

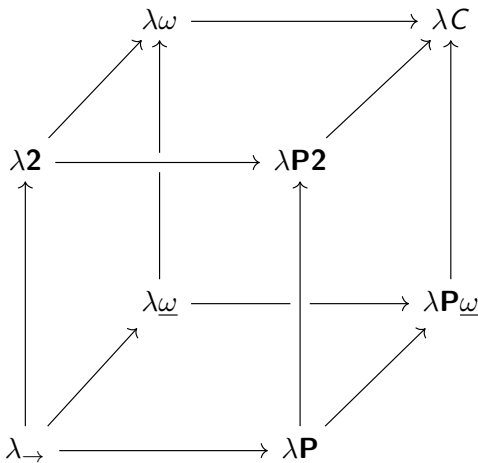
A general Subject Reduction for Pure Type Systems

COMP527 – Logic and Computation

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The Cube



Lots of Systems

In this diagram, eight systems are defined. The arrow represents \subseteq , so the weakest system is the simply typed lambda calculus, at the bottom left. Other named systems are $\lambda 2$, which is equivalent to system **F**, which is the polymorphic lambda calculus. $\lambda\omega$ is equivalent to $F\omega$, which was proposed by Girard, λP is equivalent to the "automath" language, and is sometimes called LF .

Kinds of Dependencies

(\star, \star)	Terms can depend on terms
(\square, \star)	Terms can depend on types
(\star, \square)	Types can depend on terms
(\square, \square)	Types can depend on types

Parameterized & Homogeneous

$\lambda \rightarrow$	(\star, \star)			
$\lambda 2$	(\star, \star)	(\square, \star)		
$\lambda \underline{\omega}$	(\star, \star)		(\square, \square)	
$\lambda \omega$	(\star, \star)	(\square, \star)	(\square, \square)	
$\lambda \mathbf{P}$	(\star, \star)			(\star, \square)
$\lambda \mathbf{P}2$	(\star, \star)	(\square, \star)		(\star, \square)
$\lambda \mathbf{P}\underline{\omega}$	(\star, \star)		(\square, \square)	(\star, \square)
$\lambda \mathbf{P}\omega$	(\star, \star)	(\square, \star)	(\square, \square)	(\star, \square)

Just change two two rules!

$$\Pi\text{-rule} \quad \frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash (\Pi x : A. B) : s_3}$$

$$\lambda\text{-rule} \quad \frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash b : B \quad \Gamma, x : A \vdash B : s_2}{\Gamma(\lambda x : A. b) : (\Pi x : A. B)}$$

What are they

In addition to \square and \star , we can add arbitrary sorts and axiom relations. A *pure type system* $\lambda(S, A, R)$ can be described as

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We abbreviate ‘pure type system’ by PTS.

Even more systems

A surprising number of systems can be defined as PTSs:

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- $\lambda C' = (\star^t, \star^p, \square), (\star^t : \square, \star^p : \square), (S^2, \text{ that is, all pairs})$
- $\lambda U = [\star, \square, \Delta], [\star : \square, \square : \Delta], [(\star, \star), (\square, \star), (\square, \square), (\Delta, \square), (\Delta, \star)]$

Mechanize Proof for Each PTS

Since so many systems are easily describable as PTSs, it would be nice to prove some properties about the systems in general.

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- Prove Subject Reduction for arbitrary PTSs.
- Mechanize arbitrary PTSs in Beluga.

Tasks

- Milton: Implementing PTSs in Beluga and mechanizing the proof of Subject Reduction.

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- Everyone else: Prove Subject Reduction for PTSs on paper.
- Zhaoshen: Write up the extended abstract.

Obstacles

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- Programming in Beluga is *hard*. Like, really hard.
- We need to first understand the proofs for System F and $\lambda\mathbf{P}$.