Mathematical Game Theory

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Contents

Preface

People have gambled and played games for thousands of years. Yet, only in the 17th century we see suddenly a serious attempt for a scientific approach to the subject. The combinatorial foundations of probability theory were developed by various mathematicians as a means to understand games of chance (mostly with dice) and to make conjectures[1].

Since then, game theory has grown into a wide field and appears at times quite removed from its mathematical roots. The notion of a *game* has been broadened to encompass all kinds of human behavior and the interactions of individuals or of groups and societies[2]. Much of current research studies humans in economic and social contexts and seeks to discover behavioral laws in analogy to physical laws.

The role of mathematics, however, has been quite limited so far in this endeavor. One major reason lies certainly the fact that players in real life behave often differently than a simple mathematical model would predict. So seemingly paradoxical situations exist where people appear to contradict the straightforward analysis of the mathematical model builder. A famous such example is SELTEN's *chain store paradox*[3].

As interesting and worthwhile as research into laws that govern psychological, social or economic behavior of humans may be, *mathematical game theory* is not about these aspects of game theory. In the center of our attention are mathematical models that may be useful for the analysis of game theoretic situations. We are concerned with the mathematics of game theoretic models but leave the question aside whether a particular model describes a particular situation in real life appropriately.

The mathematical analysis of a game theoretic model treats objects neutrally. Elements and sets have no feelings *per se* and show no psychological behavior. They are neither generous nor cost conscious unless such features are built into the model as clearly formulated mathematical properties. The

[1] see, *e.g.*, the *Ars Conjectandi* by J. BERNOULLI (1654-1705)

[2] see, *e.g.*, E. BERNE, *Games People Play: The Psychology of Human Relationships*, Grove Press, 1964

[3] R. SELTEN (1978): *The chain store paradox*, Theory and Decision 9, 127-159.

advantage of mathematical neutrality is substantial, however, because it allows us to imbed the mathematical analysis into a much wider framework.

The present introduction into mathematical game theory sees games being played on (possibly quite general) *systems*. Moves of the game then correspond to transitions of the system from one state to another. This approach reveals a close connection with fundamental physical systems *via* the same underlying mathematics. Indeed, it is hoped that mathematical game theory may eventually play a role for real world games akin to the role of theoretical physics to real world physical systems.

The reader of this introductory text is expected to have some knowledge in mathematics, perhaps at the level of an introductory course in linear algebra and real analysis. Nevertheless, the text will try to review relevant mathematical notions and properties and point to the literature for further details.

The reader is expected to read the text ”actively’. ”Ex.” marks not only an ”example” but also an ”exercise” that might deepen the understanding of the mathematical development.

The book is based on a one-term course on the subject the author has given repeatedly at the University of Cologne to pre-master level students with an interest in applied mathematics, operations research and mathematical modelling.

Part 1

Introduction

CHAPTER 1

Mathematical Models of the Real World

This introductory chapter discusses mathematical models, sketches the mathematical tools for their analysis, defines systems in general and systems of decisions in particular. Then games are introduced from a general point of view and it is indicated how the may arise in combinatorial, economic, social, physical and other contexts.

1. Mathematical modelling

Mathematics is *the* powerful human instrument to analyze and to structure observations and to possibly discover natural "laws". These laws are logical principles that allow us not only to understand observed phenomena (*i.e.*, the so-called *real world*) but also to compute possible evolutions of current situations and thus to try a "look into the future".

Why is that so? An answer to this question is difficult if not impossible. There is a wide-spread belief that *mathematics is the language of the universe*[1]. So everything can supposedly be captured by mathematics and all mathematical deductions reveal facts about the real world. I do not know whether this is true. But even if it were, one would have to be careful with real-world interpretations of mathematics, nonetheless. A simple example may illustrate the difficulty:

While apples on a tree are counted by natural numbers n, it is not true that, *for every natural number* n, there exists a tree with n apples. In other words, when we use the set of nonnegative integers to describe the number of apples, our mathematical model will comprise mathematical objects that have no real counterparts.

Theoretically, one could try to get out of the apple dilemma by restricting the mathematical model to those numbers n that are realized by apple trees. But such a restricted model would be of no practical use as the set of such apple numbers n is not explicitly known.

[1] GALILEO GALILEI (1564-1642)

In general, a mathematical model of a real-world situation is, alas, not necessarily guaranteed to be absolutely comprehensive. Mathematical conclusions are possibly only theoretical and may suggest objects and situations which do not exist in reality. One always has to double check real-world interpretations of mathematical deductions and ask whether an interpretation is ”reasonable” in the sense that it is commensurate with one’s own personal experience.

In the analysis of a game theoretic situation, for example, one may want to take the psychology of individual players into account. A mathematical model of psychological behavior, however, is typically based on assumptions whose accuracy is often not clear. Consequently, mathematically established results within such a model must be interpreted with care, of course.

Moreover, similar to physical systems with a large number of particles (like molecules *etc.*), game theoretic systems with many agents (*e.g.*, traffic systems and economies) are too complex to analyze by following each of the many agents individually. Hence a practical approach will have to concentrate on ”group behavior” and consider statistical parameters that average over individual numerical attributes.

Having cautioned the reader about the real-world interpretation of mathematical deductions, we will concentrate on mathematical models (and their mathematics) and leave the interpretation to the reader. Our emphasis is on *game theoretic* models. So we should explain what we understand by this.

A *game* involves *players* that perform actions which make a given system go through a sequence of states. When the game ends, the system is in a state according to which the players receive *rewards* (or are charged with *costs* or whatever). Many game theorists think of a ”player” as a humanoid, *i.e.*, a creature with human feelings, wishes and desires, and thus give it a human name[2].

Elements of a *mathematical* model, however, do not have humanoid feelings *per se*. If they are to represent objects with wishes and desires, these wishes and desires must be explicitly formulated as mathematical optimization challenges with specified objective functions and restrictions. Therefore, we will try to be neutral and refer to ”players” often just as *agents* with no specified sexual attributes. In particular, an agent will typically be an ”it” rather than a ”he” or ”she”.

This terminological neutrality makes it clear that mathematical game theory comprises many more models than just those with human players. As we

[2] *Alice* and *Bob* are quite popular choices

will see, many models of games, decisions, economics and social sciences have the same underlying mathematics as models of physics and informatics.

A note on continuous and differentiable functions. Real world phenomena are often modelled with continuous or even differentiable functions. However,

- *There exists no practically feasible test for the continuity or differentiability of a function*!

Continuity and differentiability, therefore, are *assumptions* of the model builder. These assumptions appear often very reasonable and produce good results in applications. Moreover, they facilitate the mathematical analysis. The reader should nevertheless be aware of this difference between a mathematical model and its physical origin.

2. Mathematical preliminaries

The reader is assumed to have some basic mathematical knowledge (at least at the level of an introductory course on linear algebra). Nevertheless, it is useful to review some of the mathematical terminology. Further basic facts are outlined in the Appendix.

2.1. Functions and data representation. A function $f : S \to W$ assigns elements $f_s = f(s)$ of a set W to the elements s of a set S. One way of looking at f is to imagine f as a measuring device which produces the result $f(s)$ upon the input s:

$$s \in S \;\longrightarrow\; \boxed{f} \;\longrightarrow\; f(s) \in W.$$

We denote the collection of all such functions as

$$W^S = \{f : S \to W\}.$$

There is a *dual* way to look at this situation where the roles of the function f and variable s are reversed. The dual viewpoint sees s as a *probe* which produces the value $f(s)$ when applied to f:

$$f \in W^S \;\longrightarrow\; \boxed{s} \;\longrightarrow\; f(s).$$

If S is small, f can be presented in table form which displays the total effect of f on S:

$$f \;\longleftrightarrow\; \begin{array}{|c|c|c|c|c|}\hline s_1 & s_2 & s_3 & \ldots & s_n \\ \hline f(s_1) & f(s_2) & f(s_3) & \ldots & f(s_n) \\ \hline \end{array}$$

The dual viewpoint would fix an element $s \in S$ and evaluate the effect of the measuring devices $f_1, \ldots, f_k$, for example, and thus represent an individual element $s \in S$ by a k-dimensional data table:

$s \longleftrightarrow$

f_1	f_2	f_3	$\ldots$	f_k
$f_1(s)$	$f_2(s)$	$f_3(s)$	$\ldots$	$f_k(s)$

The dual viewpoint is typically present when one tries to describe the state s of an economic, social or physical system *via* the data values $f_1(s), \ldots, f_k(s)$ of statistical measurements $f_1, \ldots, f_k$ with respect to k system characteristics.

The two viewpoints are logically equivalent. Indeed, the dual point of view sees the element $s \in S$ just like a function $\hat{s} : W^S \to W$ with values

$$\hat{s}(f) = f(s).$$

Also the first point of view is relevant for data representation. Consider, for example, a n-element set

$$N = \{i_1, \ldots, i_n\}.$$

In this context, a function $f : N \to \{0, 1\}$ may specify a subset S_f of N, *via* the identification

$$f \in \{0,1\}^N \quad \longleftrightarrow \quad S_f = \{i \in N \mid f(i) = 1\} \subseteq N. \tag{1}$$

REMARK 1.1. *The vector f in (1) is the* incidence vector *of the subset S_f. Denoting by $\mathcal{N}$ the collection of all subsets of N and writing $\mathbf{2} = \{0, 1\}$, the context (1) establishes the correspondence*

$$\mathbf{2}^N = \{0,1\}^N \quad \longleftrightarrow \quad \mathcal{N} = \{S \subseteq N\}.$$

NOTA BENE. $(0, 1)$-*valued functions may also have other interpretations, of course. Information theory, for example, thinks of them as* bit vectors *and thus as carriers of information. An abstract function is a purely mathematical object with no physical meaning by itself. It obtains a concrete meaning only within the context to which it refers.*

Notation. When we think of a function $f : S \to W$ as a data representative, we think of f as a coordinate vector $f \in W^S$ with coordinate components $f_s = f(s)$ and also use the notation

$$f = (f_s | s \in S).$$

In the case $S = \{s_1, s_2, s_3 \ldots\}$, we may write

$$f = (f(s_1), f(s_2), \ldots) = (f_{s_1}, f_{s_2}, \ldots) = (f_s | s \in S)$$

In the case of a direct product

$$S = X \times Y = \{(x, y) \mid x \in X, y \in Y\}$$

of sets X and Y, a function $A : X \times Y \to W$ can be imagined as a matrix with rows labeled by the elements $x \in X$ and columns labeled by the elements $y \in Y$:

$$A = \begin{bmatrix} A_{x_1y_1} & A_{x_1y_2} & A_{x_1y_3} & \ldots \\ A_{x_2y_1} & A_{x_2y_2} & A_{x_3y_3} & \ldots \\ A_{x_3y_1} & A_{x_3y_2} & A_{x_3y_3} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The function values A_{xy} are the coefficients of A. We express this point of view in the short hand notation

$$A = [A_{xy}] \in W^{X \times Y}.$$

The matrix form suggests to relate similar structures to A. The *transpose* of the matrix $A \in W^{X \times Y}$, for example, is the matrix

$$A^T = [A^T_{yx}] \in W^{Y \times X} \quad \text{with the coefficients } \; A^T_{yx} = A_{xy}.$$

In the case $X = \{1, 2, \ldots, m\}$ and $Y = \{1, 2, \ldots, n\}$, one often simply writes

$$W^{m \times n} \;\cong\; W^{X \times Y}.$$

REMARK 1.2 (Row and column vectors). *When one thinks of a coordinate vector $f \in W^X$ as a matrix having just f as the only column, one calls f a* column vector. *If f corresponds to a matrix with f as the only row, f is* row vector. *So*

$$f^T \textit{ row vector} \quad \Longleftrightarrow \quad f \textit{ column vector.}$$

Graphs. A *(combinatorial) graph* $G = G(X)$ consists of a set X of *nodes* (or *vertices*) whose ordered pairs (x, y) of elements are viewed as arrows (or *(directed) edges*) between nodes:

$$\text{x} \longrightarrow \text{y} \quad (x, y \in X).$$

Denoting, as usual, by $\mathbb{R}$ the set of all real numbers, an $\mathbb{R}$-*weighting* of G is an assignment of real number values a_{xx} to the nodes $x \in X$ and a_{xy} to the other edges (x, y) and hence corresponds to a matrix

$$A \in \mathbb{R}^{X \times X}$$

with X as its row and its column index set and coefficients $A_{xy} = a_{xy}$.

Although logically equivalent to a matrix, a graph representation is often more intuitive in dynamic contexts. A directed edge $e = (x, y)$ may, for example, represent a road along which one can travel from x to y in a traffic context. e could also indicate a possible transformation of x into y *etc.* The edge weight $a_e = a_{xy}$ could be the distance from x to y or the strength of an action exerted by x onto y *etc.*

2.2. Algebra of functions and matrices. While the coefficients of data vectors or matrices could be quite varied (colors, sounds, configurations in games, *etc. etc.*), we will typically deal with numerical data so that coordinate vectors have real numbers as components. Hence we deal with coordinate spaces of the type

$$\mathbb{R}^S = \{f : S \to \mathbb{R}\}.$$

Addition and scalar multiplication. The *sum* $f + g$ of two coordinate vectors $f, g \in \mathbb{R}^S$ is the vector of component sums $(f + g)_s = f_s + g_s$, *i.e.*,

$$f + g = (f_s + g_s | s \in S).$$

For any scalar $\lambda \in \mathbb{R}$, the *scalar product* multiplies each component by λ:

$$\lambda f = (\lambda f_s | s \in S).$$

WARNING. There are many – quite different – notions for multiplication with vectors.

Products. The *product* $f \bullet g$ of two vectors $f, g \in \mathbb{R}^S$ is the vector with the componentwise products, *i.e.*

$$f \bullet g = (f_s g_s | s \in S).$$

In the special case of matrices $A, B \in \mathbb{R}^{X \times Y}$ the function product of A and B is called the HADAMARD[3] *product*

$$A \bullet B \in \mathbb{R}^{X \times Y} \quad \text{(with coefficients } (A \bullet B)_{xy} = A_{xy} B_{xy}\text{)}.$$

WARNING: This is *not* the standard matrix multiplication rule (2)!

The *standard* product of matrices $A \in \mathbb{R}^{X \times Y}$ and $B \in \mathbb{R}^{U \times Z}$ is ONLY declared in the case $U = Y$ and, then, defined as the matrix

$$C = AB \in \mathbb{R}^{X \times Z} \quad \text{with coefficients } C_{xz} = \sum_{y \in Y} A_{xy} B_{yz}. \tag{2}$$

NOTA BENE. The standard product of two matrices may not be well-defined in cases where the index sets X and Y are infinite because infinite sums are problematic. For the purposes of this book, however, this is no obstacle:

- *Mainly finite sums are considered.*

REMARK 1.3 (HILBERT[4] spaces). *Much of game theoretic analysis can be extended to the framework of* HILBERT *spaces, namely to coordinate spaces of the form*

$$\ell_2(S) = \{f : S \to \mathbb{R} \mid \sum_{s \in S} f_s^2 < \infty\}, \tag{3}$$

where S is a countable (possibly infinite) set.

Inner product and norm. The *inner product* $\langle A|B\rangle$ of two matrices $A, B \in \mathbb{R}^{X \times Y}$ is the sum of the products of the respective components:

$$\langle A|B\rangle = \sum_{x \in X} \sum_{y \in Y} = \sum_{x \in X} \sum_{y \in Y} A_{xy} B_{xy} = \sum_{(x,y) \in X \times Y} (A \bullet B)_{xy}.$$

NOTA BENE. The inner product $\langle A|B\rangle$ is a scalar number – and not a matrix!

[3] J. HADAMARD (1865-1963)

[4] D. HILBERT (1862-1943)

In the vector case, we have for $f, g \in \mathbb{R}^S$, the inner product

$$\langle f|g\rangle = \sum_{S\in S} f_s g_s \cong f^T g,$$

where the latter expression assumes that we think of f and g as column vectors and identify the (1×1)-matrix $f^T g$ with the scalar $\langle f|g\rangle$.

EX. 1.1. *If $f, g \in \mathbb{R}^n$ are column vectors, then $f^T g \in \mathbb{R}^{1\times 1}$ is a (1×1)-matrix. This is to be clearly distinguished from the $(n\times n)$-matrix*

$$fg^T = \begin{bmatrix} f_1g_1 & f_1g_2 & \dots & f_1g_n \\ f_2g_1 & f_2g_2 & \dots & f_2g_n \\ \vdots & \vdots & \ddots & \vdots \\ f_ng_1 & f_ng_2 & \dots & f_ng_n \end{bmatrix} \in \mathbb{R}^{n\times n}$$

The *norm* of a vector (or matrix) $f \in \mathbb{R}^S$ is defined as

$$\|f\| = \sqrt{\langle f|f\rangle} = \sum_{s\in S} |f_s|^2.$$

The norm of a vector is often geometrically interpreted as its *euclidian length* . So one says that two vectors $f, g \in \mathbb{R}^S$ are *orthogonal* if they satisfy the so-called *Theorem of* PYTHAGORAS:

$$\|f+g\|^2 = \|f\|^2 + \|g\|^2. \tag{4}$$

LEMMA 1.1 (Orthogonality). *Assuming S finite, one has for the coordinate vectors $f, g \in \mathbb{R}^S$:*

$$f \text{ and } g \text{ are orthogonal} \quad \Longleftrightarrow \quad \langle f|g\rangle = 0.$$

Proof. Straightforward exercise.

2.3. Numbers and algebra. The set $\mathbb{R}$ of real numbers has an algebraic structure under the usual addition and multiplication rules for real numbers. $\mathbb{R}$ contains the set of natural numbers

$$\mathbb{N} = \{0, 1, 2, \dots, n, \dots\}.$$

So the algebraic computational rules of $\mathbb{R}$ may also be applied to $\mathbb{N}$ as the sum and the product of two natural numbers yields a natural number[5]. Similar algebraic rules can be defined on other sets. We give two examples.

[5]though the same is not guaranteed for subtractions and divisions

Complex numbers. The computational rules of $\mathbb{R}$ can be extended to the set $\mathbb{R}\times\mathbb{R}$ of pairs of real numbers when one defines

$$\begin{aligned}(a,b)+(c,d) &= (a+c,b+d)\\ (a,b)\cdot(c,d) &= (ab-bd,ad+bd).\end{aligned}$$

A convenient notation with respect to this algebra is the form

$$(a,b)=a(1,0)+b(0,1) \quad\longleftrightarrow\quad a+\mathrm{i}b,$$

with the so-called *imaginary unit* $\mathrm{i} \;\leftrightarrow\; (0,1)$. Notice that algebra then yields

$$\mathrm{i}^2 \;\leftrightarrow\; (0,1)\cdot(0,1)=(-1,0) \;\leftrightarrow\; -1+\mathrm{i}\cdot 0=-1.$$

We define the set of *complex numbers* as the set

$$\mathbb{C}=\{z=a+\mathrm{i}b \mid (a,b)\in\mathbb{R}\times\mathbb{R}\}$$

and identify $\mathbb{R}$ as a subset of $\mathbb{C}$:

$$a\in\mathbb{R} \quad\longleftrightarrow\quad (a,0)\in\mathbb{R}\times\mathbb{R} \quad\longleftrightarrow\quad a+\mathrm{i}\cdot 0\in\mathbb{C}.$$

Algebra in $\mathbb{C}$ follows the same rules as algebra in $\mathbb{R}$ with the additional rule

$$\mathrm{i}^2=-1.$$

Binary algebra. On the 2-element set $\mathcal{B}=\{0,1\}$, define addition $\oplus$ and multiplication $\otimes$ according to the following tables:

$$\begin{array}{c|cc} \oplus & 0 & 1\\ \hline 0 & 0 & 1\\ 1 & 1 & 0\end{array} \quad\text{and}\quad \begin{array}{c|cc} \otimes & 0 & 1\\ \hline 0 & 0 & 0\\ 1 & 0 & 1\end{array} \quad .$$

In this binary algebra, also division is possible in the sense that the equation

$$x\otimes y=1$$

has a unique solution y ”for every” $x\neq 0$. (There is only one such case : $y=x=1$.)

Vector algebra. Complex numbers allow us the define sums and products of vectors with complex coefficients in analogy with real sums and products. Applications of this algebraic technique will be discussed in Chapter 8.

The same is true for vectors with $(0, 1)$-coefficients under the binary rules. An application of binary algebra is the analysis of winning strategies for nim games in Chapter 2.

REMARK. *Are there clearly defined "correct" or "optimal" addition and multiplication rules on data structures that would reveal their real-world structure mathematically?*

The answer is "no" *in general. The imposed algebra is always a choice of the mathematical analyst – and not of "mother nature". It often requires care and ingenuity. Moreover, different algebraic setups may reveal different structural aspects and thus lead to additional insight.*

2.4. Probabilities, information and entropy. Consider n mutually exclusive events $E_1, \ldots, E_n$, and expect that any one of these, say E_i, indeed occurs "with probability" $p_i = \Pr(E_i)$. Then the parameters p_i form a *probability distribution* on the set $\mathcal{E} = \{E_1, \ldots, E_n\}$, *i.e.*, the p_i are non-negative real numbers that sum to 1:

$$p_1 + \ldots + p_n = 1 \quad \text{and} \quad p_1, \ldots, p_n \geq 0.$$

If we have furthermore a measuring or observation device f that produces the number f_i if E_i occurs, then these numbers have the *expected value*

$$\mu(f) = f_1 p_1 + \ldots + f_n p_n = \sum_{k=1}^{n} f_i p_i = \langle f | p \rangle. \tag{5}$$

In a game theoretic context, a probability is often a *subjective* evaluation of the likelihood for an event to occur. The gambler, investor, or general player may not know in advance what the future will bring, but has more or less educated guesses on the likelihood of certain events. There is a close connection with the notion of *information*.

Intensity. We think of the *intensity* of an event E as a numerical parameter that is inversely proportional to its probability $p = \Pr(E)$: the smaller p, the more intensely felt is the actual occurrence of E. For simplicity, let us take $1/p$ as our subjective intensity measure.

REMARK 1.4 (FECHNER's law). *According to* FECHNER[6], *the intensity of physical stimulations is physiologically felt on a logarithmic scale. Well-known examples are the Richter scale for earthquakes or the decibel scale for the sound.*

Following FECHNER, we *feel* the intensity of an event E that we expect with probability p on a logarithmic scale and hence according to the function

$$I_a(p) = \log_a(1/p) = -\log_a p, \tag{6}$$

where $\log_a p$ is the logarithm of p relative to the basis $a > 0$ (see Ex. 1.2). In particular, the occurrence of an "impossible" event, which we expect with zero probability, has infinite intensity

$$I_a(0) = -\log_a 0 = +\infty.$$

NOTA BENE. *The mathematical intensity of an event depends* only *on the probability p with which it occurs – and* not *on its interpretation within a modelling context or its "true nature" in a physical environment.*

EX. 1.2 (Logarithm). *Recall: For any given positive numbers $a, x > 0$, there is a unique number $y = \log_a x$ such that*

$$x = a^y = a^{\log_a x}.$$

Where $e = 2.718281828...$ is EULER*'s number*[7], *the notation $\ln x = \log_e x$ is commonly used. $\ln x$ is the so-called* natural logarithm *with the function derivative*

$$(\ln x)' = 1/x \quad \textit{for all } x > 0.$$

Two logarithm functions $\log_a x$ and $\log_b x$ differ just by a multiplicative constant. Indeed, one has

$$a^{\log_a x} = x = b^{\log_b x} = a^{(\log_b a)\log_b x}$$

and hence

$$\log_a x = (\log_b a) \cdot \log_b x \quad \textit{for all } x > 0.$$

[6] G.TH. FECHNER (1801-1887)

[7] L. EULER (1707-1783)

Information. In the fundamental theory of information[8], the parameter $I_2(p) = -\log_2 p$ is the quantity of information provided by an event E that occurs with probability p. Note that the probability value p can be regained from the information quantity $I_2(p)$:

$$p = 2^{\log_2 p} = 2^{-I_2(p)}.$$

This relationship shows that "probabilities" can be understood as parameters that capture the amount of information (or lack of information) we have on the occurrence of events.

Entropy. The expected quantity of information provided by the family

$$\mathcal{E} = \{E_1, \ldots, E_n\}$$

of events with the probability distribution $\pi = (p_1, \ldots, p_n)$ is known as its *entropy*

$$H_2(\mathcal{E}) = H_2(\pi) = \sum_{k=1}^{n} p_k I_2(p_k) = -\sum_{k=1}^{n} p_k \log_2 p_k, \tag{7}$$

where, by convention, one sets $0 \cdot \log_2 0 = 0$. Again, it should be noticed:

> $H_2(\mathcal{E})$ just depends on the parameter vector π – and *not* on a real-world interpretation of $\mathcal{E}$.

REMARK 1.5. *Entropy is a also a fundamental notion in thermodynamics, where it serves, for example, to define the temperature of a system. Physicists prefer to work with base* e *rather than base* 2 *and thus with* $\ln x$ *instead of* $\log_2 x$*,* i.e.*, with the accordingly scaled entropy*

$$H(\pi) = -\sum_{k=1}^{n} p_k \ln p_k = (\ln 2) \cdot H_2(\pi).$$

3. Systems

A *system* is a physical, economic, or other entity that is in a certain *state* at any given moment. Denoting by $\mathfrak{S}$ the collection of all possible states σ, we identify the system with $\mathfrak{S}$. This is, of course, a very abstract definition. In practice, one will have to describe the system states in a way that is suitable for a concrete mathematical analysis. To get a first idea of what is meant, let us look at some examples.

[8]due to C.E. SHANNON (1916-2001)

Chess. A system arises from a game of chess as follows: A state of chess is a particular configuration of the chess pieces on the chess board, together with the information which of the two players (”B” or ”W”) is to draw next. If $\mathfrak{C}$ is the collection of all possible chess configurations, a state could thus be described as a pair

$$\sigma = (C, p) \quad \text{with } C \in \mathfrak{C} \text{ and } p \in \{B, W\}.$$

In a similar way, a card game takes place in the context of a system whose states are the possible distributions of cards among the players together with the information which players are to move next.

Economies. The model of an *exchange economy* involves a set N of agents and a set $\mathcal{G}$ of certain specified goods. A *bundle* for agent $i \in N$ is a data vector

$$b \in \mathbb{R}^{\mathcal{G}}$$

where the component b_G indicates that the bundle b comprises b_G units of the good $G \in \mathcal{G}$. Denote by $\mathcal{B}$ the set of all possible bundles. A state of the exchange economy is now described by a map

$$\beta : N \to \mathcal{B} \quad \text{(or vector } \beta \in \mathcal{B}^N\text{)}$$

that specifies agent i's particular bundle $\beta(i) \in \mathcal{B}$.

Closely related is the description of the state of a general economy. One considers a set $\mathcal{E}$ of economic statistics E. Assuming that these statistics take numerical values ϵ_E at a given moment, the corresponding economic state is given by the data vector

$$\epsilon \in \mathbb{R}^{\mathcal{E}}$$

having the statistical values ϵ_E as its components.

Decisions. In a general *decision system* $\mathfrak{D}$, we are given a set $N = \{n_1, n_2, \ldots\}$ of agents and assume that each agent $n_i \in N$ has to make a decision of a given type, *i.e.*, we assume that n_i has to choose an element d_i in its ”decision set” D_i. The joint decision of N is then a vector

$$d = (d_1, d_2, \ldots) = (d_i | i \in N) \in D_1 \times D_2 \times \cdots = \prod_{i \in N} D_i$$

and thus describes a *decision state* of the set N. In the context of game theory, decisions of agents often correspond to choices of strategies from certain feasible strategy sets.

Decision systems are ubiquitous. In the context of a traffic situation, for example, N can be a set N of persons who want to travel from individual starting points to individual destinations. Suppose that each person i selects a path P_i from a set $\mathcal{P}_i$ of possible paths in order to do so. Then a state of the associated traffic system is a definite selection π of paths of members of the group N and thus a data vector with path-valued components:

$$\pi = (P_i | i \in N).$$

4. Games

A *game* Γ involves a set N of *agents* (or *players*) and a system $\mathfrak{S}$ relative to which the game is played. A concrete game instance γ starts with some initial state σ_0 and consists in a sequence of *moves*, *i.e.*, state transitions

$$\sigma_t \to \sigma_{t+1}$$

that are feasible according to the rules of Γ. After t steps, the system has evolved from state σ_0 into a state σ_t in a sequence of (feasible) moves

$$\sigma_0 \to \sigma_1 \to \ldots \to \sigma_{t-1} \to \sigma_t.$$

We refer to the associated sequence $\gamma_t = \sigma_0\sigma_1 \cdots \sigma_{t-1}\sigma_t$ as the *stage* of Γ at time t and denote the set of all possible stages after t steps by

$$\Gamma_t = \{\gamma_t \mid \gamma_t \text{ is a possible stage of } \Gamma \text{ at time } t\}. \tag{8}$$

If the game instance γ ends in stage $\gamma_t = \sigma_0\sigma_1 \cdots \sigma_t$, then σ_t is the *final* state of γ. It is important to note that there may be many state sequence in $\mathfrak{S}$ that are not necessarily stages of Γ because they are not feasible according to the rules of Γ.

This informal discussion indicates how a general game can be defined from an abstract point of view:

- *A* game Γ *is, by definition, a collection of finite state sequences* γ *with the property*

  $$\sigma_0\sigma_1 \cdots \sigma_{t-1}\sigma_t \in \Gamma \quad \Longrightarrow \quad \sigma_0\sigma_1 \cdots \sigma_{t-1} \in \Gamma.$$

  The members $\gamma \in \Gamma$ *are called the* stages *of* Γ.

Chess would thus be abstractly defined as the set of all possible finite sequences of legal chess moves. This set, however, is infinitely large and impossibly difficult to handle computationally.

In concrete practical situations, a game Γ is characterized by a set of *rules* that allow us to check whether a state sequence γ is feasible for Γ, *i.e.*, belongs to that potentially huge set Γ. The rules typically involve also a set N of *players* (or *agents*) that "somehow" influence the evolution of a game by exerting certain actions and making certain choices at subsequent points in time $t = 0, 1, 2, \ldots$.

Let us remain a bit vague on the precise mathematical meaning of "influence" at this point. It will become clear in special game contexts later.

In an instance of chess, for example, one knows which of the players is to move at a given time t. This player can then *move the system* deterministically from the current state σ_t into a next state σ_{t+1} according to the rules of chess. Many games, however, involve stochastic procedures (like rolling dice or shuffling cards) whose outcome is not known in advance and make it impossible for a player to select a desired subsequent state with certainty.

REMARK 1.6. *When a game starts in a state σ_0 at time $t = 0$, it is usually not clear in what stage γ it will end (or whether it ends at all).*

Objectives and utilities. The players in a game have typically certain *objectives* according to which they try to influence the evolution of a game. A rigorous mathematical model requires these objectives to be clearly formulated in mathematical terms, of course. A typical example of such objectives is a set u of *utility functions*

$$u_i : \Gamma \to \mathbb{R} \quad (i \in N)$$

which associates with each player $i \in N$ a real number $u_i(\gamma) \in \mathbb{R}$ as its *utility value* once the stage $\gamma \in \Gamma$ is realized.

Its expected utility is, of course, of importance for the strategic decision of a player in a game. We illustrate this with an example in a betting context.

EX. 1.3. *Consider a single player with a capital of 100 euros in a situation where a bet can be placed on the outcome of a $(0,1)$-valued stochastic variable X with probabilities*

$$p = \Pr\{X = 1\} \quad \textit{and} \quad q = \Pr\{X = 0\} = 1 - p.$$

Assume:

- *If the player invests f euros into the game and the event $\{X = 1\}$ occurs, the player will receive $2f$ euros. In the event $\{X = 0\}$ the investment f will be lost.*

Question: *What is the optimal investment amount f^* for the player?*

To answer it, observe that the player's total portfolio after the bet is

$$x = x(f) = \begin{cases} 100 + f & \textit{with probability } p \\ 100 - f & \textit{with probability } q \end{cases}$$

For the sake of the example, let us suppose that the player has a utility function $u(x)$ and wants to maximize the expected utility of x, that is the function

$$g(f) = p \cdot u(100 + f) + q \cdot u(100 - f).$$

Let us consider two scenarios:

(i) $u(x) = x$ *and hence*

$$g(f) = p(100 + f) + 1(100 - f)$$

with derivative

$$g'(f) = p - q = 2p - 1.$$

If $p < 1/2$*,* $g(f)$ *is monotonically decreasing in* f*. Consequently* $f^* = 0$ *would be the best decision.*

In the case $p > 1/2$*,* $g(f)$ *is monotonically increasing and, therefore, the full investment* $f^* = 100$ *is optimal.*

(ii) $u(x) = \ln x$ *and hence*

$$g(f) = p\ln(100 + f) + q\ln(100 - f)$$

with the derivative

$$g'(f) = \frac{p}{100 + f} - \frac{q}{100 - f} = \frac{100(p - q) - f}{10000 + f^2} \quad (0 \le f \le 100).$$

In this case, $g(f)$ *increases monotonically until* $f = 100(p - q)$ *and decreases monotonically afterwards. So the best investment choice is*

$$f^* = 100(p - q) \quad \textit{if } p \ge q.$$

If $p < q$*, we have* $100(p - q) < 0$*. Hence* $f^* = 0$ *would be the best investment choice.*

NOTA BENE. The player in Ex. 1.3 with utility function $u(x) = x$ risks a complete loss of the capital in the case $p > 1/2$ with probability q.

A player with utility function $u(x) = \ln x$ will never experience a complete loss of the capital.

EX. 1.4. *Analyze the betting problem in Ex. 1.3 for an investor with utility function* $u(x) = x^2$.

