

# BIOS:4120 – Introduction to Biostatistics

## Unit 6: Theoretical Probability Distributions

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# Learning Objectives (1/2)

At the end of this unit, you should be able to:

- Define what a random variable is and distinguish between discrete and continuous random variables.
- Explain differences between probability mass and density functions and the properties of each.
- Describe situations under which data from either the binomial or Poisson distribution may arise.
- Calculate probabilities from binomial and Poisson distributions; and also the mean, variance, and standard deviation of each.

## Learning Objectives (2/2)

At the end of this unit, you should be able to:

- List the criteria that characterize a Poisson process.
- Explain when the Poisson distribution approximates the binomial distribution.
- Apply the additive property of Poisson distributions.
- Describe the basic properties of the standard normal distribution, and list the common critical values and percentiles.
- Calculate standardized values and calculate probabilities from the general normal distribution.

Distributions

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Binomial Distribution

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Poisson Distribution

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Normal Distribution

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# Overview

- Theoretical Probability Distributions
- The Binomial (Bernoulli) Distribution
- The Poisson Distribution
- The Normal (Gaussian) Distribution

# Variables

- A *Variable* is a generic term we use to describe any quantitative or qualitative measure we record.
- A *Random Variable* is a numeric variable that assumes a value based on the outcome of a random experiment.
- Formal definition:  
Consider a random experiment with a sample space  $\mathcal{S}$ .  
A function  $X$ , which assigns to each element  $s \in \mathcal{S}$  one and only one number  $X(s) = x$ , is called a *Random Variable*.

## Variables

- *Discrete Random Variable*: may assume only specific numeric values (often integers).
  - *Continuous Random Variable*: may assume any value over some interval or continuum.

## Examples of Random Variables

- $X$  = the # of heads that appear when a coin is tossed 3 times.
  - $X$  is
  - Sample space for  $X$  :  $\mathcal{S} = \{$
  - $T$  = the high temperature in Iowa City (in Fahrenheit) on a randomly selected summer day.
  - $T$  is
  - Sample space for  $T$  :  $\mathcal{S} = \{$

## Notation Convention

- We will use capital letters (e.g.,  $T, X, Y, Z$ ) to denote random variables, and lower-case letters (e.g.,  $t, x, y, z$ ) to denote observed values of random variables.

Distributions

Binomial Distribution

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Poisson Distribution

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Normal Distribution

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# Definitions

- The *Probability Distribution* of a discrete random variable  $X$  is a function  $p(x)$  which assigns every possible value of  $X$  a probability.
- A *Probability Histogram* is a graphical representation of a probability distribution.

## Example: Coin Toss

- Consider tossing an unbiased coin three times.  
Let  $X$  count the number of heads that appear.
- Sample space for the random experiment:

$$\mathcal{S} = \{$$

- $X$  is
- Sample space for  $X : \mathcal{S} = \{$

Distributions  
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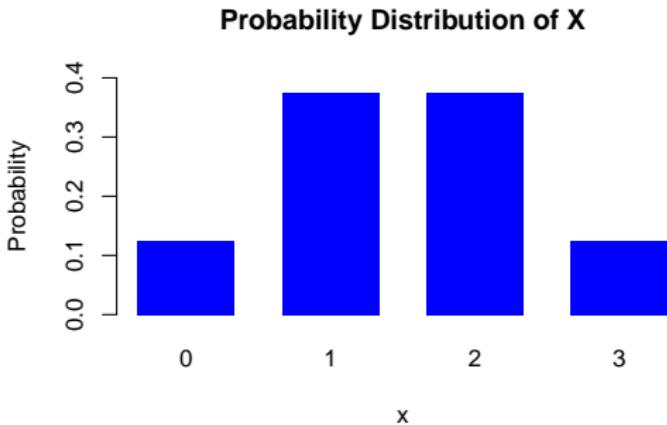
Binomial Distribution  
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Poisson Distribution  
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Normal Distribution  
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## Example: Coin Toss

Probability, $p(x)$ or $P(X = x)$	
$x$	
0	$1/8 = 0.125$
1	$3/8 = 0.375$
2	$3/8 = 0.375$
3	$1/8 = 0.125$



## Application of $p(x)$ to Example

- Let  $A$  denote the event that 2 heads are obtained
- Let  $B$  denote the event that at least 2 heads are obtained.
- $P(A) = P(X = 2) = 0.375.$
- $P(B) = P(X = 2 \cup X = 3) = P(X = 2) + P(X = 3) = 0.5.$
- Suppose we want to find  $P(A|B)$ :

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)} = \frac{0.375}{0.5} = 0.75.$$

# Probability Mass Function

- The probability function of a discrete random variable (RV):
  1. defines all possible values of the variable;
  2. displays the probabilities with which the random variable takes on those values;
  3. can sometimes be described using a formula.
- For discrete random variables the function  $p(x)$  is often referred to as the *probability mass function* (or *pmf*).
- For each value of the RV, the pmf gives the probability of that value happening.

# Mean and Variance of a Random Variable

- Suppose a random variable is repeatedly measured, resulting in an increasingly large collection of observed values.
- The *Mean* of a random variable,  $\mu$ , is the mean of a large set of measurements taken on  $X$ .
- The *Variance* of a random variable,  $\sigma^2$ , is the variance of a large set of measurements taken on  $X$ .
- The *Standard Deviation* of a random variable,  $\sigma$ , is the square root of the variance,  $\sqrt{\sigma^2}$ .

# Factorials and Combinations

- Let  $n$  be a positive integer.
- The *Factorial* of  $n$ , denoted  $n!$ , is defined as

$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 2 \times 1.$$

- For example,

$$3! = 3 \times 2 \times 1 = 6.$$

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$$

$$10! = 10 \times 9 \times 8 \times \cdots \times 2 \times 1 = 3,628,800.$$

- So what does,  $0! = ?$ .

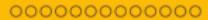
Distributions



Binomial Distribution



Poisson Distribution



Normal Distribution



# Factorials and Combinations

- Consider a group of  $n$  objects. The number of ways in which these  $n$  objects can be ordered is given by  $n!$
- Consider a group of  $n$  objects. How many ways can we select  $x$  of the  $n$  objects ( $0 \leq x \leq n$ ), without regard to order?
- The *Binomial Coefficient* provides the answer to this question:

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$

- We say “ $n$  choose  $x$ ”.

Distributions  
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Binomial Distribution  
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Poisson Distribution  
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Normal Distribution  
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## Combination Examples

$$\binom{3}{2} = \frac{3!}{2!(3-2)!} = 3$$

$$\binom{6}{3} = \frac{6!}{3!(6-3)!} = 20$$

$$\binom{6}{6} = \frac{6!}{6!(6-6)!} = 1$$

Distributions  
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Binomial Distribution  
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Poisson Distribution  
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Normal Distribution  
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## Example

Consider a list of 8 patients.

- a) In how many ways can the names on this list be ordered?
  
  
  
  
  
  
- b) In how many ways can a sample of 4 of these patients be drawn (without regard to order)?

Distributions  
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Binomial Distribution  
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Poisson Distribution  
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Normal Distribution  
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## Bernoulli Trials

Consider a sequence of random experiments that satisfy the following criteria.

1. Each trial has only 2 possible outcomes. These outcomes are often referred to as 'successes' and 'failures.'
2. The trials are independent.
3. The probability of a success,  $p$ , remains the same from trial to trial.

Distributions  
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Binomial Distribution  
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Poisson Distribution  
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Normal Distribution  
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# Bernoulli Trials

## Examples:

- 25 tosses of a coin.
- 12 tosses of a die, where the outcomes are classified as 'even numbers' or 'odd numbers.'
- 1000 ELISA test results for HIV status (positive or negative), conducted on 1000 blood samples randomly drawn from a large blood bank.

