Uncovering Fold and Transcritical Normal Form Bifurcations in Modern Climate Data

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Abstract

We conduct a study on modern climate data from thirteen Canadian cities using a Long Short Term Memory Machine Learning model. The purpose of this study to prove the existence of Fold and Transcritical Normal Form Bifurcations in the data and to identify Early Warning Signals before each detected bifurcation or cluster of bifurcations. We detect both Fold and Transcritical Normal Form Bifurcations with the machine learning model and identify Uniform, distinct Early Warning Signals before each identified bifurcation and bifurcation cluster. This research serves as motivation to pursue the search for bifurcations on a larger sample of climate data, and on more covariates within the climate data.

1 Introduction

Extreme temperature transitions, such as Heat Waves and Cold Snaps, have the potential to wreak havoc on infrastructure, human life, and the functioning of local ecosystems (Wang et al. 2020). Today's most up-to-date climate models offer a high-resolution picture of the local and global climate, but these models are often very complicated, requiring computationally intensive simulation to generate numerical solutions and predictions (Crucifix & Rougier, 2009). Although these models offer high spatial resolution, it is still notoriously difficult to predict the temporally abrupt transitions, such as Heat Waves and Cold Snaps with these models (Wang et al. 2020).

An alternative approach to modeling and predicting these extreme temperature transitions involves using the mathematical theory behind dynamical systems. A dynamical system consists of one or more ordinary differential equations (ODEs) that mathematically represent how particular quantities, for instance temperature or other weather indices, change over time. When parameters in the system change, it is possible for the dynamical system to undergo a bifurcation, in which the stability of its equilibrium points change. This can alter the trajectories of the quantities being modelled by the dynamical system, causing phenomena like large jumps or oscillations. Because of the potential for bifurcations in dynamical systems to cause unexpected and disruptive changes, Early Warning Signals (EWS) have been identified and developed so upcoming bifurcations can be spotted before state changes occur in the system, based on patterns in data (Bury et al. 2021).

The dynamical systems approach and the use of EWS to identify incoming bifurcations in climate data was used in two recent studies, which serve as motivation for this project. The first was Wang et al. 2020, which identified clear EWS before todays global warming as well as and several recent heat waves using a pitchfork bifurcation. This pitchfork bifurcation can be represented as a simple dynamical system, as done in the study. The second was Bury et al. 2021, which proved that Machine Learning can be used to detect EWS for a verity of normal form of dynamical systems in the presence of noisy data. This included bifurcations in Paleoclimate data.

We plan to build of off Wang et al and Bury et al, and prove the existence of normal form bifurcation in modern climate data. More specifically, our research goals are to identify Fold and Transcritical Normal Form Bifurcations in temperature data and identify common EWS before the bifurcations, if the bifurcations exist. We conducted this study using a Long Short Term Memory Machine Learning model on modern climate data from thirteen Canadian cities. We specifically looked to identify these bifurcations in the temperature measurements. We were able to detect both Fold and Transcritical Normal Form Bifurcations with the Machine Learning model and verify these detection by inspection. Fold Normal Form Bifurcations were found to accurately represented abrupt temperature spikes and drops in the data, as well as cold snaps. Transcritical Normal Form Bifurcations were detected as a rapid but steady decline in temperature and may be used to represent cold snaps, but more data is needed to verify this result. Lastly we were able to successfully identify common and clear EWS before each identified bifurcation in the bifurcation probability data generated by the model, which was verified by inspection.

Overall, the results of this study are important for several reasons. The first is that by proving these normal form bifurcation and EWS exist in our small data set, there is now motivation for this analysis to be conducted on a larger sample size with with more climate covariance such as pressure, wind speed, and dew point. The second reason builds off of the first. If the larger larger study is successful, it proves the theory necessary for the development of a novel, standardized method that can extreme predict temperature transitions using bifurcation theory. This novel and standardized method would provide a way for cities, municipalities, and scientist to predict these damaging transitions far in advance with ideally higher accuracy and less financial and computational cost. Being able to predict these transitions further in advance allows for more successful mitigation of the financial, ecologic, life, and overall resource cost of these event. Lastly, this study open the door for research into low-dimensional dynamical system that can accurately represent climate data from local measurements.

