BIOS:4120 – Introduction to Biostatistics
Unit 7: Sampling Distribution of the Mean

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Learning Objectives

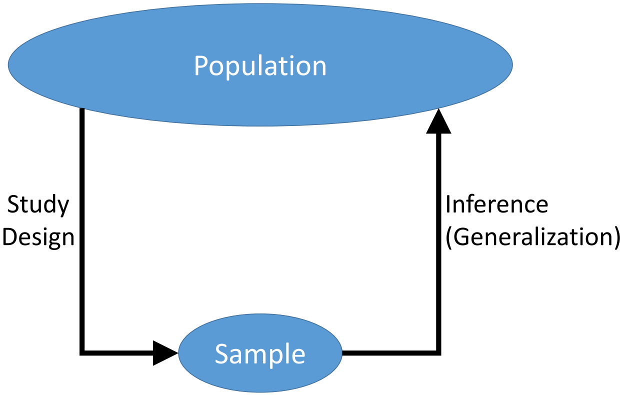
At the end of this session, you should be able to:

- Describe the sampling distribution of the mean in the context of repeated sampling.
- Calculate the expected value and standard error of the sample mean.
- Describe the central limit theorem and the conditions when it is valid.
- Perform probability calculations on the sample mean.

Overview

- Sampling Distributions
- The Central Limit Theorem
- Applications of the Central Limit Theorem

Sampling Distributions

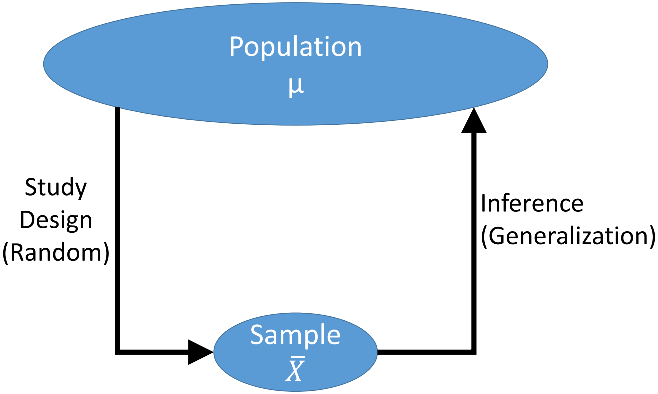


Sampling Distributions

- *Inferential Statistics* deals with methods for making generalizations about a population based on information contained in the sample.
- A *Statistic* is a numerical value describing a sample characteristic.
- A *Parameter* is a numerical value describing a population characteristic.

Sampling Distributions

- Perhaps we draw a random sample from a population, and use the statistic $\bar{X}$ (the sample mean), to estimate the parameter μ (the population mean).



Sampling Distributions

- One can view $\bar{X}$ as a numeric variable that assumes a value based on the outcome of a random experiment: i.e., the process of drawing a sample at random from the population.
- $\bar{X}$ is therefore, by definition, a random variable. It will vary from sample to sample.
- By the same reasoning, so is any statistic.
- The probability distribution of a statistic is referred to as a *Sampling Distribution*.
- The sampling distribution reflects which values of the statistic are likely and which values are improbable.

Sampling Distributions

- The mean of a statistic is often called the *Expected Value*, for example, $E(\bar{X})$.
- The standard deviation of a statistic is often called the *Standard Error*, e.g., the square root of $Var(\bar{X})$.
- The sampling distribution of a statistic is used in developing inferential procedures.

Sampling Distributions: Example 1

- Let X be the number of medications a person is taking.
- Suppose we have a population of five people to sample from, and the number of medications they take is as follows:

Person	# Medications
A	1
B	4
C	4
D	7
E	10

- So the population mean, $E[X] = \mu = 5.2$.

Sampling Distributions: Example 1

- How many different ways could we sample three people from this population?