REMARK 1.7 (Concavity). *Utility functions which represent a* gain *are typically "concave", which intuitively means that the marginal utility gain is higher when the reference quantity is small than when it is big.*

As an illustration, assume that $u : (0, \infty) \to \mathbb{R}$ *is a differentiable utility function. Then the derivative* $u'(x)$ *represents the marginal utility value at* x*.* u *is concave if the derivative function*

$$x \mapsto u'(x)$$

decreases monotonically with x*.*

The logarithm function $f(x) = \ln x$ has the strictly decreasing derivative $f'(x) = 1/x$ and is thus an (important) example of a concave utility.

Profit and cost. In a *profit game* the players i are assumed to aim at maximizing their utility u_i. In a *cost game* one tries to minimize one's utility to the best possible.

REMARK 1.8. *The notions of profit and cost games are closely related: A profit game with the utilities u_i is formally equivalent to a cost game with utilities $c_i = -u_i$.*

Terminology. A game with a set N of players is a so-called *N-person* game[9]. If N has cardinality $n = |N|$, the N-person game is also simply termed a *n-person* game. The particular case of 2-person games is fundamental as we will see later.

Decisions and strategies. In order to pursue an agent's $i \in N$ objective in a game, the agent may choose a *strategy* s_i from a set S_i of possible strategies. The joint strategic choice

$$s = (s_i | i \in N)$$

typically influences the evolution of the game. We illustrate the situation with a well-known game theoretic puzzle:

EX. 1.5 (Prisoner's dilemma). *There are two agents A, B and the data matrix*

$$U = \begin{bmatrix} (u_{11}^A, u_{11}^B) & (u_{12}^A, u_{12}^B) \\ (u_{21}^A, u_{21}^B) & (u_{22}^A, u_{22}^B) \end{bmatrix} = \begin{bmatrix} (7,7) & (1,9) \\ (9,1) & (3,3) \end{bmatrix}. \tag{9}$$

A and B play a game with these rules:

1. *A chooses a row i and B a column j of U.*
2. *The choice (i, j) entails that A is "penalized" with the value u_{ij}^A and B with the value u_{ij}^B.*

This 2-person game has an initial state σ_0 and 4 other states $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$, which correspond to the four entry positions of U. The agents have to decide on strategies $i, j \in \{1, 2\}$. Their joint decision (i, j) will move the game from σ_0 into the final state $\sigma_1 = (i, j)$. The game ends at time $t = 1$. The utility of player A is then the value u_{ij}^A. B has the utility value u_{ij}^B.

This game is usually understood as a cost game, i.e., *A and B aim at minimizing their utilities. What should A and B do optimally?*

[9]even if the players are not real "persons"

REMARK 1.9 (Prisoner's Dilemma). *The utility matrix U in (9) yields a version of the so-called* Prisoner's dilemma*, which is told as the story of two prisoners A and B who can either individually "confess" or "not confess" to the crime they are jointly accused of. Depending on their joint decision, they supposedly face prison terms as specified in U. Their "dilemma" is:*

- *no matter what they do, at least one of them will feel to have taken the wrong decision in the end.*

Part 2

2-Person-Games

CHAPTER 2

Combinatorial Games

A look is taken at general games from the standpoint of two alternating players. This aspect reveals a recursive character of games. Finite games are combinatorial. Under the normal winning rule, combinatorial games are seen to behave like generalized numbers. Game algebra allows one to explicitly compute winning strategies for nim games, for example.

1. Alternating players

Let Γ be a game that is played on a system $\mathfrak{S}$ and recall that Γ represents the collection of all possible stages in an abstract sense. Assume that a concrete instance of Γ starts in initial state $\sigma_0 \in \mathfrak{S}$. Then we may imagine that the evolution of the game is caused by two ”superplayers” that alternate with the following moves:

(1) The beginning player chooses a stage $\gamma_1 = \sigma_0\sigma_1 \in \Gamma$.
(2) Then the second player chooses a stage $\gamma_2 = \sigma_0\sigma_1\sigma_2 \in \Gamma$.
(3) Now it is again the turn of the first player to realize the next feasible stage $\gamma_3 = \sigma_0\sigma_2\sigma_3 \in \Gamma$ and so on.
(4) The game stops if the player which would be to move next cannot find a feasible extension $\gamma_{t+1} \in \Gamma$ of the current stage γ_t.

This point of view allows us to interpret the evolution of a game as the evolution of a so-called *alternating* 2-person game. For such a game $\mathcal{A}$, we assume

(A_0) There is a set $\mathcal{G}$ and two players L and R and an initial element $G_0 \in \mathcal{G}$.
(A_1) For every $G \in \mathcal{G}$, there are subsets $G^L \subseteq \mathcal{G}$ and $G^R \subseteq \mathcal{G}$.

The two sets G^L and G^R in (A_1) are the sets of *options* of the respective players relative to G.

The rules of the alternating game $\mathcal{A}$ are:

(A_3) The beginning player chooses an option G_1 relative to G_0. Then the second player chooses an option G_2 relative to G_1. Now the first player may select an option G_3 relative to G_2 and so on.
(A_4) The game stops with G_t if the player whose turn it is has no option relative to G_t (*i.e.*, the corresponding option set is empty).

EX. 2.1 (Chess). *Chess is an obvious example of an alternating* 2*-person game. Its stopping rule* (A_4) *says that the game ends when a player's king has been taken (*"checkmate"*).*

REMARK 2.1. While a chess game always starts with a move of the white player, notice that we have not specified whether L or R is the first player in the general definition of an altenating 2-person game. This specification will offer the necessary flexibility in the recursive analysis of games later.

2. Recursiveness

An alternating 2-person game $\mathcal{A}$ as above has a *recursive* structure:

(R) *A feasible move $G \to G'$ of a player reduces the current game to a new alternating* 2*-player game with initial element G'.*

To make this conceptually clear, we denote the options of the players L and R relative to G as

$$G = \{G_1^L, G_2^L, \ldots | G_1^R, G_2^R, \ldots\} \tag{10}$$

and think of G as the (recursive) description of a game that could possibly be reduced by L to a game G_i^L or by R to a game G_j^L, depending on whose turn it is to make a move.

3. Combinatorial games

Consider an alternating 2-person game in its recursive form (10):

$$G = \{G_1^L, G_2^L, \ldots | G_1^R, G_2^R, \ldots\}.$$

Denoting by $|G|$ the maximal number subsequent moves that are possible in G, we say that G is a *combinatorial game* if

$$|G| < \infty,$$

i.e., if G is guaranteed to stop after a finite number of moves (no matter which player starts). Clearly, all the options G_i^L and G_j^R of G must then be combinatorial games as well:

$$|G| < \infty \quad \Longrightarrow \quad |G_i^L|, |G_j^R| \leq |G| - 1 < \infty.$$

EX. 2.2 (Chess). *According to its standard rules, chess is* not *a combinatorial game because the players could move pieces back and forth and thus create a never ending sequence of moves. In practice, chess is played with an additional rule that ensures finiteness and thus makes it combinatorial (in the sense above). The use of a timing clock, for example, limits the the number of moves.*

EX. 2.3 (Nim). *The* nim game $G = G(N_1, \ldots, N_k)$ *has two alternating players and starts with the initial configuration of a collection of k finite and pairwise disjoint sets $N_1, \ldots, N_k$. A move of a player is:*

- *Select one of these sets, say N_j, and remove one or more of the elements from N_j.*

Clearly, one has $|G(N_1, \ldots, N_k)| \leq |N_1| + \ldots + |N_k| < \infty$. So nim is a combinatorial game[1].

EX. 2.4 (Frogs). *Having fixed numbers n and k, the two frogs L and R sit n positions apart. A move of a frog consists in taking a leap of at least 1 but not more than k positions toward the other frog:*

$$L \rightarrow \bullet \; \bullet \; \bullet \cdots \bullet \; \bullet \; \bullet \leftarrow R$$

The frogs are not allowed to jump over each other. Obviously, the game ends after at most n moves.

[1] a popular version of nim starts from four sets N_1, N_2, N_3, N_4 of pieces (pebbles or matches *etc.*) with $|N_1| = 1, |N_2| = 3, |N_3| = 5$ and $|N_4| = 7$ elements

REMARK 2.2. *The game of frogs in Ex. 2.4 can be understood as a nim game with an additional move restriction. Initially, there is a set N with n elements (which correspond to the positions separating the frogs). A player must remove at least 1 but not more than k elements.*

Creation of combinatorial games. The class $\mathfrak{R}$ of all combinatorial games can be created systematically. We first observe that there is exactly one combinatorial game G with $|G| = 0$, namely the game

$$O = \{\cdot \mid \cdot\}$$

in which no player has an option to move. Recall, furthermore, that all options G^L and G^R of a game G with $|G| = t$ must satisfy $|G^L| \leq t-1$ and $|G^R| \leq t-1$. So we can imagine that $\mathfrak{R}$ is "created" in a never ending process from day to day:

DAY 0: The game $O = \{\cdot \mid \cdot\}$ is created and yields $\mathfrak{R}_0 = \{O\}$.

DAY 1: The games $\{O|\cdot\}, \{\cdot \mid O\}, \{O \mid O\}$ are created and one obtains the class

$$\mathfrak{R}_1 = \{O, \{O|\cdot\}, \{\cdot \mid O\}, \{O \mid O\}\}$$

of all combinatorial games G with $|G| \leq 1$.

DAY 2: The creation of the class $\mathfrak{R}_2$ of those combinatorial games with options in $\mathfrak{R}_1$ is completed. These include the games already in $\mathfrak{R}_1$ and the new games

$$\begin{array}{l}
\{\cdot|\{O|\cdot\}\}, \{\cdot \mid \{O|\cdot\}\}, \{\cdot \mid \{\cdot \mid O\}\} \ldots \\
\{O|\{O|\cdot\}\}, \{O \mid \{O|\cdot\}\}, \{O \mid \{\cdot \mid O\}\} \ldots \\
\{O, \{\cdot \mid O\}|\{O|\cdot\}\}, \{O, \{O \mid \cdot\} \mid \{O, \{O \mid \cdot\}|\cdot\} \ldots \\
\{O, \{\cdot \mid O\}|\{O|\cdot\}\} \ldots \\
\vdots
\end{array}$$

DAY t: The class $\mathfrak{R}_t$ of all those combinatorial games G with options in $\mathfrak{R}_{t-1}$ is created.

So one has $\mathfrak{R}_0 \subset \mathfrak{R}_1 \subset \ldots \subset \mathfrak{R}_t \subset \ldots$ and

$$\mathfrak{R} = \mathfrak{R}_0 \cup \mathfrak{R}_1 \cup \ldots \cup \mathfrak{R}_t \cup \ldots$$

EX. 2.5. *The number of combinatorial games grows rapidly:*

(1) *List all the combinatorial games in $\mathfrak{R}_2$.*
(2) *Argue that many more than 6000 combinatorial games exist at the end of* DAY 3 *(see Ex. 2.6).*

EX. 2.6. *Show that* $r_t = |\mathfrak{R}_t|$ *grows super-exponentially fast:*

$$r_t > 2^{r_{t-1}} \quad (t = 1, 2, \ldots)$$

(Hint: A finite set S *with* $n = |S|$ *elements admits* 2^n *subsets.)*

4. Winning strategies

A combinatorial game is started with either L or R making the first move. This determines the *first player*. The other player is the *second* player. The *normal* winning rule for an alternating 2-person games is:

(NR) If a player $i \in \{L, R\}$ cannot move, player i has *lost* and the other player is declared the *winner*.

Chess matches, for example, are played under the normal rule: A loss of the king means a loss of the match (see Ex. 2.1).

REMARK 2.3 (Misère). *The* misère *rule declares the player with no move to be the winner of the game.*

A *winning strategy* for player i is a move (option) selection rule for i that ensures i to be the winner.

THEOREM 2.1. *In any combinatorial game* G*, an overall winning strategy exists for either the first or the second player.*

Proof. We prove the theorem by mathematical induction on $t = |G|$. In the case $t = 0$, we have

$$G = O = \{\cdot \mid \cdot\}.$$

Because the first player has no move in O, the second player is automatically the winner in normal play and hence has a winning strategy trivially guaranteed. Under the misère rule, the first player wins.

Suppose now $t \geq 1$ and that the Theorem is true for all games that were created on DAY $t - 1$ or before. Consider the first player in G and assume that it is R. (The argument for L would go exactly the same way!)

If R has no option, L is the declared winner in normal play while R is the declared the winner in misère play. Either way, G has a guaranteed winner.

If options G^R exist, the induction hypothesis says that each of R's options leads to a situation in which either the first or the second player would have a winning strategy.

If there is (at least) one option G^R with the second player as the winner, R can take this option and win as the second player in G^R.

On the other hand, if all of R's options have their first player as the winner, there is nothing R can do to prevent L from winning. So the originally second player L has an overall strategy to win the game guaranteed.

$\diamond$

Note that the proof of Theorem 2.1 is constructive in the following sense:

(1) Player i marks by $v(G^i) = +1$ all the options G^i in G that would have i as the winner and sets $v(G^i) = -1$ otherwise.
(2) Player i follows the strategy to move to an option with the highest v-value.
(3) Provided a winning strategy exists at all for i, strategy (2) is a winning strategy for i.

The reader must be cautioned, however. The concrete computation of a winning strategy may be a very difficult task in real life,

EX. 2.7 (De Bruin's game). *Two players choose a natural number $n \geq 1$ and write down all the numbers*

$$1, 2, 3, \ldots, n-1, n.$$

A move of a player consists in selecting one of the numbers still present and erasing it together with all its (proper or improper) divisors.

Note that a winning strategy exists for the first player in normal play. Indeed, if it existed for the second player, the first player could simply erase "1" on the first move and afterwards (being now the second player) follow that strategy and win. Alas, no practically efficient method for the computation of a winning strategy is known.

REMARK 2.4. *If chess is played with a finiteness rule, then a winning strategy exists for one of the two players. Unfortunately, it is not known what it looks like. It is not even known which player is the potential guaranteed winner.*

4.1. Playing in practice. While winning strategies can be computed in principle (see the proof of Theorem 2.1), the combinatorial structure of many games is so complex that even today's computers cannot perform the computation efficiently.

In practice, a player i will proceed according to the following v-*greedy* strategy[2]:

> $(vg_i\,)$ *Assign a quality estimate* $v(G^i) \in \mathbb{R}$ *to all the options* G^i *and move to an option with a highest* v*-value.*

A quality estimate v is not necessarily completely pre-defined by the game in absolute terms but may reflect previous experience and other considerations. Once quality measures are accepted as ”reasonable”, it is perhaps natural to expect that the game would evolve according to greedy strategies relative to these measures.

EX. 2.8. *A popular rule of thumb evaluates the quality of a chess configuration* σ *for a player* W*, say, by assigning a numerical weight* v *to the white pieces on the board. For example:*

$$\begin{aligned} v(\text{pawn}) &= 1 \\ v(\text{bishop}) &= 3 \\ v(\text{knight}) &= 3 \\ v(\text{castle}) &= 4.5 \\ v(\text{queen}) &= 9. \end{aligned}$$

Where $v(\sigma)$ *is the total weight of the white pieces, a* v*-greedy player* W *would choose a move to a configuration* σ *with a maximal value* $v(\sigma)$*. (Player* B *can, of course, evaluate the black pieces similarly.)*

5. Algebra of games

For the rest of the chapter we will (unless explicitly said otherwise) assume:

- *The combinatorial games under consideration are played with the normal winning rule.*

The set $\mathfrak{R}$ of combinatorial games has an algebraic structure which allows us to view games as generalized numbers. This section will give a short sketch of the idea [3].

[2] also chess computer programs follow this idea

[3] (much) more can be found in the highly recommended treatise: J.H. CONWAY, *On Numbers and Games.* A.K. Peters, 2000

Negation. We first define the *negation* for the game

$$G = \{G_1^L, G_2^L, \ldots \mid G_1^R, G_2^R, \ldots\} \in \mathfrak{R}$$

as the game $(-G)$ where the players L and R interchange their roles: L becomes the "right" and R the "left" player.

So we obtain the negated games recursively as

$$\begin{aligned} -O &= O \\ -G &= \{-G_1^R, -G_2^R \ldots \mid -G_1^L, -G_2^L, \ldots\} \quad \text{if } G \neq O. \end{aligned}$$

Also $-G$ is a combinatorial game and one has the algebraic rule

$$G = -(-G).$$

Addition. The *sum* $G+H$ of the games G and H is the game in which a player $i \in \{L, R\}$ may choose to play either on G or on H. This means that i chooses an option G^i in G *or* an option H^i in H and accordingly reduces the game

$$\text{either to} \quad G^i + H \quad \text{or to} \quad G + H^i.$$

The reader is invited to verify the further algebraic rules:

$$\begin{aligned} G + H &= H + G \\ (G + H) + K &= G + (H + K) \\ G + O &= G. \end{aligned}$$

Moreover, we write

$$G - H = G + (-H).$$

EX. 2.9. The second player wins $G - G$ in normal play with the obvious strategy:

- *Imitate every move of the first player. When the first player chooses the option G^i in G, the second player will answer with the option $(-G^i)$ in $(-G)$ etc.*

5.1. Congruent games. Motivated by Ex. 2.9, we say that combinatorial games G and H are *congruent* (notation: "$G \equiv H$") if

(C) $G - H$ is won by the second player (in normal play).

In particular, $G \equiv O$ means that G is won by the second player.

THEOREM 2.2 (Congruence theorem). *For all $G, H, K \in \mathfrak{R}$ one has:*

(a) *If $G \equiv H$, then $H \equiv G$.*
(b) *If $G \equiv H$, then $G + K \equiv H + K$.*
(c) *If $G \equiv H$ and $H \equiv K$, then $G \equiv K$.*

Proof. The verification of the commutativity rule (a) is left to the reader. To see that (b) is true, we consider the game

$$\begin{aligned} M &= (G+K)-(H+K) \\ &= G+K-H-K \\ &= (G-H)+(K-K). \end{aligned}$$

The game $K-K$ can always be won by the second player (Ex. 2.9). Hence, if the second player can win $G-H$, then clearly M as well:

- It suffices for the second player to apply the respective winning strategies to $G-H$ and to $K-K$.

The proof of the transitivity rule (c) is similar. By assumption, the game

$$T = (G-K)+(-H+H) = (G-H)+(H-K)$$

can be won by the second player. We must show that the second player can therefore win $G-K$.

Suppose to the contrary that $G-K \not\equiv O$ were true and that the game $G-K$ could be won by the first player. Then the first player could win T by beginning with a winning move in $G-K$ and continuing with the win strategy whenever the second player moves in $G-K$. If the second player moves in $K-K$, the first player becomes second there and thus is assured to win on $K-K$! So the first player would win T, which would contradict the assumption however.

Hence we conclude that $G-K \equiv O$ must hold.

$\diamond$

Congruence classes. For any $G \in \mathfrak{R}$, the class of congruent games is

$$[G] = \{H \in \mathfrak{R} \mid G \equiv H\}.$$

Theorem 2.2 says that addition and subtraction can be meaningfully defined for congruence classes:

$$[G]+[H] = [G+H] \quad \text{and} \quad [G]-[H] = [G-H].$$

In particular, we obtain the familiar algebraic rule

$$[G] - [G] = [G - G] = [O],$$

where $[O]$ is the class of all combinatorial games that are won by the second player. Hence we can re-cast the optimal strategy for a player (under the normality rule):

- *Winning strategy:*
  Make a move $G \mapsto G'$ to an option $G' \in [O]$.

5.2. Strategic equivalence. Say that the combinatorial games G and H are *strategically equivalent* (denoted "$G \sim H$") if one of the following statements is true:

(SE_1) G and H can be won by the first player (*i.e.*, $G \not\equiv O \not\equiv H$).
(SE_2) G and H can be won by the second player (*i.e.*, $G \equiv O \equiv H$).

THEOREM 2.3 (Strategic equivalence). *Congruent games $G, H \in \mathfrak{R}$ are strategically equivalent,* i.e.*,*

$$G \equiv H \quad \Longrightarrow \quad G \sim H.$$

Proof. We claim that strategically non-equivalent games G and H cannot be congruent.

So assume, for example, that the first players wins G (*i.e.*, $G \not\equiv O$), and the second player wins H (*i.e.*, $H \equiv O$ and hence $(-H) \equiv O$). We will argue that the first player has a winning strategy for $G - H$, which means $G \not\equiv H$.

Indeed, the first player can begin with a winning strategy on G. Once the second player moves on $(-H)$, the first player, being now the second player on $(-H)$, wins there. Thus an overall victory is guaranteed for the first player.

$\diamond$

6. Impartial games

A combinatorial game G is said to be *impartial* (or *neutral*) if both players have the same options. The formal definition is recursive:

- $O = \{\cdot \mid \cdot\}$ is impartial.
- $G = \{A, B, \ldots, T \mid A, B, \ldots\}$ is impartial if all the options $A, B, \ldots, T$ are impartial.

Notice the following rules for impartial games G and H:

(1) $G = -G$ and hence $G + G = G - G \in [O]$.

(2) $G + H$ is impartial.

Nim is the prototypical impartial game (as we will see with the Sprague-Grundy Theorem 2.4 below).

To formalize this claim, we use the notation $*n$ for a nim game relative to just one single set N_1 with $n = |N_1|$ elements. The options of $*n$ are the nim games

$$*0, *1, \ldots, *(n-1).$$

Moreover,

$$G = *n_1 + *n_2 + \ldots + *n_k$$

is the nim game described in Ex. 2.3 with k piles of sizes $n_1, n_2, \ldots, n_k$.

EX. 2.10. *Show that the frog game of Ex. 2.4 is impartial.*

We now define the *mex* ("minimal excluded") of numbers $a, b, c \ldots, t$ as the smallest natural number g that equals none of the numbers $a, b, c, \ldots, t$:

$$\boxed{\mathrm{mex}\{a, b, c, \ldots, t\} = \min\{g \geq 0 \mid g \notin \{a, b, c, \ldots, t\}\},} \tag{11}$$

The crucial observation is stated in Lemma 2.1.

LEMMA 2.1. *Let $a, b, c, \ldots t$ be arbitrary natural numbers. Then one has*

$$G = \{*a, *b, *c, \ldots, *t \mid *a, *b, *c, \ldots, *t\} \equiv *\mathrm{mex}\{a, b, c, \ldots, \},$$

i.e., *the impartial game G with the nim options $*a, *b, *c, \ldots, *t$ is equivalent to the simple nim game $*m$ with $m = \mathrm{mex}\{a, b, c, \ldots, \}$.*

Proof. In view of $*m = - * m$, we must show: $G + *m \equiv O$, *i.e.*, the second player wins $G + *m$. Indeed, if the first player chooses an option j from

$$*m = \{*0, *1, \ldots, *(m-1)\},$$

then the second player can choose $j*$ from G (which must exist because of the definition of m as the minimal excluded number) and continue to win $*j + *j$ as the second player.

If the first player selects an option from G, say $*a$, we distinguish two cases. If $a > m$ then the second player reduces $*a$ to $*m$ and wins. If $a < m$, then the second player can reduce $*m$ to $*a$ and win. (Note that $a = m$ is impossible by the definition of mex.)

$\diamond$

THEOREM 2.4 (SPRAGUE-GRUNDY). *Every impartial combinatorial game*

$$G = \{A, B, C, \ldots \mid A, B, C \ldots, T\}$$

is equivalent to a unique nim game of type $*m$. *The number* m *is the so-called* GRUNDY number $\mathcal{G}(G)$ *and can be computed recursively:*

$$m = \mathcal{G}(G) = \operatorname{mex}\{\mathcal{G}(A), \mathcal{G}(B), \mathcal{G}(C), \ldots, \mathcal{G}(T)\}. \tag{12}$$

Proof. We prove the Theorem by induction on $|G|$ and note $G \equiv O$ if $|G| = 0$. By induction, we now assume that the Theorem is true for all options of G, *i.e.*, $A \equiv *a$, $B \equiv *b$ *etc.* with $a = \mathcal{G}(A)$, $b = \mathcal{G}(B)$ *etc.*

Hence we can argue $G \equiv *m = *\mathcal{G}(G)$ exactly as in the proof of Lemma 2.1. G cannot be equivalent to another nim game $*k$ since (as Ex. 2.11 below shows):

$$*k \equiv *m \quad \Longrightarrow \quad k = m.$$

$\diamond$

EX. 2.11. *Show for all natural numbers* k *and* n*:*

$$*k \equiv *m \quad \Longleftrightarrow \quad k = m.$$

EX. 2.12 (GRUNDY number of frogs). *Let* $F(n, k)$ *be the frog game of Ex. 2.4 and* $\mathcal{G}(n, k)$ *its* GRUNDY *number. For* $k = 3$, $F(n, k)$ *has the options*

$$F(n-1, 3), F(n-2, 3), F(n-3, 3).$$

So the associated GRUNDY *number* $\mathcal{G}(n, 3)$ *has the recursion*

$$\mathcal{G}(n, 3) = \operatorname{mex}\{\mathcal{G}(n-1, 3), \mathcal{G}(n-2, 3), \mathcal{G}(n-3)\}.$$

Clearly, $\mathcal{G}(0, 3) = 0, \mathcal{G}(1, 3) = 1$ *and* $\mathcal{G}(2, 3) = 2$. *The recursion then produces the subsequent* GRUNDY *numbers:*

n	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$\mathcal{G}(n, 3)$	0	1	2	3	0	1	2	3	0	1	2	$\cdots$

The second player wins the nim game $*m$ if and only if $m = 0$. So the first player can win exactly the impartial games G with GRUNDY number $\mathcal{G}(G) \neq 0$. In general, we note:

- *Winning strategy for impartial games:*
  Make a move $G \mapsto G'$ to an option G'
  with a GRUNDY number $\mathcal{G}(G') = 0$.

6.1. Sums of GRUNDY numbers. If G and H are impartial games with GRUNDY numbers $m = \mathcal{G}(G)$ and $n = \mathcal{G}(H)$, the GRUNDY number of their sum is

$$\mathcal{G}(G+H) = \mathcal{G}(*n + *m)$$

Indeed, if $G \equiv *m$ and $H \equiv *n$, then $G + H \equiv *m + *n$ must hold[4]. For the study of sums, we may therefore restrict ourselves to nim games. Moreover, the fundamental property

$$\mathcal{G}(G+G) = \mathcal{G}(*n + *n) = \mathcal{G}(O) = 0$$

suggests to study sums in the context of *binary algebra*.

Binary algebra. Recall that every natural number n has a unique binary representation in terms of powers of 2,

$$n = \sum_{j=0}^{\infty} \alpha_j 2^j,$$

with binary coefficients $\alpha_j \in \{0,1\}$. We define binary addition of 0 and 1 according to the rules

$$0 \oplus 0 = 0 = 1 \oplus 1 \quad \text{and} \quad 0 \oplus 1 = 1 = 1 \oplus 0$$

and extend it to natural numbers:

$$\Big(\sum_{j=0}^{\infty} \alpha_j 2^j\Big) \;\oplus\; \Big(\sum_{j=0}^{\infty} \beta_j 2^j\Big) = \sum_{j=0}^{\infty} (\alpha_j \oplus \beta_j) 2^j.$$

REMARK 2.5. *Notice that* $\alpha_j = 0$ *must hold for all* $j > \log_2 n$ *if*

$$n = \sum_{j=0}^{\infty} \alpha_j 2^j \quad \text{with } \alpha_j \in \{0,1\}.$$

EX. 2.13. *Show for the binary addition of natural numbers* m, n, k*:*

$$\begin{aligned} n \oplus m &= m \oplus n \\ n \oplus (m \oplus k) &= (n \oplus m) \oplus k \\ n \oplus m \oplus k = 0 &\iff n \oplus m = k. \end{aligned}$$

[4] recall Theorem 2.2!

The sum theorem. We consider nim games with three piles of n, m and k objects, *i.e.*, sums of 3 single nim games $*n$, $*m$, and $*k$.

LEMMA 2.2. *For all natural numbers n, m, k, one has:*

(1) *If $n \oplus m \oplus k \neq 0$, then the first player wins $*n + *m + *k$.*
(2) *If $n \oplus m \oplus k = 0$, then the second player wins $*n + *m + *k$.*

Proof. We prove the Lemma by induction on $n + m + k$ and note that the statements (1) and (2) are obviously true in the case $n + m + k = 0$. By induction, we now assume that the Lemma is true for all natural numbers n', m', k' such that

$$n' + m' + k' < n + m + k.$$

We must now show that the Lemma holds for n, m, k with the binary representations

$$n = \sum_{j=0}^{\infty} \alpha_j 2^j,\ m = \sum_{j=0}^{\infty} \beta_j 2^j,\ k = \sum_{j=0}^{\infty} \gamma_j 2^j.$$

In the case (1) with $n \oplus m \oplus k \neq 0$, there must be at least one j such that

$$\alpha_j \oplus \beta_j \oplus \gamma_j = 1.$$

Let J be the largest such index j. Two of these coefficients $\alpha_J, \beta_J, \gamma_J$ must be equal and the third one must have value 1. So suppose $\alpha_J = \beta_J$ and $\gamma_J = 1$, for example, which implies

$$n \oplus m < k \quad \text{and} \quad n + m + (n \oplus m) < n + m + k.$$

Let $k' = n \oplus m$. We claim that the first player can win by reducing $*k$ to $*k'$. Indeed, the induction hypothesis says that the Lemma is true for n, m, k'. Since

$$n \oplus m \oplus k' = n \oplus m \oplus n \oplus m = 0,$$

property (2) guarantees a winning strategy for the second player in the reduced nim game

$$*n + *m + *k'.$$

But the latter is the originally first player! So statement (1) is found to be true.

In case (2), when $n \oplus m \oplus k = 0$, the first player must make a move on one of the three piles. Let us say that $*n$ is reduced to $*n'$. Because $n = m \oplus k$, we have

$$n' \neq m \oplus k \quad \text{and therefore} \quad n' \oplus m \oplus k \neq 0.$$

Because the Lemma is assumed to be true for n', m, k, statement (1) guarantees a winning strategy for the first player in the reduced game

$$*n' + *m * k,$$

which is the originally second player.

$\diamond$

THEOREM 2.5 (Sums of impartial games). *For any impartial combinatorial games G and H, one has*

$$\mathcal{G}(G+H) = \mathcal{G}(G) \oplus \mathcal{G}(H).$$

Proof. Let $n = \mathcal{G}(G)$ and $m = \mathcal{G}(H)$ and $k = n \oplus m$. Then $n \oplus m + k = 0$ holds. So Lemma 2.2 says that the second player wins $n * + * m * (n \oplus m)$, which yields

$$G + H \equiv *n + *m \equiv *(n \oplus m).$$

Consequently, $n \oplus m$ must be the GRUNDY number of $G + H$.

$\diamond$

We illustrate Theorem 2.5 with the nim game

$$G = *1 + *3 * + * 5 + *7$$

of four piles with $1, 3, 5$ and 7 objects respectively. The binary representations of the pile sizes are

$$\begin{aligned} 1 &= 1 \cdot 2^0 \\ 3 &= 1 \cdot 2^0 + 1 \cdot 2^1 \\ 5 &= 1 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 \\ 7 &= 1 \cdot 2^0 + 1 \cdot 2^1 + 1 \cdot 2^2. \end{aligned}$$

So the GRUNDY number of G is

$$\mathcal{G}(*1 + *3 * + * 5 + *7) = 1 \oplus 3 \oplus 5 \oplus 7 = 0.$$

Hence G can be won by the second player in normal play.

EX. 2.14. *Suppose that the first player removes* 3 *objects from the pile of size* 7 *in* $G = *1 + *3 * + * 5 + *7$. *How should the second player respond?*

EX. 2.15. *There is a pile of* 10 *red and another pile of* 10 *black pebbles. Two players move alternatingly with the following options:*

- EITHER: *take at least* 1 *but not more than* 3 *of the red pebbles*
  OR: *take at least* 1 *but not more than* 2 *of the black pebbles.*

Which of the players has a winning strategy in normal play? (Hint: Compute the GRUNDY *numbers for the red and black piles separately (as in Ex. 2.4) and apply Theorem 2.5.)*

CHAPTER 3

Zero-sum Games

Zero-sum games abstract the model of combinatorial games. They arise naturally as LAGRANGE games from mathematical optimization problems and thus furnish an important link between game theory and mathematical optimization theory. In particular, strategic equilibria in games correspond to optimal solutions of optimization problems. Conversely, mathematical optimization techniques are important tools for the analysis of game theoretic situations.

As in the previous chapter, we consider games Γ with 2 agents (or players). However, rather than having to compute explicit strategies from scratch, the players are assumed to have sets X and Y of possible strategies (or decisions, or actions *etc.*) already at their disposal. One player now chooses a strategy $x \in X$ and the other player a strategy $y \in Y$. So Γ can be viewed as being played on the system

$$\mathfrak{G} = X \times Y = \{(x, y) \mid x \in X, y \in Y\} \cup \{\sigma_0\}$$

of all possible joint strategy choices of the players together with an initial state σ_0. Γ is a *zero-sum game* if there is a function

$$U : X \times Y \to \mathbb{R}$$

that encodes the *utility* of the strategic choice (x, y) in the sense that the utility values of the individual players add up to zero:

(1) $u_1(x, y) = U(x, y)$ is the gain of the X-player;
(2) $u_2(x, y) = -U(x, y)$ is the gain of the Y-player.

So the two players have opposing goals:

(X) *The x-player wants to choose $x \in X$ as to maximize $U(x, y)$.*
(Y) *The y-player wants to choose $y \in Y$ as to minimize $U(x, y)$.*

We denote the corresponding zero-sum game by $\Gamma = \Gamma(X, Y, U)$.