2 Important Concepts

2.1 Bifurcations

A bifurcation can be defined as a change in equilibrium or stability state within a dynamical system. A bifurcation happens when a parameter or set of parameters withing the dynamical system passes its or their critical points. A Normal Form is any model where the parameter(s) critical point is at zero. An example of a bifurcation can be given with the Hoph Normal Form, which can be represented by (1)

$$\begin{aligned}\frac{\partial f}{\partial x} &= x\mu - y - x(x^2 + y^2)\\ \frac{\partial f}{\partial y} &= y\mu - x - y(x^2 + y^2)\end{aligned} \tag{1}$$

When the bifurcation parameter $\mu < 0$, then the system has one stable state (Figure 1C). When μ passes its critical point $\mu = 0$ and becomes positive, then the system bifurcates and enters an oscillatory state (Figure 1 (c)).

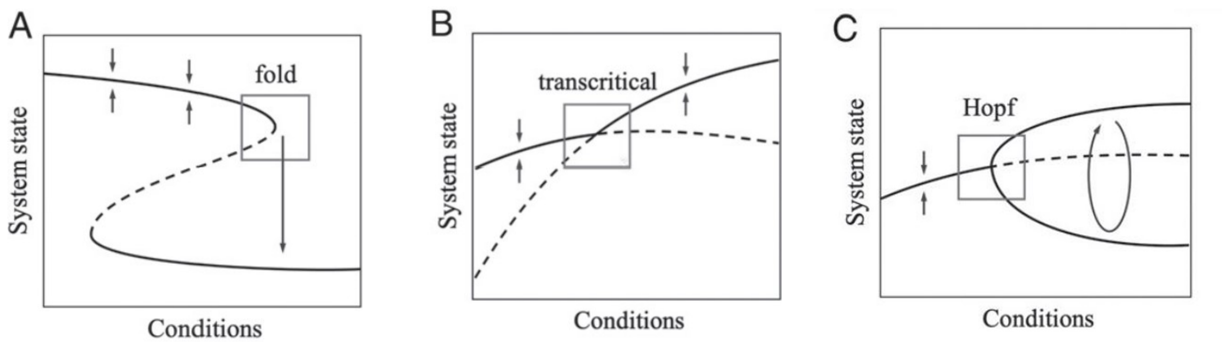
2.2 Fold Bifurcation

We will be looking to find evidence of Fold Normal Form Bifurcations in temperature data. The Fold Normal Form can be represented by (2).

$$\frac{\partial f}{\partial x} = \mu - x^2 \tag{2}$$

When the bifurcation parameter $\mu < 0$, the system has two equilibrium points, $x = \pm\sqrt{\mu}$ where $-\sqrt{\mu}$ is unstable and $\sqrt{\mu}$ is stable. As the bifurcation parameter reaches zero, the equilibrium points get closer together and meet to form one stable equilibrium point at $x = 0$. Finally when $\mu > 0$, the system no longer has any equilibrium points and becomes chaotic. This chaotic state often causes the system to transition to make a crash or a sudden spike (Figure 1 (a)).