Sampling Distributions: Example 1

- Possible samples of size 3 that we could select from this population and the resulting sample mean are:

Sample (person)	Sample (values)	$\bar{x}$
(C,D,E), (B,D,E)	(4,7,10), (4,7,10)	7
(B,C,E), (A,D,E)	(4,4,10), (1,7,10)	6
(B,C,D), (A,C,E), (A,B,E)	(4,4,7), (1,4,10), (1,4,10)	5
(A,C,D), (A,B,D)	(1,4,7), (1,4,7)	4
(A,B,C)	(1,4,4)	3

- Note that *none* of the possible sample means are equal to the population mean.

Sampling Distributions: Example 1

- The sampling distribution of $\bar{X}$ for a sample of size three from these data is given by:

	Probability
$P(\bar{X} = 7)$	0.2
$P(\bar{X} = 6)$	0.2
$P(\bar{X} = 5)$	0.3
$P(\bar{X} = 4)$	0.2
$P(\bar{X} = 3)$	0.1

- So what is $E[\bar{X}]$, the expected value of $\bar{X}$?

Sampling Distributions: Example 1

- If we took samples over and over again, we would end up with 20% with a mean of 7; 20% with a mean of 6; 30% with a mean of 5; and so on.

- So the average (expected) value we would observe if we took repeated samples would be:

$$20\% \times 7 + 20\% \times 6 + 30\% \times 5 + 20\% \times 4 + 10\% \times 3$$

  which is

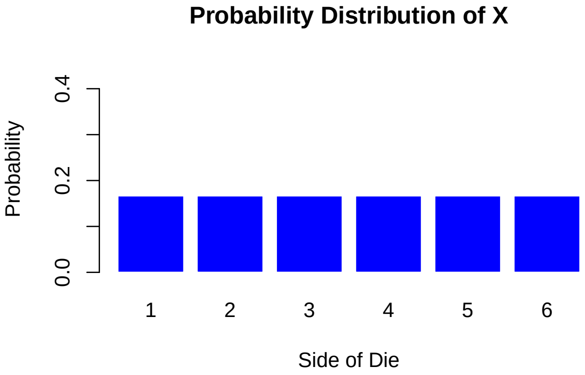
$$0.2 \times 7 + 0.2 \times 6 + 0.3 \times 5 + 0.2 \times 4 + 0.1 \times 3 = 5.2$$

- That is, $E[\bar{X}] = \mu = 5.2$.

- Because of this property, we say that $\bar{X}$ is an *unbiased estimator* of μ.

Sampling Distributions: Example 2

- Let X = the outcome of the toss of a fair (unbiased) six-sided die.
- Sample space for $X : \mathcal{S} = \{1, 2, 3, 4, 5, 6\}$.



Sampling Distributions: Example 2

- Consider measuring X twice, and averaging the two observed values of X to obtain a sample mean, $\bar{X}$.
- We can view the two measurements on X as a sample, and the sample mean $\bar{X}$ as a statistic.
- We can view the population mean μ as the mean of the random variable X. Thus, $\mu = 3.5$.
- What would be the sampling distribution of $\bar{X}$?

Sampling Distributions: Example 2

To determine the sampling distribution, we would need to determine the following:

i) every possible sample mean,

ii) the samples of size two corresponding to every possible sample mean,

iii) the probability corresponding to every possible sample mean.

Sampling Distributions: Example 2

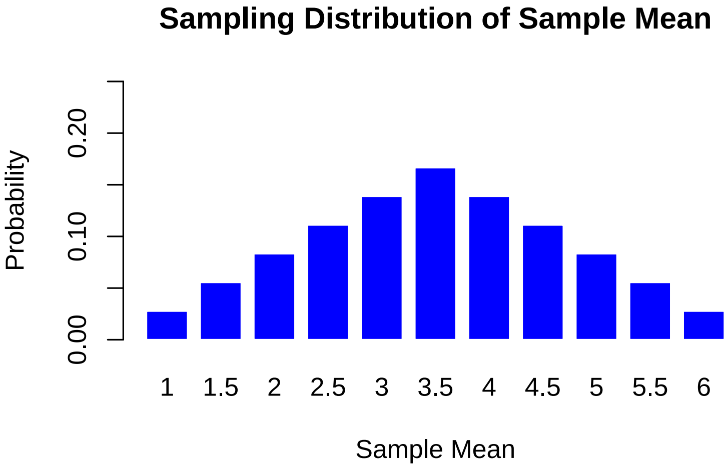
- Sampling distribution of mean of two throws of a six-sided die.