EX. 3.1. *A combinatorial game with respective strategy sets X and Y for the two players is, in principle, a zero-sum game with the utility function*

$$U(x,y) = \begin{cases} +1 & \text{if } x \text{ is a winning strategy for the } x\text{-player} \\ -1 & \text{if } y \text{ is a winning strategy for the } x\text{-player} \\ 0 & \text{otherwise.} \end{cases}$$

1. Matrix games

In the case of finite strategy sets, say $X = \{1, \ldots, m\}$ and $Y = \{1, \ldots, n\}$, a function $U : X \times Y \to \mathbb{R}$ can be given in matrix form:

$$U = \begin{bmatrix} u_{11} & u_{21} & \ldots & u_{1n} \\ u_{21} & u_{22} & \ldots & u_{2n} \\ \vdots & \vdots & & \vdots \\ u_{m1} & u_{m2} & \ldots & u_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}.$$

$\Gamma = (X, Y, U)$ is the game where the x-player chooses a row i and the y-player a column j. This joint selection (i, j) has the utility value u_{ij} for the row player and the value $(-u_{ij})$ for the column player. As an example, consider the game with the utility matrix

$$U = \begin{bmatrix} +1 & -2 \\ -1 & +2 \end{bmatrix}. \tag{13}$$

There is no obvious overall ”optimal” choice of strategies. No matter what row i or column j is selected by the players, one of the players will find[1] that the other choice would have been more profitable. In this sense, this game has no ”solution”.

Before pursuing this point further, we will introduce a general concept for the *solution* of a zero-sum game in terms of an equilibrium between both players.

2. Equilibria

Let us assume that both players in the zero-sum game $\Gamma = (X, Y, U)$ are risk avoiding and want to ensure themselves optimally against the worst case. So they consider the worst case functions

$$\begin{aligned} U_1(x) &= \min_{y \in Y} u(x,y) \in \mathbb{R} \cup \{-\infty\} \\ U_2(y) &= \max_{x \in X} u(x,y) \in \mathbb{R} \cup \{+\infty\}. \end{aligned} \tag{14}$$

The x-player thus faces the *primal problem*

[1] in hindsight!

$$\max_{x \in X} U_1(x) = \max_{x \in X} \min_{y \in Y} U(x,y), \tag{15}$$

while the y-player has to solve the *dual problem*

$$\min_{y \in Y} U_2(y) = \min_{y \in Y} \max_{x \in X} U(x,y). \tag{16}$$

From the definition, one immediately deduces for any $x \in X$ and $y \in Y$ the *primal-dual inequality*:

$$U_1(x) \leq U(x,y) \leq U_2(y) \tag{17}$$

We say that $(x^*, y^*) \in X \times Y$ is an *equilibrium* of the game Γ if it yields the equality $U_1(x^*) = U_2(y^*)$, *i.e.*, if the primal-dual inequality is, in fact, an equality:

$$\boxed{\max_{x \in X} \min_{y \in Y} U(x,y) = U(x^*, y^*) = \min_{y \in Y} \max_{x \in X} U(x,y)} \tag{18}$$

In the equilibrium (x^*, y^*), none of the risk avoiding players has an incentive to deviate from the chosen strategy. In this sense, equilibria represent *optimal* strategies for risk avoiding players.

EX. 3.2. *Determine the best worst-case strategies for the two players in the matrix game with the matrix U of (13) and show that the game has no equilibrium.*

Give furthermore an example of a matrix game that possesses at least one equilibrium.

If the strategy sets X and Y are finite and hence $\Gamma = (X, Y, U)$ is a matrix game, the question whether an equilibrium exists, can – in principle – be answered in finite time by a simple procedure:

- *Check each strategy pair $(x^*, y^*) \in X \times Y$ for the property (18).*

If X and Y are infinite, the existence of equilibria can usually only be decided if the function $U : X \times Y \to \mathbb{R}$ has special properties. From a theoretical point of view, the notion of convexity is very helpful and important.

3. Convex zero-sum games

Recall[2] that a *convex combination* of points $x_1, \ldots, x_k \in \mathbb{R}^n$ is a linear combination

$$\overline{x} = \sum_{i=1}^{k} \lambda_i x_i \quad \text{with coefficients } \lambda_i \geq 0 \text{ such that } \sum_{i=1}^{k} \lambda_i = 1.$$

An important interpretation of $\overline{x}$ is based on the observation that the coefficient vector $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a probability distribution:

- If a point x_i is selected from the set $\{x_1, \ldots, x_k\}$ with probability λ_i, then the convex combination $\overline{x}$ has as components exactly the expected component values of the stochastically selected point.

Another way of looking at $\overline{x}$ is:

- If weights of size λ_i are placed on the points x_i, then $\overline{x}$ is their center of gravity.

A set $X \subseteq \mathbb{R}^n$ is *convex* if X contains all convex combinations of all possible finite subsets $\{x_1, \ldots, x_k\} \subseteq X$.

EX. 3.3. *Let $S = \{s_1, \ldots, s_m\}$ an arbitrary set with $m \geq 1$ elements. Show that the set $\overline{S}$ of all probability distributions λ on S forms a compact convex subset of $\mathbb{R}^m$.*

A function $f : X \to \mathbb{R}$ is *convex* (or *convex up*) if X is a convex subset of some coordinate space $\mathbb{R}^n$ and for every $x_1, \ldots, x_k \in X$ and probability distribution $\lambda = (\lambda_1, \ldots, \lambda_k)$, one has

$$f(\lambda_1 x_1 + \ldots + \lambda_k x_k) \leq \lambda_1 f(x_1) + \ldots + \lambda_k f(x_k).$$

f is *concave* (or *convex down*) if $g = -f$ is convex (up).

With this terminology, we say that the zero-sum game $\Gamma = (X, Y, U)$ is *convex* if

(1) X and Y are non-empty convex strategy sets;
(2) the utility $U : X \times Y \to \mathbb{R}$ is such that
   (a) For every $y \in Y$, the map $x \mapsto U(x, y)$ is concave.
   (b) For every $x \in X$, the map $y \mapsto U(x, y)$ is convex.

[2] see also Section 2 of the Appendix for more details

The main theorem on general convex zero-sum-games guarantees the existence of at least one equilibrium in the case of compact strategy sets:

> THEOREM 3.1. *A convex zero-sum game $\Gamma = (X, Y, U)$ with compact strategy sets X and Y and a continuous utility U admits a strategic equilibrium $(x^*, y^*) \in X \times Y$.*

Proof. Since X and Y are convex and compact, so is $Z = X \times Y$ and hence also $Z \times Z$. Consider the continuous function $G : Z \times Z \to \mathbb{R}$ where

$$G((x', y'), (x, y)) = U(x, y') - U(x', y).$$

Since U is concave in the first variable x and $(-U)$ concave in the second variable y, we find that G is concave in the second variable (x, y). So we deduce from Corollary A.1 in the Appendix the existence of an element $(x^*, y^*) \in Z$ that satisfies for all $(x, y) \in Z$ the inequality

$$0 = G((x^*, y^*), (x^*, y^*)) \ \geq \ G((x^*, y^*), (x, y)) = U(x, y^*) - U(x^*, y)$$

and hence

$$U(x, y^*) \leq U(x^*, y) \quad \text{for all } x \in X \text{ and all } y \in Y.$$

This shows that x^* is the best strategy for the x-player if the y-player chooses $y^* \in Y$. Similarly, y^* is optimal against x^*. In other words, (x^*, y^*) is an equilibrium of (X, Y, U).

$\diamond$

Theorem 3.1 has important consequences not only in game theory but also in the theory of mathematical optimization in general, which we will sketch in Section 4 in more detail below. To illustrate the situation, let us first look at the special case of randomizing the strategic decisions in finite zero-sum games.

3.1. Randomized matrix games. The utility U of a zero-sum game $\Gamma = (X, Y, U)$ with finite sets $X = \{1, \ldots, m\}$ can be described as a matrix $U = [u_{ij}] \in \mathbb{R}^{m \times n}$ with coefficients u_{ij}. Such a matrix game Γ does not necessarily admit an equilibrium.

Suppose the players *randomize* the choice of their respective strategies. That is to say, the x- player decides on a probability distribution $\overline{x}$ on X and chooses an $i \in X$ with probability $\overline{x}_i$. Similarly, the y-player chooses a probability distribution $\overline{y}$ on Y and chooses $j \in Y$ with probability $\overline{y}_j$. Then the x-player's expected gain is

$$\overline{U}(\overline{x}, \overline{y}) = \sum_{i=1}^{m} \sum_{j=1}^{n} u_{ij} \overline{x}_i \overline{y}_j.$$

So we arrive at a zero-sum game $\overline{\Gamma} = (\overline{X}, \overline{Y}, \overline{U})$, where $\overline{X}$ is the set of probability distributions on X and $\overline{Y}$ the set of probability distributions on Y. $\overline{X}$ and $\overline{Y}$ are compact convex sets (*cf.* Ex. 3.3). The function $\overline{U}$ is linear – and thus both concave and convex – in both components. It follows that $\overline{\Gamma}$ is a convex game that satisfies the hypothesis of Theorem 3.1 and therefore admits an equilibrium. This proves VON NEUMANN's Theorem[3]:

THEOREM 3.2 (VON NEUMANN). *Let $U \in \mathbb{R}^{m\times n}$ be an arbitrary matrix with coefficients u_{ij}. Then there exist $x^* \in X$ and $y^* \in Y$ such that*

$$\max_{x\in X}\min_{y\in Y}\sum_{ij} u_{ij}x_iy_i = \sum_{ij} u_{ij}x_i^*y_i^* = \min_{y\in Y}\max_{x\in X}\sum_{ij} u_{ij}x_iy_i.$$

where X is the set of all probability distributions on $\{1,\ldots,m\}$ and Y the set of all probability distributions on $\{1,\ldots,n\}$. ◇

3.2. Computational aspects. While it is generally not easy to compute equilibria in zero-sum games, the task becomes tractable for randomized matrix games. Consider, for example, the two sets $X = \{1,\ldots,m\}$ and $Y = \{1,\ldots,n\}$ and the utility matrix

$$U = \begin{bmatrix} u_{11} & u_{21} & \ldots & u_{1n} \\ u_{21} & u_{22} & \ldots & u_{2n} \\ \vdots & \vdots & & \vdots \\ u_{m1} & u_{m2} & \ldots & u_{mn} \end{bmatrix} \in \mathbb{R}^{m\times n}$$

For the probability distributions $x \in \overline{X}$ and $y \in \overline{Y}$, the expected utility for the X-player is

$$\overline{U}(x,y) = \sum_{i=1}^m\sum_{j=1}^n u_{ij}x_iy_j = \sum_{j=1}^n y_j\big(\sum_{i=1}^m u_{ij}x_i\big).$$

So the worst case for the x-player happens when the y-player selects a probability distribution that puts the full weight 1 on $k \in Y$ such that

$$\sum_{i=1}^m u_{ik}x_i = \min\left\{\sum_{i=1}^m u_{ij}x_i \mid j = 1,\ldots,n\right\} = \overline{U}_1(x).$$

Hence

$$\max_{x\in\overline{X}} U_1(x) = \max_{z\in\mathbb{R},x\in\overline{X}}\{z \mid z \le \sum_{i=1}^m u_{ij}x_i \ \text{ for all } i = 1,\ldots,m\}. \tag{19}$$

[3] J. VON NEUMANN (1928): *Zur Theorie der Gesellschaftsspiele*, Math. Annalen 100

Similarly, the worst case for the y-player it attained when the x-player puts the full probability weight 1 onto $\ell \in X$ such that

$$\sum_{j=1}^{n} u_{\ell j} y_j = \max\left\{\sum_{j=1}^{n} u_{ij} y_j \mid i = 1, \ldots, m\right\} = \overline{U}_2(y).$$

This yields

$$\min_{y \in \overline{Y}} \overline{U}_2(y) = \min_{w \in \mathbb{R}, y \in \overline{Y}} \{w \mid w \geq \sum_{j=1}^{n} u_{ij} y_j \ \text{ for all } j = 1, \ldots, n\}. \tag{20}$$

This analysis shows:

PROPOSITION 3.1. *If (z^*, x^*) is an optimal solution of (19) and (w^*, y^*) and optimal solution of (20), then*

1. (x^*, y^*) *is an equilibrium of* $\overline{\Gamma} = (\overline{X}, \overline{Y}, \overline{U})$.
2. $z^* = \max\limits_{x \in \overline{X}} U_1(x) = \min\limits_{y \in \overline{Y}} U_2(y) = w^*$.

REMARK 3.1. *As further outlined in Section 4.4 below, the optimization problems (19) and (19) are so-called* linear programs *that are dual to each other. They can be solved very efficiently in practice. For explicit solution algorithms, we refer the interested reader to the standard literature on mathematical optimization*[4].

4. LAGRANGE games

The analysis of zero-sum games is very closely connected with a fundamental technique in mathematical optimization. A very general form of an optimization problem is

$$\max_{x \in \mathcal{F}} f(x),$$

where $\mathcal{F}$ could be any set and $f : \mathcal{F} \to \mathbb{R}$ an arbitrary *objective function*. In our context, however, we will look at more concretely specified problems and understand by a *mathematical optimization problem* a problem of the form

$$\max_{x \in X} f(x) \quad \text{such that} \quad g(x) \geq 0, \tag{21}$$

where X is a subset of some coordinate space $\mathbb{R}^n$ with an objective function $f : X \to \mathbb{R}$. The vector valued function $g : X \to \mathbb{R}^m$ is a *restriction*

[4] *e.g.*, U. FAIGLE, W. KERN and G. STILL, *Algorithmic Principles of Mathematical Programming*, Springer (2002)

function and combines m real-valued restriction functions $g_i : X \to \mathbb{R}$ as its components. The set of *feasible solutions* of (21) is

$$\mathcal{F} = \{x \in X \mid g_i(x) \geq 0 \text{ for all } i = 1, \ldots, m\}.$$

REMARK 3.2. *The model (21) formulates an optimization problem as a maximization problem. Of course, minimization problems can also be formulated within this model because of*

$$\min_{x \in \mathcal{F}} f(x) = -\max_{x \in \mathcal{F}} \tilde{f}(x) \quad \textit{with the objective function } \tilde{f}(x) = -f(x).$$

The optimization problem (21) gives rise to a zero-sum game $\Lambda = (X, \mathbb{R}^m_+, L)$ with the so-called LAGRANGE *function*

$$L(x, y) = f(x) + y^T g(x) = f(x) + \sum_{i=1}^{m} y_i g_i(x) \tag{22}$$

as its utility. We refer to Λ as a LAGRANGE *game*[5].

EX. 3.4 (Convex LAGRANGE games). *If X is convex and the objective function $f : X \to \mathbb{R}$ in (21) as well as the restriction functions $g_i : X \to \mathbb{R}$ are concave, then the* LAGRANGE *game $\Lambda = (X, \mathbb{R}^m_+, L)$ is a convex zero-sum game. Indeed, $L(x, y)$ is concave in x for every $y \geq 0$ and linear in y for every $x \in X$. Since linear functions are in particular convex, the game Λ is convex.*

Complementary slackness. The choice of an element $x \in X$ with at least one restriction violation $g_i(x) < 0$ would allow the y-player in the LAGRANGE game $\Lambda = (X, \mathbb{R}^m_+, L)$ to increase its utility value infinitely with $y_i \approx \infty$. So the risk avoiding x-player will always try to select a feasible x.

On the other hand, if $g_i(x) \geq 0$ holds for all i, the best the y-player can do is the selection of $y \in \mathbb{R}^m_+$ such that the so-called *complementary slackness condition*

$$\sum_{i=1}^{m} y_i g_i(x) = y^T g(x) = 0 \quad \text{and hence} \quad L(x, y) = f(x) \tag{23}$$

[5] the idea goes back to J.-L. LAGRANGE (1736-1813)

is satisfied. Consequently, one finds:

The primal LAGRANGE problem is identical with the original problem:

$$\max_{x \in X} \min_{y \geq 0} L(x, y) = \max_{x \in \mathcal{F}} L_1(x) = \max_{x \in \mathcal{F}} f(x). \tag{24}$$

The dual LAGRANGE worst case function is

$$L_2(y) = \max_{x \in X} f(x) + y^T g(x). \tag{25}$$

LEMMA 3.1. *If (x^*, y^*) is an equilibrium of the* LAGRANGE *game Λ, then x^* is an optimal solution of problem (21).*

Proof. For every feasible $x \in \mathcal{F}$, we have

$$f(x) \leq L_2(y^*) = L_1(x^*) = f(x^*).$$

So x^* is optimal.

$\diamond$

4.1. The KKT-conditions. Lemma 3.1 indicates the importance of being able to identify equilibria in LAGRANGE games. In order to establish necessary conditions, *i.e.*, conditions which candidates for equilibria must satisfy, we impose further assumptions on problem (21):

(1) $X \subseteq \mathbb{R}^n$ is a convex set, *i.e.*, X contains with every x, x' also the whole line segment

$$[x, x'] = \{x + \lambda(x' - x) \mid 0 \leq \lambda \leq 1\}.$$

(2) The functions f and g_i in (21) have continuous partial derivatives $\partial f(x)/\partial x_j$ and $\partial g_i(x)/\partial x_j$ for all $j = 1, \ldots, n$.

It follows that also the partial derivatives of the LAGRANGE function L exist. So the marginal change of L into the direction d of the x-variables is

$$\begin{aligned} \nabla_x L(x, y)d &= \nabla f(x)d + \sum_{i=1}^m y_i \nabla g_i(x)d \\ &= \sum_{j=1}^n \frac{\partial f(x)}{\partial x_j} d_j + \sum_{i=1}^m \sum_{j=1}^n \frac{\partial g_i(x)}{\partial x_j} y_i d_j. \end{aligned}$$

REMARK 3.3 (JACOBI matrix). *The* $(m \times n)$ *matrix* $Dg(x)$ *having as coefficients*

$$Dg(x)_{ij} = \frac{\partial g_i(x)}{\partial x_j}$$

the partial derivatives of a function $g : \mathbb{R}^n \to \mathbb{R}^m$ *is known as a* functional matrix *or* JACOBI[6] matrix. *It allows a compact matrix notation for the marginal change of the* Lagrange *function:*

$$\nabla_x L(x, y)d = \nabla f(x) + y^T Dg(x)d.$$

LEMMA 3.2 (KKT-conditions). *The pair* $(x, y) \in X \times \mathbb{R}^m_+$ *cannot be an equilibrium of the* LAGRANGE *game* Λ *unless:*

(K_0) $g(x) \geq 0$, i.e., x *is feasible.*
(K_1) $y^T g(x) = 0$.
(K_3) $\nabla_x L(x, y)d \leq 0$ *holds for all* d *such that* $x + d \in X$.

Proof. We already know that the feasibility condition (K_0) and the complementary slackness condition (K_1) are necessarily satisfied by an equilibrium. If (K_3) were violated and $\nabla_x L(x, y)d > 0$ were true, the x-player could improve the L-value by moving a bit into direction d. This would contradict the definition of an "equilibrium".

$\diamond$

REMARK 3.4. *The three conditions of Lemma 3.2 are the so-called* KKT-conditions[7]. *Although they are always necessary, they are not always sufficient to conclude that a candidate* (x, y) *is indeed an equilibrium.*

4.2. Shadow prices. The optimization problem

$$\max_{x \in \mathbb{R}^n_+} f(x) \quad \text{s.t.} \quad a_1(x) \leq b_1, \ldots, a_n(x) \leq b_m \tag{26}$$

is of type (21) with the m restriction functions $g_i(x) = b_i - a_i(x)$ and has the LAGRANGE function

[6] C.G. JACOBI (1804-1851)
[7] named after the mathematicians KARUSH, KUHN and TUCKER

$$
\begin{aligned}
L(x,y) &= f(x) + \sum_{i=1}^{m} y_i(b_i - a_i(x)) \\
&= f(x) - \sum_{i=1}^{m} y_i a_i(x) + \sum_{i=1}^{m} y_i b_i .
\end{aligned}
$$

For an intuitive interpretation of the problem (26), think of the data vector

$$x = (x_1, \ldots, x_n)$$

as a plan for n products to be manufactured in quantities x_j and of $f(x)$ as the market value of x.

Assume that x requires the use of m materials in respective quantities $a_i(x)$, for $i = 1, \ldots, m$, and that the b_i are the quantities of the materials already in the possession of the manufacturer.

If the y_i represent the market prices (per unit) of the m materials, $L(x,y)$ is the market value of the production x plus the value of the materials left in stock after the production of x. The manufacturer would, of course, like to have that value as high as possible.

"The market" is an opponent of the manufacturer and looks at the value

$$-L(x,y) = \sum_{i=1}^{m} y_i(a_i(x) - b_i) - f(x),$$

which is the value of the materials the manufacturer must still buy on the market for the production of x minus the value of the production that the market would have to pay to the manufacturer for the production x. The market would like to set the prices y_i so that $-L(x,y)$ is as large as possible.

Hence:

- *The manufacturer and the market play a* LAGRANGE *game* Λ.
- An equilibrium (x^*, y^*) of Λ reflects an economic balance: *Neither the manufacturer nor the market have a guaranteed way to improve their value by changing the production plan or by setting different prices.*

In this sense, the production plan x^* is *optimal*. The (from the market point of view) optimal prices $y_1^*, \ldots, y_m^*$ are the so-called *shadow prices* of the m materials.

The complementary slackness condition (K_1) says that a material which is in stock but not completely used by x^* has zero market value:

$$a_i(x^*) < b_i \quad \Longrightarrow \quad y_i^* = 0.$$

The condition (K_2) implies that x is a production plan of optimal value $f(x^*) = L(x^*, y^*)$ under the given restrictions. Moreover, one has

$$\sum_{i=1}^{m} y_i^* a_i(x^*) = \sum_{i=1}^{m} y_i^* b_i,$$

which says that the price of the materials used for the production x^* equals the value of the inventory under the shadow prices y_i^*.

Property (K_3) says that the marginal change $\nabla_x L(x^*, y^*)d$ of the manufacturer's value L is negative in any feasible production modification from x^* to $x^* + d$ and only profitable for the market because

$$\nabla_x(-L(x^*, y^*)) = -\nabla_x L(x^*, y^*).$$

We will return to production games in the context of cooperative game theory in Section 1.3.

4.3. Equilibria of convex LAGRANGE games. Remarkably, the KKT-conditions turn out to be not only necessary but also sufficient for the characterization of equilibria in convex LAGRANGE games with differentiable objective functions. This gives a way to compute such equilibria and hence to solve optimization problems of type (21) in practice[8]:

- *Find a solution $(x^*, y^*) \in X \times \mathbb{R}^m_+$ for the KKT-inequalities. (x^*, y^*) will yield an equilibrium in $\Lambda = (X, \mathbb{R}^m_+, L)$ and x^* will be an optimal solution for* (21).

THEOREM 3.3. *A pair $(x^*, y^*) \in X \times \mathbb{R}^m_+$ is an equilibrium of the convex* LAGRANGE *game $\Lambda = (X, \mathbb{R}^m_+, L)$ if and only if (x^*, y^*) satisfies the KKT-conditions.*

Proof. From Lemma 3.2, we know that the KKT-conditions are necessary. To show sufficiency, assume that $(x^*, y^*) \in X \times \mathbb{R}^m_+$ satisfies the

[8] it is not our current purpose to investigate in detail further computational aspects, which can be found in the established literature on mathematical programming

KKT-conditions. We must demonstrate that (x^*, y^*) is an equilibrium of the LAGRANGE game $\Lambda = (X, \mathbb{R}^m_+, L)$, *i.e.*, satisfies

$$\max_{x\in X} L(x, y^*) = L(x^*, y^*) = \min_{y\geq 0} L(x^*, y) \tag{27}$$

for $L(x, y) = f(x) + y^T g(x)$. Since $x \mapsto L(x, y)$ is concave for every $y \geq 0$ we find for every $x \in X$,

$$L(x, y^*) \leq L(x, y^*) + \nabla_x L(x, y^*)(x - x^*) \leq L(x^*, x^*)$$

because (K_2) guarantees $\nabla_x L(x, x^*)(x - x^*) \leq 0$. So the first equality in (27) follows. From (K_0) and (K_1), we have $g(x^*) \geq 0$ and $(y^*)^T g(x^*) = 0$ and therefore deduce the second equality:

$$\begin{aligned} \min_{y\geq 0} L(x^*, y) &= f(x^*) + \min_{y\geq 0} y^T g(x^*) = f(x^*) + 0 \\ &= f(x^*) + (y^*)^T g(x^*) = L(x^*, y^*). \end{aligned}$$

$\diamond$

4.4. Linear programs. A *linear program* (*LP*) is an optimization problem of the form

$$\max_{x\in\mathbb{R}^n_+} c^T x \quad \text{s.t.} \quad Ax \leq b, \tag{28}$$

where $c \in \mathbb{R}^n$ is an n-dimensional coefficient vector, $A \in \mathbb{R}^{m\times n}$ a matrix and $b \in \mathbb{R}^m$ an m-dimensional coefficient vector. The problem type (28) is a special case of (21) with the parameters

(1) $X = \mathbb{R}^n_+$,
(2) $f(x) = c^T x = \sum_{j=1}^n c_j x_j$,
(3) $g(x) = b - Ax$,

and the LAGRANGE function

$$L(x, y) = c^T x + y^T (b - Ax) = y^T b + (c^T - y^T A)x,$$

which is concave and convex in both variables x and y. The worst-case functions are

$$L_1(x) = \begin{cases} c^T x & \text{if } Ax \leq b \\ -\infty & \text{if } Ax \not\leq b. \end{cases}$$

$$L_2(y) = \begin{cases} y^T b & \text{if } y^T A \geq c^T \\ +\infty & \text{if } y^T A \not\geq c^T. \end{cases}$$

To mark this special case, we refer to a LAGRANGE game relative to the linear program (28) as an *LP-game* and denote it by $LP(c; A, b)$.

As we already know, the problem of maximizing $L_1(x)$ corresponds to the original problem (28). The dual problem of minimizing $L_2(y)$ corresponds to the optimization problem

$$\min_{y \in \mathbb{R}^m_+} b^T y \quad \text{s.t.} \quad A^T y \geq c. \tag{29}$$

EX. 3.5. *Formulate a linear program of type (28) which is equivalent to the optimization problem (29).*

EX. 3.6. *Formulate the problem of finding an optimal strategy in a randomized matrix game $(\overline{X}, \overline{Y}, \overline{U})$ as a linear program of the form (28)* i.e.*, find a suitable matrix A and coefficient vectors c and b.*

We know from Theorem 3.3 that equilibria of LP-games can be computed as solutions of the KKT-conditions. As to their existence, the fundamental Theorem 3.4 provides a compactness-free characterization:

THEOREM 3.4 (Main Theorem on Linear Programming). *The LP-game (c, A, b) has an equilibrium if and only if both problems (28) and (29) have feasible solutions.*

LP-games are not only interesting as zero-sum games in their own right. In the theory of *cooperative games* with possibly more than two players (see Chapter 7) linear programming is a structurally analytical tool.

Linear programming problems are particularly important in applications because they can be solved efficiently. We do not go into algorithmic details here but refer to the standard mathematical optimization literature[9]. (See also Section 4 in the Appendix.)

[9] *e.g.*, U. FAIGLE, W. KERN and G. STILL, *Algorithmic Principles of Mathematical Programming*, Springer, 2002

CHAPTER 4

Investing and Betting

> The opponent of a gambler is usually a player with no specific optimization goal. The opponent's strategy choices are determined by chance. Therefore, the gambler will have to decide on strategies with good expected returns. Information plays an important role in the quest for the best decision. Hence the problem how to model information exchange and common knowledge among (possibly more than two) players deserves to be addressed as well.

Assume that an *investor* (or *bettor* or *gambler* or simply *player*) is considering a financial engagement in a certain venture. Then the obvious – albeit rather vague – big question for the investor is:

- *What decision should best be taken*?

More specifically, the investor wants to decide whether an engagement is worthwhile at all and, if so, how much money should be how invested. Obviously, the answer depends a lot on additional information: What is the likelihood of a success, what gain can be expected? What is the risk of a loss? *etc.*

The investor is thus about to participate as a player in a 2-person game with an opponent whose strategies and objective are not always clear or known in advance. Relevant information is not completely (or reliably) available to the investor so that the decision must be made under uncertainties. Typical examples are gambling and betting where the success of the engagement depends on events that may or may not occur and hence on ”fortune” or ”chance”. But also investments in the stock market fall into this category when it is not clear in advance whether the value of a particular investment will go up or down.

We will not be able to answer the big question above completely but discuss various aspects of it. Before going into further details, let us illustrate the difficulties of the subject with a classical – and seemingly paradoxical – gambling situation.

The St. Petersburg paradox. Imagine yourself as a potential player in the following game of chance.

EX. 4.1 (St. Petersburg game). *A coin (with faces "H" and "T") is tossed repeatedly until "H" shows. If this happens at the* n*th toss, a participating player will receive* $\alpha_n = 2^n$ *euros. There is a participation fee of* a_0 *euros, however. So the net gain of the player is*

$$a = \alpha_n - a_0 = 2^n - a_0$$

if the game stops at the n*th toss. At what fee* a_0 *would a participation in the game be attractive?*

Assuming a fair coin in the St. Petersburg game, the probability to go through more than n tosses (and hence to have the first n results as "T") is

$$q_n = \left(\frac{1}{2}\right)^n = \frac{1}{2^n} \ \to\ 0 \quad (n \to \infty).$$

So the game ends almost certainly after a finite number of tosses. The expected return to a participant is nevertheless infinite:

$$E_P = \sum_{n=1}^{\infty} 2^n q_n = \frac{2^1}{2^1} + \frac{2^2}{2^2} + \ldots + \frac{2^n}{2^n} + \ldots = +\infty,$$

which might suggest that a player should be willing to pay any finite amount a_0 for being allowed into the game. In practice, however, this could be a risky venture (see Ex. 4.2).

EX. 4.2. *Show that the probability of receiving a return of* 100 *euros or more in the St. Petersburg game is less than* 1%*. So a participation fee of* $a_0 = 100$ *euros or more appears to be not attractive because it will not be recovered with a probability of more than* 99%*.*

Paradoxically, when we evaluate the utility of the return 2^n not directly but by its logarithm $\log_2 2^n = n$, the St. Petersburg payoff has a finite utility expectation:

$$G_P = \frac{\log_2 2^1}{2^1} + \frac{\log_2 2^2}{2^2} + \ldots + \frac{\log_2 2^n}{2^n} + \ldots = \sum_{n=1}^{\infty} \frac{n}{2^n} < 2.$$

It suggests that one should expect an utility value of less than 2 and hence a return of less than $2^2 = 4$ euros.

REMARK 4.1 (Logarithmic utilities). *The logarithm function as a measure for the utility value of a financial gain was introduced by* BERNOULLI[1] *in his analysis of the St. Petersburg game. This concave function plays an important role in our analysis as well.*

Whether one uses $\log_2 x$*, the logarithm base* 2*, or the natural logarithm* $\ln x$*, does not make any essential difference since the two functions differ just by a scaling factor:*

$$\ln x = (\ln 2) \cdot \log_2 x.$$

1. Proportional investing

Our general model consists of a potential investor with an initial portfolio of $B > 0$ euros (or dollars or...) and an investment opportunity A. The investor is to decide what portion aB (with scaling factor $0 \leq a \leq 1$) of B should be invested and considers k possible scenarios $A_1, \ldots, A_k$ for its development. The investor believes that one of these scenarios will be realized and furthermore assumes:

(S1) If A_i occurs, then each invested euro returns $\rho_i \geq 0$ euros.
(S2) Scenario A_i occurs with probability p_i.

Expected gain. Under the investor's assumptions, the expected return on every invested euro is

$$\overline{\rho} = \rho_1 p_1 + \ldots + \rho_k p_k.$$

If the investor's decision is motivated by the maximization of the expected return, the *naiv investment rule* applies:

(NIR) If $\overline{\rho} > 1$, invest all of B in A and expect the return $B\overline{\rho} > B$.
If $\overline{\rho} \leq 1$, invest nothing since no proper gain is expected.

In spite of its intuitive appeal, rule (NIR) can be quite risky (see Ex. 4.3).

EX. 4.3. *For* $k = 2$*, assume the return rates* $\rho_1 = 0$ *and* $\rho_2 = 100$*. If* $p_1 = 0.9$ *and* $p_2 = 0.1$*, the investor expects a tenfold return on the investment:*

$$\overline{\rho} = 0 \cdot 0.9 + 100 \cdot 0.1 = 10.$$

However, with probability $p_1 = 90\%$*, the investment can be expected to result in a total loss.*

[1] D. BERNOULLI (1700-1782)

Expected utility. With respect to the logarithmic utility function $\ln x$, the expected utility of an investment of size aB would be

$$\begin{aligned} U(a) &= \sum_{i=1}^{k} p_i \ln[(1-a)B + \rho_i aB)] \\ &= \sum_{i=1}^{k} p_i \ln[1 + (\rho_i - 1)a] + \ln B. \end{aligned}$$

The derivative of $U(a)$ is

$$\boxed{U'(a) = \sum_{i=1}^{k} \frac{p_i(\rho_i - 1)}{1 + (\rho_i - 1)a} = \sum_{i=1}^{k} \frac{p_i}{1/r_i + a}}$$

with $r_i = \rho_i - 1$ being the investor's expected surplus over each invested euro in scenario A_i.

The investment rate a^* with the optimal utility value would have to satisfy $U'(a^*) = 0$ and can thus be computed by solving the equation $U'(a) = 0$.

