BIOS:4120 – Introduction to Biostatistics
Unit 4: Probability (Introduction)

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Learning Objectives

At the end of this session, you should be able to:

- Describe and use the three basic operations on events and probability, (intersection, union, and complement).
- Construct a Venn diagram.
- List the properties of probabilities, e.g. $0 \leq P(A) \leq 1$.
- State the additive rule of probability, the probability of a general union, and the multiplicative rule of probability; and use these properties.
- Determine if events are mutually exclusive or independent.
- Define and calculate conditional probabilities.

Overview

- Introduction to Probability
- Operations on Events and Probability
- Conditional Probability

Introduction to Probability

What is Probability?

Introduction to Probability

Today we will discuss four different types of probability:

1) Theoretical Probability
2) Relative Frequency
3) Personal or Subjective Probability
4) Due to Ignorance

Introduction to Probability

Theoretical Probability

- For theoretical reasons we assume that we know the relative likelihood of all possible outcomes of a particular experiment, and assign probabilities accordingly.
- Classical Rule for Assigning Probabilities:
  If a sample space consists of b equally likely outcomes and the event A consists of a of these outcomes, the probability of A is given by $P(A) = \frac{a}{b}$.

Introduction to Probability

Relative Frequency

- The probability of a particular outcome happening when a process is carried out is equal to the proportion of times that outcome will occur when the process is repeated over and over again, under the same circumstances.
- If a random experiment is repeated n times under the same circumstances, and the event A occurs m times, the probability of A is estimated as $P(A) \approx m/n$.

Introduction to Probability

Relative Frequency

- The probability of a particular outcome happening when a process is carried out is equal to the proportion of times that outcome will occur when the process is repeated over and over again, under the same circumstances.
- If a random experiment is repeated n times under the same circumstances, and the event A occurs m times, the probability of A is estimated as $P(A) \approx m/n$.
  - The outcome of one trial does not influence the outcome of any other trial.
  - The trials are identical.
  - The probability of a specified outcome is the limit of its relative frequency of occurrence as the number of trials becomes infinitely large.

Introduction to Probability

Personal or Subjective Belief Probability

- The probability associated with an outcome measures one's subjective belief as to the likelihood of the occurrence of the outcome.
- These are values (between 0 and 1, or 0 and 100%) assigned by individuals based on how likely they think events are to occur.

Introduction to Probability

Due to Ignorance

- The outcome has already occurred but the is not known.
- Any of the previous three descriptions of probability may be used.

Introduction to Probability

What type of probability is each of the following?

- The probability of drawing a queen from a standard deck of 52 cards?
- The probability that the coin I tossed this morning and placed on my desk is showing heads?
- The probability of rolling a four from a weighted six-sided dice?
- The probability that the Hawkeyes will win their next basketball game?

Definitions

- A *Random Experiment* (*Random Phenomenon*) refers to any act or process that results in an outcome that cannot be predicted with certainty.
- The *Sample Space* of a random experiment is the set of all its possible outcomes. (Often denoted by $\mathcal{S}$.)
- An *Event* is a collection of outcomes for a random experiment; i.e., a subset of the sample space.

Introduction to Probability

- Example: Suppose we toss a fair coin three times.

  $\mathcal{S} = \{$

- Possible events of interest:

  $A =$ obtaining exactly two heads (in any order)
  $= \{$

  $B =$ obtaining heads on the first toss
  $= \{$

  $C =$ obtaining all heads
  $= \{$

Operations on Events and Probability

- *Intersection*:
  For two events A and B, the intersection $A \cap B$ represents the event that both A and B occur.

- *Union*:
  For two events A and B, the union $A \cup B$ represents the event that A or B occurs: i.e., A occurs without B, B occurs without A, or A and B both occur.