$\bar{x}$	Sample	Probability
1.0	$(1, 1)$	$1/36$
1.5	$(1, 2), (2, 1)$	$2/36$
2.0	$(1, 3), (3, 1), (2, 2)$	$3/36$
2.5	$(1, 4), (4, 1), (2, 3), (3, 2)$	$4/36$
3.0	$(1, 5), (5, 1), (2, 4), (4, 2), (3, 3)$	$5/36$
3.5	$(1, 6), (6, 1), (3, 4), (4, 3), (2, 5), (5, 2)$	$6/36$
4.0	$(2, 6), (6, 2), (3, 5), (5, 3), (4, 4)$	$5/36$
4.5	$(3, 6), (6, 3), (4, 5), (5, 4)$	$4/36$
5.0	$(4, 6), (6, 4), (5, 5)$	$3/36$
5.5	$(5, 6), (6, 5)$	$2/36$
6.0	$(6, 6)$	$1/36$

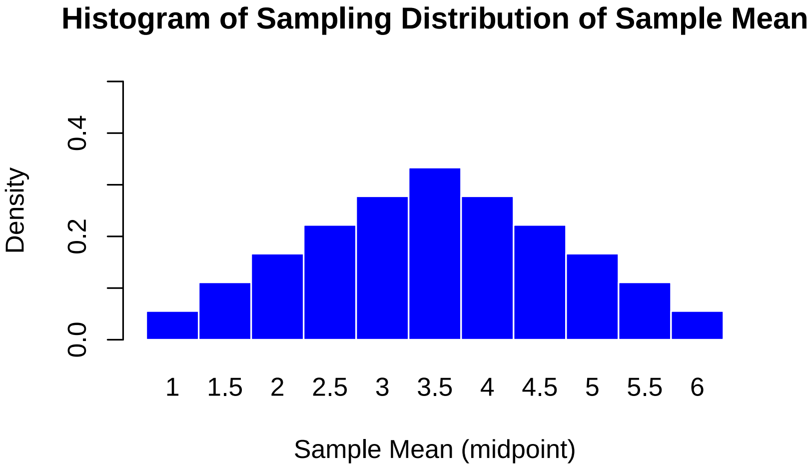
Sampling Distributions: Example 2

- The sampling distribution reflects which values of $\bar{X}$ are likely and which values are improbable.
- The most likely value of $\bar{X}$ is $\mu = 3.5$. This is meaningful, since we are viewing $\bar{X}$ as an estimator of μ.
- Values of $\bar{X}$ near μ are more likely than values more distant from μ.

Sampling Distributions: Example 2



Sampling Distributions: Example 2



Sampling Distributions

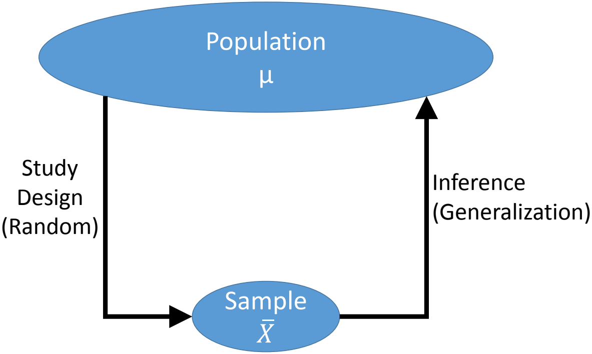
- How do you think the sampling distribution of $\bar{X}$ would change if we based $\bar{X}$ on a sample of size 3? A sample of size 30? A sample of size 1000?
- The sampling distribution reflects the shape of a histogram that would result if a long set of measurements on $\bar{X}$ was compiled, and a histogram was constructed based on the relative frequency distribution of these measurements.