To identify the Fold Normal Form Bifurcation in weather data, we look for the temperature to go from fairly consistent stable state, with moderate temperature fluctuations, to a sudden spike or drop to a much lower or higher temperature state. We also look for the temperature to go from a relatively stable state



(a) Fold Bifurcation

(b) Transcritical Bifurcation

(c) Hoph Bifurcation

Figure 1: Diagram of the Fold, Transcritical, and the Hoph Normal Form Bifurcations (Bury et al. 2021). (a) Diagram of the Fold Normal Form Bifurcation. As the bifurcation parameter increases, the system makes a sudden transition from a stable equilibrium state to a significantly lower state when the bifurcation parameter passes its critical point. One should note that the transition can also be to a chaotic state, where no identifiable pattern is apparent. (b) Diagram of the the Transcritical Normal form Bifurcation. As the bifurcation parameter increases, we see that the system transitions from one stable state to a steeply increasing equilibrium state when the bifurcation parameter passes its critical point. (c) Diagram of the Hoph Normal Form Bifurcation. As the bifurcation parameter increases, we can see that the system transitions from a stable equilibrium state, denoted by the arrows in the diagram, to an oscillatory state when the bifurcation parameter passes its critical point. This oscillatory state is known as limit cycles.

to a completely chaotic state, where the system displays massive temperature spikes and drops with no evident pattern.

2.3 Transcritical Bifurcation

We will also be looking to find evidence of the Transcritical Normal Form Bifurcation in weather data. The Transcritical Normal Form can be represented by (3).

$$\frac{\partial f}{\partial x} = x\mu - x^2 \tag{3}$$

When the bifurcation parameter $\mu < 0$, the system has two equilibrium points. They are located at $x = 0$, which is stable, and $x = \mu$, which is unstable. When $\mu = 0$, the system displays one stable equilibrium point at $x = 0$. When $\mu > 0$, the system returns to having two equilibrium points, $x = 0$ and $x = -\mu$. Often, a system will follow equilibrium points and transition from one stable state to another stable state (Figure 1 (c)),

To identify the Transcritical Normal Form Bifurcation in the weather data, we look for a temperature to transition from a fairly consistent stable state, with moderate temperature fluctuations, to a increasing or decreasing temperature state. This increasing or decreasing state is where the temperature quickly drops or rises to a much lower or higher temperature. This happens when the bifurcation parameter passes zero from negative to positive within the system, and the system transitions from a non moving stable equilibrium state, to an increasing or decreasing equilibrium state.

2.4 Early Warning Signals (EWS)

An EWS is any indication that a dynamical system is approaching a bifurcation or state transition. It can be an increase or decrease in variance, or lag one auto correlation, which was used in Bury et al 2021 to identify the incoming of bifurcations in multiple different system from data. Another EWS that is commonly used is the domination of a certain behavior within a system before it system bifurcates. For example, a more chaotic behavior of the parameter being measured, or the steady increasing or decreasing of measurable parameters within a dynamical system, that generally lead to a bifurcation.

2.5 Heat Wave

A Heat Wave occurs when local temperature quickly increases to levels significantly above what is expected, and remains there for an extended period. Heat Waves are caused when a high pressure system becomes trapped over a region, creating an environment with no wind. Due to the high pressure, air is forced downward which acts as a barrier preventing warm air from rising. This phenomenon prevents precipitation, which allows the soil to dry because the warm air cannot rise high enough to condensate. The result of these unique environmental conditions is a spike in temperature, which creates a stagnant warm pocket of hot air at ground level with little temperature fluctuation. This is the Heat Wave. We believe that the temperature transition to be a heat wave can be modeled by a Normal Form Bifurcation due to the fact that a heat wave is a change in temperature state caused by very specific changes in a unique set of climate system parameters, as mentioned above. Due to the fact that a Heat Wave should be able to be represented by a bifurcation, EWS should exist before the transition.

