

SUBJECT REDUCTION FOR PURE TYPE SYSTEMS

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ABSTRACT. Following [GN91] and [Bar91], we study the basics of *pure type systems*, which abstract many of the constructs found in the eight systems of the λ -cube. We start with a brief introduction to the systems of the λ -cube, discuss their expressive power, and introduce pure type systems as a unifying framework in which they can be studied. We then give a detailed proof of subject reduction for arbitrary pure type systems.

Subject reduction is a crucial property of a type system that guarantees its ‘computational consistency’ by ensuring that reductions of a well-typed expression remains well-typed, and also supports the slogan that ‘well-typed programs do not go wrong’. It is thus desirable that it can be stated and proven uniformly across many different type systems, and this is the goal of the present note.

To this end, we follow [Bar91] and work within the framework of *pure type systems*, which is an abstraction of type systems based on the idea of ‘dependencies’ between terms and types. In the simply-typed λ -calculus λ_{\rightarrow} , terms depend on terms: Fx is a term depending on the term x , and we can abstract this dependency to a function $\lambda x:\alpha_1.(Fx)$ of type $\alpha_1 \rightarrow \alpha_2$, where $Fx:\alpha_2$. The other three ways that terms and types can be mutually dependent are present in other type systems, all extending λ_{\rightarrow} .

- In the polymorphic λ -calculus λ_2 , the term $I_\alpha := \lambda x:\alpha.x$ depends on the type α , and this abstracts to $\lambda\alpha:*.I_\alpha$ of type $\forall\alpha:*.(\alpha \rightarrow \alpha)$. Here, ‘ $\alpha:*$ ’ formalizes ‘ α is a type’ within λ_2 .
- In the (weak) higher-order λ -calculus $\lambda\omega$, the type $\alpha \rightarrow \alpha$ depends on the type α , and this abstracts to a *constructor* $\lambda\alpha:*.(\alpha \rightarrow \alpha)$ of *kind* $* \rightarrow *$. Similarly, in the λ -calculus λP of dependent types, the type $\alpha_1^n \rightarrow \alpha_2$ depends on the term n , and abstracts to a constructor $\lambda n:\mathbb{N}.(\alpha_1^n \rightarrow \alpha_2)$ of kind $\mathbb{N} \rightarrow *$.

Imposing that the type systems that we care about all extend λ_{\rightarrow} , and hence terms can depend on terms, we obtain a total of $8 = 2^3$ type systems with all possible combinations of dependencies, called the λ -cube:

Type system		Dependencies			
λ_{\rightarrow}		$(*, *)$			
λ_2		$(*, *)$	$(\square, *)$		
$\lambda\omega$		$(*, *)$		(\square, \square)	
$\lambda\omega = \lambda_2\omega$		$(*, *)$	$(\square, *)$	(\square, \square)	
λP		$(*, *)$			$(*, \square)$
λP_2		$(*, *)$	$(\square, *)$		$(*, \square)$
$\lambda P\omega$		$(*, *)$		(\square, \square)	$(*, \square)$
$\lambda C = \lambda P\omega$		$(*, *)$	$(\square, *)$	(\square, \square)	$(*, \square)$

The systems $\lambda P\omega$, λP_2 , and $\lambda\omega := \lambda_2\omega$ have three kinds of dependencies, while the strongest system of them all, the *calculus of constructions* $\lambda C := \lambda P\omega$, have all four kinds of dependencies.

To explain the table on the right, we observe due to the mutual dependencies between terms and types, it is no longer natural to separate their definitions. Instead, we propose a uniform object, called a *pseudoterm*, and consider the judgement $M:N$ between pseudoterms M and N . These pseudoterms include terms, types, and kinds, so this begs the question: what is $*$? If $*:*$, then one might encounter a Russell-like paradox of the ‘type of all types’, so instead, we introduce a new symbol \square , the ‘sort of all kinds’, and assert that $*:\square$. Similarly, $(* \rightarrow *):\square$ and $(\mathbb{N} \rightarrow *):\square$, so sorts also capture the identity of constructors.