EX. 4.4. *In the situation of Ex. 4.3, one has*

$$U(a) = \frac{9}{10} \ln(1-a) + \frac{1}{10} \ln[(1 + 99a]) + \ln B$$

with the derivative

$$U'(a) = \frac{-9}{10(1-a)} + \frac{99}{10(1+99a)}.$$

$U'(a) = 0$ *implies* $a = 1/11$*. So the portion* $B/11$ *of* B *should be invested in order to maximize the expected utility. The rest* $B = B - B/11$ *of the portfolio should be retained and not invested.*

2. The fortune formula

We turn to a fundamental question:

- *Should one invest in an opportunity A that offers an expected return at the rate of ρ with probability p but also a complete loss (*i.e.*, a zero return) with probability $q = 1-p$?*

A special case of this situation was already encountered in Ex. 4.4. Denoting by $r = \rho - 1$ the expected surplus rate of the investment, the associated expected logarithmic utility in general is

$$U(a) = q\ln(1-a) + p\ln(1+ra) + \ln B \tag{30}$$

with the derivative

$$U'(a) = \frac{-q}{1-a} + \frac{p}{1/r+a}\ . \tag{31}$$

If a loss is to be expected with positive probability $q > 1$, and the investor decides on a full investment, *i.e.*, chooses $a = 1$, then the utility value

$$U(1) = -\infty$$

must be expected – no matter how big the surplus rate r might be.

On the other hand, the choice $a = 0$ of no investment has the utility

$$U(0) = \ln B.$$

The investment rate a^* with the optimal utility lies somewhere between these extremes.

LEMMA 4.1. *Let $U'(a)$ be as in (31) and $0 < a^* < 1$. Then*

$$U'(a^*) = 0 \quad\Longleftrightarrow\quad a^* = p - q/r.$$

Proof. (Exercise left to the reader.)

◇

The choice of the investment rate a^* with optimal expected logarithmic utility $U(a^*)$, *i.e.*, according to the so-called *fortune formula* of KELLY [2]:

$$a^* = p - \frac{q}{r} \quad \text{if } 0 < p - \frac{q}{r} < 1 \tag{32}$$

Betting one's belief. It is important to keep in mind that the probability p in the fortune formula (32) is the subjective evaluation of an investment success by the investor.

The "true" probability is often unknown at the time of the investment. However, if p reflects the investor's best knowledge about the true probability, there is nothing better the investor could do. This truism is known as the investment advice

Bet your belief !

[2] J.L KELLY (1956): *A new interpretation of information rate*, The Bell System Technical Journal

3. Fair odds

An investment into an opportunity A offering a return of $\rho \geq 1$ euros per euro invested with a certain probability $\Pr(A)$ or returning nothing (with probability $1 - \Pr(A)$) is called a *bet* on A. The investor is then a *bettor* (or a *gambler*) and the return ρ is the payoff. The payoff is assumed to be guaranteed by a *bookmaker* (or *bank*). The payoff rate is also denoted by $1 : \rho$ and known as the *odds* of the bet.

The expected gain (per euro) of the gambler is

$$E = \rho \Pr(A) + (-1)(1 - \Pr(A) = (\rho + 1) \Pr(A) - 1.$$

So $(-E)$ is the expected gain of the bookmaker.

The odds $1 : \rho$ are considered to be *fair* if the gambler and the bookmaker have the same expected gain, *i.e.*, if $E = -E$ and hence $E = 0$ holds. In other words:

$$\boxed{1 : \rho \quad \text{is fair} \quad \Longleftrightarrow \quad \rho = \frac{1 - \Pr(A)}{\Pr(A)}}$$

If the true probability $\Pr(A)$ is not known to the bettor, it needs to be estimated. Suppose the bettor's estimate for $\Pr(A)$ is p. Then the bet appears (subjectively) advantageous if and only if

$$E(p) > 0 \quad \textit{i.e.}, \text{ if } \rho + 1 > 1/p. \tag{33}$$

The bettor will consider the odds $1 : \rho$ as fair if

$$E(p) = 0 \;\text{ and hence }\; \rho + 1 = 1/p.$$

In the case $E(p) < 0$, of course, the bettor would not expect a gain but a loss on the bet – on the basis of the information that has lead to the probability estimate p for $\Pr(A)$.

3.1. Examples.

Let us look at some examples.

EX. 4.5 (DE MÉRÉ's game[3]). *Let A be the event that* no "6" *shows if a single 6-sided die is rolled four times. Suppose the odds $1 : 1$ are offered on A. If the gambler considers all results as equally likely, the gambler's estimate of the probability for A is*

$$p = \frac{5^4}{6^4} = \frac{625}{1296} \approx 0.482 < 0.5$$

[3] mentioned to B. PASCAL (1623-1662)

because there are $6^4 = 1296$ *possible result sequences on* 4 *rolls of the die, of which* $5^4 = 625$ *correspond to* A. *So the player should expect a negative return:*

$$E(p) = (\rho + 1)p - 1 = 2p - 1 = 2(5/6)^4 - 1 < 0.$$

In contrast, let $\tilde{A}$ *be the event that* no *double* 6 *shows if a pair of dice is rolled* 24 *times. Now the prospective gambler estimates* $\Pr(\tilde{A})$ *as*

$$\tilde{p} = (35/36)^{24} > 0.5.$$

Consequently, the odds $1 : 1$ *on* $\tilde{A}$ *would let the gambler expect a proper gain:*

$$\tilde{E} = 2\tilde{p} - 1 > 0.$$

EX. 4.6 (Roulette). *Let* $W = \{0, 1, 2, \ldots, 36\}$ *represent a roulette wheel and assume that* $0 \in W$ *is colored* green *while eighteen numbers in* W *are* red *and the remaining eighteen numbers* black. *Assume that a number* $X \in W$ *is randomly determined by spinning the wheel and allowing a ball come to rest at one of these numbers.*

(a) Fix $w \in W$ *and the odds* $1 : 18$ *on the event* $A_w = \{X = w\}$. *Should a gambler expect a positive return when placing a bet on* A_w*?*

(b) Suppose the bank offers the odds $1 : 2$ *on the event* $R = \{X = \text{red}\}$. *Should a gambler consider these odds on* R *to be fair?*

3.2. The doubling strategy. For the game of roulette (see Ex. 4.6) and for similar betting games with odds $1 : 2$, a popular wisdom[4] recommends repeated betting according to the following strategy:

> (R) Bet the amount 1 on $R = \{X = \text{red}\}$. If R does not occur, continue with the double amount 2 on R. If R does not show, double again and bet 4 on R and so on – until the event R happens.

Once R shows, one has a *net gain* of 1 on the original investment of size 1 (see Ex. 4.7). The probability for R *not* to happen in one spin is $19/37$. So the probability of seeing red in one of the first n spins of an equally balanced roulette wheel is high:

$$1 - (19/37)^n \;\to\; 1 \quad (n \to \infty).$$

Hence: *Strategy (R) achieves a net gain of* 1 *with high probability.*

[4] I have learned strategy (R) myself as a youth from my uncle Max

Paradoxically(?), the expected net gain for betting any amount $x > 0$ on the event R is always strictly negative however:

$$E_R = 2x\left(\frac{18}{37}\right) - x = -\frac{x}{37} \ < \ 0.$$

EX. 4.7. *Show for the game of roulette with a well-balanced wheel:*

(1) *If* $\{X = \text{red}\}$ *shows on the fifth spin of the wheel only, strategy* (R) *has lost a total of* 15 *on the first* 4 *spins. However, having invested* 16 *more and then winning* $2^5 = 32$ *on the fifth spin, yields the overall net return*

$$32 - (15 + 16) = 1.$$

(2) *The probability for* $\{X = \text{red}\}$ *to happen on the first* 5 *spins is more than* 95%.

COMMENT. The problem with strategy (R) is its risk management. A player has only a limited amount of money available in practice. If the player wants to limit the risk of a loss to B euros, then the number of iterations in the betting sequence is limited to at most k, where

$$2^{k-1} \leq B < 2^k \quad \text{and hence} \quad k = \lfloor \log_2 B \rfloor.$$

Consequently:

- The available budget B is lost with probability $(19/37)^k$.
- The portfolio grows to $B + 1$ with probability $1 - (19/37)^k$.

4. Betting on alternatives

Consider k mutually exclusive events $A_0, \ldots, A_{k-1}$ of which one will occur and a bank that offers the odds $1 : \rho_i$ on the k events A_i, which means:

(1) The bank offers a scenario with $1/\rho_i$ being the probability for A_i to occur.
(2) The bank guarantees a payoff of ρ_i euros for each euro bet on A_i if the event A_i occurs.

Suppose a gambler estimates that the events A_i to occur with probabilities $p_i > 0$ and decides to invest the capital $B = 1$ fully. Under this condition, a *(betting) strategy* is a k-tuple $a = (a_0, a_1, \ldots, a_{k-1})$ of numbers $a_i \geq 0$ such that

$$a_0 + a_1 + \ldots + a_{k-1} = 1$$

with the interpretation that the portion a_i of the capital will be bet onto the occurrence of event A_i for $i = 0, 1, \ldots, k-1$. So the gambler's expected logarithmic utility of strategy a is

$$\begin{aligned} U(a,p) &= \sum_{i=0}^{k-1} p_i \ln(a_i \rho_i) \\ &= \sum_{i=0}^{k-1} p_i \ln a_i + \sum_{i=0}^{k-1} p_i \ln \rho_i. \end{aligned}$$

Notice that $p = (p_0, p_1, \ldots, p_{k-1})$ is a strategy in its own right and that the second sum term in the expression for $U(a,p)$ does not depend on the choice of a. So only the first sum term is of interest when one searches a strategy with optimal expected utility.

THEOREM 4.1. *Let $p = (p_0, p_1, \ldots, p_{k-1})$ be the gambler's probability assessment. Then:*

$$U(a,p) < U(p,p) \quad \Longleftrightarrow \quad a \neq p.$$

Consequently, $a^* = p$ is the strategy with the optimal logarithmic utility under the gambler's expectations.

Proof. The function $f(x) = x - 1 - \ln x$ is defined for all $x > 0$. Its derivative

$$f'(x) = 1 - 1/x$$

is negative for $x < 1$ and positive for $x > 1$. So $f(x)$ is strictly decreasing for $x < 1$ and strictly increasing for $x > 1$ with the unique minimum $f(1) = 0$. This yields BERNOULLI's inequality

(34) $$\ln x \leq x - 1 \quad \text{and} \quad \ln x = x - 1 \iff x = 1.$$

Applying the BERNOULLI inequality, we find

$$\begin{aligned} U(a,p) - U(p,p) &= \sum_{i=0}^{k-1} p_i \ln a_i - \sum_{i=0}^{k-1} p_i \ln p_i = \sum_{i=0}^{k-1} p_i \ln(a_i/p_i) \\ &\leq \sum_{i=1}^{k-1} p_i(a_i/p_i - 1) = \sum_{i=1}^{k-1} a_i - \sum_{i=1}^{k-1} p_i \; = \; 0 \end{aligned}$$

with equality if and only if $a_i = p_i$ for all $i = 0, 1, \ldots, k_1$.

$\diamond$

Theorem 4.1 leads to the betting rule with the optimal expected logarithmic utility:

> (BR) *For all $i = 0, 1, \ldots, k-1$, bet the portion $a_i = p_i$ of the capital B on the event A_i.*

NOTA BENE. *The proportional rule (BR) only depends on the gambler's probability estimate p. It is independent of the particular odds $1 : \rho_i$ the bank may offer*!

Fair odds. As in the proof of Theorem 4.1, one sees:

$$\sum_{i=0}^{k-1} p_i \ln \rho_i = -\sum_{i=0}^{k-1} p_i \ln(1/\rho_i) \geq -\sum_{i=0}^{k-1} p_i \ln p_i$$

with equality if and only if $\rho_i = 1/p_1$ holds for all $i = 0, 1, \ldots, k-1$. It follows that the best odds for the bank (and worst for the gambler) are given by

$$\rho_i = 1/p_i \quad (i = 0, 1, \ldots, k-1). \tag{35}$$

In this case, the gambler expects the logarithmic utility of the optimal strategy p as

$$U(p) = \sum_{i=0}^{k-1} p_i \ln p_i - \sum_{i=0}^{k-1} p_i \ln(1/\rho_i) = 0.$$

We understand the odds as in (35) to be *fair* in the context of betting with alternatives.

4.1. Statistical frequencies. Assume the gambler of the previous sections has observed that in n consecutive instances of the bet the event A_i has popped up s_i times.

So, under the strategy a, the original portfolio $B = 1$ would have developed into

$$B_n(a) = (a_0\rho_0)^{s_0}(a_1\rho_1)^{s_1} \cdots (a_{k-1}\rho_{k-1})^{s_{k-1}}$$

with the logarithmic utility

$$\begin{aligned} U_n(a) = \ln B_n(a) &= \sum_{i=0}^{k-1} s_i \ln(a_i\rho_i) \\ &= \sum_{i=0}^{k-1} s_i \ln a_i + \sum_{i=0}^{k-1} s_i \ln \rho_i. \end{aligned}$$

Based on the observed frequencies s_i the gambler might reasonably estimate the events A_i to occur with probabilities according to the relative frequencies

$$p_i = s_i/n \quad (i = 0, 1, \ldots, k-1).$$

As in the proof of Theorem4.1, we now find in hindsight:

COROLLARY 4.1. *The strategy $a^* = (s_0/n), \ldots, s_{k-1}/n)$ would have lead to the maximal logarithmic utility value*

$$U_n(a^*) = \sum_{i=0}^{k-1} s_i \ln(s_i/t) + \sum_{i=0}^{k-1} p_i \ln \rho_i$$

and hence to the maximal growth

$$B_n(a^*) = \frac{(s_0\rho_0)^{s_0}(s_1\rho_1)^{s_1}\cdots(s_{k-1}\rho_{k-1})^{s_{k-1}}}{n^n}$$

5. Betting and information

Assuming a betting situation with the k alternatives $A_0, A_1, \ldots, A_{k-1}$ and the odds $1 : \rho_x$ as before, suppose however that the event A_x is already established – but the bettor has no such information before placing the bet.

Suppose further that information now arrives through some (human or technical) *communication channel* K so that the outcome A_x is reported to the bettor (perhaps incorrectly) as A_y:

$$x \;\to\; \boxed{\quad K \quad} \;\to\; y.$$

Having received the ("insider") information "y", how should the bettor place the bet?

To answer this question, let

$p(x|y) =$ probability for the true result to be x when receiving y.

Note that these parameters $p(x|y)$ are typically subjective evaluations of the bettor's trust in the channel K.

A *betting strategy* in this information setting is now a $(k \times k)$-matrix A with coefficients $a(x|y) \geq 0$ which satisfy

$$\sum_{x=0}^{k-1} a(x|y) = 1 \quad \text{for } \; y = 0, 1, \ldots, k-1.$$

$a(x|y)$ is the fraction of the budget that is bet on the event A_x when y is received. In particular, the bettor's trust matrix P with coefficients $p(x|y)$ is a strategy. For the case that A_x is the true result, one therefore expects the logarithmic utility

$$U_x(A) = \sum_{y=0}^{k-1} p(x|y)\ln[a(x|y)\rho_x] = \sum_{y=0}^{k-1} p(x|y)\ln a(x|y) + \ln \rho_x.$$

As in Corollary 4.1, we find for all $x = 0, 1, \ldots, k-1$:

$$U_x(A) < U_x(P) \quad \Longleftrightarrow \quad a(x|y) = p(x|y) \;\; \forall y = 0, 1, \ldots, k-1.$$

So the strategy P is optimal (under the given trust in K on part of the bettor) and confirms the betting rule:

Bet your belief!

Information transmission. Let $p = (x_0, \ldots, x_{k-1})$ be the bettor's probability estimates on the k events $A_0, A_1, \ldots, A_{k-1}$ or, equivalently, on the index set $\{0, 1, \ldots, k-1\}$. Then the expected logarithmic utility of strategy A is relative to base 2:

$$U_2^{(p)}(A) = \sum_{x=0}^{k-1}\sum_{y=0}^{k-1} p_x p(x|y)\log_2 a(x|y) + \sum_{x=0}^{k-1} p_x \log_2 \rho_x.$$

NOTA BENE. *The probabilities p_x are estimates on the likelihood of the events A_i, while the probabilities $p(x|y)$ are estimates on the trust of the bettor into the reliability of the communication channel K. They are logically* not *related.*

Setting

$$\begin{aligned} H(X) &= -\sum_{i=1}^{k-1} p_x \log_2 x \\ H(\rho) &= -\sum_{i=1}^{k-1} \rho_x \log_2 x \\ H(X,Y) &= -\sum_{x=0}^{k-1}\sum_{y=0}^{k-1} p_x p(x|y)\log_2 a(x|y), \end{aligned}$$

we thus have

$$U_2^{(p)} = -H(X|Y) - H(\rho) = U_2(p) + T(X|Y)$$

where

$$T(X|Y) = H(X) - H(X|Y)$$

is the increase of the bettor's expected logarithmic utility due to the communication *via* channel K.

REMARK 4.2 (Channel capacity). *Given the channel K as above with transmission probabilities $p(s|r)$ and the distribution $p = (p_0, p_1, \ldots, p_{k-1})$ on the channel inputs x, the parameter $T(X,Y)$ is the* (information) transmission rate *of K.*

Maximizing over all possible input distributions p, one obtains the channel capacity *$C(K)$ as the smallest upper bound on the achievable transmission rates:*

$$C(K) = \sup_p \; T(X,Y).$$

The parameter $C(K)$ plays an important role in the theory of information and communication in general[5].

EX. 4.8. *A bettor expects the event A_0 with probability 80% and the alternative event A_1 with probability 20%. What bet should be placed?*

Suppose now that an expert tells the bettor that A_1 is certain to happen. What bet should the bettor place under the assumption that the expert is believed to be right with probability 90%?

6. Common knowledge

Having discussed information with respect to betting, let us digress a little and take a more general view on information and knowledge. Given a system $\mathfrak{S}$, we ask:

> *To what extent does common knowledge in a group of agents influence individual conclusions about the state of $\mathfrak{S}$?*

To explain what is meant here, we first discuss a well-known riddle.

[5] C.E. SHANNON (1948): *A mathematical theory of communication.* Bell System Technical Journal

6.1. Red and white hats. Imagine the following situation:

(I) Three girls, G_1, G_2 and G_3, with *red* hats are sitting in a circle.
(II) They all know that their hats are either *red* or *white*.
(III) Each can see the color of all hats except her own.

Now the teacher comes and announces:

(1) *There is at least one red hat.*
(2) *I will start counting slowly. As soon as someone knows the color of her hat, she should raise her hand.*

What will happen? Does the teacher provide information that goes beyond the common knowledge the girls already have? After all, each girl sees two red hats – and hence *knows* that each of the other girls sees at least one red had as well.

Because of (III), the girls know their hat universe $\mathfrak{S}$ is in one of the 8 states of possible color distributions:

	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8
G_1	R	R	R	W	R	W	W	W
G_2	R	R	W	R	W	R	W	W
G_3	R	W	R	R	W	W	R	W

None of these states can be jointly ruled out. The entropy H_2^0 of their common knowledge is:

$$H_2^0 = \log_2 8 = 3.$$

The teacher's announcement, however, rules out the state σ_8 and reduces the entropy to

$$H_2^1 = \log_2 7 < H_2^0,$$

which means that the teacher has supplied proper additional information.

At the teacher's first count, no girl can be sure about her own hat because none sees *two* white hats. So no hand is raised, which rules out the states σ_5, σ_6 and σ_7 as possibilities.

Denote now by $P_i(\sigma)$ the set of states thought possible by girl G_i when the hat distribution is actually σ. So we have, for example,

$$P_1(\sigma_3) = \{\sigma_3\}, P_2(\sigma_2) = \{\sigma_2\}, P_3(\sigma_4) = \{\sigma_4\}.$$

Consequently, in each of the states $\sigma_2, \sigma_3, \sigma_4$, at least one girl would raise her hand at the second count and conclude confidently that her hat is *red*, which would signal the state (and hence the hat distribution) to the other girls.

If no hand goes up at the second count, *all* girls know that they are in state σ_1 and will raise their hands at the third count.

In contrast, consider the other extreme scenario and assume:

(I') Three girls, G_1, G_2 and G_3, with *white* hats are sitting in a circle.
(II) They all know that their hats are either *red* or *white*.
(III) Each can see the color of all hats except her own.

The effect of the teacher's announcement is quite different:

- Each girl will immediately conclude that her hat is red and raise her hand because she sees only white hats on the other girls.

This analysis shows:

(i) The information supplied by the teacher is *subjective*: Even when the information ("there is at least one red hat") is false, the girls will eventually conclude with confidence that they know their hat's color.
(ii) When a girl *thinks* she knows her hat's color, she may nevertheless have arrived at a factually wrong conclusion.

EX. 4.9. *Assume an arbitrary distribution of red and white hats among the three girls. Will the teacher's announcement nevertheless lead the girls to the belief that they know the color of their hats?*

6.2. Information and knowledge functions. An *event* in the system $\mathfrak{S}$ is a subset $E \subseteq \mathfrak{S}$ of states. We say that E *occurs* when $\mathfrak{S}$ is in a state $\sigma \in E$. Denoting by $\mathbf{2}^{\mathfrak{S}}$ the collection of all possible events, we think of a function $P : \mathfrak{S} \to \mathbf{2}^{\mathfrak{S}}$ with the property

$$\sigma \in P(\sigma) \quad \text{for all } \sigma \in \mathfrak{S}$$

as an *information function*. P has the interpretation:

- *If $\mathfrak{S}$ is in the state σ, then P provides the information that the event $P(\sigma)$ has occurred.*

Notice that P is not necessarily sharp: any state $\tau \in P(\sigma)$ is a candidate for the true state under the information function P.

The information function P defines a *knowledge function* $K : \mathbf{2}^{\mathfrak{S}} \to \mathbf{2}^{\mathfrak{S}}$ *via*

$$K(E) = \{\sigma \mid P(\sigma) \subseteq E\}$$

with the interpretation:

- *$K(E)$ is the set of states $\sigma \in \mathfrak{S}$ where P suggests that the event E has certainly occurred.*

LEMMA 4.2. *The knowledge K of the information function P has the properties:*

(K.1) $K(\mathfrak{S}) = \mathfrak{S}$.
(K.2) $E \subseteq F \quad \Longrightarrow \quad K(E) \subseteq K(F)$.
(K.3) $K(E \cap F) = K(E) \cap K(F)$.
(K.4) $K(E) \subseteq E$.

Proof. Straightforward exercise, left to the reader.

$\diamond$

Property (K.4) is the so-called *reliability axiom*: If one knows (under K) that E has occurred, then E really has occurred.

EX. 4.10 (Transparency). *Verify the* transparency axiom

(K.5) $K(K(E) = K(E)$ *for all events* E.

Interpretation: When one knows with certainty that E has occurred, then one knows with certainty that one considers E as having occurred.

We say that E is *evident* if $E = K(E)$ is true, which means:

- The knowledge function K considers an evident event E as having occurred if and only if E really has occurred.

EX. 4.11. *Show: The set* $\mathfrak{S}$ *of all possible states constitutes always an evident event.*

EX. 4.12 (Wisdom). *Verify the* wisdom axiom

(K.6) $\mathfrak{S} \setminus K(E) = K(\mathfrak{S} \setminus E)$ *for all events* E.

Interpretation: When one does not know with certainty that E has occurred, then one is aware of one's uncertainty.

6.3. Common knowledge. Consider now a set $N = \{p_1, \ldots, p_n\}$ of n players p_i with respective information functions P_i. We say that the event $E \subseteq \mathfrak{S}$ is *evident* for N if E is evident for each of the members of N, *i.e.*, if

$$E = K_1(E) = \ldots = K_n(E).$$

More generally, an event $E \subseteq \mathfrak{S}$ is said to be *common knowledge* of N in the state σ if there is an event $F \subseteq E$ such that

$$F \text{ is evident for } N \quad \text{and} \quad \sigma \in F.$$

PROPOSITION 4.1. *If the event $E \subseteq \mathfrak{S}$ is common knowledge for the n players p_i with information functions P_i in state σ, then*

$$\sigma \in K_{i_1}(K_{i_2}(\ldots(K_{i_m}(E))\ldots)))$$

holds for all sequences $i_1 \ldots i_m$ of indices $1 \leq i_j \leq n$.

Proof. If the event E is common knowledge, it comprises an evident event $F \subseteq E$ with $\sigma \in F$. By definition, we have

$$\in K_{i_1}(K_{i_2}(\ldots(K_{i_m}(F))\ldots))) = F$$

By property (K.2) of a knowledge function (Lemma 4.2), we thus conclude

$$K_{i_1}(K_{i_2}(\ldots(K_{i_m}(E))\ldots))) \supseteq K_{i_1}(K_{i_2}(\ldots(K_{i_m}(F))\ldots))) = F \ni \sigma.$$

$\diamond$

As an illustration of Proposition 4.1, consider the events

$$K_1(E), K_2(K_1(E)), K_3(K_2(K_1(E))).$$

$K_1(E)$ are all the states where player p_1 is sure that E has occurred. The set $K_2(K_1(E))$ comprises those states where player p_2 is sure that player $p_1(E)$ is sure that E has occurred. In $K_3(K_2(K_1(E))$ are all the states where player p_3 is certain that player p_2 is sure that player p_1 believes that E has occurred. *And so on.*

6.4. Different opinions. Let p_1 and p_2 be two players with information functions P_1 and P_2 relative to a finite system $\mathfrak{S}$ and assume:

- Both players have the same probability estimates $\Pr(E)$ on the occurrence of events $E \subseteq \mathfrak{S}$.

We turn to the question:

- *Can there be common knowledge among the two players in a certain state σ^* that they* differ *in their estimate on the likelihood of an event E having occurred*?

Surprisingly(?), the answer can be "*yes*" as Ex. 4.13 shows. For the analysis in the example, recall that the *conditional probability* of an event E given the event A, is

$$\Pr(E|A) = \begin{cases} \Pr(E\cap A)/\Pr(A) & \text{if } \Pr(A) > 0 \\ 0 & \text{if } \Pr(A) = 0. \end{cases}$$

EX. 4.13. *Let* $\mathfrak{S} = \{\sigma_1, \sigma_2\}$ *and assume* $\Pr(\sigma_1) = \Pr(\sigma_2) = 1/2$. *Consider the information functions*

$$\begin{aligned} P_1(\sigma_1) &= \{\sigma_1\} \quad \textit{and} \quad P_1(\sigma_2) = \{\sigma_2\} \\ P_2(\sigma_1) &= \{\sigma_1, \sigma_2\} = P_2(\sigma_2). \end{aligned}$$

For the event $E = \{\sigma_1\}$*, one finds*

$$\begin{aligned} \Pr(E|P_1(\sigma_1)) &= 1 \quad \textit{and} \quad \Pr(E|P_1(\sigma_2)) = 0 \\ \Pr(E|P_2(\sigma_1)) &= 1/2 \quad \textit{and} \quad \Pr(E|P_2(\sigma_2)) = 1/2. \end{aligned}$$

The ground set $\mathfrak{S} = \{\sigma_1, \sigma_2\}$ *corresponds to the event* "the two players differ in their estimates on the likelihood that E has occurred". $\mathfrak{S}$ *is (trivially) common knowledge in each of the two states* σ_1, σ_2.

For a large class of information functions, however, our initial question has the guaranteed answer "*no*". For example, let us call an information function P *strict* if

- Every evident event E is a union of *pairwise disjoint* sets $P(\sigma)$.

PROPOSITION 4.2. *Assume that both information functions* P_1 *and* P_2 *are strict. Let* $E \subseteq \mathfrak{S}$ *be arbitrary event. Then there is* no *state* σ^* *in which it could be common knowledge of the players that their likelihood estimates* η_1 *resp.* η_2 *on the occurrence of* E *are different.*

Proof. Consider the events

$$E_i = \{\sigma \mid \Pr(E|P_i(\sigma) = \eta_1\} \quad (i = 1, 2).$$

The event $E_1 \cap E_2$ is then the event that player p_1 estimates the probability for the occurrence of E with η_1 while player p_2's estimate is η_2.

Suppose $E_1 \cap E_2$ is common knowledge in state σ, *i.e.*, there exists an event $F \subseteq E_1 \cap E_2$ such that

$$\sigma^* \in F \quad \text{and} \quad K_1(F) = F = K_2(F).$$

Because the information function P_1 is strict, F is the union of pairwise disjoint sets $P_1(\sigma_1), \ldots, P_1(\sigma_k)$, say. Because of $F \subseteq E_1 \cap E_2$, one has

$$\Pr(E|P_1(\sigma_1)) = \ldots = \Pr(E|P_1(\sigma_k)) = \eta_1.$$

Taking Ex. 4.14 into account, we therefore find

$$\Pr(E|F) = \Pr(E|P_1(\sigma_1)) = \eta_1.$$

Similarly, $\Pr(E|F) = \eta_2$ is deduced and hence $\eta_2 = \eta_1$ follows.

$\diamond$

EX. 4.14. *Let A, B be events such that $A \cap B = \emptyset$. Then the conditional probability satisfies:*

$$\Pr(E|A) = \Pr(E|B) \quad \Longrightarrow \quad \Pr(E|A \cup B) = \Pr(E|A).$$

Part 3

n-Person Games

CHAPTER 5

Utilities, Potentials and Equilibria

Before discussing n-person games *per se*, it is useful to go back to the fundamental model of a game Γ being played on a system $\mathfrak{S}$ of states and look at characteristic features of Γ. The aim is a numerical assessment of the worth of states and strategic decisions from a general perspective.

1. Utilities and Potentials

1.1. Utilities. A *utility* on the system $\mathfrak{S}$ is an ensemble

$$U = \{u^\sigma \mid \sigma \in \mathfrak{S}\}$$

of functions $u^\sigma : \mathfrak{S} \to \mathbb{R}$, the so-called *local utility functions* of U.

We think of U as a measuring instrument which allows us to evaluate a possible move $\sigma \mapsto \tau$ numerically by the ensuing *marginal difference*

$$\boxed{\partial U(\sigma,\tau) = u^\sigma(\tau) - u^\sigma(\sigma)}$$

If the quality of any move $\sigma \mapsto \tau$ in the game Γ is evaluated *via* the utility U, we call U the *characteristic utility* of Γ.

1.2. Potentials. Having a 'potential' means to have the capability to enact something. In physics, the term *potential* refers to a characteristic quantity of a system whose change results in a dynamic behavior of the system. Potential energy, for example, may allow a mass to be set into motion. The resulting *kinetic energy* corresponds to the change in the potential energy. Gravity is thought to result from changes in a corresponding potential, the so-called *gravitational field etc.*

Mathematically, a potential is represented as a real-valued numerical parameter. In other words: A *potential* on the system $\mathfrak{S}$ is just a function

$$v : \mathfrak{S} \to \mathbb{R}$$

which assigns to a state $\sigma \in \mathfrak{S}$ a numerical value $v(\sigma)$.

The potential v gives rise to an associated utility $V = \{v^\sigma \mid \sigma \in \mathfrak{S}\}$, where

$$v^\sigma = v \quad \text{for all } \sigma \in \mathfrak{S}.$$

If V is the utility of the game Γ on $\mathfrak{S}$, then the potential v is called the *characteristic function* of Γ. The value of a move $\sigma \mapsto \tau$ is then given by the marginal difference

$$\boxed{\partial V(\sigma,\tau) = \partial v(\sigma,\tau) = v(\tau) - v(\sigma)}$$

Path independence. Given the utility U on $\mathfrak{S}$, a path

$$\gamma = \sigma_0 \mapsto \sigma_1 \mapsto \sigma_2 \mapsto \ldots \mapsto \sigma_{k-1} \mapsto \sigma_k$$

of system transitions has the total utility weight

$$\partial U(\gamma) = \partial U(\sigma_0,\sigma_1) + \partial U(\sigma_1,\sigma_2)\ldots + \partial U(\sigma_{k-1},\sigma_k).$$

We say that U is *path independent* if the utility weight of any path depends only on its initial state σ_0 and the final state (but not on the states $\sigma_i \neq \sigma_0, \sigma_k$ in between):

$$\partial U(\sigma_0 \mapsto \ldots \mapsto \sigma_i \mapsto \ldots \mapsto \sigma_k) = \partial U(\sigma_0,\sigma_k).$$

PROPOSITION 5.1. *The utility U is path independent on $\mathfrak{S}$ if and only if U is derived from a potential on $\mathfrak{S}$.*

Proof. If U is derived from the potential $u : \mathfrak{S} \to \mathbb{R}$, we have

$$\partial U(\sigma_{i-1},\sigma_i) = u(\sigma_i) - u(\sigma_{i-1})$$

and, therefore, for any $\gamma = \sigma_0 \mapsto \sigma_1 \mapsto \ldots \mapsto \sigma_k$:

$$\begin{aligned}\partial U(\gamma) &= \sum_{i=1}^{k}(u(\sigma_i) - u(\sigma_{i-1})) = \sum_{i=1}^{k} u(\sigma_i) - \sum_{i=0}^{k-1} u(\sigma_i)\\ &= u(\sigma_k) - u(\sigma_0)\\ &= \partial U(\sigma_k,\sigma_0),\end{aligned}$$

which shows that U is path independent.

Conversely, assume that $U = \{u^\sigma \mid \sigma \in \mathfrak{S}\}$ is a path independent utility. Fix a state σ_0 and notice that the utility function $u = u^{\sigma_0}$ is a potential in its own right. Since U is path independent, we have for all $\sigma,\tau \in \mathfrak{S}$,

$$\partial U(\sigma_0,\sigma) + \partial U(\sigma,\tau) = \partial U(\sigma_0,\tau)$$

and, therefore,

$$\partial U(\sigma,\tau) = \partial U(\sigma_0,\tau) - \partial U(\sigma_0,\sigma) = u(\tau) - u(\sigma).$$

So U is identical with the utility derived from the potential u.