Distributions  
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ooooooooooooooooooooNormal Distribution  
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# Binomial Distribution

- In a sequence of  $n$  Bernoulli trials, let  $X$  count the number of successes.
- $X$  is said to be a *Binomial Random Variable*.
- The probability distribution of  $X$  is called the *Binomial Distribution*, often denoted as  $\text{Bin}(n, p)$  or  $B(n, p)$ .
- This distribution is given by

$$p(x) = P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x}$$

- The possible values of  $X$  are  $0, 1, 2, \dots, n$ .

## Justification of $p(x)$

- The sample space for the set of  $n$  Bernoulli trials is comprised of  $2^n$  success/failure sequences, where each sequence has a length  $n$ .
  - The number of these sequences in which we have  $x$  successes and  $(n - x)$  failures is given by  $\binom{n}{x}$ .
  - The probability associated with each sequence in which we have  $x$  successes and  $(n - x)$  failures is given by  $p^x(1 - p)^{n-x}$ .

## Properties of the Binomial Distribution

If  $Y$  is distributed  $\text{Bin}(n, p)$ , that is  $Y \sim \text{Bin}(n, p)$ , then:

- the mean of  $Y$ , denoted  $E(Y)$  for the ‘expectation of  $Y$ ,’ is equal to  $np$ ;
  - the variance of  $Y$ , denoted  $Var(Y)$  or  $V(Y)$ , is  $np(1 - p)$ ;
  - the standard deviation of  $Y$ , usually denoted  $sd(Y)$  or  $SD(Y)$ , is the square root of the variance and therefore equals  $\sqrt{np(1 - p)}$ .

Distributions



Binomial Distribution



Poisson Distribution



Normal Distribution



# The Bernoulli Distribution

If  $n = 1$  (one trial only), this distribution is also known as the *Bernoulli* distribution, for which:

- $E(Y) = \mu = p$
- $Var(Y) = \sigma^2 = p(1 - p)$
- $sd(Y) = \sigma = \sqrt{p(1 - p)}$

In other words, the Binomial random variable  $Y$  is just the count (sum) of successes in  $n$  independent Bernoulli trials.

Distributions  
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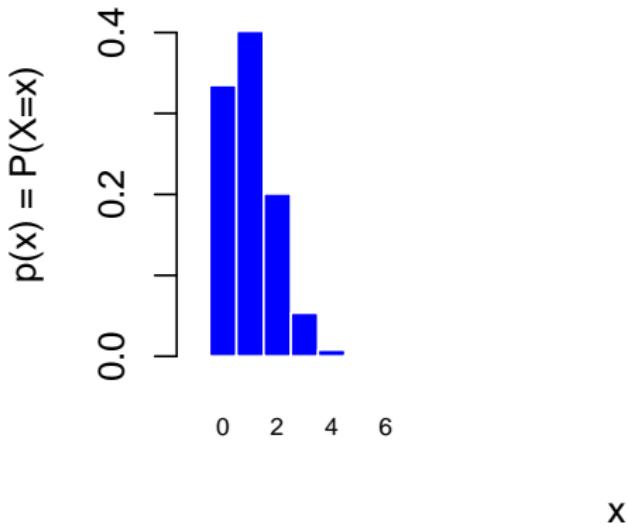
Binomial Distribution  
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Poisson Distribution  
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Normal Distribution  
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## Probability Histograms for Some Binomial Distributions

**Bin(6, 1/6)**



Distributions  
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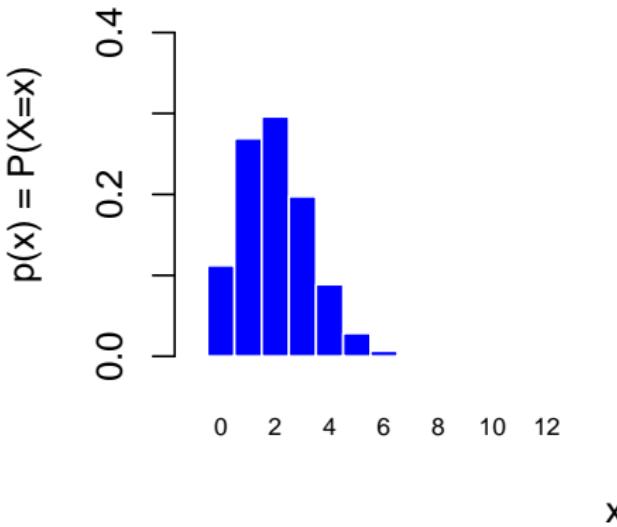
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Poisson Distribution  
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Normal Distribution  
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## Probability Histograms for Some Binomial Distributions

**Bin(12, 1/6)**



Distributions  
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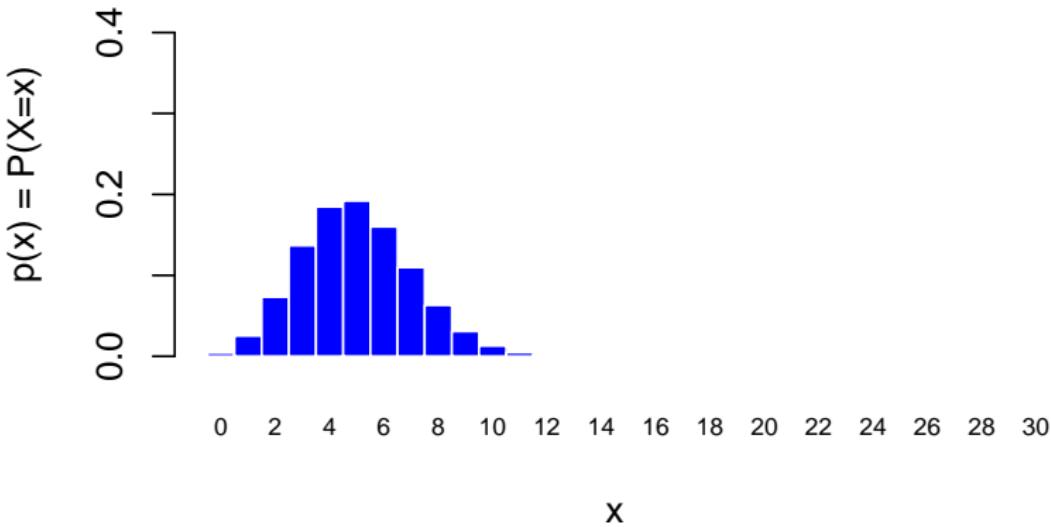
Binomial Distribution  
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Poisson Distribution  
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Normal Distribution  
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## Probability Histograms for Some Binomial Distributions

**Bin(30, 1/6)**



## Example: Blood Types

- Recall (from Chapter 6) the probability assignments for O, A, B, and AB blood types for a randomly chosen person in the United States.

Blood type	O	A	B	AB
U.S. probability	0.45	0.40	0.11	0.04

- Consider a random sample of 10 Americans. Let the random variable  $X$  count the number of individuals sampled with the blood type A.
  - What is the mean of  $X$ ? What is the standard deviation of  $X$ ?

Distributions  
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Binomial Distribution  
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Poisson Distribution  
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Normal Distribution  
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## Example: Blood Types

- b) What is the probability that exactly 4 of the 10 individuals sampled will have type A blood?
  
- c) What is the probability that at least 2 of the 10 individuals sampled will have type A blood?

## Example: Blue or Brown Eyes

- A newly married couple is planning to have 6 children. Assume that the probability the couple will conceive a child with blue eyes is 0.25, and the probability they will conceive a child with brown eyes is 0.75.
  - Let  $X$  count the number of children conceived by the couple who will have blue eyes; let  $Y$  count the number of children who will have brown eyes.
- a) What is the mean and standard deviation of  $X$ ?

