1981) using a 22-sector model for the rural Evros region in Greece, 18 of 484 possible coefficients (3.7 percent) were important – only here, because of the less-developed nature of the economy, the critical values used were $\alpha = 20$ and $\beta = 1$. With those same values, a 22-sector model for the Greek national economy had 38 important coefficients (7.9 percent).

12.3.5 Impacts on Gross Outputs

Early applications of these ideas to the impact of coefficient change on gross outputs are found in Sekulić (1968) and Jílek (1971).[26] In matrix terms, $\Delta\mathbf{x}_{(ij)} = \mathbf{x}^*_{(ij)} - \mathbf{x} =$

[25] In each of the eight cases the largest change caused by that Δa_{ij} was in the associated l_{ij}, as expected, but four of those were below the $\beta = 10$ threshold.

[26] Sekulić (1968) and later Jílek (1971) attribute the approach to E. B. Yershof who contributed a chapter of a 1965 Moscow publication on planning (in Russian). The bases of the approach are in the work of SMW, some 15 years earlier.

$\mathbf{L}^*_{(ij)}\mathbf{f} - \mathbf{L}\mathbf{f} = \Delta\mathbf{L}_{(ij)}\mathbf{f}$. From (12.31), we see that row r of $\Delta\mathbf{L}_{(ij)}$ is

$$\begin{bmatrix}\Delta l_{r1(ij)} & \cdots & \Delta l_{rn(ij)}\end{bmatrix} = \frac{l_{ri}\Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}}\begin{bmatrix}l_{j1} & \cdots & l_{jn}\end{bmatrix}$$

and therefore

$$\Delta x_{r(ij)} = \begin{bmatrix}\Delta l_{r1(ij)} & \cdots & \Delta l_{rn(ij)}\end{bmatrix}\begin{bmatrix}f_1\\ \vdots \\ f_n\end{bmatrix} = \left[\frac{l_{ri}\Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}}\right]\begin{bmatrix}l_{j1} & \cdots & l_{jn}\end{bmatrix}\begin{bmatrix}f_1\\ \vdots \\ f_n\end{bmatrix}$$

But since $\begin{bmatrix}l_{j1} & \cdots & l_{jn}\end{bmatrix}\begin{bmatrix}f_1\\ \vdots \\ f_n\end{bmatrix} = x_j$, this is just

$$\Delta x_{r(ij)} = \frac{l_{ri}x_j\Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}} = l_{ri}k^2_{(ij)} \tag{12.37}$$

where $k^2_{(ij)} = x_j\Delta a_{ij}/(1 - l_{ji}\Delta a_{ij})$. Compared with the expression for $\Delta l_{rs(ij)}$ in (12.32), l_{js} has been replaced on the right-hand side by x_j. Again, from Observation 1, $l_{ii} > l_{ri}$ (for $r = 1, \ldots, n;\ r \neq i$), so (12.37) indicates that the largest gross output change from Δa_{ij} will be in sector i (that is, when $r = i$).[27]

The *relative* change in x_r is then

$$\frac{\Delta x_{r(ij)}}{x_r} = \frac{l_{ri}x_j\Delta a_{ij}}{x_r(1 - l_{ji}\Delta a_{ij})} = \left[\frac{l_{ri}}{x_r}\right]k^2_{(ij)} \tag{12.38}$$

Here the largest *relative* change in gross output for a given Δa_{ij} will be in sector s for which $l_{si}/x_s = \underset{r=1,\ldots,n}{\text{Max}}\ (l_{ri}/x_r)$, and from Observation 3, this will be for sector i. The interested reader can easily show this to be true for the numerical example, above. Table 12.11 presents the same calculations for the 2003 US seven-sector data from Chapter 2, showing that the largest ratios (in bold) are on the main diagonal.

Finally, multiplication in (12.38) by 100 creates a *percentage* change,

$$100\left[\frac{\Delta x_{r(ij)}}{x_r}\right] = 100\left[\frac{\Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}}\right]\left[\frac{l_{ri}x_j}{x_r}\right] \tag{12.39}$$

Table 12.12 contains these percentages for x_1, x_2, and x_3 from the hypothetical example as a result of $\Delta a_{ij} = (0.2)a_{ij}$ for all nine direct input coefficients ($i, j = 1,\ 2,\ 3$).

As expected, for any Δa_{ij}, the largest changes are found in x_i; in the row for $i = 1$, this means $\Delta x_1 > \Delta x_2$ and $\Delta x_1 > \Delta x_3$, and so on in the rows for $i = 2$ and $i = 3$. Also, with $\alpha = 20$, if the criterion for "importance" is that one or more *outputs* changes by

[27] If x_i is small relative to other outputs, then a large Δx_i may not have much economy-wide importance. There have been attempts to take this aspect of relative output size into account, but we do not consider this level of detail. The interested reader might speculate on how this could be done.

Table 12.11 $(l_{ri}/x_r) \times 10^6$ for the 2003 US Seven-Sector Model

	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$
$r=1$	**4.5868**	0.0211	0.0477	0.2093	0.0136	0.0253	0.0263
$r=2$	0.0381	**4.4187**	0.0501	0.1408	0.0794	0.0136	0.0301
$r=3$	0.0071	0.0032	**0.9449**	0.0060	0.0061	0.0105	0.0235
$r=4$	0.0589	0.0306	0.0672	**0.3449**	0.0178	0.0220	0.0324
$r=5$	0.0523	0.0297	0.0480	0.0547	**0.3811**	0.0209	0.0298
$r=6$	0.0261	0.0321	0.0295	0.0319	0.0297	**0.1544**	0.0343
$r=7$	0.0107	0.0105	0.0102	0.0162	0.0124	0.0131	**0.4566**

Table 12.12 Percentage Change in **x** Resulting from $\Delta a_{ij} = (0.2)a_{ij}$

	$j=1$	$j=2$	$j=3$
$i=1$	$\begin{bmatrix} 4.27 \\ 0.82 \\ 1.78 \end{bmatrix}$	$\begin{bmatrix} \mathbf{14.02} \\ 2.71 \\ 5.85 \end{bmatrix}$	$\begin{bmatrix} 1.37 \\ 0.27 \\ 0.57 \end{bmatrix}$
$i=2$	$\begin{bmatrix} 1.73 \\ 2.74 \\ 1.99 \end{bmatrix}$	$\begin{bmatrix} 0.86 \\ 1.37 \\ 0.99 \end{bmatrix}$	$\begin{bmatrix} 3.54 \\ 5.61 \\ 4.07 \end{bmatrix}$
$i=3$	$\begin{bmatrix} 1.53 \\ 1.81 \\ 7.85 \end{bmatrix}$	$\begin{bmatrix} 2.59 \\ 3.07 \\ \mathbf{13.28} \end{bmatrix}$	$\begin{bmatrix} 0.25 \\ 0.30 \\ 1.31 \end{bmatrix}$

$\beta = 10$ percent, then a_{12} and a_{32} would be labeled most and second-most important. These percentage changes are indicated in bold in the table. [The interested reader might speculate on why it is not surprising that coefficients judged important by the criterion in (12.35) are likely to be tagged as important by the criterion in (12.39).]

Again, the "importance" of any a_{ij} could be defined in terms of the impact of *relative* or *percentage* changes in a_{ij} on the associated relative or percentage changes in each x_r. Using γ_r for the (user-specified) threshold on percentage changes in x_r,

$$\left[\frac{100\Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}}\right]\left[\frac{l_{ri}x_j}{x_r}\right] \geq \gamma_r$$

[Compare (12.35).] Again, with $\Delta a_{ij} = [\alpha/100]\, a_{ij}$, we have

$$\left[\frac{\alpha a_{ij}}{1 - l_{ji}\left[\frac{\alpha}{100}\right] a_{ij}}\right]\left[\frac{l_{ri}x_j}{x_r}\right] = \left[\frac{100\alpha a_{ij}}{100 - l_{ji}\alpha a_{ij}}\right]\left[\frac{l_{ri}x_j}{x_r}\right] \geq \gamma_r$$

Table 12.13 Upper Threshold on $\Delta a_{ij}/a_{ij}$ for $\gamma = 1$ Percent

	$j=1$	$j=2$	$j=3$
$i=1$	4.90	1.47	14.71
$i=2$	7.46	14.92	3.73
$i=3$	2.58	1.55	15.50

Much of the empirical work in this area is based on a rearrangement of (12.39). Putting Δa_{ij} on the left and converting to *relative* change in a_{ij}, we have

$$\frac{\Delta a_{ij}}{a_{ij}} = \frac{\Delta x_{r(ij)}/x_r}{a_{ij}[(l_{ji}\Delta x_{r(ij)}/x_r) + (l_{ri}x_j/x_r)]}$$

Define an allowable error limit, γ, for *all* sectors r which is just fulfilled by positive relative deviations $\Delta a_{ij}/a_{ij}$. This is often called a "tolerable limit, TL" and hence the name "tolerable limits approach." As is frequently done, let $\gamma = 100(\Delta x_{r(ij)}/x_r) = 1$ percent; then in percentage terms

$$\frac{\Delta a_{ij}}{a_{ij}} = \frac{100\Delta x_{r(ij)}/x_r}{a_{ij}[(l_{ji}100\Delta x_{r(ij)}/x_r) + 100(l_{ri}x_j/x_r)]} = \frac{1}{a_{ij}[l_{ji} + 100(l_{ri}/x_r)x_j]}$$

Expressed in this way, we see that the larger the denominator on the right-hand side, the smaller $\Delta a_{ij}/a_{ij}$. So the upper threshold on $\Delta a_{ij}/a_{ij}$ will be determined by $\underset{r=1,\ldots,n}{\text{Max}}\ l_{ri}/x_r$,

$$\frac{\Delta a_{ij}}{a_{ij}} \le \frac{1}{a_{ij}[l_{ji} + 100 \underset{r=1,\ldots,n}{\text{Max}} (l_{ri}/x_r)x_j]}$$

As noted (Observation 3) $\underset{r=1,\ldots,n}{\text{Max}}\ l_{ri}/x_r = l_{ii}/x_i$, so

$$\frac{\Delta a_{ij}}{a_{ij}} \le \frac{1}{a_{ij}[l_{ji} + 100(l_{ii}/x_i)x_j]} \tag{12.40}$$

establishes an *upper limit* on the relative change in a_{ij} that assures that *no* gross output will be changed by more than one percent.[28] The smaller $\Delta a_{ij}/a_{ij}$, the more important the coefficient a_{ij}. Table 12.13 shows the right-hand sides of (12.40) for our small numerical example.

From the upper-left element in the table, we learn that a_{11} could change by as much as 4.9 percent before any output would be changed by more than one percent. Similarly,

[28] This can be found in Sekulić (1968). Forssell (1989, p. 431) describes it as a measure "developed by Mäenpää (1981)" but it seems to have been suggested much earlier. The Jugoslav journal in which the Sekulić paper appeared may not be well known, but the paper was also presented at the Fourth International Conference on input–output Techniques in Geneva in 1968.

Table 12.14 Average Values in US Total Requirements Matrices

Number of Sectors	$\mathbf{i}'\hat{\mathbf{L}}\mathbf{i}/n$	$\mathbf{i}'\check{\mathbf{L}}\mathbf{i}/(n^2-n)$
$n = 7$	1.1739	0.0868
$n = 16$	1.1290	0.0429
$n = 61$	1.1113	0.0133

a_{12} (1.47) is identified as the most important coefficient (smallest value in Table 12.13), followed by a_{32} (1.55), a_{31}(2.58) and so on. While perhaps not of much interest, we would also conclude that a_{33} is least important, since it could change by as much as 15.5 percent before any gross output would be changed by more than one percent. [Since the result in (12.40) comes directly from the result in (12.39), it should not be surprising that the importance rankings of the nine coefficients in our numerical example that are shown in Tables 12.12 and 12.13 are exactly the same.][29]

The denominator on the right in (12.40),

$$a_{ij}[l_{ji} + 100(l_{ii}/x_i)x_j]$$

has been described as a measure of the "degree of importance" of a_{ij} (for example, by Schintke and Stäglin, 1984). In real-world applications it turns out that $l_{ji} \ll 100(l_{ii}/x_i)x_j$, especially for relatively disaggregated input–output models, again because of Observation 1 ($l_{ii} > 1 > l_{ij}$). In fact, there are usually quite large differences between the l_{ii} and the l_{ij}. For example, average values of on-diagonal elements (in $\hat{\mathbf{L}}$) and off-diagonal elements (in $\check{\mathbf{L}}$) in Leontief inverses for 2003 US input–output data are shown in Table 12.14.

This suggests that, for any given a_{ij} and irrespective of x_i and x_j,[30] the first term can be ignored and the measure can be approximated as

$$a_{ij}[l_{ji} + 100(l_{ii}/x_i)x_j] \approx 100a_{ij}(l_{ii}/x_i)x_j$$

Using $b_{ij} = z_{ij}/x_i = a_{ij}x_j/x_i$ (the usual "output coefficient" from the Ghosh model) this has also been expressed as

$$a_{ij}[l_{ji} + 100(l_{ii}/x_i)x_j] \approx 100b_{ij}l_{ii}$$

[29] Empirical examples identifying important coefficients for a variety of tolerable limits can be found in Aroche-Reyes (1996, 2002) for Mexico (1970, 1980) in the first case and for Mexico (1971, 1990), Canada (1972, 1990) and the US (1971, 1990) in the second.

[30] Of course one could generate counter examples with very large x_i and very small x_j so that $l_{ji} > 100(l_{ii}/x_i)x_j$. The point is that this does not seem to happen in real-world applications.

12.3.6 Fields of Influence

In a number of articles, Sonis and Hewings and their colleagues have developed and applied the concept of a "field of influence" associated with each coefficient in an **A** matrix.[31] This is essentially an extension of the Sherman–Morrison approach that generates in one operation the entire matrix of changes in the Leontief inverse associated with a given change in a particular a_{ij}. Recall that $\Delta l_{rs(ij)}$ is related to Δa_{ij} through

$$\Delta l_{rs(ij)} = l^*_{rs(ij)} - l_{rs} = \frac{l_{ri} l_{js} \Delta a_{ij}}{1 - l_{ji} \Delta a_{ij}} = l_{ri} l_{js} k^1_{(ij)}$$

[This is (12.32), above.] Finding all the n^2 elements in the $n \times n$ matrix $\Delta \mathbf{L}_{(ij)} = [\Delta l_{rs(ij)}]$ would require $[n^2 - (2n-1)]$ operations, as we saw above (footnote 21). Instead, Sonis and Hewings propose an efficient alternative.

Let column i and row j of $\mathbf{L}$ be denoted $\mathbf{L}_{.i} = \begin{bmatrix} l_{1i} \\ l_{2i} \\ \vdots \\ l_{ni} \end{bmatrix}$ and $\mathbf{L}_{j.} = \begin{bmatrix} l_{j1} & l_{j2} & \cdots & l_{jn} \end{bmatrix}$.

Then the first order (direct) field of influence of the incremental change Δa_{ij} is defined by Sonis and Hewings as the matrix[32]

$$\mathbf{F}[i,j] = \mathbf{L}_{.i}\mathbf{L}_{j.} = \begin{bmatrix} l_{1i} \\ l_{2i} \\ \vdots \\ l_{ni} \end{bmatrix} \begin{bmatrix} l_{j1} & l_{j2} & \cdots & l_{jn} \end{bmatrix} = \begin{bmatrix} l_{1i}l_{j1} & l_{1i}l_{j2} & \cdots & l_{1i}l_{jn} \\ l_{2i}l_{j1} & l_{2i}l_{j2} & \cdots & l_{2i}l_{jn} \\ \vdots & \vdots & & \vdots \\ l_{ni}l_{j1} & l_{ni}l_{j2} & \cdots & l_{ni}l_{jn} \end{bmatrix}$$

Thus, $\mathbf{F}[i,j] = [l_{ri} l_{js}]$ for $r, s = 1, \ldots, n$ is the expanded version of the product $l_{ri} l_{js}$ on the right-hand side of (12.32), and the *matrix* showing the change in each element of $\mathbf{L}$ caused by Δa_{ij} is just $\Delta \mathbf{L}_{(ij)} = \mathbf{F}[i,j] k^1_{(ij)}$. Therefore

$$\mathbf{L}^*_{(ij)} = \mathbf{L} + \Delta \mathbf{L}_{(ij)} = \mathbf{L} + [(\Delta a_{ij})/(1 - l_{ji} \Delta a_{ij})]\mathbf{F}[i,j] = \mathbf{L} + \mathbf{F}[i,j] k^1_{(ij)}$$

Since $k^1_{(ij)}$ is a constant for any specific Δa_{ij}, corresponding elements of $\Delta \mathbf{L}_{(ij)}$ and $\mathbf{F}[i,j]$ are proportional and will have the same ordering – for example, largest to smallest.

In the numerical example, $\mathbf{L}_{.i} = \mathbf{L}_{.1} = \begin{bmatrix} 1.3651 \\ 0.5273 \\ 0.5698 \end{bmatrix}$ and

[31] The publications are numerous, going back at least to Sonis and Hewings (1989). A fairly compact statement can be found in Sonis and Hewings (1992) and an application (to the Chicago economy) is presented in Okuyama *et al.* (2002).

[32] Sonis and Hewings used $\mathbf{F}\begin{pmatrix} i \\ j \end{pmatrix}$ to indicate a field of influence in early publications; later (for example, Sonis and Hewings, 1999) this became $\mathbf{F}[i,j]$.

$\mathbf{L}_{j\cdot} = \mathbf{L}_{2\cdot} = \begin{bmatrix} 0.5273 & 1.3481 & 0.5954 \end{bmatrix}$ so

$$\mathbf{F}[1,2] = \mathbf{L}_{\cdot 1}\mathbf{L}_{2\cdot} = \begin{bmatrix} 0.7198 & 1.8402 & 0.8127 \\ 0.2781 & 0.7109 & 0.3139 \\ 0.3005 & 0.7682 & 0.3393 \end{bmatrix}$$

Further, $\Delta a_{12} = 0.05$ and $k^1_{(12)} = [(\Delta a_{12})/(1 - l_{21}\Delta a_{12})] = 0.0514$ so

$$\Delta\mathbf{L}_{(12)} = \mathbf{F}[1,2](0.0514) = \begin{bmatrix} 0.0370 & 0.0945 & 0.0417 \\ 0.0143 & 0.0365 & 0.0161 \\ 0.0154 & 0.0395 & 0.0174 \end{bmatrix}$$

and it is easily verified that $\mathbf{L}^*_{(12)} = \mathbf{L} + \Delta\mathbf{L}_{(12)}$.

Sonis and Hewings suggest that inverse-important coefficients can be identified by comparing their fields of influence.[33] The problem is how to reduce the n^2 pieces of information in each $\mathbf{F}[i,j]$ in order to make comparisons across the Δa_{ij}.[34] The norms of these matrices offer one possible compact measure; the trouble is that there are many different definitions of a matrix norm. Among those that they mention (Sonis and Hewings, 1992, p. 147) are

$\|\mathbf{F}\| = \max\limits_{ij} \left|f_{ij}\right|$ (largest individual element)[35]

$\|\mathbf{F}\| = \sum\limits_{ij} \left|f_{ij}\right|$ (sum of all elements)

$\|\mathbf{F}\| = \left[\sum\limits_{ij} \left|f_{ij}\right|\right]^{1/2}$

In Chapter 2 we used a largest column sum norm; $\|\mathbf{F}\| = \max\limits_j \sum\limits_i \left|f_{ij}\right|$. Further,

> [t]he choice of norm $\|\mathbf{F}\|$ is the basis of the construction of the rank-size sequence of the elements a_{ij} of the matrix $\mathbf{A}$ according to the numerical sizes of the norms $\|\mathbf{F}[i,j]\|$. The decision or cutting rule must be formulated in such a way that only a relatively small number of the elements of the rank-size sequence will comprise the set of inverse-important coefficients. (p. 147).

Returning to our numerical example, we generated fields of influence for each of the nine coefficients in $\mathbf{A}$ using $\Delta a_{ij} = (0.2)a_{ij}$ – namely, $\Delta a_{11} = (0.2)a_{11}$, then $\Delta a_{12} = (0.2)a_{12}$, and so on. Tables 12.15 and 12.16 present two summary measures (norms) from these nine $\mathbf{F}[i,j]$ matrices. Table 12.15 contains the column sums, and the $\|\mathbf{F}\| =$

[33] In other publications they also propose higher-order fields of influence when two or more coefficients change (with associated mathematical representations that are much more complicated), and they also use some of these concepts to characterize the fundamental structures of economies and provide alternative kinds of model decompositions.

[34] Generally, for comparability, each a_{ij} is changed by the same percentage, α, so that $\Delta a_{ij} = (\alpha/100)a_{ij}$ for all i,j. In presenting applications identifying important coefficients Sonis and Hewings (1992) do not specify either their choice of norm or their coefficient alteration mechanism.

[35] There is no need to generate the entire field of influence matrix if one is then going to summarize the information by using the $\max\limits_{ij} \left|f_{ij}\right|$ norm of that matrix. We know that the largest $\Delta l_{rs(ij)}$ is $\Delta l_{ij(ij)}$ and that $f_{rs(ij)}$ is proportional to $\Delta l_{rs(ij)}$ so this can be found using the Sherman–Morrison results in (12.32) for $\Delta l_{ij(ij)}$ only.

Table 12.15 Column Sums of $|\mathbf{F}[i,j]|$ for Numerical Example

a_{ij}	$j=1$	$j=2$	$j=3$
$i=1$	$[3.3612 \quad 1.0471 \quad 0.6178]$	$[1.2984 \quad 3.3193 \quad 1.4659]$	$[1.4031 \quad 1.2042 \quad 3.1727]$
$i=2$	$[3.0884 \quad 0.9621 \quad 0.5676]$	$[1.1930 \quad 3.0499 \quad 1.3469]$	$[1.2892 \quad 1.1064 \quad 2.9152]$
$i=3$	$[2.9142 \quad 0.9078 \quad 0.5356]$	$[1.1257 \quad 2.8779 \quad 1.2710]$	$[1.2165 \quad 1.0440 \quad 2.7508]$

Table 12.16 Sum of all Elements in $|\mathbf{F}[i,j]|$ $\left(\|\mathbf{F}\| = \sum_{ij} |f_{ij}|\right)$

a_{ij}	$j=1$	$j=2$	$j=3$
$i=1$	5.0261	6.0837	5.7800
$i=2$	4.6181	5.5898	5.3108
$i=3$	4.3577	5.2746	5.0113

$\max_j \sum_i |f_{ij}|$ norm is obvious by inspection in each case. Table 12.16 contains the $\|\mathbf{F}\| = \sum_{ij} |f_{ij}|$ norm for the nine coefficients.

12.3.7 Additional Measures of Coefficient Importance

Converting Output to Employment, Income, etc. As noted many times earlier in this book, gross outputs may not ultimately be the most important measure of economic impact. Gross output requirements can be translated into employment (for example, person-years) using employment coefficients (for example, person-hours per dollar's worth of each sector's output). If these coefficients are denoted $\mathbf{e}_c$ and total employment in each sector is represented by $\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{bmatrix}$, then $\Delta\boldsymbol{\varepsilon} = \hat{\mathbf{e}}_c \Delta\mathbf{x}$ converts changes in outputs to changes in employment. For example, from (12.36),

$$\Delta\varepsilon_r = (e_c)_r \Delta x_{r(ij)} = \frac{(e_c)_r l_{ri} x_j \Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}} = (e_c)_r l_{ri} k^2_{(ij)} \quad \text{where} \quad k^2_{(ij)} = \frac{x_j \Delta a_{ij}}{1 - l_{ji}\Delta a_{ij}}$$

The largest employment impact of Δa_{ij} will thus be in the sector with the largest $(e_c)_r l_{ri}$, and this is no longer assured to be sector i. Numerous other conversions are also possible – for example, to changes in income, value added, energy use, environmental impacts, and so forth.[36]

Elasticity Coefficient Analysis Several authors have suggested a variation of the measure of relative change that parallels the concept of *elasticity* in economics (see section 6.6), namely the relative change in $l_{rs(ij)}$ divided by the relative change in a_{ij}

$$\eta_{l_{rs(ij)}} = \frac{\frac{\Delta l_{rs(ij)}}{l_{rs}}}{\frac{\Delta a_{ij}}{a_{ij}}} = \frac{\frac{\Delta l_{rs(ij)}}{\Delta a_{ij}}}{\frac{l_{rs}}{a_{ij}}} = \left(\frac{\Delta l_{rs(ij)}}{\Delta a_{ij}}\right)\left(\frac{a_{ij}}{l_{rs}}\right) \tag{12.41}$$

From (12.33), this is

$$\eta_{l_{rs(ij)}} = \frac{l_{ri}l_{js}a_{ij}}{l_{rs}(1 - l_{ji}\Delta a_{ij})} = \frac{l_{ri}l_{js}}{l_{rs}}k^3_{(ij)} \tag{12.42}$$

where $k^3_{(ij)} = a_{ij}/(1 - l_{ji}\Delta a_{ij})$. Notice that this differs from the expression for $\Delta l_{rs(ij)}/l_{rs}$ in (12.33) only in that Δa_{ij} has been replaced by a_{ij} in the numerator. For any a_{ij}, there will be n^2 of these elasticities. Then Maaß (1980; cited in Schnabl, 2003) proposed the maximum of these elasticities as another measure of the importance of a_{ij} – $\text{Max}_{rs}(\eta_{lrs(ij)})$. From Observation 2, again, it is clear that

$$\text{Max}_{rs}(\eta_{l_{rs(ij)}}) = \frac{l_{ii}l_{jj}a_{ij}}{l_{ij}(1 - l_{ji}\Delta a_{ij})}$$

So, as noted by Schnabl, this elasticity analysis generates the same results as the important coefficient analysis above.

Replacement of $\Delta l_{rs(ij)}/l_{rs}$ in the numerator in (12.41) by $\Delta x_{r(ij)}/x_r$ will lead to an expression for the elasticity of gross output with respect to Δa_{ij}. And, just as gross output impacts can be translated into employment, income, value-added, etc. effects, these variations too can be converted to elasticity measures.

Relative Changes in All Gross Outputs A straightforward error measure that takes into account changes in *all* outputs is

$$E_{(ij)} = \mathbf{i}'\left|\Delta \mathbf{x}_{(ij)}\right| = \sum_{k=1}^{n}\left|\Delta x_{k(ij)}\right|$$

Or, to take account of the relative sizes of the sectors, Siebe (1996) suggests

$$\text{SUM}_{(ij)} = \sum_{k=1}^{n}\left|\Delta x_{k(ij)}/x_k\right|$$

[36] Tarancón *et al.* (2008) discuss in some detail the identification of important coefficients using alternative measures of economic welfare.

as a measure of importance of each coefficient, a_{ij}. As with previous measures, this could be transformed into an aggregate effect on employment, income, value-added, etc.

Impacts of Changes in more than One Element of the A *Matrix* Assessing the importance of each a_{ij} relative to all the others is carried out using one or more of the one-at-a-time approaches that we explored above. There has also been considerable work on the impacts of simultaneous changes (errors) in many or all a_{ij} coefficients. Indeed Sherman and Morrison (1949) considered cases with more than one change, but concentrated in a single row (or column). This was also explored in many publications by Schintke (1979 and elsewhere) and Schintke and Stäglin (1984 and elsewhere). Since this is somewhat peripheral to our "important coefficient" interests, we confine some of the background and results to Web Appendix 12W.2.

12.4 Summary

Initially in this chapter we explored the supply-side (Ghosh) model with both its early and later interpretations, in terms of quantity and price models, respectively. Various approaches to measuring linkages in an input–output system were the topic of section 12.2. Early approaches identified backward and forward linkages through appropriate row and column sums of the Leontief and Ghosh coefficient matrices (**A** and **B**) or their counterpart inverses, **L** and **G**. An alternative and more comprehensive view of linkage measurement grew out of the notion of hypothetical extraction, which can be implemented for backward, forward or total linkage measures. A detailed classification of hypothetical extraction possibilities is presented in Web Appendix 12W.1. The final topic considered in this chapter is the problem of how to define (conceptually) and identify (mathematically) "important" coefficients in an input–output system. Many approaches have been suggested. A major reason for interest in this topic is that it helps to identify where one might concentrate resources when trying to improve (for example, update) an input–output model's data base. Some historical background and details on this issue are relegated to Appendix 12.1 and Web Appendix 12W.2.

Appendix 12.1 The Sherman–Morrison–Woodbury Formulation

A12.1.1 Introduction

Given a nonsingular matrix, $\mathbf{M}$, and its inverse, suppose that one or more elements of $\mathbf{M}$ are changed, producing $\mathbf{M}^*$. The question is: can we find $(\mathbf{M}^*)^{-1} = [\mu^*_{ij}]$ by "adjusting" $\mathbf{M}^{-1} = [\mu_{ij}]$, which is already known? This is addressed by Sherman and Morrison (1949, 1950) for the case in which only one element is changed and by Woodbury (1950) for the case in which more than one element is changed. The answer is "yes," and the adjustment is relatively simple.[37] (Hereafter we will refer to the "SMW" results.)

Here is an illustration for the case of a change in one element only (Miller, 2000, pp. 281–286). Given

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 6 \\ 3 & 7 & 1 \end{bmatrix} \text{ and } \mathbf{M}^{-1} = \begin{bmatrix} 3.5 & -0.5 & -0.5 \\ -1.3333 & 0.1667 & 0.3333 \\ -1.1667 & 0.3333 & 0.1667 \end{bmatrix}$$

consider an $\mathbf{M}^*$ that differs from $\mathbf{M}$ only in that 3 has been added to m_{23}, changing it from a 6 to a 9. Let $\mathbf{M}^* = \mathbf{M} + \Delta\mathbf{M}$ where, in this case, $\Delta\mathbf{M} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. For later reference, we can easily find

$$(\mathbf{M}^*)^{-1} = \begin{bmatrix} 2.625 & -0.25 & -0.375 \\ -1.0417 & 0.0833 & 0.2917 \\ -0.5833 & 0.1667 & 0.0833 \end{bmatrix}$$

The idea is to find an alternative to the direct computation of $(\mathbf{M}^*)^{-1}$, making use only of $\mathbf{M}^{-1}$ and of the size of the change (here $\Delta m_{23} = 3$).[38]

The heart of the procedure is contained in two matrices (for this example, these are vectors). Let $\mathbf{C} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{R} = \begin{bmatrix} 0 & 0 & 3 \end{bmatrix}$; then $\Delta\mathbf{M} = \mathbf{CR}$. The trick is to let $\mathbf{C}$ be the ith column of an identity matrix (the same size as $\mathbf{M}$), where i identifies the *row* in $\mathbf{M}$ in which the change occurs, and where $\mathbf{R}$ is an appropriately sized null row vector with the jth element replaced by Δm_{ij}. The fundamental result is

$$(\mathbf{M}^*)^{-1} = \mathbf{M}^{-1} - \Delta\mathbf{M}^{-1} = \mathbf{M}^{-1} - \frac{(\mathbf{M}^{-1}\mathbf{C})(\mathbf{R}\mathbf{M}^{-1})}{(1 + \mathbf{R}\mathbf{M}^{-1}\mathbf{C})} \tag{A12.1.1}$$

This is not as complex as it might appear. The numerator of $\Delta\mathbf{M}^{-1}$ is the product of a column vector $\mathbf{M}^{-1}\mathbf{C}$ and a row vector $\mathbf{R}\mathbf{M}^{-1}$ and the denominator is simply a scalar.[39]

The expression for an individual element in $(\mathbf{M}^*)^{-1}$ follows directly from (A12.1.1). For a matrix $\mathbf{M}$ in which element m_{ij} is changed (increased or decreased) by Δm_{ij}, the value of the element in row r and column s of the new inverse, μ^*_{rs}, is

$$\mu^*_{rs} = \mu_{rs} - \frac{\mu_{ri}\mu_{js}\Delta m_{ij}}{1 + \mu_{ji}\Delta m_{ij}} \tag{A12.1.2}$$

[37] Henderson and Searle (1981) is an important reference on inverses of sums of matrices that seems generally ignored in the input–output literature. It includes at least six different variations on the SMW results and an extensive set of references.

[38] If changes in each of several a_{ij} are to be examined, it is helpful to use the notation $\mathbf{M}^*_{ij}$ in order to identify the specific case under consideration.

[39] A similar result can be derived with the roles of $\mathbf{R}$ and $\mathbf{C}$ interchanged (see Miller, 2000, Appendix 5.2).

The new elements in column i and row j of $(\mathbf{M}^*)^{-1}$ will be strictly proportional to the corresponding elements in $\mathbf{M}^{-1}$. For column i, when $s = i$,

$$\mu^*_{ri} = \mu_{ri} - \frac{\mu_{ri}\mu_{ji}\Delta m_{ij}}{1 + \mu_{ji}\Delta m_{ij}} = \frac{\mu_{ri} + \mu_{ri}\mu_{ji}\Delta m_{ij} - \mu_{ri}\mu_{ji}\Delta m_{ij}}{1 + \mu_{ji}\Delta m_{ij}} = \mu_{ri}k_{ij}$$

where $k_{ij} = 1/(1 + \mu_{ji}\Delta m_{ij})$ is a constant for a given Δm_{ij}, and exactly similar algebra shows that when $r = j$, $\mu^*_{js} = \mu_{js}k_{ij}$.

For the numerical example,

$$\mathbf{M}^{-1}\mathbf{C} = \begin{bmatrix} -0.5 \\ 0.1667 \\ 0.3333 \end{bmatrix}, \quad \mathbf{R}\mathbf{M}^{-1} = \begin{bmatrix} -3.5 & 1 & 0.5 \end{bmatrix} \text{ and } \mathbf{R}\mathbf{M}^{-1}\mathbf{C} = 1$$

so that, from (A12.1.1),

$$\Delta\mathbf{M}^{-1} = (0.5)\begin{bmatrix} 1.75 & -0.5 & -0.25 \\ -0.5833 & 0.1667 & 0.0833 \\ -1.1667 & 0.3333 & 0.1667 \end{bmatrix} = \begin{bmatrix} .875 & -0.25 & -0.125 \\ -0.2917 & 0.0833 & 0.0417 \\ -0.5833 & 0.1667 & 0.0833 \end{bmatrix}$$

and

$$\begin{aligned}(\mathbf{M}^*)^{-1} &= \mathbf{M}^{-1} - \Delta\mathbf{M}^{-1} \\ &= \begin{bmatrix} 3.5 & -0.5 & -0.5 \\ -1.3333 & 0.1667 & 0.3333 \\ -1.1667 & 0.3333 & 0.1667 \end{bmatrix} - \begin{bmatrix} .875 & -0.25 & -0.125 \\ -0.2917 & 0.0833 & 0.0417 \\ -0.5833 & 0.1667 & 0.0833 \end{bmatrix} \\ &= \begin{bmatrix} 2.625 & -0.25 & -0.375 \\ -1.0417 & 0.0833 & 0.2917 \\ -0.5833 & 0.1667 & 0.0833 \end{bmatrix}\end{aligned}$$

This is exactly the inverse that was found directly earlier in this Appendix. The reader can easily check the results in (A12.1.2) for any of the elements in $(\mathbf{M}^*)^{-1}$.

The (obvious) point is that a change (here an increase of 50 percent) in the value of just one element in $\mathbf{M}$ leads to changes in *all* elements in $\mathbf{M}^{-1}$. Note that some changes are increases, as with μ_{12} (and three other elements), and some are decreases, as with μ_{11} (and four other elements). Absolute values of the percentage changes can be found as[40] $|p_{ij}| = 100\left|(\mu^*_{ij} - \mu_{ij})/\mu_{ij}\right|$, or

$$|\mathbf{P}| = 100\left|[(\mathbf{M}^*)^{-1} - \mathbf{M}^{-1}] \oslash \mathbf{M}^{-1}\right|$$

where "$\oslash$" indicates element-by-element division. Here

$$|\mathbf{P}| = \begin{bmatrix} 25 & 50 & 25 \\ 21.875 & 50 & 12.5 \\ 50 & 50 & 50 \end{bmatrix}$$

40 Frequently the changes are expressed as $(\mu_{ij} - \mu^*_{ij})/\mu_{ij}$. This simply reverses signs. If absolute values are used, it makes no difference.

As expected for this example with a change in m_{23}, the elements in column 2 and row 3 of $(\mathbf{M}^*)^{-1}$are proportional to the corresponding elements in $\mathbf{M}^{-1}$, and hence the percentage changes are all the same.[41]

A12.1.2 Application to Leontief Inverses

The relevance to input–output models is that one can investigate the influence of changes (or "errors") in one or more elements of an **A** matrix on the associated Leontief inverse, $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$. Here we begin with $\mathbf{A}^* = \mathbf{A} + \Delta\mathbf{A}$ but since our interest is in $\mathbf{L}^* = (\mathbf{I} - \mathbf{A}^*)^{-1}$, the parallel to $\mathbf{M}^* = \mathbf{M} + \Delta\mathbf{M}$ is

$$(\mathbf{I} - \mathbf{A}^*) = [\mathbf{I} - (\mathbf{A} + \Delta\mathbf{A})] = (\mathbf{I} - \mathbf{A}) + (-\Delta\mathbf{A})$$

and the result in (A12.1.1) becomes

$$\mathbf{L}^* = \mathbf{L} + \frac{(\mathbf{LC})(\mathbf{RL})}{1 - \mathbf{RLC}} \tag{A12.1.3}$$

Notice that negative and positive signs are interchanged, compared to (A12.1.1).

In terms of an individual element in the new inverse, l^*_{rs}, the parallel to (A12.1.2) for a change Δa_{ij} is (with notation to remind us of which element in **A** is changed)

$$l^*_{rs(ij)} = l_{rs} + \frac{l_{ri} l_{js} \Delta a_{ij}}{1 - l_{ji} \Delta a_{ij}} \tag{A12.1.4}$$

Again, note the changes in signs, this time compared to (A12.1.2). Define percentage differences in Leontief inverse elements as $\Delta l_{rs(ij)} = (l^*_{rs(ij)} - l_{rs})/l_{rs}$; then

$$100\left[\frac{\Delta l_{rs(ij)}}{l_{rs}}\right] = 100\left[\frac{l_{ri} l_{js} \Delta a_{ij}}{1 - l_{ji} \Delta a_{ij}}\right]\left[\frac{1}{l_{rs}}\right] \tag{A12.1.5}$$

As before, all elements in row j and in column i of the matrix of absolute percentage differences will be the same.

Problems

12.1 The centrally planned economy of Czaria is involved in its planning for the next fiscal year. The technical coefficients and total industry outputs for Czaria are given below:

a. Compute the output inverse for this economy.

b. If next year's value-added inputs for agriculture, mining, military manufactured products, and civilian manufacturing in Czaria are projected to be \$4,558 million, \$5,665 million, \$2,050 million and \$5,079 million, respectively, compute the projected GDP for Czaria next year.

[41] The fact that all these changes are 50 percent (the same as the increase in m_{23}) is a coincidence of this example only. Moreover, some of the changes are 50 percent increases (μ_{12} and μ_{31}) and some are 50 percent decreases (μ_{22}, μ_{32}, and μ_{33}).

	1	2	3	4	Total Output
1. Agriculture	0.168	0.155	0.213	0.212	12,000
2. Mining	0.194	0.193	0.168	0.115	15,000
3. Military Manufacturing	0.105	0.025	0.126	0.124	12,000
4. Civilian Manufacturing	0.178	0.101	0.219	0.186	16,000

c. Compute the new total gross production for each economic sector. Note that this is the "old view" of the Ghosh model as described in section 12.1.1.

12.2 Consider a case where $\mathbf{Z} = \begin{bmatrix} 13 & 75 & 45 \\ 53 & 21 & 48 \\ 67 & 68 & 93 \end{bmatrix}$ and $\mathbf{f} = \begin{bmatrix} 130 \\ 150 \\ 220 \end{bmatrix}$ for base year.

a. If final demands for the next year are projected to be $\mathbf{f}^1 = \begin{bmatrix} 200 \\ 300 \\ 500 \end{bmatrix}$ and the change in interindustry transactions is expected to be $\Delta\mathbf{Z} = \begin{bmatrix} 0 & 5 & 0 \\ 10 & 0 & 0 \\ 0 & 0 & 15 \end{bmatrix}$ what is the mean absolute percentage difference (MAPD) between the output coefficients for the base year and next year?

b. Now compute MAPD between the corresponding output inverses.

12.3 For input–output transactions matrix of $\mathbf{Z} = \begin{bmatrix} 384 & 520 & 831 \\ 35 & 54 & 530 \\ 672 & 8 & 380 \end{bmatrix}$ and total outputs of $\mathbf{x} = \begin{bmatrix} 2500 \\ 1200 \\ 3000 \end{bmatrix}$ for a base year, if additional growth in value added for the next year is projected to result in $\mathbf{v}^{new} = \begin{bmatrix} 2000 \\ 1000 \\ 1500 \end{bmatrix}$, what are the price changes of output for the three industries for the new year relative to the base year?

12.4 For the economy shown in problem 12.3, compute the value-added coefficients for next year using the supply model. Compute $\mathbf{L}$ and show that the Leontief price model from Chapter 2 produces the same relative price changes of industrial output for the new year relative to the base year as found in problem 12.3.

12.5 Consider the case of $\mathbf{Z} = \begin{bmatrix} 418 & 687 & 589 & 931 \\ 847 & 527 & 92 & 654 \\ 416 & 702 & 911 & 763 \\ 263 & 48 & 737 & 329 \end{bmatrix}$ and $\mathbf{f} = \begin{bmatrix} 2000 \\ 3000 \\ 2500 \\ 1500 \end{bmatrix}$.

a. Compute the direct and total backward linkages.

b. Compute the direct and total forward linkages.

12.6 Consider the three-region IRIO table for Japan given in Table A4.1.1. Using the measure of spatial backward linkage of $B(d)^{rr} = (1/n)\mathbf{i}'\mathbf{A}^{rr}\mathbf{i}$ (and analogous measures for direct forward and total backward and forward linkage), which of the three regions is the "least backward linked" to the other regions and, similarly, which region is the least "forward linked"?

12.7 Consider the 2005 US input–output table provided in Appendix B.

a. If the agriculture sector were hypothetically extracted from the economy, what would be the decrease in total output of the economy?

b. Which of the sectors would create the largest decrease in total output if it were hypothetically extracted?

12.8 Consider an economy with $\mathbf{Z} = \begin{bmatrix} 8 & 64 & 89 \\ 28 & 44 & 77 \\ 48 & 24 & 28 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 300 \\ 250 \\ 200 \end{bmatrix}$. Examine element a_{13} for "inverse importance" if the criteria are:

a. $\alpha = 30$ and $\beta = 5$ – that is, if a 30 percent change in a_{13} generates a 5 percent change in one or more elements in the associated Leontief inverse.

b. $\alpha = 20$ and $\beta = 10$.

c. $\alpha = 10$ and $\beta = 10$.

This illustrates the sensitivity of the results to the values of α and β specified by the analyst.

12.9 Create a supply-driven model for the US economy for 2005 using the data that are presented in Appendix B. Determine the sensitivity of the national economy to an interruption in a scarce-factor input – for example, a strike – in one of the sectors.

12.10 Using the input–output data for the United States presented in Appendix B, find both the direct and the total forward and backward linkages for the sectors in the US economy and examine how these linkages may have changed over time.

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13 Structural Decomposition, Mixed and Dynamic Models

13.1 Structural Decomposition Analysis

When there are two or more sets of input–output data for an economy, analysts are often interested in trying to disaggregate the total amount of change in some aspect of that economy into contributions made by its various components. For example, the total change in gross outputs between two periods could be broken down into that part associated with changes in technology (as reflected, initially, in the changes in the Leontief inverse for the economy over the period) and that part related to changes in final demand over the period.

At the next level, the total change in the Leontief inverse matrix could be disaggregated into a part that is associated with changes in technology within each sector (as reflected in changes in the direct input coefficients matrix) and that part associated with changes in product mix within each sector. Similarly, the change in final demand could be further disaggregated into a part that reflects changes in the overall level of final demand and a part that captures changes in the composition of final demand. And there are numerous additional options – for example, there is no need to use only two contributing factors; changes in employment, value added, energy use, etc. may be of more economic interest than changes in gross outputs; and so on. For a general overview of this literature, see Rose and Casler (1996) or Dietzenbacher and Los (1997, 1998). Two early empirical examples of this kind of work can be found in Feldman, McClain and Palmer (1987) for the USA and Skolka (1989) for Austria.[1]

13.1.1 Initial Decompositions: Changes in Gross Outputs

To get a general idea of the structural decomposition analysis (SDA) approach, we initially explore gross output changes. Assume that there are two time periods for which input–output data are available. Using superscripts 0 and 1 for the two different years (0 earlier than 1), our illustration of structural decomposition in an input–output

[1] Schumann (1994), expanding on Schumann (1990), argues for the superiority of semi-closed models (for example, to household consumption) in general but also claims that structural decomposition analyses with such models lead to inferior results because they isolate sources of structural change that are less clear cut and more complex.

model focuses on the differences in the gross output vectors for those two years. As usual, gross outputs in year t, $\mathbf{x}^t (t = 0, 1)$, are found in an input–output system as

$$\mathbf{x}^1 = \mathbf{L}^1\mathbf{f}^1 \text{ and } \mathbf{x}^0 = \mathbf{L}^0\mathbf{f}^0 \tag{13.1}$$

where $\mathbf{f}^t$ = the vector of final demands in year t, and $\mathbf{L}^t = (\mathbf{I} - \mathbf{A}^t)^{-1}$. Then the observed change in gross outputs over the period is

$$\Delta\mathbf{x} = \mathbf{x}^1 - \mathbf{x}^0 = \mathbf{L}^1\mathbf{f}^1 - \mathbf{L}^0\mathbf{f}^0 \tag{13.2}$$

The task is to decompose the total change in outputs into changes in the various components – in (13.2) that would (at least initially) mean separation into changes in $\mathbf{L}$ ($\Delta\mathbf{L} = \mathbf{L}^1 - \mathbf{L}^0$) and changes in $\mathbf{f}$ ($\Delta\mathbf{f} = \mathbf{f}^1 - \mathbf{f}^0$).[2] In order to remove the influence of price changes, we assume that all data are expressed in prices for a common year.

A number of alternative expansions and rearrangements of the terms in (13.2) can be derived. For example, using only year-1 values for $\mathbf{L}$ and only year-0 values for $\mathbf{f}$ – replacing $\mathbf{L}^0$ with $(\mathbf{L}^1 - \Delta\mathbf{L})$ and $\mathbf{f}^1$ with $(\mathbf{f}^0 + \Delta\mathbf{f})$ in (13.2) – we have

$$\Delta\mathbf{x} = \mathbf{L}^1(\mathbf{f}^0 + \Delta\mathbf{f}) - (\mathbf{L}^1 - \Delta\mathbf{L})\mathbf{f}^0 = (\Delta\mathbf{L})\mathbf{f}^0 + \mathbf{L}^1(\Delta\mathbf{f}) \tag{13.3}$$

This simple algebra produces a straightforward decomposition of the total change in gross outputs into (1) a part that is attributable to changes in technology, $\Delta\mathbf{L}$, in this case weighted by year-0 final demands ($\mathbf{f}^0$), and (2) a part that reflects final-demand changes, $\Delta\mathbf{f}$, which are here weighted by year-1 technology ($\mathbf{L}^1$).

Notice that each term on the right-hand side of (13.3) has a certain amount of intuitive appeal – for example, $(\Delta\mathbf{L})\mathbf{f}^0 = \mathbf{L}^1\mathbf{f}^0 - \mathbf{L}^0\mathbf{f}^0$. The first term quantifies the output that would be needed to satisfy old (year-0) demand with new (year-1) technology; the second term is, of course, the output needed to satisfy old demand with old technology. So the difference is one reasonable measure of the effect of technology change. And $\mathbf{L}^1(\Delta\mathbf{f})$ in (13.3) has a similar kind of interpretation.

Alternatively, using only year-0 values for $\mathbf{L}$ and only year-1 values for $\mathbf{f}$, which means replacing $\mathbf{L}^1$ with $(\mathbf{L}^0 + \Delta\mathbf{L})$ and $\mathbf{f}^0$ with $(\mathbf{f}^1 - \Delta\mathbf{f})$, (13.2) becomes

$$\Delta\mathbf{x} = (\mathbf{L}^0 + \Delta\mathbf{L})\mathbf{f}^1 - \mathbf{L}^0(\mathbf{f}^1 - \Delta\mathbf{f}) = (\Delta\mathbf{L})\mathbf{f}^1 + \mathbf{L}^0(\Delta\mathbf{f}) \tag{13.4}$$

In this case, the technology change contribution is weighted by year-1 final demands and the final-demand change contribution is weighted by year-0 technology.

These alternatives, in (13.3) and (13.4), are equally valid in the sense that both are "mathematically correct," given (13.2) and the definitions $\Delta\mathbf{L} = \mathbf{L}^1 - \mathbf{L}^0$ and $\Delta\mathbf{f} = \mathbf{f}^1 - \mathbf{f}^0$. Yet clearly the measures in (13.3) of the individual contributions from changed technology and from changed final demands will be different from those in (13.4), except in the totally uninteresting and implausible case where $\mathbf{L}^1 = \mathbf{L}^0$ and/or $\mathbf{f}^1 = \mathbf{f}^0$ –

[2] In section 7.2.1 we explored some of the most frequently used approaches to assessing overall structural change. One frequently used measure was to compare $\mathbf{x}^1 = \mathbf{L}^1\mathbf{f}^1$ with $\mathbf{L}^0\mathbf{f}^1$, the output that $\mathbf{f}^1$ would have generated with $\mathbf{L}^0$ technology.

no change in technology or no change in demand (or no change in either) over the period. The results in (13.3) and (13.4) can be derived from (13.2) in another way. For example, adding and subtracting $\mathbf{L}^1\mathbf{f}^0$ to (13.2), and rearranging, gives (13.3). Similarly, adding and subtracting $\mathbf{L}^0\mathbf{f}^1$ (outputs needed if year-1 demands were satisfied using year-0 technology) to (13.2) gives (13.4), after rearrangement.

And there is more. Other expressions emerge if only year-0 or only year-1 values are used for weights on both change terms. If we use year-0 weights exclusively, so that $\mathbf{L}^1$ and $\mathbf{f}^1$ are replaced by $(\mathbf{L}^0 + \Delta\mathbf{L})$ and $(\mathbf{f}^0 + \Delta\mathbf{f})$, then (13.2) becomes

$$\Delta\mathbf{x} = (\mathbf{L}^0 + \Delta\mathbf{L})(\mathbf{f}^0 + \Delta\mathbf{f}) - \mathbf{L}^0\mathbf{f}^0 = (\Delta\mathbf{L})\mathbf{f}^0 + \mathbf{L}^0(\Delta\mathbf{f}) + (\Delta\mathbf{L})(\Delta\mathbf{f}) \qquad (13.5)$$

In this case, both technology and final-demand changes are weighted by year-0 values, but an additional ("interaction") term – $(\Delta\mathbf{L})(\Delta\mathbf{f})$ – has appeared. Unlike the first two terms in (13.5), this new interaction term does not have an intuitively appealing interpretation.[3]

Finally, using only year-1 weights means putting $\mathbf{L}^0 = (\mathbf{L}^1 - \Delta\mathbf{L})$ and $\mathbf{f}^0 = (\mathbf{f}^1 - \Delta\mathbf{f})$ into (13.2), which becomes

$$\Delta\mathbf{x} = \mathbf{L}^1\mathbf{f}^1 - (\mathbf{L}^1 - \Delta\mathbf{L})(\mathbf{f}^1 - \Delta\mathbf{f}) = (\Delta\mathbf{L})\mathbf{f}^1 + \mathbf{L}^1(\Delta\mathbf{f}) - (\Delta\mathbf{L})(\Delta\mathbf{f}) \qquad (13.6)$$

again with the same interaction term, only this time it is subtracted rather than added.[4]

Various researchers have worked with one or more of these four alternatives. For example, Skolka (1989) presented the first three decompositions;[5] Rose and Chen (1991) work only with the expression in equation (13.5), although ultimately in an expanded form. Vaccara and Simon (1968) used the factorizations in (13.3) and (13.4), then averaged the two measures of final-demand change and the two measures of coefficient change. This is also the approach of Feldman, McClain and Palmer (1987), Miller and Shao (1994) and others. Dietzenbacher and Los (1998) examine a wide variety of possible decompositions and conclude that using an average of results from (13.3) and (13.4) is often an acceptable approach.[6]

We can view this as follows. Adding (13.3) and (13.4) gives

$$2\Delta\mathbf{x} = (\Delta\mathbf{L})\mathbf{f}^0 + \mathbf{L}^1(\Delta\mathbf{f}) + (\Delta\mathbf{L})\mathbf{f}^1 + \mathbf{L}^0(\Delta\mathbf{f})$$

and so

$$\Delta\mathbf{x} = (1/2)\underbrace{(\Delta\mathbf{L})(\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Technology change}} + (1/2)\underbrace{(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{f})}_{\text{Final-demand change}} \qquad (13.7)$$

[3] Derivation of this result by adding and subtracting like terms in (13.2) is possible but more complicated. In fact, it requires that $\mathbf{L}^1\mathbf{f}^0$, $\mathbf{L}^0\mathbf{f}^1$, and $\mathbf{L}^0\mathbf{f}^0$ all be both added and subtracted and then (considerably) rearranged.

[4] This result can be derived by adding and subtracting $\mathbf{L}^1\mathbf{f}^0$, $\mathbf{L}^0\mathbf{f}^1$, and $\mathbf{L}^1\mathbf{f}^1$ in (13.2) and (again) extensive algebraic rearrangement.

[5] He also classifies much of the pre-1989 work in this area according to which version of the decomposition was used.

[6] Not everyone would agree. Fromm (1968) discusses the index number issues that are involved in finding averages of measures with weights from different years. In terms of (13.3), the $(\Delta\mathbf{L})\mathbf{f}^0$ term is a kind of Laspeyres index (original year weights, in $\mathbf{f}^0$) and the $\mathbf{L}^1(\Delta\mathbf{f})$ term is a kind of Paasche index (terminal year weights in $\mathbf{L}^1$); in (13.4) the Laspeyres and Paasche terms are reversed. He suggests that averaging the two – (13.3) and (13.4) – gives a "... bastard measure of beginning- and end-point quantities and prices" (p. 65).

