A general Subject Reduction for Pure Type Systems

COMP527 – Logic and Computation

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The Cube

$$\begin{tikzcd}
 & \lambda\omega \arrow[rr] & & \lambda C \\
\lambda 2 \arrow[ur] \arrow[rr] & & \lambda \mathbf{P2} \arrow[ur] & \\
 & \lambda\underline{\omega} \arrow[uu] \arrow[rr] & & \lambda \mathbf{P}\underline{\omega} \arrow[uu] \\
\lambda_{\to} \arrow[uu] \arrow[ur] \arrow[rr] & & \lambda \mathbf{P} \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

Lots of Systems

In this diagram, eight systems are defined. The arrow represents $\subseteq$, so the weakest system is the simply typed lambda calculus, at the bottom left. Other named systems are $\lambda 2$, which is equivalent to system **F**, which is the polymorphic lambda calculus. $\lambda\omega$ is equivalent to $F\omega$, which was proposed by Girard, λP is equivalent to the "automath" language, and is sometimes called LF.

Kinds of Dependencies

$(\star, \star)$	Terms can depend on terms
$(\Box, \star)$	Terms can depend on types
$(\star, \Box)$	Types can depend on terms
$(\Box, \Box)$	Types can depend on types

Parameterized & Homogeneous

$\lambda\rightarrow$	$(\star,\star)$			
$\lambda\mathbf{2}$	$(\star,\star)$	$(\square,\star)$		
$\lambda\underline{\omega}$	$(\star,\star)$		$(\square,\square)$	
$\lambda\omega$	$(\star,\star)$	$(\square,\star)$	$(\square,\square)$	
$\lambda\mathbf{P}$	$(\star,\star)$			$(\star,\square)$
$\lambda\mathbf{P2}$	$(\star,\star)$	$(\square,\star)$		$(\star,\square)$
$\lambda\mathbf{P}\underline{\omega}$	$(\star,\star)$		$(\square,\square)$	$(\star,\square)$
$\lambda\mathbf{P}\omega$	$(\star,\star)$	$(\square,\star)$	$(\square,\square)$	$(\star,\square)$

Just change two two rules!

$$\Pi{-}rule \qquad \frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash (\Pi x : A.B) : s_3}$$

$$\lambda{-}rule \qquad \frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash b : B \quad \Gamma, x : A \vdash B : s_2}{\Gamma(\lambda x : A.b) : (\Pi x : A.B)}$$

What are they

In addition to $\Box$ and $\star$, we can add arbitrary sorts and axiom relations. A *pure type system* $\lambda(S, A, R)$ can be described as

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We abbreviate 'pure type system' by PTS.

Even more systems

A surprising number of systems can be defined as PTSs:

- $\lambda C' = (\star^t, \star^p, \square), (\star^t : \square, \star^p : \square), (S^2,$ that is, all pairs)

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- $\lambda C' = (\star^t, \star^p, \Box), (\star^t : \Box, \star^p : \Box), (S^2,$ that is, all pairs)
- $\lambda U = [\star, \Box, \Delta], [\star : \Box, \Box : \Delta], [(\star, \star), (\Box, \star), (\Box, \Box), (\Delta, \Box), (\Delta, \star)]$

Mechanize Proof for Each PTS

Since so many systems are easily describable as PTSs, it would be nice to prove some properties about the systems in general.

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- Prove Subject Reduction for arbitrary PTSs.
- Mechanize arbitrary PTSs in Beluga.

Tasks

- Milton: Implementing PTSs in Beluga and mechanizing the proof of Subject Reduction.

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- Everyone else: Prove Subject Reduction for PTSs on paper.
- Zhaoshen: Write up the extended abstract.

Obstacles

- Programming in Beluga is *hard*.

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- We need to first understand the proofs for System F and $\lambda\mathbf{P}$.