$\diamond$

2. Equilibria

When we talk about an *equilibrium* of a utility $U = \{u^\sigma \mid \sigma \in \mathfrak{S}\}$ on the system $\mathfrak{S}$, we assume that each state σ has associated a *neighborhood*

$$\mathcal{F}^\sigma \subseteq \mathfrak{S} \quad \text{with } \sigma \in \mathcal{F}^\sigma.$$

This means that we concentrate state transitions to neighbors, *i.e.*, to transitions $\sigma \mapsto \tau$ with $\tau \in \mathcal{F}^\sigma$. We now say that a system state σ is an "equilibrium" if it yields a locally extreme value of the utility function from where a no transition to a neighbor appears attractive. To be precise, we distinguish maximal extreme values and minimal extreme values and, therefore, define:

(1) σ is a *gain* (or *profit*) *equilibrium* of U if

$$u^\sigma(\tau) \leq u^\sigma(\sigma) \quad \text{holds for all } \tau \in \mathcal{F}^\sigma;$$

(2) σ is a *cost equilibrium* of U if

$$u^\sigma(\tau) \geq u^\sigma(\sigma) \quad \text{holds for all } \tau \in \mathcal{F}^\sigma.$$

Many real-world systems are assumed to be subject to a dynamic process that eventually settles in an equilibrium state (or at least approximates an equilibrium) according to some utility measure. This phenomenon is strikingly observed in physics. But also economic theory has long suspected that economic systems may tend towards equilibrium states[1].

If the utility measure suggests to maximize the value, gain equilibria are of interest. If a minimal value is desirable, one is investigates cost equilibria.

REMARK 5.1 (Gains and costs). *Denote by $C = -U$ the utility with local utility functions $c^\sigma = -u^\sigma$. Then one has*

$$\sigma \textit{ is a gain equilibrium of } U \iff \sigma \textit{ is a cost equilibrium of } C$$

From an abstract point of view, the theory of gain equilibria is, therefore, equivalent to the theory of cost equilibria.

2.1. Existence of equilibria. In practice, the determination of an equilibrium is typically a very difficult computational task. In fact, many utilities do not have equilibria. It is generally not easy to just find out whether an equilibrium for a given utility exists at all. So one is interested in conditions that allow one to conclude that at least one equilibrium exists.

[1] A.A. COURNOT (1838): *Recherche sur les principes mathématiques de la théorie de la richesse*, Paris

2.1.1. *Utilities from potentials.* Consider a potential $u : \mathfrak{S} \to \mathbb{R}$ with the derived utility values

$$\partial u(\sigma, \tau) = u(\tau) - u(\sigma).$$

Here, one has conditions that are obviously sufficient:

(1) If $u(\sigma) = \max_{\tau \in \sigma} u(\tau)$, then σ is a gain equilibrium.
(2) If $u(\sigma) = \min_{\tau \in \sigma} u(\tau)$, then σ is a cost equilibrium.

Since every function on a finite set attains a maximum and a minimum, we find

PROPOSITION 5.2. *If $\mathfrak{S}$ is finite, then every potential function yields a utility with at least one gain and one cost equilibrium.*

Similarly, we can derive the existence of equilibria on systems that are represented in a coordinate space.

PROPOSITION 5.3. *If $\mathfrak{S}$ can be represented as a compact set $\mathcal{S} \subseteq \mathbb{R}^m$ such that $u : \mathcal{S} \to \mathbb{R}$ is continuous, then u implies a utility on $\mathcal{S}$ with at least one gain and one cost equilibrium.*

Indeed, it is well-known that a continuous function on a compact set attains a maximum and a minimum.

REMARK 5.2. *Notice that the conditions given in this section, are sufficient to guarantee the existence of equilibria – no matter what neighborhood structure on $\mathfrak{S}$ is assumed.*

2.1.2. *Convex and concave utilities.* If the utility is not implied by a potential function, not even the finiteness of $\mathfrak{S}$ may be sufficient to guarantee the existence of an equilibrium (see Ex. 5.1).

EX. 5.1. *Give the example of a utility U relative to a finite state state $\mathfrak{S}$ with no gain and no cost equilibrium.*

We now derive sufficient conditions for utilities U on systems whose states are represented by a nonempty convex set $\mathcal{S} \subseteq \mathbb{R}^m$.

We say:

- U is *convex* if every local function $u^s : \mathcal{S} \to \mathbb{R}$ is convex.
- U is *concave* if every local function $u^s : \mathcal{S} \to \mathbb{R}$ is concave.

THEOREM 5.1. *Let U be a utility with continuous local utility functions $u^s : \mathcal{S} \to \mathbb{R}$ on the nonempty compact set $\mathcal{S} \subseteq \mathbb{R}^m$. Then*

(1) *If U is convex, a cost equilibrium exists.*
(2) *If U is concave, a gain equilibrium exists.*

Proof. Define the function $G : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ with values

$$G(s,t) = u^s(t) \quad \text{for all } s,t \in \mathcal{S}.$$

Then the hypothesis of the Theorem says that G satisfies the conditions of Corollary A.1 of the Appendix. Therefore, an element $s \in \mathcal{S}$ exists such that

$$u^s(t) = G(s,t) \leq G(s,s) = u^s(s) \quad \text{holds for all } s \in \mathcal{S}.$$

Consequently, s^* is a gain equilibrium of U (The convex case is proved in the same way.)

$\diamond$

CHAPTER 6

n-Person Games

> n-person games generalize 2-person games. Yet, it turns out that the special techniques for the analysis of 2-person games apply in this seemingly wider context as well. Traffic systems, for example, fall into this category naturally.

The model of a *n-person game* Γ assumes the presence of a finite set N with $n = |N|$ elements together with a family $X = \{X_i \mid i \in N\}$ of n further nonempty sets X_i. The elements $i \in N$ are thought of as *players* or *agents etc.* A member $X_i \in X$ represents the collection of *resources* (or *actions*, *strategies*, *decisions etc.*) that are available to agent $i \in N$.

A *state* of Γ is a particular selection $\mathbf{x} = (x_i \mid i \in N)$ of individual resources $x_i \in X_i$ by the n agents i. So the collection of all states $\mathbf{x}$ of Γ is represented by the direct product

$$\mathfrak{X} = \prod_{i \in N} X_i.$$

REMARK 6.1. *It is often convenient to label the elements of N by natural numbers and assume $N = \{1, 2, \ldots, n\}$ for simplicity of notation. In this case, a state $\mathbf{x}$ of Γ can be denoted in the form*

$$\mathbf{x} = (x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \ldots \times X_n \ \ (= \mathfrak{X}).$$

We furthermore assume that each player $i \in N$ has an individual utility

$$U_i = \{u_i^{\mathbf{x}} \mid \mathbf{x} \in \mathfrak{X}\}$$

so that $u_i^{\mathbf{x}}(\mathbf{y})$ assesses the value of a state transition $\mathbf{x} \mapsto \mathbf{y}$ for i. The whole context

$$\Gamma = \Gamma(U_i \mid i \in N)$$

of the set players and their utilities now describes the n-person game under consideration.

EX. 6.1. *The matrix game Γ with a row player R and a column player C and the payoff matrix*

$$P = \begin{bmatrix} (p_{11}, q_{11}) & (p_{12}, q_{12})) \\ (p_{21}, q_{21}) & (p_{22}, q_{22}) \end{bmatrix} = \begin{bmatrix} (+1, -1) & (-1, +1) \\ (-1, +1) & (+1, -1) \end{bmatrix}.$$

is a 2-person game with the player set $N = \{R, C\}$ and the strategy sets $X_R = \{1, 2\}$ and $X_C = \{1, 2\}$. Accordingly, the set of states is

$$\mathfrak{X} = X_R \times X_C = \{(1,1), (1,2), (2,1), (2,2)\}.$$

The individual utility functions $u_R^{(i,j)}, u_C^{(i,j)} : \mathfrak{X} \to \mathbb{R}$ take the values

$$u_R^{(i,j)}(s,t) = p_{st} \quad \textit{and} \quad u_C^{(i,j)}(s,t) = q_{st} \quad \textit{for all } (s,t) \in \mathfrak{X}.$$

Potential games. The n-person game $\Gamma = (U_i \mid i \in N)$ is called a *potential game* if there is a potential $v : \mathfrak{X} \to \mathbb{R}$ such that, for all $\in N$ and $\mathbf{x}, \mathbf{y} \in \mathfrak{X}$ the marginal utility change equals the change in the potential:

$$u_i^{\mathbf{x}}(\mathbf{y}) - u_i^{\mathbf{x}}(\mathbf{x}) = \partial v(\mathbf{x}, \mathbf{y}) = v(\mathbf{y}) - v(\mathbf{x})$$

Cooperation. The basic game model with a set N of players is readily generalized to a model where groups of players (and not just individuals) derive a utility value from a certain state $\mathbf{x} \in \mathfrak{X}$. To this end, we call a subset $S \subseteq N$ of players a *coalition* and assume an individual utility function $u^S : \mathfrak{X} \to \mathbb{R}$ to exist for each coalition S.

From an abstract mathematical point of view, however, this generalized model can be treated like a standard $|\mathcal{N}|$-person game, having the set

$$\mathcal{N} = \{S \subseteq N\}$$

of coalitions as its set of "superplayers". In fact, we may allow each coalition S to be endowed with its own set X_S of resources.

In this chapter, we therefore retain the basic model with respect to an underlying set N of players. A special class of potential games with cooperation, so-called *cooperative games*, will be studied in more detail in Chapter 7.

Probabilistic models. There are many probabilistic aspects of n-person games. One consists in having a probabilistic model to start with (see Ex.6.2).

EX. 6.2 (Fuzzy games). *Assume a game Γ where any player $i \in N$ has to decide between two alternatives, say "0" and "1", and chooses "1" with probability x_i. Then Γ is a $|N|$-person game in which each player i has the unit interval*

$$X_i = [0,1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$$

as its set of resources. A joint strategic choice

$$\mathbf{x} = (x_1, \ldots, x_i, \ldots, x_n) \in [0,1]^N$$

can be interpreted as a "fuzzy" decision to form a coalition $X \subseteq N$:

- *Player i will be a member of X with probability x_i.*

$\mathbf{x}$ *is thus the description of a* fuzzy coalition. Γ *is a* fuzzy cooperative game *in the sense of* AUBIN[1] *if it is a potential game in our terminology.*

A further model arises from the *randomization* of a n-person game (see Section 3). Other probabilistic aspects of n-person games are studied in Chapter 7 and in Chapter 8.

1. Dynamics of n-person games

If the game $\Gamma = (U_i \mid i \in N)$ is played, a game instance yields a sequence of state transitions. The transitions are thought to result from changes in the strategy choices of the players.

Suppose $i \in N$ replaces its current strategy x_i by the strategy $y \in X_i$ while all other players $j \neq i$ retain their choices $x_j \in X_j$. Then a state transition $\mathbf{x} \mapsto \mathbf{y} = \mathbf{x}_{-i}(y)$ results, where the new state has the components

$$y_j = \begin{cases} y & \text{if } j = i \\ x_j & \text{if } j \neq i. \end{cases}$$

Note in particular that $\mathbf{x}_{-i}(x_i) = \mathbf{x}$ holds under this definition. Let us take the set

$$\mathcal{F}_i(\mathbf{x}) = \{\mathbf{x}_{-i}(y) \mid y \in X_i\}$$

as the *neighborhood* of the state $\mathbf{x} \in \mathfrak{X}$ for the player $i \in N$. So the *neighbors* of $\mathbf{x}$ from i's perspective are those states that could be achieved by i with a change of its current strategy x_i, provided all other players $j \neq i$ retain their current strategies x_j.

2. Equilibria

A *gain equilibrium* of $\Gamma = \Gamma(U_i \mid i \in N)$ is a joint strategic choice $\mathbf{x} \in \mathfrak{X}$ such that no player has an utility incentive to switch to another strategy, *i.e.*,

$$u_i^{\mathbf{x}}(\mathbf{x}) \geq u_i^{\mathbf{x}}(\mathbf{z}) \quad \text{holds for } \textit{all } i \in N \text{ and } \mathbf{z} \in \mathcal{F}_i(\mathbf{x}).$$

Completely analogously, a *cost equilibrium* is defined *via* the reverse condition:

$$u_i^{\mathbf{x}}(\mathbf{x}) \leq u_i^{\mathbf{x}}(\mathbf{z}) \quad \text{holds for } \textit{all } i \in N \text{ and } \mathbf{z} \in \mathcal{F}_i(\mathbf{x}).$$

[1] J.-P. AUBIN (1981): *Fuzzy cooperative games*, Math. Operations Research 6, 1-13

This notion of an equilibrium can be brought in line with the general definition in Chapter 2). Given the state $\mathbf{x}$, imagine that each player i considers an alternative y_i to its current strategy x_i. The aggregated sum of the resulting utility values is

$$G(\mathbf{x},\mathbf{y}) = \sum_{i\in N} u^{\mathbf{x}}(\mathbf{x}_{-i}(y_i)) \quad (\mathbf{y} = (y_i \mid y_i \in X_i)).$$

LEMMA 6.1. *$\mathbf{x} \in \mathfrak{X}$ is a gain equilibrium of $\Gamma(U_i \mid i \in N)$ if and only if*

$$G(\mathbf{x},\mathbf{y}) \leq G(\mathbf{x},\mathbf{x}) \quad \textit{holds for all } \mathbf{y} \in \mathfrak{X}.$$

Proof. If $\mathbf{x}$ is a gain equilibrium and $\mathbf{y} = (y_i \mid i \in N) \in \mathfrak{X}$, we have

$$u_i^{\mathbf{x}}(\mathbf{x}) \geq u^{\mathbf{x}}(\mathbf{x}_{-i}(y_i)) \quad \text{for all } y_i \in X_i,$$

which implies $G(\mathbf{x},\mathbf{x}) \geq G(\mathbf{x};\mathbf{y})$. Conversely, if $\mathbf{x}$ is not a gain equilibrium, there is an $i \in N$ and a $y \in X_i$ such that

$$0 < u_i^{\mathbf{x}}(\mathbf{x}_{-i}(y)) - u_i^{\mathbf{x}}(\mathbf{x}) = G(\mathbf{x},\mathbf{x}_i(y)) - G(\mathbf{x},\mathbf{x}).$$

which means that $\mathbf{y} = \mathbf{x}_{-i}(y) \in \mathfrak{X}$ violates the inequality.

$\diamond$

Lemma 6.1 reduces the quest for an equilibrium of Γ to the quest for an equilibrium of the utility $G = \{g^{\mathbf{x}} \mid \mathbf{x} \in \mathfrak{X}\}$ with values

$$g^{\mathbf{x}}(\mathbf{y}) = G(\mathbf{x},\mathbf{y}).$$

It follows that we can immediately carry over the general sufficient conditions in Chapter 5 for the existence of equilibria to the n-person game $\Gamma = (U_i \mid i \in N)$ with utility aggregation function G:

(1) If Γ is a potential game with a finite set $\mathfrak{X}$ of states, then the existence of a gain and of a cost equilibrium is guaranteed.
(2) If $\mathfrak{X}$ is represented as a nonempty compact and convex set in a finite-dimensional real parameter space, and all the maps $\mathbf{y} \mapsto G(\mathbf{x},\mathbf{y})$ are continuous and concave, then Γ admits a gain equilibrium.
(3) If $\mathfrak{X}$ is represented as a nonempty compact and convex set in a finite-dimensional real parameter space, and all the maps $\mathbf{y} \mapsto G(\mathbf{x},\mathbf{y})$ are continuous and convex, then Γ admits a cost equilibrium.

EX. 6.3. *Show that the matrix game in Ex. 6.1 is not a potential game (Hint: The set of states is finite.)*

3. Randomization of matrix games

A n-person game $\Gamma = (U_i \mid i \in N)$ is a *matrix game* if

(i) The set X_i of resources of any player $i \in N$ is finite.
(ii) Each player $i \in N$ has just *one* utility function $u_i : \mathfrak{X} \to \mathbb{R}$.

For a motivation of the terminology, assume $N = \{1, \ldots, n\}$ and think of the sets X_i as index sets for the coordinates of a multidimensional matrix U. A particular index vector

$$\mathbf{x} = (x_1, \ldots, x_i, \ldots, x_n) \in \mathfrak{X} = X_1 \times \ldots \times X_i \times \ldots X_n$$

thus specifies a position in U with the n-dimensional coordinate entry

$$U_{\mathbf{x}} = (u_1(\mathbf{x}), \ldots, u_i(\mathbf{x}), \ldots, u_n(\mathbf{x})) \in \mathbb{R}^n.$$

Let us now change the rules of the matrix game Γ in the following way:

(R) For each $i \in N$, player i chooses a probability distribution $p^{(i)}$ on X_i and selects the element $x \in X_i$ with probability $p^{(i)}_x$.

Under rule (R), the players are really playing the related n-person game $\overline{\Gamma} = (\overline{U}_i \mid i \in N)$ with resource sets P_i and utility functions $\overline{u}_i : P_i \to \mathbb{R}$, where

(1) P_i is the set of all probability distributions on X_i.
(2) $\overline{u}_i(p)$ is the expected value of u_i relative to the joint probability distribution $p = (p^{(i)} \mid i \in N)$ of the players.

The n-person game $\overline{\Gamma}$ is the *randomization* of the matrix game Γ. Assuming $N = \{1, \ldots, n\}$, one has the expected utility values given as

$$\boxed{\overline{u_i}(p^{(1)}, \ldots, p^{(n)})) = \sum_{x_1 \in X_1} \cdots \sum_{x_n \in X_n} u_i(x_1, \ldots, x_n) p^{(1)}_{x_1} \cdots p^{(n)}_{x_n}.}$$

As Ex. 6.1 shows, a (non-randomized) matrix game Γ does not necessarily have equilibria. On the other hand, notice that the coordinate product function

$$(t_1, \ldots, t_n) \in \mathbb{R}^n \mapsto t_1 \cdots t_n \in \mathbb{R}$$

is continuous and linear in each variable. Each utility function $\overline{u}_i$ of the randomized game $\overline{\Gamma}$ is a linear combination of such functions and, therefore, also continuous and linear in each variable.

Since linear functions are both concave and convex and the state set

$$\mathfrak{P} = P_1 \times \ldots \times P_n$$

is convex and compact, we conclude:

> THEOREM 6.1 (NASH). *The randomization $\overline{\Gamma}$ of a n-person matrix game Γ admits both a gain and a cost equilibrium.*

REMARK 6.2. *An equilibrium of a randomized matrix game is also known as a* NASH equilibrium[2].

4. Traffic flows

A fundamental model for the analysis of flows in traffic networks goes back to WARDROP[3]. It is based on a graph $G = (V, E)$ with a (finite) set V of nodes and set E of (directed) edges e between nodes,

$$\text{(v)} \xrightarrow{e} \text{(w)},$$

representing directed connections from nodes to other nodes. The model assumes:

> (W) There is a set N of players. A player $i \in N$ wants to travel along a path in G from a starting point s_i to a destination t_i and has a set $\mathcal{P}_i$ of paths to choose from.

Game-theoretically speaking, a strategic action s of player $i \in N$ means a particular choice of a path $P \in \mathcal{P}_i$. Let us identify a path $P \in \mathcal{P}_i$ with its incidence vector in $\mathbb{R}^E$ with the components

$$P_e = \begin{cases} 1 & \text{if } P \text{ passes through } e \\ 0 & \text{otherwise.} \end{cases}$$

The joint travel path choice $\mathbf{s}$ of the players generates the *traffic flow*

$$\mathbf{x}^{\mathbf{s}} = \sum_{i \in N} \sum_{P \in \mathcal{P}_i} \lambda_P^{\mathbf{s}} P \quad \text{of size} \quad |\mathbf{x}^{\mathbf{s}}| = \sum_P \lambda_P^{\mathbf{s}} \leq n,$$

[2]J. NASH (1950): *Equilibrium points in n-person games*, Proc. National Academy of Sciences 36, 48-49

[3]J.G. WARDROP (1952): *Some theoretical aspects of road traffic research.* Institution of Civil Engineers 1, 325–378

where $\lambda_P^{\mathbf{s}}$ is the number of players that choose path P in $\mathbf{s}$. The component $x_e^{\mathbf{s}}$ of $\mathbf{x}^{\mathbf{s}}$ is the amount of traffic on edge e caused by the choice $\mathbf{s}$.

We assume that a traffic flow $\mathbf{x}$ produces congestion costs $c(x_e)$ along the edges e and hence results in the total congestion cost

$$C(\mathbf{x}) = \sum_{e \in E} c_e(x_e)x_e$$

across all edges. An individual player i has the congestion cost just along its chosen path P:

$$C(P, \mathbf{x}) = \sum_{e \in P} c_e(x_e)x_e.$$

If we associate with the flow $\mathbf{x}$ the potential of aggregated costs

$$\Phi(\mathbf{x}) = \sum_{e \in E} \sum_{t=1}^{x_e} c_e(t),$$

we find that player i's congestion cost along path P in $\mathbf{x}$ equals the marginal potential:

$$C(P, \mathbf{x}) = \sum_{e \in P} c_e(x_e) = \Phi(\mathbf{x}) - \Phi(\mathbf{x} - P) = \partial_P \Phi(\mathbf{x} - P).$$

It follows that the players in the WARDROP traffic model play a n-person potential game on the finite set $\mathfrak{X}$ of possible traffic flows.

$\mathbf{x} \in \mathfrak{X}$ is said to be a NASH *flow* if no player i can improve its congestion cost by switching from the current path $P \in \mathcal{P}_i$ to the use of another path $Q \in \mathcal{P}_i$. In other words, the NASH flows are the cost equilibrium flows. Since the potential function Φ is defined on a finite set, we conclude

- The WARDROP traffic flow model admits a NASH flow.

BRAESS' paradox. If one assumes that traffic in the WARDROP model eventually settles in a NASH flow, *i.e.*, that the traffic flow evolves toward a cost equilibrium situation, the well-known observation of BRAESS[4] is counter-intuitive:

(B) *It can happen that a reduction of the congestion along a particular connection* increases(!) *the total congestion cost.*

As an example of BRAESS' paradox, consider the network $G = (V, E)$ with

$$V = \{s, r, q, t\} \quad \text{and} \quad E = \{(s, r), (s, q), (r, t), (q, t), (r, q)\}.$$

[4] D. BRAESS (1968): Über ein Paradoxon aus der Verkehrsplanung

Assume that the cost functions on the edges are

$$c_{sr}(x) = x, c_{sq}(x) = 4, c_{rt}(x) = 4, c_{qt}(x) = x, c_{rq} = 10$$

and that there are four network users, which choose individual paths from the starting point s to the destination t and want to minimize their individual travel times.

Because of the high congestion cost, no user will travel along (r, q). As a consequence, a NASH flow will have two users of path $P = (s \to r \to t)$ while the other two users would travel along $\tilde{P} = (s \to q \to t)$.

The overall cost is:

$$C(2P + 2\tilde{P}) = 2 \cdot 2 + 4 \cdot 2 + 4 \cdot 2 + 2 \cdot 2 = 24.$$

If road improvement measures are taken to reduce the congestion on (r, q) to $c'_{rq} = 0$, a user of path P can lower its current cost $C(P) = 6$ to $C(Q) = 5$ by switching to the path

$$Q = (s \to r \to q \to t).$$

The resulting traffic flow, however, causes a higher overall cost:

$$C'(P + Q + 2\tilde{P}) = 2 \cdot 2 + 4 \cdot 1 + 3 \cdot 1 + 4 \cdot 2 + 3 \cdot 2 = 25.$$

CHAPTER 7

Cooperative Games

Players in a cooperative game strive for a common goal, from which they possibly profit. Mathematically, such games are special potential games and best studied within the context of linear algebra. Central is the question how to distribute the achieved goal's profit appropriately. The core of a cooperative game is an important analytical notion. It provides a link to the theory of discrete optimization and greedy algorithms in particular. Moreover, natural models for the dynamics of coalition formation are closely related to thermodynamical models in statistical physics. Consequently, the notion of a temperature in a potential game can be made precise, for example.

While the agents in the 2-person games of the previous chapters typically have opposing utility goals, the model of a *cooperative game* refers to a finite set N of $n = |N|$ players that may or may not be active towards a common goal. A subset $S \subseteq N$ of potentially active players is traditionally called a *coalition*. Mathematically, there are several ways of looking at the system of coalitions:

From a set-theoretic point of view, one has the system of the 2^n coalitions

$$\mathcal{N} = \{S \mid S \subseteq N\}.$$

On the other hand, one may represent a subset $S \in \mathcal{N}$ by its incidence vector $x^{(S)} \in \mathbb{R}^N$ with the coordinates

$$x_i^{(S)} = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S. \end{cases}$$

The incidence vector $x^{(S)}$ suggests the interpretation of $i \in N$ being *active* if $x_i^{(S)} = 1$. The coalition S is thus the collection of active players.

A further interpretation imagines every player $i \in N$ to have a binary strategy set $X_i = \{0, 1\}$ from which to choose one element. An incidence vector

$$x = (x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n = \{0,1\}^N \subseteq \mathbb{R}^N$$

represents the joint strategy decision of the n players and we have the correspondence

$$\mathcal{N} \quad \longleftrightarrow \quad \{0,1\}^N = 2^N$$

By a *cooperative game* we will just understand a n-person game Γ with player set N and state set

$$\mathfrak{X} = \mathcal{N} \quad \text{or} \quad \mathfrak{X} = 2^N,$$

depending on a set theoretic of vector space point of view.

1. Cooperative TU-games

A *transferable utility* relative to a set N of players is a quantity v whose value $v(S)$ depends on the coalition S of active players and hence is a potential

$$v : \mathcal{N} \mapsto \mathbb{R}.$$

The resulting potential game $\Gamma = (N, v)$ represents a *cooperative TU-game* with *characteristic function* v. $v(\emptyset)$ is the utility value if no member of N is active in the game Γ.

Typically, (N, v) is assumed to be *zero-normalized*, *i.e.*, to have $v(\emptyset) = 0$. In the case $v(\emptyset) \neq 0$, one considers the TU-game $(N, v^{(0)})$ instead of (N, v) with the zero-normalized characteristic function values

$$v^{(0)}(S) = v(S) - v(\emptyset).$$

In the sequel, we will concentrate on TU-games and therefore just talk about a *cooperative game* (N, v).

REMARK 7.1 (Terminology). *Often the characteristic function v of a cooperative game (N, v) is already called a* cooperative game. *In discrete mathematics and computer science a function*

$$v : \{0,1\}^n \to \mathbb{R}$$

is also known as a pseudo-boolean *function. Decision theory refers to pseudo-boolean functions as* set functions[1].

The characteristic function v of a cooperative game can represent a *cost* utility or a *profit* utility. The real-world interpretation of the mathematical analysis, of course, depends on whether a cost or a gain model is assumed. Usually, the modeling context makes it clear however.

[1]see, *e.g.*, M. GRABISCH (2016): *Set functions, Games and Capacities in Decision Making*, Springer-Verlag

1.1. Vector spaces of TU-games. Identifying a TU-game (N, v) with its characteristic function v, we think of the function space

$$\mathbb{R}^{\mathcal{N}} = \{v : \mathcal{N} \to \mathbb{R}\} \quad \text{with} \quad \mathcal{N} = \{S \subseteq N\}$$

as the vector space of all TU-games on N. $\mathbb{R}^{\mathcal{N}}$ is isomorphic with coordinate space $\mathbb{R}^{2^n}$ and has dimension

$$\dim \mathbb{R}^{\mathcal{N}} = |\mathcal{N}| = 2^n = \dim \mathbb{R}^{2^n}.$$

The 2^n unit vectors of $\mathbb{R}^{\mathcal{N}}$ correspond to the so-called DIRAC functions $\delta_S \in \mathbb{R}^{\mathcal{N}}$ with the values

$$\delta_S(T) = \begin{cases} 1 & \text{if } T = S \\ 0 & \text{if } T \neq S. \end{cases}$$

The set $\{\delta_S \mid S \in \mathcal{N}\}$ is a basis of $\mathbb{R}^{\mathcal{N}}$. Any $v \in \mathbb{R}^{\mathcal{N}}$ has the representation

$$v = \sum_{S \in \mathcal{N}} = v(S)\delta_S.$$

1.1.1. **Duality.** It is advantageous to retain $\mathcal{N}$ as the index set explicitly because one can use the set theoretic structure of $\mathcal{N}$ for game theoretic analysis. One such example is the *duality operator* $v \mapsto v^*$ on $\mathbb{R}^{\mathcal{N}}$, where

$$v^*(S) = v(N) - v(N \setminus S) \quad \text{for all } S \subseteq N. \tag{36}$$

We say that the game (N, v^*) is the *dual* of (N, v). For any possible coalition $S \in \mathcal{N}$, the numerical value

$$v^*(N \setminus S) = v(N) - v(S)$$

is the "surplus" of the "grand coalition" N *vs.* S in the game (N, v). So duality expresses a balance

$$v(S) + v^*(N \setminus S) = v(N) \quad \text{for all coalitions } S.$$

EX. 7.1. *Show:*

(1) $v \mapsto v^*$ *is a linear operator on* $\mathbb{R}^{\mathcal{N}}$.
(2) *The dual* $v^{**} = (v^*)^*$ *of the dual* v^* *of* v *yields exactly the zero-normalization of* v.

1.1.2. **MÖBIUS transformation.** For any $v \in \mathbb{R}^{\mathcal{N}}$, let us define its MÖBIUS[2] *transform* as the function $\hat{v} \in \mathbb{R}^{\mathcal{N}}$ with values

$$\hat{v}(S) = \sum_{T \subseteq S} v(T) \quad (S \in \mathcal{N}).$$

[2] A.F. MÖBIUS (1790-1868)

$\hat{v}(S)$ sums up the v-values of all subcoalitions $T \subseteq S$. In this sense, the MÖBIUS transformation is a kind of "discrete integral" on the function space $\mathbb{R}^{\mathcal{N}}$.

EX. 7.2 (Unanimity games). *The* MÖBIUS *transform* $\widehat{\delta}_S$ *of the* DIRAC *function* δ_S *is known as a* unanimity game *and has the values*

$$\widehat{\delta}_S(T) = \begin{cases} 1 & \text{if } S \subseteq T \\ 0 & \text{if } S \not\subseteq T. \end{cases}$$

A coalition T has a non-zero value $\widehat{\delta}_S(T) = 1$ exactly when the coalition T includes all *members of S. Unanimity games appear to be quite simple and yet are basic (Corollary 7.1 below). Many concepts in cooperative game theory are tested against their performance on unanimity games.*

Clearly, the MÖBIUS *transformation* $v \mapsto \hat{v}$ is a linear operator on $\mathbb{R}^{\mathcal{N}}$. The important observation concerns an inverse property: every characteristic function v arises as the transform of a uniquely determined other characteristic function w.

THEOREM 7.1 (MÖBIUS inversion). *For each $v \in \mathbb{R}^{\mathcal{N}}$, there is a unique $w \in \mathbb{R}^{\mathcal{N}}$ such that $v = \hat{w}$.*

Proof. Recall from linear algebra that it suffices to show that $\hat{z} = O$ implies $z = O$, *i.e.*, that the kernel of the MÖBIUS transform contains just the zero vector $O \in \mathbb{R}^{\mathcal{N}}$.

So let us consider an arbitrary function $z \in \mathbb{R}^{\mathcal{N}}$ with transform $\hat{z} = O$. Let $S \in \mathcal{N}$ be a coalition and observe in the case $S = \emptyset$:

$$z(\emptyset) = \hat{z}(\emptyset) = 0.$$

Assume now, by induction, that $z(T) = 0$ holds for all $T \in \mathcal{N}$ of size $|T| < |S|$. Then the conclusion

$$z(S) = \hat{z}(S) - \sum_{T \subset S} z(T) = 0 - 0 = 0$$

follows and completes the inductive step of the proof. So $z(S) = 0$ must be true for all coalitions S.

$\diamond$

Since the MÖBIUS operator is linear, Theorem 7.1 implies that it is, in fact, an automorphism of $\mathbb{R}^{\mathcal{N}}$, which maps bases onto bases. So we find in particular:

COROLLARY 7.1 (Unanimity basis). *The unanimity games $\widehat{\delta}_S$ form a basis of $\mathbb{R}^{\mathcal{N}}$,* i.e.*, each $v \in \mathbb{R}^{\mathcal{N}}$ admits a unique representation of the form*

$$v = \sum_{S \in \mathcal{N}} \lambda_S \widehat{\delta}_S \quad \textit{with coefficients} \quad \lambda_S \in \mathbb{R}.$$

EX. 7.3 (HARSANYI dividends). *Where $v = \hat{w}$, the values $w(S)$ are known in cooperative game theory as the* HARSANYI dividends *of the coalitions S in the game (N, v). It follows that the value $v(S)$ of any coalition S is the sum of the* HARSANYI *dividends of its subcoalitions T:*

$$v(S) = \hat{w}(S) = \sum_{T \subseteq S} w(T).$$

REMARK 7.2. *The literature is not quite clear on the terminology and often refers to the inverse transformation $\hat{v} \mapsto v$ as the* MÖBIUS *transformation. Either way, the* MÖBIUS *transformation is a classical and important tool also in in number theory and in combinatorics*[3].

1.1.3. **Potentials and linear functionals.** A potential $f : \mathcal{N} \to \mathbb{R}$, interpreted as a vector $f \in \mathbb{R}^{\mathcal{N}}$ defines a linear functional $\tilde{f} : \mathbb{R}^{\mathcal{N}} \to \mathbb{R}$, where

$$\tilde{f}(g) = \langle f|g \rangle = \sum_{S \in \mathcal{N}} f_S g_S \quad \text{for all } g \in \mathbb{R}^{\mathcal{N}}.$$

If $g^{(S)}$ is the $(0,1)$-incidence vector of a particular coalition $S \in \mathcal{N}$, we have

$$\tilde{f}(g^{(S)}) = \langle f|g^{(S)} \rangle = f_S \cdot 1 = f_S.$$

which means that $\tilde{f}$ extends the potential f on 2^N $(= \mathcal{N})$ to all of $\mathbb{R}^{\mathcal{N}}$.

