BIOS:4120 – Introduction to Biostatistics
Unit 8: Confidence Intervals

Knute D. Carter

Department of Biostatistics
The University of Iowa

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Overview

- Two-Sided Confidence Intervals
- One-Sided Confidence Intervals
- Student's t Distribution

Confidence Intervals

- There are two general approaches to estimating a parameter:
  - point estimation,
  - interval estimation.
- A *Point Estimate* of a parameter is a single statistic computed from the sample data used to estimate the parameter. (E.g., $\bar{x} \to \mu$, $s \to \sigma$)
- An *Interval Estimate* of a parameter is an interval computed from the sample data used to estimate the parameter. The interval is formulated so that we can have a high degree of confidence that the interval covers the parameter.

Two-Sided Confidence Interval for the Population Mean

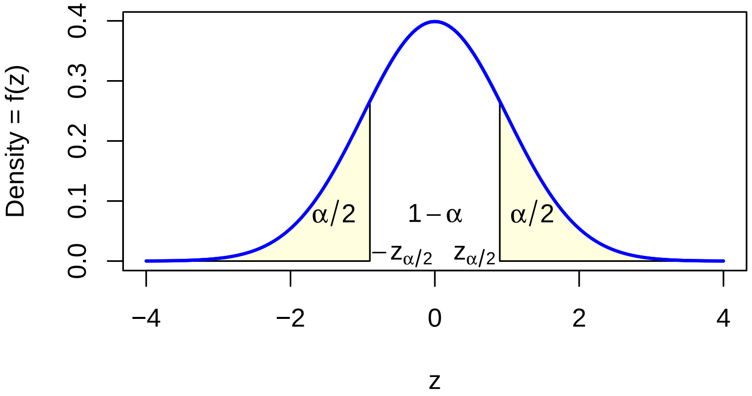
- Assume we draw a random sample of size n from a population with unknown mean μ and *known* standard deviation σ.
- Let $\bar{x}$ denote the sample mean.
- Our goal is to construct an interval estimator for μ that will cover μ with 'high' probability.

Two-Sided Confidence Interval for the Population Mean

- We typically denote the coverage probability by $(1-\alpha)$. Here, α represents a small number chosen by the researcher (typically 0.10, 0.05, or 0.01).
- The percentage $100(1-\alpha)\%$ is called the *Confidence Level*.
- Let $z_{\alpha/2}$ denote a cut-off point along the horizontal axis under the standard normal curve such that

$$P(-z_{\alpha/2} \leq Z \leq +z_{\alpha/2}) = (1-\alpha)$$

Two-Sided Confidence Interval for the Population Mean



Two-Sided Confidence Interval for the Population Mean

- The cut-off point $z_{\alpha/2}$ is often called a *Critical Value*.
- Common critical values:

$100(1-\alpha)\%$	$\alpha/2$	$z_{\alpha/2}$
80%	0.100	1.282
90%	0.050	1.645
95%	0.025	1.960
99%	0.005	2.576

Two-Sided Confidence Interval for the Population Mean

- If $\bar{Y}_n \sim N(\mu, \sigma^2/n)$, (at least approximately using the Central Limit Theorem and based on a random sample of size n), then,
$$P\left(-1.96 < \frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} < 1.96\right) = 0.95.$$
- Note that the quantity in the middle is a random variable; while -1.96 and $+1.96$ are just constants.
- We will next rearrange the terms so that μ is in the center.

Two-Sided Confidence Interval for the Population Mean

$$
\begin{array}{rrcl}
 & -1.96 & < \quad \frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} \quad < & 1.96 \\
\Rightarrow & -1.96\frac{\sigma}{\sqrt{n}} & < \quad \bar{Y}_n - \mu \quad < & 1.96\frac{\sigma}{\sqrt{n}} \\
\Rightarrow & -1.96\frac{\sigma}{\sqrt{n}} - \bar{Y}_n & < \quad -\mu \quad < & 1.96\frac{\sigma}{\sqrt{n}} - \bar{Y}_n \\
\Rightarrow & 1.96\frac{\sigma}{\sqrt{n}} + \bar{Y}_n & > \quad \mu \quad > & -1.96\frac{\sigma}{\sqrt{n}} + \bar{Y}_n \\
\Rightarrow & \bar{Y}_n + 1.96\frac{\sigma}{\sqrt{n}} & > \quad \mu \quad > & \bar{Y}_n - 1.96\frac{\sigma}{\sqrt{n}} \\
\Rightarrow & \bar{Y}_n - 1.96\frac{\sigma}{\sqrt{n}} & < \quad \mu \quad < & \bar{Y}_n + 1.96\frac{\sigma}{\sqrt{n}}
\end{array}
$$