[The average in (13.7) is the same as the average of the results in (13.5) and (13.6), as the reader can easily show.][7]

Numerical Example Here is a small numerical illustration of these decompositions. Let

$$\mathbf{Z}^0 = \begin{bmatrix} 10 & 20 & 25 \\ 15 & 5 & 30 \\ 30 & 40 & 5 \end{bmatrix}, \quad \mathbf{f}^0 = \begin{bmatrix} 45 \\ 30 \\ 25 \end{bmatrix}, \quad \mathbf{Z}^1 = \begin{bmatrix} 12 & 15 & 35 \\ 24 & 11 & 30 \\ 36 & 50 & 8 \end{bmatrix}, \quad \mathbf{f}^1 = \begin{bmatrix} 50 \\ 35 \\ 26 \end{bmatrix}$$

From $\mathbf{x}^0 = \mathbf{Z}^0\mathbf{i} + \mathbf{f}^0$ and $\mathbf{x}^1 = \mathbf{Z}^1\mathbf{i} + \mathbf{f}^1$, $\mathbf{L}^0$ and $\mathbf{L}^1$ are easily found, as are

$$\Delta\mathbf{L} = \begin{bmatrix} .0649 & -.0941 & .0320 \\ .1447 & .0607 & .0116 \\ .1448 & .0342 & .0586 \end{bmatrix}, \quad \Delta\mathbf{f} = \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix} \text{ and } \quad \Delta\mathbf{x} = \begin{bmatrix} 12 \\ 20 \\ 20 \end{bmatrix}$$

The alternative decompositions of $\Delta\mathbf{x}$, for this example, are shown in Table 13.1.[8]

It should be noted at the outset that input–output structural decomposition studies generate, by definition, results at the sectoral level. For an n-sector model, each element in the n-element vector of changes – $\Delta\mathbf{x}$ in the case of gross outputs – will be decomposed into two or more constituent elements. This means that there is an inherent problem in finding appropriate summary measures of results in these studies. One obvious solution is to use total (economy-wide) figures – in the case of the decomposition in (13.7), this would be[9]

$$\mathbf{i}'(\Delta\mathbf{x}) = \underbrace{\mathbf{i}'[(1/2)(\Delta\mathbf{L})(\mathbf{f}^0 + \mathbf{f}^1)]}_{\text{Economy-wide technology change effect}} + \underbrace{\mathbf{i}'[(1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{f})]}_{\text{Economy-wide final-demand change effect}}$$

Alternatives include grouping sectors into categories and then finding averages (simple or weighted) over the smaller numbers of elements in these groupings. For example: "fastest growing sectors" (say the top x percent), "slowest growing (fastest declining) sectors" (the bottom x percent) and other sectors [the middle $(100 - 2x)$ percent], or primary (natural resource related), secondary (manufacturing and processing) and tertiary (support and service oriented) sectors. As will be clear from this small example and from the empirical studies examined in section 13.2.5, any such economy-wide or

[7] There is some not very illuminating discussion in the literature about terms in (13.3) or (13.4) "absorbing" the interaction term. Starting with a rearranged (13.5), $[(\Delta\mathbf{L})\mathbf{f}^0 + (\Delta\mathbf{L})(\Delta\mathbf{f})] + \mathbf{L}^0(\Delta\mathbf{f}) \Rightarrow (\Delta\mathbf{L})\mathbf{f}^1 + \mathbf{L}^0(\Delta\mathbf{f})$, which is (13.4), and so $(\Delta\mathbf{L})\mathbf{f}^1$ incorporates the interaction term $[+(\Delta\mathbf{L})(\Delta\mathbf{f})]$. Equally plausible, however, is viewing (13.5) as $(\Delta\mathbf{L})\mathbf{f}^0 + [\mathbf{L}^0(\Delta\mathbf{f}) + (\Delta\mathbf{L})(\Delta\mathbf{f})] \Rightarrow (\Delta\mathbf{L})\mathbf{f}^0 + \mathbf{L}^1(\Delta\mathbf{f})$ which is (13.3), and now it is $\mathbf{L}^1(\Delta\mathbf{f})$ that has absorbed $[+(\Delta\mathbf{L})(\Delta\mathbf{f})]$. Similar rearrangements of (13.6) will show that $(\Delta\mathbf{L})\mathbf{f}^0$ in (13.3) or $\mathbf{L}^0(\Delta\mathbf{f})$ in (13.4) could be viewed as absorbing $[-(\Delta\mathbf{L})(\Delta\mathbf{f})]$. Mathematically, the result in (13.7) allocates one-half of the interaction term to technical change and one-half to final-demand change. See also Casler (2001) for thoughts on the interaction term.

[8] The reader can easily identify the various "absorptions" in the previous footnote in terms of the results in this table.

[9] Dividing both sides by n would generate one kind of "average" figure.

Table 13.1 Alternative Structural Decompositions

	Technology Change Contribution	Final-Demand Change Contribution	Interaction Term
Equation (13.3)	$\begin{bmatrix} 0.90 \\ 8.62 \\ 9.01 \end{bmatrix}$	$\begin{bmatrix} 11.10 \\ 11.38 \\ 10.99 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
Equation (13.4)	$\begin{bmatrix} 0.78 \\ 9.66 \\ 9.96 \end{bmatrix}$	$\begin{bmatrix} 11.22 \\ 10.34 \\ 10.04 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
Equation (13.5)	$\begin{bmatrix} 0.90 \\ 8.62 \\ 9.01 \end{bmatrix}$	$\begin{bmatrix} 11.22 \\ 10.34 \\ 10.04 \end{bmatrix}$	$+\begin{bmatrix} -0.12 \\ 1.04 \\ 0.95 \end{bmatrix}$
Equation (13.6)	$\begin{bmatrix} 0.78 \\ 9.66 \\ 9.96 \end{bmatrix}$	$\begin{bmatrix} 11.10 \\ 11.38 \\ 10.99 \end{bmatrix}$	$-\begin{bmatrix} -0.12 \\ 1.04 \\ 0.95 \end{bmatrix}$
Equation (13.7)	$\begin{bmatrix} 0.84 \\ 9.14 \\ 9.49 \end{bmatrix}$	$\begin{bmatrix} 11.16 \\ 10.86 \\ 10.51 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Table 13.2 Sector-Specific and Economy-Wide Decomposition Results [Equation (13.7)]

	Output Change	Technology Change Contribution	Final-Demand Change Contribution
Sector 1	12	0.84 (7)	11.16 (93)
Sector 2	20	9.14 (46)	10.86 (54)
Sector 3	20	9.49 (47)	10.51 (53)
Economy-wide Total	52	19.47 (37)	32.53 (63)

averaging figures sweep an enormous amount of detail (and, usually, variation) under the rug.

Table 13.2 emphasizes the results from (13.7). Figures in parentheses indicate percentages of the total output change in each row. (Since these are hypothetical figures for illustration only, there is no need to be compulsive about detail in the percentages. We use no places to the right of the decimal.)

Of the economy-wide total output change in this example, 37 percent is seen to be attributable to technological change and 63 percent results from changes in final demand. But variation across sectors is large. The technology change contribution to individual sector output growth varies from 7 to 47 percent and (therefore) the final demand contribution varies from 53 to 93 percent.

13.1.2 Next-Level Decompositions: Digging Deeper into $\Delta\mathbf{f}$ *and* $\Delta\mathbf{L}$

Of course the story need not and does not end with the decompositions in (13.3)–(13.7). Changes in final demands, for example, may be the result of a change in the overall *level* of final demand or of a change in the relative proportions of expenditure on the various goods and services in the final-demand vector (the final-demand *mix*). Or, indeed, final-demand data may be collected and presented in several vectors, one for each final-demand *category*, such as household consumption, exports, government spending (federal, state, and local), and so on, and the relative importance of these categories may change.

Similarly, changes in the Leontief inverse result from changes in the economy's **A** matrix – which, in turn, may reflect various aspects of technology change, such as changes in production recipes (replacing metals with plastics in automobiles), substitutions caused by relative price changes (for domestically produced inputs and also for imports), reductions in a sector's materials inputs per unit of output brought about by economies of scale, and so on – as noted in section 7.2. We examine some approaches to account for these "next-level" effects. Before doing that, we need to generalize the decomposition results.

Additive Decompositions with Products of more than Two Terms The results in (13.7), above, can be looked at in the following way, which lends itself to generalization. Let $y^t = x_1^t x_2^t$ represent the general case in which the product of two variables (scalars, vectors, matrices or appropriate combinations) defines a dependent variable; the particular example here is $\mathbf{x}^t = \mathbf{L}^t \mathbf{f}^t$. Then the decompositions of $\Delta y = x_1^1 x_2^1 - x_1^0 x_2^0$ in (13.3) and (13.4) are seen to be of the form $\Delta y = (\Delta x_1)x_2^0 + x_1^1(\Delta x_2)$ and $\Delta y = (\Delta x_1)x_2^1 + x_1^0(\Delta x_2)$, respectively. Specifically, year-0 weights are to the right of a change term and year-1 weights are to the left in (13.3), and the year-0 and year-1 terms are reversed for (13.4).

An approach for the case of more than two terms, as in $y^t = x_1^t x_2^t \ldots x_n^t$, is to extend the logic of these two alternatives.[10] We begin with the case of $n = 3$, where $y^t = x_1^t x_2^t x_3^t$ and hence $\Delta y = x_1^1 x_2^1 x_3^1 - x_1^0 x_2^0 x_3^0$. Persistent and tedious substitutions from $x_1^1 = x_1^0 + \Delta x_1$, $x_2^1 = x_2^0 + \Delta x_2$ and $x_3^1 = x_3^0 + \Delta x_3$ will lead to

$$\Delta y = (\Delta x_1)x_2^0 x_3^0 + x_1^1(\Delta x_2)x_3^0 + x_1^1 x_2^1(\Delta x_3) \tag{13.8}$$

Alternative substitutions and rearrangements will generate

$$\Delta y = (\Delta x_1)x_2^1 x_3^1 + x_1^0(\Delta x_2)x_3^1 + x_1^0 x_2^0(\Delta x_3) \tag{13.9}$$

[10] These are not the only options. See Dietzenbacher and Los (1998) for a very thorough discussion of alternatives.

The usual averaging leads to

$$\Delta y = (1/2)(\Delta x_1)(x_2^0 x_3^0 + x_2^1 x_3^1) + (1/2)[x_1^0(\Delta x_2)x_3^1 + x_1^1(\Delta x_2)x_3^0] + (1/2)(x_1^0 x_2^0 + x_1^1 x_2^1)(\Delta x_3) \tag{13.10}$$

[Notice that the (1/2) terms result from averaging the *two* expressions for Δy in (13.8) and (13.9). They are unrelated to the *number* of elements in each of the terms on the right-hand sides of Δy.]

There are similar results for $n > 3$. The pattern is the same in the equations parallel to (13.8) and (13.9) – year-0 (year-1) weights always appear on the right of the Δx term and year-1 (year-0) weights always appear on the left. The generalization is straightforward but, again, the algebra is tedious. The parallel to (13.8) is

$$\Delta y = (\Delta x_1)(x_2^0 \dots x_n^0) + x_1^1(\Delta x_2)(x_3^0 \dots x_n^0) + \cdots + (x_1^1 \dots x_{n-2}^1)(\Delta x_{n-1})x_n^0 + (x_1^1 \dots x_{n-1}^1)(\Delta x_n) \tag{13.11}$$

The parallel to (13.9) has exactly the structure of (13.11) with superscripts "0" and "1" reversed. We write out the n-variable extension of (13.10), for completeness.

$$\begin{aligned}\Delta y = {} & (1/2)(\Delta x_1)[(x_2^0 \dots x_n^0) + (x_2^1 \dots x_n^1)] \\ & + (1/2)[x_1^0(\Delta x_2)(x_3^1 \dots x_n^1) + x_1^1(\Delta x_2)(x_3^0 \dots x_n^0)] \\ & + \cdots + (1/2)[(x_1^0 \dots x_{n-2}^0)(\Delta x_{n-1})x_n^1 + (x_1^1 \dots x_{n-2}^1)(\Delta x_{n-1})x_n^0] \\ & + (1/2)[(x_1^0 \dots x_{n-1}^0) + (x_1^1 \dots x_{n-1}^1)](\Delta x_n)\end{aligned} \tag{13.12}$$

Changes in Final Demand Among the factors that may contribute to changes in final demands between two periods are: (1) the total amount of all expenditures for final demands – the final-demand *level*; (2) the *distribution* of total expenditure across final-demand categories – for example, the total value of household consumption, exports (possibly broken down by countries of destination), government expenditures (possibly separated into federal, state, and local), and other final demands, as proportions of total final-demand expenditure; and (3) the *product mix* within each particular final-demand category – for example, the proportion of total household consumption expenditure that goes to computers and computer services. This is reflected in the coefficients in the bridge matrix (see below).

In an n-sector input–output model, if there are p categories of final demand – instead of a single final-demand vector, $\underset{(n\times 1)}{\mathbf{f}^t}$ – then we have a final-demand matrix, $\underset{(n\times p)}{\mathbf{F}^t} = [\mathbf{f}_1^t, \dots, \mathbf{f}_p^t]$, where $\mathbf{f}_k^t = \begin{bmatrix} f_{1k}^t \\ \vdots \\ f_{nk}^t \end{bmatrix}$, and f_{ik}^t records the amount of expenditure by final-demand category k on the product of sector i in year t. In particular,

a. $\mathbf{F}^t\mathbf{i} = \mathbf{f}^t$, the n-element vector of total final-demand deliveries from each sector in year t.
b. $\mathbf{i}'\mathbf{F}^t\mathbf{i} = \mathbf{i}'\mathbf{f}^t = f^t$, the *level* (total amount) of final-demand expenditure over all sectors in year t.
c. Let $\mathbf{y}^t = (\mathbf{i}'\mathbf{F}^t)' = \begin{bmatrix} y_1^t \\ \vdots \\ y_p^t \end{bmatrix}$, where y_k^t = total final-demand expenditure by final-demand category k in year t).

The vector that indicates the *distribution* of f^t across the p final-demand categories is found as the column sums of $\mathbf{F}^t$ divided by f^t, or

$$\underset{(p\times 1)}{\mathbf{d}^t} = [d_k^t] = (1/f^t)\mathbf{y}^t = \begin{bmatrix} y_1^t/f^t \\ \vdots \\ y_p^t/f^t \end{bmatrix} \tag{13.13}$$

So d_k^t represents the *proportion* of total final-demand expenditure in year t that originated in category k. Finally, the bridge (*product mix*) matrix, $\underset{(n\times p)}{\mathbf{B}^t}$, is

$$\mathbf{B}^t = [b_{ik}^t] = (\mathbf{F}^t)(\hat{\mathbf{y}}^t)^{-1} \tag{13.14}$$

So $\mathbf{B}^t$ is $\mathbf{F}^t$ normalized by its column sums – $b_{ik}^t = f_{ik}^t/y_k^t$ indicates the *proportion* of total expenditures by final-demand category k that was spent on the product of sector i in year t.[11]

With these definitions,

$$\mathbf{f}^t = f^t\mathbf{B}^t\mathbf{d}^t = \mathbf{B}^t\mathbf{y}^t \tag{13.15}$$

and so

$$\Delta\mathbf{f} = \mathbf{f}^1 - \mathbf{f}^0 = f^1\mathbf{B}^1\mathbf{d}^1 - f^0\mathbf{B}^0\mathbf{d}^0 = \mathbf{B}^1\mathbf{y}^1 - \mathbf{B}^0\mathbf{y}^0 \tag{13.16}$$

This holds for data with either only one final-demand vector ($p = 1$) or with several final-demand categories ($p > 1$). In the former case, $\mathbf{F}^t = \mathbf{f}^t = \begin{bmatrix} f_1^t \\ \vdots \\ f_n^t \end{bmatrix}$, $f^t = \mathbf{y}^t$ (a scalar), $\mathbf{B}^t$ is a column vector ($b_i^t = f_i^t/f^t = f_i^t/\mathbf{y}^t$) and $\mathbf{d}^t = 1$. In the latter case, the final-demand matrix, disaggregated by categories, is seen to be $\mathbf{F}^t = \mathbf{B}^t\hat{\mathbf{y}}^t$.

[11] This use of **B** is not to be confused with the output coefficients matrix in the Ghosh model.

Decomposing the final-demand change in (13.16) as in (13.8), (13.9), and (13.10) gives

$$\Delta \mathbf{f} = (\Delta f)\mathbf{B}^0\mathbf{d}^0 + f^1(\Delta \mathbf{B})\mathbf{d}^0 + f^1\mathbf{B}^1(\Delta \mathbf{d}) \tag{13.17}$$

$$\Delta \mathbf{f} = (\Delta f)\mathbf{B}^1\mathbf{d}^1 + f^0(\Delta \mathbf{B})\mathbf{d}^1 + f^0\mathbf{B}^0(\Delta \mathbf{d}) \tag{13.18}$$

and

$$\begin{aligned}\Delta \mathbf{f} = &\underbrace{(1/2)(\Delta f)(\mathbf{B}^0\mathbf{d}^0 + \mathbf{B}^1\mathbf{d}^1)}_{\text{Final-demand } \textit{level} \text{ effect}} \\ &+ \underbrace{(1/2)[f^0(\Delta \mathbf{B})\mathbf{d}^1 + f^1(\Delta \mathbf{B})\mathbf{d}^0]}_{\text{Final-demand } \textit{mix} \text{ effect}} \\ &+ \underbrace{(1/2)(f^0\mathbf{B}^0 + f^1\mathbf{B}^1)(\Delta \mathbf{d})}_{\text{Final-demand } \textit{distribution} \text{ effect}}\end{aligned} \tag{13.19}$$

When $p = 1, \Delta \mathbf{d} = 0$, and the third terms disappear from (13.17)–(13.19); in fact, (13.19) is simplified to

$$\Delta \mathbf{f} = \underbrace{(1/2)(\Delta f)(\mathbf{B}^0 + \mathbf{B}^1)}_{\text{Final-demand level effect}} + \underbrace{(1/2)(f^0 + f^1)(\Delta \mathbf{B})}_{\text{Final-demand mix effect}} \tag{13.20}$$

13.1.3 Numerical Examples

One Category of Final Demand ($p = 1$) Continuing with the same numerical illustration,[12]

$$\mathbf{B}^0 = \begin{bmatrix} .45 \\ .3 \\ .25 \end{bmatrix}, \quad \mathbf{B}^1 = \begin{bmatrix} .4505 \\ .3153 \\ .2342 \end{bmatrix}, \quad \Delta \mathbf{B} = \begin{bmatrix} .0005 \\ .0153 \\ -.0158 \end{bmatrix}, \quad f^1 = 111, \quad f^0 = 100$$

Notice that (by definition) the column sums in $\mathbf{B}^0$ and $\mathbf{B}^1$ must be one and so the column sum in $\Delta\mathbf{B}$ must be zero; there must be one or more negative elements in $\Delta\mathbf{B}$ to balance one or more positive elements. This means that the final-demand mix effect for at least one sector – the second term in (13.20) – must be negative. In this numerical illustration, sector 3 has become relatively less important in total final-demand spending. Using (13.20) leads to the results shown in Table 13.3.

Two Categories of Final Demand ($p = 2$) Suppose that data are available on two categories of final demand – for example, households and all other final demand.

[12] It is necessary to work with more than two decimal places in these calculations, but results will continue to be rounded to two.

Table 13.3 Sector-Specific and Economy-Wide Decomposition Results (with Two-Factor Final-Demand Decomposition Detail)[a]

	Output Change	Final-Demand Change Contribution		
		Level	Mix	Total
Sector 1	12	11.05 (92)	.11 (1)	11.16 (93)
Sector 2	20	9.35 (47)	1.51 (7)	10.86 (54)
Sector 3	20	11.45 (57)	−.94 (−5)	10.51 (53)
Total	52	31.85 (61)	.68 (1)	32.53 (63)

[a] In this and later tables, percentages are shown with no decimal places, so there may be (small) discrepancies between the total effect and the sum of its parts.

Consistent with the numerical illustration, let

$$\mathbf{F}^0 = [\mathbf{f}_1^0\ \mathbf{f}_2^0] = \begin{bmatrix} 20 & 25 \\ 10 & 20 \\ 15 & 10 \end{bmatrix} \text{ and } \mathbf{F}^1 = [\mathbf{f}_1^1\ \mathbf{f}_2^1] = \begin{bmatrix} 25 & 25 \\ 15 & 20 \\ 18 & 8 \end{bmatrix}$$

Then

$$\mathbf{d}^0 = \begin{bmatrix} 45/100 \\ 55/100 \end{bmatrix} = \begin{bmatrix} 0.4500 \\ 0.5500 \end{bmatrix} \text{ and } \mathbf{d}^1 = \begin{bmatrix} 58/111 \\ 53/111 \end{bmatrix} = \begin{bmatrix} 0.5225 \\ 0.4775 \end{bmatrix}$$

and the bridge matrices are

$$\mathbf{B}^0 = \begin{bmatrix} 20 & 25 \\ 10 & 20 \\ 15 & 10 \end{bmatrix} \begin{bmatrix} 1/45 & 0 \\ 0 & 1/55 \end{bmatrix} = \begin{bmatrix} 0.4444 & 0.4545 \\ 0.2222 & 0.3636 \\ 0.3333 & 0.1818 \end{bmatrix} \text{ and}$$

$$\mathbf{B}^1 = \begin{bmatrix} 25 & 25 \\ 15 & 20 \\ 18 & 8 \end{bmatrix} \begin{bmatrix} 1/58 & 0 \\ 0 & 1/53 \end{bmatrix} = \begin{bmatrix} 0.4310 & 0.4717 \\ 0.2586 & 0.3774 \\ 0.3103 & 0.1509 \end{bmatrix}$$

Finally,

$$\Delta\mathbf{d} = \begin{bmatrix} .0725 \\ -.0725 \end{bmatrix}, \quad \Delta\mathbf{B} = \begin{bmatrix} -.0134 & .0172 \\ .0364 & .0137 \\ -.0230 & -.0309 \end{bmatrix}, \quad \Delta f = 11$$

The decomposition in (13.19) generates the results in Table 13.4. Notice that, again by definition, column sums in $\Delta\mathbf{d}$ (as with $\Delta\mathbf{B}$) must be zero. This introduces negative elements into both the final-demand mix and distribution effects [the second and third terms in (13.19)].

13.1.4 Changes in the Direct Inputs Matrix

Decomposition of $\Delta\mathbf{L}$ Changes in the Leontief inverse between two time periods reflect, of course, changes in the underlying direct inputs matrices. One

Table 13.4 Sector-Specific and Economy-Wide Decomposition Results (with Three-Factor Final-Demand Decomposition Detail)

	Output Change	Final-Demand Change Contribution			
		Level	Mix	Distribution	Total
Sector 1	12	11.05 (92)	.31 (3)	−.21 (−2)	11.16 (93)
Sector 2	20	9.35 (47)	2.42 (12)	−.91 (−5)	10.86 (54)
Sector 3	20	11.45 (57)	−1.65 (−8)	.71 (4)	10.51 (53)
Total	52	31.85 (61)	1.08 (2)	−.41 (−1)	32.53 (63)

approach to translating $\Delta\mathbf{A}$ into $\Delta\mathbf{L}$ proceeds as follows. Given $\mathbf{L}^1 = (\mathbf{I} - \mathbf{A}^1)^{-1}$ and $\mathbf{L}^0 = (\mathbf{I} - \mathbf{A}^0)^{-1}$, postmultiply $\mathbf{L}^1$ through by $(\mathbf{I} - \mathbf{A}^1)$

$$\mathbf{L}^1(\mathbf{I} - \mathbf{A}^1) = \mathbf{I} = \mathbf{L}^1 - \mathbf{L}^1\mathbf{A}^1 \tag{13.21}$$

and premultiply $\mathbf{L}^0$ through by $(\mathbf{I} - \mathbf{A}^0)$

$$(\mathbf{I} - \mathbf{A}^0)\mathbf{L}^0 = \mathbf{I} = \mathbf{L}^0 - \mathbf{A}^0\mathbf{L}^0 \tag{13.22}$$

Rearrange (13.21) and postmultiply by $\mathbf{L}^0$

$$\mathbf{L}^1 - \mathbf{I} = \mathbf{L}^1\mathbf{A}^1 \Rightarrow \mathbf{L}^1\mathbf{L}^0 - \mathbf{L}^0 = \mathbf{L}^1\mathbf{A}^1\mathbf{L}^0 \tag{13.23}$$

Similarly, rearrange (13.22) and premultiply by $\mathbf{L}^1$

$$\mathbf{L}^0 - \mathbf{I} = \mathbf{A}^0\mathbf{L}^0 \Rightarrow \mathbf{L}^1\mathbf{L}^0 - \mathbf{L}^1 = \mathbf{L}^1\mathbf{A}^0\mathbf{L}^0 \tag{13.24}$$

Finally, subtract (13.24) from (13.23)

$$\Delta\mathbf{L} = \mathbf{L}^1 - \mathbf{L}^0 = \mathbf{L}^1\mathbf{A}^1\mathbf{L}^0 - \mathbf{L}^1\mathbf{A}^0\mathbf{L}^0 = \mathbf{L}^1(\Delta\mathbf{A})\mathbf{L}^0 \tag{13.25}$$

This expression relates the change in the Leontief inverse to the change in $\mathbf{A}$; the decomposition is a multiplicative one in which $\Delta\mathbf{A}$ is "doubly weighted" – in this case by $\mathbf{L}^1$ on the left and by $\mathbf{L}^0$ on the right. The reader can verify that changing each premultiplication to a postmultiplication, and vice versa, in deriving (13.21) through (13.24) will generate the (possibly surprising[13]) result that, in addition,

$$\Delta\mathbf{L} = \mathbf{L}^1 - \mathbf{L}^0 = \mathbf{L}^0\mathbf{A}^1\mathbf{L}^1 - \mathbf{L}^0\mathbf{A}^0\mathbf{L}^1 = \mathbf{L}^0(\Delta\mathbf{A})\mathbf{L}^1 \tag{13.26}$$

Since there is only one term on the right in either (13.25) or (13.26), there is no need to express $\Delta\mathbf{L}$ as the average of the two expressions; either one will do.

Again, interaction terms will appear if we choose to have only year-0 ($\mathbf{L}^0$) or only year-1 ($\mathbf{L}^1$) weights. For example, replacing $\mathbf{L}^1$ with $\mathbf{L}^0 + \Delta\mathbf{L}$ in (13.25) leads to

[13] The result is surprising in the sense that the order in which matrices appear in matrix multiplication usually makes a difference in the outcome (in contrast to scalar multiplication).

$\Delta\mathbf{L} = \mathbf{L}^0(\Delta\mathbf{A})\mathbf{L}^0 + (\Delta\mathbf{L})(\Delta\mathbf{A})\mathbf{L}^0$. Making the same replacement in (13.26) generates $\Delta\mathbf{L} = \mathbf{L}^0(\Delta\mathbf{A})\mathbf{L}^0 + \mathbf{L}^0(\Delta\mathbf{A})(\Delta\mathbf{L})$. This identifies another instance in which the general "order makes a difference" rule in matrix algebra is violated; since the second terms must be equal, we see that $(\Delta\mathbf{L})(\Delta\mathbf{A})\mathbf{L}^0 = \mathbf{L}^0(\Delta\mathbf{A})(\Delta\mathbf{L})$. Also, substituting $\mathbf{L}^1 - \Delta\mathbf{L}$ for $\mathbf{L}^0$ in both (13.25) and (13.26) will produce $\Delta\mathbf{L} = \mathbf{L}^1(\Delta\mathbf{A})\mathbf{L}^1 - \mathbf{L}^1(\Delta\mathbf{A})(\Delta\mathbf{L})$ and $\Delta\mathbf{L} = \mathbf{L}^1(\Delta\mathbf{A})\mathbf{L}^1 - (\Delta\mathbf{L})(\Delta\mathbf{A})\mathbf{L}^1$, respectively. In these two cases, we also find that the interaction terms are equal – $(\Delta\mathbf{L})(\Delta\mathbf{A})\mathbf{L}^1 = \mathbf{L}^1(\Delta\mathbf{A})(\Delta\mathbf{L})$.

In what follows, we will use the result in (13.25) to convert changes in the Leontief inverse into changes in the $\mathbf{A}$ matrix.[14]

Decomposition of $\Delta\mathbf{A}$ There are many ways to create decompositions of $\Delta\mathbf{A}$. For example, the RAS procedure has been proposed as a descriptive device to identify the underlying causes of coefficient change between $\mathbf{A}_0$ and $\mathbf{A}_1$ when both matrices are known. This is entirely different from the usual use of RAS, which is to estimate an unknown $\mathbf{A}_1$ when only $\mathbf{u}_1$, $\mathbf{v}_1$, and $\mathbf{x}_1$ are known.[15] [A general introduction to RAS is given in Chapter 7; there we used $\mathbf{A}(0)$, $\mathbf{u}(1)$ and so on rather than $\mathbf{A}_0$, $\mathbf{u}_1$, etc.] From $\tilde{\mathbf{A}}_1 = \hat{\mathbf{r}}\mathbf{A}_0\hat{\mathbf{s}}$ and when $\tilde{\mathbf{A}}_1 \neq \mathbf{A}_1$ (as is generally the case), let $\mathbf{D} = \mathbf{A}_1 - \tilde{\mathbf{A}}_1 = \mathbf{A}_1 - \hat{\mathbf{r}}\mathbf{A}_0\hat{\mathbf{s}}$ or $d_{ij} = a^1_{ij} - \tilde{a}^1_{ij} = a^1_{ij} - r_i a^0_{ij} s_j$; so $\mathbf{A}_1 = \tilde{\mathbf{A}}_1 + \mathbf{D} = \hat{\mathbf{r}}\mathbf{A}_0\hat{\mathbf{s}} + \mathbf{D}$ and $a^1_{ij} = r_i a^0_{ij} s_j + d_{ij}$. This allows the separation of coefficient changes into those that are column-specific (fabrication effects for sector j, captured in s_j), row-specific (substitution effects in sector i, reflected in r_i) and cell-specific (that part of the change in a_{ij} that is caused by other circumstances, d_{ij}).[16]

Here we illustrate a straightforward disaggregation into column-specific changes only. Since each column in $\mathbf{A}$ reflects a sector's production recipe, identifying the changes column by column is one way of disentangling the effects of input changes in each of the sectors in the economy. For expositional simplicity, we denote these as technology change. (The interested reader might refer again to section 7.2 where alternative and more aggregate approaches to measuring changes in coefficients were explored.)

For an n-sector economy,

$$\mathbf{A}^1 = \mathbf{A}^0 + \Delta\mathbf{A} = \begin{bmatrix} a^0_{11} + \Delta a_{11} & \cdots & a^0_{1n} + \Delta a_{1n} \\ \vdots & & \vdots \\ a^0_{n1} + \Delta a_{n1} & \cdots & a^0_{nn} + \Delta a_{nn} \end{bmatrix}$$

[14] A continuous version of this approach has been noted (for example, Afrasiabi and Casler, 1991, Rose and Casler, 1996). As in (13.21), with $\mathbf{L}(\mathbf{I} - \mathbf{A}) = \mathbf{L} - \mathbf{L}\mathbf{A} = \mathbf{I}$, use the product rule for differentiation, $(d\mathbf{L}/dt) - (d\mathbf{L}/dt)\mathbf{A} - \mathbf{L}(d\mathbf{A}/dt) = 0$ or $(d\mathbf{L}/dt)(\mathbf{I} - \mathbf{A}) = \mathbf{L}(d\mathbf{A}/dt)$. Postmultiplying by $\mathbf{L}$, $(d\mathbf{L}/dt) = \mathbf{L}(d\mathbf{A}/dt)\mathbf{L}$.

[15] For example, see van der Linden and Dietzenbacher, 2000, Dietzenbacher and Hoekstra, 2002; also de Mesnard, 2004, 2006.

[16] As argued in van der Linden and Dietzenbacher (2000, pp. 2208–2209), a poor RAS performance simply indicates that other [cell specific] determinants need to be taken into account. These provide the necessary corrections whenever the fabrication effects and substitution effects alone do not adequately capture the coefficient changes.

Let $\Delta\mathbf{A}^{(j)} = \begin{bmatrix} 0 & \cdots & \Delta a_{1j} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & \Delta a_{nj} & \cdots & 0 \end{bmatrix}$ represent changes in sector j's technology – the superscript "(j)" identifies the sector (column) in which coefficients change.[17] Then

$$\Delta\mathbf{A} = \Delta\mathbf{A}^{(1)} + \cdots + \Delta\mathbf{A}^{(j)} + \cdots + \Delta\mathbf{A}^{(n)} = \sum_{j=1}^{n} \underbrace{\Delta\mathbf{A}^{(j)}}_{\substack{\text{Technology change} \\ \text{in sector } j}} \tag{13.27}$$

This decomposition of $\Delta\mathbf{A}$ can be introduced into (13.25), and the resulting expression for $\Delta\mathbf{L}$ can then be used in (13.7), which now looks like this:

$$\begin{aligned} \Delta\mathbf{x} &= (1/2)(\Delta\mathbf{L})(\mathbf{f}^0 + \mathbf{f}^1) + (1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{f}) \\ &= [(1/2)\mathbf{L}^1(\Delta\mathbf{A})\mathbf{L}^0](\mathbf{f}^0 + \mathbf{f}^1) + (1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{f}) \\ &= [(1/2)\mathbf{L}^1(\Delta\mathbf{A}^{(1)} + \cdots + \Delta\mathbf{A}^{(n)})\mathbf{L}^0](\mathbf{f}^0 + \mathbf{f}^1) + (1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{f}) \\ &= \underbrace{(1/2)[\mathbf{L}^1(\Delta\mathbf{A}^{(1)})\mathbf{L}^0](\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Effect of technology change in sector 1}} + \cdots + \underbrace{(1/2)[\mathbf{L}^1(\Delta\mathbf{A}^{(n)})\mathbf{L}^0](\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Effect of technology change in sector } n} \\ &\quad + \underbrace{(1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{f})}_{\text{Effect of final-demand change}} \end{aligned} \tag{13.28}$$

Numerical Illustration (continued) For our numerical example,

$$\mathbf{A}^0 = \begin{bmatrix} .1000 & .2500 & .2500 \\ .1500 & .0625 & .3000 \\ .3000 & .5000 & .0500 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^1 = \begin{bmatrix} .1071 & .1500 & .2917 \\ .2143 & .1100 & .2500 \\ .3214 & .5000 & .0667 \end{bmatrix}$$

so

$$\Delta\mathbf{A} = \begin{bmatrix} .0071 & -.1 & .0417 \\ .0643 & .0475 & -.0500 \\ .0214 & 0 & .0167 \end{bmatrix}$$

and, in particular,

$$\Delta\mathbf{A}^{(1)} = \begin{bmatrix} .0071 & 0 & 0 \\ .0643 & 0 & 0 \\ .0214 & 0 & 0 \end{bmatrix} \quad \mathbf{A}^{(2)} = \begin{bmatrix} 0 & -.1 & 0 \\ 0 & .0475 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{A}^{(3)} = \begin{bmatrix} 0 & 0 & .0417 \\ 0 & 0 & -.0500 \\ 0 & 0 & .0167 \end{bmatrix}$$

[17] The superscript parentheses serve to distinguish $\mathbf{A}^1$, the direct inputs matrix in period 1, from $\Delta\mathbf{A}^{(1)}$, the matrix that reflects the technology *change* in sector 1 only.

Table 13.5 Sector-Specific and Economy-Wide Decomposition Results (with Additional Technology and Final-Demand Decomposition Detail)

		Technology Change Contribution				Final-Demand Change Contribution			
	Output Change	Sector 1	Sector 2	Sector 3	Total	Level	Mix	Distri-bution	Total
Sector 1	12	6.64	−10.25	4.45	.84	11.05	.31	−.21	11.16
		(55)	(−85)	(37)	(7)	(92)	(3)	(−2)	(93)
Sector 2	20	12.42	1.28	−4.56	9.14	9.35	2.42	−.91	10.86
		(62)	(6)	(−23)	(46)	(47)	(12)	(−5)	(54)
Sector 3	20	11.37	−2.85	.97	9.49	11.45	−1.65	.71	10.51
		(57)	(−14)	(5)	(47)	(57)	(−8)	(4)	(53)
Total	52	30.43	−11.82	.86	19.47	31.85	1.08	−.41	32.53
		(59)	(−23)	(2)	(37)	(61)	(2)	(−1)	(63)

Table 13.5 indicates the additional results from using this technology change decomposition for our numerical illustration. As usual, percentages of row totals are in parentheses. (Final-demand results repeat those in Table 13.4.)

13.1.5 Decompositions of Changes in Some Function of $\mathbf{x}$

A number of studies have decomposed not simply gross output change but rather changes in some variable that depends on output. For example, if we have a set of labor input coefficients – employment per dollar of output in sector j at time t (e_j^t) – let $(\mathbf{e}^t)' = [e_1^t, \ldots, e_n^t]$. Then the vector of employment, by sector, associated with output at t will be $\boldsymbol{\varepsilon}^t = \hat{\mathbf{e}}^t\mathbf{x}^t = \hat{\mathbf{e}}^t\mathbf{L}^t\mathbf{f}^t$, and the vector of changes in employment is

$$\Delta\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^1 - \boldsymbol{\varepsilon}^0 = \hat{\mathbf{e}}^1\mathbf{L}^1\mathbf{f}^1 - \hat{\mathbf{e}}^0\mathbf{L}^0\mathbf{f}^0 \tag{13.29}$$

Decomposition into contributions by the three elements now follows the standard pattern shown in (13.10). Here this means

$$\begin{aligned}\Delta\boldsymbol{\varepsilon} = (1/2)\ &\underbrace{(\Delta\hat{\mathbf{e}})(\mathbf{L}^0\mathbf{f}^0 + \mathbf{L}^1\mathbf{f}^1)}_{\text{Labor input coefficient change}}\\ &+ (1/2)\underbrace{[\hat{\mathbf{e}}^0(\Delta\mathbf{L})\mathbf{f}^1 + \hat{\mathbf{e}}^1(\Delta\mathbf{L})\mathbf{f}^0]}_{\text{Technology change}}\\ &+ (1/2)\underbrace{(\hat{\mathbf{e}}^0\mathbf{L}^0 + \hat{\mathbf{e}}^1\mathbf{L}^1)(\Delta\mathbf{f})}_{\text{Final-demand change}}\end{aligned} \tag{13.30}$$

Of course, additional decompositions of $\Delta\mathbf{L}$ and/or $\Delta\mathbf{f}$ as in section 13.1.2 are possible. Exactly the same principles apply for any economic variable that is related to output by a similar set of coefficients per dollar of sectoral output – pollution generation, energy consumption, value added, etc.

13.1.6 Summary for $\Delta \mathbf{x}$

For $\Delta \mathbf{x}$ we assemble both the final-demand decomposition (including distribution across final-demand categories) and the technology change decomposition in the same expression, primarily for completeness. The expression includes all six of the change components.

$$\begin{aligned}
\Delta \mathbf{x} &= (1/2)(\Delta \mathbf{L})(\mathbf{f}^0 + \mathbf{f}^1) + (1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta \mathbf{f}) \\
&= \underbrace{(1/2)[\mathbf{L}^1(\Delta \mathbf{A}^{(1)})\mathbf{L}^0](\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Effect of technology change in sector 1}} + \underbrace{(1/2)[\mathbf{L}^1(\Delta \mathbf{A}^{(2)})\mathbf{L}^0](\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Effect of technology change in sector 2}} \\
&\quad + \underbrace{(1/2)[\mathbf{L}^1(\Delta \mathbf{A}^{(3)})\mathbf{L}^0](\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Effect of technology change in sector 3}} + \underbrace{(1/4)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta f)(\mathbf{P}^0\mathbf{d}^0 + \mathbf{P}^1\mathbf{d}^1)}_{\text{Effect of change in final-demand level}} \\
&\quad + \underbrace{(1/4)(\mathbf{L}^0 + \mathbf{L}^1)[f^0(\Delta \mathbf{P})\mathbf{d}^1 + f^1(\Delta \mathbf{P})\mathbf{d}^0]}_{\text{Effect of change in final-demand mix}} + \underbrace{(1/4)(\mathrm{f}^0\mathbf{P}^0 + \mathrm{f}^1\mathbf{P}^1)(\Delta \mathbf{d})}_{\text{Effect of change in final-demand distribution}}
\end{aligned} \tag{13.31}$$

13.1.7 SDA in a Multiregional Input–Output (MRIO) Model

The standard form of the MRIO model (Chapter 3) is $\mathbf{x} = (\mathbf{I} - \mathbf{CA})^{-1}\mathbf{Cf} = \tilde{\mathbf{L}}\mathbf{Cf}$, where $\tilde{\mathbf{L}} = (\mathbf{I} - \mathbf{CA})^{-1}$, $\mathbf{A}$ is a technical coefficients matrix indicating intermediate inputs for each region from both within and outside of the region and $\mathbf{C}$ contains input proportions (both intraregional and interregional shipments). The distinctive feature of this formulation is that the Leontief-like inverse contains both technical coefficients and trade proportions.

Following (13.10), for $\mathbf{x} = \tilde{\mathbf{L}}\mathbf{Cf}$ we have

$$\begin{aligned}
\Delta \mathbf{x} &= (1/2)(\Delta \tilde{\mathbf{L}})(\mathbf{C}^0\mathbf{f}^0 + \mathbf{C}^1\mathbf{f}^1) + (1/2)[\tilde{\mathbf{L}}^0(\Delta \mathbf{C})\mathbf{f}^1 + \tilde{\mathbf{L}}^1(\Delta \mathbf{C})\mathbf{f}^0] \\
&\quad + (1/2)(\tilde{\mathbf{L}}^0\mathbf{C}^0 + \tilde{\mathbf{L}}^1\mathbf{C}^1)(\Delta \mathbf{f})
\end{aligned} \tag{13.32}$$

To disentangle the trade proportions and technical coefficients in $\tilde{\mathbf{L}}$ we follow (13.25) and then (13.7), namely

$$\Delta \tilde{\mathbf{L}} = \tilde{\mathbf{L}}^1(\Delta \mathbf{CA})\tilde{\mathbf{L}}^0$$

and

$$\Delta \mathbf{CA} = (1/2)(\Delta \mathbf{C})(\mathbf{A}^0 + \mathbf{A}^1) + (1/2)(\mathbf{C}^0 + \mathbf{C}^1)(\Delta \mathbf{A}) \tag{13.33}$$

First, using $\Delta \tilde{\mathbf{L}} = \tilde{\mathbf{L}}^1(\Delta \mathbf{CA})\tilde{\mathbf{L}}^0$ in (13.32),

$$\begin{aligned}
\Delta \mathbf{x} &= (1/2)[\tilde{\mathbf{L}}^1(\Delta \mathbf{CA})\tilde{\mathbf{L}}^0](\mathbf{C}^0\mathbf{f}^0 + \mathbf{C}^1\mathbf{f}^1) + (1/2)[\tilde{\mathbf{L}}^0(\Delta \mathbf{C})\mathbf{f}^1 + \tilde{\mathbf{L}}^1(\Delta \mathbf{C})\mathbf{f}^0] \\
&\quad + (1/2)(\tilde{\mathbf{L}}^0\mathbf{C}^0 + \tilde{\mathbf{L}}^1\mathbf{C}^1)(\Delta \mathbf{f})
\end{aligned}$$

and then using (13.33) (and rearranging)

$$\Delta\mathbf{x} = \underbrace{(1/4)[\tilde{\mathbf{L}}^1(\mathbf{C}^0 + \mathbf{C}^1)(\Delta\mathbf{A})\tilde{\mathbf{L}}^0](\mathbf{C}^0\mathbf{f}^0 + \mathbf{C}^1\mathbf{f}^1)}_{\text{Effect of technology change}}$$
$$+ \underbrace{(1/4)[\tilde{\mathbf{L}}^1(\Delta\mathbf{C})(\mathbf{A}^0 + \mathbf{A}^1)\tilde{\mathbf{L}}^0](\mathbf{C}^0\mathbf{f}^0 + \mathbf{C}^1\mathbf{f}^1)}_{\text{One effect of trade coefficient change}}$$
$$+ \underbrace{(1/2)[\tilde{\mathbf{L}}^0(\Delta\mathbf{C})\mathbf{f}^1 + \tilde{\mathbf{L}}^1(\Delta\mathbf{C})\mathbf{f}^0]}_{\text{A second effect of trade coefficient change}} + \underbrace{(1/2)(\tilde{\mathbf{L}}^0\mathbf{C}^0 + \tilde{\mathbf{L}}^1\mathbf{C}^1)(\Delta\mathbf{f})}_{\text{Effect of final-demand change}} \quad (13.34)$$

Notice in particular that the change in trade proportions exerts influence in conjunction with both the technical coefficients ($\mathbf{A}^0$ and $\mathbf{A}^1$) and also the final demands ($\mathbf{f}^0$ and $\mathbf{f}^1$). This is logical, since in the MRIO model both $\mathbf{A}$ and $\mathbf{f}$ are transformed – into $\mathbf{CA}$ and $\mathbf{Cf}$.

Embellishments are possible. For example, the final-demand effect might be further decomposed into level, mix and/or distribution, as in section 13.1.2. Furthermore, some models may feature (or at least propose) separate trade proportions for intermediate inputs and for final demands, leading to $\mathbf{x} = (\mathbf{I} - \mathbf{C}_a\mathbf{A})^{-1}\mathbf{C}_f\mathbf{f} = \tilde{\mathbf{L}}^*\mathbf{C}_f\mathbf{f}$. In that case, $\Delta\mathbf{C}_a$ and $\Delta\mathbf{C}_f$ must be treated separately. This simply leads to more complexity (more terms) in (13.34). In Appendix 13.1 we explore the implications of alternative groupings of the terms in $\mathbf{x} = \tilde{\mathbf{L}}\mathbf{C}\mathbf{f}$ (as has been done in some published studies) into either $\mathbf{x} = \mathbf{M}\mathbf{f}$, where $\mathbf{M} = \tilde{\mathbf{L}}\mathbf{C}$, or $\mathbf{x} = \tilde{\mathbf{L}}\mathbf{y}$, where $\mathbf{y} = \mathbf{C}\mathbf{f}$.

13.1.8 Empirical Examples

Analysts are generally interested in structural decompositions because they offer a means of quantifying the relative importance of various components in an "explanation" of some observed economic change – in early studies this was usually changes in industry outputs; more recently, changes in labor use, value added, energy use, pollution emissions, service industry outputs, etc. have also been decomposed. The results of empirical SDA studies are often used to inform policy decisions – the relative importance of trade (and hence trade policy) to an economy, the relative importance of one or more components of final demand (and hence tax or subsidy policy), and so forth. As noted earlier, decompositions generate results at a sectoral level and summary measures are needed. In Tables 13.6, 13.7, and 13.9, below, virtually all of the rich detail in each of the studies cited has been foregone in favor of simple averages in order to present figures that are comparable across studies.

Studies Using National Models The first study known to us that uses this approach is Chenery, Shishido and Watanabe (1962), for Japan over the periods 1914–1935 and 1935–1954.[18] The authors were interested in deviations of later year output

[18] This builds on earlier work by Chenery (for example, Chenery, 1960). A thorough summary of this kind of analysis in the economic development literature can be found in Syrquin (1988). Illustrative examples include

from what it would have been under a regime of proportional growth from an earlier year. These deviations were decomposed into the effects of (1) changes in domestic final demand, (2) changes in exports, (3) changes in imports and (4) changes in technology (as represented by changes in elements of the **A** matrix).

A study by Vaccara and Simon (1968), to the best of our knowledge, represents the first application of this kind of decomposition approach to the US economy. Using 42 industry groups, they measured the amount of output change that was attributable to final-demand change and the amount due to coefficient change over the 1947–1958 period. As a (very) general conclusion, they found final-demand changes somewhat more important than changes in technical coefficients in contributing to overall output change over the period.

Bezdek and Wendling (1976) continued this kind of analysis. They factored $\Delta\mathbf{x}$ into final-demand and coefficient change in a 75-sector model of the US economy for the 1947–1958, 1958–1963, and 1963–1966 periods. In addition, they compared their decomposition results for 1958–1963 with those reported for Germany (1958–1962) in Stäglin and Wessels (1972) at a 35-sector level. They found similarity in the industry-specific influences of final-demand change but not of coefficient change.

The late 1980s and early 1990s saw the beginnings of an explosion of empirical studies using SDA. The work of Feldman, McClain and Palmer (1987) is frequently cited.[19] This study also examined the relative importance in the US economy of changes in final demands and changes in technology – this time over the 1963–1978 period using a very disaggregated 400-sector level of analysis. (The 1978 table was an updated version of the 1972 survey-based national table.)

They use the form $\mathbf{x} = \mathbf{Ax} + \mathbf{Bf} \Rightarrow \mathbf{x} = \mathbf{LBf}$ and then define $\mathbf{C} = \mathbf{LB}$ so that $\mathbf{x} = \mathbf{Cf}$, where $\underset{(n\times p)}{\mathbf{B}}$ is the bridge matrix that connects the outputs of some $n =$ 400 sectors to $p = 160$ categories of final demand.[20] Thus their decomposition takes the form $\Delta\mathbf{x} = (\Delta\mathbf{C})\mathbf{f}^0 + \mathbf{C}^1(\Delta\mathbf{f})$ or $\Delta\mathbf{x} = (\Delta\mathbf{C})\mathbf{f}^1 + \mathbf{C}^0(\Delta\mathbf{f})$ – as in (13.3) and (13.4). They define structural change broadly – "including changes in the structure of production (technical change, reflected in changes in **A**) and in the microstructure of expenditure (reflected in changes in **B**)" (p. 504).[21] Generally speaking, the contribution made by coefficient change was larger than the contribution made by final-demand change for many of the fastest growing (termed "emerging") and fastest declining industries. At the same time, for most industries (almost 80 percent), the coefficient change component accounted (in absolute terms) for less than half of the gross output change.[22]

Fujita and James (1990) – and many other publications by these authors – at the national level and Siegel, Alwang and Johnson (1995) for a "growth accounting" study at the regional level.

19 And, less frequently, Feldman and Palmer (1985).

20 This use of **C** is not to be confused with the trade proportions matrix of the MRIO model.

21 They recognize that an alternative would be to group **B** with **f** and to use $\mathbf{x} = \mathbf{L}(\mathbf{Bf})$, leading to $\Delta\mathbf{x} = [\Delta\mathbf{L}](\mathbf{B}^0\mathbf{f}^0) + \mathbf{L}^1(\Delta\mathbf{Bf})$ and $\Delta\mathbf{x} = [\Delta\mathbf{L}](\mathbf{B}^1\mathbf{f}^1) + \mathbf{L}^0(\Delta\mathbf{Bf})$. See comments on the effect of alternative groupings on decompositions in Appendix 13.1.

22 Wolff (1985) used the same mode of analysis to study trends in productivity in the US economy.

A second frequently cited study from this period is that by Skolka (1989). It describes the structural decomposition methodology in some detail and applies it to a 19-sector data set for Austria (1964–1976). Both net output (value added) change and employment change were decomposed into an intermediate demand (technology) component (with separate domestic and imports parts) and a final-demand component (with separate domestic and exports parts).

In what follows, we present brief overviews of several (from among many) additional empirical SDA studies concerned with identifying components of total output change (in chronological order). The main characteristics of these (and other studies) are summarized in Table 13.6.

1. Fujimagari (1989). Fujimagari suggests that bundling $\mathbf{L}$ and $\mathbf{B}$ together (as in Feldman, McClain and Palmer) is inappropriate. Instead he uses two tripartite decompositions and averages their results. These are

$$\Delta\mathbf{x} = (\Delta\mathbf{L})\mathbf{B}^0\mathbf{f}^0 + \mathbf{L}^1(\Delta\mathbf{B})\mathbf{f}^0 + \mathbf{L}^1\mathbf{B}^1(\Delta\mathbf{f}) \text{ and}$$
$$\Delta\mathbf{x} = (\Delta\mathbf{L})\mathbf{B}^1\mathbf{f}^1 + \mathbf{L}^0(\Delta\mathbf{B})\mathbf{f}^1 + \mathbf{L}^0\mathbf{B}^0(\Delta\mathbf{f})$$

   [as in (13.8) and (13.9)] in a 189-sector Canadian model for 1961–1971 and 1971–1981. This approach has been used by others in later studies.
2. Barker (1990). Changes over 1979–1984 in the output of market service industries in the UK are investigated – including distribution, transport, communications, business services, and others. The decomposition – into changes internal to the services group, external to the group in the rest of manufacturing and external in the rest of industry – uses partitioned matrices extensively. Each of these is further decomposed into changes in: input–output coefficients, total final demand (level) and the structure of final demand (the distribution, as reflected in the bridge matrix).
3. Martin and Holland (1992). Changes over 1972–1977 in the output of some 477 US industries are decomposed from the defining equation

$$\mathbf{x}^t = (\mathbf{I} - \hat{\mathbf{u}}^t\mathbf{A}^t)^{-1}(\hat{\mathbf{u}}^t\mathbf{f}^t + \mathbf{e}^t) = \mathbf{L}^t(\hat{\mathbf{u}}^t\mathbf{f}^t + \mathbf{e}^t)$$

   in which (all for year t) $\hat{\mathbf{u}}$ is a diagonal matrix containing the domestic supply ratio for each sector, $\mathbf{A}$ is the technical coefficient matrix (including imports), $\mathbf{f}$ is the domestic final-demand vector and $\mathbf{e}$ is a vector of exports. Thus $\hat{\mathbf{u}}^t\mathbf{A}^t$ is an estimate of the domestic direct input coefficients matrix and $\hat{\mathbf{u}}^t\mathbf{f}^t$ is an estimate of the vector of domestic final demand that is satisfied from domestic sources. The decomposition used is essentially that in (13.9), namely

$$\Delta\mathbf{x} = (\Delta\mathbf{L})(\hat{\mathbf{u}}^1\mathbf{f}^1 + \mathbf{e}^1) + \mathbf{L}^0(\Delta\mathbf{u})\mathbf{f}^1 + \mathbf{L}^0\hat{\mathbf{u}}^0(\Delta\mathbf{f}) + \mathbf{L}^0(\Delta\mathbf{e})$$

After a good deal of algebra this can be expressed as

$$\Delta \mathbf{x} = \mathbf{L}^0\hat{\mathbf{u}}^0(\Delta \mathbf{f}) + \mathbf{L}^0(\Delta \mathbf{e}) + \mathbf{L}^0(\Delta \mathbf{u})(\mathbf{f}^1 + \mathbf{A}^1\mathbf{x}^1) + \mathbf{L}^0\hat{\mathbf{u}}^0(\Delta \mathbf{A})\mathbf{x}^1$$

(No alternative decompositions were used and so the results were not averages.) The decomposition is thus apportioned to changes due to: domestic final demand, export demand, import substitution, and input–output coefficients. With results for 477 sectors, groupings were necessary – these included aggregations into: (1) primary (25 natural resource related industries), secondary (409 manufacturing and processing industries) and tertiary (43 support and service oriented industries); (2) nine sectors that represent the BEA one-digit aggregation level; and (3) the 30 fastest growing and the 30 slowest growing industries.