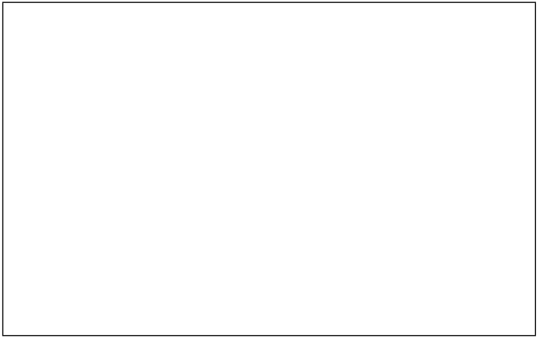
- *Complement*:
  For an event A, the complement of A represents the event that occurs if A does not occur. It is typically denoted A^C or $\overline{A}$.

Operations on Events and Probability

- Example: In the previous coin tossing example,
- $A =$ obtaining exactly two heads (in any order) $= \{HHT, HTH, THH\}$
- $B =$ obtaining heads on the first toss $= \{HHH, HHT, HTH, HTT\}$
- $A \cap B = \{HHT, HTH\}$ = obtaining exactly two heads and obtaining heads on the first toss
- $A \cup B = \{HHT, HTH, THH, HHH, HTT\}$ = obtaining exactly two heads or obtaining heads on the first toss
- $B^C = \{THH, THT, TTH, TTT\}$ = obtaining tails on the first toss

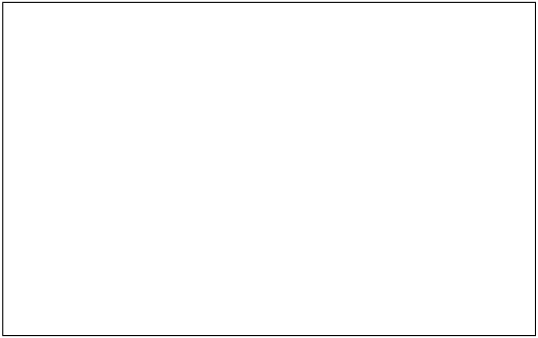
Venn Diagrams

- $A \cap B$



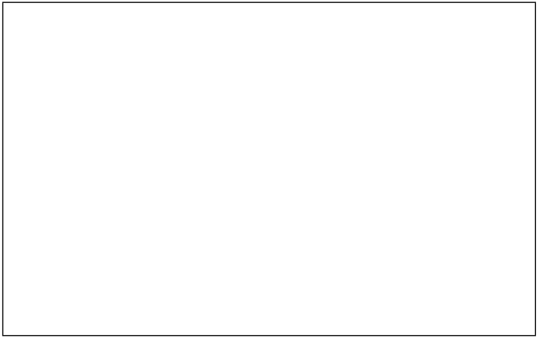
Venn Diagrams

- $A \cup B$



Venn Diagrams

- A^C or $\overline{A}$



Properties of Probability

- For any event A, $0 \leq P(A) \leq 1$
- An event which cannot occur has probability 0. Such an event is often called the *Null Event*, and is denoted $\varnothing$.
- The sum of the probabilities for all of the outcomes in the sample space $\mathcal{S}$ must equal 1.
- For any event A, $P(A)$ is the sum of the probabilities for all of the outcomes which comprise A.
- For any event A, $P(A^C) = 1 - P(A)$.
- Two events A and B are said to be *Mutually Exclusive* if they both cannot occur: i.e., if

$$A \cap B = \varnothing \text{ or } P(A \cap B) = 0.$$

Additive Rule of Probability

- If events A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

Probability of a General Union

- For any two events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Additive Rule of Probability

- If $A_1, A_2, \ldots, A_k$ represent k mutually exclusive events, then,

$$P(A_1 \cup A_2 \cup \cdots \cup A_k) = P(A_1) + P(A_2) + \cdots + P(A_k)$$

Example:

- Human blood can be 'ABO-typed' as one of O, A, B, or AB. The distribution of the types varies somewhat among different populations. For a randomly chosen person in the United States, the probabilities for the various blood types are as follows.