Two-Sided Confidence Interval for the Population Mean

- Thus,

$$P\left(-1.96 < \frac{\bar{Y}_n - \mu}{\sigma/\sqrt{n}} < 1.96\right) = 0.95$$

  is equivalent to

$$P\left(\bar{Y}_n - 1.96\frac{\sigma}{\sqrt{n}} < \mu < \bar{Y}_n + 1.96\frac{\sigma}{\sqrt{n}}\right) = 0.95.$$

- Note that now the endpoints of the interval have become the random variables, and μ in the center is just an (unknown) constant.

Two-Sided Confidence Interval for the Population Mean

- The endpoints will change from sample to sample, but the probability statement (based on the sampling distribution of $\bar{Y}_n$) is a true statement regardless of the actual value of $\bar{y}_n$ from a particular sample.
- For a sample with mean $\bar{y}_n$, the interval

$$\left(\bar{y}_n - 1.96\frac{\sigma}{\sqrt{n}},\ \ \bar{y}_n + 1.96\frac{\sigma}{\sqrt{n}}\right)$$

is called a 95% *confidence interval* for μ.

Two-Sided Confidence Interval for the Population Mean

Correct interpretation of a confidence interval:

- If you were to repeat this process an infinite number of times, 95% of interval estimates for μ created this way will contain the true parameter value μ.
- We treat the population mean μ as being fixed. Any particular interval may or may not contain the true population mean μ.
- We say we are '95% confident' the interval does contain μ because the procedure used to construct this interval produces a correct interval estimate 95% of the time.
- We do **not** say there is a 95% probability that μ lies between these values.

Two-Sided Confidence Interval for the Population Mean

- The general formula for a $100(1-\alpha)\%$ confidence interval is

$$\left(\bar{y}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}},\ \ \bar{y}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right).$$

- This formula assumes that σ is <u>known</u>, which is an unlikely situation. We will relax this assumption later.

- The confidence level $100(1-\alpha)\%$ represents the 'success rate' of the confidence interval procedure; i.e., the percentage of time the procedure produces an interval that covers the parameter of interest.

Two-Sided Confidence Interval for the Population Mean

- The quantity
$$z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$
is often called the *Margin of Error*.

- The general form of a two-sided confidence interval is as follows:
$$\text{Point Estimate} \pm \text{Margin of Error}$$

- In the media, margins of error are generally based on a 95% confidence level. In which case, the critical value $z_{0.025} = 1.960$ is used.

Two-Sided Confidence Interval: Trauma Example

- Trauma patients are classified as either blunt or penetrating trauma victims.
- In a random sample of 107 blunt trauma victims for the state of Missouri (1994), the mean hospital charges incurred was \$18,300.
- The population standard deviation of the charges is assumed to be \$19,300.
- Use this sample information to construct a 95% two-sided confidence interval for the mean hospital charges incurred by all blunt trauma victims in Missouri (in 1994).

Two-Sided Confidence Interval: Trauma Example

The setup:

- $100(1-\alpha)\% = 95\% \quad \Rightarrow \quad \alpha = 0.05$,
- $z_{0.025} = 1.960$,
- $n = 107$,
- $\bar{x} = \$18,300$,
- $\sigma = \$19,300$.

Two-Sided Confidence Interval: Trauma Example

95% Confidence Interval:

$$\left(\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \quad \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right)$$

$$\left(18300 - 1.96\,\frac{19300}{\sqrt{107}}, \quad 18300 + 1.96\,\frac{19300}{\sqrt{107}}\right)$$

$$(18300 - 3657, \quad 18300 + 3657)$$

$$(\$14643, \quad \$21957)$$

Conclusion: We can be 95% confident that the true mean hospital charges incurred by all blunt trauma victims in Missouri (in 1994) is between \$14,643 and \$21,957.

Note: $\mu = \$20{,}490$ based on $N = 5{,}247$ patients.

Two-Sided Confidence Interval for the Population Mean

Two-Sided Confidence Interval Facts:

- For a fixed confidence level $100(1-\alpha)\%$, as the sample size n is increased, the width of the confidence interval is decreased.

$$\text{Width } = 2 \times \text{Margin of Error} = 2 \times z_{\alpha/2} \frac{\sigma}{\sqrt{n}}.$$

- For a fixed sample size n, as the confidence level $100(1-\alpha)\%$ is increased, the width of the confidence interval is increased.
- The interval is symmetric and centered at $\bar{x}$, the point estimate of μ.