When commodity sectors were categorized according to 1972–1977 growth rates, the importance of the technical change contribution was seen to increase with categories of increasing growth or decline – results consistent with those in Feldman, McClain and Palmer (1987). At the same time, examination of the specific decompositions for the 30 fastest growing and 30 fastest declining sectors indicated that final demand was the dominant component in output change in 60 and 67 percent of the cases, respectively, whereas technical coefficient change was dominant in about 30 percent of the cases (both for rapidly growing and rapidly declining sectors). This view of their results is at variance with those of Feldman, McClain and Palmer.

4. Liu and Saal (2001). This study examines changes in gross outputs in South Africa over 1975–1993. It employs essentially the same decomposition as Martin and Holland (1992), except that final demand is separated into changes in private consumption, investment spending, government spending, exports, and import substitution.
5. Dietzenbacher and Hoekstra (2002). This study focuses on output change for 25 sectors in the Netherlands over 1975–1985. The Netherlands data are embedded in an intercountry model for the European Union, and final-demand categories include separate columns for exports to each of five EU member countries (Germany, France, Italy, Belgium, and Denmark), the rest of the EU, the rest of the world, household consumption, and other final demand. As might be expected, large differences were observed across sectors, countries, and final-demand categories.
6. Roy, Das and Chakraborty (2002). The particular interest of this study is to identify sources of growth in the information sectors in a 31-sector input–output model of the Indian economy over 1983–1984 to 1989–1990. Instead of partitioning the matrices into quadrants of information and non-information sectors (as in some of the energy studies noted below), the authors simply define a matrix $\hat{\mathbf{z}}$, created from an identity matrix by replacing the main-diagonal ones with zeros for all non-information sectors (so the remaining on-diagonal elements – and all off-diagonal elements – are 0). Then $\hat{\mathbf{z}}\mathbf{x}$ selects only the information rows from the results of various decompositions.

Table 13.6 Selected Empirical Structural Decomposition Studies

Author(s) and Source	Details (country; dates; changed variable(s); aggregation level)	Decomposition Components (percentage of total change[a])	
		Technology	Final Demand
Feldman McClain and Palmer (1987, Table 1)	US; 1963–1978; $\Delta\mathbf{x}$; 400 sectors (results for 15 fastest growing industries)	62	38
Skolka (1989, pp. 59–60)	Austria; 1964–76; Δ (value added) and also Δ (employment); 19 industries	26 (v.a.), 34 (emp.)	74 (v.a.), 66 (emp.) *Domestic*: 18 (v.a.), 46 (emp.) *Foreign*: 56 (v.a.), 20 (emp.)
Fujimagari (1989, Tables 1 and 2)	Canada; 1961–71 and 1971–81; $\Delta\mathbf{x}$; 189 industries (results for 15 fastest and 15 slowest growing industries)	1961–71 28 (top 15), –86 (bottom 15) 1971–81 22 (top 15), 159 (bottom 15)	1961–71 72 (top 15) 186 (bottom 15) 1971–81 78 (top 15) –59 (bottom 15) *Level* 1961–71: 38 (top 15), 69 (bottom 15) 1971–81: 61 (top 15), –120 (bottom 15) *Mix* 1961–71: 34 (top 15), 117 (bottom 15) 1971–81: 17 (top 15), 61 (bottom 15)
Barker (1990, Table 4)	UK; 1979–84; $\Delta\mathbf{x}$ (service industries); 101 ind., 13 serv. ind. (aggregated to 5 serv. ind.)	63	18 *Level*: –1 *Mix*: 20
Martin and Holland (1992, Table 1)	US; 1972–77; $\Delta\mathbf{x}$; 477 sectors	6	94 *Domestic*: 81 *Export*: 23 *Import Use*: –10
Liu and Saal (2001, Table 5)	South Africa; 1975–93; $\Delta\mathbf{x}$; 34 and 10 sectors; results for 10 sectors only	28	72 (Pvt. Cons., 61; Gov. Cons., 7; Inv., –32; Exp. 29, Imp. Subs., 7)

Table 13.6 (*cont.*)

Author(s) and Source	Details (country; dates; changed variable(s); aggregation level)	Decomposition Components (percentage of total change[a]): Technology		Final Demand		
Dietzenbacher and Hoekstra (2002, Table 10.2)	The Netherlands; 1975–85; $\Delta\mathbf{x}$; 25 sectors	21[b] (–201, 135)		79[b] (–35, 301)		
				Level 78 (–118, 2458)	*Category* 1 (–2594, 257)	*Product Mix* 0 (–39, 236)
Roy, Das and Chakraborty (2002, Table 4)	India; 1983/4–89/90; $\Delta\mathbf{x}$ (information sectors); 30 non-information sectors plus 5 information sectors	3		97		
		Info. coeffs. 3	*Non-info. coeffs.* 0	*Domestic* 91 (Level 65, Mix 26)	*Exports* 6	*Import Substitution* 0

[a]Figures may not add to 100 percent due to rounding.
[b]Figures in parentheses indicate boundaries in the range of values across the 25 sectors in the study.

There have been many SDA studies concerned with energy and environmental issues; some are noted in Table 13.7.[23] Brief overviews of some of these are given below.

1. Office of Technology Assessment (US Congress, OTA, 1990). The primary interest of this study is to investigate the components of the change in energy use in the USA between 1972 and 1985. Final-demand level and mix along with changes in technology, disaggregated into energy inputs and non-energy inputs, are investigated. The decompositions are carried out for five energy types: coal, crude oil and gas, refined petroleum, primary electricity, and utility gas.

   The calculation of the change in energy use due to different economic factors was achieved by using 1985 as a base year and systematically varying one factor over time while holding all other factors constant in their 1985 form. The model separated energy sectors and other sectors and uses hybrid-units form (Chapter 9). The first k sectors (here $k = 5$) are energy commodities and energy industries. In partitioned form, the units in the four quadrants of the model are

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11}(\text{BTU/BTU}) & \mathbf{A}_{12}(\text{BTU/\$}) \\ \mathbf{A}_{21}(\text{\$/BTU}) & \mathbf{A}_{22}(\text{\$/\$}) \end{bmatrix},$$

23 Early energy-use decomposition studies can be found in Casler and Hannon (1989), or Casler, Afrasiabi and McCauley (1991) who studied changes in energy input–output coefficients. There are many other energy-related studies in which Casler is a contributor.

$$\mathbf{L} = \begin{bmatrix} \mathbf{L}_{11}(\text{BTU/BTU}) & \mathbf{L}_{12}(\text{BTU/\$}) \\ \mathbf{L}_{21}(\text{\$/BTU}) & \mathbf{L}_{22}(\text{\$/\$}) \end{bmatrix},$$

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}_e(\text{BTU}) \\ \mathbf{f}_\$(\$) \end{bmatrix}$$

So $\mathbf{x} = \begin{bmatrix} \mathbf{x}_e \\ \mathbf{x}_\$ \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}$, where $\underset{(k\times 1)}{\mathbf{x}_e}$ represents output of the energy sectors and $\underset{[(n-k)\times 1]}{\mathbf{x}_\$}$ is a vector of outputs of the other sectors. In particular, $\mathbf{x}_e = \begin{bmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} \end{bmatrix} \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}$. For example, to assess the influence on energy sectors of changing final demands, the authors use

$$\Delta\mathbf{x}_e^{1985/1972} = [(\mathbf{L}_{11}^{1985})\mathbf{f}_1^{1985} + (\mathbf{L}_{12}^{1985})\mathbf{f}_2^{1985}] - [(\mathbf{L}_{11}^{1985})\mathbf{f}_1^{1972} + (\mathbf{L}_{12}^{1985})\mathbf{f}_2^{1972}]$$

Further decompositions into final-demand level and mix and technology change in energy and in non-energy inputs are found following the framework in earlier sections of this chapter. There is an interaction term at each decomposition level. The authors admit that there is no consistent set of guidelines regarding what to do with it, so they just report it as a separate component.[24]

2. Rose and Chen (1991). This study is also concerned with changes in energy use. Here final-demand contributions were also broken down into level and mix, and changes in technology were decomposed into a large number of either individual or interactive effects involving capital (K), labor (L), energy (E), and materials (M), along the lines of a two-tier KLEM production function. Coal, petroleum, natural gas, and electricity are examined separately. (There were 14 change components in all.)
3. Lin and Polenske (1995). This study focuses on changes in energy use in China over 1981–1987. The usual input–output accounting equation $\mathbf{x} = \mathbf{A}\mathbf{x}+\mathbf{f}$ is accompanied by the energy accounting identity $\mathbf{E} = \mathbf{E}_g + \mathbf{E}_d$ [total energy consumption equals intermediate energy consumption (used in production activities) plus final energy consumption]. This is expressed as $\mathbf{mx} = \mathbf{mAx} + \mathbf{mf}$, where $\mathbf{m}$ is created from an identity matrix by keeping a 1 only in those column locations that correspond to energy sectors; that is, $\mathbf{m}$ selects the energy rows in $\mathbf{Ax}$, $\mathbf{f}$, and $\mathbf{x}$. This approach was already noted, above, in Roy, Das and Chakraborty (2002), but the Lin and Polenske study precedes that. [This is an alternative to rearranging (renumbering) sectors so that the energy sectors are all together – for example, the first k, as in the OTA study,

[24] They cite (a) Wolff (1985), who ignores it; (b) Feldman, McClain and Palmer (1987) and others, who allocate it equally among the other sources of change; and (c) Casler and Hannon (1989) and others, who "treat it separately and report its magnitude" (US Congress, OTA, 1990, p. 56).

above.] A little algebra shows that

$$\mathbf{E}_g = \mathbf{mAx} = \mathbf{m}[(\mathbf{I} - \mathbf{A})^{-1} - \mathbf{I}]\mathbf{f}$$

Decompositions are then carried out in the usual way.

4. Wier (1998). The concern of this study is to identify environmental effects of production, in particular the sources of emissions of carbon dioxide (CO_2), sulphur dioxide (SO_2), and nitrogen oxide (NO_x) in the Danish economy between 1966 and 1988, using a 117-sector input–output model for that country. The decompositions identify the following contributors: changes in energy intensity, changes in fuel-mix in production sectors, changes in fuel-mix in energy production sectors, input coefficient change (the **A** matrix), changes in final-demand level, and final-demand mix.
5. Kagawa and Inamura (2001). This model, for Japan, is in commodity-by-commodity format (Chapter 5), so the defining equation takes the form $\mathbf{q} = (\mathbf{I} - \mathbf{BC}^{-1})^{-1}\mathbf{e}$, relating commodity final demand to commodity output. Changes in total energy requirements over 1985–1990 are analyzed. Partitioned matrices are used (as in OTA, 1990) to distinguish between energy-supplying industries and other industries. The commodity technology assumption (where the simple technical coefficients matrix, **A**, is replaced by $\mathbf{BC}^{-1}$) allows for an additional decomposition into both $\Delta\mathbf{B}$ and $\Delta\mathbf{C}^{-1}$ components (for both energy-supplying and non-energy sectors) – thereby reflecting changes in input structure and in product mix, respectively.

Many of the studies noted in Tables 13.6 and 13.7 were published in *Economic Systems Research*, and often they contain, in their references, a number of additional examples of SDA applications to which the interested reader can turn. It is important to remember that the figures presented in these tables are aggregates over all (frequently very many) sectors, or a subset of sectors, and of course all the rich sectoral detail is lost in such summary measures.[25] In general, analysts will often be interested in the more detailed results for individual sectors (or groups of sectors). This is the reason for including the detail on range of values in the Dietzenbacher and Hoekstra (2002) study in Table 13.6, where each of the figures in the table is an average over 25 values.

It also should be noted that percentage figures (as in these tables) are extremely sensitive to the differences between various changes. When a large positive effect (for example, final demand contribution) is nearly offset by a large negative effect (for example, technology change contribution), the percentages can be enormous. A simple table with several hypothetical results illustrates this fairly obvious fact.

Studies Using a Single-Region or Connected-Region Model

Washington State. Holland and Cooke (1992) used the structural decomposition framework at a regional (state) level to study the sources of change in the economy of Washington over 1963–1982, using the survey-based Washington input–output tables

[25] Most of the numbers in the tables were obtained by (simple) averaging over the more disaggregated results – either in value terms or in percentages – presented in the studies.

Table 13.7 Selected Empirical Structural Decompositions of Changes in Energy Use or Pollution Emissions

Author(s) and Source	Details (country; dates; changed variable(s); aggregation level)	Decomposition Components (percentage of total change[a]): Technology	Decomposition Components (percentage of total change[a]): Final Demand
US Congress, OTA (1990, Tabs. 2, 3, 6)	US; 1972–85; Δ (primary energy use); 88 sectors	–975 *Energy inputs* –770; *Non-energy inputs* –185; *Interaction* –20	720[b] *Level* 885; *Mix* –290; *Interaction* 125
Rose and Chen (1991)	US; 1972–82; Δ (energy use); 80 sectors	Coal, 64; Petroleum, 231; Natural gas, 65; Electricity, 56	Coal, 9; Petroleum, –370; Natural gas, –50; Electricity, 65[c] *Level*: Coal, 60; Petr.,–520; Nat. gas, –92; Elec., 70 *Mix*: Coal, –51; Petr., 150; Nat. gas, 42; Elec., –5
Lin and Polenske (1995, Table 3)	China; 1981–87; Δ (energy use); 18 sectors	–85 *Energy inputs* –106; *Non-energy inputs* 21	185 *Level* 196; *Mix* 3; *Distribution* –13
Wier (1998, Tables 3–5)	Denmark; 1966–88; Δ (pollution emissions); 117 sectors	–53 (CO_2) 373 (SO_2) 6 (NO_x)	153 (CO_2); –274 (SO_2); 95 (NO_x) *Level*: 175 (CO_2); –308 (SO_2) 112 (NO_x) *Mix*: –22 (CO_2); 34 (SO_2) –17 (NO_x)
Kagawa and Inamura (2001, Table 5)	Japan; 1985–90; Δ (total energy requirements); 94 sectors	0 *Energy inputs* 4; *Non-energy inputs* –4	100 *Energy* 8; *Non-energy* 92[d]

[a] Figures may not add to 100 percent due to rounding.

[b] Technology plus final-demand figures do not add to 100 percent because an interaction term between those two components is included in this study; in this case the interaction term is not small at 355%.

[c] Again, technology plus final-demand figures do not add to 100 percent because of an interaction term between the two components. This term is: Coal, 27; Petroleum, 39; Natural gas, 15; Electricity, −21.

[d] This figure is further decomposed into the following percentages: Household consumption (49), Non-household consumption (3), Capital formation, public (10), Capital formation, private (52); Other (−22).

Table 13.8 SDA Percentage Change Sensitivities

Technology Change	Final-Demand Change	Total Change	Technology Change as a Percentage of Total Change	Final-Demand Change as a Percentage of Total Change
−50	51	1	−5000	5100
−50	52	2	−2500	2600
−48	52	4	−1200	1300
−55	45	10	−550	450

for 1963 and 1982. Reflecting a concern with the importance of trade for the Washington economy, they separated out the role of changes in demand (intermediate and final) within the state, within the rest of the USA (national markets), and outside the USA (international markets).

The US Multiregional Model (Miller and Shao, 1994). Two implementations of a multiregional input–output (MRIO) model for the US economy are available – for 1963 ($t = 0$) and 1977 ($t = 1$). The 1963 model takes the form

$$\mathbf{x}^0 = (\mathbf{I} - \mathbf{C}^0\mathbf{A}^0)^{-1}\mathbf{C}^0\mathbf{f}^0$$

and the 1977 model is

$$\mathbf{x}^1 = (\mathbf{I} - \mathbf{D}^1\mathbf{C}^1\mathbf{B}^1)^{-1}\mathbf{C}^1\mathbf{f}^1$$

The $\mathbf{C}^0$ and $\mathbf{C}^1$ matrices contain the interregional trade proportions for the two years. However, matrices $\mathbf{D}^1$ and $\mathbf{B}^1$ reflect technology in the 1977 model (only), which is based on commodity–industry input–output accounting.[26] Similarly, $\mathbf{A}^0$ is a matrix of technical coefficients in the 1963 model (only). Therefore, for simplicity, the superscripts on $\mathbf{D}$, $\mathbf{B}$, and $\mathbf{A}$ can be eliminated, giving the following equation for gross output change over the period:

$$\Delta\mathbf{x} = \mathbf{x}^1 - \mathbf{x}^0 = (\mathbf{I} - \mathbf{D}\mathbf{C}^1\mathbf{B})^{-1}\mathbf{C}^1\mathbf{f}^1 - (\mathbf{I} - \mathbf{C}^0\mathbf{A})^{-1}\mathbf{C}^0\mathbf{f}^0 \tag{13.35}$$

The two total requirements matrices (transforming final demands into outputs) can be denoted $\tilde{\mathbf{L}}^1 = (\mathbf{I} - \mathbf{D}\mathbf{C}^1\mathbf{B})^{-1}\mathbf{C}^1$ and $\tilde{\mathbf{L}}^0 = (\mathbf{I} - \mathbf{C}^0\mathbf{A})^{-1}\mathbf{C}^0$.[27] Then

$$\Delta\mathbf{x} = \tilde{\mathbf{L}}^1\mathbf{f}^1 - \tilde{\mathbf{L}}^0\mathbf{f}^0 \tag{13.36}$$

This parallels (13.2), only now the two total requirements matrices are more complicated than the usual Leontief inverses, $\mathbf{L}^t = (\mathbf{I} - \mathbf{A}^t)^{-1}$. In particular, they incorporate both

[26] To be consistent with the 1963 model, in which industry final demands drive industry outputs, the 1977 model is in industry-by-industry format under the industry-based technology assumption.

[27] Appendix 13.1 indicates alternative ways of decomposing $\mathbf{x} = (\mathbf{I} - \mathbf{C}\mathbf{A})^{-1}\mathbf{C}\mathbf{f}$.

technology coefficients (**D** and **B** in one case, **A** in the other) and trade proportions ($\mathbf{C}^1$ and $\mathbf{C}^0$, respectively). In any event, following (13.7),

$$\Delta\mathbf{x} = (1/2)(\Delta\tilde{\mathbf{L}})(\mathbf{f}^0 + \mathbf{f}^1) + (1/2)(\tilde{\mathbf{L}}^0 + \tilde{\mathbf{L}}^1)(\Delta\mathbf{f}) \tag{13.37}$$

where now $(\Delta\tilde{\mathbf{L}}) = \tilde{\mathbf{L}}^1 - \tilde{\mathbf{L}}^0$.

In (13.7) the two terms on the right captured the effects of technology change and final-demand change, respectively. Here, where

$$\Delta\tilde{\mathbf{L}} = (\mathbf{I} - \mathbf{D}\mathbf{C}^1\mathbf{B})^{-1}\mathbf{C}^1 - (\mathbf{I} - \mathbf{C}^0\mathbf{A})^{-1}\mathbf{C}^0 \tag{13.38}$$

the $(1/2)(\Delta\tilde{\mathbf{L}})(\mathbf{f}^0 + \mathbf{f}^1)$ term encompasses changes in both technology and trade.

Digging Deeper into $\Delta\tilde{\mathbf{L}}$: Technical Coefficients, Trade Structure

Decomposition 1. Create $\mathbf{M} = (\mathbf{I} - \mathbf{D}\mathbf{C}^0\mathbf{B})^{-1}\mathbf{C}^0$. This represents a kind of hybrid total requirements matrix that combines 1977 technology (in **B** and **D**) with 1963 trade structure (in $\mathbf{C}^0$). By subtracting and adding this term in (13.38),

$$\begin{aligned}\Delta\tilde{\mathbf{L}} = {} & [(\mathbf{I} - \mathbf{D}\mathbf{C}^1\mathbf{B})^{-1}\mathbf{C}^1 - (\mathbf{I} - \mathbf{D}\mathbf{C}^0\mathbf{B})^{-1}\mathbf{C}^0] \\ & + [(\mathbf{I} - \mathbf{D}\mathbf{C}^0\mathbf{B})^{-1}\mathbf{C}^0 - (\mathbf{I} - \mathbf{C}^0\mathbf{A})^{-1}\mathbf{C}^0]\end{aligned} \tag{13.39}$$

The first term is a measure of the contribution to $\Delta\tilde{\mathbf{L}}$ made by changing trade proportions (with constant 1977 technology) and the second measures the effect on $\Delta\tilde{\mathbf{L}}$ of changing technology (with constant 1963 trade proportions). Then (13.39) can be written as

$$\Delta\tilde{\mathbf{L}} = \underbrace{(\tilde{\mathbf{L}}^1 - \mathbf{M})}_{\substack{\text{Trade change,} \\ \text{1977 technology}}} + \underbrace{(\mathbf{M} - \tilde{\mathbf{L}}^0)}_{\substack{\text{Technology change,} \\ \text{1963 trade patterns}}} \tag{13.40}$$

Decomposition 2. Consider, instead, $\mathbf{N} = (\mathbf{I} - \mathbf{C}^1\mathbf{A})^{-1}\mathbf{C}^1$. This is a kind of total requirements matrix that combines 1963 technology (in **A**) with 1977 trade structure (in $\mathbf{C}^1$). Subtracting and adding this term in (13.38) gives

$$\begin{aligned}\Delta\tilde{\mathbf{L}} = {} & [(\mathbf{I} - \mathbf{D}\mathbf{C}^1\mathbf{B})^{-1}\mathbf{C}^1 - (\mathbf{I} - \mathbf{C}^1\mathbf{A})^{-1}\mathbf{C}^1] \\ & + [(\mathbf{I} - \mathbf{C}^1\mathbf{A})^{-1}\mathbf{C}^1 - (\mathbf{I} - \mathbf{C}^0\mathbf{A})^{-1}\mathbf{C}^0]\end{aligned} \tag{13.41}$$

In this case, the first term is a measure of the influence on $\Delta\tilde{\mathbf{L}}$ that is due to technology change (with constant 1977 trade proportions) and the second captures the effect on $\Delta\tilde{\mathbf{L}}$ of trade proportions change (assuming 1963 technology). Now, (13.41) can be written as

$$\Delta\tilde{\mathbf{L}} = \underbrace{(\tilde{\mathbf{L}}^1 - \mathbf{N})}_{\substack{\text{Technology change,} \\ \text{1977 trade patterns}}} + \underbrace{(\mathbf{N} - \tilde{\mathbf{L}}^0)}_{\substack{\text{Trade change,} \\ \text{1963 technology}}} \tag{13.42}$$

Averaging. Averaging the results in (13.40) and (13.42) in the usual way gives

$$\Delta\tilde{\mathbf{L}} = \underbrace{(1/2)(\tilde{\mathbf{L}}^1 + \mathbf{M} - \tilde{\mathbf{L}}^0 - \mathbf{N})}_{\text{Technology change effect}} + \underbrace{(1/2)(\tilde{\mathbf{L}}^1 + \mathbf{N} - \tilde{\mathbf{L}}^0 - \mathbf{M})}_{\text{Trade change effect}} \tag{13.43}$$

and, putting this result into (13.37)

$$\begin{aligned}\Delta\mathbf{x} = {} & \underbrace{(1/4)(\tilde{\mathbf{L}}^1 + \mathbf{M} - \tilde{\mathbf{L}}^0 - \mathbf{N})(\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Technology change effect}} + \underbrace{(1/4)(\tilde{\mathbf{L}}^1 + \mathbf{N} - \tilde{\mathbf{L}}^0 - \mathbf{M})(\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Trade change effect}} \\ & + \underbrace{(1/2)(\tilde{\mathbf{L}}^0 + \tilde{\mathbf{L}}^1)(\Delta\mathbf{f})}_{\text{Final-demand change effect}}\end{aligned} \tag{13.44}$$

Digging Deeper into $\Delta\mathbf{f}$: Level and Mix

The decompositions of $\Delta\mathbf{f}$ given in (13.20) – into level and mix – were also carried out. The final expression for $\Delta\mathbf{x}$ is

$$\begin{aligned}\Delta\mathbf{x} = {} & \underbrace{(1/4)(\tilde{\mathbf{L}}^1 + \mathbf{M} - \tilde{\mathbf{L}}^0 - \mathbf{N})(\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Technology change effect}} + \underbrace{(1/4)(\tilde{\mathbf{L}}^1 + \mathbf{N} - \tilde{\mathbf{L}}^0 - \mathbf{M})(\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Trade change effect}} \\ & + \underbrace{(1/4)(\tilde{\mathbf{L}}^0 + \tilde{\mathbf{L}}^1)(\Delta f)(\mathbf{B}^0 + \mathbf{B}^1)}_{\text{Final-demand level effect}} + \underbrace{(1/4)(\tilde{\mathbf{L}}^0 + \tilde{\mathbf{L}}^1)(f^0 + f^1)(\Delta\mathbf{B})}_{\text{Final-demand mix effect}}\end{aligned} \tag{13.45}$$

This was used originally for a 70-sector, 51-region version of the model. The article presents results for a version aggregated to 10 sectors and nine regions. This means that in the original study there were 3,570 separate results for each of the decompositions. The results from this study noted in Table 13.9 below are averages over 90 outcomes for each decomposition. This illustrates again that a structural decomposition analysis for a reasonably large sized model generates an enormous amount of detail.

*A Multicountry Model for the European Community (*Oosterhaven and van der Linden, 1997*).* Here the authors are concerned with changes in value added that are associated with changes in output in a multicountry input–output setting. The model is a variant of the MRIO model, with 25 sectors, 8 countries and 4 categories of final demand in each country. Their decomposition follows the general structure of (13.34), with embellishments. Letting $\mathbf{v}^t$ and $\mathbf{c}^t$ represent column vectors of value added and value added per dollar of output at t, they work with

$$\mathbf{x}^t = \mathbf{L}^t\mathbf{f}^t = \mathbf{L}^t\mathbf{B}^t\mathbf{y}^t \text{ and } \mathbf{v}^t = \hat{\mathbf{c}}^t\mathbf{x}^t = \hat{\mathbf{c}}^t\mathbf{L}^t\mathbf{B}^t\mathbf{y}^t$$

(The bridge matrix, $\mathbf{B}^t$, and $\mathbf{y}^t$, which contains final-demand expenditures by final-demand category k in year t, were examined in section 13.1.2.)

Table 13.9 Selected Empirical Structural Decompositions at a Regional, Interregional or Multiregional Level

Author(s) and Source		Details Decomposition Components (percentage of total change[a])				
		Technology/Trade			Final Demand	
Holland and Cooke (1992, Table 2)	Washington state; 1963–82; $\Delta\mathbf{x}$; 51 sectors	5			95	
					Washington 39	*Rest of US and world* 56
Miller and Shao (1994, Table 4)	US MRIO model; 1963–77; $\Delta\mathbf{x}$; 51 regions, 70 sectors; (aggregated to 9 regions, 10 sectors)	34			67	
		Intraregional coefficients 28 (19, 59)[b]	*Interregional*[b] *coefficients* 6 (–43, 19)		*Level* 65 (53, 79)	*Mix* 2 (–5, 13)
Oosterhaven and van der Linden (1997)	Intercountry model for EC; 1975–85; Δ (value added); 8 countries, 25 sectors	–2			102	
		Intra-regional coeff. 4	*Inter-regional coeff.* –2	*Value-added coeff.* –3	*Level* 102[c]	*Mix* –1

[a] Figures may not add to 100 percent due to rounding.
[b] Figures in parentheses indicate the range of values across the nine regions in the study.
[c] This figure is further decomposed into the following percentages: Household consumption, 47; Government consumption, 20; Investment, 13; Exports to other EC countries, 9; Exports outside the EC, 12.

Then, following (13.12) for $n = 4$,

$$\Delta\mathbf{v} = (1/2)(\Delta\hat{\mathbf{c}})(\mathbf{L}^0\mathbf{B}^0\mathbf{y}^0 + \mathbf{L}^1\mathbf{B}^1\mathbf{y}^1) + (1/2)[(\hat{\mathbf{c}}^0)(\Delta\mathbf{L})(\mathbf{B}^1\mathbf{y}^1) + (\hat{\mathbf{c}}^1)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{y}^0)]$$
$$+ (1/2)[(\hat{\mathbf{c}}^0\mathbf{L}^0)(\Delta\mathbf{B})(\mathbf{y}^1) + (\hat{\mathbf{c}}^1\mathbf{L}^1)(\Delta\mathbf{B})\mathbf{y}^0)] + (1/2)(\hat{\mathbf{c}}^0\mathbf{L}^0\mathbf{B}^0 + \hat{\mathbf{c}}^1\mathbf{L}^1\mathbf{B}^1)(\Delta\mathbf{y}) \quad (13.46)$$

This accounts for the four components that contribute to the change in value added. The embellishments come from further decompositions of $\Delta\mathbf{L}$ and $\Delta\mathbf{B}$.

The European Union. The Dietzenbacher and Hoekstra study (Table 13.6) also has a spatial component because the data used came from intercountry input–output tables for the European Union (EU). This made it possible to disaggregate their final-demand component into: household consumption, other domestic final demands (government consumption, capital stock formation, inventory stock changes) and exports – to Germany, France, Italy, Belgium, Denmark, the rest of the EU, and the rest of the world.

Results from some of these studies are collected together in Table 13.9.

13.2 Mixed Models

In the usual form of the standard demand-side input–output model – $(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{f}$ and $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}$ – the final-demand elements, $\mathbf{f}$, are the exogenous components. Changes in the f_j come about as a result of forces that are outside the model (e.g., changes in consumer tastes, government purchases), and it is the effects of these changes on the economy's gross outputs, $\mathbf{x}$, that are quantified through the input–output model.

In certain situations a mixed type of input–output model may be appropriate, in which final demands for some sectors and gross outputs for the remaining sectors are specified exogenously. For example, due to a strike of a major supplier, output from a particular sector might be fixed at the amounts currently on hand in warehouses, awaiting transportation and delivery to buyers. Or, in a planned economy, a target might be to increase agricultural output by 12 percent by the end of the next planning period.

Mixed input–output models have often been applied in empirical studies in agricultural and resource economics. Some examples (discussed in Steinback, 2004) are:

- Agriculture [Johnson and Kulshreshtha, 1982 (economic importance of different farm types); Findeis and Whittlesey, 1984 (impacts of two irrigation development projects); Tanjuakio, Hastings and Tytus, 1996 (contribution of agriculture to the Delaware economy); Papadas and Dahl, 1999 (relative importance of 16 different US farm commodities); Roberts, 1994 (effects of milk production quotas)],
- Mining [Petkovich and Ching, 1978 (effects of partial elimination of mining in Nevada due to ore depletion)],
- Forestry [Eiser and Roberts, 2002 (relative economic importance of four different woodland types)],
- Fisheries [Leung and Pooley, 2002 (impacts of reduction in fishing areas in order to protect certain turtle populations)].

Most of these contain references to numerous additional studies.

All of the analysis in what follows is equally valid if we wish to model exogenous *changes in* some final demands and *changes in* gross outputs of the remaining sectors – that is, if the model is represented in $\Delta\mathbf{f}$ and $\Delta\mathbf{x}$ terms. We illustrate both scenarios below.

13.2.1 Exogenous Specification of One Sector's Output

Rearranging the Basic Equations As an example, in a three-sector model, assume that $f_1, f_2,$ and x_3 are treated as exogenous. (Since the numbering of sectors is arbitrary, we can always assume that sector n is the one whose output, not final demand, is fixed.) The basic input–output relationships are still embodied in the following three equations:

$$\begin{aligned}(1-a_{11})x_1 - a_{12}x_2 - a_{13}x_3 &= f_1\\ -a_{21}x_1 + (1-a_{22})x_2 - a_{23}x_3 &= f_2\\ -a_{31}x_1 - a_{32}x_2 + (1-a_{33})x_3 &= f_3\end{aligned}$$

Rearrange all three equations in order to have the exogenous variables ($f_1, f_2,$ and x_3) on the right-hand side and the endogenous variables ($x_1, x_2,$ and f_3) on the left. This gives

$$\begin{aligned}(1-a_{11})x_1 - a_{12}x_2 + 0f_3 &= f_1 + a_{13}x_3\\ -a_{21}x_1 + (1-a_{22})x_2 + 0f_3 &= f_2 + a_{23}x_3\\ -a_{31}x_1 - a_{32}x_2 - f_3 &= -(1-a_{33})x_3\end{aligned}$$

It is clear that not only f_1 but now also $a_{13}x_3$ (for a fixed x_3) serve as exogenous "demand" for sector 1 (first equation) and similarly both f_2 and $a_{23}x_3$ are now exogenous drivers for sector 2. To facilitate later generalization, we rewrite these equations to include all variables in each equation. This gives

$$\begin{aligned}(1-a_{11})x_1 - a_{12}x_2 + 0f_3 &= f_1 + 0f_2 + a_{13}x_3\\ -a_{21}x_1 + (1-a_{22})x_2 + 0f_3 &= 0f_1 + f_2 + a_{23}x_3\\ -a_{31}x_1 - a_{32}x_2 - f_3 &= 0f_1 + 0f_2 - (1-a_{33})x_3\end{aligned}$$

In matrix form (we use partitioned matrices and vectors to emphasize differences from the standard input–output model) these two equations are

$$\begin{bmatrix}(1-a_{11}) & -a_{12} & 0\\ -a_{21} & (1-a_{22}) & 0\\ -a_{31} & -a_{32} & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ f_3\end{bmatrix} = \begin{bmatrix}f_1 + a_{13}x_3\\ f_2 + a_{23}x_3\\ -(1-a_{33})x_3\end{bmatrix} \tag{13.47}$$

and

$$\begin{bmatrix}(1-a_{11}) & -a_{12} & 0\\ -a_{21} & (1-a_{22}) & 0\\ -a_{31} & -a_{32} & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ f_3\end{bmatrix} = \begin{bmatrix}1 & 0 & a_{13}\\ 0 & 1 & a_{23}\\ 0 & 0 & -(1-a_{33})\end{bmatrix}\begin{bmatrix}f_1\\ f_2\\ x_3\end{bmatrix} \tag{13.48}$$

Let $\mathbf{M} = \begin{bmatrix} (1-a_{11}) & -a_{12} & 0 \\ -a_{21} & (1-a_{22}) & 0 \\ -a_{31} & -a_{32} & -1 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 1 & 0 & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & -(1-a_{33}) \end{bmatrix}$. Then (13.47) and (13.48) can be expressed as

$$\mathbf{M}\begin{bmatrix} x_1 \\ x_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} f_1 + a_{13}x_3 \\ f_2 + a_{23}x_3 \\ -(1-a_{33})x_3 \end{bmatrix} \tag{13.49}$$

and

$$\mathbf{M}\begin{bmatrix} x_1 \\ x_2 \\ f_3 \end{bmatrix} = \mathbf{N}\begin{bmatrix} f_1 \\ f_2 \\ x_3 \end{bmatrix} \tag{13.50}$$

with solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ f_3 \end{bmatrix} = \mathbf{M}^{-1}\begin{bmatrix} f_1 + a_{13}x_3 \\ f_2 + a_{23}x_3 \\ -(1-a_{33})x_3 \end{bmatrix} \tag{13.51}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \\ f_3 \end{bmatrix} = \mathbf{M}^{-1}\mathbf{N}\begin{bmatrix} f_1 \\ f_2 \\ x_3 \end{bmatrix} \tag{13.52}$$

Using results on partitioned matrix inverses (Appendix A), it can be shown that

$$\mathbf{M}^{-1} = \begin{bmatrix} l_{11}^{(2)} & l_{12}^{(2)} & 0 \\ l_{21}^{(2)} & l_{22}^{(2)} & 0 \\ \beta_1 & \beta_2 & -1 \end{bmatrix}$$

where $\mathbf{L}^{(2)} = \begin{bmatrix} l_{11}^{(2)} & l_{12}^{(2)} \\ l_{21}^{(2)} & l_{22}^{(2)} \end{bmatrix} = \begin{bmatrix} (1-a_{11}) & -a_{12} \\ -a_{21} & (1-a_{22}) \end{bmatrix}^{-1}$, the Leontief inverse for a *two-sector* model.[28] The important result to notice is that the inverse of the smaller model is a component in $\mathbf{M}^{-1}$. Carrying out the multiplication $\mathbf{M}^{-1}\mathbf{N}$, (13.52) is

$$\begin{bmatrix} x_1 \\ x_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} \mathbf{L}^{(2)} & \mathbf{L}^{(2)}\begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \\ \begin{bmatrix} \beta_1 & \beta_2 \end{bmatrix} & \gamma \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \\ x_3 \end{bmatrix} \tag{13.53}$$

[28] Here and in Appendix 13.2 we will sometimes find it helpful to use $\mathbf{A}^{(k)}$ and $\mathbf{L}^{(k)} = (\mathbf{I} - \mathbf{A}^{(k)})^{-1}$ to identify coefficient and Leontief inverse matrices for a k-sector input–output model.

The exact values of β_1, β_2, and γ need not concern us at this point.

Of particular interest is the result for the endogenous outputs, x_1 and x_2,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{L}^{(2)} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \mathbf{L}^{(2)} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} x_3 = \mathbf{L}^{(2)} \begin{bmatrix} f_1 + a_{13} \\ f_2 + a_{23} \end{bmatrix} x_3 \tag{13.54}$$

Suppose that a decision has been made to increase sector 3 output to some amount, $\bar{x}_3$, for whatever reason (for example, to fill back orders, or because of *anticipated* new demand, etc.). Using (13.54), we have $f_1 = 0,\ f_2 = 0,$ and $x_3 = \bar{x}_3$, and the effects on sectors 1 and 2 are found as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{L}^{(2)} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \bar{x}_3 = \begin{bmatrix} l_{11}^{(2)} & l_{12}^{(2)} \\ l_{21}^{(2)} & l_{22}^{(2)} \end{bmatrix} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \bar{x}_3 \tag{13.55}$$

The vector $\begin{bmatrix} a_{13}\bar{x}_3 \\ a_{23}\bar{x}_3 \end{bmatrix}$ translates the new sector 3 output into sector 3's increased demands for inputs from sectors 1 and 2, and the inverse for the two-sector model converts these input demands into total necessary gross outputs from those two sectors.

"Extracting" the Sector There is an alternative approach that leads to precisely the same algebraic results for the impact of $\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ and x_3 on $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. If we modify the $\mathbf{A}$ matrix for the three-sector model by setting all the coefficients in row 3 equal to zero – $\tilde{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 \end{bmatrix}$ – we generate $(\mathbf{I} - \tilde{\mathbf{A}}) = \begin{bmatrix} 1 - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & 1 - a_{22} & -a_{23} \\ 0 & 0 & 1 \end{bmatrix}$ and, importantly,[29]

$$(\mathbf{I} - \tilde{\mathbf{A}})^{-1} = \begin{bmatrix} \mathbf{L}^{(2)} & \mathbf{L}^{(2)} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} & 1 \end{bmatrix}$$

This result depends, again, on properties of inverses to partitioned matrices (Appendix A). It is explored further in Appendix 13.2 to this chapter.

Consequently,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \mathbf{L}^{(2)} & \mathbf{L}^{(2)} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ x_3 \end{bmatrix}$$

and the results for $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ are identical to those in (13.53) or (13.54). (We show in Appendix 13.2 that this approach is valid for the general case of $n - k$ sectors with exogenized outputs.)

[29] Notice that although $\tilde{\mathbf{A}}$ is singular (a row of all zeros), $(\mathbf{I} - \tilde{\mathbf{A}})$ is not; and it is the latter matrix whose inverse is needed.

In a regional context, this approach seems to have first been discussed in Tanjuakio, Hastings and Tytus (1996); it is also featured in Steinback (2004). The economic logic, at the regional level, is that the regional purchase coefficients for the exogenized sectors are set equal to zero, thereby creating zero rows in **A** and eliminating those sectors as suppliers of (additional) interindustry inputs. It may be particularly helpful in regional situations where the **A** matrix of a ready-made input–output model is available (e.g., IMPLAN) and can be easily altered by zeroing out appropriate rows.[30]

13.2.2 *An Alternative Approach When $f_1, \ldots, f_{n-1}$ and x_n Are Exogenously Specified*[31]

This alternative makes use of the concept of an "output-to-output" multiplier (section 6.5.3). Recall that $\mathbf{L}^* = [l^*_{ij}] = \mathbf{L}\hat{\mathbf{L}}^{-1}$, where

$$l^*_{ij} = l_{ij}/l_{jj} = [\Delta x_i/\Delta f_j]/[\Delta x_j/\Delta f_j] = \Delta x_i/\Delta x_j$$

These elements, l^*_{ij}, are viewed as "output–to–output" multipliers. Each of the elements in column j of $\mathbf{L}^*$ indicates the amount of sector i output (the row label) that would be required if the *output* of sector j were one dollar.

If sector j increases its output to some new amount, $\bar{x}_j$, then $\mathbf{L}^*\bar{\mathbf{x}}$ (where $\bar{\mathbf{x}} = [0, \ldots, 0, \bar{x}_j, 0, \ldots, 0]'$) will generate a vector of total new outputs necessary from each sector in the economy because of the exogenously determined output in sector j. That is,

$$\mathbf{x}^* = \mathbf{L}^*\bar{\mathbf{x}} \tag{13.56}$$

This calculation gives the same result for the endogenous x_i as found using the approach in (13.55), above, as is demonstrated in the following examples. This result is shown to hold for the general case in Appendix 13.2.

The structure of $\mathbf{L}^*$ makes clear that a standard Leontief inverse, $\mathbf{L}$, can easily be used to capture impacts when any sector's output is made exogenous. If the output of sector j is specified exogenously, then all that is needed is that the elements in column j of $\mathbf{L}$ (known) be divided by l_{jj} (known). Put otherwise, standard demand-driven output multipliers for sector j will *uniformly overestimate* output–to–output multipliers for sector j by $[(l_{jj}-1)\times 100]$ percent. [The reader can easily show that $(l_{ij}-l^*_{ij})/l^*_{ij} = l_{jj}-1$, given that $l^*_{ij} = l_{ij}/l_{jj}$.][32]

[30] This approach is closely related to variants of the "hypothetical extraction" method for assessing a sector's importance to an economy through measures of sectoral "linkage" (a topic explored in section 12.2.5).

[31] This approach is apparently first discussed in Evans and Hoffenberg (1952) and again in Ritz and Spaulding (1975, p. 14).

[32] Roberts (1994) provides a numerical illustration in an empirical application with both standard output multipliers (final demand driven, from **L**) and those from $\mathbf{L}^*$ (output driven) for the case in which the milk sector's output is made exogenous. The (constant) percentage overestimation in the **L** model is 8.09, and the milk sector's on-diagonal element in **L** is 1.0809.

13.2.3 *Examples with x_n Exogenous*[33]

Suppose, as above, that we have a three-sector model in which f_1, f_2, and x_3 are treated as exogenous. Let $\mathbf{A} = \begin{bmatrix} .15 & .25 & .30 \\ .20 & .05 & .18 \\ .20 & .20 & .10 \end{bmatrix}$ (the first two rows and columns repeat the two-sector example in Chapter 2). In the format of (13.50), we have

$$\begin{bmatrix} .85 & -.25 & 0 \\ -.20 & .95 & 0 \\ -.20 & -.20 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & .30 \\ 0 & 1 & .18 \\ 0 & 0 & -.9 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ x_3 \end{bmatrix}$$

In particular,

$$\mathbf{M} = \begin{bmatrix} .85 & -.25 & 0 \\ -.20 & .95 & 0 \\ -.20 & -.20 & -1 \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} 1 & 0 & .30 \\ 0 & 1 & .18 \\ 0 & 0 & -.9 \end{bmatrix} \text{ and}$$

$$\mathbf{M}^{-1} = \begin{bmatrix} 1.2541 & .3300 & 0 \\ .2640 & 1.1221 & 0 \\ -.3036 & -.2904 & -1 \end{bmatrix}$$

so

$$\mathbf{M}^{-1}\mathbf{N} = \begin{bmatrix} 1.2541 & .3300 & .4356 \\ .2640 & 1.1221 & .2812 \\ -.3036 & -.2904 & .7566 \end{bmatrix}$$

Notice that the 2×2 upper-left submatrix in $\mathbf{M}^{-1}$ *and* in $\mathbf{M}^{-1}\mathbf{N}$ is indeed just the inverse of $\begin{bmatrix} .85 & -.25 \\ -.20 & .95 \end{bmatrix}$ from the two-sector model in Chapter 2:

$$\mathbf{L}^{(2)} = \begin{bmatrix} 1.2541 & .3300 \\ .2640 & 1.1221 \end{bmatrix}$$

Example 1: $f_1 = 100{,}000$, $f_2 = 200{,}000$, $x_3 = 150{,}000$ In this case, from (13.53),

$$\begin{bmatrix} x_1 \\ x_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1.2541 & .3300 & .4356 \\ .2640 & 1.1221 & .2812 \\ -.3036 & -.2904 & .7566 \end{bmatrix} \begin{bmatrix} 100{,}000 \\ 200{,}000 \\ 150{,}000 \end{bmatrix} = \begin{bmatrix} 256{,}750 \\ 293{,}000 \\ 25{,}050 \end{bmatrix}$$

[33] Even though we will use four figures to the right of the decimal in the numerical illustrations, comparisons of alternative techniques will still display small differences due to rounding, especially when matrices are inverted.

If we are only interested in the effects on the gross outputs of sectors 1 and 2, then, from (13.53),

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{L}^{(2)} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \mathbf{L}^{(2)} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} x_3 = \mathbf{L}^{(2)} \begin{bmatrix} f_1 + a_{13}x_3 \\ f_2 + a_{23}x_3 \end{bmatrix}$$

and for this example, $\begin{bmatrix} f_1 + a_{13}x_3 \\ f_2 + a_{23}x_3 \end{bmatrix} = \begin{bmatrix} 145,000 \\ 227,000 \end{bmatrix}$, so

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.2541 & .3300 \\ .2640 & 1.1221 \end{bmatrix} \begin{bmatrix} 145,000 \\ 227,000 \end{bmatrix} = \begin{bmatrix} 256{,}755 \\ 292{,}997 \end{bmatrix}$$

(The differences between these values and those for x_1 and x_2 in the three-sector version come about because of rounding in the computation of $\mathbf{M}^{-1}\mathbf{N}$, in particular in the elements in the third column of that matrix.)

Example 2: $f_1 = f_2 = 0$, $x_3 = 150,000$

Approach I. Suppose that only $x_3 = 150,000$ is exogenously specified; then $f_1 = f_2 = 0$, and (13.53) leads to

$$\begin{bmatrix} x_1 \\ x_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1.2541 & .3300 & .4356 \\ .2640 & 1.1221 & .2812 \\ -.3036 & -.2904 & .7566 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 150,000 \end{bmatrix} = \begin{bmatrix} 65,340 \\ 42,180 \\ 113,490 \end{bmatrix}$$

Again, if only the gross outputs of sectors 1 and 2 are of interest, and since $f_1 = f_2 = 0$,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{L}^{(2)} \begin{bmatrix} a_{13}x_3 \\ a_{23}x_3 \end{bmatrix} = \begin{bmatrix} 1.2541 & .3300 \\ .2640 & 1.1221 \end{bmatrix} \begin{bmatrix} 45,000 \\ 27,000 \end{bmatrix} = \begin{bmatrix} 65,345 \\ 42,177 \end{bmatrix}$$

(Differences are again due to rounded elements in the third column of $\mathbf{M}^{-1}\mathbf{N}$.)

Approach II. Continuing with the same numerical example but using the alternative approach, we create $\mathbf{L}^*$ for our three-sector illustration. Here,

$$\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.4289 & .4973 & .5758 \\ .3769 & 1.2300 & .3716 \\ .4013 & .3838 & 1.3216 \end{bmatrix}$$

and so

$$\mathbf{L}^* = \mathbf{L}\hat{\mathbf{L}}^{-1} = \begin{bmatrix} 1 & .4043 & .4356 \\ .2637 & 1 & .2812 \\ .2808 & .3121 & 1 \end{bmatrix}$$

Consider again the case in which sector 3's output is set at \$150,000. Here, then,

$$\bar{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \\ 150,000 \end{bmatrix}$$

and, as in (13.56),

$$\mathbf{x}^* = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & .4043 & .4356 \\ .2637 & 1 & .2812 \\ .2808 & .3121 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 150,000 \end{bmatrix} = \begin{bmatrix} 65,340 \\ 42,180 \\ 150,000 \end{bmatrix}$$

These values for x_1 and x_2 are the same as in our earlier results for the three-sector model (Approach I), and of course $x_3 = 150,000$, which is part of the stipulation of the problem and is assured by the fact that $l^*_{33} = 1$. (By definition, all $l^*_{jj} = 1$.)

In Appendix 13.2 we demonstrate that these two approaches for the case when x_n is exogenously specified must always give the same results for the outputs of x_1 through x_{n-1}. (Again, results on the inverse of a partitioned matrix turn out to be useful.)

Example 3: $f_1 = 100,000$, $f_2 = 200,000$, $x_3 = 100,000$ Consider the same three-sector model, with exogenous values $f_1 = 100,000$, $f_2 = 200,000$ (both as before), but $x_3 = 100,000$ (instead of 150,000). Using (13.53), we have

$$\begin{bmatrix} x_1 \\ x_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1.2541 & .3300 & .4356 \\ .2640 & 1.1221 & .2812 \\ -.3036 & -.2904 & .7566 \end{bmatrix} \begin{bmatrix} 100,000 \\ 200,000 \\ 100,000 \end{bmatrix} = \begin{bmatrix} 234,970 \\ 278,940 \\ -12,780 \end{bmatrix}$$

This simply means that the exogenously specified values of f_1, f_2, and x_3 in this example cannot possibly be satisfied unless f_3 is negative. If all variables represent *changes in* ($\Delta\mathbf{x}$ and $\Delta\mathbf{f}$) then to *increase* final demand for sectors 1 and 2 by 100,000 and 200,000, while *increasing* output of sector 3 by only 100,000, can only be accomplished by *decreasing* final demand for sector 3 by 12,780. This is not unusual in planned economies; increased *production* targets ($\Delta x_i > 0$) may be attainable only through decreases in allocations to *consumption* ($\Delta f_i < 0$). Similarly, in the case of a shortage (due to a strike, for example), increases in consumption in other sectors may require a decrease in consumption of the product that is in short supply. Whether or not negative values for f_3 make sense depends entirely on the context of the problem. If all x's and f's are *not* changes in, it may still be possible to attach meaning to a negative f_j. For example, if the exports component of final demand is defined as *net* exports, then a negative value here for f_j would mean net *imports* of j-type goods.

Example 4: The Critical Value of x_3 From the solution in (13.53), using the example values of $f_1 = 100,000$ and $f_2 = 200,000$, we can find the critical value of x_3 (call it $\bar{x}^c_3$) that makes $f_3 = 0$. (For x_3 above this value, f_3 will be positive; for x_3 below this value, f_3 will be negative.) Replacing 100,000 by $\bar{x}^c_3$ and setting $f_3 = 0$, we have

$$\begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.2541 & .3300 & .4356 \\ .2640 & 1.1221 & .2812 \\ -.3036 & -.2904 & .7566 \end{bmatrix} \begin{bmatrix} 100,000 \\ 200,000 \\ \bar{x}^c_3 \end{bmatrix}$$

From the third equation, $0 = (-.3036)(100,000) + (-.2904)(200,000) + (.7566)\bar{x}_3^c$ or $\bar{x}_3^c = 116,891$.

Multipliers From the discussion thus far and from these numerical examples, we recognize that $\mathbf{M}^{-1}\mathbf{N}$ is a *multiplier matrix* that relates the exogenously given values, $\mathbf{x}^{ex} = [x_3]$ and $\mathbf{f}^{ex} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ in our examples, to those remaining x's and f's that are endogenous, $\mathbf{x}^{en} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{f}^{en} = [f_3]$ in our examples. The elements in this matrix have the same kind of multiplier interpretation as we explored in Chapter 6 for the usual input–output system – $\mathbf{x} = \mathbf{L}\mathbf{f}$. In this example,

$$\mathbf{M}^{-1}\mathbf{N} = \begin{bmatrix} 1.2541 & .3300 & .4356 \\ .2640 & 1.1221 & .2812 \\ -.3036 & -.2904 & .7566 \end{bmatrix}$$

So, for example, if $\Delta f_1 = 1$, $\Delta f_2 = \Delta x_3 = 0$, we see that $\Delta x_1 = 1.2541$, $\Delta x_2 = 0.2640$ and $\Delta f_3 = -0.3036$; if only final demand for sector 1 increases, then output in sectors 1 and 2 must increase while final demand for sector 3 goods must decrease. The elements in the second column have a similar interpretation. The third column contains elements that multiply changes in sector 3's *output* to generate consequent changes in outputs of sectors 1 and 2 (and final demand for sector 3). Specifically, $\Delta f_1 = \Delta f_2 = 0$, $\Delta x_1 = (.4356)\Delta x_3$ and $\Delta x_2 = (.2812)\Delta x_3$. These third column elements in $\mathbf{M}^{-1}\mathbf{N}$ are thus exactly the "output–to–output" multipliers that we created in deriving $\mathbf{L}^*$. Notice in particular (Example 3, above) that $l_{13}^* = 0.4356$ and $l_{23}^* = 0.2812$; these are precisely the elements in corresponding positions in $\mathbf{M}^{-1}\mathbf{N}$. This is no accident; Appendix 13.2 demonstrates why this will always be the case.