## Example: Blue or Brown Eyes

- b) Find the probability that the couple will have exactly one child with blue eyes.
  
- c) Find the probability that the couple will have at least one child with blue eyes.

## Example: Blue or Brown Eyes

- d) Find the probability that the couple will have at least one child with brown eyes.

# Estimation for the Binomial Distribution

- In the Binomial distribution, the parameter  $n$  (number of independent trials) will always be known, so there is no need to estimate it.
- The parameter  $p$  is often unknown as must be estimated from the sample data.

$$\hat{p} =$$

- Sometimes the Greek letter  $\pi$  is used to denote the underlying (often unknown) parameter value, and then either  $\hat{\pi}$  or the lower case letter  $p$  to denote the sample estimate of that parameter (which can vary from sample to sample and thus is a random variable itself).

# Estimation for the Binomial Distribution

## Case 1:

- If the random variable of interest is the *proportion* of successes in a Binomial data situation.
- Estimated proportion of successes:  $\hat{p}$
- Estimated variance of  $\hat{p} = \frac{\hat{p}(1-\hat{p})}{n}$
- Estimated std. dev. of  $\hat{p} = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

# Estimation for the Binomial Distribution

Case 2:

- The random variable of interest is the *number* of successes.
- Estimated mean number of success:  $n\hat{p}$
- Estimated variance of the # of successes =  $n\hat{p}(1 - \hat{p})$
- Estimated std. dev. of the # of successes =  $\sqrt{n\hat{p}(1 - \hat{p})}$

# The Poisson Distribution

- Consider a process which results in a specific event that occurs periodically over time or space.
- The Poisson distribution is often used to assign probabilities to the number of events that take place over a certain interval of time or region of space.

# The Poisson Distribution

A *Poisson Process* is characterized by the following criteria.

- An event occurs periodically over time (or space).
- The expected number of events in an interval is proportional to the length of the interval.
- Within a single interval, an infinite number of occurrences of the event are theoretically possible.
- The events occur independently, both within the same interval and between consecutive intervals.

# The Poisson Distribution

- Given a Poisson process, let  $X$  count the number of occurrences of the event of interest over a certain interval. Assume that the mean number of occurrences over an interval of this length is denoted by  $\lambda$ .
- $X$  is said to be a *Poisson Random Variable*.

# The Poisson Distribution

- The probability distribution of  $X$  is the *Poisson Distribution*, and may be denoted as  $\text{Poisson}(\lambda)$ ,  $\text{Poi}(\lambda)$  or  $\text{P}(\lambda)$ .
- This distribution is characterized by

$$p(x) = P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!}$$

where the possible values of  $X$  are  $0, 1, 2, \dots, \infty$ .

- Here,  $e = 2.71828$ , the base of the natural logarithm.
- The mean of  $X$  is given by  $\mu = \lambda$ .
- The variance of  $X$  is given by  $\sigma^2 = \lambda$ .

Distributions  
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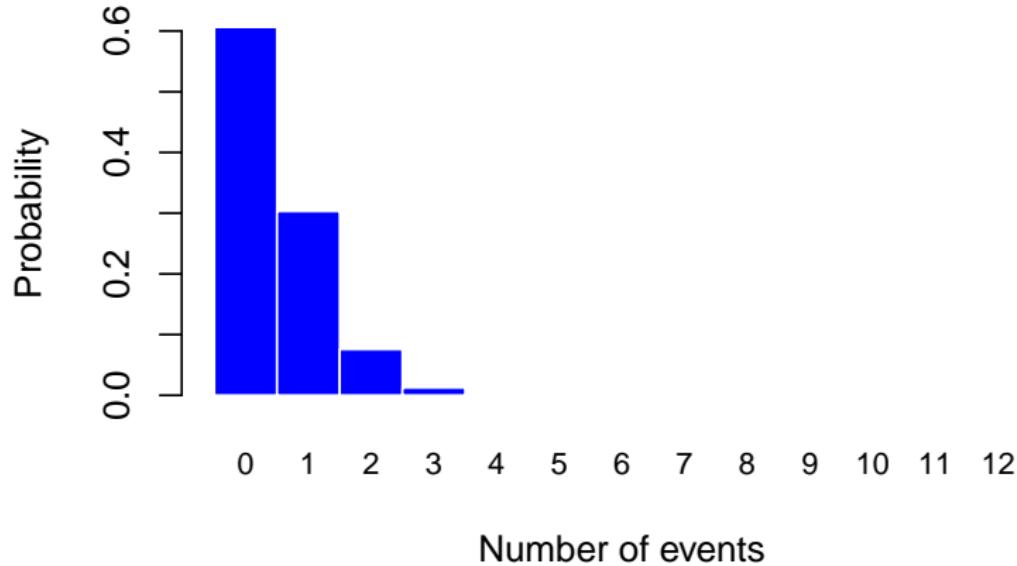
Binomial Distribution  
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Poisson Distribution  
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Normal Distribution  
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# The Poisson Distribution

**Poisson: Lambda = 0.5**



Distributions  
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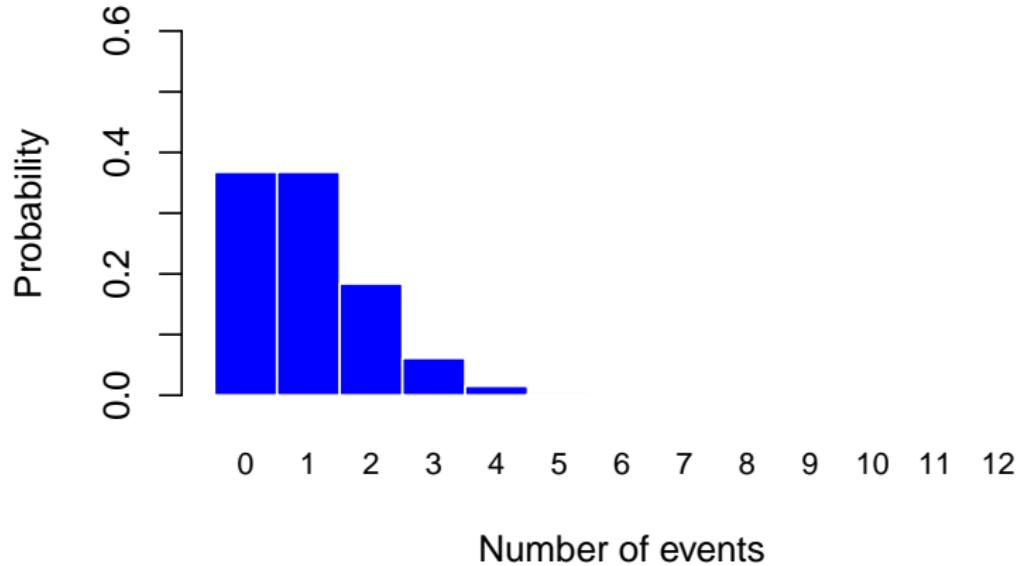
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Poisson Distribution  
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Normal Distribution  
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# The Poisson Distribution