Two-Sided Confidence Interval: Sample Size Example

- Suppose we want to find the sample size n needed so that the margin of error does not exceed some value k. Accordingly, we want to find the sample size n needed so that the width of a two-sided confidence interval does not exceed $2k$.
- So we need to find the n that will ensure:

$$z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \leq k.$$

- The solution is to take:

$$n \geq \left(\frac{z_{\alpha/2}\, \sigma}{k} \right)^2.$$

Two-Sided Confidence Interval: Sample Size Example

If σ is unknown, the value of σ can be based on

- s from a previous study or a 'pilot' study,
- an 'educated guess,'
- one-fourth the plausible range of the variable being measured:

$$\frac{1}{4}(\max - \min) \approx \sigma.$$

Two-Sided Confidence Interval: Sample Size Example

- In the trauma example, find the sample size required to obtain a 95% two-sided confidence interval for μ that has a width of \$2,000 (or less).
- We will assume that σ is given by \$19,300.
- $100(1-\alpha)\% = 95\% \quad \Rightarrow \quad \alpha = 0.05,$
- $z_{0.025} = 1.960,$
- $k = \$1{,}000.$

Two-Sided Confidence Interval: Sample Size Example

- So we have

$$n \geq \left(\frac{z_{\alpha/2}\, \sigma}{k} \right)^2 = \left(\frac{1.96 \times 19300}{1000} \right)^2 = 1430.96$$

  and we take $n = 1,431$.

- Note that this sample size is over 13 times as large as the sample size used in constructing the original confidence interval.

One-Sided Confidence Interval

- In certain applications where we need to estimate a population mean μ, we are only interested in an upper bound or a lower bound for μ.
- Examples:

- What is a lower bound on the mean birth weight of infants?
- What is an upper bound on the mean adult BMI?
- What is an upper bound for the mean ACT score for UI freshmen?

One-Sided Confidence Interval

- A level $100(1-\alpha)\%$ *One-Sided Confidence Interval* for μ is given by

$$\left(-\infty\,, \quad \bar{x} + z_\alpha \frac{\sigma}{\sqrt{n}}\right) \qquad \text{for an upper bound}$$

$$\left(\bar{x} - z_\alpha \frac{\sigma}{\sqrt{n}}\,, \quad +\infty\right) \qquad \text{for a lower bound}$$

- Where z_α denotes a cut-off point along the horizontal axis under the standard normal curve such that the area to the right of z_α is α.

One-Sided Confidence Interval

- These formulas are valid when the population is normal and σ is known.
- Otherwise, the formulas are approximately valid provided that n is 'large.' (If n is 'large' and σ is unknown, s can be used in place of σ.)
- Common critical values:

$100(1-\alpha)\%$	α	z_α
90%	0.10	1.282
95%	0.05	1.645
99%	0.01	2.327

One-Sided Confidence Interval: Trauma Example

- In the trauma example, find a 95% upper confidence bound for μ.
- $100(1-\alpha)\% = 95\% \quad \Rightarrow \quad \alpha = 0.05$,
- $z_{0.05} = 1.645$,
- $n = 107$,
- $\bar{x} = \$18,300$,
- $\sigma = \$19,300$.

One-Sided Confidence Interval: Trauma Example

- 95% Upper Confidence Bound

$$\begin{aligned} \bar{x} + z_\alpha \frac{\sigma}{\sqrt{n}} &= 18300 + 1.645 \frac{19300}{\sqrt{107}} \\ &= 18300 + 3069 \\ &= \$21370 \end{aligned}$$

- Conclusion: We can be 95% confident that the true mean hospital charges incurred by all blunt trauma victims in Missouri (in 1994) does not exceed $21,370.

Student's t Distribution

- So far we have focused on an interval estimate of μ (confidence interval) assuming we already knew the value of σ.
- In most situations, not knowing the mean while already knowing the standard deviation is pretty unrealistic.
- If we don't know σ, but do have an estimate of it, s, can we just substitute s for σ and do the same thing anyway?
- Answer: almost, but not exactly.

Student's t Distribution

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

- has a standard normal distribution if the Xs are normally distributed
- is approximately standard normal if n is 'large' enough for the Central Limit Theorem to take effect

Student's t Distribution

- For 'large' n the sample standard deviation, s should be close to the population standard deviation σ.
- Thus for 'large' n, if we replace σ with s the previous result should still be approximately true.
- That is to say,
$$\frac{\bar{X} - \mu}{s/\sqrt{n}}$$
will be approximately standard normal if n is 'large enough'.