13.2.4 Exogenous Specification of $f_1, \ldots, f_k, x_{k+1}, \ldots, x_n$

The reader can easily work out the matrix representation of, say, a four-sector model with f_1, f_2, x_3, and x_4 exogenous, starting from the basic $(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{f}$ relationships to generate the parallel to, say, (13.48). For the general n-sector case, assume that sectors have been labeled so that the outputs of the first k sectors are endogenous:[34]

$$\mathbf{x}^{en} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix}$$

[34] Sectors in an n-sector model can always be numbered so that the first k are those with endogenous gross outputs and the remaining $(n - k)$ have exogenous gross outputs.

and the corresponding final demands are exogenous:

$$\mathbf{f}^{ex} = \begin{bmatrix} f_1 \\ \vdots \\ f_k \end{bmatrix}$$

Similarly the last $(n-k)$ sectors are those whose gross outputs are exogenous:

$$\mathbf{x}^{ex} = \begin{bmatrix} x_{k+1} \\ \vdots \\ x_n \end{bmatrix}$$

and corresponding final demands are endogenous:

$$\mathbf{f}^{en} = \begin{bmatrix} f_{k+1} \\ \vdots \\ f_n \end{bmatrix}$$

Partition the coefficients matrix as $\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$, where $\mathbf{A}_{11} = \mathbf{A}^{(k,k)}$ denotes the submatrix made up of the first k rows and the first k columns of $\mathbf{A}$ [this can also be denoted $\mathbf{A}^{(k)}$ (see footnote 24)], $\mathbf{A}_{12} = \mathbf{A}^{[k,-(n-k)]}$ denotes the submatrix made up of the *first* k *rows* and the *last* $(n-k)$ *columns* of $\mathbf{A}, \mathbf{A}_{21} = \mathbf{A}^{[-(n-k),k]}$ denotes the submatrix made up of the *last* $(n-k)$ *rows* and the *first* k *columns* of $\mathbf{A}$, and $\mathbf{A}_{22} = \mathbf{A}^{[-(n-k),-(n-k)]}$ denotes the submatrix containing the *last* $(n-k)$ *rows and columns* of $\mathbf{A}$, and where the $\mathbf{I}$ and $\mathbf{0}$ matrices are of appropriate dimension in each case. The notation in the last three cases is necessary in order to distinguish *specific* row and column composition of a matrix from the general notation $\mathbf{A}^{(k)}$ for the coefficient matrix of a k-sector input–output model.

The generalization of (13.48) for $(n-k)$ exogenous outputs is

$$\begin{bmatrix} (\mathbf{I}-\mathbf{A}^{(k)}) & \mathbf{0} \\ -\mathbf{A}_{21} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}^{en} \\ \mathbf{f}^{en} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{A}_{12} \\ \mathbf{0} & -(\mathbf{I}-\mathbf{A}_{22}) \end{bmatrix} \begin{bmatrix} \mathbf{f}^{ex} \\ \mathbf{x}^{ex} \end{bmatrix} \tag{13.57}$$

The solution procedure is the same as for any square set of linear equations. Using the same notation as earlier, in the case when only x_n was exogenous, we have $\mathbf{M} = \begin{bmatrix} (\mathbf{I}-\mathbf{A}^{(k)}) & \mathbf{0} \\ -\mathbf{A}_{21} & -\mathbf{I} \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} \mathbf{I} & \mathbf{A}_{12} \\ \mathbf{0} & -(\mathbf{I}-\mathbf{A}_{22}) \end{bmatrix}$, so the solution to $\mathbf{M} \begin{bmatrix} \mathbf{x}^{en} \\ \mathbf{f}^{en} \end{bmatrix} =$

$\mathbf{N}\begin{bmatrix}\mathbf{f}^{ex}\\ \mathbf{x}^{ex}\end{bmatrix}$, namely $\begin{bmatrix}\mathbf{x}^{en}\\ \mathbf{f}^{en}\end{bmatrix} = \mathbf{M}^{-1}\mathbf{N}\begin{bmatrix}\mathbf{f}^{ex}\\ \mathbf{x}^{ex}\end{bmatrix}$, becomes[35]

$$\begin{bmatrix}\underset{(k\times 1)}{\mathbf{x}^{en}}\\ \underset{[(n-k)\times 1]}{\mathbf{f}^{en}}\end{bmatrix} = \begin{bmatrix}\mathbf{L}^{(k)} & \mathbf{L}^{(k)}\mathbf{A}_{12}\\ -\mathbf{A}_{21}\mathbf{L}^{(k)} & (\mathbf{I}-\mathbf{A}_{22})-\mathbf{A}_{21}\mathbf{L}^{(k)}\mathbf{A}_{12}\end{bmatrix}\begin{bmatrix}\underset{(k\times 1)}{\mathbf{f}^{ex}}\\ \underset{[(n-k)\times 1]}{\mathbf{x}^{ex}}\end{bmatrix} \tag{13.58}$$

where $(\mathbf{I}-\mathbf{A}^{(k)})^{-1} = \mathbf{L}^{(k)}$. (We indicate the dimensions of exogenous and endogenous vectors as an aid for what follows.) The parallel result for the earlier case is in (13.53).

As a check on the logic of the results in (13.57), notice that the two "extreme cases" correspond exactly to the basic input–output model.

Case 1: No exogenous outputs. Here $k = n$, $\mathbf{L}^{(k)} = \mathbf{L}^{(n)}$, $\mathbf{x}^{en} = \begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$, $\mathbf{f}^{ex} = \begin{bmatrix}f_1\\ \vdots\\ f_n\end{bmatrix}$, and $\mathbf{A}_{21}, \mathbf{A}_{12}, (\mathbf{I}-\mathbf{A}_{22}), \mathbf{x}^{ex}$, and $\mathbf{f}^{en}$ do not exist, so (13.57) is just the standard input–output model $(\mathbf{I}-\mathbf{A}^{(n)})\mathbf{x}^{en} = \mathbf{f}^{ex}$.

Case 2: All outputs exogenous. Here $k = 0, (\mathbf{I}-\mathbf{A}_{22}) = (\mathbf{I}-\mathbf{A}^{(n)}), \mathbf{x}^{ex} = \begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$, $\mathbf{f}^{en} = \begin{bmatrix}f_1\\ \vdots\\ f_n\end{bmatrix}$, and $\mathbf{L}^{(k)}, \mathbf{A}_{21}, \mathbf{A}_{12}, \mathbf{x}^{en}$, and $\mathbf{f}^{ex}$ disappear from (13.57), leaving $-\mathbf{I}\mathbf{f}^{en} = -(\mathbf{I}-\mathbf{A}^{(n)})\mathbf{x}^{ex}$ or $(\mathbf{I}-\mathbf{A}^{(n)})\mathbf{x}^{ex} = \mathbf{f}^{en}$. In words, if you specify all n outputs in the standard model, the n final demands are uniquely determined.

Consider also the two "less extreme" cases, which make more sense when we are dealing with the model in "changes in" (Δ) form, namely in which either $\Delta\mathbf{x}^{ex} = \mathbf{0}$ or $\Delta\mathbf{f}^{ex} = \mathbf{0}$.

Case 3: $\Delta\mathbf{x}^{ex} = \mathbf{0}$. Here the model is driven only by changes in final demands for sectors $1, \ldots, k$; $\Delta\mathbf{f}^{ex} \neq \mathbf{0}$. Then $\Delta\mathbf{x}^{en} = \mathbf{L}^{(k)}\Delta\mathbf{f}^{ex}$, which is a standard k-sector input–output model. As a consequence, $\Delta\mathbf{f}^{en} = -\mathbf{A}_{21}\mathbf{L}^{(k)}\Delta\mathbf{f}^{ex} = -\mathbf{A}_{21}\Delta\mathbf{x}^{en}$. This is a perfectly logical but not very interesting case. Here if $\Delta\mathbf{f}^{ex} > \mathbf{0}$, then $\Delta\mathbf{x}^{en} \geq \mathbf{0}$ and $\Delta\mathbf{f}^{en} = -\mathbf{A}_{21}\Delta\mathbf{x}^{en} \leq \mathbf{0}$, meaning that the changes in at least some of the $n-k$ endogenous final demands are necessarily negative. In words, since $\Delta\mathbf{x}^{ex} = \mathbf{0}$, the needs of sectors $1, \ldots, k$ for inputs from sectors $k+1, \ldots, n$, as itemized in $\mathbf{A}_{21}$, must be met by reductions in the amounts available for final demands for those remaining sectors.

Case 4: $\Delta\mathbf{f}^{ex} = \mathbf{0}$. Here the model is driven only by changes in outputs of sectors $k+1, \ldots, n$; $\Delta\mathbf{x}^{ex} \neq \mathbf{0}$. In this case,

$$\Delta\mathbf{x}^{en} = \mathbf{L}^{(k)}\mathbf{A}_{12}\Delta\mathbf{x}^{ex} \tag{13.59}$$

[35] This depends on results for the inverse of a partitioned matrix and also on the straightforward rules for multiplication of partitioned matrices. See Appendix A.

where $\mathbf{A}_{12} = \mathbf{A}^{[k,-(n-k)]} = \begin{bmatrix} a_{1,k+1} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k,k+1} & \cdots & a_{kn} \end{bmatrix}$. For example, the first element in $\mathbf{A}_{12}\Delta\mathbf{x}^{ex}$ will be $a_{1,k+1}\Delta x_{k+1} + a_{1,k+2}\Delta x_{k+2} + \cdots + a_{1n}\Delta x_n$; this represents the inputs that are needed from endogenous sector 1 to allow production of the fixed amounts of output in sectors $k+1, \ldots, n$. [In an early application of input–output analysis at the regional level, Tiebout (1969) specified (projected) the outputs of 13 out of 57 local sectors exogenously and found the consequent outputs of the remaining 44 sectors in the regional economy in just this way.]

At the same time, in this scenario $\Delta\mathbf{f}^{en} = [(\mathbf{I} - \mathbf{A}_{22}) - \mathbf{A}_{21}\mathbf{L}^{(k)}\mathbf{A}_{12}]\Delta\mathbf{x}^{ex}$. This is exactly the structure as we saw earlier in examining interregional feedback effects in an interregional input–output model (Chapter 3) and in multiplier decompositions (Chapter 6). Here the logic is essentially the same: (a) $\mathbf{A}_{12}\Delta\mathbf{x}^{ex}$ identifies inputs from endogenous sectors to satisfy $\Delta\mathbf{x}^{ex}$; (b) $\mathbf{L}^{(k)}\mathbf{A}_{12}\Delta\mathbf{x}^{ex}$ converts those needs into *total* endogenous sector production (direct plus indirect effects); (c) $\mathbf{A}_{21}\mathbf{L}^{(k)}\mathbf{A}_{12}\Delta\mathbf{x}^{ex}$ then translates that production into necessary inputs from exogenous sectors; and (d) since $\Delta\mathbf{x}^{ex}$ has already been fixed, this added amount must be netted out of what would have otherwise been available for final demands in sectors $k+1, \ldots, n$, $(\mathbf{I} - \mathbf{A}_{22})\Delta\mathbf{x}^{ex}$.

We will see in section 13.4 that a mix of x's and f's in the endogenous and exogenous categories can also be a useful framework for assessing the impact of a new industry on an economy.

13.2.5 An Example with x_{n-1} and x_n Exogenous

Example 5 (Example 2 expanded)

Approach I. Suppose now that both x_2 and x_3 are exogenous; along with $f_1 = 0$ and $x_3 = 150,000$ (as in Example 2), let $x_2 = 100,000$. Then, in terms of (13.57) (the reader might want, for practice, to check each of these submatrices as well as the subsequent matrix multiplications),

$$\mathbf{M} = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^{(1)}) & \mathbf{0} \\ -\mathbf{A}_{21} & -\mathbf{I} \end{bmatrix} = \begin{bmatrix} .85 & 0 & 0 \\ -.2 & -1 & 0 \\ -.2 & 0 & -1 \end{bmatrix}$$

and

$$\mathbf{N} = \begin{bmatrix} \mathbf{I} & \mathbf{A}_{12} \\ \mathbf{0} & -(\mathbf{I} - \mathbf{A}_{22}) \end{bmatrix} = \begin{bmatrix} 1 & .25 & .3 \\ 0 & -.95 & .18 \\ 0 & .2 & -.9 \end{bmatrix}$$

giving

$$\begin{bmatrix} x_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1.1765 & .2941 & .3530 \\ -.2535 & .8912 & -.2506 \\ -.2535 & -.2588 & .8294 \end{bmatrix} \begin{bmatrix} 0 \\ 100,000 \\ 150,000 \end{bmatrix} = \begin{bmatrix} 82,360 \\ 51,530 \\ 98,530 \end{bmatrix} \quad (13.60)$$

These results can be verified by noticing that $\begin{bmatrix} x_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 82,360 \\ 51,530 \\ 98,530 \end{bmatrix}$, along with $f_1 = 0, x_2 = 100,000$ and $x_3 = 150,000$, satisfy (except for rounding) the basic input–output equations, at the beginning of section 13.2.1.

Approach II. Alternatively, if we tried using $\mathbf{L}^*$, we would find

$$\mathbf{x}^* = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{L}^*\bar{\mathbf{x}} = \begin{bmatrix} 1 & .4043 & .4356 \\ .2637 & 1 & .2812 \\ .2808 & .3121 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 100,000 \\ 150,000 \end{bmatrix} = \begin{bmatrix} 105,770 \\ 142,180 \\ 181,210 \end{bmatrix} \quad (13.61)$$

which is totally wrong – neither x_2 nor x_3 is at its prespecified exogenous value and x_1 is wildly different from the result in (13.60). As already mentioned, we indicate in Appendix 13.2 why the $\mathbf{L}^*$ approach is only possible when just one sector's output is specified exogenously.

13.3 New Industry Impacts in the Input–Output Model

The input–output model provides a framework within which to assess the economic impact associated with the introduction of a new industry into an economy – for example, a basic manufacturing activity in a less-developed country, an export-oriented industry in a region, and so on. A quantitative approach to this kind of problem is extremely important. Individuals responsible for planning economic development (for a nation or a region) need to be able to make quantitative estimates of the total amount of economic benefit that can be expected from policies designed to attract certain kinds of industry to an area. Then the costs associated with attracting the activity – for example, reduced business taxes as an incentive, possible environmental degradation – can be weighed against the benefits of the new economic activity associated with the new industry. For convenience, in this section we will consider that the in-movement of the new industry is to a region, whether studied in isolation or as part of an interregional or multiregional system. It will be clear that the same principles apply if the "region" is in fact an entire country. In the input–output literature, one finds discussions of essentially two ways of introducing a new production activity into an economic area – through a new final-demand vector only and through the addition of new elements into the technical coefficients table for the economy. We examine these in turn.

13.3.1 New Industry: The Final-Demand Approach

For illustration, we again consider a two-sector regional economy, for which we have a 2×2 input coefficient matrix, $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. If a firm in a different industry, which we will denote sector 3, were to locate in the region, one way of attempting to quantify the impact of this in-movement on the region is as follows.[36] From an input–output coefficient table for another region of the country, or from a national table, or from surveys, assume that it is possible to estimate what the inputs will be from sectors 1 and 2 per dollar's worth of output of the new sector 3; that is, a_{13} and a_{23}.

In order to quantify the impact of the in-movement of sector 3 to the economy, we must have some measure of the *magnitude* of new economic activity associated with sector 3. In input–output terms, this means that either sector 3's level of production (gross output), x_3, or of sales to final demand, f_3, must be specified. For this example, assume that the measure of new activity by sector 3 is gross output; denote this proposed level of sector 3 production by $\bar{x}_3$. This is often the case. A new firm plans to build, say, a \$2.5 million plant with a planned annual output of \$850,000, for example. Then the new demand on sectors 1 and 2 that arises because of production by the new sector 3 is $a_{13}\bar{x}_3$ and $a_{23}\bar{x}_3$, respectively. That is, we can view these new demands as an *exogenous* change imposed on the original two sectors; $\Delta\mathbf{f} = \begin{bmatrix} a_{13}\bar{x}_3 \\ a_{23}\bar{x}_3 \end{bmatrix}$, and so the impacts, in terms of the outputs from these two sectors, will be given by $\Delta\mathbf{x} = \mathbf{L}\Delta\mathbf{f}$:

$$\Delta\mathbf{x} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} a_{13}\bar{x}_3 \\ a_{23}\bar{x}_3 \end{bmatrix} = \begin{bmatrix} l_{11}a_{13}\bar{x}_3 + l_{12}a_{23}\bar{x}_3 \\ l_{21}a_{13}\bar{x}_3 + l_{22}a_{23}\bar{x}_3 \end{bmatrix} \tag{13.62}$$

Given that there are also the usual kinds of final demands, $\bar{f}_1$ and $\bar{f}_2$, for the products of the two sectors, total gross outputs in sectors 1 and 2 will be

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} \bar{f}_1 + a_{13}\bar{x}_3 \\ \bar{f}_2 + a_{23}\bar{x}_3 \end{bmatrix} = \begin{bmatrix} l_{11}(\bar{f}_1 + a_{13}\bar{x}_3) + l_{12}(\bar{f}_2 + a_{23}\bar{x}_3) \\ l_{21}(\bar{f}_1 + a_{13}\bar{x}_3) + l_{22}(\bar{f}_2 + a_{23}\bar{x}_3) \end{bmatrix} \tag{13.63}$$

This is exactly the structure of the model in (13.55), and for the same reason. We are specifying $\bar{f}_1$ and $\bar{f}_2$ and, in addition, the value of x_3. When $\bar{x}_3 = 0$, that is, without the new sector in the region, this is a standard input–output exercise. When $\bar{f}_1 = 0$ and $\bar{f}_2 = 0$, then in (13.63) we find the impact of the new industry alone – as in (13.62).

For example, using the same illustration, let $\mathbf{A} = \begin{bmatrix} .15 & .25 \\ .20 & .05 \end{bmatrix}$. Then $(\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} 1.253 & .330 \\ .264 & 1.122 \end{bmatrix}$. Assume that our estimates of the direct input coefficients for the new sector 3 are $a_{13} = 0.30$ and $a_{23} = 0.18$, and that the plant in the new sector 3 that is moving into the region expects to produce at a level of \$100,000 per year. So $\bar{x}_3 =$

[36] This is essentially the approach used by Isard and Kuenne (1953) and by Miller (1957) in early applications of the input–output framework at a regional level.

100,000, $\Delta\mathbf{f} = \begin{bmatrix} 30,000 \\ 18,000 \end{bmatrix}$, and, as in (13.62),

$$\Delta\mathbf{x} = \begin{bmatrix} 1.253 & .330 \\ .264 & 1.122 \end{bmatrix} \begin{bmatrix} 30,000 \\ 18,000 \end{bmatrix} = \begin{bmatrix} 43,560 \\ 28,116 \end{bmatrix} \tag{13.64}$$

Sector 1, in satisfying the new demand for \$30,000 worth of its product, will ultimately have to increase its output by \$43,560. Similarly, the new demands on sector 2 from sector 3 are \$18,000, but in the end sector 2 will need to produce a total of \$28,116 more output. These figures represent one way of measuring the impact on an economy that comes about from the in-movement of new industrial activity.

With a_{13} and a_{23} assumed known, but $a_{31} = a_{32} = a_{33} = 0$, the basic equations in this approach are

$$\begin{aligned} (1 - a_{11})x_1 - a_{12}x_2 - a_{13}x_3 &= f_1 \\ -a_{21}x_1 + (1 - a_{22})x_2 - a_{23}x_3 &= f_2 \\ 0x_1 + 0x_2 + x_3 &= f_3 \end{aligned}$$

The first two equations reflect the fact that sector 1 and 2 outputs are used as inputs to (the new) sector 3. The third equation shows that all of sector 3's output can be used to satisfy final demand, since it is not used as an input to production in the region. (For example, a sector may move to a region to be closer to the sources of inputs, while continuing to produce a product for export.)

In matrix terms, with

$$\bar{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & 0 & 0 \end{bmatrix} \text{ and } (\mathbf{I} - \bar{\mathbf{A}}) = \begin{bmatrix} (1 - a_{11}) & -a_{12} & -a_{13} \\ -a_{21} & (1 - a_{22}) & -a_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

we have partially included the new sector in the **A** matrix. To assess the impact of new sector 3 production, $\bar{x}_3$, we let $f_1 = 0$ and $f_2 = 0$. Also, $f_3 = x_3 = \bar{x}_3$, from the third equation above. Thus

$$\mathbf{x} = \bar{\mathbf{L}} \begin{bmatrix} 0 \\ 0 \\ \bar{x}_3 \end{bmatrix}$$

where $\bar{\mathbf{L}} = [\bar{l}_{ij}] = (\mathbf{I} - \bar{\mathbf{A}})^{-1}$. Because of the zeros in $\mathbf{f}$, $x_1 = \bar{l}_{13}\bar{x}_3$, $x_2 = \bar{l}_{23}\bar{x}_3$, and $x_3 = \bar{l}_{33}\bar{x}_3$. That is, only the third column of the inverse is of interest. Using results on the inverse of a partitioned matrix (Appendix A) it is easily shown that

$$\begin{bmatrix} \bar{l}_{13} \\ \bar{l}_{23} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \text{ and } \bar{l}_{33} = 1$$

In particular, then,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \bar{x}_3$$

exactly as in (13.62), above. Note also that, as expected, $x_3 = (1)\bar{x}_3$.

13.3.2 New Industry: Complete Inclusion in the Technical Coefficients Matrix

The estimate of the impact of the new industry that was given above is clearly conservative; the complete impact of a new sector of an economy would reflect the fact that not only would the new industry buy inputs from existing sectors, but it would probably also sell its own product as an input to other producing sectors in the economy and ultimately the entire technical structure of the economy may change. In the first place, there will be a new column and row of direct-input coefficients associated with purchases by and sales of the new sector. In addition, there may be changes in the elements of the original **A** matrix, reflecting, for example, substitution of the newly available input for one previously used.

To completely "close" the previous 2×2 coefficient matrix with respect to the new industry, we need a_{13} and a_{23} (which we have already assumed can be estimated), and we also need a_{31} and a_{32}, estimates of how much each of the old industries (1 and 2) will buy from the new sector (3) per dollar's worth of their outputs, plus a_{33}, the intrasectoral input coefficient for the new industry. For in-movement of a new industry into a region with n original sectors, the previous approach required that we estimate n new coefficients (a column for the new sector, except for the last element). For the present approach we need an additional $(n+1)$ coefficients (a row for the new sector, including intraindustry use per dollar's worth of output); we need $(2n+1)$ new coefficients in all.

Again, assuming that x_3 is known, our three-equation model, relating the endogenous variables $x_1, x_2,$ and f_3 to the values $\bar{f}_1$, $\bar{f}_2$, and $\bar{x}_3$, is still

$$\begin{aligned}(1-a_{11})x_1 - a_{12}x_2 - a_{13}\bar{x}_3 &= \bar{f}_1\\ -a_{21}x_1 + (1-a_{22})x_2 - a_{23}\bar{x}_3 &= \bar{f}_2\\ -a_{31}x_1 - a_{32}x_2 + (1-a_{33})\bar{x}_3 &= f_3\end{aligned} \tag{13.65}$$

Rearranging, to put exogenous variables on the right-hand side,

$$\begin{aligned}(1-a_{11})x_1 - a_{12}x_2 + 0f_3 &= \bar{f}_1 + a_{13}\bar{x}_3\\ -a_{21}x_1 + (1-a_{22})x_2 + 0f_3 &= \bar{f}_2 + a_{23}\bar{x}_3\\ -a_{31}x_1 - a_{32}x_2 - f_3 &= -(1-a_{33})\bar{x}_3\end{aligned} \tag{13.66}$$

The matrix representation for (13.66) is

$$\begin{bmatrix}(1-a_{11}) & -a_{12} & 0\\ -a_{21} & (1-a_{22}) & 0\\ -a_{31} & -a_{32} & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ f_3\end{bmatrix} = \begin{bmatrix}\bar{f}_1 + a_{13}\bar{x}_3\\ \bar{f}_2 + a_{23}\bar{x}_3\\ -(1-a_{33})\bar{x}_3\end{bmatrix} \tag{13.67}$$

This is exactly the structure of the model in (13.48), in the previous section, and so solution possibilities are the same as we saw in the examples of that section. In particular, there is no guarantee that the f_3 associated with given values $\bar{f}_1, \bar{f}_2$, and $\bar{x}_3$ will be positive.

Instead of specification of the level of gross output of the new sector, one could quantify the magnitude of the new operation by exogenously fixing the level of sales to final demand – that is, by specifying f_3 at $\bar{f}_3$, instead of x_3 at $\bar{x}_3$. But then, from (13.65), with x_3 now a variable to be determined (not specified at $\bar{x}_3$), we see that this is a standard kind of input–output problem. Whether $\bar{f}_1 = 0$ or not, and whether $\bar{f}_2 = 0$ or not, given some $\bar{f}_3 > 0$, we find the associated values of the necessary gross outputs, x_1, x_2, and x_3, through the use of the Leontief inverse to the 3×3 $(\mathbf{I} - \mathbf{A})$ matrix in (13.67). Thus, when the level of new sector activity is specified in terms of sales to final demand rather than gross output, no new principles are involved in assessing the impact on the economy into which the industry moves.

For example, for our illustrative problem, the 3×3 technical coefficients matrix is

$$\bar{\mathbf{A}} = \begin{bmatrix} .15 & .225 & .30 \\ .20 & .05 & .18 \\ .20 & .20 & .10 \end{bmatrix}$$

(using an overbar to distinguish this from the original 2×2 $\mathbf{A}$ matrix). Thus the matrix of coefficients in the equations in (13.67) is

$$(\mathbf{I} - \bar{\mathbf{A}}) = \begin{bmatrix} .85 & -.25 & -.30 \\ -.20 & .95 & -.18 \\ -.20 & -.20 & .90 \end{bmatrix} \tag{13.68}$$

and the corresponding inverse is

$$\bar{\mathbf{L}} = \begin{bmatrix} 1.429 & .497 & .576 \\ .377 & 1.230 & .372 \\ .401 & .384 & 1.322 \end{bmatrix} \tag{13.69}$$

Given $\bar{f}_1 = 100,000$, $\bar{f}_2 = 200,000$, and, say, $\bar{f}_3 = 50,000$, we find that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1.429 & .497 & .576 \\ .377 & 1.230 & .372 \\ .401 & .384 & 1.322 \end{bmatrix} \begin{bmatrix} 100,000 \\ 200,000 \\ 50,000 \end{bmatrix} = \begin{bmatrix} 271,100 \\ 302,300 \\ 183,000 \end{bmatrix} \tag{13.70}$$

in standard input–output fashion.

13.3.3 A New Firm in an Existing Industry

If the firm that moves into a region belongs to a sector that is already established in the region, so that the effect is to augment the production capacity of a particular existing industry, *not* introduce it into the local economy for the first time, the assessment of its impact is fairly straightforward. In particular, an input–output table for the economy in question will already include interindustry and intraindustry relationships for the sector in which the new firm is classified.

Assume that we have a three-sector economy and that the new firm is classified as a member of sector 3. Thus 3×3 $\mathbf{A}$ and $\mathbf{L}$ matrices are known. If the level of activity in the new firm is specified as a certain total amount of production, then we have a positive x_3^*, and the relationships among sectors are exactly those shown in (13.65), above, where now we use x_3^* in place of $\bar{x}_3$ to distinguish the two cases ($\bar{x}_3$ when the industry was new to the region, x_3^* when the new firm only represents an increase in capacity of the existing sector). The new demands on the three original sectors are found as

$$\begin{bmatrix} a_{13}x_3^* \\ a_{23}x_3^* \\ a_{33}x_3^* \end{bmatrix} \tag{13.71}$$

and impacts on all three sectors are found in the standard input–output way:

$$\Delta\mathbf{x} = \mathbf{L}\begin{bmatrix} a_{13}x_3^* \\ a_{23}x_3^* \\ a_{33}x_3^* \end{bmatrix} \tag{13.72}$$

If the level of new capacity in sector 3 is specified through an additional amount of sales to final demand, that is, as Δf_3, then the impact is found in the usual input–output way. The new final-demand vector is $\begin{bmatrix} 0 \\ 0 \\ \Delta f_3 \end{bmatrix}$ and

$$\Delta\mathbf{x} = \mathbf{L}\begin{bmatrix} 0 \\ 0 \\ \Delta f_3 \end{bmatrix} \tag{13.73}$$

which is just

$$\Delta x_1 = l_{13}\Delta f_3,\ \Delta x_2 = l_{23}\Delta f_3,\ \Delta x_3 = l_{33}\Delta f_3 \text{ or } \Delta\mathbf{x} = \begin{bmatrix} l_{13} \\ l_{23} \\ l_{33} \end{bmatrix}(\Delta f_3) \tag{13.74}$$

For example, assume that the Leontief inverse for the three-sector economy is as shown in (13.69). If a new firm in sector 3 moves into the economy and its projected level of annual production is \$120,000 ($x_3^* = 120,000$), then, using the elements in the third column of the technical coefficients matrix, we find the new final demands in (13.71) as

$$\begin{bmatrix} (.30)(120,000) \\ (.18)(120,000) \\ (.10)(120,000) \end{bmatrix} = \begin{bmatrix} 36,000 \\ 21,600 \\ 12,000 \end{bmatrix}$$

and, as in (13.72)

$$\Delta\mathbf{x} = \begin{bmatrix} 1.429 & .497 & .576 \\ .377 & 1.230 & .372 \\ .401 & .384 & 1.322 \end{bmatrix}\begin{bmatrix} 36,000 \\ 21,600 \\ 12,000 \end{bmatrix} = \begin{bmatrix} 69,127 \\ 44,604 \\ 38,594 \end{bmatrix}$$

Notice that the *total* new output from sector 3 is \$158,594. This figure includes the \$120,000 from the new firm and \$38,594 of additional output from the old (existing) firms in sector 3. On the other hand, if increased capacity in sector 3 is specified as, say, \$70,000 more sales to final demand for sector 3 goods, then, as in (13.73),

$$\Delta \mathbf{x} = \begin{bmatrix} 1.429 & .497 & .576 \\ .377 & 1.230 & .372 \\ .401 & .384 & 1.322 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 70,000 \end{bmatrix} = \begin{bmatrix} 40,320 \\ 26,880 \\ 92,540 \end{bmatrix}$$

which is just $\begin{bmatrix} .576 \\ .372 \\ 1.322 \end{bmatrix}$(70,000), as in (13.74).

13.3.4 Other Structural Changes

As already mentioned, when a new industry moves into an economic area, or when the capacity of an existing sector is increased, it is entirely possible that current transaction patterns for existing sectors in the region will change. For example, sector j, which formerly bought input i from a firm located outside the region, may now purchase some (or all) of input i from the new local establishment. Or, indeed, sector j may replace formerly used input k, bought from a producer in the region, with input i bought from the new establishment in the region. Such changes in transactions, the elements of the **Z** matrix, will generate changes in direct-input coefficients in columns and rows other than those for the new sector (or for the sector whose capacity has been increased.)

It should be clear that out-movement of a firm or an entire sector from a local economy can be treated in much the same way. Usually output, income, employment or value-added multipliers provide an adequate approach to quantifying such decreases in economic activity – particularly if, say, one plant closes but other plants in the same sector remain. If all economic activity in a sector is stopped – for example, all shoe manufacturing leaves Massachusetts and moves to the South – then the column and row for that sector disappear from the Massachusetts **A** matrix, and local producers in other sectors that use the product as an input will either have to import the good that has disappeared from the local economy or else they will substitute alternative locally produced inputs. Similarly, local firms that previously supplied inputs to the now-absent sector will find their sales patterns altered. Again, changes will occur in other columns and/or rows of the **A** matrix. However, it is extremely difficult to predict exactly where these changes will be and exactly what their magnitude will be.

13.4 Dynamic Considerations in Input–Output Models

13.4.1 General Relationships

Thus far, we have considered analysis using the **A** matrix of technical coefficients derived from measured *flows* of goods between sectors, purchased to serve current production needs during a particular period of time. Each of the flows, z_{ij}, is viewed as serving as an input for current output, x_j, and these relations are reflected in the technical coefficients, $a_{ij} = z_{ij}/x_j$. Actually, however, some input goods contribute to the production process but are not immediately used up during that production – machines, buildings, and so on. In other words, a sector has a certain capital stock that is also necessary for production. If one could measure the value of the output of sector i that is held by sector j as *stock*, k_{ij}, then one could estimate a "capital coefficient," by dividing this holding of stock by the output of sector j, over some period. Along with fixed investment items such as buildings and machinery, goods bought as inventory by sector j, to use as inputs to later production, may also be included in the k_{ij} term. Let $b_{ij} = k_{ij}/x_j$; this coefficient is interpreted as the amount of sector i's product (in dollars) held as capital stock for production of one dollar's worth of output by sector j.[37]

For example, if sector i is the construction industry and sector j is automobiles, b_{ij} might represent the dollars' worth of factory space per dollar's worth of automobiles produced. Clearly, for current production, the machinery, buildings, and so forth must already be in place. But if an economy is growing, then anticipated production (next year) is different from current production (this year), and the amount of supporting capital may change: one simple assumption (often used) is that the amount of new production from sector i for capital stocks in sector j in time period $t+1$ (say next year) will be given by $b_{ij}(x_j^{t+1} - x_j^t)$, where the superscripts denote time periods (here years); that is, the amount of sector i production necessary to satisfy the added demand in sector j for goods from sector i as capital stocks for next year's production is given by the observed capital coefficient, b_{ij}, times the change in sector j output between this year and next year, $(x_j^{t+1} - x_j^t)$. This use of the capital coefficients assumes that production is at or near effective capacity in sector j, since the anticipated increase in production, if $(x_j^{t+1} - x_j^t)$ is positive, requires new capital goods.[38]

The typical equation for the output of sector i in period t would become

$$x_i^t = \sum_{j=1}^{n} a_{ij}x_j^t + \sum_{j=1}^{n} b_{ij}(x_j^{t+1} - x_j^t) + f_i^t \tag{13.75}$$

or

$$x_i^t - \sum_{j=1}^{n} a_{ij}x_j^t + \sum_{j=1}^{n} b_{ij}x_j^t - \sum_{j=1}^{n} b_{ij}x_j^{t+1} = f_i^t \tag{13.76}$$

[37] It has become traditional to use b_{ij}, and later $\mathbf{B} = [b_{ij}]$, for capital coefficients in a dynamic input–output model. It is also traditional to use $\mathbf{B}$ in the Ghosh model, as we saw in section 12.1, and to represent a "bridge" matrix, as in section 13.1.8. The context should make clear which meaning is intended.

[38] The $x_j^{t+1} - x_j^t$ term could also be negative or zero. Thus, if $b_{ij} = 0.02$ and $x_j^{t+1} - x_j^t = \$100$, there will be a need for \$2 more output from sector i for sector j; if $x_j^{t+1} - x_j^t = -\$300$, the model would forecast a decrease of \$6 in purchases from i by j. In general, we are usually concerned with sectoral consequences of economic growth, so that the usual setting in which the dynamic model is used is when $x_j^{t+1} - x_j^t$ is strictly positive.

The matrix form, using an $n \times n$ capital coefficients matrix $\mathbf{B} = [b_{ij}]$, is

$$(\mathbf{I} - \mathbf{A})\mathbf{x}^t - \mathbf{B}(\mathbf{x}^{t+1} - \mathbf{x}^t) = \mathbf{f}^t \text{ or } (\mathbf{I} - \mathbf{A} + \mathbf{B})\mathbf{x}^t - \mathbf{B}\mathbf{x}^{t+1} = \mathbf{f}^t \qquad (13.77)$$

One rearrangement of this result is

$$\mathbf{B}\mathbf{x}^{t+1} = (\mathbf{I} - \mathbf{A} + \mathbf{B})\mathbf{x}^t - \mathbf{f}^t \qquad (13.78)$$

for $t = 0, 1, \ldots, T$. For example, if the time superscripts denote years, this represents a set of relationships between gross outputs and final demands starting now (year $t = 0$) and extending T years into the future.[39]

These are linear *difference equations*, since the values of the variables – the x_j – are related for different periods of time via the coefficients in **A** and **B** and the final demands. Solution methods for sets of difference equations, and analysis of the values of the variables over time, are topics that go beyond the level of this text. The intention here is primarily to acquaint the reader with the notion of capital coefficients and with one of the ways in which the existence of stocks of capital goods for production have been incorporated into input–output analysis.[40] Clearly, the assumptions inherent in this model – for example, the stability of capital coefficients over time – deserve just as careful scrutiny as those in the static model. Moreover, data and measurement problems for estimating capital coefficients are even more severe than those for technical coefficients.

From (13.77) it is possible to derive either a "forward looking" or a "backward looking" expression. Solving for $\mathbf{x}^t$ in terms of $\mathbf{x}^{t+1}$ gives $\mathbf{x}^t = (\mathbf{I} - \mathbf{A} + \mathbf{B})^{-1}(\mathbf{B}\mathbf{x}^{t+1} + \mathbf{f}^t)$; letting $\mathbf{G} = (\mathbf{I} - \mathbf{A} + \mathbf{B})$, this is $\mathbf{x}^t = \mathbf{G}^{-1}(\mathbf{B}\mathbf{x}^{t+1} + \mathbf{f}^t)$.[41] Each period's outputs depend on the outputs of the following period (and current period final demands). This kind of solution is possible as long as $\mathbf{G}^{-1}$ exists, and in practice $(\mathbf{I} - \mathbf{A} + \mathbf{B})$ is not likely to be singular. On the other hand, from (13.77) or (13.78) we can equally well find $\mathbf{x}^{t+1}$ as a function of $\mathbf{x}^t$, namely $\mathbf{x}^{t+1} = \mathbf{B}^{-1}(\mathbf{G}\mathbf{x}^t - \mathbf{f}^t)$, and now each period's outputs depend on the outputs from the previous period (and, again, current period final demands). This approach requires that **B** be nonsingular, and, in fact, singularity of the **B** matrix is a problem in dynamic input–output models. It is easy to see why it might be that $|\mathbf{B}| = 0$. In a model with a fairly large number of sectors (a relatively disaggregated model), it is very likely that there will be sectors that do not supply capital goods to any sectors – that is, sectors whose row in the **B** matrix will contain all zeros. (For example, if there were a sector labeled "Agriculture, potatoes.") When one or more rows of a

[39] In some discussions of dynamic input–output models, the time superscripts are shifted "backward" by one period, leading to $(\mathbf{I} - \mathbf{A} + \mathbf{B})\mathbf{x}^{t-1} - \mathbf{B}\mathbf{x}^t = \mathbf{f}^{t-1}$. There have also been differing labeling suggestions – "backward-lag" vs. "forward-lag" models – which need not concern us.

[40] For the reader who is familiar with differential calculus, there is a continuous version of this model. As the time interval between periods becomes very small, the difference $x_j^{t+1} - x_j^t$ approaches the derivative $\mathrm{d}x_j/\mathrm{d}t$. The continuous analog to (13.75) is thus $x_i = \sum_{j=1}^{n} a_{ij}x_j + \sum_{j=1}^{n} b_{ij}(\mathrm{d}x_j/\mathrm{d}t) + f_i$, and, denoting the time derivative of the vector $\mathbf{x}$ by $\dot{\mathbf{x}}$, we would have $\mathbf{B}\dot{\mathbf{x}} = (\mathbf{I} - \mathbf{A})\mathbf{x} - \mathbf{f}$. These are linear *differential equations* for which solution procedures and stability analysis are also possible but beyond the level of this text.

[41] Recall than **G** is also used in the Ghosh model, but the context should make clear which meaning is intended.

matrix contains all zeros, the determinant of the matrix is zero and so the matrix has no inverse.[42] In later examples we will see that even when $\mathbf{B}$ is nonsingular, it may be somewhat "ill-conditioned" and contain unusually large elements in its inverse.

In developing capital coefficients, one may also wish to distinguish between "replacement capital" – for example, investment for replacing depreciated equipment – which is a function of current production, $\mathbf{x}^t$, and "expansion capital" – for example, investment in new equipment for expanded production capacity – which is a function of industry growth (the difference between current and past production, $\mathbf{x}^{t+1} - \mathbf{x}^t$). In this case we might write the analog to (13.77) as

$$(\mathbf{I} - \mathbf{A} - \mathbf{D} + \mathbf{B})\mathbf{x}^t - \mathbf{B}\mathbf{x}^{t+1} = \mathbf{f}^t$$

where $\mathbf{D}$ is the newly added matrix of replacement capital coefficients and $\mathbf{B}$ is now the matrix of expansion capital coefficients.

At a regional level, several operational models have been formulated, such as those found in Miernyk *et al.* (1970), which examines alternative economic development strategies for the state of West Virginia, and Miernyk and Sears (1974), where the impacts of pollution-control technologies on regional economies are analyzed, using a dynamic input–output model.

13.4.2 A Three-Period Example

Consider (13.77) again with $\mathbf{G} = (\mathbf{I} - \mathbf{A} + \mathbf{B})$ and let $T = 3$. Then the difference equation relationships are

$$\begin{aligned}
\mathbf{G}\mathbf{x}^0 - \mathbf{B}\mathbf{x}^1 &= \mathbf{f}^0 \\
\mathbf{G}\mathbf{x}^1 - \mathbf{B}\mathbf{x}^2 &= \mathbf{f}^1 \\
\mathbf{G}\mathbf{x}^2 - \mathbf{B}\mathbf{x}^3 &= \mathbf{f}^2 \\
\mathbf{G}\mathbf{x}^3 - \mathbf{B}\mathbf{x}^4 &= \mathbf{f}^3
\end{aligned}$$

or

$$\begin{bmatrix} \mathbf{G} & -\mathbf{B} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} & -\mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G} & -\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G} & -\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{x}^0 \\ \mathbf{x}^1 \\ \mathbf{x}^2 \\ \mathbf{x}^3 \\ \mathbf{x}^4 \end{bmatrix} = \begin{bmatrix} \mathbf{f}^0 \\ \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \end{bmatrix} \tag{13.79}$$

Notice that there are four matrix equations involving five unknown vectors, $\mathbf{x}^0$ through $\mathbf{x}^4$. If there are n sectors in the economy, we have $4n$ linear equations in $5n$ variables. An issue that arises in many dynamic models, including the input–output system, is which values to specify as fixed in the dynamic process. Generally, there are *initial* values, at the beginning ($t = 0$), when one starts with a given amount of, say,

[42] There is a large literature on singularity in the dynamic Leontief model and variations in the model that attempt to avoid the problem. This subject is vast and beyond the scope of this book. An interested reader might want to refer to Leontief (1970), Duchin and Szyld (1985), Leontief and Duchin (1986) or to Steenge and Thissen (2005) for critical summaries of many of these attempts to avoid or counteract the singularity problem.

output in the economy, or else there are *terminal* values, specifying desired characteristics of the system at the end of the period over which the model is being used ($t = T$ or $T + 1$). We investigate several possibilities in the case where $T = 3$.

Terminal Conditions In (13.79), when $T = 3$, this means $\mathbf{x}^{T+1} = \mathbf{x}^4$. In some versions of the dynamic input–output model (for example, Leontief, 1970), it is simply assumed that we cannot (or don't care to) see beyond year T; it is the last year that is of interest, and so $\mathbf{x}^{T+1} = \mathbf{0}$.[43] In that case, the equations in (13.79) become

$$\begin{bmatrix} \mathbf{G} & -\mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} & -\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G} & -\mathbf{B} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{x}^0 \\ \mathbf{x}^1 \\ \mathbf{x}^2 \\ \mathbf{x}^3 \end{bmatrix} = \begin{bmatrix} \mathbf{f}^0 \\ \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \end{bmatrix} \tag{13.80}$$

Since $\mathbf{x}^4 = \mathbf{0}$, it disappears from the $\mathbf{x}$ vector in (13.79), and the last column of the coefficient matrix in (13.79) is also unnecessary.

Given a set of final demands in the current year and in the next three years – $\mathbf{f}^0, \mathbf{f}^1, \mathbf{f}^2,$ and $\mathbf{f}^3$ – we could find the associated gross outputs in each of those years – $\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2,$ and $\mathbf{x}^3$ – using the inverse of the matrix on the left in (13.80), provided that it exists. In fact, it can be shown – using results for the inverses of partitioned matrices (Appendix A) and letting $\mathbf{R} = \mathbf{G}^{-1}\mathbf{B}$ – that

$$\begin{bmatrix} \mathbf{G} & -\mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} & -\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G} & -\mathbf{B} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{G}^{-1} & \mathbf{R}\mathbf{G}^{-1} & \mathbf{R}^2\mathbf{G}^{-1} & \mathbf{R}^3\mathbf{G}^{-1} \\ \mathbf{0} & \mathbf{G}^{-1} & \mathbf{R}\mathbf{G}^{-1} & \mathbf{R}^2\mathbf{G}^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}^{-1} & \mathbf{R}\mathbf{G}^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}^{-1} \end{bmatrix} \tag{13.81}$$

For an n-sector economy this will be a square matrix of order $4n$. For a time horizon of T years, this matrix will be of order $(T + 1)n$; that is, it can become fairly large for "reasonable" problems. For a ten-year planning problem in a 100-sector economy this matrix will be 1100×1100.

The particular structure of these equations when $\mathbf{x}^{T+1} = \mathbf{0}$, as in (13.80), allows for a simple recursive solution procedure. Given $\mathbf{f}^3$, find $\mathbf{x}^3$ from

$$\mathbf{x}^3 = \mathbf{G}^{-1}\mathbf{f}^3 \tag{13.82}$$

Using this value for $\mathbf{x}^3$, find $\mathbf{x}^2$ from the third equation in (13.79) as

$$\mathbf{x}^2 = \mathbf{G}^{-1}(\mathbf{B}\mathbf{x}^3 + \mathbf{f}^2) = \mathbf{G}^{-1}(\mathbf{B}\mathbf{G}^{-1}\mathbf{f}^3 + \mathbf{f}^2) = \mathbf{R}\mathbf{G}^{-1}\mathbf{f}^3 + \mathbf{G}^{-1}\mathbf{f}^2 \tag{13.83}$$

[43] Bródy (1995) calls this the "doom" or "doomsday" scenario (meaning, essentially, that the world ends at the end of period T). This reference includes an examination of alternative "truncations" of the matrix in (13.79) and discusses the alternative scenarios that they reflect.

In similar fashion, knowing $\mathbf{x}^3$ and $\mathbf{x}^2$,

$$\begin{aligned}\mathbf{x}^1 &= \mathbf{G}^{-1}(\mathbf{B}\mathbf{x}^2 + \mathbf{f}^1) = \mathbf{G}^{-1}[\mathbf{B}(\mathbf{R}\mathbf{G}^{-1}\mathbf{f}^3 + \mathbf{G}^{-1}\mathbf{f}^2) + \mathbf{f}^1] \\ &= \mathbf{R}^2\mathbf{G}^{-1}\mathbf{f}^3 + \mathbf{R}\mathbf{G}^{-1}\mathbf{f}^2 + \mathbf{G}^{-1}\mathbf{f}^1 \end{aligned} \tag{13.84}$$

and finally

$$\begin{aligned}\mathbf{x}^0 &= \mathbf{G}^{-1}(\mathbf{B}\mathbf{x}^1 + \mathbf{f}^0) = \mathbf{G}^{-1}[\mathbf{B}(\mathbf{R}^2\mathbf{G}^{-1}\mathbf{f}^3 + \mathbf{R}\mathbf{G}^{-1}\mathbf{f}^2 + \mathbf{G}^{-1}\mathbf{f}^1) + \mathbf{f}^0] \\ &= \mathbf{R}^3\mathbf{G}^{-1}\mathbf{f}^3 + \mathbf{R}^2\mathbf{G}^{-1}\mathbf{f}^2 + \mathbf{R}\mathbf{G}^{-1}\mathbf{f}^1 + \mathbf{G}^{-1}\mathbf{f}^0 \end{aligned} \tag{13.85}$$

This approach moves backward in time, starting at the end ($\mathbf{x}^3$) and finishing at the beginning ($\mathbf{x}^0$).[44] As the reader can see, this sequential solution procedure simply carries out the computations embedded in the upper triangular inverse matrix (zeros below the main diagonal).

Instead of assuming that $\mathbf{x}^{T+1} = \mathbf{0}$ in (13.79), we could have some target value of $\mathbf{x}$ for the first post-terminal year; that is, we could specify that $\mathbf{x}^4 = \bar{\mathbf{x}}^4$. Then the matrix structure in (13.80) would be altered only in that $\mathbf{f}^3$ on the right-hand side would be replaced by $\mathbf{f}^3 + \mathbf{B}\bar{\mathbf{x}}^4$. The solution could still be found using the inverse of the matrix on the left of (13.80), or the recursive solution, as in (13.82)–(13.85), could proceed as before.

Alternatively, one can specify that $\mathbf{x}^{T+1} = \mathbf{H}\mathbf{x}^T$, where $\mathbf{H}$ is a diagonal matrix whose elements are exogenously set growth rates for each of the sectors in the first post-terminal year. In that case, the last equation in (13.79) would be $\mathbf{G}\mathbf{x}^3 - \mathbf{B}\mathbf{H}\mathbf{x}^3 = \mathbf{f}^3$. The matrix structure would be

$$\begin{bmatrix} \mathbf{G} & -\mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} & -\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G} & -\mathbf{B} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{G} - \mathbf{B}\mathbf{H}) \end{bmatrix} \begin{bmatrix} \mathbf{x}^0 \\ \mathbf{x}^1 \\ \mathbf{x}^2 \\ \mathbf{x}^3 \end{bmatrix} = \begin{bmatrix} \mathbf{f}^0 \\ \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \end{bmatrix} \tag{13.86}$$

and solution procedures would be as above.

Initial Conditions Alternatively in assessing future impacts of current events, it is often assumed that the initial ($t = 0$) values of all elements in the system are known and then the usefulness of the model comes from its description of the values to be taken by the variables of interest in subsequent years. From that point of view, we would assume that both $\mathbf{f}^0$ *and* $\mathbf{x}^0$ have given initial values. This reduces the system in (13.79)

[44] A particular special case emerges from (13.85). If we are interested in a τ-year planning period with production to satisfy a constant level of final demand, $\mathbf{f}^*$, each year, an extension of the result in (13.85) leads to $\mathbf{x}^0 = [\mathbf{I} + \mathbf{R} + \mathbf{R}^2 + \cdots + \mathbf{R}^\tau]\mathbf{G}^{-1}\mathbf{f}^*$. If, as τ gets large, the power series in brackets converges – as we saw in Chapter 2 for the case of $(\mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots + \mathbf{A}^m)$ – then $\mathbf{x}^0 = (\mathbf{I} - \mathbf{R})^{-1}\mathbf{G}^{-1}\mathbf{f}^*$. Using $\mathbf{N}^{-1}\mathbf{M}^{-1} = (\mathbf{M}\mathbf{N})^{-1}$, and since $\mathbf{R} = \mathbf{G}^{-1}\mathbf{B}$, this is $\mathbf{x}^0 = [\mathbf{G}(\mathbf{I} - \mathbf{R})]^{-1}\mathbf{f}^* = (\mathbf{G} - \mathbf{B})^{-1}\mathbf{f}^*$ and so, with $\mathbf{G} = (\mathbf{I} - \mathbf{A} + \mathbf{B})$, $\mathbf{x}^0 = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}^*$. Finally, as $\tau \to \infty$, $\mathbf{x}^0 = \mathbf{x}^1 = \cdots = \mathbf{x} = \mathbf{x}^*$, so $\mathbf{x}^* = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}^*$. This reflects the logical limiting case. When final demand is constant and the time horizon infinite, the output level is constant and there is no need for capital growth.

to $4n$ linear equations in $4n$ variables. Then, given exogenous values for $\mathbf{f}^1, \mathbf{f}^2$, and $\mathbf{f}^3$, we could proceed sequentially from $\mathbf{x}^1$ to $\mathbf{x}^4$. As opposed to the backward sequence in (13.82) through (13.85), this one moves forward in time. From (13.79),

$$\begin{aligned}
\mathbf{x}^1 &= \mathbf{B}^{-1}(\mathbf{G}\mathbf{x}^0 - \mathbf{f}^0) \\
\mathbf{x}^2 &= \mathbf{B}^{-1}(\mathbf{G}\mathbf{x}^1 - \mathbf{f}^1) \\
\mathbf{x}^3 &= \mathbf{B}^{-1}(\mathbf{G}\mathbf{x}^2 - \mathbf{f}^2) \\
\mathbf{x}^4 &= \mathbf{B}^{-1}(\mathbf{G}\mathbf{x}^3 - \mathbf{f}^3)
\end{aligned} \tag{13.87}$$

This sequential solution procedure depends on the existence of $\mathbf{B}^{-1}$.

The results found sequentially in (13.87) can also be found in matrix form if the system in (13.79) is written as

$$\begin{bmatrix} -\mathbf{B} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{G} & -\mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} & -\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G} & -\mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \mathbf{x}^3 \\ \mathbf{x}^4 \end{bmatrix} = \begin{bmatrix} \mathbf{f}^0 - \mathbf{G}\mathbf{x}^0 \\ \mathbf{f}^1 \\ \mathbf{f}^2 \\ \mathbf{f}^3 \end{bmatrix} \tag{13.88}$$

This reflects the fact that $\mathbf{x}^0$ is now exogenously determined; it disappears from the top of the $\mathbf{x}$ vector on the left and hence the first column in the coefficient matrix in (13.79) also is removed. Then $\mathbf{x}^1$ through $\mathbf{x}^4$ can be found by premultiplying both sides of (13.88) by the inverse of the coefficient matrix on the left, provided that inverse exists. As before, the matrix on the left-hand side of (13.88) will be nonsingular if and only if the matrix on its main diagonal, here $\mathbf{B}$, is nonsingular. Again, repeated use of the results on inverses of partitioned matrices will demonstrate that (letting $\mathbf{S} = \mathbf{B}^{-1}\mathbf{G}$)

$$\begin{bmatrix} -\mathbf{B} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{G} & -\mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} & -\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{G} & -\mathbf{B} \end{bmatrix}^{-1} = \begin{bmatrix} -\mathbf{B}^{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{S}\mathbf{B}^{-1} & -\mathbf{B}^{-1} & \mathbf{0} & \mathbf{0} \\ -\mathbf{S}^2\mathbf{B}^{-1} & -\mathbf{S}\mathbf{B}^{-1} & -\mathbf{B}^{-1} & \mathbf{0} \\ -\mathbf{S}^3\mathbf{B}^{-1} & -\mathbf{S}^2\mathbf{G}\mathbf{B}^{-1} & -\mathbf{S}\mathbf{B}^{-1} & -\mathbf{B}^{-1} \end{bmatrix} \tag{13.89}$$

As opposed to the inverse matrix in the terminal conditions example, above, this inverse is lower triangular (zeros above the main diagonal), and this feature also suggests a recursive approach to solution that is illustrated by the sequence in (13.87), above.