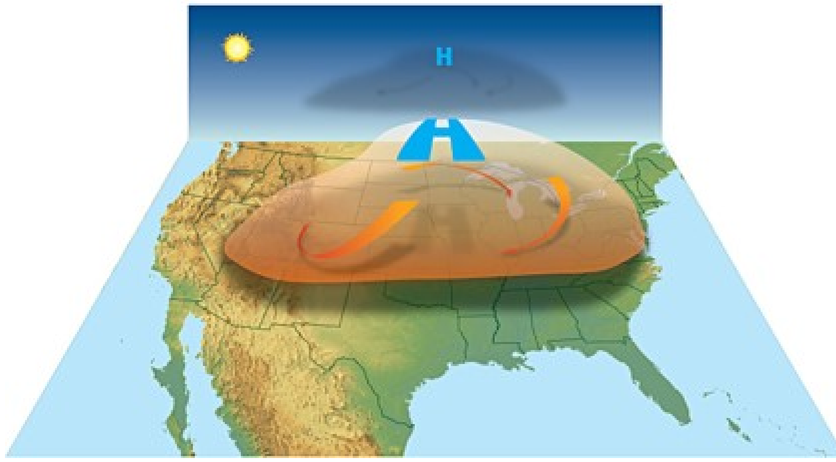


Figure 2: Diagram of a Heat Wave over the United States. The high pressure system prevents air from rising by forcing it downward, preventing precipitation. This creates an environment where air cannot escape the region and becomes trapped, causing the temperature to rise.

2.6 Cold Snap

A Cold Save occurs when local temperature quickly decrease to levels significantly below what is expected, and remains there for a period of time. Cold Snaps are caused when the polar vortex, a circulating current of cold air over the Arctic, gets disrupted or blocked by high pressure system in the upper atmosphere. These high pressure blockages direct cold air to areas of lower pressure and latitude. This cold air over the region of lower latitude is the Cold Snap. We believe that the temperature transition to a cold snap can be modeled by a Normal Form Bifurcation due to the fact that a cold snap is a change in temperature state caused by a very specific change in a unique set of climate system parameters, as mentioned above. Due to the fact that a Cold Snaps should be able to be represented by a bifurcation, EWS should exist before the transition.

3 Research Goals and Methods

The objective of this study is to prove the existence of Fold and Transcritical Normal Fold Bifurcations in temperature data. We also look to identify common EWS before the Normal Form Bifurcations if they exist.

3.1 Data

Data was obtained from The National Oceanic and Atmosphere Administration (NOAA) for thirteen cities across Canada (Table 1). We used Python to generate code to download public hourly data in the form of csv files from the NOAAs website (Website URL: https://www.ncei.noaa.gov/data/global-hourly/access/). Once the data was obtained, we uploaded it it R for data filtering. We filtered the data to include only FM-12 measurements (observations from a land based stations), had three hour gaps between measurements, and passed all quality control checks in R. The data that was left was organized into unbroken (no gaps in time) time series length 500. We were able to obtain 26041 unbroken time series measurements across the 13 cities. This data was exported from R as a csv file, back to Python where it was detrended using Gaussian smoothing to account for noise and normal temperature fluctuations caused by the time of day. This data was analysed and used for the study.

USAF	WBAN	STATION NAME	BEGIN DATE	END DATE	CITY
718920	99999	VANCOUVER INTL	1955-07-02	2022-05-12	Vancouver
718960	99999	PRINCE GEORGE	1955-07-02	2022-05-12	Prince George
711230	99999	EDMONTON INTL	1960-11-14	2022-05-12	Edmonton
718770	99999	CALGARY INTL	1955-07-02	2022-05-12	Calgary
718660	99999	SASKATOON J.G.D INTL	1955-07-02	2022-05-12	Saskatoon
718520	99999	WINNIPEG INTL	1955-07-02	2022-05-12	Winnipeg
719360	99999	YELLOWKNIFE	1942-07-01	2022-05-12	Yellowknife
717490	99999	THUNDER BAY AREA	1955-07-02	2022-05-12	ThunderBay
712650	99999	TORONTO CITY CENTRE	1979-05-07	2022-05-12	Toronto
716270	94792	MONTREAL/TRUDEAU INT	1955-01-01	2022-05-12	Montreal
717270	99999	BAGOTVILLE	1950-01-02	2022-05-12	Chicoutimi
713950	99999	HALIFAX INTL	1960-07-01	2022-05-12	Halifax
718010	99999	ST.JOHNS INTL	1941-09-29	2022-05-12	St.John's

Table 1: Brief overview of the data set being analysed containing thirteen cross Canada stations representing a unique Canadian city. USAF is the United States Air Force station code and WBAN is the Weather Bureau Army Navy Identifier, which is 99999 if the station is not contained in the United States of America. STATION NATE contains the name of the station providing the data, most being international airports. BEGIN DATE is the earliest date the NOAA records contains data from the selected station and END DATE contains the most recent date that we used for this study. CITY contains the city where the station is located.