Student's t Distribution

- What about when n is 'small'?
- If the population is normally distributed. Then,

$$T = \frac{\bar{X} - \mu}{s/\sqrt{n}}$$

  has a *Student's t Distribution* or simply a *t distribution*.

- The T statistic should have more variability than the Z statistic, since we are now estimating σ with s. The t distribution accounts for this additional uncertainty.

Student's t Distribution

The t distribution

- is bell-shaped like a standard normal distribution,
- is always centered at 0,
- has one parameter to characterize it, called the *degrees of freedom* (df) which are equal to $n - 1$ (one less than the sample size on which $\bar{X}_n$ is calculated), and
- has heavier tails than the $N(0, 1)$ distribution.

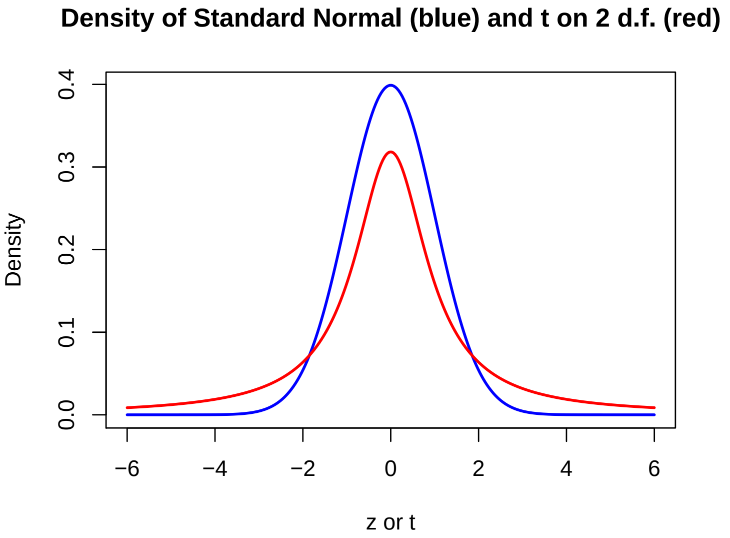
Student's t Distribution

- That is
$$T = \frac{\bar{X}_n - \mu}{s/\sqrt{n}} \sim t(n-1).$$

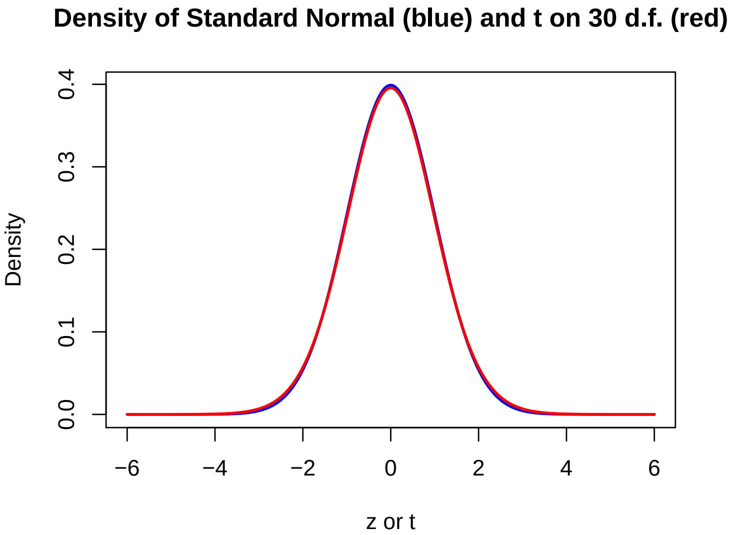
- The higher the degrees of freedom (smallest df $= 1$), the closer the t distribution is to the $N(0, 1)$ distribution.

- For $n \geq 30$, the curves are very similar.

Student's t Distribution



Student's t Distribution



Critical Values for the t Distribution

- Let $100(1-\alpha)\%$ be a confidence level.
- Let $t_{n-1,\alpha/2}$ denote a cut-off point along the horizontal axis under the t curve such that

$$P(-t_{n-1,\alpha/2} \leq T \leq t_{n-1,\alpha/2}) = (1-\alpha)$$

  for a sample of size n.
- Critical values for the t curve are in the tables on ICON.