13.4.3 Numerical Example 1

We illustrate the general workings of the dynamic input–output model using hypothetical figures for a two-sector economy. Let

$$\mathbf{A} = \begin{bmatrix} .1 & .2 \\ .3 & .4 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} .05 & .001 \\ .001 & .05 \end{bmatrix}; \text{ then } \mathbf{G} = \begin{bmatrix} .95 & -.199 \\ -.299 & .65 \end{bmatrix}.$$

For simplicity, let $T = 2$.

Terminal Conditions Suppose that $\mathbf{f}^0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$, $\mathbf{f}^1 = \begin{bmatrix} 120 \\ 150 \end{bmatrix}$, and $\mathbf{f}^2 = \begin{bmatrix} 140 \\ 200 \end{bmatrix}$. If we assume that $\mathbf{x}^3 = \mathbf{0}$, then, as in (13.82)–(13.85) – but with $T = 2$ rather than $T = 3$ – we can find the backward sequence $\mathbf{x}^2, \mathbf{x}^1, \mathbf{x}^0$. Here $\mathbf{G}^{-1} = \begin{bmatrix} 1.1649 & 0.3566 \\ 0.5358 & 1.7025 \end{bmatrix}$, so

$$\mathbf{x}^2 = \mathbf{G}^{-1}\mathbf{f}^2 = \begin{bmatrix} 234.41 \\ 415.51 \end{bmatrix} \tag{13.90}$$

Then

$$\mathbf{x}^1 = \mathbf{G}^{-1}(\mathbf{f}^1 + \mathbf{B}\mathbf{x}^2) = \begin{bmatrix} 214.91 \\ 361.94 \end{bmatrix} \tag{13.91}$$

and

$$\mathbf{x}^0 = \mathbf{G}^{-1}(\mathbf{f}^0 + \mathbf{B}\mathbf{x}^1) = \begin{bmatrix} 171.62 \\ 260.96 \end{bmatrix} \tag{13.92}$$

Alternatively, using the full matrix form, as in (13.80), where

$$\begin{bmatrix} \mathbf{G} & -\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} & -\mathbf{B} \\ \mathbf{0} & \mathbf{0} & \mathbf{G} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{G}^{-1} & \mathbf{G}^{-1}\mathbf{B}\mathbf{G}^{-1} & (\mathbf{G}^{-1}\mathbf{B})^2\mathbf{G}^{-1} \\ \mathbf{0} & \mathbf{G}^{-1} & \mathbf{G}^{-1}\mathbf{B}\mathbf{G}^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} 1.1649 & .3566 & .0784 & .0532 & .0061 & .0061 \\ .5358 & 1.7025 & .0791 & .1560 & .0090 & .0149 \\ 0 & 0 & 1.1649 & .3566 & .0784 & .0532 \\ 0 & 0 & .5358 & 1.7025 & .0791 & .1560 \\ 0 & 0 & 0 & 0 & 1.1649 & .3566 \\ 0 & 0 & 0 & 0 & .5358 & 1.7025 \end{bmatrix} \tag{13.93}$$

we could find, simultaneously, these same values for $\mathbf{x}^0, \mathbf{x}^1$, and $\mathbf{x}^2$.

If, instead of $\mathbf{x}^3 = \mathbf{0}$, we specify $\mathbf{x}^3 = \begin{bmatrix} 250 \\ 450 \end{bmatrix}$ (target values for outputs in the first post-terminal year), then $\mathbf{B}\mathbf{x}^3 = \begin{bmatrix} 12.95 \\ 22.75 \end{bmatrix}$, so that only the equation for $\mathbf{x}^2$ changes slightly from the sequence in (13.90)–(13.92), and

$$\mathbf{x}^2 = \mathbf{G}^{-1}(\mathbf{f}^2 + \mathbf{B}\mathbf{x}^3) = \begin{bmatrix} 257.60 \\ 461.18 \end{bmatrix}$$

$$\mathbf{x}^1 = \mathbf{G}^{-1}(\mathbf{f}^1 + \mathbf{B}\mathbf{x}^2) = \begin{bmatrix} 217.13 \\ 366.52 \end{bmatrix}$$

$$\mathbf{x}^0 = \mathbf{G}^{-1}(\mathbf{f}^0 + \mathbf{B}\mathbf{x}^1) = \begin{bmatrix} 171.83 \\ 261.41 \end{bmatrix}$$

In comparison with the $\mathbf{x}^0$, $\mathbf{x}^1$, and $\mathbf{x}^2$ found above when $\mathbf{x}^3 = \mathbf{0}$, the initial-year outputs are affected very little by this change in post-terminal year conditions. However, $\mathbf{x}^1$ is changed more than $\mathbf{x}^0$ and $\mathbf{x}^2$ more than $\mathbf{x}^1$. In matrix form,

$$\begin{bmatrix} \mathbf{G} & -\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} & -\mathbf{B} \\ \mathbf{0} & \mathbf{0} & \mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{x}^0 \\ \mathbf{x}^1 \\ \mathbf{x}^2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}^0 \\ \mathbf{f}^1 \\ \mathbf{f}^2 + \mathbf{B}\mathbf{x}^3 \end{bmatrix}$$

and, using (13.93), the same values of the gross outputs from both sectors in each period can be found simultaneously.

Using the $\mathbf{x}^3 = \mathbf{0}$ example again, let $\mathbf{f}^0 = \mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$. Then, from the inverse in (13.93), or from the backward recursive procedure, as above, we can find

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}^0 \\ \mathbf{x}^1 \\ \mathbf{x}^2 \end{bmatrix} = \begin{bmatrix} 166.53 \\ 249.73 \\ \\ 165.31 \\ 247.34 \\ \\ 152.15 \\ 223.83 \end{bmatrix} \tag{13.94}$$

Recall (footnote 39) that with constant final demands, $\mathbf{f}^*$, as the time period lengthens, the results in each $\mathbf{x}^t$ approach $(\mathbf{I} - \mathbf{A})^{-1}\mathbf{f}^*$. Here $\mathbf{A} = \begin{bmatrix} .1 & .2 \\ .3 & .4 \end{bmatrix}$, so

$$(\mathbf{I} - \mathbf{A})^{-1} \begin{bmatrix} 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 1.2500 & .4167 \\ .6250 & 1.8750 \end{bmatrix} \begin{bmatrix} 100 \\ 100 \end{bmatrix} = \begin{bmatrix} 166.67 \\ 250.00 \end{bmatrix}$$

which is closely approximated by $\mathbf{x}^0$ in (13.94), the outputs in the earliest year. As T gets larger, subsequent values of $\mathbf{x}^t$ will also approach $\begin{bmatrix} 166.67 \\ 250.00 \end{bmatrix}$. (The interested reader can confirm this by letting $T = 3, T = 4$, and so on, using the same $\mathbf{A}$ and $\mathbf{B}$ and constant final demand of 100 for both sectors.)

Initial Conditions Taking an alternative point of view, suppose

$$\mathbf{f}^0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}, \mathbf{f}^1 = \begin{bmatrix} 120 \\ 150 \end{bmatrix}, \mathbf{f}^2 = \begin{bmatrix} 140 \\ 200 \end{bmatrix}$$

as before, but let $\mathbf{x}^0 = \begin{bmatrix} 180 \\ 270 \end{bmatrix}$. Originally, with these final demands and $\mathbf{x}^0$ determined endogenously, we found $\mathbf{x}^0 = \begin{bmatrix} 171.61 \\ 260.96 \end{bmatrix}$, as in (13.92). We now select an $\mathbf{x}^0$ that is larger. Here, using the forward recursive procedure of (13.87), with $\mathbf{B}^{-1} = \begin{bmatrix} 20.008 & -0.4 \\ -0.4 & 20.008 \end{bmatrix}$, we find

$$\mathbf{x}^1 = \mathbf{B}^{-1}(\mathbf{G}\mathbf{x}^0 - \mathbf{f}^0) = \begin{bmatrix} 336.87 \\ 426.87 \end{bmatrix}$$

and

$$\mathbf{x}^2 = \mathbf{B}^{-1}(\mathbf{G}\mathbf{x}^1 - \mathbf{f}^1) = \begin{bmatrix} 2291.66 \\ 488.91 \end{bmatrix}$$

Essentially the same values are found, using the inverse of $\begin{bmatrix} -\mathbf{B} & \mathbf{0} \\ \mathbf{G} & -\mathbf{B} \end{bmatrix}$, as in (13.88). Here this inverse is

$$\begin{bmatrix} -20.008 & 0.4 & 0 & 0 \\ 0.4 & -20.008 & 0 & 0 \\ -384.395 & 92.522 & -20.008 & 0.4 \\ 132.538 & -264.347 & 0.4 & -20.008 \end{bmatrix}$$

This example illustrates that the dynamic input–output model, at least in the simplified form presented here, is very sensitive to the specification of initial conditions. We return to this point in Numerical Example 2, below.

If we use the same structure as in (13.87) and (13.88), but with $\mathbf{x}^0 = \begin{bmatrix} 171.62 \\ 260.96 \end{bmatrix}$, which is the actual initial output found in (13.92) when $\mathbf{x}^0$ is endogenous, we will generate exactly the values of $\mathbf{x}^1$ and $\mathbf{x}^2$ that were found initially in (13.91) and (13.90). Similarly, if we use $\mathbf{x}^0 = \begin{bmatrix} 166.53 \\ 249.73 \end{bmatrix}$ from (13.94), in conjunction with $\mathbf{f}^0 = \mathbf{f}^1 = \mathbf{f}^2 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$, we generate exactly the sequence of outputs already found for that example – $\mathbf{x}^1$ and $\mathbf{x}^2$ in (13.94). For either of these earlier examples, if we take the forward sequential approach but with an initial $\mathbf{x}^0$ that is less than that found with $\mathbf{x}^0$ endogenous (and using the same final demands), we will generate one or more negative gross outputs in years after $t = 0$. The values for $\mathbf{x}^0$ in (13.92) and (13.94) represent what is necessary to satisfy the specified sequences of final demands with an economy whose structure is reflected in the given $\mathbf{A}$ and $\mathbf{B}$ matrices, so any initial output that is less than that $\mathbf{x}^0$

will produce a sequence of additions to capital stock that eventually become inadequate for future production. (Recall that, unlike the static input–output case, in the dynamic model it is assumed that all sectors are producing at full capacity.)

13.4.4 Numerical Example 2

In order to illustrate a particularly sensitive feature of the dynamic input–output model in its forward sequential form (starting from initial conditions), we select an alternative capital coefficients matrix. In this new case, sector 1 is far more important as a supplier of capital goods than is sector 2; here $\mathbf{B} = \begin{bmatrix} .05 & .06 \\ .0004 & .0007 \end{bmatrix}$. Using the same $\mathbf{A}$ matrix as in the preceding example, we find $\mathbf{G} = \begin{bmatrix} .95 & -.14 \\ -.2996 & .6007 \end{bmatrix}$. Note that while $\mathbf{B}$ is quite different from the preceding example, the current $\mathbf{G}$ matrix is close to that in Example 1. This is because $\mathbf{G} = (\mathbf{I} - \mathbf{A} + \mathbf{B})$, and $\mathbf{A}$ is unchanged in the two examples.

Terminal Conditions We use the same sequence of final demands – namely

$$\mathbf{f}^0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix},\ \mathbf{f}^1 = \begin{bmatrix} 120 \\ 150 \end{bmatrix},\ \text{and } \mathbf{f}^2 = \begin{bmatrix} 140 \\ 200 \end{bmatrix}$$

Again, letting $\mathbf{x}^3 = \mathbf{0}$, we can find $\mathbf{x}^2,\ \mathbf{x}^1,\ \mathbf{x}^0$ sequentially, exactly as in (13.90) through (13.92). Here $\mathbf{G}^{-1} = \begin{bmatrix} 1.1361 & .2648 \\ .5667 & 1.7968 \end{bmatrix}$ (which is not a great deal different from $\mathbf{G}^{-1}$ in the previous example) and

$$\mathbf{x}^2 = \begin{bmatrix} 212.01 \\ 438.70 \end{bmatrix},\ \mathbf{x}^1 = \begin{bmatrix} 218.10 \\ 359.15 \end{bmatrix},\ \mathbf{x}^0 = \begin{bmatrix} 177.05 \\ 255.35 \end{bmatrix}$$

These results are different from those in the previous example, as is to be expected, but not by much.

Initial Conditions Using the same $\mathbf{f}^0,\ \mathbf{f}^1$ and $\mathbf{f}^2$ along with $\mathbf{x}^0 = \begin{bmatrix} 180 \\ 270 \end{bmatrix}$ from the previous example illustrates the sensitivity problem. Here, because $\mathbf{B}$ has a row of elements that are smaller than any of the elements in the previous capital coefficients matrix, its inverse can be expected to contain at least some larger elements. And indeed it does; here $\mathbf{B}^{-1} = \begin{bmatrix} 63.636 & -5454.545 \\ -36.364 & 4545.455 \end{bmatrix}$, which is very different from its counterpart in the previous example. Thus

$$\mathbf{x}^1 = \mathbf{B}^{-1}(\mathbf{G}\mathbf{x}^0 - \mathbf{f}^0) = \mathbf{B}^{-1}\begin{bmatrix} 33.20 \\ 8.26 \end{bmatrix} = \begin{bmatrix} -42942 \\ 36338 \end{bmatrix}$$

and, much worse (the results have been rounded),

$$\mathbf{x}^2 = \mathbf{B}^{-1}(\mathbf{G}\mathbf{x}^1 - \mathbf{f}^1) = \mathbf{B}^{-1}\begin{bmatrix} -45882 \\ 24694 \end{bmatrix} = \begin{bmatrix} -192{,}000{,}000 \\ 159{,}000{,}000 \end{bmatrix}$$

This illustrates that as the elements in one or more rows of $\mathbf{B}$ become small, $\mathbf{B}^{-1}$ contains very large numbers. Here $|\mathbf{B}| = 0.000011$; if one were working with four-decimal accuracy, one would conclude that $\mathbf{B}$ was singular.

Consider the determination of $\mathbf{x}^1$. Rewriting, $\mathbf{B}\mathbf{x}^1 = \mathbf{G}\mathbf{x}^0 - \mathbf{f}^0$, and with $\mathbf{A}$ and $\mathbf{B}$ (and hence $\mathbf{G}$) given, along with $\mathbf{f}^0$, the choice of $\mathbf{x}^0$ then specifies the right-hand side vector for this set of two linear equations in two unknowns. Denote a specific right-hand side vector as $\mathbf{r}^0$. In the easily visualized two-variable case, we could explore the solution-space geometry of the pair of equations. Here

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \ \mathbf{x}^1 = \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix}, \ \mathbf{r}^0 = \begin{bmatrix} r_1^0 \\ r_2^0 \end{bmatrix}$$

so

$$\begin{aligned} b_{11}x_1^1 + b_{12}x_2^1 &= r_1^0 \\ b_{21}x_1^1 + b_{22}x_2^1 &= r_2^0 \end{aligned}$$

We leave it to the interested reader to make sketches in solution space. However, it is easy to show that both lines will have positive intercepts on the vertical axis (when $\mathbf{r}^0 > \mathbf{0}$, which by definition it must be) and that both will have negative slopes. Then the conditions for the intersection of the two lines to be in the positive quadrant or on its boundaries (that is, $\mathbf{x}^1 \geq \mathbf{0}$) can be derived. The values of x_1^0 and x_2^0 must be chosen so that r_1^0/r_2^0 lies within the bounds set by b_{11}/b_{21} and b_{12}/b_{22}. The generalization to more sectors and to non-negativity of outputs further in the future – $\mathbf{x}^2, \mathbf{x}^3$, and so on – is beyond this text. The point of the illustration is simply to highlight the kinds of problems that can arise in the dynamic model when one wants to calculate forward from initial conditions, using $\mathbf{B}^{-1}$.

Note that in the first numerical example, $b_{11}/b_{21} = 50$ and $b_{12}/b_{22} = 0.02$. In that example, in fact, $(\mathbf{G}\mathbf{x}^0 - \mathbf{f}^0) = \mathbf{r}^0 = \begin{bmatrix} 18.77 \\ 21.71 \end{bmatrix}$ so that $r_1^0/r_2^0 = 0.86$, which is indeed within the bounds. In the second example, $b_{11}/b_{21} = 125$ and $b_{12}/b_{22} = 85.7$. For our initial choice of $\mathbf{x}^0 = \begin{bmatrix} 180 \\ 270 \end{bmatrix}$, $r_1^0/r_2^0 = 33.2/8.26 = 4.02$, which is outside the admissible range. A choice of $\mathbf{x}^0 = \begin{bmatrix} 180 \\ 256.8 \end{bmatrix}$, however, would lead to $\mathbf{x}^1 \geq \mathbf{0}$, since $r_1^0/r_2^0 = 105.6$, while an initial $\mathbf{x}^0 = \begin{bmatrix} 180 \\ 256.7 \end{bmatrix}$ generates $r_1^0/r_2^0 = 129.1$, which means that $\mathbf{x}^1$ will not be non-negative. By any reasonable definition, this would appear to be extreme sensitivity to initial values.

13.4.5 "Dynamic" Multipliers

The structure of the inverse in (13.81) suggests the possibility of distributing impacts backward over time. (This is described in Leontief, 1970, and it is also discussed in C. K. Liew, 1977, for a regional model and further elaborated in C. J. Liew, 2000 and

2005.) In these cases, it is usual to designate the current (or "target") period as period 0 and the preceding periods as −1, −2, etc. For example, consider the model in (13.80) and (13.81) in "Δ" form

$$\begin{bmatrix} \Delta\mathbf{x}^{-3} \\ \Delta\mathbf{x}^{-2} \\ \Delta\mathbf{x}^{-1} \\ \Delta\mathbf{x}^{0} \end{bmatrix} = \begin{bmatrix} \mathbf{G}^{-1} & \mathbf{R}\mathbf{G}^{-1} & \mathbf{R}^2\mathbf{G}^{-1} & \mathbf{R}^3\mathbf{G}^{-1} \\ \mathbf{0} & \mathbf{G}^{-1} & \mathbf{R}\mathbf{G}^{-1} & \mathbf{R}^2\mathbf{G}^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{G}^{-1} & \mathbf{R}\mathbf{G}^{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{G}^{-1} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{f}^{-3} \\ \Delta\mathbf{f}^{-2} \\ \Delta\mathbf{f}^{-1} \\ \Delta\mathbf{f}^{0} \end{bmatrix}$$

Let $\Delta\mathbf{f}^0 \neq \mathbf{0}$, $\Delta\mathbf{f}^{-1} = \Delta\mathbf{f}^{-2} = \Delta\mathbf{f}^{-3} = \mathbf{0}$; then the last column of the inverse on the right is seen to distribute the direct and indirect input requirements backward over time from period 0 in which the deliveries are made to final users. Here, $\Delta\mathbf{x}^{-3} = \mathbf{R}^3\mathbf{G}^{-1}\Delta\mathbf{f}^0$, $\Delta\mathbf{x}^{-2} = \mathbf{R}^2\mathbf{G}^{-1}\Delta\mathbf{f}^0$ and $\Delta\mathbf{x}^{-1} = \mathbf{R}\mathbf{G}^{-1}\Delta\mathbf{f}^0$; present demands require both current inputs and adequate capital stock to support production of those inputs, meaning production of capital goods in the preceding period, which in turn depends in part on production two periods back, etc.

Notice that this intertemporal influence is not a result of the fact that production takes time, it is entirely the result of the capital goods component of the model in which production for those goods depends on the *changes* in outputs over time, as reflected in $\mathbf{B}(\mathbf{x}^{t+1} - \mathbf{x}^t)$ in (13.77). Approaches to incorporating *production* lags in an input–output model will be explored below, in section 13.4.6.

13.4.6 Turnpike Growth and Dynamic Models

In Chapter 2 we introduced the notion of a completely closed input–output model as $(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{0}$ or $\mathbf{A}\mathbf{x} = \mathbf{x}$. Recall that such an input–output model is in fact a homogeneous system of linear equations which has a nontrivial solution (one other than $\mathbf{x} = \mathbf{0}$) if and only if $|\mathbf{I} - \mathbf{A}| = 0$.

The corresponding closed dynamic model is

$$\mathbf{A}\mathbf{x}^t + \mathbf{B}(\mathbf{x}^{t+1} - \mathbf{x}^t) = \mathbf{x}^t \tag{13.95}$$

If we assume for simplicity that we can find an $\mathbf{x}^{t+1}$ and $\mathbf{x}^t$ such that all industries grow at the same rate in the economy, say, at rate λ, then

$$\mathbf{x}^{t+1} = \lambda\mathbf{x}^t \tag{13.96}$$

This rate, λ, is often referred to as a *turnpike growth rate* (all industries are growing or declining on the same path – the "turnpike"), and it is interpreted as a general indicator of the "health" of the economy, that is, $\lambda > 1$ indicates that the economy is expanding, $0 < \lambda < 1$ indicates that the economy is contracting, and $\lambda < 0$ indicates that the economy is unstable, that is, experiencing periods of both decline and growth over time. Since λ is really only a theoretical number, how can it be computed? Substituting

(13.96) into (13.95), we obtain

$$\begin{aligned}\mathbf{A}\mathbf{x}^t + \mathbf{B}(\lambda\mathbf{x}^t - \mathbf{x}^t) &= \mathbf{x}^t\\ \mathbf{B}\lambda\mathbf{x}^t &= (\mathbf{I} - \mathbf{A} + \mathbf{B})\mathbf{x}^t\\ \mathbf{B}^{-1}(\mathbf{I} - \mathbf{A} + \mathbf{B})\mathbf{x}^t &= \lambda\mathbf{x}^t\end{aligned}$$

or

$$\mathbf{Q}\mathbf{x}^t = \lambda\mathbf{x}^t \tag{13.97}$$

where $\mathbf{Q} = \mathbf{B}^{-1}(\mathbf{I} - \mathbf{A} + \mathbf{B})$. Note that (13.97) has the very interesting feature that a *scalar*, λ, multiplied by $\mathbf{x}^t$, yields precisely the same value as a matrix, $\mathbf{Q}$, postmultiplied by $\mathbf{x}^t$.

Such a problem is well known in applied mathematics as an eigenvalue problem where λ is the *eigenvalue* (sometimes called a characteristic value or latent root), and $\mathbf{x}^t$ corresponding to λ in (13.97), is the *eigenvector* (sometimes called characteristic vector or latent vector). This problem is closely related to the solution of systems of homogeneous linear equations. Note that we can rewrite (13.97) as

$$(\mathbf{Q} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0} \tag{13.98}$$

for which there is a nontrivial solution if and only if

$$|\mathbf{Q} - \lambda\mathbf{I}| = 0 \tag{13.99}$$

We consider the 2×2 case, with $\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$ so that

$$|\mathbf{Q} - \lambda\mathbf{I}| = \left|\begin{bmatrix} q_{11} - \lambda & q_{12} \\ q_{21} & q_{22} - \lambda \end{bmatrix}\right| = (q_{11} - \lambda)(q_{22} - \lambda) - q_{12}q_{21} = 0 = \lambda^2 + b\lambda + c$$

where $b = -(q_{11} + q_{22})$ and $c = q_{11}q_{22} - q_{12}q_{21}$. We find the solution to $\mathbf{Q}\mathbf{x} = \lambda\mathbf{x}$ by solving $|\mathbf{Q} - \lambda\mathbf{I}| = 0$ or $\lambda^2 + b\lambda + c = 0$. This is a polynomial (sometimes called the characteristic polynomial) which, when set equal to zero, is called the characteristic equation; in this case it has two solutions, given by

$$\lambda = \frac{-(q_{11} + q_{22}) \pm [(q_{11} + q_{22})^2 - 4(q_{11}q_{22} - q_{12}q_{21})]^{1/2}}{2}$$

Denote these solutions as λ_1 and λ_2. The turnpike growth rate is defined to be the largest eigenvalue found (see Carter, 1974), which we define as $\lambda_{\max}$.

Example

$$\text{Suppose } \mathbf{Q} = \mathbf{B}^{-1}(\mathbf{I} - \mathbf{A} + \mathbf{B}) = \begin{bmatrix} 1.0 & .5 \\ 2.0 & 1.0 \end{bmatrix}, \text{ then}$$

$$|\mathbf{Q} - \lambda\mathbf{I}| = (1 - \lambda)(1 - \lambda) - 1 = 0$$

$$\lambda^2 - 2\lambda = 0$$

$$\lambda(\lambda - 2) = 0$$

so $\lambda_1 = 0$ and $\lambda_2 = 2$. The turnpike growth rate is $\lambda_{max} = \lambda_2 = 2$. As mentioned earlier, if $\lambda_{max} < 0$ then the economy is unstable, that is, oscillating. The interpretation of negative λ's can be specified more precisely by relating it to the solution of a system of ordinary differential equations, but this is beyond the scope of this text. Carter (1974) and Leontief and Duchin (1986) examine the notion of turnpike growth as an indicator economic stability resulting from changes in technology in the United States.

13.4.7 Alternative Input–Output Dynamics

In the standard input–output model, $\mathbf{x} = \mathbf{Lf}$, there is no consideration of the fact that production takes time; results are independent of time in the sense that $\mathbf{f}^{new}$ leads to $\mathbf{x}^{new}$ (via $\mathbf{x}^{new} = \mathbf{Lf}^{new}$). This is generally interpreted in something like the following fashion: "new demands, $\mathbf{f}^{new}$, next period will lead to new outputs, $\mathbf{x}^{new}$, next period," ignoring that sectors will generally have production lags (of differing length for different sectors). This missing temporal characteristic of the input–output model was noted in the early work of Dorfman, Samuelson and Solow (1958, pp. 253–254) where the authors comment on the absence of a "time" aspect to the round-by-round process of the power series for the Leontief inverse.

> Needless to say, the rounds of which we speak do not take place in calendar time, with the second round following the first . . . Artificial computational time is involved, and if we insist on giving a calendar-time interpretation we must think of the . . . process as showing how much production must be started many periods back if we are to meet the new consumption targets today.

However, as observed by Mules (1983, p. 197), with respect to the assumptions implicit in using input–output multipliers,

> The traditional multiplier does not stipulate the time taken to realize effects, assuming instead that they usually occur almost immediately or within the space of one year (a year being the usual accounting period for which input–output data is compiled).

Starting around the mid-1980s, research emerged on ways to incorporate the notion of time lags in production in an input–output framework. [References include Mules, 1983; ten Raa, 1986, 2005 (Chapter 13); Romanoff and Levine, 1986, 1990; and Cole, 1988, 1997, 1999b.] Mules (p. 199) makes the assumption that each round of the power series process does in fact take a finite period of (calendar) time, suggesting that a typical period may be a month or a quarter. He further assumes that each sector is able to respond in each period to the demands made upon it in the previous period, but with varying lags in this production response. As an illustration, he suggests a five-period lag for primary sectors, one period for manufacturing and no lag (that is, delivery next period) for services. Simulation exercises lead to the conclusion that "... on some occasions there may be a significant proportion of multiplier effects still outstanding after one year has elapsed. We may be in error if we assume that all effects have occurred near the time of the original stimulus" (Mules, 1983, p. 204).

This problem was also addressed in the work of Romanoff and Levine. A good deal of their work on what they call the sequential interindustry model (SIM) appears in unpublished discussion papers from the Regional Science Research Center (initially in Cambridge, Mass. and later in Lexington, Mass.). Possibly the first is dated 1980, so it precedes (and is cited by) Mules. The authors recognize the fact that "... it takes time for each industry to produce its product beforehand and supply its own final demand and that of the directly demanding industries, for the latter to use as inputs to their own production" (Romanoff and Levine, 1990, pp. 1–2). A given a_{ij} is modeled as distributed backward (over discrete time intervals); $a_{ij}(k)(k = 0, -1, -2, \ldots)$ is the fraction of a_{ij} (per-unit input of i by j) that occurs k periods before completion of production by j.

Ten Raa (1986) and Cole (1988) identify technical coefficients as non-negative (continuous) distributions along the negative time axis (that is, backwards in time from "now"). As in the dynamic models in sections 13.4.1–13.4.5, ten Raa also considers capital accumulation. The specific nature, characteristics, and properties of the assumed distributions are beyond the level of this text. The interested reader is referred to the cited literature and additional references in those articles. Cole has successfully applied his distributed-lag framework in a number of studies, especially at the small-area level. In Cole (1989) the illustration is a plant closure in Western New York. Assumed lags are: 3 months for production sectors, 4 months for households, 18 months for local government activities and 36 months for investments.[45] In Cole (1999a) there is a stylized illustration, including Miyazawa interrelational multiplier aspects, for a community (an inner-city neighborhood in Buffalo, New York).[46]

13.5 Summary

In this chapter we have explored several applications and variations of the input–output framework. Structural decomposition analysis presents an approach to disentangling the sources of change in some aspect of an economy into its component parts – for example relating output changes to changes in demand and technology. We saw that it is further possible to decompose the demand and technology changes into further underlying components. And further layers of decomposition are also possible. In a large (many-sector) input–output model this approach rapidly generates a very large set of results which are generally difficult to interpret without some kind of aggregation

[45] In comparing his approach with that of ten Raa or Romanoff and Levine, Cole (1989, p. 106) suggests that the required computations needed in either of those approaches "... are still complex in any practical situation."

[46] A vigorous exchange in print – Jackson, Madden and Bowman (1997) → Cole (1997) → Jackson and Madden (1999) → Cole (1999b) – provides several illustrations of Cole's approach and comparisons of Cole's work with that of ten Raa and of Romanoff and Levine.

(for example, finding averages); this, as usual, removes much of the detail which the input–output model provides.

We also explored the variations that arise when the model is used to assess the impact of exogenously specified outputs for one or more sectors (rather than final demands), or when a new sector is introduced into the economy. Finally, we sketched the basic features of the dynamic version of an input-model, where production for current input use is coupled with production for capital goods. The dynamic model has been much less widely embraced in real-world applications, although there have been notable exceptions – including the work of Miernyk and his associates at the regional level in the 1970s, Almon (1970, and other publications associated with the long-running INFORUM project at the University of Maryland) and Leontief and Duchin as well as Duchin and her associates (for example, Leontief and Duchin, 1986; Duchin and Szyld, 1985). An alternative approach to dynamics is represented in the sequential input–output model and its variants that include the recognition of production lags in an economy.

Appendix 13.1 Alternative Decompositions of $\mathbf{x} = \mathbf{LBf}$

Alternative views of an input–output equation like $\mathbf{x} = \mathbf{LBf}$ will generate somewhat different decompositions. We explore three variations in this Appendix.

1. Using (13.10) directly on $\mathbf{x} = \mathbf{LBf}$ gives

$$\Delta\mathbf{x} = \underbrace{(1/2)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0 + \mathbf{B}^1\mathbf{f}^1)}_{\text{Effect of } \Delta\mathbf{L}} + \underbrace{(1/2)[\mathbf{L}^0(\Delta\mathbf{B})\mathbf{f}^1 + \mathbf{L}^1(\Delta\mathbf{B})\mathbf{f}^0]}_{\text{Effect of } \Delta\mathbf{B}}$$
$$+ \underbrace{(1/2)(\mathbf{L}^0\mathbf{B}^0 + \mathbf{L}^1\mathbf{B}^1)(\Delta\mathbf{f})}_{\text{Effect of } \Delta\mathbf{f}}$$

2. If we combine $\mathbf{L}$ and $\mathbf{B}$, so that $\mathbf{M} = \mathbf{LB}$ and $\mathbf{x} = \mathbf{Mf}$, and then use (13.7),

$$\Delta\mathbf{x} = (1/2)(\Delta\mathbf{M})(\mathbf{f}^0 + \mathbf{f}^1) + (1/2)(\mathbf{M}^0 + \mathbf{M}^1)(\Delta\mathbf{f})$$

Since $\mathbf{M} = \mathbf{LB}$,

$$\Delta\mathbf{M} = (1/2)(\Delta\mathbf{L})(\mathbf{B}^0 + \mathbf{B}^1) + (1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{B})$$

so that

$$\begin{aligned}\Delta\mathbf{x} &= (1/2)[(1/2)(\Delta\mathbf{L})(\mathbf{B}^0 + \mathbf{B}^1) + (1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{B})](\mathbf{f}^0 + \mathbf{f}^1) \\ &\quad + (1/2)(\mathbf{M}^0 + \mathbf{M}^1)(\Delta\mathbf{f}) \\ &= \underbrace{(1/4)(\Delta\mathbf{L})(\mathbf{B}^0 + \mathbf{B}^1)(\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Effect of } \Delta\mathbf{L}} + \underbrace{(1/4)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{B})(\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Effect of } \Delta\mathbf{B}} \\ &\quad + \underbrace{(1/2)(\mathbf{M}^0 + \mathbf{M}^1)(\Delta\mathbf{f})}_{\text{Effect of } \Delta\mathbf{f}}\end{aligned}$$

And, again since $\mathbf{M} = \mathbf{LB}$, the last term is $\underbrace{(1/2)(\mathbf{L}^0\mathbf{B}^0 + \mathbf{L}^1\mathbf{B}^1)(\Delta\mathbf{f})}_{\text{Effect of } \Delta\mathbf{f}}$, as in (1).

3. If we combine $\mathbf{B}$ and $\mathbf{f}$, so that $\mathbf{y} = \mathbf{Bf}$ and $\mathbf{x} = \mathbf{Ly}$, and then use (13.7),

$$\Delta\mathbf{x} = (1/2)(\Delta\mathbf{L})(\mathbf{y}^0 + \mathbf{y}^1) + (1/2)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{y})$$

Since $\mathbf{y} = \mathbf{Bf}$,

$$\Delta\mathbf{y} = (1/2)(\Delta\mathbf{B})(\mathbf{f}^0 + \mathbf{f}^1) + (1/2)(\mathbf{B}^0 + \mathbf{B}^1)(\Delta\mathbf{f})$$

so that

$$\begin{aligned}\Delta\mathbf{x} &= (1/2)(\Delta\mathbf{L})(\mathbf{y}^0 + \mathbf{y}^1) + (1/2)(\mathbf{L}^0 + \mathbf{L}^1)[(1/2)(\Delta\mathbf{B})(\mathbf{f}^0 + \mathbf{f}^1) \\ &\quad + (1/2)(\mathbf{B}^0 + \mathbf{B}^1)(\Delta\mathbf{f})] \\ &= \underbrace{(1/2)(\Delta\mathbf{L})(\mathbf{y}^0 + \mathbf{y}^1)}_{\text{Effect of } \Delta\mathbf{L}} + \underbrace{(1/4)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{B})(\mathbf{f}^0 + \mathbf{f}^1)}_{\text{Effect of } \Delta\mathbf{B}} \\ &\quad + \underbrace{(1/4)(\mathbf{L}^0 + \mathbf{L}^1)(\mathbf{B}^0 + \mathbf{B}^1)(\Delta\mathbf{f})}_{\text{Effect of } \Delta\mathbf{f}}\end{aligned}$$

and since $\mathbf{y} = \mathbf{Bf}$, the first term is $\underbrace{(1/2)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0 + \mathbf{B}^1\mathbf{f}^1)}_{\text{Effect of } \Delta\mathbf{L}}$, as in (1).

Table A13.1.1 summarizes these results. Terms that do not appear in Equation (1) are boxed. For example, in Equation (2), $\Delta\mathbf{L}$ appears in two terms – $(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0+\mathbf{B}^1\mathbf{f}^1)$ and $(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^1+\mathbf{B}^1\mathbf{f}^0)$ – but each is weighted by (1/4) instead of the (1/2) in Equation (1). The amount by which $(1/2)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0+\mathbf{B}^1\mathbf{f}^1)$ differs from $(1/4)[(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0+\mathbf{B}^1\mathbf{f}^1)+(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^1+\mathbf{B}^1\mathbf{f}^0)]$ depends entirely on the difference between $(\mathbf{B}^0\mathbf{f}^0+\mathbf{B}^1\mathbf{f}^1)$ and $(\mathbf{B}^0\mathbf{f}^1+\mathbf{B}^1\mathbf{f}^0)$. Similar observations can be made for the weightings on $\Delta\mathbf{B}$ in Equations (2) and (3) vs. Equation (1) and on the weighting on $\Delta\mathbf{f}$ in Equation (3) vs. Equations (1) and (2).

Appendix 13.2 Exogenous Specification of Some Elements of x

A13.2.1 The General Case: An n-sector Model with k Endogenous Outputs

The general representation for an n-sector model with (the first) k gross outputs and (the last) $(n-k)$ final demands endogenous was given in (13.57) in the text as

$$\begin{bmatrix} (\mathbf{I} - \mathbf{A}^{(k)}) & \mathbf{0} \\ -\mathbf{A}_{21} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \underset{(k\times 1)}{\mathbf{x}^{en}} \\ \underset{[(n-k)\times 1]}{\mathbf{f}^{en}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{A}_{12} \\ \mathbf{0} & -(\mathbf{I} - \mathbf{A}_{22}) \end{bmatrix} \begin{bmatrix} \underset{(k\times 1)}{\mathbf{f}^{ex}} \\ \underset{[(n-k)\times 1]}{\mathbf{x}^{ex}} \end{bmatrix} \qquad \text{(A13.2.1)}$$

Table A13.1.1 Alternative Decompositions of $\mathbf{x} = \mathbf{LBf}$

Alternative	Effect of $\Delta\mathbf{L}$	Effect of $\Delta\mathbf{B}$	Effect of $\Delta\mathbf{f}$
(1)	$(1/2)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0 + \mathbf{B}^1\mathbf{f}^1)$	$(1/2)[\mathbf{L}^0(\Delta\mathbf{B})\mathbf{f}^1 + \mathbf{L}^1(\Delta\mathbf{B})\mathbf{f}^0]$	$(1/2)(\mathbf{L}^0\mathbf{B}^0 + \mathbf{L}^1\mathbf{B}^1)(\Delta\mathbf{f})$
(2)	$(1/4)(\Delta\mathbf{L})(\mathbf{B}^0 + \mathbf{B}^1)(\mathbf{f}^0 + \mathbf{f}^1) =$ $(1/4)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0 + \mathbf{B}^1\mathbf{f}^1) +$ $\boxed{(1/4)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^1 + \mathbf{B}^1\mathbf{f}^0)}$	$(1/4)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{B})(\mathbf{f}^0 + \mathbf{f}^1) =$ $(1/4)[\mathbf{L}^0(\Delta\mathbf{B})\mathbf{f}^1 + \mathbf{L}^1(\Delta\mathbf{B})\mathbf{f}^0] +$ $\boxed{(1/4)[\mathbf{L}^0(\Delta\mathbf{B})\mathbf{f}^0 + \mathbf{L}^1(\Delta\mathbf{B})\mathbf{f}^1]}$	$(1/2)(\mathbf{M}^0 + \mathbf{M}^1)(\Delta\mathbf{f}) =$ $(1/2)(\mathbf{L}^0\mathbf{B}^0 + \mathbf{L}^1\mathbf{B}^1)(\Delta\mathbf{f})$
(3)	$(1/2)(\Delta\mathbf{L})(\mathbf{y}^0 + \mathbf{y}^1) =$ $(1/2)(\Delta\mathbf{L})(\mathbf{B}^0\mathbf{f}^0 + \mathbf{B}^1\mathbf{f}^1)$	$(1/4)(\mathbf{L}^0 + \mathbf{L}^1)(\Delta\mathbf{B})(\mathbf{f}^0 + \mathbf{f}^1) =$ $(1/4)[\mathbf{L}^0(\Delta\mathbf{B})\mathbf{f}^1 + \mathbf{L}^1(\Delta\mathbf{B})\mathbf{f}^0] +$ $\boxed{(1/4)[\mathbf{L}^0(\Delta\mathbf{B})\mathbf{f}^0 + \mathbf{L}^1(\Delta\mathbf{B})\mathbf{f}^1]}$	$(1/4)(\mathbf{L}^0 + \mathbf{L}^1)(\mathbf{B}^0 + \mathbf{B}^1)(\Delta\mathbf{f}) =$ $(1/4)(\mathbf{L}^0\mathbf{B}^0 + \mathbf{L}^1\mathbf{B}^1)(\Delta\mathbf{f}) +$ $\boxed{(1/4)(\mathbf{L}^0\mathbf{B}^1 + \mathbf{L}^1\mathbf{B}^0)(\Delta\mathbf{f}]}$

Letting $\mathbf{M} = \begin{bmatrix} (\mathbf{I}-\mathbf{A}^{(k)}) & \mathbf{0} \\ -\mathbf{A}_{21} & -\mathbf{I} \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} \mathbf{I} & \mathbf{A}_{12} \\ \mathbf{0} & -(\mathbf{I}-\mathbf{A}_{22}) \end{bmatrix}$, and using results from Appendix A on inverses of partitioned matrices,

$$\mathbf{M}^{-1} = \begin{bmatrix} \mathbf{L}^{(k)} & \mathbf{0} \\ -\mathbf{A}_{21}\mathbf{L}^{(k)} & -\mathbf{I} \end{bmatrix}$$

[where $(\mathbf{I}-\mathbf{A}^{(k)})^{-1} = \mathbf{L}^{(k)}$] and so

$$\mathbf{M}^{-1}\mathbf{N} = \begin{bmatrix} \mathbf{L}^{(k)} & \mathbf{L}^{(k)}\mathbf{A}_{12} \\ -\mathbf{A}_{21}\mathbf{L}^{(k)} & (\mathbf{I}-\mathbf{A}_{22}) - \mathbf{A}_{21}\mathbf{L}^{(k)}\mathbf{A}_{12} \end{bmatrix}$$

This product reflects not only the results on inverses of partitioned matrices but also the specific structure of those matrices in (A13.2.1) – especially the locations of **0** and **I** submatrices and their influence in the partitioned matrix multiplication. Thus

$$\begin{bmatrix} \mathbf{x}^{en} \\ \mathbf{f}^{en} \end{bmatrix} = \begin{bmatrix} \mathbf{L}^{(k)} & \mathbf{L}^{(k)}\mathbf{A}_{12} \\ -\mathbf{A}_{21}\mathbf{L}^{(k)} & (\mathbf{I}-\mathbf{A}_{22}) - \mathbf{A}_{21}\mathbf{L}^{(k)}\mathbf{A}_{12} \end{bmatrix} \begin{bmatrix} \mathbf{f}^{ex} \\ \mathbf{x}^{ex} \end{bmatrix} \tag{A13.2.2}$$

[This is (13.58) in the text.]

If $\mathbf{f}^{ex} = \mathbf{0}$, the influence of the specified exogenous outputs, $\mathbf{x}^{ex}$, on the endogenous outputs, $\mathbf{x}^{en}$, is given by

$$\mathbf{x}^{en} = \mathbf{L}^{(k)}\mathbf{A}_{12}\mathbf{x}^{ex} \tag{A13.2.3}$$

which was (in "Δ" form) (13.59) in the text.

A13.2.2 The Output-to-Output Multiplier Matrix

For a three-sector model, we saw that an "output–to–output" multiplier matrix was created from $\mathbf{L}^{(3)}$ through division of each element in a column by the on-diagonal element for that column, namely

$$\mathbf{L}^{(3)} = \begin{bmatrix} l_{11}^{(3)} & l_{12}^{(3)} & l_{13}^{(3)} \\ l_{21}^{(3)} & l_{22}^{(3)} & l_{23}^{(3)} \\ l_{31}^{(3)} & l_{32}^{(3)} & l_{33}^{(3)} \end{bmatrix} \text{ and } \mathbf{L}^{(3)*} = \begin{bmatrix} \frac{l_{11}^{(3)}}{l_{11}^{(3)}} & \frac{l_{12}^{(3)}}{l_{22}^{(3)}} & \frac{l_{13}^{(3)}}{l_{33}^{(3)}} \\ \frac{l_{21}^{(3)}}{l_{11}^{(3)}} & \frac{l_{22}^{(3)}}{l_{22}^{(3)}} & \frac{l_{23}^{(3)}}{l_{33}^{(3)}} \\ \frac{l_{31}^{(3)}}{l_{11}^{(3)}} & \frac{l_{32}^{(3)}}{l_{22}^{(3)}} & \frac{l_{33}^{(3)}}{l_{33}^{(3)}} \end{bmatrix} = \begin{bmatrix} 1 & \frac{l_{12}^{(3)}}{l_{22}^{(3)}} & \frac{l_{13}^{(3)}}{l_{33}^{(3)}} \\ \frac{l_{21}^{(3)}}{l_{11}^{(3)}} & 1 & \frac{l_{23}^{(3)}}{l_{33}^{(3)}} \\ \frac{l_{31}^{(3)}}{l_{11}^{(3)}} & \frac{l_{32}^{(3)}}{l_{22}^{(3)}} & 1 \end{bmatrix}$$

A13.2.3 *The Inverse of a Partitioned* $(\mathbf{I}-\mathbf{A}^{(n)})$ *Matrix*

Let

$$(\mathbf{I}-\mathbf{A}^{(n)}) = \begin{bmatrix} (\mathbf{I}-\mathbf{A}^{(k)}) & -\mathbf{A}_{12} \\ -\mathbf{A}_{21} & (\mathbf{I}-\mathbf{A}_{22}) \end{bmatrix} = \begin{bmatrix} \underset{(k\times k)}{\mathbf{E}} & \underset{[k\times(n-k)]}{\mathbf{F}} \\ \underset{[(n-k)\times k]}{\mathbf{G}} & \underset{[(n-k)\times(n-k)]}{\mathbf{H}} \end{bmatrix} \qquad \text{(A13.2.4)}$$

Then, again using results on inverses of partitioned matrices,

$$(\mathbf{I}-\mathbf{A}^{(n)})^{-1} = \mathbf{L}^{(n)} = \begin{bmatrix} \underset{(k\times k)}{\mathbf{S}} & \underset{[k\times(n-k)]}{\mathbf{T}} \\ \underset{[(n-k)\times k]}{\mathbf{U}} & \underset{[(n-k)\times(n-k)]}{\mathbf{V}} \end{bmatrix} \qquad \text{(A13.2.5)}$$

The important result from Appendix A is that $\mathbf{T} = -\mathbf{E}^{-1}\mathbf{F}\mathbf{V}$, or

$$-\mathbf{E}^{-1}\mathbf{F} = \mathbf{T}\mathbf{V}^{-1} \qquad \text{(A13.2.6)}$$

A13.2.4 *The Case of* $k = 2, n = 3$

We now use the results in the preceding sections of this Appendix to examine the specific case of a three-sector model with $x_3 = \bar{x}_3$. This was the subject matter of the examples in section 13.2.3 in the text. In this case, (A13.2.3) becomes

$$\mathbf{x}^{en} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{L}^{(2)}\mathbf{A}_{12}\bar{x}_3 = \begin{bmatrix} l_{11}^{(2)} & l_{12}^{(2)} \\ l_{21}^{(2)} & l_{22}^{(2)} \end{bmatrix} \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} \bar{x}_3 \qquad \text{(A13.2.7)}$$

In terms of (A13.2.4), $\mathbf{E} = (\mathbf{I}-\mathbf{A}^{(2)})$ and $\mathbf{F} = \begin{bmatrix} -a_{13} \\ -a_{23} \end{bmatrix}$, so (A13.2.7) can be written

$$\mathbf{x}^{en} = -\mathbf{E}^{-1}\mathbf{F}\bar{x}_3 \qquad \text{(A13.2.8)}$$

In the alternative approach, the exogenous specification of $x_3 = \bar{x}_3$ is represented by the 3×1 vector $\bar{\mathbf{x}} = \begin{bmatrix} 0 \\ 0 \\ \bar{x}_3 \end{bmatrix}$ and

$$\underset{(3\times 1)}{\mathbf{x}^*} = \underset{(3\times 3)}{\mathbf{L}^{(3)*}} \; \underset{(3\times 1)}{\bar{\mathbf{x}}} \qquad \text{(A13.2.9)}$$

where $\mathbf{x}^* = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Here

$$\mathbf{L}^{(3)*} = \begin{bmatrix} 1 & \frac{l_{12}^{(3)}}{l_{22}^{(3)}} & \frac{l_{13}^{(3)}}{l_{33}^{(3)}} \\ \frac{l_{21}^{(3)}}{l_{11}^{(3)}} & 1 & \frac{l_{23}^{(3)}}{l_{33}^{(3)}} \\ \frac{l_{31}^{(3)}}{l_{11}^{(3)}} & \frac{l_{32}^{(3)}}{l_{22}^{(3)}} & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{11}^{(3)*} & \mathbf{L}_{12}^{(3)*} \\ \mathbf{L}_{21}^{(3)*} & \mathbf{L}_{22}^{(3)*} \end{bmatrix}$$

and (A13.2.9) can be alternatively expressed as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{11}^{(3)*} & \mathbf{L}_{12}^{(3)*} \\ \mathbf{L}_{21}^{(3)*} & \mathbf{L}_{22}^{(3)*} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \bar{x}_3 \end{bmatrix}$$

In particular,

$$\mathbf{x}^{en} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{L}_{12}^{(3)*} \bar{x}_3 \tag{A13.2.10}$$

and

$$x_3 = \mathbf{L}_{22}^{(3)*} \bar{x}_3 = \bar{x}_3$$

From (A13.2.5), $\mathbf{T} = \begin{bmatrix} l_{13}^{(3)} \\ l_{23}^{(3)} \end{bmatrix}$, $\mathbf{V} = [l_{33}^{(3)}]$ and so $\mathbf{TV}^{-1} = \begin{bmatrix} \frac{l_{13}^{(3)}}{l_{33}^{(3)}} \\ \frac{l_{23}^{(3)}}{l_{33}^{(3)}} \end{bmatrix} = \begin{bmatrix} l_{13}^{(3)*} \\ l_{23}^{(3)*} \end{bmatrix}$. Therefore, the results in (A13.2.10) can be expressed as

$$\mathbf{x}^{en} = \begin{bmatrix} l_{13}^{(3)*} \\ l_{23}^{(3)*} \end{bmatrix} \bar{x}_3 = \mathbf{TV}^{-1} \bar{x}_3 \tag{A13.2.11}$$

Conclusion: since $-\mathbf{E}^{-1}\mathbf{F} = \mathbf{TV}^{-1}$ (A13.2.6), the results in (A13.2.8) and (A13.2.11) are equivalent. This will be true for an input–output model of any size in which x_n is made endogenous. It will *not* be true in an input–output model in which more than one output is made exogenous. We examine why in the next section.

A13.2.5 The Case of $k = 1$, $n = 3$

The case in which more than one output is exogenous can be illustrated for a three-sector model in which $x_2 = \bar{x}_2$ and $x_3 = \bar{x}_3$. The results generalize to any n with $k < (n - 1)$.

For this example, where $(n-k)=2$,

$$\mathbf{M}=\begin{bmatrix}(1-a_{11}) & 0 & 0\\ -a_{21} & -1 & 0\\ -a_{31} & 0 & -1\end{bmatrix},\quad \mathbf{N}=\begin{bmatrix}1 & a_{12} & a_{13}\\ 0 & -(1-a_{22}) & a_{23}\\ 0 & a_{32} & -(1-a_{33})\end{bmatrix},$$

$$\text{and}\quad \mathbf{M}^{-1}=\begin{bmatrix}l_{11}^{(1)} & 0 & 0\\ -a_{21}l_{11}^{(1)} & -1 & 0\\ -a_{31}l_{11}^{(1)} & 0 & -1\end{bmatrix}$$

The parallel to (A13.2.7) is

$$\mathbf{x}^{en}=[x_1]=\mathbf{L}^{(1)}\mathbf{A}_{12}\begin{bmatrix}\bar{x}_2\\ \bar{x}_3\end{bmatrix}=(1-a_{11})^{-1}\begin{bmatrix}a_{12} & a_{13}\end{bmatrix}\begin{bmatrix}\bar{x}_2\\ \bar{x}_3\end{bmatrix}$$

From section A13.2.3, it is easily established that $\mathbf{E}^{-1}=(1-a_{11})^{-1}$ and $\mathbf{F}=\begin{bmatrix}-a_{12} & -a_{13}\end{bmatrix}$ and so

$$[x_1]=-\mathbf{E}^{-1}\mathbf{F}\begin{bmatrix}\bar{x}_2\\ \bar{x}_3\end{bmatrix}$$

Here we find

$$\mathbf{T}=\begin{bmatrix}l_{12}^{(3)} & l_{13}^{(3)}\end{bmatrix},\quad \mathbf{V}=\begin{bmatrix}l_{22}^{(3)} & l_{23}^{(3)}\\ l_{32}^{(3)} & l_{33}^{(3)}\end{bmatrix}\ \text{and}\ \mathbf{TV}^{-1}=\begin{bmatrix}l_{12}^{(3)} & l_{13}^{(3)}\end{bmatrix}\begin{bmatrix}l_{22}^{(3)} & l_{23}^{(3)}\\ l_{32}^{(3)} & l_{33}^{(3)}\end{bmatrix}^{-1}$$

Notice how the dimensions of $\mathbf{T}$ and $\mathbf{V}$ have been altered in this case. Because of (A13.2.6), we can write

$$[x_1]=\mathbf{TV}^{-1}\begin{bmatrix}\bar{x}_2\\ \bar{x}_3\end{bmatrix}=\begin{bmatrix}l_{12}^{(3)} & l_{13}^{(3)}\end{bmatrix}\begin{bmatrix}l_{22}^{(3)} & l_{23}^{(3)}\\ l_{32}^{(3)} & l_{33}^{(3)}\end{bmatrix}^{-1}\begin{bmatrix}\bar{x}_2\\ \bar{x}_3\end{bmatrix}\tag{A13.2.12}$$

On the other hand, now

$$\mathbf{L}^{(3)*}=\begin{bmatrix}\frac{l_{11}^{(3)}}{l_{11}^{(3)}} & \frac{l_{12}^{(3)}}{l_{22}^{(3)}} & \frac{l_{13}^{(3)}}{l_{33}^{(3)}}\\ \frac{l_{21}^{(3)}}{l_{11}^{(3)}} & \frac{l_{22}^{(3)}}{l_{22}^{(3)}} & \frac{l_{23}^{(3)}}{l_{33}^{(3)}}\\ \frac{l_{31}^{(3)}}{l_{11}^{(3)}} & \frac{l_{32}^{(3)}}{l_{22}^{(3)}} & \frac{l_{33}^{(3)}}{l_{33}^{(3)}}\end{bmatrix}=\begin{bmatrix}1 & \frac{l_{12}^{(3)}}{l_{22}^{(3)}} & \frac{l_{13}^{(3)}}{l_{33}^{(3)}}\\ \frac{l_{21}^{(3)}}{l_{11}^{(3)}} & 1 & \frac{l_{23}^{(3)}}{l_{33}^{(3)}}\\ \frac{l_{31}^{(3)}}{l_{11}^{(3)}} & \frac{l_{32}^{(3)}}{l_{22}^{(3)}} & 1\end{bmatrix}=\begin{bmatrix}\mathbf{L}_{11}^{(3)*} & \mathbf{L}_{12}^{(3)*}\\ \mathbf{L}_{21}^{(3)*} & \mathbf{L}_{22}^{(3)*}\end{bmatrix}$$

(notice how the matrix partitions have moved), and the parallel to the result in (A13.2.10) becomes

$$\mathbf{x}^{en}=[x_1]=\mathbf{L}_{12}^{(3)*}\begin{bmatrix}\bar{x}_2\\ \bar{x}_3\end{bmatrix}=\begin{bmatrix}l_{12}^{(3)*} & l_{13}^{(3)*}\end{bmatrix}\begin{bmatrix}\bar{x}_2\\ \bar{x}_3\end{bmatrix}=\begin{bmatrix}\frac{l_{12}^{(3)}}{l_{22}^{(3)}} & \frac{l_{13}^{(3)}}{l_{33}^{(3)}}\end{bmatrix}\begin{bmatrix}\bar{x}_2\\ \bar{x}_3\end{bmatrix}\tag{A13.2.13}$$

Clearly, the results in (A13.2.12) differ from those in (A13.2.13). Only the results in (A13.2.12) are valid, because they are derived from the fundamental input–output model for mixed exogenous/endogenous variables – (13.57) in the text.