**Poisson: Lambda = 1.0**



Distributions  
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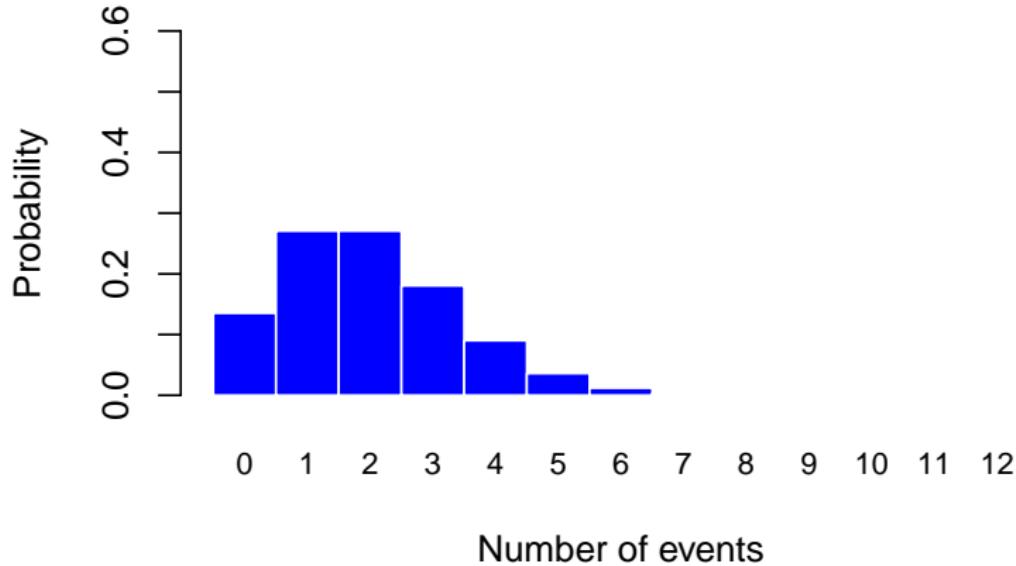
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Poisson Distribution  
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Normal Distribution  
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# The Poisson Distribution

**Poisson: Lambda = 2.0**



Distributions

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Binomial Distribution

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Poisson Distribution

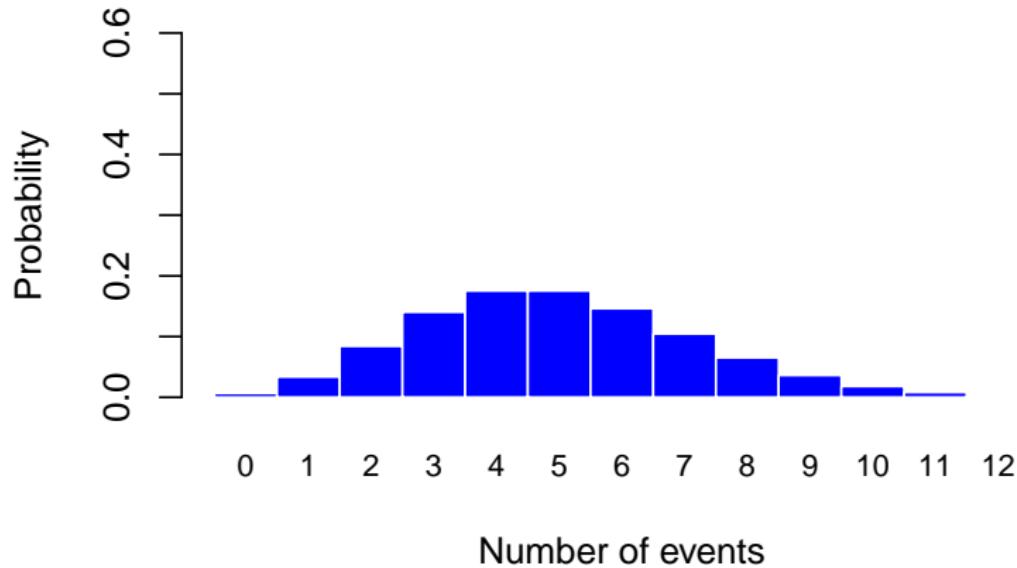
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Normal Distribution

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# The Poisson Distribution

**Poisson: Lambda = 5.0**



## Examples of Poisson Random Variables

- The number of babies delivered in the maternity ward of UIHC on a particular day.
- The number of car accidents that occur at a busy intersection in Iowa City during a certain month.
- The number of calls received by Quitline Iowa (a statewide smoking cessation telephone counseling hotline) during a specific week.

# The Poisson Distribution

- Let  $X$  denote the number of patients that arrive at the UIHC Emergency Treatment Center (ETC) during the early morning hours of a certain day (between midnight and 6:00 a.m.).
  - Assume that on average, 4.5 patients arrive at the UIHC ETC during the early morning hours.
- a) What is the probability that no patients will arrive at UIHC ETC during the given time period?

# The Poisson Distribution

- b) What is the probability that at least one patient will arrive at UIHC ETC during the given time period?
  
- c) What is the probability that four or five patients will arrive at UIHC ETC during the given time period?

# The Poisson Distribution

- Note: The Poisson distribution can be used to approximate probabilities for binomial random variables where  $n$  is large and  $p$  is small.
- In such applications,  $\lambda$  is set equal to the product  $np$ .
- Thus in such situations, if

$$X \sim \text{Bin}(n, p) \text{ then } X \stackrel{\text{approx}}{\sim} \text{Poisson}(np).$$

- An example of such an application will be covered in a lab session.

# The Poisson Distribution

- Suppose  $X_1$  and  $X_2$  are independent Poisson random variables, such that  $X_1 \sim \text{Poisson}(\lambda_1)$  and  $X_2 \sim \text{Poisson}(\lambda_2)$  then

$$Y = X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$$

# Probability Mass Function (Review)

- The probability function of a **discrete random variable**:
  1. defines all possible values of the variable;
  2. displays the probabilities with which the random variable takes on those values;
  3. can sometimes be described using a formula.
- For discrete random variables the function  $p(x)$  is often referred to as the **probability mass function** (or **pmf**).
- For each value of the RV, the pmf gives the probability of that value happening.

Distributions

Binomial Distribution

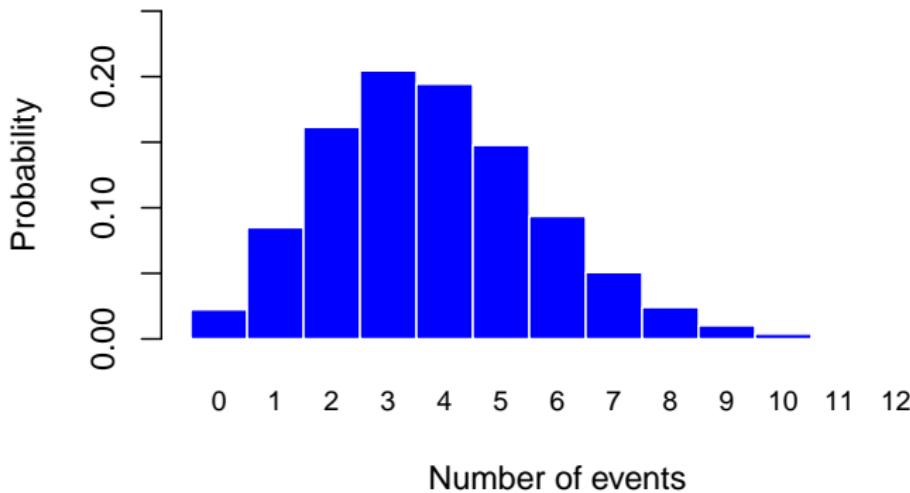
Poisson Distribution

Normal Distribution

## Probability Mass Function: Example

If  $X \sim \text{Poisson}(3.8)$  then  $P(X = k) = \frac{e^{-3.8} 3.8^k}{k!}$  for  $k = 0, 1, 2, \dots$

**Poisson: Lambda = 3.8**



# Probabilities for Continuous Random Variables

- How do we calculate probabilities for continuous random variables?
- What is the probability of any specific value of a continuous random variable?
- What about the probability of being within a specified range of values?