Definition 1. A *pure type system* is a tuple $\sigma := (\mathcal{C}, \mathcal{V}, \mathcal{S}, \mathcal{A}, \mathcal{R})$ consisting of a set \mathcal{C} of *constants*, a set \mathcal{V} of *variables*, a set $\mathcal{S} \subseteq \mathcal{C}$ of *sorts*, a set $\mathcal{A} \subseteq \mathcal{C}^2$ of *axioms*, and a set $\mathcal{R} \subseteq \mathcal{S}^3$ of *rules*.

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Notation 2. Throughout, let σ be denote an arbitrary pure type system.

TODO: link this with the λ -cube by interpreting λ_{\rightarrow} as a PTS.

Definition 3. The collection of σ -pseudoterms is defined by $T := \mathcal{V} \mid \mathcal{C} \mid (TT) \mid (\lambda\mathcal{V}:T.T) \mid (\Pi\mathcal{V}:T.T)$. Pairs $(A, B) \in T^2$ are called σ -assignments, written $A:B$, and a finite sequence thereof is called a σ -pseudocontext.

Definition 4. The β -reduction relation is the least relation on σ -terms satisfying the following for all σ -terms A, A', A'' : the *principal reduction rule* $(\lambda x:A.A')A'' \rightarrow_{\beta} A'[A''/x]$, and the *congruence rules* $A A' \succ A A''$, $A' A \succ A'' A$, $\lambda x:A.A' \succ \lambda x:A.A''$, $\lambda x:A'.A \succ \lambda x:A''.A$, $\Pi x:A.A' \succ \Pi x:A.A''$, and $\Pi x:A'.A \succ \Pi x:A''.A$.

Notation 5. We write \rightarrow_{β} for the reflexive and transitive closure of \rightarrow_{β} , and $=_{\beta}$ for the equivalence relation generated by \rightarrow_{β} . A σ -term of the form $(\lambda x:A.A')A''$ is called a β -redex.

Definition 6. Let Γ be a σ -pseudocontext and let M, N be σ -pseudoterms. We say that Γ *proves* $M:N$, and write $\Gamma \vdash M:N$, if there is a finite well-founded tree \mathcal{D} , called a *derivation*, such that the following hold.

1. Vertices of \mathcal{D} are of the form $\Delta \vdash A:B$, where A and B are σ -pseudoterms and Δ is a σ -pseudocontext.
2. The root of \mathcal{D} is $\Gamma \vdash M:N$ and the leaves of \mathcal{D} are instances of $\vdash c:c'$, where $(c, c') \in \mathcal{A}$.
3. Each interior vertex of \mathcal{D} is a conclusion of an *inference rule*, whose successors are exactly the premises.

The inference rules of σ are as follows. Below, $s \in \mathcal{S}$, $x \in \mathcal{V} \setminus \text{dom } \Gamma$, $(s_1, s_2, s_3) \in \mathcal{R}$, and $C =_{\beta} C'$.

$$\begin{array}{c} \frac{\Gamma \vdash A:s}{\Gamma, x:A \vdash x:A} \text{ INIT} \quad \frac{\Gamma \vdash A:s \quad \Gamma \vdash B:C}{\Gamma, x:A \vdash B:C} \text{ WEAK} \quad \frac{\Gamma \vdash B:C \quad \Gamma \vdash C':s}{\Gamma \vdash B:C'} \text{ CONV} \quad \frac{\Gamma \vdash B_1:s_1 \quad \Gamma, x:B_1 \vdash B_2:s_2}{\Gamma \vdash (\Pi x:B_1.B_2):s_3} \text{ II-RULE} \\ \frac{\Gamma \vdash B_1:s_1 \quad \Gamma, x:B_1 \vdash B_2:s_2 \quad \Gamma, x:B_1 \vdash C:B_2}{\Gamma \vdash (\lambda x:B_1.C):(\Pi x:B_1.B_2)} \lambda\text{-RULE} \quad \frac{\Gamma \vdash B_1:(\Pi x:C_1.C_2) \quad \Gamma \vdash B_2:C_1}{\Gamma \vdash B_1 B_2:C_2[B_2/x]} \text{ APP} \end{array}$$

Definition 7. If $\Gamma \vdash A:B$, then Γ is a σ -context and A, B are σ -terms.

