

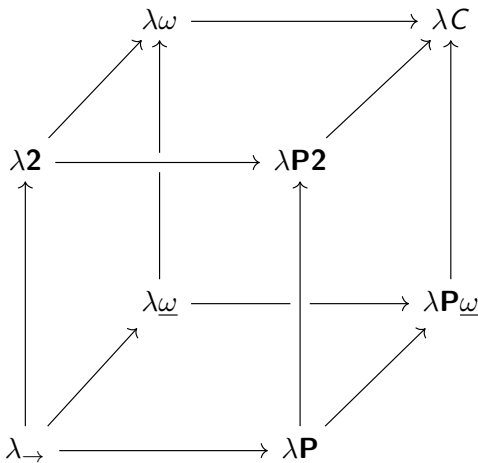
# A general Subject Reduction for Pure Type Systems

COMP527 – Logic and Computation

Charlotte Marchal, Dashiell Rich, Milton Rosenbaum, Zhaoshen Zhai

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# The Cube



# Lots of Systems

In this diagram, eight systems are defined. The arrow represents  $\subseteq$ , so the weakest system is the simply typed lambda calculus, at the bottom left. Other named systems are  $\lambda 2$ , which is equivalent to system **F**, which is the polymorphic lambda calculus.  $\lambda\omega$  is equivalent to  $F\omega$ , which was proposed by Girard,  $\lambda P$  is equivalent to the "automath" language, and is sometimes called  $LF$ .

# Kinds of Dependencies

$(\star, \star)$	Terms can depend on terms
$(\square, \star)$	Terms can depend on types
$(\star, \square)$	Types can depend on terms
$(\square, \square)$	Types can depend on types

# Parameterized & Homogeneous

$\lambda \rightarrow$	$(\star, \star)$			
$\lambda 2$	$(\star, \star)$	$(\square, \star)$		
$\lambda \underline{\omega}$	$(\star, \star)$		$(\square, \square)$	
$\lambda \omega$	$(\star, \star)$	$(\square, \star)$	$(\square, \square)$	
$\lambda \mathbf{P}$	$(\star, \star)$			$(\star, \square)$
$\lambda \mathbf{P}2$	$(\star, \star)$	$(\square, \star)$		$(\star, \square)$
$\lambda \mathbf{P}\underline{\omega}$	$(\star, \star)$		$(\square, \square)$	$(\star, \square)$
$\lambda \mathbf{P}\omega$	$(\star, \star)$	$(\square, \star)$	$(\square, \square)$	$(\star, \square)$

# Just change two two rules!

$$\Pi\text{-rule} \quad \frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash (\Pi x : A. B) : s_3}$$

$$\lambda\text{-rule} \quad \frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash b : B \quad \Gamma, x : A \vdash B : s_2}{\Gamma(\lambda x : A. b) : (\Pi x : A. B)}$$

# What are they

In addition to  $\square$  and  $\star$ , we can add arbitrary sorts and axiom relations. A *pure type system*  $\lambda(S, A, R)$  can be described as

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We abbreviate 'pure type system' by PTS.

# Even more systems

A surprising number of systems can be defined as PTSs:

- $\lambda C' = (\star^t, \star^p, \square), (\star^t : \square, \star^p : \square), (S^2, \text{ that is, all pairs})$

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- $\lambda U = [\star, \square, \Delta], [\star : \square, \square : \Delta], [(\star, \star), (\square, \star), (\square, \square), (\Delta, \square), (\Delta, \star)]$

# Mechanize Proof for Each PTS

Since so many systems are easily describable as PTSs, it would be nice to prove some properties about the systems in general.

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In our project, we plan to:

- Prove Subject Reduction for arbitrary PTSs.
- Mechanize arbitrary PTSs in Beluga.

# Tasks

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- Everyone else: Prove Subject Reduction for PTSs.
- Zhaoshen: Write up the extended abstract.

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- We need to first understand the proofs for System F and  $\lambda\mathbf{P}$ .