3.2 Data Analysis For Existence of Bifurcation

A Long Short Term Memory Machine Learning Model was used to identify the probability of both Fold and Transcritical Normal Form Bifurcations in each 500 point time series being analysed. The model was trained with 500,000 bifurcating time series of each Normal Form Bifurcation to insure a high probability of correct bifurcation identification. The training data was generated by a previous study, Bury et al 2021, which identified bifurcations and EWS in Paleoclimate data, so we assumed that this training data would be a good base to build the model from. The model generation and time series analysis was done in Python. Each of the 26041 time series was analysed by the model, which assigned a probability that the time series would contain either a Fold or a Transcritical Normal Form Bifurcation. This data was exported back to R as a CSV file for furthur analysis. Each time series (or cluster of time series) where the model predicted $P(bifurcation) \geq 90\%$ that ether a Fold or Hoph Normal Form Bifurcation existed was analysed by inspection for evidence of a bifurcation or bifurcation cluster. To identify bifurcations, we looked for data where the temperature displayed large drops, spikes, or very chaotic behavior compared to the temperature before it.

3.3 Data Analysis For Existence of EWS

We analysed the one month period before every properly identified bifurcation or bifurcation cluster where the data existed using R. The EWS we specifically looked for in the temperature data by inspection was an increase in temperature variance before the identified bifurcation or cluster, similar to what was done in Bury et al. The EWS we specifically looked for by inspection in the probability data was an increase in probability variance before the bifurcation. We also looked for an increase in bifurcation probability leading up to the bifurcation.

4 Results and Analysis

4.1 Fold Bifurcation

21 independent time-series clusters where the model identified $P(bifurcation) \geq 90\%$ were analysed. The length of these time series clusters ranged from one prediction over 90% to clusters of predictions over 90% spanning up to 16 days. Each cluster showed evidence of multiple distinct bifurcations in the temperature data while singular predictions over 90% percent reveled one distinct bifurcation. Transitions to both very cold and very hot temperature states were identified Fold Normal Form Bifurcations. We were also able to identify distinct cold snaps as Fold Normal Form Bifurcations (Figure 3) but we were not able to identify any distinct heat waves as Fold Normal Form Bifurcations within our data set. We were able to identify rapid spikes to very hot temperatures in the data with the model, but no bifurcations stayed in this elevated state long enough to be classified as a Heat Wave. We can conclude from this result that the model can be used to identify Cold Snaps and temperature transitions within the data.

We were able to predict the phenomena of flickering with the Long Short Term Machine Learning Model and verify this by inspection in the temperature data (Figure 3). Flickering is when a dynamical system makes quick transition between two equilibrium states as it makes the slow official transition from the first state to the second state. This can be seen in Figure 4, where the temperature makes sudden jumps from the warmer state to the cold state as the whole system slowly transitions to the cold state state. Each one of the flickers is considered to be a Fold Bifurcation because they are sudden transition from a warmer state to a colder state and are identified with high probability by the model.