The problem occurs because when more than one output is made exogenous – $k < (n-1)$ [or $(n-k) > 1$] – the immediate consequence is that $\mathbf{T}$ changes from a column vector to a matrix (with $n-k$ columns) and $\mathbf{V}^{-1}$ changes from the reciprocal of a scalar to the inverse of an $(n-k) \times (n-k)$ matrix. As a result, the operation $\mathbf{TV}^{-1}$ no longer produces a column of elements that have been divided by the on-diagonal element in that column but rather a matrix of elements that differ from the elements in $\mathbf{L}^*$. (Notice that if $\mathbf{V}$ were a *diagonal* matrix then the operation $\mathbf{TV}^{-1}$ would in fact produce a matrix with elements from $\mathbf{L}^*$; but since $\mathbf{V}$ is a submatrix from $\mathbf{L}^{(n)}$ it will *not* be diagonal.)

A13.2.6 "Extracting" the Last $(n-k)$ Sectors

Assume, again, that outputs for the last $(n-k)$ sectors in an n-sector input–output model have been made exogenous. Then modify the $\mathbf{A}^{(n)}$ coefficient matrix by replacing all coefficients in the last $(n-k)$ rows with zeros, creating $\tilde{\mathbf{A}}^{(n)} = \begin{bmatrix} \underset{(k\times k)}{\mathbf{A}_{11}} & \underset{[k\times(n-k)]}{\mathbf{A}_{12}} \\ \underset{[(n-k)\times k]}{\mathbf{0}} & \underset{[(n-k)\times(n-k)]}{\mathbf{0}} \end{bmatrix}$ with an associated

$$(\mathbf{I} - \tilde{\mathbf{A}}^{(n)}) = \begin{bmatrix} (\mathbf{I} - \mathbf{A}^{(k)}) & -\mathbf{A}_{12} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{bmatrix} \text{ and}$$

$$(\mathbf{I} - \tilde{\mathbf{A}}^{(n)})^{-1} = \tilde{\mathbf{L}}^{(n)} = \begin{bmatrix} \mathbf{S} & \mathbf{T} \\ \mathbf{U} & \mathbf{V} \end{bmatrix}$$

Using results from Appendix A, $\mathbf{G} = \mathbf{0}$ means that $\mathbf{U} = \mathbf{0}$ and $\mathbf{S} = (\mathbf{I} - \mathbf{A}^{(k)})^{-1} = \mathbf{L}^{(k)}$. Also, since $\mathbf{H} = \mathbf{I}$, $\mathbf{V} = \mathbf{I}$. Finally, $\mathbf{T} = \mathbf{L}^{(k)}\mathbf{A}_{12}$ and so

$$\tilde{\mathbf{L}}^{(n)} = \begin{bmatrix} \mathbf{L}^{(k)} & \mathbf{L}^{(k)}\mathbf{A}_{12} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

Finally, then,

$$\begin{bmatrix} \mathbf{x}^{en} \\ \mathbf{f}^{en} \end{bmatrix} = \tilde{\mathbf{L}}^{(n)} \begin{bmatrix} \mathbf{f}^{ex} \\ \mathbf{x}^{ex} \end{bmatrix} = \begin{bmatrix} \mathbf{L}^{(k)} & \mathbf{L}^{(k)}\mathbf{A}_{12} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{f}^{ex} \\ \mathbf{x}^{ex} \end{bmatrix}$$

and the results for $\mathbf{x}^{en}$ will be exactly the same as given in (A13.2.2) since the upper two submatrices here are identical to those in that equation. [We leave it as an exercise for the interested reader to show that it is immaterial whether the sectors with exogenous outputs are represented as the *last* $(n-k)$ sectors, as here, or as the *first* $(n-k)$ sectors.]

Problems

13.1 Consider two input–output economies specified by

$$\mathbf{Z}^0 = \begin{bmatrix} 10 & 20 & 30 \\ 5 & 2 & 25 \\ 20 & 40 & 60 \end{bmatrix}, \quad \mathbf{f}^0 = \begin{bmatrix} 60 \\ 40 \\ 55 \end{bmatrix}, \quad \mathbf{Z}^1 = \begin{bmatrix} 15 & 25 & 40 \\ 12 & 7.5 & 30 \\ 10 & 30 & 40 \end{bmatrix}, \quad \mathbf{f}^1 = \begin{bmatrix} 75 \\ 55 \\ 40 \end{bmatrix}$$

We seek to measure how the economy has changed in structure in one year, specified by $\mathbf{Z}^1$ and $\mathbf{f}^1$, relative to an earlier year for the same economy, specified by $\mathbf{Z}^0$ and $\mathbf{f}^0$. Compute for each sector the change in total output between the two years that was attributable to changing final demand or to changing technology.

13.2 Consider an input–output economy specified by $\mathbf{Z} = \begin{bmatrix} 14 & 76 & 46 \\ 54 & 22 & 5 \\ 68 & 71 & 94 \end{bmatrix}$ and $\mathbf{f} = \begin{bmatrix} 100 \\ 200 \\ 175 \end{bmatrix}$ where the three industrial sectors are manufacturing, oil, and electricity.

a. Suppose economic forecasts determine that total domestic output for oil and electricity will remain unchanged in the next year and final demand for manufactured goods will increase by 30 percent. What would be the input–output projections of final demand for oil and electricity and the total output of manufacturing?
b. If instead the final demand for manufactured goods increased by 50 percent instead of 30 percent, what are the new projections of final demand for oil and electricity and the total output of manufacturing?

13.3 Consider the impact on the economy of Problem 2.1 of the establishment of a new economic sector, finance, and insurance (sector 3).

a. Suppose you know that the total output of this new sector will be $900 during the current year (its first year of operation), and that its needs for agricultural and manufactured goods are represented by $a_{13} = 0.001$ and $a_{23} = 0.07$. In the absence of any further information, what would you estimate to be the impact of this new sector on the economy?
b. You later learn (1) that the agriculture and manufacturing sectors bought $20 and $40 in finance and insurance services last year from foreign firms (i.e., that they imported these inputs), and (2) that sector 3 will use $15 of its own product for each $100 worth of its output. Assuming that they will now buy from the domestic sector, how might you now assess the impact of the new sector on this economy?

13.4 Recall the Czaria economy from problem 12.1. Next year's projected total outputs in millions of dollars for agriculture, mining, and civilian manufacturing in Czaria are 4,558, 5,665, and 5,079, respectively, and final demand of military manufactured products is projected to be $2,050 million. Compute the GDP and total gross production of the economy next year.

13.5 Consider an input–output economy with technical coefficients defined as $\mathbf{A} = \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & 0.5 \end{bmatrix}$ and capital coefficients defined as $\mathbf{B} = \begin{bmatrix} .01 & .003 \\ .005 & .020 \end{bmatrix}$. Current final demand is $\mathbf{f}^0 = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$ and the projections for the next three years for final demand are given by $\mathbf{f}^1 = \begin{bmatrix} 125 \\ 160 \end{bmatrix}$, $\mathbf{f}^2 = \begin{bmatrix} 150 \\ 175 \end{bmatrix}$ and $\mathbf{f}^3 = \begin{bmatrix} 185 \\ 200 \end{bmatrix}$. We are not interested in total output for beyond the projection three years, but what would be the projections of total output for this economy in the next three years?

13.6 Consider the following closed dynamic input–output model, $\mathbf{Ax} + \mathbf{B}(\mathbf{x}^t - \mathbf{x}) = \mathbf{x}$ where $\mathbf{x}^t$ = future outputs, $\mathbf{x}$ = current outputs, and where $\mathbf{A} = \begin{bmatrix} 0.5 & 0.1 \\ 0.1 & 0.5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0 \end{bmatrix}$. Assume that $\mathbf{x}^t = \lambda\mathbf{x}$, where λ is some scalar (the turnpike growth rate); compute λ.

13.7 Given the closed dynamic input–output model $\mathbf{Ax} + \mathbf{B}(\mathbf{x}^t - \mathbf{x}) = \mathbf{x}$, where

$$\mathbf{A} = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.4 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

a. Compute the turnpike growth rate for this example.
b. If both the capital coefficients for the first industry (the first column of **B**) are changed to 0.1, then what is the new turnpike growth rate and what has happened to the apparent "health" of the economy?

13.8 Consider an input–output economy with technical coefficients defined as $\mathbf{A} = \begin{bmatrix} 0.2 & 0.1 \\ 0.3 & 0.5 \end{bmatrix}$ and capital coefficients defined as $\mathbf{B} = \begin{bmatrix} .02 & .002 \\ .003 & .01 \end{bmatrix}$. Current final demand is $\mathbf{f}^0 = \begin{bmatrix} 185 \\ 200 \end{bmatrix}$ and final demands for the previous three years are given by $\mathbf{f}^{-1} = \begin{bmatrix} 150 \\ 175 \end{bmatrix}$, $\mathbf{f}^{-2} = \begin{bmatrix} 125 \\ 160 \end{bmatrix}$, and $\mathbf{f}^{-3} = \begin{bmatrix} 100 \\ 100 \end{bmatrix}$. Compute the "dynamic" multipliers for this economy that show how direct and indirect input requirements for final demands in period 0 are distributed backward over time for the previous three years.

13.9 Consider **A**, **L**, and **f** for the US economy provided in Appendix B for the years 1972 and 2002. Compute the changes in total output between 1972 and 2002 for all sectors attributed to changes in final demand and to changes in technology.

13.10 Consider the 2005 US input–output table provided in Appendix B. Suppose our economic forecast projects for 2010 a 10 percent growth in final demand for agriculture, mining, and construction, a 5 percent growth in final demand for manufactured goods, and a 6 percent growth in total output for the trade, transportation, utilities, services, and other economic sectors. What are the corresponding input–output estimates of total output for agriculture, mining, construction, and manufacturing as well as the estimates of final demand for trade, transportation, utilities, services, and other economic sectors?

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14 Additional Topics

14.1 Introduction

There has been an enormous output of research, developments, extensions, and applications of input–output models in the decades since the late 1940s and early 1950s, when the subject first entered an academic curriculum at Harvard University. The increasing capacity and speed of modern computers has contributed to this work by facilitating large-scale data-intensive experiments that were unthinkable in the early days. Throughout the text we have identified many of these extensions and applications, but there are many topics that we have not been able to include or reference in the preceding chapters. So, in addition to those topics to which we have referred but not developed further for one reason or another in previous chapters, here we explore several additional areas, primarily to give some possible guidance to the relevant literature, should the reader wish to explore one or more of these topics. (To sort out the appropriate literature, we depart from the practice in earlier chapters and include references at the end of each individual section.)

In particular, in this chapter, we survey a number of topics that presently are either frontier areas in input–output analysis or that relate input–output to other types of economic analysis that we have not included or only briefly addressed in previous chapters. We do not cover these topics in as much detail as those covered in previous chapters, in some cases because the topic involves methods beyond the scope of the present volume, such as statistics, econometrics, or mathematical programming. In other cases we consider the topics difficult to place logically in other chapters or to be emerging research or application areas where a substantial body of literature has not yet appeared but we felt it important to identify the topic as a frontier area in input–output analysis.

The range of topics which we have covered throughout this volume along with the overview of those included in this chapter underscore how much input–output analysis has matured in the last half century and how broadly input–output has influenced other areas of economic analysis, including often providing a fundamental point of departure for collecting economic data used in the construction of economic models addressing a wide range of policy questions.

14.2 Input–Output and Measuring Economic Productivity

A key source of growth and health in many economies is the rate of growth in its economic productivity, broadly defined as the level of output of an industry or of the economy as a whole per unit of input. Exploring different methods of measuring this economic productivity has been an active area of analysis for the last two decades (Jorgenson and Griliches, 1967). A number of measures of productivity can be expressed easily in input–output terms – see, for example, Baumol and Wolff (1984), Wolff (1985, 1994, 1997) and ten Raa (2005). In this section we explore one such formulation, the concept of total factor productivity (TFP), which is defined generally as the growth in total output that is not attributable to growth in inputs.[1]

14.2.1 Total Factor Productivity

We begin with the matrix of technical coefficients, a_{ij}, and value-added coefficients,[2] v_j, and total industry outputs, x_j, and recalling the fundamental input accounting relationship

$$x_j = \sum_{i=1}^{n} a_{ij}x_j + v_jx_j = \left(\sum_{i=1}^{n} a_{ij} + v_j\right)x_j \tag{14.1}$$

Using the rule for the differential of a product

$$\mathrm{d}x_j = \mathrm{d}\left[\left(\sum_{i=1}^{n} a_{ij} + v_j\right)x_j\right] = \left(\sum_{i=1}^{n} a_{ij} + v_j\right)\mathrm{d}x_j + \left(\sum_{i=1}^{n} \mathrm{d}a_{ij} + \mathrm{d}v_j\right)x_j \tag{14.2}$$

The rate of TFP growth is often defined as

$$\tau_j = -\left(\sum_{i=1}^{n} \mathrm{d}a_{ij} + \mathrm{d}v_j\right) \tag{14.3}$$

so that (14.2) becomes

$$\mathrm{d}x_j = \left(\sum_{i=1}^{n} a_{ij} + v_j\right)\mathrm{d}x_j - \tau_j x_j \tag{14.4}$$

Often in the TFP literature, expressions in continuous (differential) form are transformed into logarithmic terms, from the calculus rule that $\mathrm{d}\ln(z) = (1/z)(\mathrm{d}z)$ or $\mathrm{d}z = z(\mathrm{d}\ln z)$. This generates

$$\tau_j = -\left[\sum_{i=1}^{n} a_{ij}(\mathrm{d}\ln a_{ij}) + v_j(\mathrm{d}\ln v_j)\right] \tag{14.5}$$

[1] *Economic Systems Research* (2007) contains articles summarizing developments in this area up until that date.
[2] In previous chapters we have used $\mathbf{v}_c$ to denote value-added coefficients. In this section we drop the subscript to simplify the notation.

Also, in TFP analyses v_j is usually decomposed into at least its labor and capital components, l_j and k_j. This is then cited as a continuous version of a measure of sectoral technical change first proposed by Leontief in Leontief *et al.* (1953); for example, see Wolff (1994, p. 77) or Aulin-Ahmavaara (1999, p. 352).

In order to make use of available input–output data, it is usual to express the relationships in (14.2) and (14.3) in finite-difference form, where $\mathrm{d}x_j \cong \Delta x_j = x_j^1 - x_j^0$, $\mathrm{d}a_{ij} \cong \Delta a_{ij} = a_{ij}^1 - a_{ij}^0$ and $\mathrm{d}v_j \cong \Delta v_j = v_j^1 - v_j^0$. Ignoring "second-order" effects,[3] (14.2) becomes

$$x_j^1 - x_j^0 = \Delta x_j = \Delta\left[\left(\sum_{i=1}^n a_{ij} + v_j\right)x_j\right] = \left(\sum_{i=1}^n a_{ij}^0 + v_j^0\right)\Delta x_j + \left(\sum_{i=1}^n \Delta a_{ij} + \Delta v_j\right)x_j^0 \tag{14.6}$$

or

$$x_j^1 - x_j^0 = \Delta x_j = \underbrace{\left(\sum_{i=1}^n a_{ij}^0 + v_j^0\right)x_j^1 - \left(\sum_{i=1}^n a_{ij}^0 + v_j^0\right)x_j^0}_{\substack{\text{Portion of change accounted for by using old technology,} \\ \text{as reflected in } a_{ij}^0 \text{ and } v_j^0\text{, to meet new input needs}}}$$

$$+ \underbrace{\left(\sum_{i=1}^n a_{ij}^1 + v_j^1\right)x_j^0 - \left(\sum_{i=1}^n a_{ij}^0 + v_j^0\right)x_j^0}_{\substack{\text{Portion of change accounted for by using new technology,} \\ \text{as reflected in } a_{ij}^1 \text{ and } v_j^1\text{, to meet old input needs}}}$$

Productivity studies are often concerned with the *rate* of productivity change relative to the initial output, which can be found by normalizing (dividing) by the total initial output, x_j^0.

In finite-difference form (14.3) is

$$\tau_j = -\left(\sum_{i=1}^n \Delta a_{ij} + \Delta v_j\right) \tag{14.7}$$

so

$$\Delta x_j = \Delta\left[\left(\sum_{i=1}^n a_{ij} + v_j\right)x_j\right] = \left(\sum_{i=1}^n a_{ij} + v_j\right)\Delta x_j - \tau_j x_j^0$$

In matrix terms $\Delta\mathbf{x} = [\langle \mathbf{i}'\mathbf{A}\rangle + \hat{\mathbf{v}}]\Delta\mathbf{x} + [\langle \mathbf{i}'\Delta\mathbf{A}\rangle + \langle \Delta\mathbf{v}\rangle]\mathbf{x}$ and

$$\boldsymbol{\tau} = -[(\mathbf{i}'\Delta\mathbf{A})' + \Delta\mathbf{v}] = -\left[\left(\sum_{i=1}^n \Delta a_{ij} + \Delta v_j\right)\right] \tag{14.8}$$

[3] Such effects were labeled "interaction terms" in section 13.1 on structural decomposition methods. Much of this total factor productivity analysis bears a close resemblance to the approaches in section 13.1, especially those decompositions that include the influence of changes in the elements of **A**.

14.2.2 Numerical Example: Total Factor Productivity

Consider an input–output economy for which technical coefficients and value added in three successive years are defined by

$$\mathbf{A}^{(0)} = \begin{bmatrix} .233 & .323 & .326 \\ .116 & .242 & .13 \\ .186 & .274 & .38 \end{bmatrix} \quad \text{and} \quad \mathbf{v}^{(0)} = \begin{bmatrix} .465 \\ .161 \\ .163 \end{bmatrix} \text{ for year 0;}$$

$$\mathbf{A}^{(1)} = \begin{bmatrix} .12 & .244 & .246 \\ .06 & .183 & .098 \\ .096 & .207 & .287 \end{bmatrix} \quad \text{and} \quad \mathbf{v}^{(1)} = \begin{bmatrix} .723 \\ .366 \\ .369 \end{bmatrix} \text{ for year 1;}$$

$$\mathbf{A}^{(2)} = \begin{bmatrix} .078 & .108 & .109 \\ .039 & .081 & .043 \\ .062 & .091 & .127 \end{bmatrix} \quad \text{and} \quad \mathbf{v}^{(2)} = \begin{bmatrix} .465 \\ .161 \\ .163 \end{bmatrix} \text{ for year 2.}$$

Here $d\mathbf{v}^{(10)} \sim \Delta\mathbf{v}^{(10)} = \mathbf{v}^{(1)} - \mathbf{v}^{(0)}$ and $d\mathbf{v}^{(21)} \sim \Delta\mathbf{v}^{(21)} = \mathbf{v}^{(2)} - \mathbf{v}^{(1)}$ for the changes in value-added coefficients between years 0 and 1 and between years 1 and 2, respectively. Also $d\mathbf{A}^{(10)} \sim \Delta\mathbf{A}^{(10)} = \mathbf{A}^{(1)} - \mathbf{A}^{(0)}$ and $d\mathbf{A}^{(21)} \sim \Delta\mathbf{A}^{(21)} = \mathbf{A}^{(2)} - \mathbf{A}^{(1)}$. Hence we can rewrite (14.8) as $\boldsymbol{\tau}^{(ts)} = -\left((\Delta\mathbf{A}^{(ts)})'\mathbf{i} + \Delta\mathbf{v}^{(ts)}\right)$, and for our example $\Delta\mathbf{A}^{(10)} = \begin{bmatrix} -.112 & -.079 & -.08 \\ -.056 & -.059 & -.032 \\ -.09 & -.067 & -.094 \end{bmatrix}$, $\Delta\mathbf{v}^{(10)} = \begin{bmatrix} .258 \\ .205 \\ .206 \end{bmatrix}$, $\Delta\mathbf{A}^{(21)} = \begin{bmatrix} -.043 & -.137 & -.137 \\ -.021 & -.102 & -.055 \\ -.034 & -.116 & -.16 \end{bmatrix}$ and $\Delta\mathbf{v}^{(21)} = \begin{bmatrix} -.258 \\ -.205 \\ -.206 \end{bmatrix}$. Notice that both $\Delta\mathbf{A}^{(10)}$ and $\Delta\mathbf{A}^{(21)}$ contain only negative elements, indicating that fewer intermediate inputs were required in each subsequent year; the positive elements in $\Delta\mathbf{v}^{(10)}$ reflect increasing use of value-added inputs in year 1 compared to year 0, but the negative elements in $\Delta\mathbf{v}^{(21)}$ indicate the opposite, namely decreased use of value-added inputs in year 2 compared to year 1. For this example,

$$\boldsymbol{\tau}^{(10)} = -\left((\Delta\mathbf{A}^{(10)})'\mathbf{i} + \Delta\mathbf{v}^{(10)}\right) = -\begin{bmatrix} -.112 & -.056 & -.09 \\ -.079 & -.059 & -.067 \\ -.08 & -.032 & -.094 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} .258 \\ .205 \\ .206 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$\boldsymbol{\tau}^{(21)} = -\left((\Delta\mathbf{A}^{(21)})'\mathbf{i} + \Delta\mathbf{v}^{(21)}\right) = -\begin{bmatrix} -.043 & -.021 & -.034 \\ -.137 & -.102 & -.116 \\ -.137 & -.055 & -.16 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
$$-\begin{bmatrix} -.258 \\ -.205 \\ -.206 \end{bmatrix} = \begin{bmatrix} .357 \\ .559 \\ .558 \end{bmatrix}$$

Note that no sector experienced TFP growth from year 0 to year 1, but all sectors experienced TFP growth between years 1 and 2. It is easy to see why this is the case for this simple example by examining the underlying transactions matrices for each period:

$$\mathbf{Z}^{(0)} = \mathbf{A}^{(0)}\hat{\mathbf{x}}^{(0)} = \mathbf{Z}^{(1)} = \mathbf{A}^{(1)}\hat{\mathbf{x}}^{(1)} = \begin{bmatrix} 10 & 20 & 30 \\ 5 & 15 & 12 \\ 8 & 17 & 35 \end{bmatrix} \text{ and}$$
$$\mathbf{Z}^{(2)} = \mathbf{A}^{(2)}\hat{\mathbf{x}}^{(2)} = \begin{bmatrix} 30 & 60 & 90 \\ 15 & 45 & 36 \\ 24 & 51 & 105 \end{bmatrix}$$

while the vectors of value-added inputs are

$$\hat{\mathbf{v}}^{(0)}\mathbf{x}^{(0)} = \begin{bmatrix} 20 \\ 10 \\ 15 \end{bmatrix} \text{ and } \hat{\mathbf{v}}^{(1)}\mathbf{x}^{(1)} = \hat{\mathbf{v}}^{(2)}\mathbf{x}^{(2)} = \begin{bmatrix} 60 \\ 30 \\ 45 \end{bmatrix}$$

Between year 0 and year 1, the increase in productivity of intermediate inputs (negative elements in $\Delta\mathbf{A}^{(10)}$) were wiped out by the decreases in productivity of value-added elements (positive elements in $\Delta\mathbf{v}^{(10)}$), whereas between years 1 and 2 both intermediate inputs and value-added inputs exhibited increased productivity (negative elements in both $\Delta\mathbf{A}^{(21)}$ and $\Delta\mathbf{v}^{(21)}$).

14.2.3 Accounting for Prices

In most applications there are changes in prices between years as well, and as a consequence all data are usually deflated to prices in some reference year. Also, we can separate value added into labor and capital coefficients, l_j and k_j, respectively, with known wage rate, w, and profit rate for capital, r, which serve as prices for those value-added categories (for simplicity w and r are treated as scalars).[4] In the end, this leads to

$$\tau_j = -\frac{\left[\sum_{i=1}^{n} p_i \mathrm{d}a_{ij} + w\mathrm{d}l_j + r\mathrm{d}k_j\right]}{p_j}$$

as the parallel to (14.3), or

$$\boldsymbol{\tau} = -\hat{\mathbf{p}}^{-1}\left[(\mathrm{d}\mathbf{A})'\,\mathbf{p} + w\mathrm{d}\mathbf{l} + r\mathrm{d}\mathbf{k}\right]$$

in matrix form. The reader is referred to Wolff (1985) for numerous details. Also, it is fairly straightforward to express the formulation of TFP presented in this section in commodity-by-industry terms, as in ten Raa (2004) or Wolff (1985 or 1997).

14.2.4 References for Section 14.2

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14.3 Graph Theory, Structural Path Analysis, and Qualitative Input–Output Analysis (QIOA)

The idea that potentially interesting properties of input–output matrices can be derived without knowing individual cell values in great detail seems to go back at least as far as Solow (1952, p. 41):

[4] Here again for simplicity denote l_j and k_j as coefficients, i.e., value per unit, rather than as total values as in previous chapters.

The particular nature of these properties [e.g., indecomposability (a property of connectedness)] is illustrated by the fact that they can be investigated with no knowledge of the value of a_{ij} other than which … are zero and which are not. To test whether an [**A**] matrix is decomposable one needs only the matrix with the a_{ij} replaced by, say, + and 0.

"Qualitative" input–output analysis (QIOA) builds on this observation. The general approach is that a binary (Boolean) transformation is performed on either transactions (**Z**) or coefficients (**A**) to generate matrices with a "1" in cells for which $z_{ij} \neq 0$ (or $a_{ij} \neq 0$), and a "0" elsewhere. (See Bon, 1989, for a brief overview. Some discussions, including Bon's, use "+" instead of "1.") Frequently, a nonzero "filter" is used to determine whether to set a cell value to 0 – if $z_{ij} < f^z$ (or $a_{ij} < f^a$), set it to 0, otherwise set it to 1. The idea is that if $z_{ij} < f^z$ (or $a_{ij} < f^a$), the values are relatively insignificant in the overall economic picture and can be ignored. In the case of direct input coefficients, a filter of size $(1/n)$ is often used (Aroche-Reyes, 2001).

These Boolean (*adjacency*) matrices are frequently represented in the literature by **W**. In graph-theory terms, **W** has an associated *directed graph* (*digraph*) in which each industry (sector) is represented by a vertex in the graph and each nonzero entry in column j is represented by an arc (or arrow) pointing from the demanding sector (j in the case of z_{ij} or a_{ij}) to the supplying sector (i). These arcs indicate flows of intermediate demand that originate in the demanding sector. Graph-theoretical methods and operations can be applied to such graphs. These methods identify various direct and indirect links (*chains* or *paths*) of transmission between and among vertices. The intent is to reveal paths of influence in the transmission of economic impulses in the input–output system – to reveal the "underlying 'characteristic structure' of an input–output table" (Schnabl, 2001, p. 245).[5]

For example, let $\mathbf{A} = \begin{bmatrix} 0 & .2 & 0 \\ .2 & .3 & .1 \\ .3 & 0 & 0 \end{bmatrix}$; with a filter of 0, we have $\mathbf{W}^{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, where $\mathbf{W}^{(1)}$ denotes the Boolean matrix associated with $\mathbf{A}$ $(= \mathbf{A}^1)$. Higher powers of **W** reflect the an *indirect* connections that exist among sectors (for example, as represented in the power series approach to finding a Leontief inverse, as in Chapter 2). Here $\mathbf{A}^2 = \begin{bmatrix} .04 & .06 & .02 \\ .09 & .13 & .03 \\ 0 & .06 & 0 \end{bmatrix}$, so $\mathbf{W}^{(2)} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Notice $w_{13}^{(1)} = 0$ but $w_{13}^{(2)} = 1$, reflecting the fact that $a_{13}^1 = 0$ but $a_{13}^2 = .02$. [The notation a_{13}^2 identifies element (1,3) in $\mathbf{A}^2$; it is not $(a_{13})^2$.] This comes about because there is an *indirect* link from sector 3 to sector 1 via sector 2. This is known as a path of length 2. The matrix multiplication **AA** makes clear the exact composition of this link. Element

[5] In view of the fact that information is lost in converting data to binary form, the approach is certainly not without its critics. For example, see de Mesnard, 1995.

a_{13}^2 in $\mathbf{A}^2$ is generated by the usual matrix multiplication of corresponding elements in appropriate rows and columns, here row 1 and column 3 of $\mathbf{A}$, and then adding. In this case, $\begin{bmatrix} 0 & .2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ .1 \\ 0 \end{bmatrix} = .02$, or

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}\begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \underbrace{(a_{11})(a_{13})}_{0} + \underbrace{(a_{12})(a_{23})}_{.02} + \underbrace{(a_{13})(a_{33})}_{0} = .02$$

Thus, sector 3's demand on sector 1 is transmitted via sector 2 – 3 demands from 2 (a_{23}) and as a result 2 demands from 1 (a_{12}), thereby connecting 3 to 1.

If instead of $\mathbf{W}^{(2)}$ we calculate $\mathbf{W}$ squared we have $\mathbf{W}^2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Compared with $\mathbf{W}^{(2)}$ we see that nonzero entries are in exactly the same locations; the difference is that the elements in $\mathbf{W}^2$ indicate the *number* of different links connecting a column (demanding) sector to a row (supplying) sector. For example, $w_{21}^2 = 2$ reflects the fact that $a_{21}^2 = .09$. From matrix operations, $w_{21}^2 = 2$ comes from

$$\begin{bmatrix} .2 & .3 & .1 \end{bmatrix}\begin{bmatrix} 0 \\ .2 \\ .3 \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \end{bmatrix}\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \underbrace{a_{21}a_{11}}_{0} + \underbrace{a_{22}a_{21}}_{.06} + \underbrace{a_{23}a_{31}}_{.03} = .09$$

This identifies the *two* paths of length two that connect destination 1 with origin 2 – $\underbrace{a_{23}a_{31}}_{.03}$, a connection via sector 3, and $\underbrace{a_{22}a_{21}}_{.06}$, via a "loop" at 2 ($a_{21}$ indicates the demand from 1 to 2 and a_{22} the response from 2).

By contrast, $w_{31}^2 = 0$, while one step earlier $w_{31} = 1$. In this case, the origins of w_{31}^2 are

$$\begin{bmatrix} .3 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ .2 \\ .3 \end{bmatrix} = \begin{bmatrix} a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \underbrace{a_{31}a_{11}}_{0} + \underbrace{a_{32}a_{21}}_{0} + \underbrace{a_{33}a_{31}}_{0} = 0$$

So, while there is a direct path from 1 to 3, there are no indirect paths of length two that connect destination 1 to origin 3.

Higher powers of the Boolean matrices identify the numbers of such indirect connections at each level but they do not identify exactly what those paths are. Clearly, the size of the filter used is critical in determining how many links will be uncovered in the process of finding length-two, length-three, … paths in the structure. A filter that is too low results in graphs that are cluttered with connections, many of them very small, and a large filter generates a very sparse picture. (See Aroche-Reyes, 2001, for graphic illustrations of the influence of filter size.)

Early applications of this approach in an interindustry setting can be found in Campbell (1972, 1974, 1975) using Washington State input–output data to illustrate the ideas.[6] Also the early 1970s saw the beginnings of a very large output of work in this area in French, pioneered by Lantner and others. (Lantner and Carluer, 2004, provide references to some of this early work.) Defourny and Thorbecke (1984), in discussing their closely-related *structural path analysis* approach, include extensive historical background and references also to this early work (see also Kahn and Thorbecke, 1988). Holub and Schnabl (1985) and Schnabl (1994, 2001 and additional cited references) propose *minimal flow analysis*, a variant also based on adjacency matrices. A number of papers in Lahr and Dietzenbacher (2001) discuss variants and applications of the qualitative approach.[7] For example, Aroche-Reyes (2002) identifies "important coefficients" (section 12.3, above) in Mexican (1970, 1990), Canadian (1971, 1990) and US (1972, 1990) economies and creates Boolean matrices for those economies by setting the identified important coefficients to 1, others to 0. He then analyzes the interconnections in the three economies and their evolution over time.[8] All of this work contributes to the research area concerned with measurement of economic "interconnectedness," linkages, and structural change as illuminated by input–output data. Sonis and Hewings and their colleagues, and many others, have written much and often on the subject (see, as examples, Sonis, Hewings and Lee, 1994, and Strassert, 2001, and the many references in the relevant chapters of Lahr and Dietzenbacher, 2001).

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[6] Campbell's work seems to have been largely overlooked in subsequent discussions of QIOA (Aroche-Reyes, 2001, is an exception).

[7] Particularly, Aroche-Reyes (2001), Lantner (2001), de Mesnard (2001) and Schnabl (2001).

[8] An earlier article is concerned with Mexico only (Aroche-Reyes, 1996).

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14.4 Fundamental Economic Structure (FES)

The concept of fundamental economic structure was apparently first suggested by Jensen, West and Hewings (1988). The objective is to identify more comprehensive, whole-table concepts from the vast array of detail embodied in the n^2 elements of an input–output table. (In Jensen's terminology, what is sought is a "holistic" as opposed to "partitive" view of the data.) In a spatial context, the idea is to identify regularities across regions. To that end, cells containing flows that are consistently present at predictable levels over a range of economies are classified as "fundamental;" that is, they identify economic activity that is inevitably required in all economies. These constitute the fundamental economic structure (FES). Other cells with data for more region-specific sectors (for example, mining) define the nonfundamental economic structure (NFES).

The authors illustrate with an application using regional tables for ten regions in the 1978/1979 11-sector Queensland, Australia economy. Sectors were ordered in a consistent "primary-secondary-tertiary" continuum. Individual transactions (z_{ij}^r) were then regressed against regional total gross output or regional total value added (as a measure of regional "size"), across all regions (r). The authors report (Jensen, West and Hewings, 1988, p. 219) that about 75 percent of the cells were "…'predictable' in that a statistical relationship (linear or logarithmic) to at least the 10 percent level exists between the size of these cells and the size of the economy…" Also "…it can be reasonably concluded that the regression equations applied to a system of tables can provide very reasonable and workable estimates for significant portions of the table" (p. 215). These were cells for sectors that lie toward the tertiary end of the continuum.

The identifiable patterns of predictable cells constitute the fundamental economic structure. The authors suggest that the research implication of these findings is that predictable cells (FES) can be estimated using regression techniques and that a higher commitment of resources might be appropriate for other, unpredictable cells (NFES). The approach is also suggested in a temporal setting; that is, for updating input–output data. The reader is directed to Jensen *et al.* (1991) and to West (2001) for very comprehensive overviews of this topic.

14.4.1 References for Section 14.4

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14.5 Input–Output, Econometrics, and Computable General Equilibrium Models

Robert Kuenne in Kuenne (1963) notes: "One of the most fruitful of many economic adoptions from the field of mechanics is the concept of 'economic equilibrium,' or a specific solution characterized by a state of balance between opposed forces acting upon economic variables." Leontief (1951) notes that the concept of economic equilibrium is implicit in Quesnay's *Tableau Économique*. Kuenne, in the work cited above, provides a straightforward way of conceptualizing Leontief's open input–output model as an equilibrium model rooted in the feature that the relative prices of inputs are completely determined by the fixed technical coefficients. We provide a somewhat simplified version below following the description of Intriligator (1971) or Dorfman, Samuelson and Solow (1958).

In the nineteenth and early twentieth centuries as economists such as Walras (1874), Pareto (1906) and Cassel (1924) refined the concepts of economic equilibrium, the role of prices became central to defining a state of competitive equilibrium. (The historical context is discussed much more in Appendix C.) A more general model of economic equilibrium relies on methods for specifying elasticities – dimensionless parameters that capture behavioral responses in an economy as functions of relative prices of inputs.

The task of expanding the input–output framework as a model of economic equilibrium has been among the most active areas in the last several decades relating input–output to other types of economic analysis, especially in its connection to a variety of applications to econometric analysis as a means of specifying elasticities. A principal goal of such a connection is to relax the input–output assumption of fixed technical coefficients by specifying technical coefficients econometrically as a function of relative prices of inputs. The functional specifications are estimated statistically based on surveys of price data over some historical period. Of course such specifications carry their own limitations in terms of available data and assumptions about the functional relationship between prices and consumption.

The first implementation of an applied general equilibrium model that did not use fixed technical coefficients was Johansen (1960). In that work Johansen retained fixed coefficients for demand of intermediate goods, but employed linear logarithmic production functions in modeling capital, labor, and technical change. Since the time of Johansen's work many researchers have expanded the framework to model producer behavior via transcendental price functions as described in Christiansen, Jorgenson and Lau (1971) and Jorgenson (1982, 1983). One such application discussed already in Chapter 9, the Hudson–Jorgenson (HJ) model, was among the first attempts to include such econometric specifications in a fully integrated way. This kind of formulation has been at the heart of many national and regional econometric models in addition to those applied to specific policy areas such as energy, the subject of the HJ model outlined in Chapter 9 (Hudson and Jorgenson, 1974), and extensions of that work into areas such as the economic implications of alternative policies for limiting greenhouse gas emissions in the United States in Jorgenson and Wilcoxen (1993) and analyzing the role of information technology in the changing structure of the US economy in Jorgenson (2002).

14.5.1 The Variable Input–Output Model

In a large number of articles, Chong K. Liew and Chung J. Liew have introduced, explored and extended what they call a Variable Input–Output model (VIO), in which changes in industrial structure occur in response to changing input costs. (See C. K. Liew, 1980; C. K. Liew and C. J. Liew, 1984a, 1984b; C. J. Liew, 1984; or C. J. Liew and C. K. Liew, 1988, for representative examples.) Technical coefficients are derived from the basic duality between production and price possibility frontiers. Liew and Liew cite Hudson and Jorgenson (1974), noted above and discussed in more detail in Chapter 9, who start with a translog production function whereas the Liew and Liew models emerge from the duality of the two frontiers. Extensions of the model include household utility-maximizing behavior of consumers (Household Interaction VIO; HIVIO), versions including pollution generation, a multiregional framework (MRVIO), multi-product sectors, modal choices and dynamics (all with appropriate acronyms).

> The Dynamic VIO model is a time-varying cost sensitive single region input–output model which incorporates a computable general equilibrium model … [in] Leontief's input–output model. The MRVIO or VIO model is also a Computable General Equilibrium (CGE) model (a partial CGE model in the sense that it is a demand driven model) which enables it to be cost-sensitive … In the Dynamic VIO model, demands for intermediate goods are price-sensitive and therefore, not fixed but variable. (C. J. Liew, 2000, p. 592.)

The articles cited below in the references to this section are only a small sample of the work by these researchers and similar research by others, such as Lahiri (1976).

14.5.2 Regional Input–Output Econometric Models

Integrated input–output/econometric models at the national level have been in use for many years, such as the INFORUM family of models (Almon, 1991) or the line of applications begun with Hudson and Jorgenson (1974) discussed above, where the aim is to retain the detailed sectoral specification of an input–output framework but to include endogenous econometric relationships specifying demand elasticities. At the regional level, where data availability often constrains the model design more severely, the integrated input–output econometric models are even more common, but vary considerably in structure depending upon the region of interest and the purpose of the model, e.g., impact analysis, analysis of fiscal policy, forecasting or analysis of income distribution. One of the first examples of an integrated regional input–output/econometric model can be found in Conway (1990). A comparison of regional input–output models with integrated input–output/econometric models and computable general equilibrium (CGE) models (discussed more below) is provided in West (1995, 2002) and a survey of many regional CGE models is given in Partridge and Rickman (1998) or Haddad, Hewings and Peter (2002).

14.5.3 Computable General Equilibrium Models

All of the formulations discussed in this section are often loosely referred to as computable general equilibrium (CGE) models. It is common to build CGE models around an input–output technical coefficients table or increasingly a social accounting matrix where coefficients of interindustry, factor inputs, and final demand can all be econometrically specified.

A simplified way to conceptualize the basic input–output model as a CGE model is by defining competitive equilibrium as the situation where there can be no profits earned by production processes (which translates in realistic terms to "normal" profits of enterprises) and that the average cost of producing any good is greater than or equal to the price of that good, which we express as

$$\sum_{i=1}^{n} p_i a_{ij} + \sum_{k=1}^{m} w_k b_{kj} \geq p_j$$

for $i = 1, 2, \ldots, n$ industries, $k = 1, 2, \ldots, m$ factor or value-added inputs; a_{ij} and b_{kj} are technical and value-added coefficients, respectively; and finally p_i and w_i are commodity prices and value-added prices, respectively. In matrix terms this is $\mathbf{p}'\mathbf{A} + \mathbf{w}'\mathbf{B} \geq \mathbf{p}'$ or $\mathbf{p}'(\mathbf{I} - \mathbf{A}) \leq \mathbf{w}'\mathbf{B}$. If we define v_k as the availability of value-added factor k then we insist that use of that factor cannot exceed its availability, i.e., $\sum_{i=1}^{n} b_{kj}x_j \leq v_k$ or, in matrix terms, $\mathbf{Bx} \leq \mathbf{v}'$. And finally, we require that industry production be non-negative

– $x_j \geq 0$ or, in matrix terms, $\mathbf{x} \geq \mathbf{0}$ and define f_j as the final demand for commodity j (an element of $\mathbf{f}$).

We can formulate the competitive general equilibrium problem as a linear programming problem (introduced conceptually in Chapter 10) of maximizing the value of total final demand (or maximizing gross domestic product) subject to the technical coefficients and supply availability of value-added factors or Max $\mathbf{p}'\mathbf{f}$ subject to $\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{f}$, $\mathbf{B}\mathbf{x} \leq \mathbf{v}$ and $\mathbf{x} \geq \mathbf{0}$. Since $\mathbf{f} = (\mathbf{I} - \mathbf{A})\mathbf{x}$ we can rewrite this in standard linear programming form as

$$\begin{gathered} \text{Max } \mathbf{p}'(\mathbf{I} - \mathbf{A})\mathbf{x} \\ \text{subject to } \mathbf{B}\mathbf{x} \leq \mathbf{v} \\ \mathbf{x} \geq \mathbf{0} \end{gathered}$$

It turns out that this linear programming problem is also equivalent to the following:

$$\begin{gathered} \text{Min } \mathbf{w}'\mathbf{v} \\ \text{subject to } \mathbf{w}'\mathbf{B} \geq \mathbf{p}'(\mathbf{I} - \mathbf{A}) \\ \mathbf{w} \geq \mathbf{0} \end{gathered}$$

These so-called primal and dual linear programming problems, respectively, have solutions (the methods for which are beyond the scope of this text, although a graphical solution to a related problem is outlined in Chapter 10) such that $\mathbf{p}'\mathbf{f} = \mathbf{w}'\mathbf{v}$, i.e., the maximized value of final demand equals the minimized cost of value-added factors and that value is the gross domestic product, or the familiar equality of national product to national income. CGE models formulated this way can add constraints, such as commodity or primary factor supply constraints, relationships for distributing income (in input–output parlance, closing the model to some final-demand and value-added sectors), time lags, adjustment of capital stocks, and many others.

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14.6 Additional Resources for Input–Output Extensions and Applications

The literature of extensions and applications of the basic input–output framework and of its connections to other areas of economic analysis continues to expand. In this section we recap a number of the surveys of these extensions and applications by means of several chronological lists of major edited collections, special journal issues dedicated to overviews of applications and extensions, and special collections of journal reprints, all of which have appeared in the literature since the early 1980s. Many, but not all of these collections were referenced throughout this text as we explored various input–output extensions in preceding chapters.

In reviewing this literature, as we have in the course of writing this text, one can't help notice many of the key milestones that have appeared along the way, from which many extensions and applications followed. Some of these milestones include the connection of input–output concepts to social accounts in Stone (1961), subsequently leading the way to the now routinely applied commodity-by-industry framework; the solid foundation for regional and interregional analysis pioneered by Isard *et al.* (1960); the first applications to analyzing economic structural change by Carter (1970); the explosion of nonsurvey estimation techniques launched by Stone (1961) and Stone and Brown (1962); the path of extensions for tracing environmental pollution generation and elimination conceived by Leontief (1970a); the path toward practical implementation of physical input–output models originally conceived by Leontief in his earliest work but perhaps first implemented with Bullard and Herendeen (1975) in tracing energy use in the US economy; the connection to social accounting matrices envisioned by Stone and developed by Pyatt and Round (1985); the connections to planning formulations and linear programming outlined in Dorfman, Samuelson and Solow (1958); the extension to dynamic models first conceived by Leontief (1970b) and then extended by connections to econometric models advanced by Jorgenson (1982); and, of course, many others. We examine the basic history of the field in Appendix C, but the collections in this section provide an extensive, albeit not exhaustive, overview of how the field has evolved over the past half century.

14.6.1 Edited Collections[9]

Ciaschini, Maurizio (ed.). 1988. *Input-Output Analysis: Current Developments*. London: Chapman and Hall.

Harrigan, Frank and Peter G. McGregor (eds.). 1988. *Recent Advances in Regional Economic Modelling*. London: Pion.

Miller, Ronald E., Karen R. Polenske and Adam Z. Rose (eds.). 1989. *Frontiers of Input-Output Analysis*. New York: Oxford University Press.

Anselin, Luc and Moss Madden (eds.). 1990. *New Directions in Regional Analysis: Integrated and Multiregional Approaches*. London: Pinter.

Dewhurst, John H. Ll., Geoffrey J. D. Hewings and Rodney C. Jensen (eds.). 1991. *Regional Input-Output Modelling. New Developments and Interpretations*. Avebury: Aldershot.

Peterson, William (ed.). 1991. *Advances in Input-Output Analysis*. New York: Oxford University Press.

Hewings, Geoffrey J. D., Michael Sonis, Moss Madden and Yoshio Kimura (eds.). 1999. *Understanding and Interpreting Economic Structure*. Berlin: Springer.

Lahr, Michael L. and Erik Dietzenbacher (eds.). 2001. *Input-Output Analysis: Frontiers and Extensions*. Basingstoke, UK: Palgrave.

[9] Note that in this section we present the bibliographic references in chronological rather than alphabetical order.

Lahr, Michael L. and Ronald E. Miller (eds.). 2001. *Regional Science Perspectives in Economic Analysis. A Festschrift in Memory of Benjamin H. Stevens.* Amsterdam: North-Holland.

Hewings, Geoffrey J. D., Michael Sonis and David Boyce (eds.). 2002. *Trade, Networks and Hierarchies. Modeling Regional and Interregional Economies*. Heidelberg: Springer.

Dietzenbacher, Erik and Michael L. Lahr (eds.). 2004. *Wassily Leontief and Input-Output Economics*. Cambridge, UK: Cambridge University Press.

14.6.2 Journal Special Issues

Socio-Economic Planning Sciences. Vol. **18**, No. 5, 1984. Special Issue in Honor of William H. Miernyk. "Input-Output Analysis and Regional Economic Development." Guest Editor, Shelby D. Gerking. (Articles by Shelby D. Gerking; Adam Rose; Geoffrey J. D. Hewings; Sajal Lahiri; Jarvin Emerson; Eliahu Romanoff.)

Ricerche Economiche, Vol. **42**, No. 2, 1988. Special Issue on Interregional Input-Output Models. Editors, David F. Batten, Dino Martellato. (Articles by David F. Batten and Dino Martellato; Åke E. Andersson and Wei B. Zhang; Sajal Lahiri; Frederik Muller; Paolo Costa and Roberto Roson; Ronald E. Miller and Peter D. Blair; Jeffery I. Round; Frank J. Harrigan and James McGilvray; Peter W. J. Batey and Moss Madden; Shapoor Vali.)

Journal of Policy Modeling. Vol. **11**, No. 1, 1989. Special Issue in Honor of Wassily Leontief. Guest Editor, Adam Rose. (Articles by Adam Rose; Anne P. Carter and Peter A. Petri; Maria Augusztinovics; Jiří Skolka; Geoffrey J. D. Hewings, Manuel Fonseca, Joaquim Guilhoto and Michael Sonis; Jinkichi Tsukui and Hajime Hori; Graham Pyatt; Haider A. Khan and Erik Thorbecke; Boris Pleskovic.)

International Regional Science Review, Vol. **13**, Nos. 1 and 2, 1990. Special Double Issue: "The Construction and Use of Regional Input-Output Models." Guest Editors, Roger E. Bolton, Randall W. Jackson and Guy R. West. (Articles by Rodney C. Jensen; Peter W. J. Batey and Adam Z. Rose; Jan Oosterhaven and John H. Ll. Dewhurst; Moss Madden and Andrew B. Trigg; Philip J. Bourque; Richard M. Beemiller; Guy R. West; Sharon M. Brucker, Steven E. Hastings and William R. Latham III; Richard S. Conway, Jr.; Paul M. Beaumont; Steven G. Cochrane; Roy Powell, Mark McGovern and Julian Morison; Frank Giarratani.)

Regional Science and Urban Economics, Vol. **24**, No. 1, February, 1994. Special Issue on Input-Output Analysis. (Foreword by Wassily Leontief; articles by Thijs ten Raa; Vu Q. Viet; Pieter S. M. Kop Jansen; Edward N. Wolff; William J. Baumol and Edward N. Wolff; Donald A. Gilchrist and Larry V. St. Louis; Thijs ten Raa and Pierre Mohnen.)

Economic Systems Research, Vol. **7**, No. 2, 1995. Special Issue: "Regional and Interregional Interindustry Modelling." (Articles by Stefano Casini Benvenuti, Dino Martellato and Cristina Raffaelli; Dirk Stelder; Johannes Bröcker; Ping-Cheng Li and Adam Rose; Donald A. Gilchrist and Larry V. St. Louis; Peter G. McGregor, J. K. Swales and Ya Ping Yin; Guy R. West.)

Structural Change and Economic Dynamics, Vol. **6**, No. 3, August, 1995. Special Theme: In Honor of Wassily Leontief's 90th Birthday. Guest Editor, Faye Duchin. (Articles by Faye Duchin; Maria Augusztinovics; Alice H. Amsden; Adam Rose; Robert Dorfman; Karen R. Polenske; Olaf Bjerkholt; Bruce Hannon.)

Economic Systems Research. Vol. **18**, No. 4, December, 2006. Special Issue: "The History of Input-Output Analysis, Leontief's Path and Alternative Tracks." Guest Editors: Olav Bjerkholt and Heinz D. Kurz. (Articles by Olav Bjerkholt and Heinz D. Kurz; Svetlana A. Kaladina and Natal'ia Iu. Pavlova; Svetlana A. Kaladina; Gilbert Abraham-Frois and Emeric Lendjel; Heinz D. Kurz and Neri Salvadori; Olav Bjerkholt and Mark Knell; Christian Langer.)

14.6.3 Collections of Reprints

Sohn, Ira (ed.). 1986. *Readings in Input-Output Analysis*. New York: Oxford University Press.

Kurz, Heinz D., Erik Dietzenbacher and Christian Lager (eds.). 1998. *Input-Output Analysis*. Three volumes. (International Library of Critical Writings in Economics, Vol. 92.) Cheltenham, UK: Edward Elgar.

14.6.4 References for Section 14.6

Bullard, Clark and Robert Herendeen. 1975. "The Energy Costs of Goods and Services," *Energy Policy*, **1**, 268–277.

Carter, Anne P. 1970. *Structural Change in the American Economy*. Cambridge, MA: Harvard University Press.

Dorfman, Robert, Paul A. Samuelson and Robert M. Solow. 1958. *Linear Programming and Economic Analysis*. New York: McGraw-Hill.

Isard, Walter, David F. Bramhall, Gerald A. P. Carrothers, John H. Cumberland, Leon N. Moses, Daniel O. Price and Eugene W. Schooler. 1960. *Methods of Regional Analysis: An Introduction to Regional Science*. New York: The Technology Press of MIT and John Wiley & Sons.

Jorgenson, Dale W. 1982. "An Econometric Approach to General Equilibrium Analysis," in M. Hazewinkel and A. H. G. Rinnoy Kan (eds.), *Current Developments in the Interface: Economics, Econometrics, Mathematics*. Boston, MA: Reidel, pp. 125–157.

Leontief, Wassily. 1970a. "Environmental Repercussions and the Economic Structure: An Input-Output Approach," *Review of Economics and Statistics*, **52**, 262–271.

1970b. "The Dynamic Inverse," in Anne P. Carter and Andrew Bródy (eds.), *Contributions to Input-Output Analysis*. Vol. 1 of *Proceedings of the Fourth International Conference on Input-Output Techniques*. Geneva, 1968. Amsterdam: North-Holland, pp. 17–43.

Pyatt, Graham and Jeffrey I. Round (eds.). 1985. *Social Accounting Matrices: A Basis for Planning*. Washington, DC: The World Bank.

Stone, Richard. 1961. *Input-Output and National Accounts*. Paris: Organization for European Economic Cooperation.

Stone, Richard and Alan Brown. 1962. *A Computable Model of Economic Growth*. London: Chapman and Hall.