# Probabilities for Continuous Random Variables

- The *Probability Distribution* of a *continuous random variable*  $X$  is an unbroken curve or function  $f(x)$  which is used to assign probabilities to  $X$ .
- Probabilities are assigned in the following manner: the area under the curve over an interval is equal to the probability the random variable will assume a value within that interval.
- $f(x)$  is often called a *Density Curve* or a *probability density function (pdf)*.

# Probability Density Function

- A probability density function of a **continuous random variable**  $X$  is a function which satisfies:
  1. The probability that  $X$  falls between values  $a$  and  $b$ , that is  $P(a \leq X \leq b)$ , is equal to the area under the curve between  $a$  and  $b$ .
  2. The function always takes on values greater than or equal to zero.
  3. The total area under the curve is equal to 1.
- In general, calculation of probabilities for continuous random variables requires calculus.
- For a continuous random variable, the probability of the random variable assuming any specific value is zero.

## Example: Study Hours

- Let  $X$  denote the time spent by a randomly selected college student studying each week.
- $X$  is continuous.
- Suppose that the sample space for  $X$  is given by  $\mathcal{S} = \{0 \leq X \leq 50\}$ .
- Assume that the density curve for  $X$  is as illustrated in the next figure. (The vertical axis is scaled so that the entire area under the curve is equal to 1.)

Distributions  
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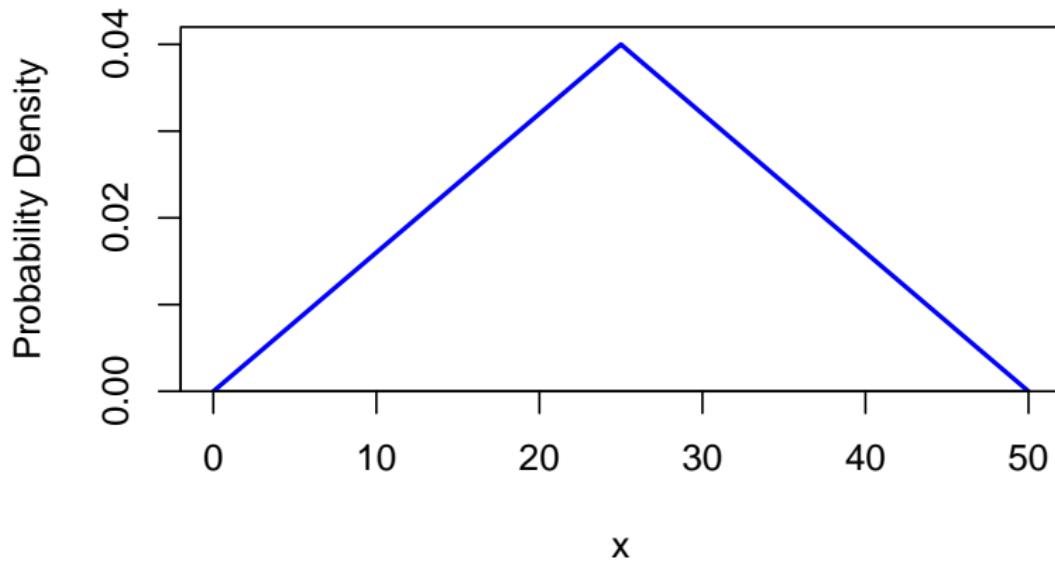
Binomial Distribution  
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Poisson Distribution  
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Normal Distribution  
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## Example: Study Hours

### Probability Density Function of X



Distributions  
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Binomial Distribution  
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Poisson Distribution  
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Normal Distribution  
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## Example: Study Hours

- $P(X \leq 25) =$
- $P(37.5 \leq X \leq 50) =$
- $P(X \leq 37.5) =$
- $P(X = 37.5) =$
- $P(0 \leq X \leq 50) =$

Distributions

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Binomial Distribution

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Poisson Distribution

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Normal Distribution

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## Example: Time to Death from Diagnosis



Distributions

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Binomial Distribution

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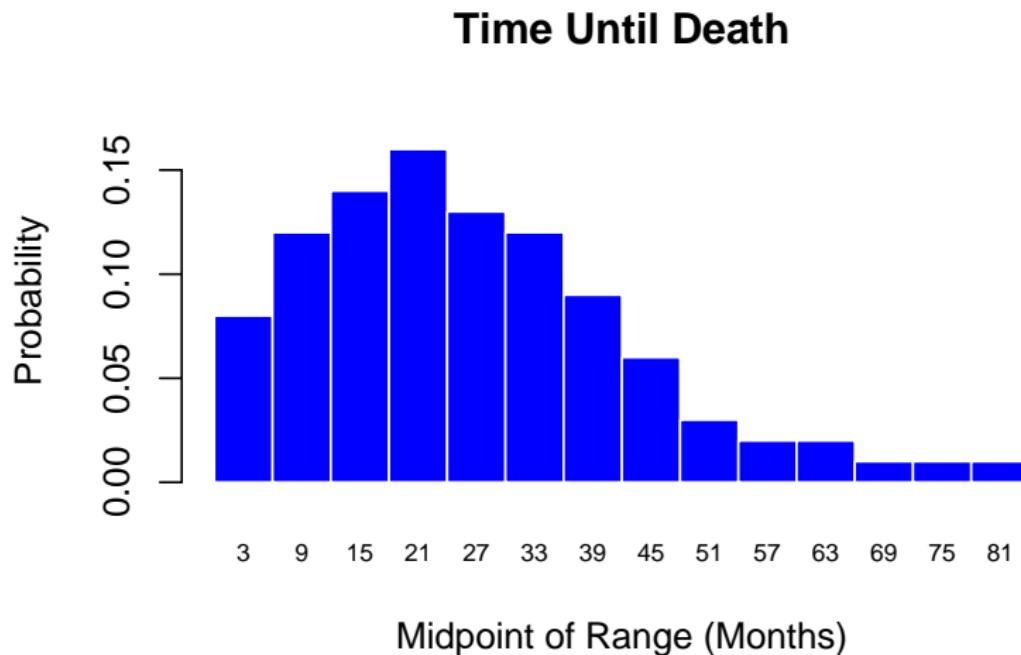
Poisson Distribution

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Normal Distribution

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## Example: Time to Death from Diagnosis



Distributions

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Binomial Distribution

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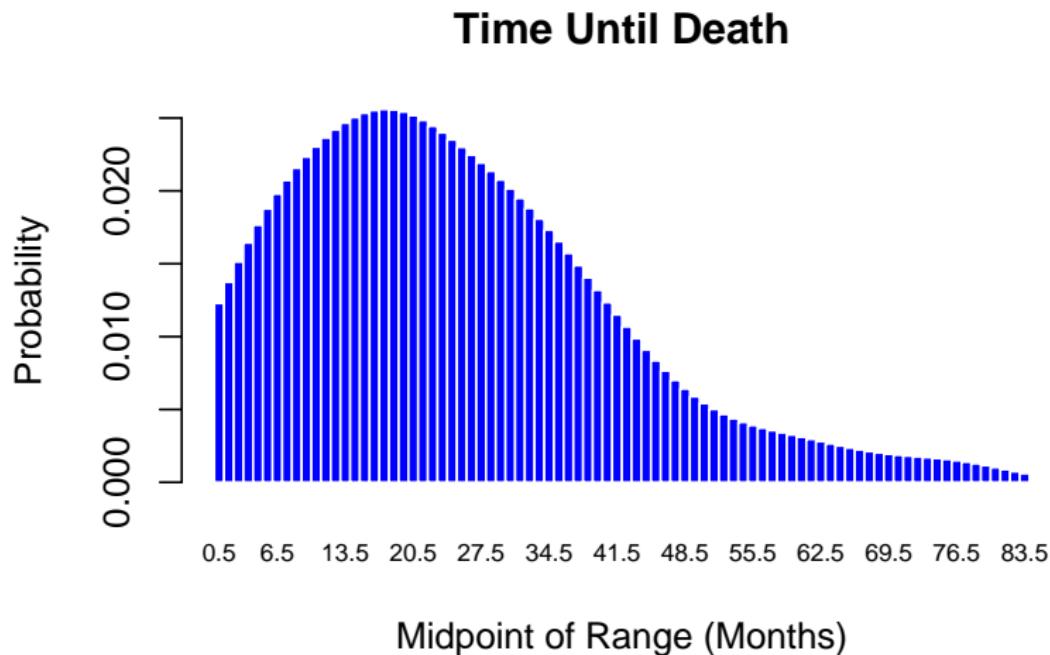
Poisson Distribution