Conversely, every linear functional $g \mapsto \langle f|g \rangle$ on $\mathbb{R}^{\mathcal{N}}$ defines a unique potential f on $\mathcal{N}$ *via*

$$f(S) = \langle f|g^{(S)} \rangle \quad \text{for all } S \in \mathcal{N}.$$

[3]see, *e.g.*, G.-C. ROTA (1964): *On the foundations of combinatorial theory I. Theory of* MÖBIUS *functions.* Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete

These considerations show that potentials (characteristic functions) on $\mathcal{N}$ and linear functionals on $\mathbb{R}^{\mathcal{N}}$ are two sides of the same coin. From the point of view of linear algebra, one can therefore equivalently define:

- *A cooperative TU-game is a pair $\Gamma = (N, v)$, where N is a set of players and $v \mapsto \langle v|g\rangle$ is a linear functional on the vector space $\mathbb{R}^{\mathcal{N}}$.*

1.2. Marginal values. The characteristic function v of the cooperative game $\Gamma = (N, v)$ is a utility relative to the system $\mathcal{N}$ of coalitions of N. Individual players $i \in N$ will assess their value with respect to v by evaluating the change in v that they can effect by being active or inactive.

For a player $i \in N$, we thus define its *marginal value* with respect to the coalition S as

$$\partial_i v(S) = \begin{cases} v(S \cup i) - v(S) & \text{if } i \in N \setminus S \\ v(S) - v(S \setminus i) & \text{if } i \in S. \end{cases}$$

Additive games. The marginal value $\partial_i v(S)$ of a player $i \in N$ depends on the coalition S it refers to. Different coalitions may yield different marginal values for the player i.

The game $\Gamma = (N, v)$ is said to be *additive* if every player's marginal values are the same relative to all possible coalitions. So there are numbers v_i such that

$$\partial_i v(S) = v_i \quad \text{for all } S \in \mathcal{N} \text{ and } i \in N.$$

Hence, if v is additive, we have

$$v(S) = v(\emptyset) + \sum_{i \in S} v_i.$$

Conversely, every vector $a \in \mathbb{R}^N$ defines a zero-normalized additive game (N, a) with the understanding

$$a(\emptyset) = 0 \quad \text{and} \quad a(S) = \sum_{s \in S} w_i \quad \text{for all } S \neq \emptyset. \tag{37}$$

Ex. 7.4. *Which unanimity games (see Ex. 7.2) are additive? Show that the vector space of all additive games on N has dimension $|N| + 1$. The subspace of all zero-normalized additive games on N has dimension $|N|$.*

We turn now to more examples of cooperative games.

1.3. Production games. As in Section 4.2, consider a factory that produces k different types of goods from m raw materials $M_1, \ldots, M_m$. Let $x = (x_1, \ldots, x_k)$ be a plan that proposes the production of x_j units of the jth good and assume

(1) x would need $a_i(x)$ units of material M_i for $i = 1, \ldots, m$;
(2) the production x could be sold for the price of $f(x)$ (euros, dollars or whatever);
(3) there is a set N of suppliers and each $s \in N$ owns b_{is} units of material M_i.

As in Section 4.2, the quest for an optimal production plan x^* leads to the optimization problem

$$\max_{x \in \mathbb{R}^k_+} f(x) \quad \text{s.t.} \quad a_i(x) \leq \sum_{s \in N} b_{is} \ (i = 1, \ldots m).$$

Assume that the market prices of the materials are $y_1^*, \ldots, y_m^*$ (per unit). Then an optimal production plan x^* needs to buy

$$v(N) = \sum_{s \in N} y_i^* a_i(x^*) = \sum_{i=1}^{m} y_i^* b_i \quad (\text{with } b_i = \sum_{s \in N} b_{is})$$

worth of materials from the suppliers.

How should the worth of an individual supplier $s \in N$ be assessed? A natural parameter is the market value of all the materials owned by s:

$$w_s^* = \sum_{i=1}^{n} y_i^* b_{is}. \tag{38}$$

Is this allocation $s \mapsto w_s^*$ to individual suppliers s "fair"? To shed more light onto this question (without answering it), let us consider an alternative approach:

Assume that a coalition S evaluates its inner worth from the shadow prices $y_i^S, \ldots, y_m^S$ of the S-restricted optimization problem

$$\max_{x \in \mathbb{R}^k_+} f(x) \quad \text{s.t.} \quad a_i(x) \leq \sum_{s \in S} b_{is} \ (i = 1, \ldots m). \tag{39}$$

An optimal solution x^S of (39) requires

$$v(S) = \sum_{i=1}^{m} y_i^S a_i(x^S) = \sum_{i=1}^{m} y_i^S b_i^S \quad (\text{with } b_i^S = \sum_{s \in S} b_{is})$$

worth of material and gives rise to a cooperative game (N, v). In this context, the worth of a supplier $s \in N \setminus T$ for a coalition T is

$$\partial_s v(T) = v(T \cup s) - v(T).$$

So one may want to argue that a "fair" assessment of the suppliers should take their marginal values into account. (This idea is studied further in Section 3.2 below.)

1.4. Linear production games. Forgetting about marginal values for the moment, the situation is particularly transparent in the case of a linear objective and linear production requirements:

$$\begin{array}{rcccl} f(x) & = & c^T x & = & c_1 x_1 + \ldots + c_n x_n \\ a_i(x) & = & a_i^T x & = & a_{i1} x_1 + \ldots + a_{in} x_n \quad (i = 1, \ldots, m). \end{array}$$

Where A denotes the matrix with the m row vectors a_i^T, the shadow price vector y^* is an optimal solution of the dual linear program

$$\min_{y \in \mathbb{R}^m_+} y^T b \quad \text{s.t.} \quad y^T A \geq c^T$$

and has the property

$$v(N) = \sum_{i=1}^{m} b_i y_i^* = \sum_{j=1}^{n} c_j x_j^* = f(x^*).$$

Note that the dual of the S-restricted production problem has the same constraints and differs only in the coefficients of the objective function:

$$\min_{y \in \mathbb{R}^m_+} y^T b^S \quad \text{s.t.} \quad y^T A \geq c^T.$$

An optimal solution x^S and a shadow price vector y^S yield

$$v(S) = \sum_{i=1}^{m} b_i^S y_i^S = \sum_{j=1}^{n} c_j x_j^S = f(x^S).$$

So the shadow price vector y^* is also feasible (but not necessarily optimal) for the S-restriction and we conclude (with w_s^* as in (38)):

$$v(S) \leq \sum_{i=1}^{m} b_i^S y_i^* = \sum_{s \in S} w_s^* = w^*(S). \tag{40}$$

REMARK 7.3. *The inequality (40) suggests that the shadow prices y^* satisfy all coalitions in the sense that every coalition S receives a material worth $w^*(S)$ that is at least as large as its pure market value $v(S)$. This is the thought behind the notion of the* core *of a game (*cf. *Section 2 below).*

1.5. Network connection games. Consider a set $N = \{p_1, \ldots, p_n\}$ of users of some utility that are to be linked, either directly or indirectly (*via* other users), to some supply node p_0. Assume that the cost of establishing a link between p_i with p_j would be c_{ij} (euros, dollars or whatever). The associated cooperative game has the utility function

$$c(S) = \text{minimal cost of connecting just } S \text{ to } p_0.$$

The relevant question is:

- *How much should a user $p_i \in N$ be charged so that a network with the desired connection can be established?*

One possible cost distribution scheme is derived from a construction method for a connection of minimal total cost $c(N)$. The *greedy algorithm* builds up a chain of coalitions

$$\emptyset = S_0 \subset S_1 \subset S_2 \subset \ldots \subset S_n = N$$

according to the following iterative procedure:

(G_0) $S_0 = \emptyset$;
(G_j) If S_j has been constructed, choose $p \in N \setminus S_j$ such that $c(S_j \cup p)$ is as small as possible and charge user p the marginal cost

$$\partial_j v(S) = c(S_j \cup p) - c(S_j).$$

(G_n) Set $S_{j+1} = S_j \cup p$ and continue until all users have been charged.

NOTA BENE. The greedy algorithm makes sure that the user set N in total is charged the minimal possible connection cost:

$$\begin{aligned}\sum_{j=1}^{n}[c(S_j) - c(S_{j-1})] &= c(S_n) - c(S_0) + \sum_{k=1}^{n-1}[c(S_k) - c(S_k)] \\ &= c(N) - c(\emptyset) \;=\; c(N).\end{aligned}$$

In this sense, the greedy algorithm is *efficient*. Nevertheless, the greedy cost allocation scheme may appear ”unfair” from the point of view of individual users (see Ex. 7.5)[4].

[4]game theorists disagree on ”the best” network cost allocation scheme

EX. 7.5. *Consider a user set* $N = \{p_1, p_2\}$ *with connection cost coefficients* $c_{01} = 100$*,* $c_{02} = 101$ *and* $c_{12} = 2$*. The greedy algorithm constructs the coalition chain*

$$\emptyset = S_0 \subset S_1 = \{p_1\} \subset S_2 = \{p_1.p_2\} = N$$

and charges $c(S_1) = 100$ *to user* p_1 *and* $c(S_2) - c(S_1) = 2$ *to user* p_2*. So* p_1 *would be to bear about* 98% *of the total cost* $c(N) = 102$*.*

1.6. Voting games. Assume there is a set N of n voters i of (not necessarily equal) voting power. Denote by w_i the number of votes voter i can cast. Given a threshold w, the associated *voting game*[5] has the characteristic function

$$v(S) = \begin{cases} 1 & \text{if} \sum_{i \in S} w_i \geq w \\ 0 & \text{otherwise.} \end{cases}$$

In the voting context, $v(S) = 1$ has the interpretation that the coalition S has the voting power to make a certain proposed measure pass. Notice that in the case $v(S) = 0$, a voter i with marginal value

$$\partial_i v(S) = v(S \cup i) - v(S) = 1$$

has the power to swing the vote by joining S. The general question is of high political importance:

- *How can (or should) one assess the overall voting power of a voter* i *in a voting context?*

REMARK 7.4. *A popular index for individual voting power is the* BANZHAF *power index (see Section 3 below). However, there are alternative evaluations that also have their merits. As in the case of network cost allocation, abstract mathematics cannot decide what the "best" method would be.*

2. The core

Note this particular feature in our current discussion:

- *In order to avoid technicalities, we assume that all cooperative games in this section are zero-normalized.*

The *core* of a (zero-normalized) cooperative profit game (N, v) is the set

$$\text{core}(v) = \{x \in \mathbb{R}^N \mid x(N) = v(N), x(S) \geq v(S)\ \forall S \subseteq N\},$$

[5] also known as a *threshold game*

with the notational understanding $x(S) = \sum_{i \in S} x_s$. Mathematically speaking, core(v) is the solution set of a finite number of linear inequalities in the euclidian space $\mathbb{R}^N$.

In a game theoretic interpretation on the other hand, a vector $x \in$ core(v) is an assignment of individual values x_i to the players $i \in N$ such that the value $v(N)$ is distributed completely and each coalition S receives at least its proper value $v(S)$.

Inequality (40) above, for example, exhibits the suppliers' market values w_s^* (see equation (38)) as the coefficients of a core vector in a linear production game.

The core of a cost game (N, c) is defined analogously:

$$\text{core}^*(c) = \{x \in \mathbb{R}^N \mid x(N) = c(N), x(S) \leq c(S)\ \forall S \subseteq N\}.$$

$x \in \text{core}^*(c)$ distributes the cost $c(N)$ among the players $i \in N$ so that no coalition S pays more than its proper cost $c(S)$.

EX. 7.6. *Show for the (zero-normalized) cooperative game (N, v) and its dual (N, v^*):*

$$\text{core}(v^*) = \text{core}^*(v).$$

EX. 7.7. *Give the example of a cooperative game (N, v) with* core$(v) = \emptyset$.

Alas, as Ex. 7.7 shows, the core is not a generally applicable concept for "fair" profit (or cost) distributions to the individual players in cooperative games because it may be empty. Therefore, further value assignments concepts are of interest. Section 3 will provide examples of such concepts. For the moment, let us continue with the study of the core.

2.1. The MONGE algorithm. We consider a fixed cooperative game (N, v) with n players and collection $\mathcal{N}$ of coalitions.

Given a parameter vector $c \in \mathbb{R}^N$ and an arrangement $\pi = i_1 i_2 \ldots i_n$ of the elements of N, the MONGE[6] *algorithm* constructs a *primal* MONGE *vector* $x^\pi \in \mathbb{R}^N$ and a *dual* MONGE *vector* $y^\pi \in \mathbb{R}^{\mathcal{N}}$ as follows:

(M$_1$) Set $S_0^\pi = \emptyset$ and $S_k^\pi = \{i_1, \ldots, i_k\}$ for $k = 1, 2, \ldots, n$.
(M$_2$) Set $x_{i_k}^\pi = v(S_k^\pi) - v(S_{k-1}^\pi)$ for $k = 1, 2, \ldots, n$.
(M$_3$) Set $y_{S_n}^\pi = c_{i_n}$ and $y_{S_\ell}^\pi = c_{i_\ell} - c_{i_{\ell+1}}$ for $\ell = 1, 2, \ldots, n-1$.
Set $y_S^\pi = 0$ otherwise.

[6]G. MONGE (1746-1818)

It is not hard to see[7] that the MONGE vectors x^π and y^π satisfy the identity

$$m^\pi(c) = \sum_{i \in N} c_i x_i^\pi = \sum_{S \in \mathcal{N}} v(S) y_S^\pi.$$

Different arrangements π and ψ of N, of course, may yield different MONGE sums $m^\pi(c)$ and $m^\psi(c)$. Important is the following observation.

LEMMA 7.1. *Let $\pi = i_1 i_2 \dots i_n$ and $\psi = j_1 j_2 \dots j_n$ be two arrangements of N such that $c_{1_1} \geq c_{i_2} \geq \ldots \geq c_{i_n}$ and $c_{j_1} \geq c_{j_2} \geq \ldots \geq c_{j_n}$. Then*

$$m^\pi(c) = \sum_{S \in \mathcal{N}} v(S) y_S^\pi = \sum_{S \in \mathcal{N}} v(S) y_S^\psi = m^\psi(c).$$

Proof. Note that $c_{i_\ell} = c_{i_{\ell+1}}$, for example, implies $y^\pi_{S_\ell} = 0$. So we may assume that the components of c have pairwise different values. But then $\pi = \psi$ holds, which makes the claim trivial.

$\diamond$

2.1.1. **The MONGE extension.** Lemma. 7.1 says that there is a well-defined real-valued function $c \mapsto [v](c)$, where

$$[v](c) = m^\pi(c) \quad \text{if } \pi = i_1 i_2 \dots i_n \text{ s.t. } c_{i_1} \geq c_{i_2} \geq \ldots \geq c_{i_n}.$$

The function $[v]$ is the MONGE *extension* of the characteristic function v.

To justify the terminology "extension", consider a coalition $S \subseteq N$ and its $(0,1)$ incidence vector $c^{(S)}$ with the component $c_i^{(S)} = 1$ in the case $i \in S$. An appropriate MONGE arrangement of the elements of N lists first all 1-components and then all 0-components:

$$\pi = i_1 \dots i_k \dots i_n \quad \text{s.t.} \quad c_{i_1}^{(S)} \dots c_{i_k}^{(S]} \dots c_{i_n}^{(S)} = 1 \dots 10 \dots 0.$$

Hence we have $y_S^\pi = 1$ and $y\pi_T = 0$ for $T \neq S$ and conclude

$$[v](c^{(S)}) = v(S) \quad \text{for all } S \subseteq N.$$

REMARK 7.5 (CHOQUET and LOVÁSZ). *Applying the idea of* MONGE *sums to nondecreasing value arrangements $f_1 \leq \ldots \leq f_n$ of nonnegative functions $f : N \to \mathbb{R}_+$, one arrives at the* CHOQUET integral

$$\int f dv = \sum_{k=1}^n f_k(v(A_k) - v(A_{k+1})),$$

where $A_k = \{k, k+1, \ldots, n\}$ and $A_{n+1} = \emptyset$.

[7] *cf.* Section 5 of the Appendix

The map $f \mapsto \int f dv$ is the so-called LOVÁSZ extension *of the function $v : \mathcal{N} \to \mathbb{R}$. Of course, mutatis mutandis, all structural properties carry over from* MONGE *to* CHOQUET *and* LOVÁSZ.

2.1.2. **Linear programming aspects.** Since core(v) is defined as the solution set of a finite number of inequalities (and one equality), it is natural to study linear programs with the core as its set of feasible solutions. Consider the linear program

$$\min_{x \in \mathbb{R}^N} c^T x \quad \text{s.t.} \quad x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ if } S \neq N. \tag{41}$$

and its dual

$$\max_{y \in \mathbb{R}^{\mathcal{N}}} v^T y \quad \text{s.t.} \quad \sum_{S \ni i} y_S \leq c_i \ \forall i \in N \text{ and } y_S \geq 0 \text{ if } S \neq N. \tag{42}$$

Observe in the case $c_{i_1} \geq \ldots\ldots \geq c_{i_n}$ that the dual MONGE vector y^π relative to c is a feasible solution since $y^\pi_S \geq 0$ is satisfied for all $S \neq N$. Hence, if core$(v) \neq \emptyset$, both linear programs have optimal solutions. Linear programming duality then furthermore shows

$$\tilde{v}(c) = \min_{x \in \textbf{core}(v)} c^T x \ \geq \ v^T y^\pi = [v](c). \tag{43}$$

THEOREM 7.2. *$\tilde{v} = [v]$ holds for the game (N, v) if and only if all primal* MONGE *x^π vectors lie in* core(v).

Proof. Assume $c_{i_1} \geq \ldots \geq c_{i_n}$ and $\pi = i_1 \ldots i_n$. If $x^\pi \in$ core(v), then x^π is a feasible solution for the linear program (41). Since the dual MONGE vector y^π is feasible for (42), we find

$$c^T x^\pi \geq \tilde{v}(c) \geq [v](v) = c^T x^\pi \quad \text{and hence} \quad \tilde{v}(c) = [v](c).$$

Conversely, $\tilde{v} = [v]$ means that the dual MONGE vector is guaranteed to yield an optimal solution for (42). So consider an arrangement $\psi = j_1 \ldots j_n$ of N and the parameter vector $c \in \mathbb{R}^N$ with the components

$$c_{j_k} = n + 1 - k \quad \text{for } k = 1, \ldots, n.$$

The dual vector y^ψ has strictly positive components $y^\psi_{S_k} = 1 > 0$ on the sets S_k. It follows from the KKT-conditions for optimal solutions that an optimal solution $x^* \in \text{core}(v)$ of the corresponding linear program (41) must satisfy the equalities

$$x^*(S_k) = \sum_{i \in S_k} x_i^* = v(S_k) \quad \text{for } k = 1, \ldots, n,$$

which means that x^* is exactly the primal MONGE vector x^π and, hence, that $x^\pi \in \text{core}(v)$ holds.

$\diamond$

2.2. Concavity. Let us call a characteristic function $v : 2^N \to \mathbb{R}$ *concave* if v arises from the restriction of a concave function to the $(0,1)$- incidence vector $c^{(S)}$ of the coalitions S, *i.e.*, if if there is a concave function $f : \mathbb{R}^N \to \mathbb{R}$ such that

$$v(S) = f(c^{(S)}) \quad \text{holds for all } S \subseteq N.$$

Accordingly, the cooperative game $\Gamma = (N, v)$ is *concave* if v is concave. We will not pursue an investigation of general conacave cooperative games here but focus on a particularly important class of concave games which are closely tied to the MONGE algorithm *via* Theorem 7.2.

PROPOSITION 7.1. *If all* MONGE *vectors of the game* (N, v) *lie in* core(v)*, then* (N, v) *is concave.*

Proof. By Theorem 7.2, the hypothesis of the Proposition says

$$\tilde{v}(c) = [v](c) \quad \text{for all } c \in \mathbb{R}^N.$$

Consequently, it suffices to demonstrate that $\tilde{v}$ is a concave function. Clearly, $\tilde{\lambda c} = \lambda \tilde{v}\langle c \rangle$ holds for all scalars $\lambda > 0$, *i.e.* $\tilde{v}$ is positively homogeneous.

Consider now arbitrary parameter vectors $c, d \in \mathbb{R}^N$ and $x \in \text{core}(v)$ such that $\tilde{v}(c + d) = (c + d)^T x$. Then

$$\tilde{v}(c + d) = c^T x + d^T x \geq \tilde{v}(c) + \tilde{v}(d),$$

which exhibits $\tilde{v}$ as concave.

$\diamond$

REMARK 7.6. *The converse of Proposition 7.1) is not true: there are concave games whose core does not include all primal* MONGE *vectors.*

A word of terminological caution. The game theoretic literature often applies the terminology "convex cooperative game" to games (N, v) having all primal MONGE vectors in core(v). In our terminology, however, such games are not *convex* but *concave*.

To avoid terminological confusion, one may prefer to refer to them as *supermodular games* (*cf.* Theorem 7.3 in the next Section 2.3).

2.3. Supermodularity. The central notion that connects the MONGE algorithm with concavity is the notion of *supermodularity*:

THEOREM 7.3. *For the cooperative game* (N, v)*, the following statements are equivalent:*

(I) *All* MONGE *vectors* $x^\pi \in \mathbb{R}^N$ *lie in* core(v).

(II) v *is* supermodular , i.e.*, satisfies the inequality*

$$v(S \cap T) + v(S \cup T) \geq v(S) + v(T) \quad \textit{for all } S, T \subseteq N.$$

Proof. Assuming (I), order the elements of N in an order $i_1 \ldots i_n$ such that

$$S \cap T = \{i_1, \ldots, i_k\}, S = \{i_1, \ldots, i_\ell\}, S \cup T = \{i_1, \ldots, i_m\}.$$

By the definition of the MONGE algorithm, x^π then satisfies

$$x^\pi(S \cap T) = v(S \cap T), x^\pi(S) = v(S), x^\pi(S \cup T) = v(S \cup T).$$

Moreover, $x^\pi(T) \geq v(T)$ holds if $x^\pi \in$ core(v). So we deduce the supermodular inequality

$$\begin{aligned} v(S \cap T) + v(S \cup T) &= x^\pi(S \cap T) + x^\pi(S \cup T) \\ &= x^\pi(S) + x^\pi(T) \\ &\geq v(S) + v(T). \end{aligned}$$

Conversely, suppose that v is not supermodular. We will exhibit a MONGE vector x^π that is not a member of core(v). Let $S, T \subseteq N$ be such that

$$v(S \cap T) + v(S \cup T) < v(S) + v(T)$$

is true and arrange the elements of N in an order $\pi = i_1 \ldots i_n$ such that

$$S \cap T = \{i_1, \ldots, i_k\}, S = \{i_1, \ldots, i_m\}, S \cup T = \{i_1, \ldots, i_\ell\}.$$

Consequently, the MONGE vector x^π satisfies

$$\begin{aligned} v(S)+v(T) &> v(S\cap T)+v(S\cup T) \\ &= x^\pi(S\cap T)+x^\pi(S\cup T) \\ &= x^\pi(S)+x^\pi(T) \\ &= v(S)+x^\pi(T) \end{aligned}$$

and therefore $v(T) > x^\pi(T)$, which shows $x^\pi \notin \text{core}(v)$.

◇

EX. 7.8. *The preceding proof uses the fact that any vector $x \in \mathbb{R}^N$ satisfies the* modular equality

$$x(S\cap T)+x(S\cup T) = x(S)+x(T) \quad \textit{for all } S,T \subseteq N,$$

with the understanding $x(S) = \sum_{i\in S} x_i$.

2.4. Submodularity. A characteristic function v is called *submodular* if the inequality

$$v(S\cap T)+v(S\cup T) \leq v(S)+v(T) \quad \text{holds for all } S,T \subseteq N.$$

EX. 7.9. *Show for the zero-normalized game (N,v) the equivalence of the statements:*

(1) *v is supermodular.*
(2) *v^* is submodular.*
(3) *$w = -v$ is submodular.*

In view of the equality core(c^*) = core$^*(c)$ (Ex. 7.6) , we find that the MONGE algorithm also constructs vectors in the core$^*(c)$ of cooperative cost games (N,c) with submodular characteristic functions c.

REMARK 7.7. *Note the fine point of Theorem 7.3, which in the language of submodularity says:* (N,c) is a submodular cost game if and only if *all* MONGE vectors x^π lie in core$^*(c)$.

Network connection games are typically not *submodular. Yet, the particular cost distribution vector discussed in Section 1.5 does lie in* core$^*(c)$*, as the ambitious reader is invited to demonstrate.*

REMARK 7.8. *Because of the* MONGE *algorithm, sub- and supermodular functions play a prominent role in the field of discrete optimiziation*[8]*. In fact, many results of discrete optimization have a direct interpretation in the*

[8] *cf.* S. FUJISHIGE (2005), *Submodular Functions and Optimization*, 2nd ed., Annals of Discrete Mathematics 58

theory of cooperative games. Conversely, the model of cooperative games often provides conceptual insight into the structure of discrete optimization problems.

REMARK 7.9 (Greedy algorithm). *The* Monge *algorithm, applied to linear programs with core-type constraints is also known as the* greedy algorithm *in discrete optimization.*

3. Values

While the marginal value $\partial_i v(S)$ of player i's decision to join resp. to leave the coalition S is intuitively clear, it is less clear how the overall strength of i should be assessed. From a mathematical point of view, there are infinitely many possibilities to do this.

In general, we understand by a *value* for the class of all TU-games (N, v) a function

$$\Phi : \mathbb{R}^{\mathcal{N}} \to \mathbb{R}^N$$

that associates with every characteristic function v a vector $\Phi(v) \in \mathbb{R}^N$. Given Φ, the number $\Phi_i(v)$ is the assessment of the strength of $i \in N$ in the game (N, v) according to the evaluation concept Φ.

3.1. Linear values. The value Φ is said to be *linear* if Φ is a linear operator, *i.e.*, if one has for all games v, w and scalars $\lambda \in \mathbb{R}$, the equality

$$\Phi(\lambda v + w) = \lambda\Phi(v) + \Phi(w).$$

In other words: Φ is linear if each component function Φ_i is a linear functional on the vector space $\mathbb{R}^{\mathcal{N}}$.

Recall from Ex. 7.2 that the unanimity games form a basis of $\mathbb{R}^{\mathcal{N}}$, which means that every game v can be uniquely expressed as a linear combination of unanimity games. Hence a linear value Φ is completely determined by the values assigned to unanimity games. The same is true for any other basis of $\mathbb{R}^{\mathcal{N}}$, of course.

Indeed, if $v_1, \ldots, v_k \in \mathcal{G}_0(N)$ are arbitrary games and $\lambda_1, \ldots, \lambda_k$ arbitrary real scalars, the linearity of Φ yields

$$\Phi(\lambda_1 v_1 + \ldots + \lambda_k v_k) = \lambda_1\Phi(v_1) + \ldots + \lambda_k\Phi(v_k).$$

We give two typical examples of linear values.

The SHAPLEY value. Consider the unanimity game $\widehat{\delta}_T$ relative to the coalition $T \in \mathcal{N}$ where

$$\widehat{\delta}_T(S) = \begin{cases} 1 & \text{if } S \supseteq T \\ 0 & \text{otherwise.} \end{cases}$$

In this case, it might appear reasonable to assess the strength of a player $s \in N \setminus T$ as null, *i.e.*, with the value $\Phi_j^{Sh}(\widehat{\delta}_T) = 0$, and the strength of each of the players $t \in T$ in equal proportion as

$$\Phi_t^{Sh}(\widehat{\delta}_T) = \frac{1}{|T|}.$$

Extending Φ^{Sh} to all games v by linearity in the sense

$$\boxed{v = \sum_{T \in \mathcal{N}} \lambda_t \widehat{\delta}_T \quad \Longrightarrow \quad \Phi^{Sh}(v) = \sum_{T \in \mathcal{N}} \lambda_T \Phi^{Sh}(\widehat{\delta}_T)}$$

one obtains a linear value $v \mapsto \Phi^{Sh}(v)$, the so-called SHAPLEY *value*.

EX. 7.10. *Show that the players in (N, v) and in its zero-normalization (N, v^{**}) are assigned the same Shapley values.*

The BANZHAF power index. The BANZHAF power index Φ^B appears at first sight quite similar to the SHAPLEY value, assessing the power

$$\Phi_s^B(\widehat{\delta}_T) = 0 \quad \text{if } s \in N \setminus T$$

while treating all $t \in T$ as equals. Assuming $T \neq \emptyset$, the mathematical difference lies in the scaling factor:

$$\Phi_i^B(\widehat{\delta}_T) = \frac{1}{2^{|T|-1}} \quad \text{for all } t \in T.$$

As was done with the SHAPLEY VALUE, the BANZHAF power index is extended by linearity from unanimity games to all games (N, v) and thus gives rise to a linear value $v \mapsto \Phi^B(v)$.

As we will see in Section 3.2, the difference between the values Φ^{Sh} and Φ^B can also be explained by two different probabilistic assumptions about the way coalitions are formed.

REMARK 7.10 (Efficiency). *If $|T| \geq 1$, the* SHAPLEY *value distributes the total amount*

$$\sum_{i \in N} \Phi_i^{Sh}(\widehat{\delta}_T) = 1 = \widehat{\delta}_T(N)$$

to the members of N and is, therefore, said to be efficient. *In contrast, we have*

$$\sum_{i\in N} \Phi_i^B((\widehat{\delta}_T) = \frac{|T|}{2^{|T|-1}} < 1 = \widehat{\delta}_T(N) \quad \text{if } |T| \geq 2.$$

So the BANZHAF *power index in not efficient.*

3.2. Random values. The concept of a random value is based on the assumption that a player $i \in N$ joins a coalition $S \subseteq N \setminus \{i\}$ with a certain probability π_S as a new member. The expected marginal value of $i \in N$ thus is

$$E_i^\pi(v) = \sum_{S\subseteq N\in\{i\}} \partial_i v(S)\pi_S.$$

The function $E^\pi : \mathcal{G}_0(N) \to \mathbb{R}^N$ with components $E_i^p i(v)$ is the associated *random value*.

Notice that marginal values are linear. Indeed, if $u = \lambda v + w$, one has

$$\partial_i u(S) = \lambda \partial_i v(S) + \partial_i w(S)$$

for all $i \in N$ and $S \subseteq N$. Therefore, the random value E^π is linear as well:

$$E^\pi(\lambda v + w) = \lambda E^\pi(v) + E^\pi(w). \tag{44}$$

REMARK 7.11. *The linearity relation (44) implicitly assumes that the probabilities π_S are independent of the particular characteristic function v. If π_S depends on v, the linearity of E^π is no longer guaranteed!*

The BOLTZMANN *value (to be discussed in Section 4 below) is a random value that is* not *linear in the sense of (44) because the associated probability distribution depends on the characteristic function.*

3.2.1. **The value of BANZHAF.** As an example, let us assume that a player i joins any of the 2^{n-1} coalitions $S \subseteq N \setminus \{i\}$ with equal likelihood, *i.e.*, with probability

$$\pi_S^B = \frac{1}{2^{n-1}}.$$

Consider the unanimity $v_T = \widehat{\delta}_T$ and observe that $\partial_i v_T(S) = 0$ holds if $i \notin T$. On the other hand, if $i \in T$, then one has

$$\partial_i v_T(S) = 1 \quad \Longleftrightarrow \quad T \setminus \{i\} \subseteq S.$$

So the number of coalitions S with $\partial_i v_T(S) = 1$ equals

$$|\{S \subseteq N \setminus \{i\} \mid T \subseteq S \cup \{i\}\}| = 2^{n-|T|-1}.$$

Hence we conclude

(45) $$E_i^{\pi^B}(v_T) = \sum_{S \subseteq N \setminus \{i\}} \partial_i v_T(S) \pi_S^B = \frac{2^{n-|T|-1}}{2^{n-1}} = \frac{1}{2^{|T|}},$$

which means that the random value E^{π^B} is identical with the BANZHAF power index. The probabilistic approach yields the explicit formula

(46) $$\Phi_i^B(v) = E_i^{\pi^B}(v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N \setminus \{i\}} (v(S \cup i) - v(S)) \quad (i \in N).$$

3.2.2. **Marginal vectors and the SHAPLEY value.** Let us imagine that the members of N build up the "grand coalition" N in a certain order

$$\sigma = i_1 i_2 \ldots i_n$$

and hence join in the sequence of coalitions

$$\emptyset = S_0^\sigma \subset S_1^\sigma \ldots \subset S_k^\sigma \subset \ldots \subset S_n^\sigma = N$$

where $S_k^\sigma = S_{k-1}^\sigma \cup \{i_k\}$ for $k = 1, \ldots, n$. Given the game (N, v), σ gives rise to the *marginal vector*[9] $\partial^\sigma(v) \in \mathbb{R}^N$ with components

$$\partial_{i_k}(v) = v(S_k^\sigma) - v(S_{k-1}^\sigma) \quad (k = 1, \ldots, n).$$

Notice that $v \mapsto \partial^\sigma(v)$ is a linear value for $\mathcal{G}_0(N)$. We can randomize this value by picking the order σ from the set Σ_N of all orders of N according to a probability distribution π. Then the expected marginal vector

$$\partial^\pi(v) = \sum_{\sigma \in \Sigma_N} \partial^\sigma(v) \pi_\sigma$$

represents, of course, also a linear value on $\mathcal{G}_0(N)$.