14.7 Some Concluding Reflections

In this text we have developed the basic framework of input–output analysis and many of the extensions and applications that followed Professor Leontief's seminal work defining the field over a half century ago. To this day input–output and its extensions endure by themselves as tools for many kinds of economic analysis. The basic framework often comprises a fundamental component of many other types of economic analysis as well, such as econometric general equilibrium and planning models. The extensions also include applications to broader social accounting problems as well as ecological analysis and tracing material and energy use and flow throughout an economy measured in physical terms. We have captured many and perhaps most but not all of these extensions and applications in this text.

Input–output is often taken for granted as a point of departure for many types of analysis because it is rooted in traditions of real data observations, around which even many basic statistical systems today are designed, and a well understood theoretical underpinning, which was perhaps Professor Leontief's key insight. The combination of these two features of the input–output framework provides for empirical verification, which was one of Leontief's principal requirements for legitimizing theoretical economic developments.

Advancing computational capabilities in the past half century have contributed most substantially to making input–output the practical tool it has become today, and at scales scarcely imaginable when Leontief was conceiving his "economy as a circular flow." Looking forward we can only expect that such capabilities will continue to advance as will conceptual developments in extending the basic input–output framework, and perhaps in particular in tying the framework even more intimately to other economic analysis tools. The discipline of insisting on empirical validation of such tools will likely continue to be a fundamental challenge for researchers and practitioners alike who should certainly not shirk from that challenge.

Appendix A Matrix Algebra for Input–Output Models

A.1 Introduction

A matrix is a collection of elements arranged in a grid – a pattern of rows and columns. In all cases that will be of interest to the topics in this book, the elements will be numbers whose values either are known or are unknown and to be determined. Matrices are defined in this "rectangular" way so that they can be used to represent systems of linear relations among variables, which is exactly the structure of an input–output model.

The general case, then, will be a matrix with m rows and n columns. If $m = 2$ and $n = 3$, and using double subscript notation, a_{ij}, to denote the element in row i and column j of the matrix, we have

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

A particular example of such a matrix might be

$$\mathbf{M} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 6 & 12 \end{bmatrix}$$

These are said to be 2×3 (read "2 by 3") matrices or matrices of *dimension* 2 by 3. Dimensions are often denoted in parentheses underneath the matrix, as in $\underset{(2\times 3)}{\mathbf{M}}$.

When $m = n$ the matrix is *square*; in this case it is often referred to as a matrix *of order m* (or *of order n*, since they are the same). If $m = 1$ (a matrix with only one row) it is called a *row vector*; if $n = 1$ (a matrix with only one column) it is called a *column vector*.[1] We adhere to the convention of using upper-case bold letters for matrices, lower-case bold letters for vectors, and italicized letters for elements of matrices and vectors. (In matrix algebra, an ordinary number is called a *scalar*.)

[1] The ultimate in shrinkage is when $m = n = 1$, a matrix with only one element. These will not be needed for input–output models.

A.2 Matrix Operations: Addition and Subtraction

A.2.1 Addition

Addition of matrices, say $\mathbf{A}+\mathbf{B}$, is accomplished by the simple rule of adding elements *in corresponding positions*. This means $a_{ij}+b_{ij}$ for all i and j; and this, in turn, means that only matrices that have exactly the same dimensions can be added. Given $\mathbf{M}$, above, and $\underset{(2\times 3)}{\mathbf{N}} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, their sum, $\mathbf{S}=\mathbf{M}+\mathbf{N}$, will be

$$\underset{(2\times 3)}{\mathbf{S}} = \begin{bmatrix} 3 & 3 & 6 \\ 7 & 8 & 13 \end{bmatrix}$$

A.2.2 Subtraction

Subtraction is defined in a completely parallel way, namely subtraction of elements *in corresponding positions*. So, again, only matrices of exactly the same dimensions can be subtracted. For example, $\mathbf{D}=\mathbf{M}-\mathbf{N}$ will be

$$\underset{(2\times 3)}{\mathbf{D}} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 4 & 11 \end{bmatrix}$$

A.2.3 Equality

The notion of equality of two (or more) matrices is also very straightforward. Two matrices are equal if they have the same dimensions and if the elements in corresponding positions are equal. So $\mathbf{A}=\mathbf{B}$ when $a_{ij}=b_{ij}$, for all i and j.

A.2.4 The Null Matrix

A zero in ordinary algebra is the number which, when added to (or subtracted from) another number leaves that number unchanged. The completely parallel notion in matrix algebra is a *null matrix*, simply defined as a matrix containing only zeros. Define $\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; then it is obvious that $\mathbf{M}+\mathbf{0}=\mathbf{M}-\mathbf{0}=\mathbf{M}$.

A.3 Matrix Operations: Multiplication

A.3.1 Multiplication of a Matrix by a Number

If a matrix is multiplied by a number (called a *scalar* in matrix algebra), each element in the matrix is simply multiplied by that number. For example

$$2\mathbf{M} = \begin{bmatrix} 4 & 2 & 6 \\ 8 & 12 & 24 \end{bmatrix}$$

A.3.2 Multiplication of a Matrix by another Matrix

Multiplication of two matrices is defined in what appears at first to be a completely illogical way. But we will see that the reason for the definition is precisely because of the way in which matrix notation is used for systems of linear relations, especially linear equations. Using **M**, again, and a 3 × 3 matrix $\mathbf{Q} = \begin{bmatrix} 2 & 0 & 4 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}$, the product $\mathbf{P} = \mathbf{MQ}$, is found as

$$\mathbf{P} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 6 & 12 \end{bmatrix} \begin{bmatrix} 2 & 0 & 4 \\ 1 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 14 & 13 & 25 \\ 50 & 54 & 88 \end{bmatrix}$$

This comes from

$$\begin{bmatrix} (4+1+9) & (0+1+12) & (8+2+15) \\ (8+6+36) & (0+6+48) & (16+12+60) \end{bmatrix}$$

The rule is: for element p_{ij} in the product, go *across row i* in the matrix on the left (here **M**) and *down column j* in the matrix on the right (here **Q**), multiplying pairs of elements and summing. So, for p_{23} we find, from row 2 of **M** and column 3 of **Q**, $(4)(4) + (6)(2) + (12)(5) = (16 + 12 + 60) = 88$. In general, then, for this example

$$p_{ij} = m_{i1}q_{1j} + m_{i2}q_{2j} + m_{i3}q_{3j} \; (i = 1, 2; \; j = 1, 2, 3)$$

This definition of matrix multiplication means that in order to be *conformable for multiplication* the number of *columns* in the matrix on the left must be the same as the number of *rows* in the matrix on the right. Look again at p_{ij} above; for the three elements in (any) row *i* of $\mathbf{M} - m_{i1}, m_{i2},$ and m_{i3} – there must be three "corresponding" elements in (any) column *j* of $\mathbf{Q} - q_{1j}, q_{2j},$ and q_{3j}.

The definition of matrix multiplication also means that the product matrix, **P**, will have the same number of rows as **M** and the same number of columns as **Q**. In general,

$$\underset{(m\times n)}{\mathbf{P}} = \underset{(m\times r)}{\mathbf{M}} \; \underset{(r\times n)}{\mathbf{Q}} \tag{A.1}$$

It also means that, in general, order of multiplication makes a difference. In this example, the product the other way around, **QM**, cannot even be found, since there are three columns in **Q** but only two rows in **M**.[2] For that reason, there is language to describe the order of multiplication in a matrix product. For example, in $\mathbf{P} = \mathbf{MQ}$, **M** is said to *premultiply* **Q** (or to multiply **Q** on the left) and, equivalently, **Q** is said to *postmultiply* **M** (or to multiply **M** on the right).

A.3.3 The Identity Matrix

In ordinary algebra, 1 is known as the *identity element for multiplication*, which means that a number remains unchanged when multiplied by it. There is an analogous concept in matrix algebra. An *identity matrix* is one that leaves a matrix unchanged when the matrix is multiplied by it.

[2] Try to carry out the multiplication in the order **QM** to easily see where the trouble arises.

If we use $\mathbf{M} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 6 & 12 \end{bmatrix}$, by what matrix could $\mathbf{M}$ be *post*multiplied so that it remained unchanged? Denote the unknown matrix by $\mathbf{I}$ (this is the standard notation for an identity matrix); we want $\mathbf{MI} = \mathbf{M}$. We know from the rule in (A.1) that $\mathbf{I}$ must be a 3×3 matrix; it needs three rows to be conformable to postmultiply $\mathbf{M}$ and three columns because the product, which will be $\mathbf{M}$ with dimensions 2 by 3, gets its second dimension from the number of columns in $\mathbf{I}$. The reader might try letting $\mathbf{I}$ be a 3×3 matrix with all 1's. It may seem logical but it is wrong. In fact, the only $\mathbf{I}$ for which $\mathbf{MI} = \mathbf{M}$ will be $\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The reader should try this and other possibilities, to be convinced that only this matrix will do the job. (Subscripts are often used, as here, to indicate the order of the identity matrix.)

An identity matrix is always square and can be of any size to satisfy the conformability requirement for the particular multiplication operation in which it appears. It has 1's along its *main diagonal*, from upper left to lower right, and 0's everywhere else. We could find another identity matrix by which to *pre*multiply $\mathbf{M}$ so that it remains unchanged. In this case we need the 2×2 identity matrix $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

A.4 Matrix Operations: Transposition

Transposition is a matrix operation for which there is no parallel in ordinary algebra. It plays a useful and important role in certain input–output operations. The *transpose* of an $m \times n$ matrix $\mathbf{M}$, denoted $\mathbf{M}'$, is an $n \times m$ matrix in which row i of $\mathbf{M}$ becomes column i of $\mathbf{M}'$. (Sometimes $\mathbf{M}^t$ or $\mathbf{M}^T$ are used to denote transposition.) For our example

$$\mathbf{M}' = \begin{bmatrix} 2 & 4 \\ 1 & 6 \\ 3 & 12 \end{bmatrix}$$

Notice that the transpose of an n-element column vector (dimensions $n \times 1$) is an n-element row vector (dimensions $1 \times n$).

A useful result, for matrices that are conformable for multiplication, is that $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$. The reader can easily see why this is the case by examining a small general example with, say, $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$.

A.5 Representation of Linear Equation Systems

Here are two linear equations in two unknowns, x_1 and x_2:

$$\begin{aligned} 2x_1 + \ x_2 &= 10 \\ 5x_1 + 3x_2 &= 26 \end{aligned} \tag{A.2}$$

Define **A** as a 2×2 matrix that contains the coefficients multiplying the x's in exactly the order in which they appear, so

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$$

Define a two-element *column* vector, **x**, containing the unknown x's and another two-element column vector, **b**, containing the values on the right-hand sides of the equations exactly in the order in which they appear, namely[3]

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \quad \mathbf{b} = \begin{bmatrix} 10 \\ 26 \end{bmatrix}$$

Then, precisely because of the way in which matrix multiplication and matrix equality are defined, the equation system in (A.2) is compactly represented as

$$\mathbf{Ax} = \mathbf{b} \tag{A.3}$$

[Writing out the system represented in (A.3) will show exactly why this is true.]

In ordinary algebra, when we have an equation like $3x = 12$, we "solve" this equation by dividing both sides by 3 – which is the same as multiplying both sides by (1/3), the reciprocal of 3 (sometimes denoted 3^{-1}); multiplication of a number by its reciprocal generates the identity element for multiplication. So, in more detail, we go from $3x = 12$ to $x = 4$ in the logical sequence

$$3x = 12 \Rightarrow (1/3)3x = (1/3)12 \text{ [or } (3^{-1})3x = (3^{-1})12] \Rightarrow (1)x = 4 \Rightarrow x = 4$$

In ordinary algebra the transition from $3x = 12$ to $x = 4$ is virtually immediate. The point here is to set the stage for a parallel approach to systems of linear equations, as in (A.2).

Given the representation in (A.3), it is clear that a way of "solving" this system for the unknowns would be to "divide" both sides by **A**, or, alternatively, multiply both sides by the "reciprocal" of **A**. Parallel to the notation for the reciprocal of a number, this is denoted $\mathbf{A}^{-1}$. If we could find such a matrix, with the property that $(\mathbf{A}^{-1})(\mathbf{A}) = \mathbf{I}$ (the identity element for matrix multiplication), we would proceed in the same way, namely

$$\mathbf{Ax} = \mathbf{b} \Rightarrow (\mathbf{A}^{-1})\mathbf{Ax} = (\mathbf{A}^{-1})\mathbf{b} \Rightarrow \mathbf{Ix} = (\mathbf{A}^{-1})\mathbf{b} \Rightarrow \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

and the values of the unknowns, in **x**, would be found as the matrix operation in which the vector **b** is premultiplied by the matrix $\mathbf{A}^{-1}$, which is usually called the *inverse* of **A**.

[3] The usual convention is to define all vectors as column vectors (as here), so row vectors are formed by transposition.

A.6 Matrix Operations: Division

In matrix algebra, "division" by a matrix is represented as multiplication by the inverse.[4] Finding inverses can be a very tedious mathematical procedure, but modern computers do it very quickly, even for relatively large matrices. Even though this can easily be done with computer software, we examine a few matrix algebra definitions involving *determinants* and their role in inverses in order to provide a rudimentary understanding of the important concept of a *singular* matrix – one that has no inverse. (The reader uninterested in mathematical details can skip directly to the result on the general definition of an inverse.)

Determinant of a matrix: the 2×2 **case**

A determinant is a number associated with any square matrix. For $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the determinant, $|\mathbf{A}|$, is defined as $|\mathbf{A}| = a_{11}a_{22} - a_{12}a_{21}$. Unfortunately, determinants of larger matrices cannot be found by obvious extensions of this simple expression, and additional definitions are needed – specifically minors, cofactors and adjoints.

Minor of an element. The minor of an element a_{ij} in a square matrix $\mathbf{A}$ (denoted m_{ij}) is the determinant of the matrix remaining when row i and column j are removed from $\mathbf{A}$. So the n^2 minors of the elements in $\underset{(n\times n)}{\mathbf{A}}$ will be determinants of $(n-1) \times (n-1)$ matrices.

Cofactor of an element. The cofactor of an element a_{ij} in a square matrix $\mathbf{A}$ (denoted $\mathbf{A}_{ij}$) is defined as $\mathbf{A}_{ij} = (-1)^{i+j}m_{ij}$. When $i+j$ is an even number, $\mathbf{A}_{ij} = m_{ij}$, when $i+j$ is an odd number, $\mathbf{A}_{ij} = -m_{ij}$.

Determinant of a matrix: the general case

For $\underset{(n\times n)}{\mathbf{A}}$, $|\mathbf{A}|$ can be found as

$$\text{(a) } |\mathbf{A}| = \sum_{j=1}^{n} a_{ij}\mathbf{A}_{ij} \text{ (for any } i\text{) or (b) } |\mathbf{A}| = \sum_{i=1}^{n} a_{ij}\mathbf{A}_{ij} \text{ (for any } j\text{).}$$

In words: $|\mathbf{A}|$ can be found by summing the products of elements and their corresponding cofactors in *any* row [from (a)] or *any* column [from (b)].

Adjoint of a matrix

The adjoint of $\mathbf{A}$ [often denoted (adj $\mathbf{A}$)] is defined as adj $\mathbf{A} = [\mathbf{A}'_{ij}]$. In words: the adjoint is the matrix whose elements are the cofactors of the transpose of $\mathbf{A}$.

[4] In this appendix we will look at inverses for square matrices only. This means that if we are dealing with the coefficient matrix for an equation system, as in (A.2) or (A.3), there are the same number of unknowns as equations in the system. There are more advanced concepts of "pseudo" inverses for nonsquare matrices, but they need not concern us at this point.

Properties of determinants

1. $|\mathbf{A}| = |\mathbf{A}'|$.
2. If any row or column of $\mathbf{A}$ contains all zeros, $|\mathbf{A}| = 0$.
3. Multiplication of all the elements in any row or column of $\mathbf{A}$ by a constant, k, creates a new matrix whose determinant is $k\,|\mathbf{A}|$.
4. If $\mathbf{A}^*$ is generated from $\mathbf{A}$ by interchanging any two rows or columns of $\mathbf{A}$, $|\mathbf{A}^*| = -|\mathbf{A}|$.
   a. If any two rows or columns in $\mathbf{A}$ are equal, $|\mathbf{A}| = 0$.
   b. If any two rows or columns in $\mathbf{A}$ are proportional, $|\mathbf{A}| = 0$.
5. a. $\sum_{j=1}^{n} a_{ij}\mathbf{A}_{i'j} = 0$ (where $i \neq i'$) and
   b. $\sum_{i=1}^{n} a_{ij}\mathbf{A}_{ij'} = 0$ (where $j \neq j'$).

In words: evaluation of a determinant using *alien cofactors* – elements from row i and cofactors from some other row (which is what makes them *alien*) or elements from column j and cofactors from some other column – always yields a value of zero. This is not difficult to show.

1. Use a_{ij} and $\mathbf{A}_{kj}$ $(i \neq k)$ and write out the alien cofactor expression $\sum_{j=1}^{n} a_{ij}\mathbf{A}_{kj}$.
2. Replace row k in $\mathbf{A}$ by row i; call this matrix $\tilde{\mathbf{A}}$. Then $\left|\tilde{\mathbf{A}}\right| = 0$ [from (4a)].
3. Find $\left|\tilde{\mathbf{A}}\right|$, which we know to be 0 [from (*ii*)], by ordinary expansion across its row k; this is $\left|\tilde{\mathbf{A}}\right| = \sum_{j=1}^{n} a_{ij}\mathbf{A}_{kj}$. But this is exactly the alien cofactor expression in (*i*), thus demonstrating (5a) for $i' = k$.

Inverse

The general expression for an inverse builds on the preceding concepts. For the $n \times n$ case, where $\text{adj}\,\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{21} & \cdots & \mathbf{A}_{n1} \\ \mathbf{A}_{12} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{A}_{1n} & \mathbf{A}_{2n} & \cdots & \mathbf{A}_{nn} \end{bmatrix}$, form the product

$$\mathbf{A}(\text{adj}\,\mathbf{A}) = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}\mathbf{A}_{1j} & \sum_{j=1}^{n} a_{1j}\mathbf{A}_{2j} & \cdots & \sum_{j=1}^{n} a_{1j}\mathbf{A}_{nj} \\ \sum_{j=1}^{n} a_{2j}\mathbf{A}_{1j} & \sum_{j=1}^{n} a_{2j}\mathbf{A}_{2j} & \cdots & \sum_{j=1}^{n} a_{2j}\mathbf{A}_{nj} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^{n} a_{nj}\mathbf{A}_{1j} & \sum_{j=1}^{n} a_{nj}\mathbf{A}_{2j} & \cdots & \sum_{j=1}^{n} a_{nj}\mathbf{A}_{nj} \end{bmatrix}$$

– the reason for doing this will soon be apparent. Each of the on-diagonal elements in this product is $|\mathbf{A}|$, found by cofactor expansions – across each of the rows in turn;

each off-diagonal element in the product is 0 because it is an expansion by alien cofactors. Therefore

$\mathbf{A}\ (\text{adj}\ \mathbf{A}) = \begin{bmatrix} |\mathbf{A}| & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & |\mathbf{A}| & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & |\mathbf{A}| \end{bmatrix} = |\mathbf{A}|\, \mathbf{I}_n$, so $\mathbf{A}(1/\,|\mathbf{A}|)(\text{adj}\ \mathbf{A}) = \mathbf{I}_n$, meaning that $\mathbf{A}^{-1} = \underbrace{(1/\,|\mathbf{A}|)}_{scalar}\ \underbrace{(\text{adj}\ \mathbf{A})}_{(n\times n)\ matrix}$.

Linear combinations; linear dependence and independence

A more general requirement for nonsingularity of **A** involves the concepts of linear dependence and independence. A complete examination of this topic, including the associated vector geometry, is beyond the level of this discussion, but the main ideas are important. We consider only the case of square matrices because our interest is in the inverses of the matrices associated with input–output models.[5] Consider a series of n vectors, either columns or rows; we deal with columns simply for illustration. Let the columns of an $n \times n$ **A** matrix be denoted $\mathbf{a}_1^{(c)},\ \mathbf{a}_2^{(c)}, \ldots,\ \mathbf{a}_n^{(c)}$. Multiply each column by a scalar and add, generating another n-element column vector;

$$s_1\mathbf{a}_1^{(c)} + s_2\mathbf{a}_2^{(c)} + \cdots + s_n\mathbf{a}_n^{(c)} = \mathbf{c} \text{ or } \sum_{i=1}^{n} s_i\mathbf{a}_i^{(c)} = \mathbf{c}$$

The vector **c** is called a *linear combination* of $\mathbf{a}_1^{(c)},\ \mathbf{a}_2^{(c)}, \ldots,\ \mathbf{a}_n^{(c)}$. If not all the scalars in the linear combination are zero and if $\mathbf{c} = \mathbf{0}$ – that is, $\sum_{i=1}^{n} s_i\mathbf{a}_i^{(c)} = \mathbf{0}$ – $\mathbf{a}_1^{(c)},\ \mathbf{a}_2^{(c)}, \ldots,\ \mathbf{a}_n^{(c)}$ are said to be *linearly dependent*. Using three-element vectors for illustration, suppose $\mathbf{a}_3^{(c)}$ is a linear combination of $\mathbf{a}_1^{(c)}$ and $\mathbf{a}_2^{(c)}$. For example, let $\mathbf{A} = \begin{bmatrix} 1 & 5 & 17 \\ 2 & 4 & 16 \\ 3 & 7 & 27 \end{bmatrix}$; $\mathbf{a}_1^{(c)} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{a}_2^{(c)} = \begin{bmatrix} 5 \\ 4 \\ 7 \end{bmatrix}$ and $2\mathbf{a}_1^{(c)} + 3\mathbf{a}_2^{(c)} = \mathbf{a}_3^{(c)}$. Then, equivalently,

$$2\mathbf{a}_1^{(c)} + 3\mathbf{a}_2^{(c)} + (-1)\mathbf{a}_3^{(c)} = \mathbf{0}$$

and the vectors $\mathbf{a}_1^{(c)},\ \mathbf{a}_2^{(c)},\ \mathbf{a}_3^{(c)}$ are (by definition) linearly dependent. Whenever some $\mathbf{a}_i^{(c)}$ can be expressed as a linear combination of the other $(n-1)$ **A** vectors, the n vectors $\mathbf{a}_1^{(c)},\ \mathbf{a}_2^{(c)}, \ldots,\ \mathbf{a}_n^{(c)}$ are linearly dependent. The important fact is that if $\underset{(n\times n)}{\mathbf{A}}$ contains linearly

[5] In input–output work, we are usually concerned with finding the inverse of $(\mathbf{I} - \mathbf{A})$. We use a generic "**A**" matrix in the discussion in this appendix for simplicity of exposition.

dependent columns, $\mathbf{A}$ is singular.[6] This provides an additional case in which $|\mathbf{A}| = 0$; it supplements the relatively simpler observations in (2), (4a), and (4b), above. Moreover, all of this holds true if "rows" are substituted for "columns" throughout the discussion; in particular, if $\mathbf{A}$ contains linearly dependent rows, $|\mathbf{A}| = 0$.

On the other hand, if the only scalars for which $\sum_{i=1}^{n} s_i \mathbf{a}_i^{(c)} = 0$ holds are (all) $s_i = 0$, the vectors are termed *linearly independent*. These ideas are used to define the important concept of the *rank* of a matrix, $\rho(\mathbf{A})$. In a nutshell, the rank of $\mathbf{A}$ is the number of linearly independent rows (or columns) in $\mathbf{A}$. And so, if $\rho(\mathbf{A}) = n$, $\mathbf{A}$ is nonsingular. Computer programs find ranks of matrices with very little effort.

One immediate application of these observations can be found with the completely closed input–output model, where $\mathbf{i}'\mathbf{A} = \mathbf{i}'$. As a consequence $\mathbf{i}'(\mathbf{I} - \mathbf{A}) = \mathbf{0}'$ [the rows of $(\mathbf{I} - \mathbf{A})$ are linearly dependent], $|(\mathbf{I} - \mathbf{A})| = 0$, and no Leontief inverse can be found.

Thus $\mathbf{A}^{-1}$ can be found only when $|\mathbf{A}| \neq 0$. This is similar to the problem with "0" in ordinary algebra; you cannot divide by it (the reciprocal of 0, 1/0, is not defined).

The matrix $\mathbf{A}$ from (A.2) is *nonsingular*; namely

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} \quad \text{and} \quad \mathbf{A}^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

which the reader can easily check. An example of a singular matrix is $\mathbf{C} = \begin{bmatrix} 2 & 4 \\ 6 & 12 \end{bmatrix}$, where $|\mathbf{C}| = 24 - 24 = 0$ (proportional rows *and* columns). There is no matrix by which $\mathbf{C}$ can be pre- or postmultiplied to generate $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Since we have found $\mathbf{A}^{-1}$ for the equations in (A.2), the solution is exactly

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 10 \\ 26 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

The reader can easily check that $x_1 = 4$ and $x_2 = 2$ are the (only) solutions to the two equations in (A.2).

An important fact about inverses is that, for nonsingular matrices $\mathbf{M}$ and $\mathbf{N}$ that are conformable for multiplication, $(\mathbf{MN})^{-1} = \mathbf{N}^{-1}\mathbf{M}^{-1}$.

A.7 Diagonal Matrices

Identity matrices are examples of *diagonal* matrices. These are always square, with elements on the diagonal from upper left to lower right and zeros elsewhere. In general,

[6] Illustration and proof of this statement is beyond the level of this text. The interested reader should turn to any good book on linear algebra.

an $n \times n$ diagonal matrix is

$$\mathbf{D} = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \vdots & \cdots & d_n \end{bmatrix}$$

A useful notational device is available for creating a diagonal matrix from a *vector*. Suppose $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; then the diagonal matrix with the elements of $\mathbf{x}$ strung out along its main diagonal is denoted by putting a "hat" over the $\mathbf{x}$ (sometimes "$\langle$" and "$\rangle$" are used to bracket the $\mathbf{x}$), so

$$\hat{\mathbf{x}} = \langle \mathbf{x} \rangle = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix}$$

A hat is also used with a square *matrix* to indicate the diagonal matrix formed from the square matrix when all off-diagonal elements are set equal to zero, and an upside down hat is used for the square matrix that is left when all diagonal elements are set equal to zero. For example, using $\mathbf{Q}$ from above,

$$\hat{\mathbf{Q}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \text{ and } \check{\mathbf{Q}} = \begin{bmatrix} 0 & 0 & 4 \\ 1 & 0 & 2 \\ 3 & 4 & 0 \end{bmatrix}$$

One useful fact about diagonal matrices is that the inverse of a diagonal matrix is another diagonal matrix, each of whose elements is just the reciprocal of the original element. For $\hat{\mathbf{x}}$ this means

$$\hat{\mathbf{x}}^{-1} = \begin{bmatrix} 1/x_1 & 0 & 0 \\ 0 & 1/x_2 & 0 \\ 0 & 0 & 1/x_3 \end{bmatrix}$$

and the reader can easily check that in this case

$$\hat{\mathbf{x}}\hat{\mathbf{x}}^{-1} = \hat{\mathbf{x}}^{-1}\hat{\mathbf{x}} = \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice also that transposition of a diagonal matrix leaves the matrix unchanged; $\hat{\mathbf{x}}' = \hat{\mathbf{x}}$.

When a diagonal matrix, $\mathbf{D}$, *post*multiplies another matrix, $\mathbf{M}$, the jth element in $\mathbf{D}$, d_j, multiplies all of the elements in the jth *column* of $\mathbf{M}$, and when a diagonal matrix *pre*multiplies $\mathbf{M}$, d_j multiplies all of the elements in the jth *row* of $\mathbf{M}$.

For example,

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 6 & 12 \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} 2d_1 & d_2 & 3d_3 \\ 4d_1 & 6d_2 & 12d_3 \end{bmatrix}$$

and

$$\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 4 & 6 & 12 \end{bmatrix} = \begin{bmatrix} 2d_1 & d_1 & 3d_1 \\ 4d_2 & 6d_2 & 12d_2 \end{bmatrix}$$

Putting the facts about inverses of diagonal matrices together with these observations about pre- and postmultiplication by a diagonal matrix, we see that postmultiplying $\mathbf{M}$ by $\mathbf{D}^{-1}$ will *divide* each element in column j of $\mathbf{M}$ by d_j, and premultiplying $\mathbf{M}$ by $\mathbf{D}^{-1}$ will *divide* each element in row j of $\mathbf{M}$ by d_j.[7]

A.8 Summation Vectors

If $\underset{(m\times n)}{\mathbf{M}}$ is postmultiplied by an n-element column vector of 1's, the results will be an m-element column vector containing the *row sums* of $\mathbf{M}$. If $\mathbf{M}$ is premultiplied by an m-element row vector of 1's, the result will be an n-element row vector containing the *column sums* of $\mathbf{M}$. For example,

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & 6 & 12 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 22 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 4 & 6 & 12 \end{bmatrix} = \begin{bmatrix} 6 & 7 & 15 \end{bmatrix}$$

Usually, a column vector of 1's is denoted by $\mathbf{i}$, and so a corresponding row vector is $\mathbf{i}'$ (sometimes $\mathbf{1}$ or $\mathbf{e}$ is used in place of $\mathbf{i}$). These are *summation vectors*.

A.9 Matrix Inequalities

A more exact characterization of vectors and matrices is often needed for more advanced matrix algebra statements when inequalities are involved. Using vectors as an example, $\mathbf{x} \geq \mathbf{0}$ ($\mathbf{x}$ is "non-negative," meaning $x_i \geq 0$ for all i; note that this allows $\mathbf{x} = \mathbf{0}$), $\mathbf{x} > \mathbf{0}$ ($\mathbf{x}$ is "semipositive," meaning $\mathbf{x} \geq \mathbf{0}$ and $\mathbf{x} \neq \mathbf{0}$; that is, at least one $x_i > 0$) and $\mathbf{x} \gg \mathbf{0}$ ($\mathbf{x}$ is "positive," meaning $x_i > 0$ for all i).[8] The definition of "semipositive" is needed for cases in which $\mathbf{x} = \mathbf{0}$ must be ruled out. The same comparisons can apply to matrices. Also, the same notation can be used to compare any pair of vectors or matrices with the same dimensions – $\mathbf{x} \geq \mathbf{y}$, $\mathbf{x} > \mathbf{y}$, and $\mathbf{x} \gg \mathbf{y}$, and so forth.

[7] This is particularly useful in defining direct input coefficients matrices (technical coefficients matrices) in input–output models.

[8] Alternative notations have been used (for example, in Lancaster, 1968, p. 250, and Takayama, 1985, p. 368). We follow the notation used in Dietzenbacher (1988 and many subsequent publications).

A.10 Partitioned Matrices

Often it is useful to divide a matrix into *submatrices*, especially if there is some logical reason to distinguish some rows and columns from others.[9] This is known as *partitioning* the matrix; the submatrices are sometimes separated by dashed or dotted lines. For example, we might create four submatrices from a 4 × 4 matrix **A**, as

$$\underset{(4\times 4)}{\mathbf{A}} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ & & & \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}$$

In the discussion of linear combinations (section A.6), we viewed **A** as composed of a series of column vectors, $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^{(c)} & \mathbf{a}_2^{(c)} & \cdots & \mathbf{a}_n^{(c)} \end{bmatrix}$. It can equally well be thought of as a "stack" of row vectors, $\mathbf{a}_i^{(r)}$ – namely, $\mathbf{A} = \begin{bmatrix} \mathbf{a}_1^{(r)} \\ \mathbf{a}_2^{(r)} \\ \vdots \\ \mathbf{a}_n^{(r)} \end{bmatrix}$.

A.10.1 Multiplying Partitioned Matrices

If matrices are partitioned so that submatrices are conformable for multiplication, then products of partitioned matrices can be found as products of these submatrices. For example, suppose that in conjunction with **A**, above, we have

$$\underset{(4\times 3)}{\mathbf{B}} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ & & \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}$$

Then

$$\underset{(4\times 3)}{\mathbf{AB}} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11}\mathbf{B}_{11} + \mathbf{A}_{12}\mathbf{B}_{21} & \mathbf{A}_{11}\mathbf{B}_{12} + \mathbf{A}_{12}\mathbf{B}_{22} \\ \mathbf{A}_{21}\mathbf{B}_{11} + \mathbf{A}_{22}\mathbf{B}_{21} & \mathbf{A}_{21}\mathbf{B}_{12} + \mathbf{A}_{22}\mathbf{B}_{22} \end{bmatrix}$$

(The reader can check that all conformability requirements for addition and for multiplication are met.)

[9] An example is in the representation of interregional or multiregional input–output models.

A.10.2 The Inverse of a Partitioned Matrix

Inverses of partitioned matrices play an important role in many input–output representations. Given a partitioned $n \times n$ matrix $\mathbf{A} = \begin{bmatrix} \underset{(p\times p)}{\mathbf{E}} & \underset{[p\times(n-p)]}{\mathbf{F}} \\ \underset{[(n-p)\times p]}{\mathbf{G}} & \underset{[(n-p)\times(n-p)]}{\mathbf{H}} \end{bmatrix}$ (note that $\mathbf{E}$ and $\mathbf{H}$ are square), elements of the inverse can be similarly partitioned as $\mathbf{A}^{-1} = \begin{bmatrix} \underset{(p\times p)}{\mathbf{S}} & \underset{[p\times(n-p)]}{\mathbf{T}} \\ \underset{[(n-p)\times p]}{\mathbf{U}} & \underset{[(n-p)\times(n-p)]}{\mathbf{V}} \end{bmatrix}$. Notice that submatrices in corresponding locations in the original matrix and the inverse have the same dimensions. This means that

$$\begin{bmatrix} \mathbf{E} & \mathbf{F} \\ \mathbf{G} & \mathbf{H} \end{bmatrix}\begin{bmatrix} \mathbf{S} & \mathbf{T} \\ \mathbf{U} & \mathbf{V} \end{bmatrix} = \mathbf{I} = \begin{bmatrix} \underset{(p\times p)}{\mathbf{I}} & \underset{[p\times(n-p)]}{\mathbf{0}} \\ \underset{[(n-p)\times p]}{\mathbf{0}} & \underset{[(n-p)\times(n-p)]}{\mathbf{I}} \end{bmatrix}$$

That is, the product (the identity matrix) can also be partitioned similarly. This matrix statement can be expanded into four *matrix* equations, using the usual rules for matrix multiplication and matrix equality. These matrix equations are

$$\begin{array}{llll} (1)\ \mathbf{ES} + \mathbf{FU} & = \mathbf{I} & (3)\ \mathbf{ET} + \mathbf{FV} & = \mathbf{0} \\ (2)\ \mathbf{GS} + \mathbf{HU} & = \mathbf{0} & (4)\ \mathbf{GT} + \mathbf{HV} & = \mathbf{I} \end{array} \tag{A.4}$$

(The reader can easily check that all matrices are conformable for the multiplications and additions in which they are involved.)

Assume that $\mathbf{E}^{-1}$ can be found; then (1) yields $\mathbf{S} = \mathbf{E}^{-1}(\mathbf{I} - \mathbf{FU})$. Putting this into (2), after considerable rearrangement, gives $\mathbf{U} = -(\mathbf{H} - \mathbf{GE}^{-1}\mathbf{F})^{-1}\mathbf{GE}^{-1}$. The important fact is that $\mathbf{U}$ is expressed as a function of only the known matrices $\mathbf{E}$, $\mathbf{F}$, $\mathbf{G}$, and $\mathbf{H}$; and once $\mathbf{U}$ is found, it can be substituted back into the expression for $\mathbf{S}$. Similarly, equations (3) and (4) can be solved to yield $\mathbf{T} = -\mathbf{E}^{-1}\mathbf{FV}$ and $\mathbf{V} = (\mathbf{H} - \mathbf{GE}^{-1}\mathbf{F})^{-1}$. As with the first pair of equations, $\mathbf{V}$ is a function of known matrices only, and once $\mathbf{V}$ is found, it can be used to find $\mathbf{T}$. Collecting these results,

$$\begin{array}{ll} \mathbf{S} = \mathbf{E}^{-1}(\mathbf{I} - \mathbf{FU}) & \mathbf{T} = -\mathbf{E}^{-1}\mathbf{FV} \\ \mathbf{U} = -\mathbf{VGE}^{-1} & \mathbf{V} = (\mathbf{H} - \mathbf{GE}^{-1}\mathbf{F})^{-1} \end{array} \tag{A.5}$$

In this way, the inverse of an $n \times n$ matrix is found from the inverses of two smaller matrices – $\underset{(p\times p)}{\mathbf{E}}$ and $\underset{[(n-p)\times(n-p)]}{\mathbf{V}}$ – along with a number of matrix multiplications.

An alternative set of results can be derived if one begins with the assumption that $\mathbf{H}^{-1}$ is known. These are

$$\begin{aligned} \mathbf{S} &= (\mathbf{E} - \mathbf{F}\mathbf{H}^{-1}\mathbf{G})^{-1} \quad & \mathbf{T} &= -\mathbf{S}\mathbf{F}\mathbf{H}^{-1} \\ \mathbf{U} &= -\mathbf{H}^{-1}\mathbf{G}\mathbf{S} & \mathbf{V} &= \mathbf{H}^{-1}(\mathbf{I} - \mathbf{G}\mathbf{T}) \end{aligned} \tag{A.6}$$

Again, inverses of two (different) smaller matrices are required – $\underset{(p\times p)}{\mathbf{S}}$ and $\underset{[(n-p)\times(n-p)]}{\mathbf{H}}$.

For $\mathbf{A}$ matrices with particular structures the solution via (A.5) or (A.6) may be particularly simple. Here are several alternatives that arise in input–output models.

1. If $\mathbf{A} = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{H} \end{bmatrix}$ then, using either (A.5) or (A.6), it is easily established that $\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{E}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}^{-1} \end{bmatrix}$; only the two smaller inverses, $\mathbf{E}^{-1}$ and $\mathbf{H}^{-1}$, are needed.
2. In the even more special case when $\mathbf{A} = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$, $\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{E}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$.
3. If $\mathbf{A} = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{G} & \mathbf{I} \end{bmatrix}$, then $\mathbf{A}^{-1} = \begin{bmatrix} \mathbf{E}^{-1} & \mathbf{0} \\ -\mathbf{G}\mathbf{E}^{-1} & \mathbf{I} \end{bmatrix}$.

The interested reader can easily construct additional variations on these special cases.

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Appendix B Reference Input–Output Tables for the United States (1919–2006)

B.1 Introduction

In this appendix we present 16 historical input–output accounts for the United States, aggregated to 7 industry/commodity sectors. The 2005 and 2003 tables are annual updates of the 1997 benchmark input–output tables prepared by the Bureau of Economic Analysis (BEA), formerly the Office of Business Economics, of the US Department of Commerce. The 2006 table was prepared as part of an annual series of updated tables based on the most recently available benchmark or comprehensive survey-based table. The tables for 2002, 1997, 1992, 1987, 1982, 1977, 1972, 1967, 1963, and 1958 are all BEA benchmark tables. The 1947 table was prepared originally by the Bureau of Labor Statistics (BLS) and reworked by Vaccara, Shapiro and Simon (1970) to conform with the industry classification and other conventions of the subsequent tables up to that time. The 1939, 1929, and 1919 tables are derived from Leontief (1941c, 1941b, and 1941a). Additional information on all these tables can be found in the references to the appendix. The original tables were all published at several different levels of aggregation – the tables since 1963 at approximately 85-, 365- and 496-sector classifications.

The transactions table provided in section B.2 are all expressed in millions of US current year dollars. With 1972 and the tables for subsequent years BEA assembled the transactions tables in the industry-by-commodity format as outlined in Chapters 4 and 5, so instead of interindustry transactions, $\mathbf{Z}$, the accounts include *Make* ($\mathbf{V}$) and *Use* ($\mathbf{U}$) tables, vectors of total industry outputs ($\mathbf{Vi}$) and of total commodity outputs ($\mathbf{V'i}$). Also included for each set of commodity-by-industry transactions is the vector of competitive commodity imports so that a domestic use table can be constructed by the methods provided in section 4.7.6.

In section B.3 we provide the matrices of technical coefficients, $\mathbf{A}$, and total requirements, $\mathbf{L}$, for all of the transactions accounts in section B.2. For 1972 and the tables for subsequent years, we employ the industry based technology assumption (see Chapter 5) and construct $\mathbf{A}$ and $\mathbf{L}$ in industry-by-industry terms, i.e., $\mathbf{A} = \mathbf{V}(\widehat{\mathbf{V'i}})^{-1}\mathbf{U}(\widehat{\mathbf{Vi}})^{-1}$ and $\mathbf{L} = (\mathbf{I} - \mathbf{A})^{-1}$. Most of these tables are available at higher levels of sector disaggregation at the internet website, www.bea.gov.

B.2 Transactions Accounts

US Use 2006	*1*	2	*3*	*4*	*5*	*6*	*7*	*Imports*
1 Agriculture	77,871	1	1,917	171,883	286	20,632	1,734	(34,939)
2 Mining	604	57,381	9,207	365,866	110,145	1,509	16,607	(259,547)
3 Construction	1,101	92	1,382	9,220	13,805	82,226	65,135	–
4 Manufacturing	59,558	42,029	374,088	1,634,097	206,055	632,799	276,795	(1,505,557)
5 Trade, Transport & Utilities	25,563	17,053	153,258	509,145	256,402	389,337	122,420	7,832
6 Other Industries	28,859	58,609	181,731	628,804	678,169	3,413,496	564,970	(60,341)
7 Other	258	1,578	2,231	73,184	49,647	102,297	39,150	(231,905)
US Make 2006	*1*	2	*3*	*4*	*5*	*6*	*7*	*Industry Output*
1 Agriculture	316,336	–	–	76	–	2,633	–	319,045
2 Mining	–	412,453	–	24,890	–	–	–	437,343
3 Construction	–	–	1,392,907	–	–	–	–	1,392,907
4 Manufacturing	–	–	–	4,876,809	–	29,214	5,846	4,911,868
5 Trade, Transport & Utilities	–	764	–	–	3,552,762	50	1,081	3,554,657
6 Other Industries	–	614	–	–	157	11,421,370	4,224	11,426,365
7 Other	5,026	1,812	–	3,948	123,486	357,785	2,201,350	2,693,408
Commodity Output	321,361	415,644	1,392,907	4,905,722	3,676,406	11,811,051	2,212,501	24,735,592

US Use 2002	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Imports*
1 Agriculture	72,028	361	2,763	145,716	997	7,406	1,653	(24,985)
2 Mining	703	8,611	9,234	140,728	63,095	2,224	7,909	(93,985)
3 Construction	1,168	6,621	718	12,208	15,697	74,277	42,456	–
4 Manufacturing	40,495	17,122	253,489	1,343,085	132,739	424,488	198,600	(1,014,741)
5 Trade, Transport & Utilities	23,899	12,207	103,373	360,139	220,365	239,088	97,013	6,614
6 Other Industries	37,151	46,606	152,787	543,989	508,815	2,513,262	432,413	(42,186)
7 Other	154	1,008	410	31,210	52,162	79,038	26,458	(166,101)
US Make 2002	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Industry Output*
1 Agriculture	269,896	–	–	42	–	1,243	–	271,182
2 Mining	–	169,188	–	15,368	–	–	–	184,556
3 Construction	–	–	1,032,363	–	–	–	–	1,032,363
4 Manufacturing	–	807	–	3,787,231	63	27,055	2,205	3,817,360
5 Trade, Transport & Utilities	–	26	–	–	2,682,398	51	961	2,683,436
6 Other Industries	–	46	–	–	21	9,114,384	487	9,114,939
7 Other	1,916	–	–	–	90,941	227,437	1,755,904	2,076,198
Commodity Output	271,812	170,068	1,032,363	3,802,641	2,773,423	9,370,171	1,759,556	19,180,034

US Use 1997	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Imports*
1 Agriculture	74,938	15	1,121	150,341	2,752	12,548	863	(23,123)
2 Mining	370	19,461	4,281	112,513	53,778	1,167	4,052	(64,216)
3 Construction	1,122	29	832	7,499	11,758	39,089	11,570	–
4 Manufacturing	49,806	19,275	178,903	1,362,660	169,915	372,193	48,133	(765,454)
5 Trade, Transport & Utilities	21,650	11,125	76,056	380,272	199,004	193,280	31,603	6,337
6 Other Industries	32,004	43,442	104,479	464,900	525,227	1,400,889	104,099	(16,682)
7 Other	999	2,573	3,666	52,691	40,322	100,066	17,903	(126,610)
US Make 1997	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Industry Output*
1 Agriculture	285,067	–	–	63	–	1,149	–	286,280
2 Mining	–	158,239	–	9,752	–	–	–	167,991
3 Construction	–	–	754,091	–	–	–	–	754,091
4 Manufacturing	–	727	–	3,718,807	259	30,289	3,669	3,753,751
5 Trade, Transport & Utilities	–	381	–	–	2,272,268	40	729	2,273,418
6 Other Industries	–	410	–	1,251	86	6,477,777	1,821	6,481,345
7 Other	–	–	–	55	78,682	16,984	1,050,280	1,146,001
Commodity Output	285,067	159,757	754,091	3,729,928	2,351,295	6,526,240	1,056,499	14,862,876

US Use 1992	*1*	2	*3*	*4*	*5*	*6*	*7*	*Imports*
1 Agriculture	55,569	43	4,027	123,104	966	13,575	317	(14,601)
2 Mining	298	25,985	5,445	94,010	54,562	38	2,688	(43,527)
3 Construction	2,895	2,670	594	18,133	41,597	72,577	21,152	–
4 Manufacturing	39,370	11,848	202,588	1,019,103	106,382	247,947	12,272	(485,599)
5 Trade, Transport & Utilities	22,226	12,911	73,711	320,070	213,175	165,013	15,043	10,385
6 Other Industries	22,004	24,412	79,571	216,081	319,142	814,529	12,730	(5,439)
7 Other	169	1,187	772	24,693	32,782	36,162	2,604	(92,856)

US Make 1992	*1*	2	*3*	*4*	*5*	*6*	*7*	*Industry Output*
1 Agriculture	235,591	–	–	1,022	11	1,038	–	237,662
2 Mining	–	147,001	–	9,716	–	–	–	156,717
3 Construction	–	–	679,330	–	–	–	–	679,330
4 Manufacturing	–	561	–	2,879,654	43	69,509	1,536	2,951,303
5 Trade, Transport & Utilities	–	–	–	–	1,965,701	28,838	542	1,995,081
6 Other Industries	–	–	–	8	53	3,881,700	521	3,882,282
7 Other	–	–	–	37	61,370	12,433	846,432	920,272
Commodity Output	235,591	147,562	679,330	2,890,437	2,027,178	3,993,518	849,031	10,822,647

US Use 1987	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Imports*
1 Agriculture	61,442	4	3,491	97,047	288	10,167	162	(6,924)
2 Mining	262	6,960	4,836	93,975	29,841	46	2,147	(31,112)
3 Construction	1,540	2,242	382	14,169	31,502	57,645	16,664	–
4 Manufacturing	27,760	8,837	181,444	841,929	76,950	210,372	10,378	(377,045)
5 Trade, Transport & Utilities	17,504	7,904	65,539	241,994	180,061	121,247	15,144	10,203
6 Other Industries	19,721	19,945	71,739	142,404	229,671	594,509	7,311	(4,800)
7 Other	203	898	383	27,005	18,544	26,417	1,566	(80,764)
US Make 1987	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Industry Output*
1 Agriculture	198,948	–	–	3,976	807	152	–	203,883
2 Mining	–	111,533	–	5,950	11,968	–	–	129,451
3 Construction	–	–	618,813	–	–	–	–	618,813
4 Manufacturing	–	554	–	2,401,769	26	57,944	2,078	2,462,371
5 Trade, Transport & Utilities	–	59	–	242	1,503,189	28,109	317	1,531,917
6 Other Industries	–	–	–	–	–	2,656,807	217	2,657,024
7 Other	–	–	–	33	50,383	12,241	508,900	571,557
Commodity Output	198,948	112,146	618,813	2,411,970	1,566,373	2,755,254	511,512	8,175,016

US Use 1982	*1*	2	*3*	*4*	*5*	*6*	*7*	*Imports*
1 Agriculture	55,652	6	486	86,494	215	6,824	2,124	(4,153)
2 Mining	254	8,956	2,578	147,741	53,080	45	1,769	(46,481)
3 Construction	1,812	4,749	452	9,623	21,428	38,372	10,177	–
4 Manufacturing	35,214	11,677	140,569	680,630	82,020	136,359	8,890	(192,754)
5 Trade, Transport & Utilities	13,712	7,999	43,943	208,400	143,047	73,291	16,777	2,126
6 Other Industries	16,483	31,102	44,475	108,534	138,442	282,916	3,388	(1,200)
7 Other	170	499	281	21,885	12,993	14,227	1,419	(89,450)
US Make 1982	*1*	2	*3*	*4*	*5*	*6*	*7*	*Industry Output*
1 Agriculture	190,104	–	–	4,262	921	103	–	195,390
2 Mining	–	178,410	–	8,970	5,235	–	–	192,615
3 Construction	–	–	438,791	–	–	–	–	438,791
4 Manufacturing	–	863	–	1,899,789	27	33,459	2,163	1,936,302
5 Trade, Transport & Utilities	–	117	–	557	1,091,181	14,614	193	1,106,662
6 Other Industries	–	–	–	–	–	1,609,434	174	1,609,608
7 Other	–	–	–	27	36,687	6,931	416,760	460,404
Commodity Output	190,104	179,391	438,791	1,913,604	1,134,051	1,664,541	419,291	5,939,773

US Use 1977	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Imports*
1 Agriculture	31, 868	8	658	62, 286	651	4, 470	163	(2, 712)
2 Mining	163	5, 562	2, 044	76, 343	19, 295	22	1, 202	(37, 565)
3 Construction	1, 383	2, 923	304	8, 706	12, 850	26, 389	4, 971	–
4 Manufacturing	26, 128	7, 261	98, 570	518, 309	42, 072	77, 391	3, 200	(110, 057)
5 Trade, Transport & Utilities	9, 459	3, 771	31, 125	118, 784	73, 176	45, 091	6, 026	2, 844
6 Other Industries	11, 594	8, 068	19, 821	67, 659	84, 188	140, 980	2, 215	(672)
7 Other	66	243	199	15, 698	6, 975	6, 231	875	(35, 991)
US Make 1977	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Industry Output*
1 Agriculture	125, 829	–	–	3, 720	15	100	–	129, 663
2 Mining	–	71, 311	–	4, 414	2, 307	–	–	78, 031
3 Construction	–	–	264, 334	–	–	–	–	264, 334
4 Manufacturing	9	441	–	1, 333, 361	44	18, 947	2, 182	1, 354, 983
5 Trade, Transport & Utilities	34	86	–	173	669, 210	7, 946	122	677, 571
6 Other Industries	–	–	–	–	–	902, 088	–	902, 088
7 Other	–	–	–	66	18, 710	4, 655	233, 893	257, 323
Commodity Output	125, 872	71, 837	264, 334	1, 341, 733	690, 285	933, 735	236, 196	3, 663, 993

US Use 1972	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Imports*
1 Agriculture	26,328	0	263	40,284	312	2,592	196	(2,043)
2 Mining	136	1,654	1,407	22,154	6,027	30	313	(4,071)
3 Construction	583	858	47	3,245	5,875	13,715	2,672	–
4 Manufacturing	11,926	2,753	58,464	287,182	14,243	44,557	1,215	(50,799)
5 Trade, Transport & Utilities	5,259	1,457	17,750	61,248	37,726	22,923	3,337	1,426
6 Other Industries	7,533	4,638	11,813	46,674	45,291	87,792	1,758	(218)
7 Other	27	143	146	7,014	3,299	4,285	432	(20,494)
US Make 1972	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Industry Output*
1 Agriculture	81,264	–	–	2,529	97	65	–	83,955
2 Mining	–	28,864	–	1,224	298	–	–	30,386
3 Construction	–	–	165,998	–	–	–	–	165,998
4 Manufacturing	5	178	–	746,059	21	13,546	1,384	761,194
5 Trade, Transport & Utilities	–	33	–	85	371,734	5,491	46	377,389
6 Other Industries	–	–	–	–	–	522,215	–	522,215
7 Other	–	–	–	28	11,756	433	148,991	161,207
Commodity Output	81,269	29,076	165,998	749,925	383,905	541,751	150,420	2,102,343

US Transactions 1967	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Total Output*
1 Agriculture	19,237	–	263	31,390	2,683	331	588	63,793
2 Mining	138	1,285	930	17,281	3,712	17	145	24,959
3 Construction	603	572	30	2,560	10,837	1,324	1,771	103,280
4 Manufacturing	8,676	2,334	37,534	240,420	18,276	23,676	8,125	617,368
5 Trade, Transport & Utilities	7,812	4,307	12,610	51,495	62,600	21,596	8,818	437,029
6 Other Industries	1,776	569	5,433	20,058	23,962	10,414	2,329	150,156
7 Other	1,169	2,402	906	25,170	19,405	4,293	1,508	33,141
								1,429,726

US Transactions 1963	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Total Output*
1 Agriculture	17,818	–	326	26,753	2,819	60	790	57,474
2 Mining	128	1,138	737	14,635	2,756	16	188	20,570
3 Construction	567	416	25	1,402	10,158	954	1,349	85,313
4 Manufacturing	7,646	1,670	31,562	185,758	12,962	15,417	6,312	466,415
5 Trade, Transport & Utilities	6,060	3,981	11,345	37,657	45,682	15,913	6,300	322,878
6 Other Industries	1,414	295	3,657	12,444	15,273	6,219	1,622	103,038
7 Other	1,139	2,020	639	16,798	14,175	3,500	931	24,518

US Transactions 1958	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Total Output*
1 Agriculture	15,577	–	237	25,077	2,362	24	733	52,732
2 Mining	102	1,131	757	13,347	1,909	41	148	18,354
3 Construction	613	11	8	751	8,894	972	1,206	69,291
4 Manufacturing	6,105	1,458	26,528	135,698	10,510	17,571	5,202	356,877
5 Trade, Transport & Utilities	5,919	2,956	9,477	31,283	35,336	10,842	4,574	248,909
6 Other Industries	1,214	426	2,968	8,760	11,235	5,176	1,266	78,187
7 Other	1,091	1,959	379	13,991	10,691	2,280	733	17,724
								842,074