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Normal Distribution

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## Example: Time to Death from Diagnosis



Distributions

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Binomial Distribution

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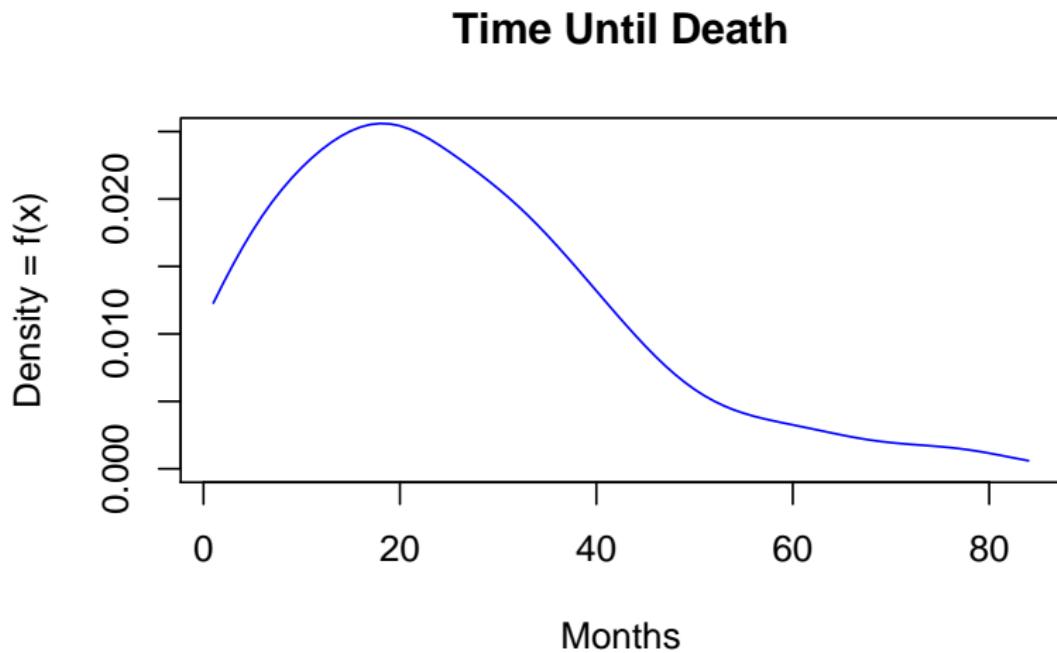
Poisson Distribution

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Normal Distribution

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## Example: Time to Death from Diagnosis



Distributions

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Binomial Distribution

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Poisson Distribution

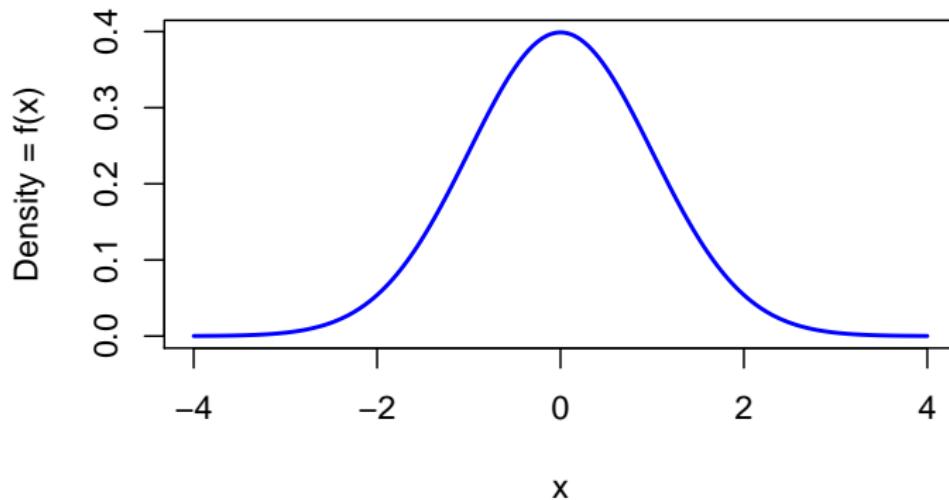
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Normal Distribution

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# The Normal Distribution

## Standard Normal Probability Density Function



- The normal (or Gaussian) distribution is the most common probability distribution for continuous random variables.

# The Normal Distribution

- The density curve for the normal distribution is called the **Normal Curve** or the **Bell Curve**.
- Has wide applicability to many different types of data, and often arises in nature.
- Widely used for theoretical properties of many statistical methods.
- The distribution depends on only two parameters:  
the mean,  $\mu$ ; and  
the standard deviation,  $\sigma$ , (or variance,  $\sigma^2$ ).

# The Normal Distribution

- The formula for the normal density function is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

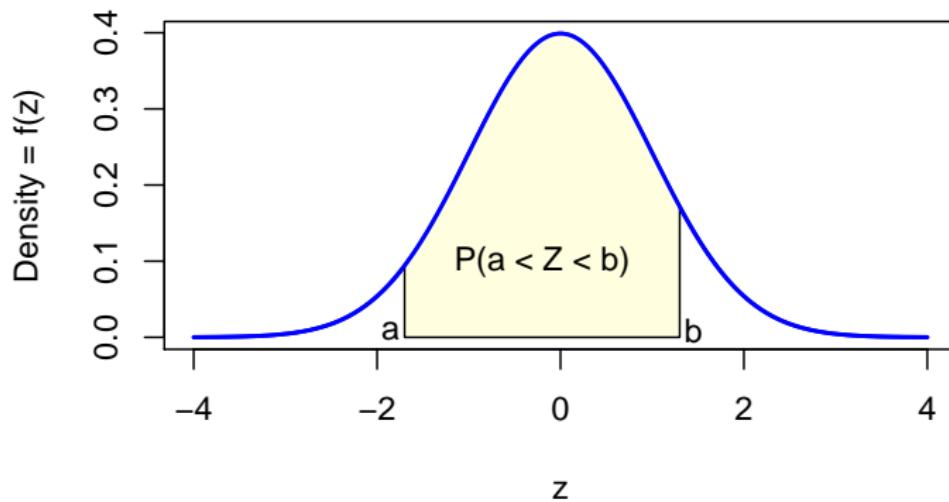
where  $-\infty < x < \infty$ , and  $\pi = 3.14159\dots$

- This is often denoted as  $X \sim N(\mu, \sigma^2)$ .
- Caution: some authors use  $X \sim N(\mu, \sigma)$ .

## The Standard Normal Distribution

- The *Standard Normal Distribution* is a normal distribution with  $\mu = 0$ , and  $\sigma = 1$ .
  - A standard normal random variable is generally denoted as  $Z$ .
  - That is,  $Z \sim N(0, 1)$ .

# Computing Probabilities for Standard Normal Variables



- The area between  $a$  and  $b$  under the standard normal density curve provides the probability that  $Z$  will assume a value over the interval  $(a, b) : P(a < Z < b)$ .