Critical Values for the t Distribution

	Area in Upper Tail			
df	0.100	0.050	0.025	$\cdots$
1	3.078	6.314	12.706	$\cdots$
2	1.886	2.920	4.303	$\cdots$
3	1.638	2.353	3.182	$\cdots$
4	1.533	2.132	2.776	$\cdots$
5	1.476	2.015	2.571	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
30	1.310	1.697	2.042	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
120	1.289	1.658	1.980	$\cdots$
∞	1.282	1.645	1.960	$\cdots$

Student's t Distribution: Two-Sided Confidence Interval

- Assume we draw a random sample of size n from a normally distributed population with unknown mean μ and unknown standard deviation σ.

- We can write

$$P\left(-t_{n-1,\alpha/2} < \frac{\bar{X}_n - \mu}{s/\sqrt{n}} < t_{n-1,\alpha/2}\right) = 1 - \alpha$$

  which is equivalent to

$$P\left(\bar{X}_n - t_{n-1,\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{X}_n + t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}\right) = 1 - \alpha.$$

- Compare with slide 10 of this unit.

Student's t Distribution: Two-Sided Confidence Interval

- A level $100(1-\alpha)\%$ Two-Sided Confidence Interval for μ is given by

$$\left(\bar{x}_n - t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}, \quad \bar{x}_n + t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}\right)$$

Student's t Distribution: One-Sided Confidence Interval

- A level $100(1-\alpha)\%$ One-Sided Confidence Interval for μ is given by

$$\left(-\infty\,, \quad \bar{x} + t_{n-1,\alpha}\,\frac{s}{\sqrt{n}}\right) \qquad \text{for an upper bound}$$

$$\left(\bar{x} - t_{n-1,\alpha}\,\frac{s}{\sqrt{n}}\,, \quad +\infty\right) \qquad \text{for a lower bound}$$

Student's t Distribution: Example 1

- Consider the following percentages of ideal body weight for 18 randomly selected insulin-dependent diabetics (Saudek et al., 1989):

107	119	99	114	120	104
124	114	88	116	101	121
152	100	95	114	125	117

- A percentage of 120 means that an individual weighs 20% more than his/her ideal body weight; a percentage of 95 means that an individual weighs 5% less than his/her ideal body weight.

Student's t Distribution: Example 1

a) Compute a 90% two-sided confidence interval for the true mean percentage of ideal body weight for the population of insulin-dependent diabetics.

Student's t Distribution: Example 1

b) Compute a 90% lower confidence bound for the true mean percentage of ideal body weight for the population of insulin-dependent diabetics.

Student's t Distribution: Example 1

c) For the intervals in (a) and (b) to be valid, what assumption must hold regarding the distribution of body weight percentages?

d) Do the intervals in a) and b) contain the value 100%? What conclusion can be drawn from the answer to this question?

Student's t Distribution: Example 2

- The following values represent the points scored by the Hawkeyes men's basketball team in 5 randomly selected games from the 2005–2006 season:

  76 70 63 59 65

a) Use this sample information to construct a 90% two-sided confidence interval for the overall mean number of points scored by the Hawkeyes during the 2005–2006 basketball season.

Student's t Distribution: Example 2

b) The actual overall mean number of points scored by the Hawkeyes during the 2005–2006 basketball season was 66.3. Does the interval in part a) capture this?

Summary Formulae for Confidence Intervals for μ

Assuming X is normally distributed $N(\mu, \sigma^2)$

- Case I: When σ is known:

a) $100(1-\alpha)\%$ two-sided confidence interval for μ:

$$\left(\bar{x}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}},\ \ \bar{x}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right).$$

b) $100(1-\alpha)\%$ lower (one-sided) bound for μ:

$$\left(\bar{x}_n - z_{\alpha}\frac{\sigma}{\sqrt{n}},\ \ \infty\right).$$

c) $100(1-\alpha)\%$ upper (one-sided) bound for μ:

$$\left(-\infty,\ \ \bar{x}_n + z_{\alpha}\frac{\sigma}{\sqrt{n}}\right).$$

Summary Formulae for Confidence Intervals for μ

- Case II: When σ is unknown and estimated by s:

a) $100(1-\alpha)\%$ two-sided confidence interval for μ:

$$\left(\bar{x}_n - t_{n-1,\alpha/2}\frac{s}{\sqrt{n}},\ \ \bar{x}_n + t_{n-1,\alpha/2}\frac{s}{\sqrt{n}}\right).$$

b) $100(1-\alpha)\%$ lower (one-sided) bound for μ:

$$\left(\bar{x}_n - t_{n-1,\alpha}\frac{s}{\sqrt{n}},\ \ \infty\right).$$

c) $100(1-\alpha)\%$ upper (one-sided) bound for μ:

$$\left(-\infty,\ \ \bar{x}_n + t_{n-1,\alpha}\frac{s}{\sqrt{n}}\right).$$