EX. 7.11. *Show that the value $v \mapsto \partial^\pi(v)$ is linear and efficient. (*HINT: *Recall the discussion of the greedy algorithm for network connection games in Section 1.5).*

[9] the marginal vectors are precisely the primal MONGE vectors of Section 2.1

PROPOSITION 7.2. *The* SHAPLEY *value results as the expected marginal vector relative to the uniform probability distribution on* Σ_N*, where all orders are equally likely:*

$$\Phi^{Sh}(v) = \frac{1}{n!} \sum_{\sigma \in \Sigma_N} \partial^\sigma(v).$$

(Recall from combinatorics that there are $n! = |\Sigma_N|$ *ordered arrangements of* N*.)*

Proof. Because of linearity, it suffices to prove the Proposition for unanimity games $v_T = \widehat{\delta}_T$. For any order $\sigma = i_1 \ldots i_n \in \Sigma_N$ and element $i_k \in N \setminus T$, we have $\partial_{i_k}(v_T) = 0$ and hence

$$\frac{1}{n!} \sum_{\sigma \in \Sigma_N} \partial_i^\sigma(v_T) = 0 \quad \text{for all } i \in N \setminus T.$$

On the other hand, the uniform distribution treats all members $i \in T$ equally and thus distributes the value $v_T(N) = 1$ equally and efficiently among the members of T:

$$\frac{1}{n!} \sum_{\sigma \in \Sigma_N} \partial_i^\sigma(v_T) = \frac{v_T(N)}{|T|} = \frac{1}{|T|} \quad \text{for all } i \in T,$$

which is exactly the concept of the SHAPLEY.

◇

We can interpret the SHAPLEY value within the framework of random values initially introduced. So we assume that an order $\sigma \in \Sigma_N$ is chosen with probability $1/n!$ and consider a coalition $S \subseteq N \setminus \{i\}$. We ask:

- What is the probability π_S^{Sh} that i would join S?

Letting $k - 1 = |S|$ be the size of S, the number of sequences σ where i would be added to S is

$$|\{\sigma \mid i = i_k \text{ and } S_{k-1}^\sigma = S\}| = (k-1)!(n-k)!$$

This is so because:

(1) The first $k-1$ elements must be chosen from S in any of the $(k-1)!$ possible orders.
(2) The remaining $n-k$ elements must be from $N \setminus (S \cup \{i\})$.

So one concludes

$$\pi_S^{Sh} = \frac{(k-1)!(n-k)!}{n!} = \frac{(|S|!(n-|S|-1)!}{n!}$$

and obtains another explicit formula for the SHAPLEY VALUE:

$$\text{(47)} \quad \boxed{\Phi_i^{Sh}(v) = \sum_{S\subseteq N\setminus\{i\}} \partial_i v(S)\pi_S^{Sh} = \sum_{S\subseteq N\setminus\{i\}} \partial_i v(S)\frac{(|S|!(n-|S|-1)!}{n!}}$$

EX. 7.12. *Consider a voting/threshold game* (cf. *Section 1.6) with four players of weights* $w_1 = 3, w_2 = 2, w_3 = 2, w_4 = 1$. *Compute the* BANZHAF *and the* SHAPLEY *values for each of the players for the threshold* $w = 4$.

4. Boltzmann values

The probabilistic analysis of the previous section shows that the value assessment concepts of the BANZHAF power index and the SHAPLEY value, for example, implicitly assume that players just join – but never leave – an existing coalition in a cooperative game $(N, v) \in \mathcal{G}(N)$.

In contrast, the model of the present section assumes an underlying probability distribution π on the set 2^N of *all* coalitions of N and assigns to player $i \in N$ its expected marginal value

$$E_i(v,\pi) = \sum_{S\subseteq N} \partial_i v(S)\pi_S = \sum_{S\subseteq N} (v(S\Delta i) - v(S))\pi_S.$$

We furthermore do allow π to depend on the particular characteristic function v under consideration. So the functional $v \mapsto E_i(v,\pi)$ is not guaranteed to be linear.

What probability distribution π should one reasonably expect in this model? To answer this question, we consider the associated expected characteristic value

$$\mu = \sum_{S\subseteq N} v(S)\pi_S$$

as a relevant parameter and ask:

- Given just μ, which probability distribution $\hat{\pi}$ would be the best unbiased guess for (the unknown) π?

From the information-theoretic point of view, the best unbiased guess $\hat{\pi}$ is the one with the highest entropy among those probability distributions π yielding the expected value μ. So we seek a solution to the optimization problem

$$
\begin{array}{lrcl}
\max_{x_S \geq 0} & H(x) & = & -\sum_{S \subseteq N}^{n} x_S \ln x_S \\
\text{s.t.} & \mu & = & \sum_{S \subseteq N} v(S) x_S \\
& 1 & = & \sum_{S \subseteq N} x_S.
\end{array} \tag{48}
$$

THEOREM 7.4. *For every potential $v : \mathcal{N} \to \mathbb{R}$ and possible expected value μ of v, there exists a unique parameter $-\infty \leq T \leq +\infty$ such that*

(1) $\mu = \sum_{S \subseteq N} v(S) e^{v(S)T} / Z_T$, *where* $Z_T = \sum_{S \subseteq N} e^{v(S)T}$*;*

(2) *the numbers $b_S^T = e^{-v(S)T}/Z_T$ are strictly positive and yield the unique optimal solution of the entropy optimization problem (48).*

A proof of Theorem 7.4 can be found in Section 6.2 of the Appendix.

The probabilities b_S^T define the BOLTZMANN *distribution* of $v : \mathcal{N} \to \mathbb{R}$ relative to the parameter T. So we obtain a BOLTZMANN *value* $\Phi^{\mathcal{B}}$ for every v and parameter T:

$$
\boxed{\Phi_i^{\mathcal{B}}(v;T) = \sum_{S \subseteq N} \partial_i v(S) b_S^T = \frac{1}{Z_T} \sum_{S \subseteq N} \partial_i v(S) e^{v(S)T} \quad (i \in N)} \tag{49}
$$

Let us look at some extreme cases. For $T = 0$, the BOLTZMANN distribution is just the uniform distribution on $\mathcal{N}$,

$$
b_S^0 = \frac{1}{|\mathcal{N}|} = \frac{1}{2^n} \quad \text{for all } S \subseteq N,
$$

and the BOLTZMANN value of a player $i \in N$ is the average over all its marginal values:

$$
\Phi_i^{\mathcal{B}}(v;0) = \frac{1}{2^n} \sum_{S \subseteq N} \partial_i v(S).
$$

In the case $T = +\infty$, b^∞ becomes the uniform distribution on the set $V_{\max} \subseteq 2^N$ of all maximizers of v (see Ex. 7.13). Hence one has

$$
\Phi_i^{\mathcal{B}}(v;+\infty) = \frac{1}{|V_{\max}|} \sum_{S \in V_{\max}} \partial_i v(S).
$$

Similarly, one sees that $b^{-\infty}$ is the uniform distribution on the set $V_{\min}$ of minimizers of v.

EX. 7.13. *Let $S^* \in V_{\max}$ be a maximizer of v and $S \subseteq N$ arbitrary. Then*

$$\lim_{T\to+\infty} \frac{e^{v(S)T}}{e^{v(S^*)T}} = \lim_{T\to+\infty} \left(e^{(v(S)-v(S^*))}\right)^T = \begin{cases} 1 & \textit{if } v(S) = v(S^*) \\ 0 & \textit{otherwise.} \end{cases}$$

Conclude that b^∞ is the uniform distribution on $V_{\max}$ (all other coalitions occur with probability 0).

Temperature. In analogy with the BOLTZMANN model in statistical thermodynamical physics, we may think of v as an *energy potential* function and ask for its minimizers. So we face BOLTZMANN distributions b^T with $T < 0$. Where $k_B > 0$ is a normalizing factor[10], we define

$$\tilde{\theta} = \frac{-1}{k_B T} \geq 0 \quad \text{and hence} \quad b_S^T = \frac{1}{Z_T} e^{-v(S)/k_B\tilde{\theta}}.$$

Thermodynamics interprets $\tilde{\theta}$ as the *temperature* of the cooperative system with potential distribution b^T. Hence, if $\tilde{\theta} \to 0$, the system attains a state (coalition) of minimal potential with high probability.

In the high temperature case $\tilde{\theta} \to \infty$, the system becomes more and more unpredictable: all states (coalitions) are about equally likely (*i.e.*, are approximately uniformly distributed).

The analogy with physics suggests to take the nonnegative parameter

$$\theta = \frac{1}{|T|}$$

as a measure for the *temperature* in general game-theoretic contexts. In particular, it appears reasonable to speak of the "temperature" of an economic system, for example, if the system is assumed to be governed by a potential (like the gross national product or a similar global indicator).

The conclusions are the same:

- If $\theta \to \infty$, all joint strategic actions are approximately equally likely.
- If $\theta \to 0$, the expected potential value becomes extreme, namely maximal if $T \to +\infty$ and minimal if $T \to -\infty$.

[10] the precise physical value of the so-called BOLTZMANN *constant* k_B is irrelevant for our game theoretic purposes!

5. Coalition formation

The term *coalition formation* refers originally to cooperative TU-games. The formation of a coalition that will eventually be active is viewed as a dynamic process where coalitions evolve over time according to the behavior of players that might leave the momentary coalition and join other players for a new coalition *etc.*. The question how a decision process evolves over time is, of course, of interest for a general potential game $\Gamma = (N, v)$ and system $\mathfrak{X}$ of joint strategies.

If one assumes $|\mathfrak{X}| < \infty$ and the agents $i \in N$ to be individually greedy, *i.e.*, to switch their strategic action if a switch offers a better marginal value, then one must conclude that the agents eventually reach an action equilibrium $\mathbf{x}^* \in \mathfrak{X}$ (*cf.* Proposition 5.2).

On the other hand, if we make no *a priori* assumption on the individual agents but assume that the decision process eventually arrives at the joint action $\mathbf{x} \in \mathfrak{X}$ with a probability $\pi_{\mathbf{x}}$ and produces the expected potential value

$$\mu = \sum_{vx \in \mathfrak{X}} v(\mathbf{x})\pi_{\mathbf{x}},$$

an unbiased estimate of the probability distribution π leads to the conclusion (Theorem 7.4) that the decision process eventually produces the joint strategic choice $\mathbf{x} \in \mathfrak{X}$ with the BOLTZMANN probability

$$b_{\mathbf{x}}^T = \frac{1}{Z_T} e^{v(\mathbf{x})T} \quad \text{with } T \text{ such that} \quad \mu = \sum_{\mathbf{x} \in \mathfrak{X}} v(\mathbf{x}) b_{\mathbf{x}}^T.$$

METROPOLIS *et al.*[11] have formulated the model of a stochastic process that converges to the distribution b^T with $T \geq 0$ as follows:

(M1) If the process is currently in the state $\mathbf{x}$, an agent $i \in N$ is chosen with some probability $p_i > 0$;
(M2) i chooses an action $y \in X_i$ with probability $q_y > 0$;
(M3) If $v(\mathbf{x}_{-i})(y) > v(\mathbf{x})$, then i switches from x_i to y;
(M4) If $v(\mathbf{x}_{-i}(y)) \leq v(\mathbf{x})$, then i switches from x_i to y with probability

$$\alpha = e^{(v(\mathbf{x}_{-i}(y) - v(\mathbf{x}))T}$$

(and does not change the action otherwise).

[11] N. METROPOLIS, A. ROSENBLUTH, M. ROSENBLUTH, A. TELLER, E. TELLER: *Equation of state computations by fast computing machines*. J. Chem. Physics 21 (1953)

The algorithm of METROPOLIS *et al.* simulates a so-called MARKOV *chain* on $\mathfrak{X}$. The proof of its correctness is not too difficult but a bit technical. Therefore, we will not reproduce it here but point out the relevant features of the algorithm:

- If $\partial_y v(\mathbf{x}) = v(\mathbf{x}_{-i})(y) - v(\mathbf{x}) \geq 0$, agent i is greedy and switches the strategic action to y.
- If $\partial_y v(\mathbf{x}) < 0$, then i switches with a probability that is small when T is large:

$$\left(e^{\partial_t v(\mathbf{x})}\right)^T \to 0 \quad \text{as} \quad T \to +\infty.$$

REMARK 7.12. *The* METROPOLIS *algorithm is easily adjusted for* $T \leq 0$: *One simply replaces* v *by the potential* $w = -v$ *and proceeds with* w *and the nonnegative parameter* $T' = -T$ *as above.*

5.1. Individual greediness and public welfare. Let us assume that N is a society whose common welfare is expressed by the potential v. If all members of N act purely greedily, an equilibrium action will eventually be arrived at that does not necessarily lead to a high common welfare level. For example, if all players in a WARDROP traffic situation (*cf.* Section 4) act purely greedily, no optimal traffic flow is guaranteed.

However, if the members of N are prepared to possibly accept a momentary marginal individual deterioration (case (M4) in the algorithm), then a common welfare of level

$$\mu^T(v) = \sum_{\mathbf{x} \in \mathfrak{X}} v(\mathbf{x}) b_{\mathbf{x}}^T$$

can be expected. Moreover, the larger T is, the closer is $\mu^T(v)$ to the maximal possible level $v_{\max}$.

So the society N must offer incentives or individual rewards to induce the players to act as in (M4) in order to achieve a high public welfare level.

5.2. Simulated annealing. If there is only a single agent, $\mathfrak{X}$ is the strategy set of that agent and the METROPOLIS algorithm can be used to optimize a function $v : \mathfrak{X} \to \mathbb{R}$, *i.e.*, to find an optimal solution to the problem

$$\max_{x \in \mathfrak{X}} v(x)$$

with high probability by adding the procedural step

(M5) After each iteration, increase T slightly with the aim $T \to \infty$.

In this form, the METROPOLIS algorithm is also known as *simulated annealing*. It has proven to be a very successful optimization technique for in the field of discrete optimization[12].

REMARKS. *Notice that the description of the simulated annealing algorithm is not very specific with respect to its practical implementation. How should the probabilities q_t in (M3) be chosen? How should T be increased in (M5)? etc. So the success of the simulated annealing method will also depend on the skill and experience of its user in practice.*

6. Equilibria in cooperative games

In the previous section, we have discussed the METROPOLIS algorithm as an example of a MARKOV chain that models coalition formation according to BOLTZMANN probabilities. If we go one step back and think of the players in a cooperative game $\Gamma = (N, v)$ as a group whose behavior is guided by individual profits from the achievement of a common goal, we must assume that each $i \in N$ has an individual utility function

$$u_i : \mathcal{N} \to \mathbb{R},$$

where $u_i(S)$ is i's expected gain (or cost) in case the coalition S will be active in Γ and thus achieve the value $v(S)$ for N. Of course, u_i will naturally depend on the characteristic function v of Γ. But other considerations may play a role as well.

In the BOLTZMANN model, a player i's utility criterion is essentially its marginal gain $\partial_i v(S)$ - if it is nonnegative. If it is negative, an additional incentive is assumed to be offered to carry out a strategic switch nonetheless with a certain probability. Typically, the BOLTZMANN model does not admit coalition equilibria – unless the game Γ is played under an extreme temperature T.

Many other value concepts (like SHAPLEY and BANZHAF, for example), are based on marginal gains as fundamental criteria for the individual utility assessment of a player.

Let us consider a game $\Gamma = (N, v)$ and take into account that the game will eventually split N into a group $S \subseteq N$ and the complementary group $S^c = N \setminus S$. Suppose a player $i \in N$ evaluates the utility of the partition (S, S^c) of N by

$$v_i(S) = v_i(S^c) = \begin{cases} v(S) - v(S \setminus i) & \text{if } i \in S \\ v(S^c) - v(S^c \setminus i) & \text{if } i \in S^c, \end{cases}$$

[12] S. KIRKPATRICK, C.D. GELAT, M.P. VECCHI: *Optimization by simulated annealing.* Science 220 (1983)

EX. 7.14. *Assume that (N, v) is a supermodular game. Then one has for all players $i \neq j$,*

$$\begin{aligned} v_i(N) = v(N) - v(N \setminus i) &\geq v(N \setminus j) - v((N \setminus j) \setminus i) = v_i(N \setminus j) \\ v_i(N) = v(N) - v(N \setminus i) &\geq v(\{i\}) - v(\emptyset) = v_i(N \setminus i). \end{aligned}$$

Consequently, the grand coalition N represents a gain equilibrium relative to the utilities v_i.

EX. 7.15. *Assume that (N, c) is a zero-normalized submodular game and that the players i have the utilities*

$$c_i(S) = \begin{cases} c(S) - c(S \setminus i) & \textit{if } i \in S \\ c(S^c) - c(S^c \setminus i) & \textit{if } i \in S^c. \end{cases}$$

Show: *The grand coalition N is a cost equilibrium relative to the utilities c_i.*

CHAPTER 8

Interaction Systems and Quantum Models

This final chapter investigates a quite general model for cooperation and interaction relative to a set X. Using complex numbers, the states of this model are naturally represented as hermitian matrices with complex coefficients. This representation allows us to carry out standard spectral analysis for interaction systems and provides a link to the standard mathematical model of quantum systems in physics.

While the analysis could be extended to general HILBERT spaces, X is assumed to be finite to keep the discussing simpler.

1. Algebraic preliminaries

Since matrix algebra is the main tool in our analysis, we review some more fundamental notions from linear algebra. Further details and proofs can be found in any decent book on linear algebra[1].

Where $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ are two finite index sets, recall that $\mathbb{R}^{X \times Y}$ denotes the real vector space of all matrices A with rows indexed by X, columns indexed by Y, and coefficients $A_{xy} \in \mathbb{R}$.

The *transpose* of $A \in \mathbb{R}^{X \times X}$ is the matrix $A^T \in \mathbb{R}^{Y \times X}$ with the coefficients $A^T_{xy} = A_{xy}$. The map $A \mapsto A^T$ establishes an isomorphism between the vector spaces $\mathbb{R}^{X \times Y}$ and $\mathbb{R}^{Y \times X}$.

Viewing $A \in \mathbb{R}^{X \times Y}$ and $B \in \mathbb{R}^{Y \times X}$ as mn-dimensional parameter vectors, we have the usual euclidian inner product as

$$\langle A|B\rangle = \sum_{(x,y)\in X\times Y} A_{xy}B_{xy} = B^T A.$$

In the case $\langle A|B\rangle = 0$, A and B are said to be *orthogonal*. The associated euclidian norm is

$$\|A\| = \sqrt{\langle A|A^T\rangle} = \sqrt{\sum_{(x,y)\in X\times Y} |A_{xy}|^2}.$$

[1] *e.g.*, E.D NERING (1967), *Linear Algebra and Matrix Theory*, Wiley, New York

We think of a vector $v \in \mathbb{R}^X$ typically as a column vector. So v^T is the row vector with the same coordinates $v^T_x = v_x$. Notice the difference between the two matrix products:

$$
\begin{aligned}
v^T v &= \sum_{x \in X} |v_x|^2 = \|v\|^2 \\
v v^T &= \begin{bmatrix} v_{x_1} v_{x_1} & v_{x_1} v_{x_2} & \dots & v_{x_1} v_{x_m} \\ v_{x_2} v_{x_1} & v_{x_2} v_{x_2} & \dots & v_{x_1} v_{x_m} \\ \vdots & \vdots & \ddots & \vdots \\ v_{x_m} v_{x_1} & v_{x_m} v_{x_2} & \dots & v_{x_m} v_{x_m} \end{bmatrix}.
\end{aligned}
$$

1.1. Symmetry decomposition. Assuming now identical index sets

$$X = Y = \{x_1, \dots, x_n\}$$

a matrix $A \in \mathbb{R}^{X \times X}$ is *symmetric* if $A^T = A$. In the case $A^T = -A$, the matrix A is *skew-symmetric*. With an arbitrary matrix $A \in \mathbb{R}^{X \times X}$, we associate the matrices

$$A^+ = \frac{1}{2}(A + A^T) \quad \text{and} \quad A^- = \frac{1}{2}(A - A^T) = A - A^+.$$

Note that A^+ is symmetric and A^- is skew-symmetric. The *symmetry decomposition* of A is the representation

$$\boxed{A = A^+ + A^-} \tag{50}$$

The matrix A allows exactly one decomposition into a symmetric and a skew-symmetric matrix (see Ex. 8.1). So the symmetry decomposition is unique.

Ex. 8.1. *Let $A, B, C \in \mathbb{R}^{X \times X}$ be such that $A = B + C$. Show that the two statements are equivalent:*

(1) *B is symmetric and C is skew-symmetric.*
(2) *$B = A^+$ and $C = A^-$.*

Notice that symmetric and skew-symmetric matrices are necessarily pairwise orthogonal (see Ex.8.2).

Ex. 8.2. *Let A be a symmetric and B a skew-symmetric matrix. Show:*

$$\langle A|B \rangle = 0 \quad \textit{and} \quad \|A + B\|^2 = \|A\|^2 + \|B\|^2.$$

2. Complex matrices

In physics and engineering, complex numbers offer a convenient means to represent orthogonal structures. Applying this idea to the symmetry decomposition, one arrives at so-called *hermitian matrices*.

Recall that a *complex number* is an expression of the form $z = a + \mathrm{i}b$ where a and b are real numbers and i a special "new" number, the so-called *imaginary unit*. In particular, a complex number z of the form $a + \mathrm{i} \cdot 0$ is identified with the real number $a \in \mathbb{R}$. We denote by $\mathbb{C}$ the set of all complex numbers, *i.e.*,

$$\mathbb{C} = \mathbb{R} + \mathrm{i}\mathbb{R} = \{a + \mathrm{i}b \mid a, b \in \mathbb{R}\}.$$

Complex numbers can be added, subtracted, multiplied and divided according to the algebraic rules for real numbers with the additional computational proviso:

$$\boxed{\mathrm{i}^2 = -1.}$$

$\overline{z} = a - \mathrm{i}b$ is the *conjugate* of the complex number $z = a + \mathrm{i}b$. So one has

$$\overline{z}z = (a + \mathrm{i}b)(a - \mathrm{i}b) = a^2 + b^2 = |z|^2.$$

The *conjugate* of a complex matrix C is the matrix $\overline{C}$ with the conjugated coefficients $\overline{C}_{xy} = \overline{C_{xy}}$. The *adjoint* C^* of C is the transpose of the conjugate of C:

$$C^* = \overline{C}^T.$$

For two complex matrices $A = A_1 + \mathrm{i}A_2$ and $B = B_1 + \mathrm{i}B_2$ with the matrices $A_1, A_2, B_1, B_2 \in \mathbb{R}^{X \times Y}$, one computes

$$B^*A \;=\; (\overline{B_1}^T - \mathrm{i}\overline{B_2}^T)(A_1 + \mathrm{i}A_2) = \langle A_1|B_2\rangle + \langle A_2|B_2\rangle,$$

which means that the definition

$$\langle A|B\rangle = B^*A$$

is a natural way to extend the inner product of real matrices to complex matrices. In particular, one has the Pythagorean property

$$\|A_1 + \mathrm{i}A_2\|^2 \;=\; \langle A_1 + \mathrm{i}A_2|A_1 + \mathrm{i}A_2\rangle = \|A_1\|^2 + \|A_2\|^2.$$

2.1. Selfadjointness and spectral decomposition. A complex matrix C is called *selfadjoint* if it equals is adjoint:

$$C = C^* = \overline{C}^T$$

If C has only real coefficients, then $\overline{C} = C$, and consequently, 'selfadjoint' boils down to 'symmetric'. It is well-known that real symmetric matrices can be diagonalized. With the same arguments, one can extend this result to general selfadjoint matrices:

THEOREM 8.1 (Spectral Theorem). *For a matrix $C \in \mathbb{C}^{X \times X}$ the two statements are equivalent:*

(1) $C = C^*$.
(2) $\mathbb{C}^X$ *admits a unitary basis* $U = \{U_x \mid x \in X\}$ *of eigenvectors* U_x *of* C *with real eigenvalues* λ_x.

Unitary means for the basis U that the vectors U_x have unit norm and are pairwise orthogonal, *i.e.*,

$$\langle U_x | U_y \rangle = U_y^* U_x = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

The scalar λ_x is the *eigenvalue* of the *eigenvector* U_x of C if

$$CU_x = \lambda_x U_x.$$

It follows from Theorem 8.1 (see Ex. 8.3) that a selfadjoint matrix C admits a *spectral*[2] *decomposition*, *i.e.*, a representation in the form

$$C = \sum_{x \in X} \lambda_x U_x U_x^*, \tag{51}$$

where the U_x are pairwise orthogonal eigenvectors of C with eigenvalues $\lambda_x \in \mathbb{R}$.

EX. 8.3. *Let* $U = \{U_x \mid x \in X\}$ *be a unitary basis of* $\mathbb{C}^X$ *together with a set* $\Lambda = \{\lambda_x \mid x \in X\}$ *a set of arbitrary complex scalars. Show:*

(1) *The* U_x *are eigenvectors with eigenvalues* λ_x *of the matrix*
$$C = \sum_{x \in X} \lambda_x U_x U_x^*.$$
(2) C *is selfadjoint if and only if all the* λ_x *are real numbers.*

The spectral decomposition shows:

[2] the *spectrum* of a matrix is, by definition, its set of eigenvalues

> *The selfadjoint matrices C in $\mathbb{C}^{X\times X}$ are precisely the linear combinations of matrices of type*
> $$C = \sum_{x\in X} \lambda_x U_x U_x^*,$$
> *where the U_x are (column) vectors in $\mathbb{C}^X$ and the λ_x are real numbers.*

Spectral unity decomposition. As an illustration, consider a matrix $U \in \mathbb{C}^{X\times X}$ with pairwise orthogonal column vectors U_j of norm $\|U\|_x = 1$, which means that the identity matrix I has the representation

$$I = UU^* = U^*U.$$

The eigenvalues of I have all value $\lambda_x = 1$. Relative to U, the matrix I has the spectral decomposition

$$I = \sum_{x\in X} U_x U_x^*. \tag{52}$$

For any vector $v \in \mathbb{C}^X$ with norm $\|v\| = 1$, we therefore find

$$1 = \langle v|v\rangle = v^*Iv = \sum_{x\in X} v^*U_xU_x^*v = \sum_{x\in X} \langle v|U_x\rangle^2.$$

If follows that the (squared) inner products $p_x^v = \langle v|U_x\rangle^2$ of the vector v with the vectors U_x yield a probability distribution p^v on the set X.

Consider now, more generally, the selfadjoint matrix C with eigenvalues ρ_x of the form

$$C = \sum_{x\in X} \rho_x U_x U_x^* v.$$

Then we have

$$\langle v|Cv\rangle = v^*Cv = \sum_{x\in X} \rho_x \langle v|U_x\rangle^2 = \sum_{x\in X} \rho_x p_x^v. \tag{53}$$

In other words:

> *The inner product $\langle v|Cv\rangle$ of the vectors v and Cv is the expected value of the eigenvalues ρ_x of C with respect to the probability distribution p^v on X.*

2.2. Hermitian representation. Coming back to real matrices in the context of symmetry decompositions, associate with a matrix $A \in \mathbb{R}^{X\times X}$ the complex matrix

$$\hat{A} = A^+ + \mathrm{i}A^-.$$

$\hat{A}$ is a *hermitian*[3] *matrix*. The *hermitian map* $A \mapsto \hat{A}$ establishes an isomorphism between the vector space $\mathbb{R}^{X\times X}$ and the vector space

$$\mathbb{H}_X = \{\hat{A} \mid A \in \mathbb{R}^{X\times X}\}$$

with the set $\mathbb{R}$ as field of scalars[4]. The import in our context is the fundamental observation that the selfadjoint matrices are precisely the hermitian matrices:

LEMMA 8.1. *Let $C \in \mathbb{C}^{X\times X}$ be an arbitrary complex matrix. Then*

$$C \in \mathbb{H}_X \quad \Longleftrightarrow \quad C = C^*$$

Proof. Assume $C = A + \mathrm{i}B$ with $A, B \in \mathbb{R}^{X\times}$ and hence

$$C^* = A^T - \mathrm{i}B^T$$

So $C = C^*$ means symmetry $A = A^T$ and skew-symmetry $B = -B^T$. Consequently, one has $\hat{A} = A$ and $\hat{B} = \mathrm{i}B$, which yields

$$C = A + \mathrm{i}B = \hat{A} + \hat{B} \in \mathbb{H}_X.$$

The converse is seen as easily.

$\diamond$

The remarkable property of the hermitian representation is:

- While a real matrix $A \in \mathbb{R}^{X\times X}$ does not necessarily admit a spectral decomposition with real eigenvalues, its hermitian representation $\hat{A}$ is always guaranteed to have one.

3. Interaction systems

Let us assume that elements $x, y \in X$ can *interact* with a certain *interaction strength*, measured by a real number a_{xy}. We denote this interaction symbolically as $a_{xy}\varepsilon_{xy}$. Graphically, one may equally well think of a weighted (directed) edge in an *interaction graph* with X as its set of nodes:

$$a_{xy}\varepsilon_{xy} \quad :: \quad \text{x} \xrightarrow{a_{xy}} \text{y}.$$

[3] C. HERMITE (1822-1901)

[4] $\mathbb{H}_X$ is not a complex vector space: The product zC of a hermitian matrix C with a complex scalar z is not necessarily hermitian.

An *interaction instance* is a weighted superposition of interactions:

$$\varepsilon = \sum_{x,y \in X} a_{ax} \varepsilon_{xy}.$$

We record the interaction instance ε in the *interaction matrix* $A \in \mathbb{R}^{X \times X}$ with the *interaction coefficients* $A_{xy} = a_{xy}$. The interaction is *symmetric* if $A^T = A$ and *skew-symmetric* if $A^T = -A$.

Conversely, each matrix $A \in \mathbb{R}^{X \times X}$ corresponds to some interaction instance

$$\varepsilon = \sum_{x,y \in X} A_{xy} \varepsilon_{xy}.$$

So we may think of $\mathbb{R}^{X \times X}$ as the *interaction space* relative to the set X. Moreover, the symmetry decomposition

$$A = A^+ + A^-$$

shows:

> *Every interaction instance ε is the superposition of a symmetric interaction instance ε^+ and a skew-symmetric interaction instance ε^-. Moreover, ε^+ and ε^- are uniquely determined by ε.*

3.1. Interaction states. The *norm* of an interaction state ε with interaction matrix A is the norm of the associated interaction matrix:

$$\|\varepsilon\| = \|A\|.$$

So $\|\varepsilon\| \neq 0$ means that at least two members $s, t \in X$ interact with strength $A_{st} \neq 0$ and that the numbers

$$p_{xy} = \frac{|A_{xy}|^2}{\|A\|^2} \quad ((x,y) \in X \times X))$$

yield a probability distribution on the set of all possibly interacting pairs and offer a probabilistic perspective on ε:

> • *A pair (x, y) of members of X is interacting nontrivially with probability p_{xy}.*

Clearly, scaling ε to $\lambda\varepsilon$ with a scalar $\lambda \neq 0$, would result in the same probability distribution on $X \times X$. From the probabilistic point of view, it therefore suffices to consider interaction instances ε with norm $\|\varepsilon\| = 1$.

We thus define:

The interaction system on X is the system $\mathfrak{I}(X)$ with the set of states

$$\mathfrak{I}_X = \{\varepsilon \mid \varepsilon \text{ is an interaction instance of } X \text{ of norm } \|\varepsilon\| = 1\}$$

In terms of the matrix representation of states, we have

$$\mathfrak{I}_X \quad \longleftrightarrow \quad \mathcal{S}_X = \{A \in \mathbb{R}^{X\times X} \mid \|A\| = 1\}.$$

3.2. Interaction potentials. A potential $F : X \times X \to \mathbb{R}$ defines a matrix with coefficients $F_{xy} = F(x,y)$ and thus a scalar-valued linear functional

$$A \mapsto \langle F|A\rangle = \sum_{x,y\in X} F_{xy}A_{xy}$$

on the vector space $\mathbb{R}^{X\times X}$. Conversely, every linear functional f on $\mathbb{R}^{X\times X}$ is of the form

$$f(A) = \sum_{x,y\in X} F_{xy}A_{xy} = \langle F|A\rangle$$

with uniquely determined coefficients $F_{xy} \in \mathbb{R}$. So potentials and linear functionals correspond to each other.

On the other hand, the potential F defines a linear operator $A \mapsto F \bullet A$ on the space $\mathbb{R}^{X\times X}$, where the matrix $F \bullet A$ is the HADAMARD *product* of F and A with the coefficients

$$(F \bullet A)_{xy} = F_{xy}A_{xy} \quad \text{for all } x, y \in X.$$

With this understanding, one has

$$\langle F|A\rangle = \sum_{x,y\in X} (F \bullet A)_{xy}.$$

Moreover, one computes

$$\boxed{\langle A|F \bullet A\rangle = \sum_{x,y\in X} A_{xy}(F \bullet A)_{xy} = \sum_{x,y\in X} F_{xy}|A_{xy}|^2.} \tag{54}$$

If $A \in \mathcal{S}_X$ (*i.e.*, if A represents an interaction state $\varepsilon \in \mathfrak{I}_X$), the parameters $p^A_{xy} = |A_{xy}|^2$ define a probability distribution on $X \times X$. The expected value of the potential F in this state ε is

$$\mu^\varepsilon(F) = \sum_{x,y\in X} F_{xy}p^A_{xy} = \langle A|F \bullet A\rangle.$$

3.3. Interaction in cooperative games. The interaction model offers a considerably wider context for the analysis of cooperation. To illustrate this, consider a cooperative TU-game $\Gamma = (N, v)$ with collection $\mathcal{N}$ of coalitions. v is a potential on $\mathcal{N}$ but not on the set

$$\mathcal{N} \times \mathcal{N}$$

of possibly pairwise interacting coalitions. However, there is a straightforward extension of v to $\mathcal{N} \times \mathcal{N}$:

$$v(S,T) = \begin{cases} v(S) & \text{if } S = T \\ 0 & \text{if } S \neq T. \end{cases}$$

Relative to a state $\sigma \in \mathfrak{I}_{\mathcal{N}\times\mathcal{M}}$ with interaction matrix A, the expected value of v is

$$v(\sigma) = \sum_{S \in \mathcal{N}} v(S)|A_{SS}|^2.$$

In the special case of a state σ_S where the coalition S interacts with itself with certainty (and hence no proper interaction among coalitions takes place), we have

$$v(\sigma_S) = v(S)$$

which is exactly the potential value of the coalition S in the classical interpretation of Γ.