# Computing Probabilities for Standard Normal Variables

- Note the following equivalences:

$$\begin{aligned} P(a < Z < b) &= P(a \leq Z < b) \\ &= P(a < Z \leq b) \\ &= P(a \leq Z \leq b) \end{aligned}$$

- This occurs because  $P(Z = a) = P(Z = b) = 0$ .

Distributions

oooooooooooo

Binomial Distribution

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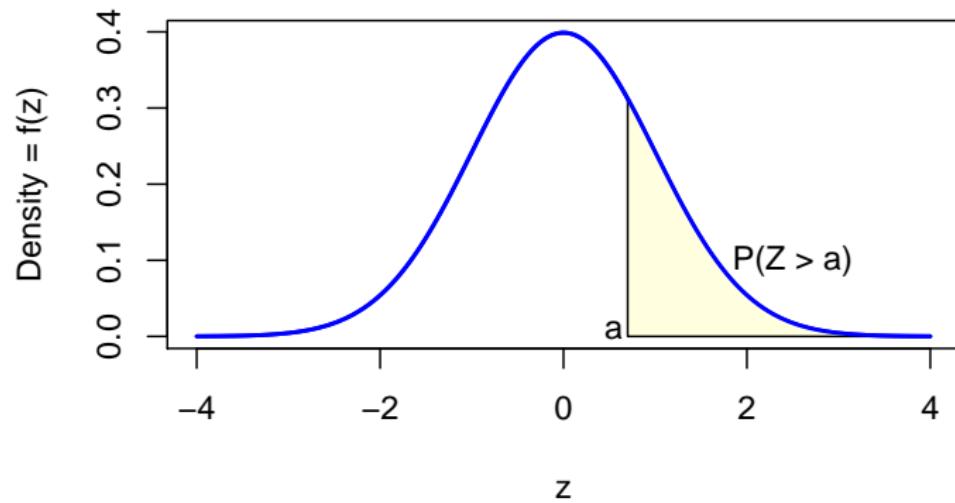
Poisson Distribution

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Normal Distribution

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- Our Normal Table provides upper tail probabilities for the standard normal distribution: i.e.,  $P(Z > a)$  where  $a$  is between 0.00 and 3.49.

# Computing Probabilities for Standard Normal Variables

- For a number  $a$  having the format  $a = w.xy$ , the digits  $w.x$  are looked up in the left-hand column of the table, and the remainder  $0.0y$  is looked up in the top-most row of the table.

$z$ (or $a$ )	0.00	0.01	0.02	...
0.0	0.500	0.496	0.492	...
0.1	0.460	0.456	0.452	...
0.2	0.421	0.417	0.413	...
0.3	0.382	0.378	0.374	...
0.4	0.345	0.341	0.337	...
⋮	⋮	⋮	⋮	⋮

# Computing Probabilities for Standard Normal Variables

Examples:

- $P(Z > 0.21) =$
- $P(Z \geq 0.21) =$
- $P(Z > 0.42) =$
- $P(Z > 0.20) =$
- $P(Z > 0.00) =$

Further examples (using Normal Table)

- $P(Z > 1.96) = 0.025$
- $P(Z > 2.15) = 0.016$
- $P(Z > 3.00) = 0.001$

# Computing Probabilities for Standard Normal Variables

- Areas that do not correspond to upper tail areas can be suitably pieced together using the symmetry of the standard normal curve.
- For instance,