Lemma 8 (Substitution Lemma; [GN91, Lemma 17]). *Let Γ and $\Gamma_1, y:A, \Gamma_2$ be σ -contexts and let A, M, N, P be σ -terms. If $\Gamma_1, y:A, \Gamma_2 \vdash M:N$ and $\Gamma \vdash P:A$, then $(\Gamma_1, \Gamma_2)[P/y] \vdash M[P/y]:N[P/y]$.*

Lemma 9 (Stripping Lemma; [GN91, Lemma 19]). *Let Γ be a σ -context and let M, N, P be σ -terms.*

1. If $\Gamma \vdash c:P$ where $c \in \mathcal{C}$, then $P =_{\beta} c'$ and $(c, c') \in \mathcal{A}$ for some $c' \in \mathcal{C}$.
2. If $\Gamma \vdash x:P$ where $x \in \mathcal{V}$, then $P =_{\beta} Q$ for some σ -term Q such that $(x:Q) \in \Gamma$.
3. If $\Gamma \vdash (\Pi x:M.N):P$, then $\Gamma \vdash M:s_1$, $\Gamma, x:M \vdash N:s_2$, and $P =_{\beta} s_3$ for some $(s_1, s_2, s_3) \in \mathcal{R}$.
4. If $\Gamma \vdash (\lambda x:M.N):P$, then $\Gamma \vdash M:s_1$, $\Gamma, x:M \vdash Q:s_2$, $\Gamma, x:M \vdash N:Q$, $\Gamma \vdash P:s_3$, and $P =_{\beta} \Pi x:M.Q$ for some $(s_1, s_2, s_3) \in \mathcal{R}$ and σ -term Q .
5. If $\Gamma \vdash M N:P$, then $\Gamma \vdash M:(\Pi x:A.B)$, $\Gamma \vdash N:A$, and $P =_{\beta} B[N/x]$ for some σ -terms A and B .

Theorem 10 (Subject Reduction; [GN91, Lemma 22]). *Let Γ, Γ' be σ -contexts and let M, M', N be σ -terms.*

1. If $\Gamma \vdash M:N$ and $M \rightarrow_{\beta} M'$, then $\Gamma \vdash M':N$.
2. If $\Gamma \vdash M:N$ and $\Gamma \rightarrow_{\beta} \Gamma'$, then $\Gamma' \vdash M:N$.

Proof. We proceed by simultaneous induction on the derivation $\mathcal{D} : \Gamma \vdash M:N$ when $M \rightarrow_{\beta} M'$ and $\Gamma \rightarrow_{\beta} \Gamma'$; the general case follows by iteration. We first prove (1), and split into cases with similar proofs.

- If \mathcal{D} ends with INIT, then there is no redex in M . If \mathcal{D} ends with CONV, then there are derivations $\mathcal{D}_1 : \Gamma \vdash M:N'$ and $\mathcal{D}_2 : \Gamma \vdash N':s$ for some $s \in \mathcal{S}$ and some σ -term N' such that $N' =_{\beta} N$. By IH₁, we have $\Gamma \vdash M':N'$, on which CONV with \mathcal{D}_2 gives $\Gamma \vdash M':N$. The case when \mathcal{D} ends with WEAK is similar.
- If \mathcal{D} ends with II-RULE, say with $M = \Pi x:B_1.B_2$, then the Stripping Lemma furnish some $(s_1, s_2, s_3) \in \mathcal{R}$ and derivations $\mathcal{D}_1 : \Gamma \vdash B_1:s_1$ and $\mathcal{D}_2 : \Gamma, x:B_1 \vdash B_2:s_2$ such that $N =_{\beta} s_3$. By definition of \rightarrow_{β} , two cases occur: if there is a σ -term B'_1 such that $B_1 \rightarrow_{\beta} B'_1$, then by IH₁ on \mathcal{D}_1 , we have $\mathcal{D}'_1 : \Gamma \vdash B'_1:s_1$. Moreover, IH₂ on \mathcal{D}_2 gives $\mathcal{D}'_2 : \Gamma, x:B'_1 \vdash B_2:s_2$, so applying II-RULE on \mathcal{D}'_1 and \mathcal{D}'_2 gives $\Gamma \vdash (\Pi x:B'_1.B_2):s_3$, on which CONV gives $\Gamma \vdash (\Pi x:B'_1.B_2):N$. The second case when $B_2 \rightarrow_{\beta} B'_2$ for some σ -term B'_2 is the same (in fact, easier). The case when \mathcal{D} ends with λ -RULE is similar (and again has two subcases).