Generalized cooperative games. A more comprehensive model for the study of cooperation among players would be structures of the type $\Gamma = (N, \mathcal{N}, v)$, where v is a potential on $\mathcal{N} \times \mathcal{N}$ (rather than just $\mathcal{N}$).

3.4. Interaction in infinite sets. Much of the current interaction analysis remains valid for infinite sets with some modifications.

For example, we admit as descriptions of interaction states only those matrices $A \in \mathbb{R}^{X\times X}$ with the property

(H1) $\text{supp}(A) = \{(x,y) \in X \times X \mid A_{xy} \neq 0\}$ is finite or countably infinite.

(H2) $\|A\|^2 = \sum_{x,y\in X} |A_{xy}|^2 = 1.$

If the conditions (H1) and (H2) are met, we factually represent interaction states in HILBERT spaces. To keep things simple, however, we retain the finiteness property of the agent set X in the current text and refer the interested reader to the literature[5] for further details.

[5] *e.g.*, J. WEIDMANN (1980): *Linear Operators in Hilbert Spaces*, Graduate Texts in Mathematics, Springer Verlag

4. Quantum systems

Without going into the physics of quantum mechanics, let us quickly sketch the basic mathematical model and then look at the relationship with the interaction model. In this context, we think of an *observable* as a mechanism α that can be applied to a system $\mathfrak{S}$,

$$\boxed{\mathfrak{S}(\sigma)} \rightsquigarrow \boxed{\alpha} \longrightarrow \alpha(\sigma)$$

with the interpretation:

- If $\mathfrak{S}$ is in the state σ, then α is expected to produce a measurement result $\alpha(\sigma)$.

4.1. The quantum model. There are two views on a *quantum system* $\mathfrak{Q}_X$ relative to a set X. They are dual to each other (reversing the roles of states and observables) but mathematically equivalent.

The SCHRÖDINGER picture. In the so-called SCHRÖDINGER[6] *picture*, the states of $\mathfrak{Q}_X$ are presented as the elements of the set

$$\mathcal{W}_X = \{v \in \mathbb{C}^X \mid \|v\| = 1\}$$

of complex vectors of norm 1 . An observable α corresponds to a selfadjoint $(n \times n)$ -matrix $A \in \mathbb{H}_X$ and produces the real number

$$\alpha(v) = \langle v|Av\rangle = v^*A^*v = v^*Av$$

when $\mathfrak{Q}_X$ is in the state $v \in \mathcal{W}$. Recall from the discussion of the spectral decomposition in Section 2.1 that $\alpha(v)$ is the expected value of the eigenvalues ρ_i of A relative to the probabilities

$$p_x^{A,v} = \langle v|U_x\rangle^2 \quad (x \in X),$$

where the vectors $U_x \in \mathcal{W}$ constitute a vector space basis of corresponding eigenvectors of A.

An interpretation of the probabilities $p^{A,v}$ could be this:

$\mathfrak{Q}_X$ is a stochastic system that shows the element $x \in X$ with probability $p_x^{A,v}$ if it is observed under A in the state v:

$$\boxed{\mathfrak{Q}_X(v)} \rightsquigarrow \boxed{A} \longrightarrow x.$$

[6]E. SCHRÖDINGER (1887-1961)

EX. 8.4. *The identity matrix $I \in \mathbb{C}^{X\times X}$ is selfadjoint and yields the distribution $p^{I,v}$ on X with probabilities*

$$p_x^{I,v} = |v_x|^2 \quad (x \in X).$$

The HEISENBERG picture. In the HEISENBERG[7] *picture* of $\mathfrak{Q}_X$, the selfadjoint matrices $A \in \mathbb{H}_X$ take over the role of states while the vectors $v \in \mathcal{W}$ induce measuring results. The HEISENBERG is dual[8] to the SCHRÖDINGER picture. In both pictures, the expected values

$$\langle v|Av\rangle \quad (v \in \mathcal{W}_X, A \in \mathbb{H}_X)$$

are thought to be the numbers resulting from measurements on the system $\mathfrak{Q}_X$.

> The HEISENBERG picture sees an element $x \in X$ according to the scheme
>
> $$\boxed{\mathfrak{Q}_X(A)} \longrightarrow \boxed{v} \rightsquigarrow x$$
>
> with probability $p_x^{A,v}$.

Densities and wave functions. The difference in the two pictures lies in the interpretation of the probability distribution $p^{A,v}$ on the index set X relative to $A \in \mathbb{H}_X$ and $v \in \mathcal{W}_X$.

In the HEISENBERG picture, $p^{A,v}$ is imagined to be implied by a possibly varying A relative to a fixed state vector v. Therefore, the elements $A \in \mathbb{H}_X$ are also known as *density matrices*.

In the SCHRÖDINGER picture, the matrix A is considered to be fixed while the state vector $v = v(t)$ may vary in time t. $v(t)$ is called a *wave function*.

4.2. Evolutions of quantum systems. A quantum evolution

$$\Phi = \Phi(M, v, A)$$

in (discrete) time t depends on a matrix-valued function $t \mapsto M_t$, a state vector $v \in \mathcal{W}$, and a density $A \in \mathbb{H}_X$. The evolution Φ produces real observation values

$$\varphi_t = v^*(M_t^* A M_t)v \quad (t = 0, 1, 2, \ldots). \tag{55}$$

Notice that the matrices $A_t = M_t^* A M_t$ are selfadjoint. So the evolution Φ can be seen as an evolution of density matrices, which is in accord with the HEISENBERG picture.

[7] W. HEISENBERG (1901-1976)

[8] in the sense of Section 2.1 of Chapter 1

If $v(t) = M_t v \in \mathcal{W}$ holds for all t, the evolution Φ can also be interpreted in the SCHRÖDINGER picture as an evolution of state vectors:

$$\varphi_t = (M_t v)^* A(M_t v) \quad (t = 0, 1, 2, \ldots). \tag{56}$$

REMARK 8.1. *The standard model of quantum mechanics assumes that evolutions satisfy the condition $M_t v \in \mathcal{W}$ at any time t, so that the* HEISENBERG *and the* SCHRÖDINGER *pictures are equivalent.*

Markov coalition formation. Let $\mathcal{N}$ be the collection of coalitions of the set N. The classical view on coalition formation sees the probability distributions p on $\mathcal{N}$ as the possible states of the formation process and the process itself as a MARKOV[9] chain.

To formalize this model, let $\mathfrak{P} = \mathfrak{P}(N)$ be the set of all probability distributions on $\mathcal{N}$. A MARKOV *operator* is a linear map

$$\mu : \mathbb{N} \to \mathbb{R}^{\mathcal{N}\times\mathcal{N}} \quad \text{such that } \mu_t p \in \mathfrak{P} \text{ holds for all } p \in \mathfrak{P}.$$

μ defines for every initial state $p^{(0)} \in \mathfrak{P}$ a so-called MARKOV *chain* of probability distributions

$$\mathcal{M}(p^{(0)}) = \{\mu^t(p^{(0)}) \mid t = 0, 1, \ldots\}.$$

Define now $P_t \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}$ as the diagonal matrix with $p^{(t)} = \mu^t(p^{(0)})$ as its diagonal coefficient vector. P_t is a real symmetric matrix and therefore a density in particular.

Any $v \in \mathcal{W}$ gives rise to a quantum evolution with observed values

$$\pi_t = v^* P_t v \quad (t = 0, 1, \ldots, n)$$

For example, if $e_S \in \mathbb{R}^{\mathcal{N}}$ is the unit vector that corresponds to the coalition $S \in \mathcal{N}$, one has

$$\pi_t^{(S)} = e_S^* P_t e_S = (P_t)_{SS} = p_S^{(t)}$$

with the usual interpretation:

- If the coalition formation proceeds according to the MARKOV chain $\mathcal{M}(p^{(0)})$, then an inspection at time t will find S is to be active with probability $\pi_t^{(S)} = p_S^{(t)}$.

REMARK 8.2. *More generally, the simulated annealing processes of Section 5.2 are* MARKOV *chains and, therefore, special cases of quantum evolutions.*

[9] A.A. MARKOV (1856-1922)

4.3. The quantum perspective on interaction. Recalling the vector space isomorphism *via* the hermitian representation

$$A \in \mathbb{R}^{X\times X} \longleftrightarrow \quad \hat{A} = A^{+}\mathrm{i}A^{-} \in H_X,$$

we may think of interaction states as manifestations of SCHRÖDINGER states of the quantum system $\mathfrak{Q}_{X\times X}$,

$$\mathcal{S}_X = \{A \in \mathbb{R}^{X\times X} \mid \|A\| = 1\} \;\leftrightarrow\; \mathcal{W}_{X\times X} = \{\hat{A} \in \mathbb{H}_X \mid \|\hat{A}\| = 1\},$$

or as normed representatives of HEISENBERG densities relative to the quantum system $\mathfrak{Q}_X$.

Principal components. An interaction instance A on X has a hermitian spectral decomposition

$$\hat{A} = \sum_{x\in X} \lambda_x U_x U_x^* = \sum_{x\in X} \lambda_x A_x$$

where the matrices $\hat{A}_x = U_x U_x^*$ the *principal components* of $\hat{A}$. The corresponding interaction instances A_x are the principal components of A:

$$A = \sum_{x\in X} \lambda_x A_x.$$

Principal components V of interaction instances arise from SCHRÖDINGER states $v = a + \mathrm{i}b \in \mathcal{W}_X$ with $a, b \in \mathbb{R}^X$ in the following way. Setting

$$\hat{V} = vv^* = (a+\mathrm{i}b)(a-\mathrm{i}b)^T = aa^T - bb^T + \mathrm{i}(ba^T - ab^T),$$

one has $V^+ = aa^T + bb^T$ and $V^- = ba^T - ab^T$ and thus

$$V = V^+ + V^- = (aa^T + bb^T) + (ba^T - ab^T).$$

The principal interaction instance V has hence the underlying structure:

(0) Each $x \in X$ has a pair (a_x, b_x) of weights $a_x, b_x \in \mathbb{R}$.
(1) The symmetric interaction between two arbitrary elements $x, y \in X$ is
$$V^+_{xy} = a_x a_y + b_x b_y.$$
(2) The skew-symmetric interaction between two arbitrary elements $x, y \in X$ is
$$V^-_{xy} = b_x a_y - a_x b_y.$$

4.4. The quantum perspective on cooperation. Let N be a (finite) set of players and family $\mathcal{N}$ of coalitions. From the quantum point view, a (SCHRÖDINGER) *state* of N is a complex vector $v \in \mathcal{W}_N$, which implies the probability distribution p^v with probabilities

$$p_i^v = |v_i|^2 \quad (i \in N)$$

on N. In the terminology of *fuzzy cooperation* (*cf.* Ex. 6.2), p^v describes a fuzzy coalition:

- Player $i \in N$ is *active* in state v i with probability p_i^v.

Conversely, if $w \in \mathbb{R}^N$ is a non-zero fuzzy coalition with component probabilities $0 \leq w_i \leq 1$, the vector

$$\sqrt{w} = (\sqrt{w_i} \mid i \in N)$$

may be normalized to a SCHRÖDINGER state

$$v = \frac{\sqrt{w}}{\|\sqrt{w}\|} \quad \text{s.t.} \quad w_i = \|\sqrt{w}\| \cdot |v_i|^2 \quad \text{for all } i \in N.$$

In the same way, a vector $\mathcal{W}_{\mathcal{N}}$ describes a SCHRÖDINGER state of interaction among the coalitions of N. It is particularly instructive to look at the interactions V of principal component type.

As we have seen in Sectionsec:quantum-interaction above, V arises a follows:

(0) The interaction V on $\mathcal{N}$ is implied by two cooperative games $\Gamma_a = (N, a)$ and $\Gamma_b = (N, b)$.
(1) Two coalitions $S, T \in \mathcal{N}$ interact symmetrically *via*

$$V_{ST}^+ = a(S)a(T) + b(S)b(T).$$

(2) Two coalitions $S, T \in \mathcal{N}$ interact skew-symmetrically *via*

$$V_{ST}^- = b(S)a(T) - a(S)b(T).$$

5. Quantum games

A large part of the mathematical analysis of game theoretic systems follows guideline

- *Represent the system in a mathematical structure, analyze the representation mathematically and re-interpret the result in the original game theoretic setting.*

When one chooses a representation of the system in the same space as the ones usually employed for the representation of a quantum system, one automatically arrives at a ”quantum game”, *i.e.*, at a quantum theoretic interpretation of a game theoretic environment.

So we understand by a *quantum game* any game on a system $\mathfrak{S}$ whose states are represented as quantum states and leave it to the reader to review game theory in this more comprehensive context.

6. Final Remarks

Why should one pass to complex numbers and the hermitian space $\mathbb{H}_X$ rather than the euclidian space $\mathbb{R}^{X\times X}$ if both spaces are isomorphic real Hilbert spaces?

The advantage lies in the algebraic structure of the field $\mathbb{C}$ of complex numbers, which yields the spectral decomposition (51), for example. It would be not impossible, but somewhat ”unnatural” to translate this structural insight back into the environment $\mathbb{R}^{X\times X}$ without appeal to complex algebra.

Another advantage becomes apparent when one studies evolutions of systems over time. In the classical situation of real vector spaces, Markov chains are an important model for system evolutions. It turns out that this model generalizes considerably when one passes to the context of HILBERT spaces[10].

The game theoretic ramifications of this approach are to a large extent unexplored at this point.

[10] U. FAIGLE AND G. GIERZ (2017): *Markovian statistics on evolving systems*, Evolving Systems, DOI 10.1007/s12530-017-9186-8

Appendix

1. Notions and facts from real analysis

The *euclidian norm* (or *geometric length*) of a vector $x \in \mathbb{R}^n$ with components x_j, is

$$\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}$$

Writing $B_r(x) = \{y \in \mathbb{R}^n \mid \|x - y\| \leq r\}$ for $r \in \mathbb{R}$ and $x \in \mathbb{R}^n$, a subset $S \subseteq \mathbb{R}^n$ is

(1) *bounded* if there is some $r > 0$ such that $S \subseteq B_r(0)$;
(2) *open* if for each $x \in S$ there is some $r > 0$ such that $B_r(x) \subseteq S$;
(3) *closed* if $\mathbb{R}^n \setminus S$ is open;
(4) *compact* if S is closed and bounded.

LEMMA A.2 (HEINE-BOREL). *$S \subseteq \mathbb{R}^n$ is compact if and only if*

(HB) *every family $\mathcal{O}$ of open sets $O \subseteq \mathbb{R}^n$ such that every $x \in S$ lies in at least one $O \in \mathcal{O}$, admits a* finite *number of sets $O_1, \ldots, O_\ell \in \mathcal{O}$ with the covering property*
- *$S \subseteq O_1 \cup O_2 \cup \ldots \cup O_\ell$.*

$\diamond$

It is important to note that compactness is preserved under forming direct products:

- If $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ are compact sets, then $X \times Y \subseteq \mathbb{R}^{n+m}$ is compact.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is *continuous* on S if for all $x \in S$, one always has

$$\lim_{d \to 0} f(x + d) = f(x).$$

LEMMA A.3 (Extreme values). *If f is continuous on the compact set S, then there exist elements $x_*, x^* \in S$ such that*

$$f(x_*) \leq f(x) \leq f(x^*) \quad \textit{holds for all } x \in S.$$

$\diamond$

The continuous function $f : S \to \mathbb{R}$ is *differentiable* on the open set $S \subseteq \mathbb{R}^n$ if for each $x \in S$ there is a (row) vector $\nabla f(x)$ such that for every $d \in \mathbb{R}^n$ of unit length $\|d\| = 1$, one has

$$\lim_{t\to 0} \frac{f(x+td) - f(x)}{t} = \lim_{t\to 0} \frac{\nabla f(x)d}{t} \quad (t \in \mathbb{R}).$$

$\nabla f(x)$ is the *gradient* of f. Its components are the partial derivatives:

$$\nabla f(x) = \big(\partial f(x)/\partial x_1, \ldots, \partial f(x)/\partial x_n\big).$$

NOTA BENE. Not all continuous functions are differentiable.

2. Convexity

A *linear combination* of elements $x_1, \ldots, x_m$ is an expression of the form

$$z = \lambda_1 x_1 + \ldots + \lambda_m x_m,$$

where $\lambda_1, \ldots, \lambda_m$ are scalars (real or complex numbers). The linear combination z is *affine* if

$$\lambda_1 + \ldots + \lambda_m = 1 \quad \text{and} \quad \lambda_1, \ldots, \lambda_m \in \mathbb{R}.$$

An affine combination is a *convex combination* if all scalars λ_i are nonnegative. The scalars $(\lambda_1, \ldots, \lambda_m)$ of a convex combination is a (m-dimensional) *probability distribution*.

The set $S \subseteq \mathbb{R}^n$ is *convex* if it contains with every $x, y \in S$ also the connecting line segment:

$$[x, y] = \{x + \lambda(y - x) \mid 0 \leq \lambda \leq 1\} \subseteq S.$$

It is easy to verify that the direct product $S = X \times Y \subseteq \mathbb{R}^{n\times m}$ is convex if $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ are convex sets.

A function $f : S \to \mathbb{R}$ is *convex (up)* on the convex set S if for all $x, y \in S$ and for all scalars $0 \leq \lambda \leq 1$,

$$f(x + \lambda(y - x)) \geq f(x) + \lambda(f(y) - f(x))).$$

This definition is equivalent to the requirement that one has for any finitely many elements $x_1, \ldots, x_m \in S$ and probability distributions $(\lambda_1, \ldots, \lambda_m)$,

$$f(\lambda_1 x_1 + \ldots + \lambda_m x_m) \geq \lambda_1 f(x_1) + \ldots + \lambda_m f(x_m).$$

The function f is *concave* (or *convex down*) if $g = -f$ is convex (up).

A differentiable function $f : S \to \mathbb{R}$ on the open set $S \subseteq \mathbb{R}^n$ is convex (up) if and only if

$$f(y) \geq f(x) + \nabla f(x)(y - x) \quad \text{holds for all } x, y \in S. \tag{57}$$

Assume, for example, that $\nabla f(x)(y-x) \geq 0$ it true for all $y \in S$, then one has

$$f(x) = \min_{y \in S} f(y).$$

On the other hand, if $\nabla f(x)(y-x) < 0$ is true for some $y \in S$, one can move from x a bit into the direction of y and find an element x' with $f(x') < f(x)$. Hence one has a criterion for minimizers of f on S:

LEMMA A.4. *If f is a differentiable convex function on the convex set S, then for any $x \in S$, the statements are equivalent:*

(1) $f(x) = \min_{y \in S} f(y)$.
(2) $\nabla f(x)(y-x) \geq 0$ *for all* $y \in S$.

If strict inequality holds in (57) for all $y \neq x$, f is said to be *stricly convex*.

In the case $n = 1$ (*i.e.*, $S \subseteq \mathbb{R}$), a simple criterion applies to twice differentiable functions:

$$f \text{ is convex} \quad \Longleftrightarrow \quad f''(x) \geq 0 \quad \text{for all } x \in S.$$

For example, the logarithm function $f(x) = \ln x$ is seen to be strictly concave on the open interval $S = (0, \infty)$ because of

$$f''(x) = -1/x^2 < 0 \quad \text{for all } x \in S.$$

3. BROUWER's fixed-point theorem

A *fixed-point* of a map $f : X \to X$ is a point $x \in X$ such that $f(x) = x$. It is usually difficult to find a fix-point (or even to decide whether a fixed-point exists). Well-known sufficient conditions were given by BROUWER[11]:

THEOREM A.2 (BROUWER (1911)). *Let $X \subseteq \mathbb{R}^n$ be a convex, compact and non-empty set and $f : X \to X$ a continuous function. Then f has a fixed-point.*

Proof. See, *e.g.*, the enyclopedic textbook of A. GRANAS and J. DUGUNDJI, *Fixed Point Theory*, Springer-Verlag 2003.

$\diamond$

For game theoretic applications, the following implication is of interest.

COROLLARY A.1. *Let $X \subseteq \mathbb{R}^n$ be a convex, compact and nonempty set and $G : X \times X \to \mathbb{R}$ a continuous map that is concave in the second variable,* i.e.,

(C) *for every $x \in X$, the map $y \mapsto G(x, y)$ is concave.*

[11] L.E.J. BROUWER (1881-1966)

Then there exists a point $x^* \in X$ such that

$$G(x^*, x^*) \geq G(x^*, y) \quad \textit{for all } y \in X.$$

Proof. We will derive a contradiction from the supposition that the Corollary is false. Indeed, if there is no x^* with the claimed property, then each $x \in X$ lies in at least one of the sets

$$O(y) = \{x \in X \mid G(x, x) < G(x, y)\} \quad (y \in X).$$

Since G is continuous, the sets $O(y)$ are open sets. Hence, since X is compact, already finitely many cover all of X, say

$$X \subseteq O(y_1) \cup O(y_2) \cup \ldots \cup O(y_h).$$

For all $x \in X$, define the parameters

$$d_\ell(x) = \max\{0, G(x, y_\ell) - G(x, x)\} \quad (\ell = 1, \ldots, h).$$

x lies in at least one of the sets $O(y_\ell)$. Therefore, we have

$$d(x) = d_1(x) + d_2(x) + \ldots + d_h(x) > 0.$$

Consider now the function

$$x \mapsto \varphi(x) = \sum_{\ell=1}^{j} \lambda_\ell(x) y_i \quad (\text{with } \lambda_\ell = d_\ell(x)/d(x)).$$

Since G is continuous, also the functions $x \mapsto d_\ell(x)$ are continuous. Therefore, $\varphi : X \to X$ is continuous. By BROUWER's Theorem A.2, φ has a fixed point

$$x^* = \varphi(x^*) = \sum_{\ell=1}^{h} \lambda_\ell(x^*) y_\ell.$$

Since $G(x, y)$ is concave in y and x^* is an affine linear combination of the y_ℓ, we have

$$G(x^*, x^*) = G(x^*, \varphi(x^*)) \geq \sum_{\ell=1}^{h} \lambda_\ell(x^*) G(x^*, y_\ell).$$

If the Corollary were false, one would have

$$\lambda_\ell G(x^*, y_\ell) \geq \lambda_\ell(x^*) G(x^*, x^*)$$

for each summand and, in at least one case, even a strict inequality

$$\lambda_\ell(x^*) G(x^*, y_\ell) > \lambda_\ell G(x^*, x^*),$$

which would produce the contradictory statement

$$G(x^*, x^*) > \sum_{\ell=1}^{h} \lambda_\ell(x^*) G(x^*, x^*) = G(x^*, x^*).$$

It follows that the Corollary must be correct.

4. Linear inequalities

Also the facts stated in this section are well-known[12]. For a coordinate vector $x \in \mathbb{R}^n$, we write $x = 0$ if $x_j = 0$ holds for all components x_j of x. $x \geq 0$ means that all components of x are nonnegative.

Assume now that the matrix $A \in \mathbb{R}^{m\times n}$ and vectors $c \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ are given and define the sets

$$\begin{aligned} X &= \{x \in \mathbb{R}^n \mid b - Ax \geq 0\} \\ Y &= \{y \in \mathbb{R}^m \mid y \geq 0, A^T y = c\}. \end{aligned}$$

Then the *main theorem of linear programming* says:

THEOREM A.3. *For X and Y as above, the following statements are true:*

(1) *For all $x \in X$ and $y \in Y$ one has: $c^T x \leq b^T y$.*
(2) *$X \neq \emptyset \neq Y$ is true exactly when there are elements $x^* \in X$ and $y^* \in Y$ such that $c^T x^* = b^T y^*$.*

◇

In terms of mathematical optimization, Theorem A.3 says that either X or Y are empty or that there are elements $x^* \in X$ and $y^* \in Y$ with the property

$$\boxed{c^T x^* = \max_{x\in X} c^T x = \min_{y\in Y} b^T y = b^T y^*.} \tag{58}$$

The optimization problems in (58) are so-called *linear programs*.

5. The MONGE algorithm

The MONGE *algorithm* with respect to coefficient vectors $c, v \in \mathbb{R}^n$ has two versions.

The *primal* MONGE algorithm constructs a vector $x(v)$ with the components

$$x_1(v) = v_1 \quad \text{and} \quad x_k(v) = v_k - v_{k-1} \quad (k = 2, 3, \ldots, n).$$

The *dual* MONGE algorithm constructs a vector $y(c)$ with the components

$$y_n(c) = c_n \quad \text{and} \quad y_\ell(c) = c_\ell - c_{\ell+1} \quad (\ell = 1, \ldots, n-1).$$

Notice:

$$\begin{aligned} c_1 \geq c_2 \geq \ldots \geq c_n &\implies y_\ell(c) \geq 0 \quad (\ell = 1, \ldots, n-1) \\ v_1 \leq v_2 \leq \ldots \leq v_n &\implies x_k(v) \geq 0 \quad (k = 2, \ldots, n). \end{aligned}$$

[12] more details can be found in, *e.g.*, U. FAIGLE, W. KERN and G. STILL, *Algorithmic Principles of Mathematical Programming*, Springer (2002)

The important property to observe is

LEMMA A.5. *The* MONGE *vectors* $x(v)$ *and* $y(c)$ *satisfy*

$$c^T x(v) = \sum_{k=1}^{n} c_k x_k(v) = \sum_{\ell=1}^{n} v_\ell y_\ell(c) = v^T y(c).$$

Proof. Writing $x = x(v)$ and $y = y(c)$, notice for all $1 \leq k, \ell \leq n$,

$$x_1 + x_2 + \ldots + x_\ell = v_\ell \quad \text{and} \quad y_k + y_{k+1} + \ldots + y_n = c_k$$

and hence

$$\sum_{k=1}^{n} c_k x_k = \sum_{k=1}^{n} \sum_{\ell=k}^{n} y_\ell x_k = \sum_{\ell=1}^{n} \sum_{k=1}^{\ell} x_k y_\ell = \sum_{\ell=1}^{n} v_\ell y_\ell.$$

$\diamond$

6. Entropy and BOLTZMANN distributions

6.1. BOLTZMANN distributions. The *partition function* Z for a given vector $v = (v_1, \ldots, v_n)$ of real numbers v_j takes the values

$$Z(t) = \sum_{j=1}^{n} e^{v_j t} \quad (t \in \mathbb{R}).$$

The associated BOLTZMANN probability distribution $b(t)$ has the components

$$b_j(t) = e^{v_j t}/Z(t) > 0$$

and yields the expected value function

$$\mu(t) = \sum_{j=1}^{n} v_j b_j(t) = \frac{Z'(t)}{Z(t)}.$$

The *variance* is defined as the expected quadratic deviation of v from $\mu(t)$:

$$\begin{aligned}
\sigma^2(t) &= \sum_{j=1}^{n} (\mu(t) - v_j)^2 b_j(t) = \sum_{j=1}^{n} v_j^2 b_j(t) - \mu^2(t) \\
&= \frac{Z''(t)}{Z(t)} - \frac{Z'(t)^2}{Z(t)^2} = \mu'(t).
\end{aligned}$$

One has $\sigma^2(t) > 0$ unless all v_j are equal to a constant K (and hence $\mu(t) = K$ for all t). Because $\mu'(t) = \sigma^2(t)$, one sees that $\mu(t)$ is strictly increasing in t unless $\mu(t)$ is constant.

Arrange the components such that $v_1 \leq v_2 \leq \ldots \leq v_n$. Then

$$\lim_{t \to \infty} \frac{b_j(t)}{b_n(t)} = \lim_{t \to \infty} e^{(v_j - v_n)t} = 0 \quad \text{unless} \quad v_j = v_n,$$

which implies $b_j(t) \to 0$ if $v_j < v_n$. It follows that the limit distribution $b(\infty)$ is the uniform distribution on the maximizers of v. Similarly, one has

$$\lim_{t\to-\infty} \frac{b_j(t)}{b_1(t)} = \lim_{t\to-\infty} e^{(v_j-v_1)t} = 0 \quad \text{unless} \quad v_j = v_1$$

and concludes that the limit distribution $b(-\infty)$ is the uniform distribution on the minimizers of v.

THEOREM A.4. *For every value $v_1 \le \xi \le v_n$, there is a unique parameter $t \in \mathbb{R} \cup \{-\infty, +\infty\}$ such that*

$$\xi = \mu(t) = \sum_{j=1}^{n} v_j b_j(t).$$

Proof. If $v_1 = \xi = v_n$, the function $\mu(t)$ is constant and the claim is trivial. In the non-constant case, $\mu(t)$ is strictly monotone and continuous on $\mathbb{R}$ and satisfies.

$$v_1 \le \mu(t) \le v_n.$$

So, for every prescribed value ξ between the extrema v_1 and v_n, there must exist precisely one t with $\mu(t) = \xi$.

$\diamond$

6.2. Entropy. The real function $h(x) = x \ln x$ is defined for all non-negative real numbers[13] and has the strictly increasing derivative

$$h'(x) = x + \ln x.$$

So h is strictly convex and satisfies the inequality

$$h(y) - h(x) > h'(x)(y - x) \quad \text{for all non-negative } y \ne x.$$

h is extended to nonnegative real vectors $x = (x_1, \ldots, x_n)$ *via*

$$h(x) = h(x_1, \ldots, x_n) = \sum_{j=1}^{n} x_j \ln x_j \ \ \Big(= \sum_{j=1}^{n} h(x_j)\Big).$$

The strict convexity of h becomes the inequality

$$h(y) - h(x) > \nabla h(x)(y - x),$$

with the gradient

$$\nabla h(x) = (h'(x_1), \ldots, h'(x_n)) = (x_1 + \ln x_1, \ldots, x_n + \ln x_n).$$

[13]with the understanding $\ln 0 = -\infty$ and $0 \cdot \ln 0 = 0$

In the case $x_1 + \ldots + x_n = 1$, the nonnegative vector x is a probability distribution on the set $\{1, \ldots, n\}$ and has[14] the *entropy*

$$H(x) = \sum_{j=1}^{n} x_j \ln(1/x_j) = -\sum_{j=1}^{n} x_j \ln x_j = -h(x_1, \ldots, x_n).$$

We want to show that BOLTZMANN probability distributions are precisely the ones with maximal entropy relative to given expected values.

THEOREM A.5. *Let $v = (v_1, \ldots, v_n)$ be a vector of real numbers and b the* BOLTZMANN *distribution on $\{1, \ldots, n\}$ with components*

$$b_j = \frac{1}{Z(t)} e^{v_j t} \quad (j = 1, \ldots, n).$$

with respect to some t. Let $p = (p_1, \ldots, p_n)$ be a probability distribution with the same expected value

$$\sum_{j=1}^{n} v_j p_j = \mu = \sum_{j=1}^{n} v_j b_j.$$

Then one has either $p = b$ or $H(p) < H(b)$.

Proof. For $d = p - b$, we have $\sum_j d_j = \sum_j p_j - \sum_j b_j = 1 - 1 = 0$, and therefore

$$\begin{aligned}
\nabla h(b) d &= \sum_{j=1}^{n} (1 + \ln b_j) = \sum_{j=1}^{n} d_j \ln b_j \\
&= \sum_{j=1}^{n} d_j (v_j t - Z(t)) = t \sum_{j=1}^{n} v_k d_j \\
&= t\big(\sum_{j=1}^{n} v_j p_j - \sum_{j=1}^{n} v_j b_j)\big) = 0.
\end{aligned}$$

In the case $p \neq b$, the strict convexity of h thus yields

$$h(p) - h(b) > \nabla h(b)(p - b) = 0 \quad \text{and hence} \quad H(p) < H(b).$$

$\diamond$

LEMMA A.6 (Divergence). *Let $a_1, \ldots, a_n, p_1, \ldots, p_n$ be arbitrary nonnegative numbers. Then*

$$\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} p_i \quad \Longrightarrow \quad \sum_{i=1}^{n} p_i \ln a_i \leq \sum_{i=1}^{n} p_i \ln p_i.$$

Equality is attained exactly when $a_i = p_i$ holds for all $i = 1, \ldots, n$.

[14] by definition!

Proof. We may assume $p_i \neq 0$ for all i and make use of the well-known fact (which follows easily from the concavity of the logarithm function):

$$\ln x \leq x - 1 \quad \text{and} \quad \ln x = x - 1 \Leftrightarrow x = 1.$$

Then we observe

$$\sum_{i=1}^{n} p_i \ln \frac{a_i}{p_i} \leq \sum_{i=1}^{n} p_i (\frac{a_i}{p_i} - 1) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} p_i \leq 0$$

and therefore

$$\sum_{i=1}^{n} p_i \ln a_i - \sum_{i=1}^{n} p_i \ln p_i = \sum_{i=1}^{n} p_i \ln \frac{a_i}{p_i} \leq 0.$$

Equality can only hold if $\ln(a_i/p_i) = (a_i/p_i) - 1$, and hence $a_i = p_i$ is true for all i.

$\diamond$