

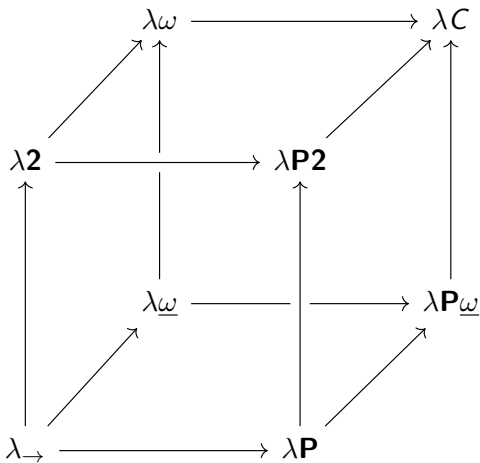
A general Subject Reduction for Pure Type Systems

COMP527 – Logic and Computation

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The Cube



Lots of Systems

In this diagram, eight systems are defined. The arrow represents \subseteq , so the weakest system is the simply typed lambda calculus, at the bottom left. Other named systems are $\lambda 2$, which is equivalent to system **F**, which is the polymorphic lambda calculus. $\lambda\omega$ is equivalent to $F\omega$, which was proposed by Girard, λP is equivalent to the "automath" language, and is sometimes called LF .

Kinds of Dependencies

(\star, \star)	Terms can depend on terms
(\square, \star)	Terms can depend on types
(\star, \square)	Types can depend on terms
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Parameterized & Homogeneous

$\lambda \rightarrow$	(\star, \star)			
$\lambda 2$	(\star, \star)	(\square, \star)		
$\lambda \underline{\omega}$	(\star, \star)		(\square, \square)	
$\lambda \omega$	(\star, \star)	(\square, \star)	(\square, \square)	
$\lambda \mathbf{P}$	(\star, \star)			(\star, \square)
$\lambda \mathbf{P}2$	(\star, \star)	(\square, \star)		(\star, \square)
$\lambda \mathbf{P}\underline{\omega}$	(\star, \star)		(\square, \square)	(\star, \square)
$\lambda \mathbf{P}\omega$	(\star, \star)	(\square, \star)	(\square, \square)	(\star, \square)

Just change two two rules!

$$\Pi\text{-rule} \quad \frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash (\Pi x : A. B) : s_3}$$

$$\lambda\text{-rule} \quad \frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash b : B \quad \Gamma, x : A \vdash B : s_2}{\Gamma(\lambda x : A. b) : (\Pi x : A. B)}$$

What are they

In addition to \square and \star , we can add arbitrary sorts and axiom relations. A *pure type system* $\lambda(S, A, R)$ can be described as

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We abbreviate 'pure type system' by PTS.

Even more systems

A surprising number of systems can be defined as PTSs:

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- $\lambda U = [\star, \square, \Delta], [\star : \square, \square : \Delta], [(\star, \star), (\square, \star), (\square, \square), (\Delta, \square), (\Delta, \star)]$

Mechanize Proof for Each PTS

Since so many systems are easily describable as PTSs, it would be nice to prove some properties about the systems in general.

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- Prove Subject Reduction for arbitrary PTSs.
- Mechanize arbitrary PTSs in Beluga.

Tasks

- Milton: Implementing PTSs in Beluga and mechanizing the proof of Subject Reduction.

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- Zhaoshen: Write up the extended abstract.

Obstacles

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- We need to first understand the proofs for System F and $\lambda\mathbf{P}$.

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