Blood type	O	A	B	AB
U.S. probability	0.45	0.40	0.11	

a) What is the probability of a randomly selected American having type AB blood?

b) Sloane has type B blood. She can safely receive blood transfusions from people with blood types O and B. What is the probability that a randomly chosen American can donate blood to Sloane?

Conditional Probability

- A *Conditional Probability* refers to the probability of one event occurring given that another event has already taken place.
- For events A and B, the conditional probability that B will occur given that A has already taken place is denoted by $P(B|A)$.

Conditional Probability

- Two events A and B are said to be *Independent* if the occurrence of one event does not alter the probability assignment for the occurrence of the other: i.e., if

$$P(B|A) = P(B) \quad \text{or} \quad P(A|B) = P(A).$$

- The preceding relations are equivalent: $P(B|A) = P(B)$ holds if and only if $P(A|B) = P(A)$ holds.

- If two events are not independent, they are said to be *Dependent*.

Multiplicative Rule of Probability

- If events A and B are independent, then

$$P(A \cap B) = P(A)\,P(B).$$

- If $A_1, A_2, \ldots, A_k$ represent k independent events, then,

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)\,P(A_2) \cdots P(A_k).$$

Multiplicative Rule of Probability

- For any two events A and B,

$$P(A \cap B) = P(A|B)\,P(B) = P(B|A)\,P(A).$$

- From the preceding relations, we can write

$$P(B|A) = \frac{P(A \cap B)}{P(A)},$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Example

- Quinn and Blake share an apartment together. Suppose that the probability that Quinn contracts the flu (event Q) is 0.25, the probability that Blake contracts the flu (event B) is 0.20, and the probability that both Quinn and Blake contract the flu (Q and B) is 0.15.

a) Is it plausible to assume that the events Q (Quinn has flu) and B (Blake has flu) are independent? Why or why not?

b) Check whether Q and B are independent using the multiplicative rule.

Example

c) What is the probability that Quinn contracts the flu or Blake contracts the flu (or both)?

d) Construct a Venn diagram to illustrate the relationship between Q and B.

Example

e) Given that Blake contracts the flu, what is the probability that Quinn contracts the flu?

f) Given that Quinn contracts the flu, what is the probability that Blake contracts the flu?

Example

- Suppose the prevalence of cigarette smoking among retirees in Windham county is 20%, and the prevalence of high blood pressure is 15%.
- If the person is a cigarette smoker, suppose that the probability they have high blood pressure is 0.30.
- For parts (a) and (b), consider a randomly selected retiree from this county.

Example

a) What is the probability the person will both smoke and have high blood pressure?

b) What is the probability the person will smoke or have high blood pressure (or both)?

Example

For parts (c), (d), and (e), consider *two* randomly selected retirees from this population, chosen independently.

c) What is the probability that both are smokers?

d) What is the probability that neither is a smoker?

e) What is the probability that at least one of the two is a smoker?

Example

For part (f), consider *six* randomly selected retirees from this population, chosen independently.

f) What is the probability that all of the six people will be smokers?

Example (2 x 2 table)

- The following table from September 2021 presents characteristics of patients hospitalized in Iowa due to COVID-19.

	Fully Vaccinated?		
	Yes	No	
In ICU?	FV	FV^C	Total
Yes (ICU)	13	124	137
No (ICU^C)	82	320	402
Total	95	444	539

Example (2 x 2 table)

Consider a randomly selected Iowa COVID-19 hospital patient:

a) What is the probability the patient is in the ICU?

b) What is the probability the patient is fully vaccinated?

Example (2 x 2 table)

c) Given the patient is in the ICU, what is the probability they are not fully vaccinated?

d) Given the patient is fully vaccinated, what is the probability that they are not in the ICU?

e) What is the probability that the patient is not fully vaccinated and is not in the ICU?

Example (2 x 2 table)

f) What is the probability that the patient is either in the ICU or not fully vaccinated?

g) What is the probability that the patient is neither in the ICU nor not fully vaccinated?

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