- If \mathcal{D} ends with APP, say with $M = B_1 B_2$, then reductions within either B_1 or B_2 are trivial. Thus, we can take $x \in \mathcal{V} \setminus \text{dom } \Gamma$ such that $B_1 = \lambda x : A_1. A_2$, and assume $M = (\lambda x : A_1. A_2) B_2 \rightarrow_\beta A_2[B_2/x]$. The Stripping Lemma then furnish σ -terms C_1 and C_2 such that $N =_\beta C_2[B_2/x]$ and derivations $\mathcal{D}_1 : \Gamma \vdash (\lambda x : A_1. A_2) : (\Pi x : C_1. C_2)$ and $\mathcal{D}_2 : \Gamma \vdash B_2 : C_1$. Again, the Stripping Lemma applied to \mathcal{D}_1 then furnish $(s_1, s_2, s_3) \in \mathcal{R}$, a σ -term C'_2 such that $\Pi x : C_1. C_2 =_\beta \Pi x : A_1. C'_2$, and derivations $\mathcal{E}_1 : \Gamma \vdash A_1 : s_1$, $\mathcal{E}_2 : \Gamma, x : A_1 \vdash C'_2 : s_2$, and $\mathcal{E}_3 : \Gamma, x : A_1 \vdash A_2 : C'_2$. Observe that $A_1 =_\beta C_1$, so CONV on \mathcal{D}_2 and \mathcal{E}_1 gives $\mathcal{D}_0 : \Gamma \vdash B_2 : A_1$, and using the Substitution Lemma with $(\mathcal{D}_0, \mathcal{E}_2)$ and $(\mathcal{D}_0, \mathcal{E}_3)$ give $\mathcal{E}'_2 : \Gamma \vdash C'_2[B_2/x] : s_2$ and $\mathcal{E}'_3 : \Gamma \vdash A_2[B_2/x] : C'_2[B_2/x]$; note that $\Gamma[B_2/x] = \Gamma$ since $x \notin \text{dom } \Gamma$. Finally, since $C_2 =_\beta C'_2$ and $N =_\beta C_2[B_2/x]$, applying CONV on \mathcal{E}'_2 and \mathcal{E}'_3 gives $\Gamma \vdash A_2[B_2/x] : N$.

For (2), if the last rule of \mathcal{D} is either APP, CONV, Π -RULE, or λ -RULE, then we are done by IH₂; indeed, Γ is unchanged for APP and CONV, and in Π -RULE and λ -RULE, reductions take place within Γ . Suppose that the last rule of \mathcal{D} is INIT or WEAK, so with the notation of Definition 6, $\Gamma = \Gamma_0, x : A$ for some σ -term A and $x \in \mathcal{V}$. If the reduction occurs within Γ_0 , then we are done by IH₂. Otherwise, $A \rightarrow_\beta A'$ for some σ -term A .

- If \mathcal{D} ends with INIT, then $\Gamma_0 \vdash A : s$. Applying IH₁, we have $\Gamma_0 \vdash A' : s$, and hence $\Gamma_0, x : A' \vdash x : A'$ from INIT. Since $A \rightarrow_\beta A'$, we see that $A =_\beta A'$, so $\Gamma_0, x : A' \vdash x : A$ by CONV, as desired.
- If \mathcal{D} ends with WEAK, then there are derivations $\mathcal{D}_1 : \Gamma_0 \vdash A : s$ and $\mathcal{D}_2 : \Gamma_0 \vdash B : C$. Applying IH₁, we obtain a derivation $\mathcal{D}'_1 : \Gamma_0 \vdash A' : s$, and applying WEAK on \mathcal{D}'_1 and \mathcal{D}_2 gives $\Gamma_0, x : A' \vdash B : C$. ■

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