Sampling Distributions

- *Repeated Sampling* refers to the process of repeatedly drawing samples from a population, and repeatedly computing a statistic (such as $\bar{X}$) based on these samples.
- Repeated sampling is not done in practice. However, understanding the behavior of a statistic in repeated sampling (i.e., understanding the sampling distribution of a statistic) helps us to characterize the properties of the statistic as an estimator of a parameter.

The Central Limit Theorem

- We will now consider the specific problem of using the statistic $\bar{X}$ (the sample mean) to estimate the parameter μ (the population mean).



The Central Limit Theorem

Properties of the sampling distribution of $\bar{X}$:

- The mean or expected value of the statistic $\bar{X}$ is the same as the mean of the sampled population, μ.
- The standard deviation or standard error of the statistic $\bar{X}$ is given by $\sigma/\sqrt{n}$, where σ is the standard deviation of the sampled population, and n is the sample size.

The Central Limit Theorem

The preceding are general properties of the sampling distribution of $\bar{X}$ that hold for any sample size n.

- The quantity $\sigma/\sqrt{n}$ is often called the *Standard Error of the Mean*.
- The magnitude of this quantity reflects the accuracy of the sample mean as an estimator of the population mean.

The Central Limit Theorem

- **Central Limit Theorem**: If the sample size is ‘large,’ the sampling distribution of $\bar{X}$ is approximately normal, regardless of the characteristics of the underlying population.
- A ‘large’ sample is generally considered to be one where $n \geq 30$.
- Suppose the random variable being measured to collect the sample data has a normal distribution. Then the sampling distribution of $\bar{X}$ is normal for *any* sample size.

The Central Limit Theorem

Note:

- If $\bar{X}$ is approximately $N(\mu, \sigma^2/n)$, then

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

  is approximately standard normal.

- Observations must be independently drawn from and be representative of the population.

- The central limit theorem applies to the sampling distribution of the mean, not necessarily to the sampling distribution of other statistics.

Central Limit Theorem: Application 1

- In Norway, the distribution of birth weights for infants whose gestational age is 40 weeks is approximately normal with mean $\mu = 3500$ grams and standard deviation $\sigma = 430$ grams.
- For parts a) and b), consider a randomly selected newborn whose gestational age is 40 weeks. Let X denote the birth weight of this infant.

a) Illustrate the probability distribution of X.

Central Limit Theorem: Application 1

b) What is the probability that the infant's birth weight is less than 3400 grams?

- For the remaining parts, consider a random sample of 25 newborns whose gestational age is 40 weeks. Let $\bar{X}$ denote the mean birth weight of these 25 infants.

c) What is the mean of $\bar{X}$? (That is, in repeated sampling, what would be the average value of $\bar{X}$?)

Central Limit Theorem: Application 1

d) What is the standard deviation (i.e., standard error) of $\bar{X}$?

e) Illustrate the sampling distribution of $\bar{X}$.

Central Limit Theorem: Application 1

f) What is the probability that $\bar{X}$ will be less than 3400 grams? (That is, in repeated sampling, what proportion of sample means will be less than 3400 grams?)

g) What is the probability that $\bar{X}$ will be between 3400 and 3600 grams? (That is, in repeated sampling, what proportion of sample means will be between 3400 and 3600 grams?)

Central Limit Theorem: Application 2

- In the Netherlands, healthy males between the ages of 65 and 79 have a distribution of serum uric acid levels that is approximately normal with mean $\mu = 341$ μmol/L and standard deviation $\sigma = 79$ μmol/L.

a) What proportion of the males have a serum uric acid level between 300 and 400 μmol/L?

Central Limit Theorem: Application 2

b) What proportion of samples of size 5 have a mean serum uric acid level between 300 and 400 μmol/L?

Central Limit Theorem: Application 2

c) What proportion of samples of size 10 have a mean serum uric acid level between 300 and 400 μmol/L?

Central Limit Theorem: Application 2

d) Construct an interval that encloses 95% of the means of samples of size 10.

Central Limit Theorem: Application 2

- How does this compare to $Pr(300 < \bar{Y}_{10} < 400)$?
- How do the lengths of the two intervals compare?
- Does it make sense? Why or why not?

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