US Transactions 1947	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Total Output*
1 Agriculture	15,285	–	92	23,946	1,800	168	111	46,712
2 Mining	47	810	277	6,601	1,205	40	25	9,700
3 Construction	568	15	7	510	5,428	259	503	29,331
4 Manufacturing	4,434	951	11,132	73,771	6,441	5,463	2,153	197,633
5 Trade, Transport & Utilities	5,044	1,058	4,289	14,522	14,945	4,009	2,397	123,342
6 Other Industries	380	85	1,280	3,185	4,768	2,109	341	31,967
7 Other	51	45	206	4,586	4,797	1,081	114	7,877
								446,562

US Transactions 1939	Agriculture	Mining	Construction	Manufacturing	Trade, Transport & Utilities	Other Industries	Other	Total Outputs
Agriculture	1,537	–	890	6,052	683	333	768	14,316
Mining	46	2,002	2,037	2,222	806	46	672	8,987
Construction	307	76	–	868	993	1,448	4,572	11,291
Manufacturing	2,280	504	2,469	17,503	5,990	3,927	10,419	75,462
Trade, Transport & Utilities	3,367	2,314	343	20,021	3,681	38	1,524	32,614
Other Industries	552	26	3	929	709	495	1,388	22,780
Other	850	1,585	94	13,293	7,967	3,945	1,414	33,161
								198,611

US Transactions 1929	Agriculture	Mining	Construction	Manufacturing	Trade, Transport & Utilities	Other Industries	Other	Total Outputs
Agriculture	6,182	45	306	5,730	195	15	871	17,969
Mining	16	630	1,180	3,350	598	22	824	7,931
Construction	11	36	–	502	291	–	3,508	6,971
Manufacturing	1,706	599	1,703	16,830	2,543	141	12,468	64,985
Trade, Transport & Utilities	1,079	1,563	–	1,819	225	32	3,316	11,625
Other Industries	39	–	102	46	–	51	683	2,238
Other	1,214	2,302	822	17,597	2,315	1,167	–	48,836
								160,555

US Transactions 1919	Agriculture	Mining	Construction	Manufacturing	Trade, Transport & Utilities	Other Industries	Other	Total Outputs
Agriculture	9,845	51	252	8,453	104	14	1,441	24,555
Mining	15	402	622	2,189	656	22	617	5,613
Construction	–	–	–	–	–	–	2,108	3,142
Manufacturing	1,831	389	1,002	13,087	2,046	6	12,180	57,533
Trade, Transport & Utilities	1,083	1,105	–	909	128	16	2,254	8,088
Other Industries	21	–	86	47	–	–	391	1,776
Other	860	2,251	330	16,846	1,674	911	–	36,257
								136,964

B.3 Matrices of Technical Coefficients and Total Requirements

US Technical Coefficients 2006	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	0.2403	0.0000	0.0014	0.0345	0.0001	0.0018	0.0007
2 Mining	0.0028	0.1307	0.0079	0.0756	0.0310	0.0004	0.0066
3 Construction	0.0035	0.0002	0.0010	0.0019	0.0039	0.0072	0.0242
4 Manufacturing	0.1858	0.0959	0.2673	0.3311	0.0581	0.0558	0.1027
5 Trade, Transport & Utilities	0.0774	0.0379	0.1063	0.1003	0.0698	0.0329	0.0439
6 Services	0.0875	0.1298	0.1262	0.1239	0.1846	0.2889	0.2029
7 Other	0.0102	0.0096	0.0095	0.0233	0.0223	0.0192	0.0225
US Total Requirements 2006	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	1.3365	0.0101	0.0238	0.0735	0.0075	0.0101	0.0118
2 Mining	0.0482	1.1716	0.0566	0.1470	0.0525	0.0162	0.0306
3 Construction	0.0091	0.0036	1.0058	0.0081	0.0079	0.0120	0.0286
4 Manufacturing	0.4275	0.2064	0.4650	1.5972	0.1424	0.1438	0.2173
5 Trade, Transport & Utilities	0.1728	0.0823	0.1826	0.2013	1.1076	0.0719	0.0911
6 Services	0.3041	0.2799	0.3294	0.3829	0.3344	1.4661	0.3698
7 Other	0.0346	0.0239	0.0323	0.0525	0.0359	0.0342	1.0382

US Technical Coefficients 2002	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	0.2638	0.0020	0.0027	0.0379	0.0004	0.0008	0.0008
2 Mining	0.0032	0.0468	0.0099	0.0381	0.0236	0.0004	0.0042
3 Construction	0.0043	0.0359	0.0007	0.0032	0.0058	0.0081	0.0204
4 Manufacturing	0.1491	0.0934	0.2450	0.3510	0.0500	0.0472	0.0959
5 Trade, Transport & Utilities	0.0852	0.0640	0.0968	0.0913	0.0794	0.0254	0.0452
6 Services	0.1333	0.2457	0.1440	0.1386	0.1844	0.2682	0.2026
7 Other	0.0087	0.0138	0.0073	0.0150	0.0267	0.0162	0.0193
US Total Requirements 2002	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	1.3780	0.0149	0.0265	0.0846	0.0077	0.0079	0.0120
2 Mining	0.0243	1.0615	0.0322	0.0704	0.0331	0.0070	0.0151
3 Construction	0.0128	0.0438	1.0076	0.0133	0.0116	0.0131	0.0257
4 Manufacturing	0.3712	0.2182	0.4325	1.6253	0.1263	0.1190	0.1996
5 Trade, Transport & Utilities	0.1795	0.1151	0.1643	0.1884	1.1140	0.0548	0.0851
6 Services	0.3854	0.4461	0.3441	0.4077	0.3301	1.4167	0.3571
7 Other	0.0295	0.0292	0.0249	0.0385	0.0383	0.0270	1.0315

US Technical Coefficients 1997	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	0.2618	0.0001	0.0015	0.0401	0.0013	0.0020	0.0008
2 Mining	0.0017	0.1150	0.0062	0.0306	0.0236	0.0003	0.0036
3 Construction	0.0039	0.0002	0.0011	0.0020	0.0052	0.0060	0.0101
4 Manufacturing	0.1740	0.1162	0.2372	0.3627	0.0758	0.0583	0.0424
5 Trade, Transport & Utilities	0.0731	0.0643	0.0975	0.0980	0.0847	0.0288	0.0267
6 Services	0.1110	0.2570	0.1376	0.1232	0.2294	0.2146	0.0902
7 Other	0.0063	0.0181	0.0086	0.0177	0.0212	0.0169	0.0167

US Total Requirements 1997	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	1.3795	0.0166	0.0269	0.0920	0.0130	0.0111	0.0068
2 Mining	0.0226	1.1433	0.0270	0.0638	0.0369	0.0070	0.0089
3 Construction	0.0094	0.0046	1.0051	0.0074	0.0089	0.0089	0.0117
4 Manufacturing	0.4361	0.2745	0.4365	1.6687	0.1849	0.1373	0.0953
5 Trade, Transport & Utilities	0.1704	0.1271	0.1683	0.2038	1.1298	0.0594	0.0472
6 Services	0.3253	0.4617	0.3094	0.3612	0.3785	1.3206	0.1521
7 Other	0.0264	0.0368	0.0262	0.0424	0.0350	0.0267	1.0226

US Technical Coefficients 1992	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	0.2339	0.0003	0.0061	0.0419	0.0005	0.0036	0.0004
2 Mining	0.0018	0.1654	0.0090	0.0329	0.0274	0.0002	0.0030
3 Construction	0.0122	0.0170	0.0009	0.0061	0.0208	0.0187	0.0230
4 Manufacturing	0.1667	0.0787	0.2992	0.3454	0.0560	0.0673	0.0135
5 Trade, Transport & Utilities	0.0914	0.0810	0.1061	0.1057	0.1048	0.0427	0.0160
6 Services	0.0900	0.1514	0.1139	0.0712	0.1555	0.2039	0.0134
7 Other	0.0038	0.0105	0.0048	0.0119	0.0201	0.0112	0.0034

US Total Requirements 1992	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	1.3284	0.0135	0.0378	0.0893	0.0104	0.0150	0.0030
2 Mining	0.0254	1.2119	0.0382	0.0712	0.0443	0.0098	0.0063
3 Construction	0.0280	0.0311	1.0143	0.0214	0.0313	0.0278	0.0247
4 Manufacturing	0.3955	0.2062	0.5204	1.6101	0.1479	0.1587	0.0391
5 Trade, Transport & Utilities	0.1993	0.1531	0.2008	0.2189	1.1559	0.0866	0.0278
6 Services	0.2336	0.2852	0.2427	0.2139	0.2536	1.2951	0.0310
7 Other	0.0169	0.0218	0.0184	0.0272	0.0286	0.0185	1.0049

US Technical Coefficients 1987	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	0.3016	0.0002	0.0062	0.0400	0.0003	0.0040	0.0003
2 Mining	0.0023	0.0541	0.0093	0.0396	0.0204	0.0006	0.0040
3 Construction	0.0076	0.0173	0.0006	0.0058	0.0206	0.0217	0.0292
4 Manufacturing	0.1376	0.0715	0.2945	0.3419	0.0533	0.0836	0.0184
5 Trade, Transport & Utilities	0.0834	0.0602	0.1029	0.0950	0.1144	0.0461	0.0256
6 Services	0.0933	0.1486	0.1118	0.0558	0.1446	0.2158	0.0123
7 Other	0.0042	0.0095	0.0045	0.0143	0.0165	0.0124	0.0036

US Total Requirements 1987	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	1.4545	0.0117	0.0397	0.0927	0.0105	0.0191	0.0039
2 Mining	0.0233	1.0672	0.0358	0.0716	0.0317	0.0115	0.0076
3 Construction	0.0233	0.0275	1.0139	0.0199	0.0313	0.0326	0.0314
4 Manufacturing	0.3641	0.1704	0.5108	1.5950	0.1448	0.1954	0.0513
5 Trade, Transport & Utilities	0.1934	0.1081	0.1911	0.1970	1.1645	0.0964	0.0408
6 Services	0.2426	0.2398	0.2279	0.1776	0.2371	1.3162	0.0334
7 Other	0.0179	0.0176	0.0185	0.0296	0.0248	0.0211	1.0057

US Technical Coefficients 1982	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	0.2853	0.0002	0.0019	0.0455	0.0005	0.0045	0.0047
2 Mining	0.0025	0.0467	0.0078	0.0780	0.0486	0.0006	0.0041
3 Construction	0.0093	0.0247	0.0010	0.0050	0.0194	0.0238	0.0221
4 Manufacturing	0.1806	0.0637	0.3201	0.3505	0.0764	0.0877	0.0194
5 Trade, Transport & Utilities	0.0683	0.0414	0.0973	0.1042	0.1255	0.0454	0.0351
6 Services	0.0816	0.1561	0.0980	0.0542	0.1210	0.1700	0.0071
7 Other	0.0035	0.0046	0.0043	0.0150	0.0164	0.0110	0.0043

US Total Requirements 1982	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	1.4305	0.0126	0.0405	0.1065	0.0148	0.0211	0.0104
2 Mining	0.0517	1.0677	0.0653	0.1469	0.0771	0.0229	0.0118
3 Construction	0.0259	0.0347	1.0149	0.0222	0.0315	0.0337	0.0246
4 Manufacturing	0.4663	0.1670	0.5691	1.6465	0.1953	0.2043	0.0559
5 Trade, Transport & Utilities	0.1843	0.0877	0.1983	0.2246	1.1868	0.0961	0.0526
6 Services	0.2108	0.2300	0.2023	0.1812	0.2056	1.2426	0.0261
7 Other	0.0177	0.0116	0.0189	0.0316	0.0253	0.0187	1.0065

US Technical Coefficients 1977	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	0.2463	0.0004	0.0035	0.0470	0.0011	0.0052	0.0007
2 Mining	0.0022	0.0712	0.0093	0.0575	0.0288	0.0005	0.0048
3 Construction	0.0107	0.0375	0.0011	0.0064	0.0190	0.0293	0.0193
4 Manufacturing	0.2021	0.0950	0.3722	0.3816	0.0645	0.0885	0.0126
5 Trade, Transport & Utilities	0.0716	0.0478	0.1148	0.0855	0.1058	0.0498	0.0228
6 Services	0.0864	0.0999	0.0724	0.0482	0.1200	0.1510	0.0083
7 Other	0.0029	0.0049	0.0043	0.0141	0.0137	0.0090	0.0040
US Total Requirements 1977	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	1.3607	0.0169	0.0490	0.1093	0.0143	0.0223	0.0038
2 Mining	0.0410	1.0951	0.0594	0.1137	0.0473	0.0176	0.0091
3 Construction	0.0289	0.0496	1.0176	0.0252	0.0307	0.0399	0.0213
4 Manufacturing	0.5156	0.2374	0.6842	1.7280	0.1774	0.2178	0.0425
5 Trade, Transport & Utilities	0.1757	0.0983	0.2131	0.1926	1.1536	0.0966	0.0344
6 Services	0.2000	0.1623	0.1680	0.1524	0.1830	1.2118	0.0204
7 Other	0.0159	0.0118	0.0190	0.0295	0.0205	0.0157	1.0055

US Technical Coefficients 1972	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	0.3141	0.0003	0.0028	0.0542	0.0010	0.0053	0.0012
2 Mining	0.0019	0.0542	0.0091	0.0296	0.0160	0.0002	0.0020
3 Construction	0.0069	0.0282	0.0003	0.0043	0.0156	0.0263	0.0166
4 Manufacturing	0.1436	0.0943	0.3522	0.3771	0.0407	0.0892	0.0078
5 Trade, Transport & Utilities	0.0616	0.0481	0.1043	0.0786	0.0980	0.0442	0.0202
6 Services	0.0865	0.1471	0.0686	0.0591	0.1157	0.1621	0.0105
7 Other	0.0023	0.0063	0.0042	0.0117	0.0118	0.0096	0.0033
US Total Requirements 1972	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	1.4913	0.0204	0.0552	0.1353	0.0125	0.0262	0.0044
2 Mining	0.0182	1.0665	0.0326	0.0563	0.0232	0.0087	0.0037
3 Construction	0.0206	0.0387	1.0117	0.0172	0.0237	0.0351	0.0179
4 Manufacturing	0.3979	0.2261	0.6256	1.6905	0.1187	0.2087	0.0292
5 Trade, Transport & Utilities	0.1503	0.0936	0.1854	0.1704	1.1327	0.0850	0.0286
6 Services	0.2078	0.2216	0.1642	0.1683	0.1723	1.2272	0.0212
7 Other	0.0121	0.0128	0.0157	0.0241	0.0168	0.0155	1.0043

US Technical Coefficients 1967	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	0.3016	0.0000	0.0025	0.0508	0.0061	0.0022	0.0177
2 Mining	0.0022	0.0515	0.0090	0.0280	0.0085	0.0001	0.0044
3 Construction	0.0095	0.0229	0.0003	0.0041	0.0248	0.0088	0.0534
4 Manufacturing	0.1360	0.0935	0.3634	0.3894	0.0418	0.1577	0.2452
5 Trade, Transport & Utilities	0.1225	0.1726	0.1221	0.0834	0.1432	0.1438	0.2661
6 Services	0.0278	0.0228	0.0526	0.0325	0.0548	0.0694	0.0703
7 Other	0.0183	0.0962	0.0088	0.0408	0.0444	0.0286	0.0455
US Total Requirements 1967	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	1.4663	0.0275	0.0581	0.1339	0.0250	0.0329	0.0744
2 Mining	0.0187	1.0666	0.0328	0.0554	0.0166	0.0133	0.0269
3 Construction	0.0274	0.0414	1.0150	0.0222	0.0363	0.0214	0.0749
4 Manufacturing	0.4126	0.2877	0.6904	1.7694	0.1643	0.3504	0.5737
5 Trade, Transport & Utilities	0.2911	0.3066	0.2624	0.2515	1.2314	0.2499	0.4478
6 Services	0.0822	0.0677	0.1039	0.0906	0.0867	1.1087	0.1367
7 Other	0.0639	0.1370	0.0586	0.0984	0.0694	0.0620	1.1019

US Technical Coefficients 1963	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	0.3100	0.0000	0.0038	0.0574	0.0087	0.0006	0.0322
2 Mining	0.0022	0.0553	0.0086	0.0314	0.0085	0.0002	0.0077
3 Construction	0.0099	0.0202	0.0003	0.0030	0.0315	0.0093	0.0550
4 Manufacturing	0.1330	0.0812	0.3700	0.3983	0.0401	0.1496	0.2574
5 Trade, Transport & Utilities	0.1054	0.1935	0.1330	0.0807	0.1415	0.1544	0.2570
6 Services	0.0246	0.0143	0.0429	0.0267	0.0473	0.0604	0.0662
7 Other	0.0198	0.0982	0.0075	0.0360	0.0439	0.0340	0.0380
US Total Requirements 1963	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	1.4900	0.0334	0.0706	0.1565	0.0329	0.0358	0.1073
2 Mining	0.0203	1.0721	0.0359	0.0631	0.0177	0.0147	0.0339
3 Construction	0.0287	0.0412	1.0163	0.0217	0.0443	0.0236	0.0787
4 Manufacturing	0.4116	0.2713	0.7085	1.7924	0.1664	0.3419	0.6041
5 Trade, Transport & Utilities	0.2621	0.3284	0.2729	0.2462	1.2290	0.2600	0.4390
6 Services	0.0699	0.0530	0.0865	0.0757	0.0745	1.0939	0.1231
7 Other	0.0628	0.1375	0.0551	0.0908	0.0678	0.0657	1.0928

US Technical Coefficients 1958	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	0.2954	0.0000	0.0034	0.0703	0.0095	0.0003	0.0414
2 Mining	0.0019	0.0616	0.0109	0.0374	0.0077	0.0005	0.0084
3 Construction	0.0116	0.0006	0.0001	0.0021	0.0357	0.0124	0.0680
4 Manufacturing	0.1158	0.0794	0.3828	0.3802	0.0422	0.2247	0.2935
5 Trade, Transport & Utilities	0.1122	0.1611	0.1368	0.0877	0.1420	0.1387	0.2581
6 Services	0.0230	0.0232	0.0428	0.0245	0.0451	0.0662	0.0714
7 Other	0.0207	0.1067	0.0055	0.0392	0.0430	0.0292	0.0414
US Total Requirements 1958	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	1.4634	0.0399	0.0850	0.1853	0.0392	0.0565	0.1410
2 Mining	0.0208	1.0806	0.0440	0.0740	0.0187	0.0231	0.0429
3 Construction	0.0325	0.0230	1.0180	0.0230	0.0505	0.0296	0.0967
4 Manufacturing	0.3680	0.2723	0.7290	1.7633	0.1839	0.4834	0.6954
5 Trade, Transport & Utilities	0.2677	0.2938	0.2877	0.2635	1.2333	0.2652	0.4670
6 Services	0.0655	0.0615	0.0874	0.0741	0.0734	1.1050	0.1343
7 Other	0.0631	0.1475	0.0579	0.0985	0.0682	0.0692	1.1050

US Technical Coefficients 1947	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	0.3272	0.0000	0.0031	0.1212	0.0146	0.0053	0.0141
2 Mining	0.0010	0.0835	0.0094	0.0334	0.0098	0.0013	0.0032
3 Construction	0.0122	0.0015	0.0002	0.0026	0.0440	0.0081	0.0639
4 Manufacturing	0.0949	0.0980	0.3795	0.3733	0.0522	0.1709	0.2733
5 Trade, Transport & Utilities	0.1080	0.1091	0.1462	0.0735	0.1212	0.1254	0.3043
6 Services	0.0081	0.0088	0.0436	0.0161	0.0387	0.0660	0.0433
7 Other	0.0011	0.0046	0.0070	0.0232	0.0389	0.0338	0.0145
US Total Requirements 1947	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	1.5446	0.0429	0.1388	0.3160	0.0617	0.0812	0.1415
2 Mining	0.0152	1.1010	0.0394	0.0658	0.0206	0.0179	0.0317
3 Construction	0.0312	0.0116	1.0189	0.0217	0.0581	0.0241	0.0915
4 Manufacturing	0.2913	0.2141	0.7037	1.7200	0.1875	0.3697	0.6016
5 Trade, Transport & Utilities	0.2322	0.1712	0.2739	0.2187	1.2006	0.2219	0.4627
6 Services	0.0306	0.0229	0.0744	0.0455	0.0589	1.0908	0.0840
7 Other	0.0191	0.0179	0.0375	0.0515	0.0544	0.0552	1.0510

US Technical Coefficients 1939	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	0.1074	0.0000	0.0788	0.0802	0.0209	0.0146	0.0232
2 Mining	0.0032	0.2228	0.1804	0.0294	0.0247	0.0020	0.0203
3 Construction	0.0214	0.0085	0.0000	0.0115	0.0304	0.0636	0.1379
4 Manufacturing	0.1593	0.0561	0.2187	0.2319	0.1837	0.1724	0.3142
5 Trade, Transport & Utilities	0.2352	0.2575	0.0304	0.2653	0.1129	0.0017	0.0460
6 Other Industries	0.0386	0.0029	0.0003	0.0123	0.0217	0.0217	0.0419
7 Other	0.0594	0.1764	0.0083	0.1762	0.2443	0.1732	0.0426

US Total Requirements 1939	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	1.2035	0.0845	0.1590	0.2025	0.1160	0.0873	0.1296
2 Mining	0.0765	1.3652	0.2859	0.1332	0.1131	0.0682	0.1241
3 Construction	0.1021	0.1118	1.0615	0.1260	0.1293	0.1305	0.2110
4 Manufacturing	0.5716	0.5152	0.5543	1.7869	0.6335	0.4931	0.7430
5 Trade, Transport & Utilities	0.5338	0.6034	0.3412	0.6596	1.4116	0.2152	0.3684
6 Other Industries	0.0815	0.0493	0.0334	0.0690	0.0664	1.0521	0.0797
7 Other	0.3458	0.5154	0.2668	0.5477	0.5179	0.3551	1.3224

US Technical Coefficients 1929	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	0.3440	0.0057	0.0439	0.0882	0.0168	0.0067	0.0178
2 Mining	0.0009	0.0794	0.1693	0.0516	0.0514	0.0098	0.0169
3 Construction	0.0006	0.0045	0.0000	0.0077	0.0250	0.0000	0.0718
4 Manufacturing	0.0949	0.0755	0.2443	0.2590	0.2188	0.0630	0.2553
5 Trade, Transport & Utilities	0.0600	0.1971	0.0000	0.0280	0.0194	0.0143	0.0679
6 Other Industries	0.0022	0.0000	0.0146	0.0007	0.0000	0.0228	0.0140
7 Other	0.0676	0.2903	0.1179	0.2708	0.1991	0.5214	0.0000

US Total Requirements 1929	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	1.5817	0.0895	0.1579	0.2411	0.1122	0.0884	0.1115
2 Mining	0.0367	1.1442	0.2339	0.1180	0.1082	0.0610	0.0751
3 Construction	0.0244	0.0525	1.0351	0.0542	0.0609	0.0555	0.0944
4 Manufacturing	0.3369	0.4072	0.5472	1.6288	0.5076	0.3899	0.5079
5 Trade, Transport & Utilities	0.1307	0.2829	0.0989	0.1216	1.0910	0.0919	0.1206
6 Other Industries	0.0076	0.0087	0.0217	0.0101	0.0073	1.0346	0.0194
7 Other	0.2416	0.5154	0.3798	0.5275	0.4047	0.6936	1.2121

US Technical Coefficients 1919	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	0.4009	0.0091	0.0802	0.1469	0.0129	0.0079	0.0397
2 Mining	0.0006	0.0716	0.1980	0.0380	0.0811	0.0124	0.0170
3 Construction	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0581
4 Manufacturing	0.0746	0.0693	0.3189	0.2275	0.2530	0.0034	0.3359
5 Trade, Transport & Utilities	0.0441	0.1969	0.0000	0.0158	0.0158	0.0090	0.0622
6 Other Industries	0.0009	0.0000	0.0274	0.0008	0.0000	0.0000	0.0108
7 Other	0.0350	0.4010	0.1050	0.2928	0.2070	0.5130	0.0000
US Total Requirements 1919	*1*	*2*	*3*	*4*	*5*	*6*	*7*
1 Agriculture	1.7558	0.2066	0.3555	0.4466	0.2091	0.1525	0.2586
2 Mining	0.0273	1.1461	0.2684	0.0935	0.1349	0.0552	0.0766
3 Construction	0.0105	0.0403	1.0280	0.0326	0.0274	0.0389	0.0740
4 Manufacturing	0.2876	0.5361	0.7294	1.6363	0.6061	0.3554	0.6541
5 Trade, Transport & Utilities	0.1002	0.2910	0.1121	0.1005	1.0920	0.0751	0.1179
6 Other Industries	0.0040	0.0092	0.0342	0.0087	0.0065	1.0087	0.0165
7 Other	0.1805	0.6930	0.4824	0.5609	0.4711	0.6686	1.2719

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Appendix C Historical Notes on the Development of Leontief's Input–Output Analysis

C.1 Conceptual Foundations

The original idea of developing a detailed accounting of interindustry activity in an economy is certainly much older than Leontief's model. Leontief himself describes input–output as an analytical formalization of basic concepts set forth over a century and three quarters earlier by the French economist François Quesnay. Quesnay, in turn, was heavily influenced by earlier eighteenth century economists dating back to the beginning of that century. Perhaps the key precursor idea was the recognition of the concept of a "circular flow" of productive interdependences in an economy, which is a notion that can be traced to as far back as the early perspectives of Sir William Petty in the mid seventeenth century. We begin the story of input–output with this "pre-history."

When British forces led by Oliver Cromwell invaded Ireland in the 1650s, Sir William Petty, a physician and Oxford professor of anatomy accompanying the British army, was assigned the task of assessing the spoils of war. In the history of economic thought Petty is often described as the first econometrician, since he portrayed his thinking as "political arithmetick," although the term econometrics was not adopted until well into the twentieth century.[1] Petty's account, documented in Petty (1690, 1691), described the characteristics of production, distribution, and disposal of the wealth of a nation as closely *interconnected*, and the problem of assessing the value of that wealth as properly reflecting the interrelationships among these characteristics.[2] He also recommended in this work that "just accounts might be kept of the People, with the respective increases and decreases of them, their wealth and foreign trade," which led to the first reported estimates of national economic accounts (Stone, 1973, p. 143).

[1] The term *econometrics* was first coined in the 1920s by Ragnar Frisch, the winner of the very first Nobel Prize in Economic Science awarded in 1969 (Frisch died in 1973, the year Leontief won the Nobel Prize). The Econometric Society was founded in 1930, at the initiative of Frisch and Yale economist Irving Fisher.

[2] This is reported in the interpretation of Kurz and Salvadori (2000a). Davenant (1699) as reported in Stone (1973) described Petty's "political arithmetick" as "the art of reasoning by figures upon things related to government."

Petty was a pupil of philosopher Thomas Hobbes[3] and became known as one of the so-called *Mercantilists* who dominated economic thinking during a substantial period of what is usually referred to as *Pre-Classical* economics (1500–1676). The Mercantilists believed that a nation's wealth came primarily from the accumulation of gold and silver. The Mercantilist view held that nations without native sources of such resources could obtain them only by selling more goods than they bought from abroad and, hence, the political leaders of such nations must intervene extensively in the marketplace, imposing import tariffs and subsidizing exports to improve the competitiveness of domestically produced goods abroad. In this sense, mercantilism represented the earliest elevation of commercial interests to the level of national policy interest, which, of course, remains an essential element of modern economic policy today. Among the most enduring concepts of the Mercantilists was Petty's concept of chronicling the details of the interdependence of industry, which Charles Davenant (Davenant, 1699), a contemporary of Petty's and a fellow Mercantilist, described as the following:[4]

> And perhaps this art alone can show the links and chains by which one business hangs upon another, and the dependence which all our various dealings have upon each other. (Pyatt, 2000, p. 426.)

While Petty was a Mercantilist in his perspectives and policies, his work included the first rudiments of what would later become the so-called *labor theory of value*. Richard Cantillon, a disciple of Petty and an Irish financier who lived in Paris in the early eighteenth century, wrote that the intrinsic value of a commodity

> is the measure of the quantity of land and of labor entering into its production, having regard to the fertility or produce of the land and to the quality of labor (Cantillon, 1755, p. 29).

However, Cantillon argued even further that market prices may deviate from the intrinsic value of a commodity due to a mismatch of demand and availability of that commodity. He attributed the gross product of an economy to proprietors of land, farmers, and artisans, emphasizing, for the most part, that all of society subsists on the basis of the production from the land. Hence, he reasoned, essentially breaking with the Mercantilists, that the source of any surplus that could account for increasing economic value can only be attributable to agriculture.

C.2 Quesnay and the Physiocrats

The primacy of agriculture became a central tenet of the Physiocrats, who were a group of eighteenth century French philosophers known in their time as *les economists* – the first economic thinkers to call themselves economists. *Physiocracy* (tr. "the rule of nature"), as their school of thought became known, was deeply influenced by "natural law." The American economist George Soule describes the Physiocrats as the first

[3] The seventeenth century political philosophy of Thomas Hobbes asserts that men in a state of nature, i.e., without civil government, are in "a war of all against all in which life is hardly worth living." Hobbes's solution to such a dismal state of affairs was to fashion a social contract that establishes the authoritarian state to keep peace and order (see Routh, 1975).

[4] As discussed in Pyatt (2000).

school of economic thinkers to consider their craft a science, i.e., to "regard their theory as objectively scientific and to develop a complete and self-contained view of the economic order as a whole" (Soule, 1952, p. 33).

The Physiocrats were led by French court physician François Quesnay.[5] The Physiocrats opposed the Mercantilist policy noted earlier of promoting trade at the expense of agriculture because they believed that agriculture was the sole source of wealth in an economy, which they termed *produit net*, or the net product of the economy. Instead of heavy government intervention advocated by the Mercantilists, the Physiocrats, like their contemporary Cantillon, advocated a policy of *laissez-faire*, which called for minimal government interference in the economy.[6]

As the Physiocrats continued to develop their economic theories into the middle of the eighteenth century, Quesnay, in 1758, conceived his seminal *Tableau Économique*, subsequently published in Quesnay (1759), which depicted income flows between economic sectors. The *Tableau* is most remembered for its diagrammatic representation of how expenditures can be traced through an economy in a systematic way (see Figure C.1). Quesnay illustrated his thinking by describing how a landowner who receives a sum of money as rent spends half of this sum on agricultural products and half on products of artisans. In turn, farmers buy industrial products, artisans buy food and raw materials, and so on.

Many of Quesnay's and the Physiocrats' views were considered quite controversial in their time. For example, as their ideas developed, they stubbornly held to the idea that the wealth of a nation lies in the size of its *produit net,* and, as a result, that manufacturing and commerce added no value to the economy, referring to them as "sterile expenditures." This meant that the value of the output of manufacturing and commerce was equal only to the value of their inputs. In modern parlance this would mean that there was no "value added" attributable to such enterprises. Virtually all economic theorists have since concluded that *produit net* is flawed reasoning. Nonetheless, one concept of lasting value advanced by the Physiocrats is the idea of the economy as a circular flow of income and output among economic sectors as reflected in Quesnay's *Tableau*. Even the *Tableau* was controversial, however, perhaps because of its association with the collection of the Physiocrats' controversial ideas, and there were mixed reactions among economic theorists for the next century and a half, ranging from "genius" (Mirabeau, 1766, and Marx, 1905) to ignoring it entirely, as it was by most economists for decades, or opining that "it should be reduced to an embarrassed footnote" (Gray, 1931). As it turned out, the key to recognizing the lasting value of the notion of circular flow and the *Tableau* lay in finding a way to express the underlying ideas mathematically.

[5] For most of his life Quesnay was a physician, including serving as the court physician to King Louis XV and his mistress, the Madame de Pompadour. Quesnay's interest in economics arose late in his life, at 63 in 1756, when he was asked as a respected physician and scientist to prepare several articles on the role of agriculture in the economy. Quesnay drew on the work of Cantillon and many others to advance his ideas. In 1757, his admirers included the Marquis de Mirabeau and Samuel DuPont de Nemours among others, who continued to champion his work for many years thereafter (Taylor, 1960, and Meek, 1965).

[6] The Physiocrats slogan, often repeated in summarizing their views, was "*Laissez faire et laissez passer, le monde va de lui-même*" or, essentially, "don't interfere, the world will take care of itself" (Soule, 1952).

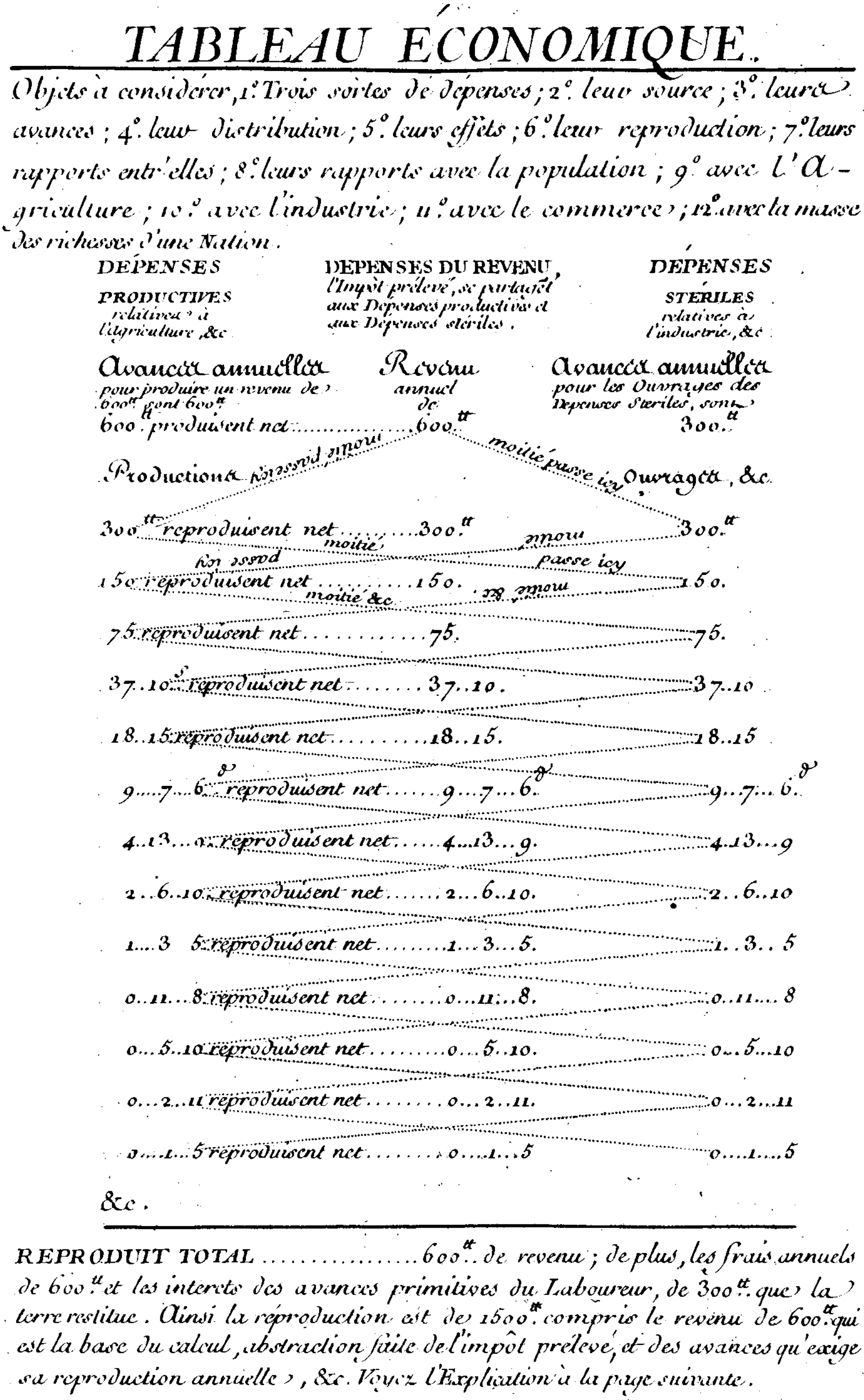

TABLEAU ÉCONOMIQUE.

Objets à considerer, 1°. Trois sortes de depenses; 2°. leur source; 3°. leurs avances; 4°. leur distribution; 5°. leurs effets; 6°. leur reproduction; 7°. leurs rapports entr'elles; 8°. leurs rapports avec la population; 9°. avec l'Agriculture; 10°. avec l'industrie; 11°. avec le commerce; 12°. avec la masse des richesses d'une Nation.

DEPENSES PRODUCTIVES relatives à l'Agriculture, &c.	DEPENSES DU REVENU, l'Impôt prélevé, se partagent aux Depenses productives et aux Depenses stériles.	DEPENSES STERILES relatives à l'industrie, &c.
Avances annuelles pour produire un revenu de 600.tt sont 600.tt	Revenu annuel de	Avances annuelles pour les Ouvrages des Depenses Steriles, sont
600. produisent net	600.tt	300.tt
Productions	moitié passe ici	Ouvrages, &c.
300.tt reproduisent net	300.tt	300.tt
150. reproduisent net	150.	150.
75. reproduisent net	75.	75.
37..10. reproduisent net	37..10.	37..10
18..15. reproduisent net	18..15.	18..15
9...7...6d reproduisent net	9...7...6d	9...7...6d
4..13...9 reproduisent net	4...13...9.	4..13...9
2..6..10. reproduisent net	2...6..10.	2..6..10
1...3...5. reproduisent net	1...3...5.	1..3..5
0..11...8. reproduisent net	0...11...8.	0..11....8
0...5..10. reproduisent net	0...5..10.	0...5...10
0...2..11 reproduisent net	0...2..11.	0...2...11
0...1...5 reproduisent net	0...1...5	0....1....5
&c.		

REPRODUIT TOTAL ....600.tt de revenu; de plus, les frais annuels de 600.tt et les interets des avances primitives du Laboureur, de 300.tt que la terre restitue. Ainsi la reproduction est de 1500.tt compris le revenu de 600.tt qui est la base du calcul, abstraction faite de l'impôt prélevé, et des avances qu'exige sa reproduction annuelle, &c. Voyez l'Explication à la page suivante.

Figure C.1 François Quesnay's *Tableau Économique*

Source: Alexander Gray. 1931. *The Development of Economic Doctrine*. London: Longman's, Green and Co. Reproduced here with permission of the publisher.

C.3 Mathematical Formalization

Achille-Nicholas Isnard, a well-known French engineer and another contemporary of the Physiocrats, was among the strongest critics of the doctrine that only agriculture was productive. In supporting his position, Isnard (1781) further developed the concept of production as a circular flow, referring to surplus value as "disposable wealth." As reported in Kurz and Salvadori (2000a), who provide detailed accounts of these developments, Isnard wrote:

> In the whole of riches, and setting aside values, there are in reality two parts, one required in production, the other destined to enjoyments .... The latter is the noble part of goods and the part which is nobly enjoyed by the proprietors. (Kurz and Salvadori, 2000a, p. 159, from Isnard, 1781, pp. 35–36.)

This notion that the accumulation of wealth depended upon the technical condition of production as well as the "exigence of nature" challenged the conclusion that industry is generally not productive. In addition, and most importantly for present purposes, Isnard was perhaps the first to represent the circular flow of income and expenditure as a system of simultaneous algebraic equations.

The framework advanced and formalized by Isnard contributed to the conceptual thinking of English classical economists Adam Smith (1776) and David Ricardo (1810–1824) in the late 1700s and early 1800s, but it was a contemporary of Ricardo's, Robert Torrens, who, in 1820, seemed to set the stage for Leontief's eventual breakthroughs. Torrens was a British army officer and owner of the influential London Globe newspaper who wrote extensively on economics and was an independent discoverer of Ricardo's principle of "comparative advantage" in international trade. Torrens (1820, 1821) postulated that the concept of economic surplus provides the key to an explanation of the share of income attributable to sources other than wages and the rate of profit.

For present purposes, the key concept in Torrens's work, described in his essay on the corn trade (Torrens, 1820), was that when one defines the agricultural rate of profit in physical terms as the ratio between net corn output and corn input (corn used as seed and consumed as food for workers) that "the exchange value of manufactured goods relative to corn is adjusted such that the same rate of profit obtains in manufacturing" (Kurz and Salvadori, 2000a, p. 161). Showing this relationship, perhaps ironically, on the one hand, essentially debunked the Physiocrats' *produit net* theory while, on the other hand, refined the analytical connection between profits and various factors of production as depicted in the *Tableau Économique*.

Later on in the century, now more than a century after Quesnay's work and nearly half a century since Torrens's ideas were put forth, another French economist, Léon Walras, applied concepts of Isaac Newton's mechanics of motion in developing the early notions of a theory of what we call today *general equilibrium* in economics, although some historians of economic thought credit Quesnay's *Tableau* as "the first method ever devised in order to convey an explicit conception of the nature of economic equilibrium" (Schumpeter, 1954, p. 217).

Walras's work, presented mostly in Walras (1874), utilized a set of production coefficients that related the quantities of factors required to produce a unit of a particular product to levels of total production of that product. Walras's ideas were heavily influenced by Isnard's earlier algebraic formulation.

At the turn of the twentieth century the published work of Karl Marx (probably the most influential socialist thinker to emerge in the nineteenth century but whose work was largely published posthumously near the turn of the century), revealed that Marx was an outspoken champion of the Physiocrats' theories in perhaps another twist of irony in this historical path – a socialist espousing *laissez-faire*. Marx considered the Physiocrats to be "the true fathers of modern political economy" (Marx, 1894 and 1905, with additional discussion in Kurz and Salvadori, 2000a). Marx argued that the concept of the *Tableau* was unduly neglected by the classical economic theorists for most of the nineteenth century and essentially resurrected it in his own work.

Marx developed a sequential or what he termed "successivist" procedure for determining profits and then prices, which was ultimately proved flawed by Russian mathematical economists Vladimir K. Dmitriev (1898) and Ladislaus von Bortkiewicz (1907), who demonstrated that the rate of profit and prices must be determined *simultaneously* rather than *successively*, consistent with the emerging ideas that would ultimately become the modern concept of *general equilibrium*.

Von Bortkiewicz, born in St. Petersburg but of Polish ancestry, was among the most ardent critics of Marx's work. He spent much of his career teaching economics and statistics at the University of Berlin, where one of his students was the young Wassily Leontief. Von Bortkiewicz (1907) was instrumental in demonstrating the concept of general equilibrium, contradicting Marx's view, and most importantly expressing his framework mathematically in an algebraic form. In particular, he assumed that commodities are produced from a fixed level of each input for each unit for commodity output, i.e., what we now often refer to as a *linear* production function.

C.4 Leontief and the "Economy as a Circular Flow"

Wassily Wassilievich Leontief was born in 1905 in Munich into an intellectual Russian family and spent his childhood in St. Petersburg during the years leading up to the Russian Revolution in 1917. In 1921, at the age of fifteen, he was arrested for opposing the communist dictatorship as it was emerging. The young Leontief was a brilliant student and was released to enter the University of Leningrad that same year to study economics following in the footsteps of his father. Following surgery on his jaw in 1925, he was permitted to leave Communist Russia under an exit visa to obtain follow-up diagnosis and treatment in Berlin (Samuelson, 2004, and Kaliadina and Pavlova, 2006).

Leontief decided not to return to Russia and entered the University of Berlin to work with von Bortkiewicz and social scientist Werner Sombart on his doctorate, which he received in 1929.[7] In the late 1920s Leontief began to assemble the ideas for his doctoral thesis, which he described as "the national economy as a circular process," drawing on Quesnay's *Tableau* and on Walras's formalization of general equilibrium, although Leontief preferred the term "interdependence," concluding that an economy is never in equilibrium (DeBresson, 2004). In 1928 he published part of his thesis in the paper, "The Economy as a Circular Flow" (Leontief, 1928), where he set forth a two-sector "input–output" system that depicted production, distribution, and consumption characteristics of an economy as a single integrated system of linear equations. Complete exposition of his analytical framework would not come for nearly another decade in Leontief (1936).

Concepts similar to Leontief's were being conceived at the time of his original work by the Italian economist Piero Sraffa (Sraffa, 1960, and described in Kurz and Salvadori, 2000b and 2003). In addition a French mathematician, Father Maurice Potron, developed similar ideas in his writings between 1911 and 1941 (Abraham-Frois and Lendjel, 2006). Despite the somewhat parallel tracks of Leontief, Sraffa, and Potron, it was likely the intense focus on empirical implementation that ultimately led to widespread use of Leontief's framework (Kurz and Salvadori, 2006, and Bjerkholt and Kurz, 2006). Some theorists characterize Leontief's model as an approximation of the Walrasian model[8] introduced a century earlier, but with several important simplifications that allowed a theory of general equilibrium to be applied and implemented empirically. Leontief felt, even very early in his career, that economists placed far too little attention on empirical verification (DeBresson, 2004).[9]

Leontief (1941, p. 9) introduces his later empirical work by stating that "this work may be best described as an attempt to construct a *Tableau Économique* of the United States." Indeed, in Quesnay's later work (discussed in Phillips, 1955, and more recently in Steenge and van den Berg, 2007), he placed his observations about circular flow transactions in the form of a table that resembles the input–output table developed by Leontief. Quesnay's original schematic is shown as Figure C.1. However, Leontief's contributions went far beyond that of constructing the *Tableau* or the table of transactions. As can be seen in this volume, in particular, Leontief devised the analytical foundations that transformed the descriptive nature of the *Tableau* into an empirical analytical tool and, today, Leontief's input–output analysis has become one of the most widely applied methods in economics (Baumol, 2000).

[7] Some fascinating anecdotes of this impressionable time in Leontief's life are provided in DeBresson (2004).

[8] In Leontief's first book (Leontief, 1941), *The Structure of American Economy*, he referred to only three other economists' works: François Quesnay, Léon Walras, and David Ricardo.

[9] In 1971 Leontief, serving as president of the American Economic Association that year, delivered his presidential address entitled "Theoretical Assumptions and Non-observed Facts," which took the economics profession to task for failing to underscore the need for empirical verification of economic theory.

C.5 Development of Input–Output Analysis

Following his graduate studies in Berlin, Leontief joined the staff of the Institute of World Economics in Kiel in 1927 where he carried out research on derivation of statistical demand and supply curves. After a year-long assignment as an advisor to the China Ministry of Railroads, Leontief moved to New York to join Simon Kuznets at the National Bureau of Economic Research in 1931. In the following year Joseph Schumpeter brought Leontief to the faculty at Harvard University where he began work on the first input–output tables for the US economy.

With Leontief's arrival at Harvard also came the university's first mathematical lectures on economics, although he seldom included his own research in his lectures (Solow, 1998, and Samuelson, 2004). In 1936 Leontief presented the theoretical framework for input–output analysis and US interindustry transactions tables for 1919 and 1929 (Leontief, 1936), followed somewhat later by his first book on the input–output structure of the US economy (Leontief, 1941).

Beginning in 1941, just prior to US entry into World War II, Leontief in collaboration with the US Government's Bureau of Labor Statistics (BLS), began preparation of a US transactions table for 1939, which was essentially completed in 1943 (Kholi, 2000 and 2001) to be used by the War Mobilization Board for planning postwar demobilization and, in particular, analyzing the implications of decreases in war spending and increases in personal consumption through detailed projections of employment by industry in the US economy.

Also during the war, Leontief was called upon to work for the Office of Strategic Services (OSS), an early predecessor of today's Central Intelligence Agency, to assemble a classified input–output table for Germany for war planning and, later, to analyze the issue of postwar German reparations.[10]

In implementing his empirical work Leontief made use of the first large scale mechanical computing machinery in 1935 and later the first commercial electro-mechanical computer, the IBM Automatic Sequence Controlled Calculator (called the Mark I), originally designed under the direction of Harvard mathematician Howard Aiken in 1939, built and operated by IBM engineers in Endicott, New York for the US Navy, and eventually moved to Harvard in 1944.

Following World War II, in 1948, as the Cold War loomed, a government interagency project funded by the Air Force's Planning Research Division, known as Scientific Computation of Optimum Programs (SCOOP), was initiated to update the 1939 US interindustry transactions table to 1947. In that same year Leontief founded the Harvard Economics Research Project (HERP), which focused on continuing to develop the input–output framework and applications. Project SCOOP's activities were greatly expanded as the Korean War erupted in 1950 to include analysis of possible obstructions

[10] According to Leontief's second protégé at Harvard, Paul Samuelson, the OSS involvement of Leontief's work during World War II was initiated with the help of Leontief's first protégé, Abram Bergson, who during the war had become head of the OSS Russian desk (Samuelson, 2005).

to wartime mobilization (Kohli, 2001) and much progress was made in the ability to work with large-scale input–output tables of more than 500 industrial sectors (Klein, 2001), although Leontief and others published only much more aggregated tables at the time. For example, Leontief's 1951 revision of his 1941 book (Leontief, 1951) was enlarged and expanded and included the US input–output table for 1939 (previously unpublished by BLS) and Evans and Hoffenberg (1952) published the 1947 table.

In the postwar period, input–output accounts began to be routinely developed in the United States and elsewhere around the world, although ironically for a period during the Cold War era, the US suspended work on constructing input–output tables because it smacked of communist "central planning" while, at around the same time, the Chinese Government shut down its preparation of input–output tables because it considered input–output to be a tool of capitalism (Polenske, 1999). The US Bureau of Economic Analysis (BEA) began preparing the US national input–output tables with the 1958 table published in 1964, and since then so-called "benchmark" tables have been published every five years corresponding to the quinquennial national economic census (every five years for years ending with 2 and 7, e.g., 1992, 1997, 2002, and 2007)[11], since the primary source of data for the input–output accounts is the national economic census. A key use of the input–output accounts in the United States since BEA began preparing them was and continues to be as a tool for, and a check on the accuracy and consistency of, a variety of other economic accounts (Landefeld and McCulla, 1999). Since 1957, input–output tables have also been routinely constructed in the United Kingdom, Norway, Denmark, the Netherlands, Italy, Canada, Japan and increasingly many other countries around the world.

Of particular importance in making input–output analysis a widely applied tool of economic analysis was the development of a standardized system of economic accounts built around input–output concepts developed under the direction of Richard Stone (Stone, 1961) in recognition of which he received the Nobel Prize in Economic Science in 1984 (Stone, 1997).

Further additional developments to Leontief's original model are presented in, among others, Leontief *et al.* (1953) and Leontief (1966a, 1966b, and 1974) and in volumes of proceedings of many international conferences on input–output techniques summarized in Table C.1. Summaries of many of these developments are included in Stone (1984) and Rose and Miernyk (1989).

Leontief's work as HERP's director continued until 1973 (Polenske, 1999) and, after 44 years, he left Harvard in 1975, but continued his research and teaching on input–output at New York University until his death at age 93 in 1999. Professor Leontief's legacy is rich and vast, as illustrated by the scale and scope of the topics that have followed from his original work that are described in this volume.

[11] Highly aggregated versions of these tables are included in Appendix B.

Table C.1 Selected International Conferences on Input–Output Analysis

	Conference			Publication (Selected Papers)		
	Date	Location	Sponsor(s)	Title	Editor(s)	Publisher, Date
1	1950	Driebergen, The Netherlands	Netherlands Economic Institute	*Input–Output Relations*	Netherlands Economic Institute	H. E. Stenfert Kroese, 1953
2	1954	Varenna, Italy	University of Pisa; Varenna Foundation	*The Structural Interdependence of the Economy*	Tibor Barna	Chapman and Hall, 1956
3	1961	Geneva, Switzerland	United Nations; Harvard Economic Research Project	*Structural Interdependence and Economic Development*	Tibor Barna	Macmillan, 1963; St. Martin's Press, 1967
4	1968	Geneva, Switzerland	United Nations; Harvard Economic Research Project	Vol. 1: *Contributions to Input–Output Analysis* Vol. 2: *Applications of Input–Output Analysis*	Anne P. Carter and András Bródy	North-Holland, 1970
5	1971	Geneva, Switzerland	Secretariat of the United Nations; Harvard Economic Research Project	*Input–Output Techniques*	András Bródy and Anne P. Carter	North-Holland, 1972
6	1974	Vienna, Austria	United Nations Industrial Development Organization (UNIDO)	*Advances in Input–Output Analysis*	Karen R. Polenske and Jiří V. Skolka	Ballinger, 1976
7	1979	Innsbruck, Austria	UNIDO	*Proceedings of the Seventh International Conference on Input–Output Techniques*		UNIDO, 1984
8	1986	Sapporo, Japan	UNIDO; University of Hokkaido	*Advances in Input–Output Analysis. Technology, Planning, & Development**	William Peterson	Oxford University Press, 1991
9	1989	Keszthely, Hungary	IIOA; Hungarian Academy of Sciences; UNIDO			
10	1993	Seville, Spain	IIOA; Economic and Finance Department of the Andalusian Government			

Table C.1 (*cont.*)

	Conference			Publication (Selected Papers)		
	Date	Location	Sponsor(s)	Title	Editor(s)	Publisher, Date
11	1995	New Delhi, India	IIOA			
12	1998	New York, USA	IIOA; C.V. Starr Center for Applied Economics, New York University; Faculty of Economics, University of Groningen; US Department of Agriculture; US Department of Commerce, Bureau of Economic Analysis	Selected papers online at IIOA website (www.iioa.org)		
13	2000	Macerata, Italy	IIOA; University of Macerata	Selected papers online at IIOA website (A few conference papers also appear in *Wassily Leontief and Input–Output Economics*)	Erik Dietzenbacher and Michael L. Lahr	Palgrave, 2004
14	2002	Montreal, Canada	IIOA; Université du Québec à Montréal; Statistics Canada	Selected papers online at IIOA website and also in *Changement Climatique, Flux Technologiques, Financiers et Commerciaux*	L. Martin Cloutier, Christian DeBresson and Erik Dietzenbacher	Presses de l'Université du Québec, 2004
15	2005	Beijing, China	IIOA; Renmin University of China; Chinese Input–Output Association	Conference program, book of abstracts and selected papers online at IIOA website		
16	2007	Istanbul, Turkey	IIOA; Department of Management, Istanbul Technical University	Conference program and selected papers online at IIOA website		

Notes: Anne Carter and Josef Richter contributed information used in compiling this table.
*At least one paper from the Sapporo Conference is included in Ronald E. Miller, Karen R. Polenske and Adam Z. Rose (eds.). 1989. *Frontiers of Input–Output Analysis*. New York: Oxford University Press – Chapter 11 by Wolff and Howell.

Table C.1 (*cont.*)

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