$$P(Z < a) = 1 - P(Z \geq a)$$

$$P(Z < -a) = P(Z > a) \text{ where } a > 0$$

$$P(-a < Z < a) = 1 - 2P(Z \geq a) \text{ where } a > 0$$

Distributions



Binomial Distribution



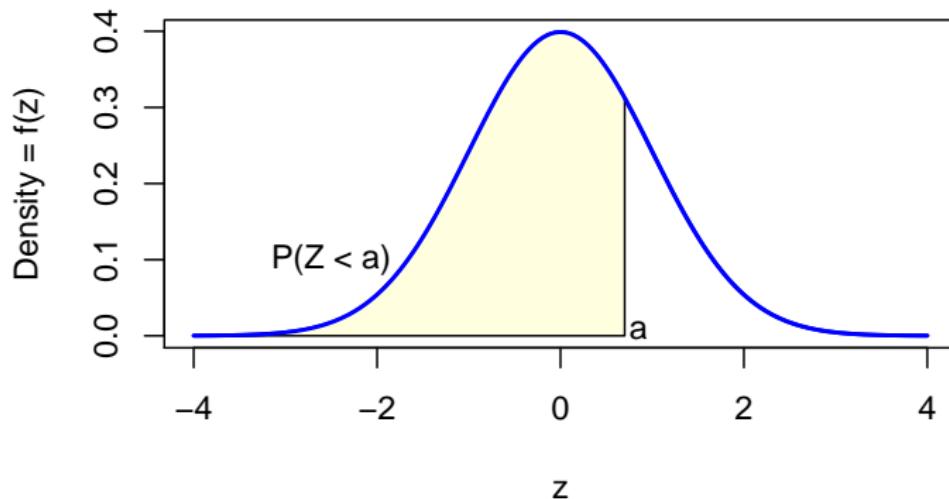
Poisson Distribution



Normal Distribution



## Computing Probabilities for Standard Normal Variables



- $P(Z < a) = 1 - P(Z \geq a)$

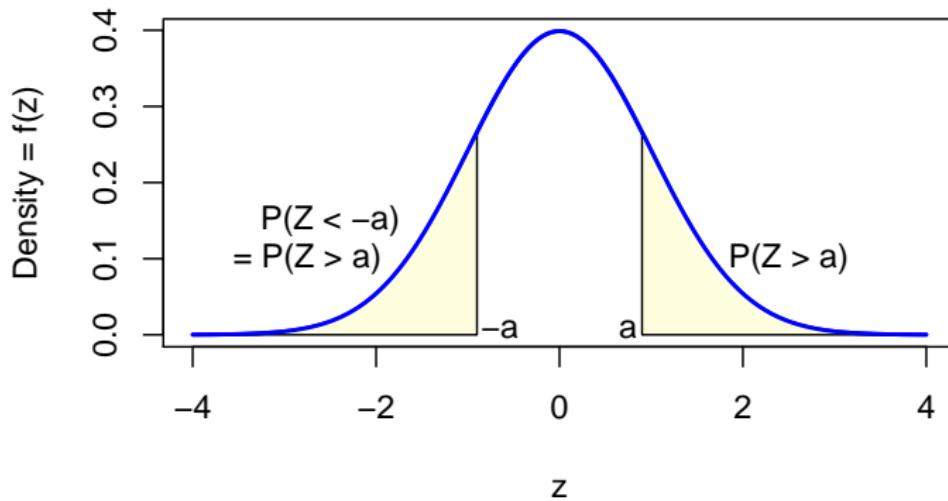
Distributions

Binomial Distribution

Poisson Distribution

Normal Distribution

# Computing Probabilities for Standard Normal Variables



- $P(Z < -a) = P(Z > a)$  where  $a > 0$

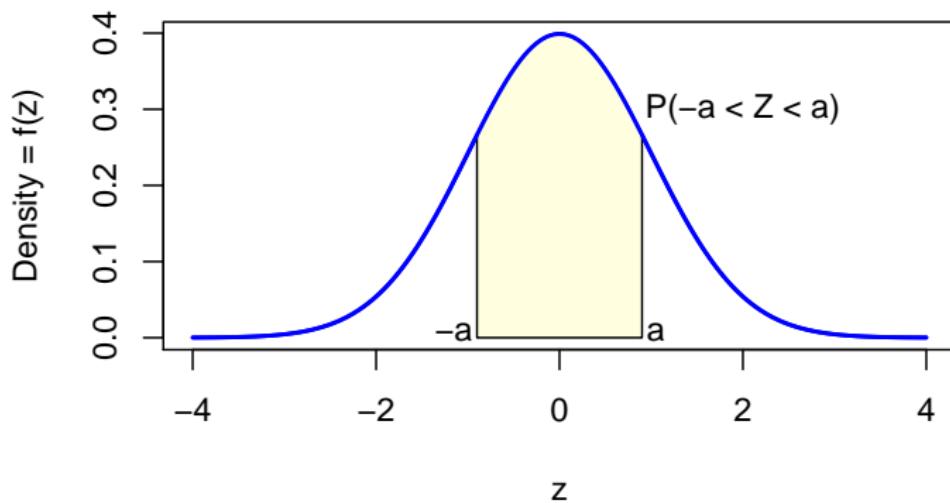
Distributions

Binomial Distribution

Poisson Distribution

Normal Distribution

# Computing Probabilities for Standard Normal Variables



- $$\begin{aligned} P(-a < Z < a) &= 1 - P(Z \leq -a) - P(Z \geq a) \\ &= 1 - 2P(Z \geq a) \text{ where } a > 0 \end{aligned}$$

## Probabilities for Standard Normal Variables: Examples

Let  $Z$  be a standard normal random variable. Find the following probabilities:

a)  $P(Z \leq 1.96)$

b)  $P(-2.00 \leq Z \leq 2.00)$

c)  $P(Z > -1.28)$

# Probabilities for Standard Normal Variables: Examples

d)  $P(-5.13 \leq Z \leq 2.00)$

e)  $P(Z = 1.71)$

# Computing Probabilities for General Normal Variables

- Probabilities of *any* normal distribution can be calculated using standard normal probabilities.
- Let  $X$  be a normal random variable with mean  $\mu$  and standard deviation  $\sigma$ : i.e.,  $X \sim N(\mu, \sigma^2)$ .
- $X$  can be converted to a standard normal random variable via the transformation

$$Z = \frac{X - \mu}{\sigma}$$

# Computing Probabilities for General Normal Variables

- Thus to solve for  $P(X > x)$ , we would transform the probability statement as follows:

$$\begin{aligned} P(X > x) &= P\left(\frac{X - \mu}{\sigma} > \frac{x - \mu}{\sigma}\right) \\ &= P\left(Z > \frac{x - \mu}{\sigma}\right) \\ &= P(Z > a) \end{aligned}$$

where  $a = \frac{x - \mu}{\sigma}$  and  $Z \sim N(0, 1)$ .

## Computing Probabilities for General Normal Variables

- More generally

$$\begin{aligned}
 P(a < X < b) &= P\left(\frac{a - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{b - \mu}{\sigma}\right) \\
 &= P\left(\frac{a - \mu}{\sigma} < Z < \frac{b - \mu}{\sigma}\right) \\
 &= P(c < Z < d)
 \end{aligned}$$

where  $c = \frac{a-\mu}{\sigma}$ ,  $d = \frac{b-\mu}{\sigma}$  and  $Z \sim N(0, 1)$ .

## Probabilities for General Normal Variables: Example

- Let  $X$  denote the pregnancy duration of an expectant female.
  - Medical records indicate that  $X$  has a normal distribution with  $\mu = 266$  days,  $\sigma = 16$  days: i.e.,  $X \sim N(266, 16^2)$ .
- What is the probability an expectant female will have a pregnancy duration lasting at least 310 days?
  - What is the probability an expectant female will have a pregnancy duration lasting between 244 days ( $\approx 8$  months) and 305 days ( $\approx 10$  months)?

## Probabilities for General Normal Variables: Example

- In a certain population, suppose that diastolic blood pressure is normally distributed with a mean of 75 mm Hg and a standard deviation of 10 mm Hg.
- a) What is the probability that a randomly selected individual from this population has a diastolic blood pressure in the range 73–77 mm Hg?

## Probabilities for General Normal Variables: Example

- b) Suppose 5 individuals are randomly selected from this population. What is the probability exactly 3 of these individuals have diastolic blood pressures in the range 73–77 mm Hg?
  
- c) What is the median of the distribution of diastolic blood pressure?

Distributions

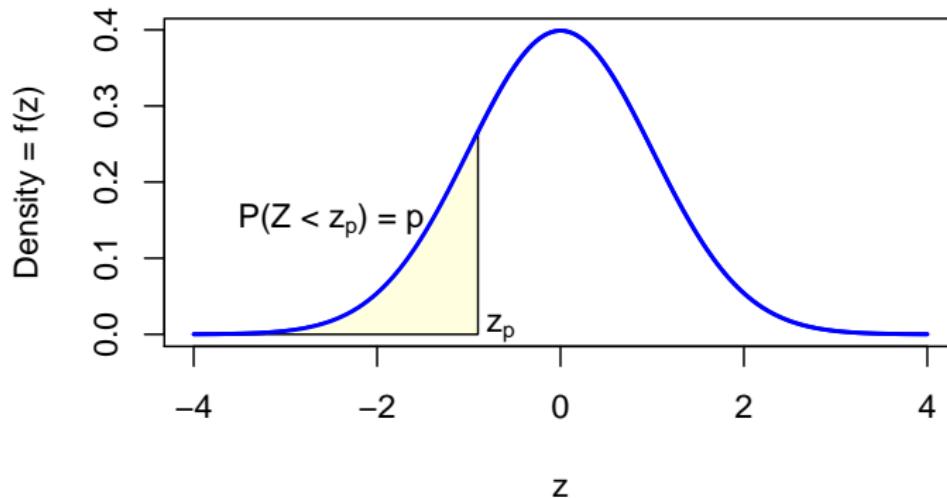
Binomial Distribution

Poisson Distribution

Normal Distribution

# Percentiles of the Standard Normal Distribution

- The  $100\% \times p^{\text{th}}$  percentile of the  $N(0, 1)$  distribution is denoted by  $z_p$ .
- $z_p$  is the value such that  $P(Z \leq z_p) = p$ .





## Percentiles of the Standard Normal Distribution: Examples

- a) Find a value such that 30% of the area under the standard normal curve lies above (to the right of)  $z$ .
  
- b) Find a value such that 2.5% of the area under the standard normal curve lies below (to the left of)  $z$ .

# The Normal Distribution

For the standard normal curve,

- about 68% of the area lies between  $-1$  and  $+1$ ,
- about 95% of the area lies between  $-2$  and  $+2$ ,
- about 99.7% of the area lies between  $-3$  and  $+3$ .

For a general normal curve,

- about 68% of the area lies between  $\mu - \sigma$  and  $\mu + \sigma$ ,
- about 95% of the area lies between  $\mu - 2\sigma$  and  $\mu + 2\sigma$ ,
- about 99.7% of the area lies between  $\mu - 3\sigma$  and  $\mu + 3\sigma$ .

# Learning Objectives (1/2)

At the end of this unit, you should be able to:

- Define what a random variable is and distinguish between discrete and continuous random variables.
- Explain differences between probability mass and density functions and the properties of each.
- Describe situations under which data from either the binomial or Poisson distribution may arise.
- Calculate probabilities from binomial and Poisson distributions; and also the mean, variance, and standard deviation of each.

## Learning Objectives (2/2)

At the end of this unit, you should be able to:

- List the criteria that characterize a Poisson process.
- Explain when the Poisson distribution approximates the binomial distribution.
- Apply the additive property of Poisson distributions.
- Describe the basic properties of the standard normal distribution, and list the common critical values and percentiles.
- Calculate standardized values and calculate probabilities from the general normal distribution.