Some of the Fold Normal Form Bifurcations were identified after they occurred. This phenomena can be denoted as "backwards identification" (Figure 5). Backwards identification happens when the model recognises the reverse image of the bifurcation. In other words, where the model recognizes the bifurcation, then EWS, instead of the other way around. This leads to an increase in Fold Normal Form Bifurcation

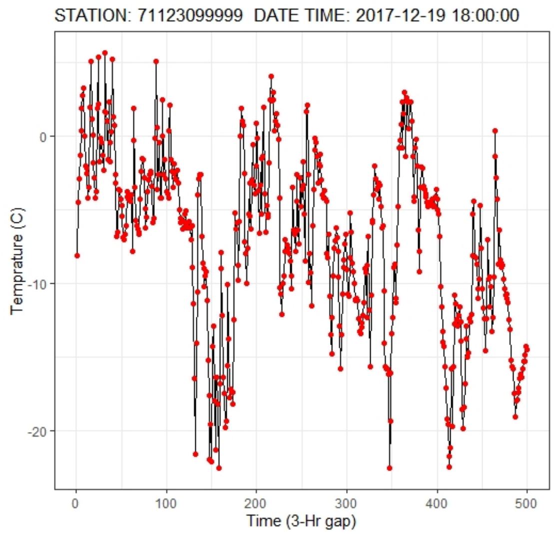


Figure 3: Cluster of Fold Bifurcations. Overall this figure has three Fold Bifurcations A Fold Bifurcation happens at Time = 100, which can be identified by a very rapid decline in temperature of about 20 degrees Celsius. A second Fold Bifurcation can be classified by the rapid 20 degree increase in temperature at about Time = 175. The Third Fold Bifurcation can be classified by the transition to a chaotic temperature state at about Time = 300.

probability after the bifurcation occurred, where the model identifies the highest probability of Fold Normal Form Bifurcation around 350-450 time series points after the bifurcation. This unusual detection is caused by two factors. The first is that the reverse image of the Fold Normal Form Bifurcation exist in the data. The second is that the model was trained to recognized the reverse image of the Fold Normal Form Bifurcation as done in Bury et al 2021. Overall, this is not use full for predicting the bifurcation in the data, but it is useful for proving the existence of the Fold Normal Form Bifurcation in the temperature measurements.

Overall we can conclude that Fold Normal Form Bifurcations exist in the temperature data from the thirteen Canadian cities from these findings.

4.2 Transcritical Bifurcation

2 independent time-series clusters where the model identified $P(bifurcation) \geq 90\%$ of a Transcritical Normal Form Bifurcation were analysed. The first detection showed a Transcritical Normal Form Bifurcation, but it was not very distinct (Figure 6(a)). This bifurcation is identified in the data as the transition from the expected temperature decline at Time $\in [0, 300]$ to a transition to a more rapidly declining state at Time $\in [301, 350]$. The second detection showed a a singular distinct Transcritical Normal Form Bifurcation where the temperature went from a stable state to a quickly decreasing state, as seen in (Figure 6(b)). The second detection accurately identifies a Cold Snap, which is shown by the 30 degree temperature drop from above 0 degrees Celsius to an extended period where the temperature is below -20 degrees Celsius.

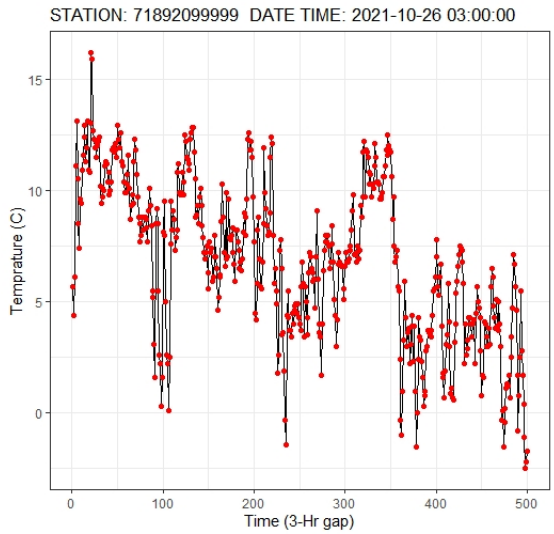


Figure 4: Flickering of Fold Bifurcation. As the system makes a steady transition from a warmer temperature state ($T > 0$ degrees Celsius) to a cold temperature state ($T < 0$ degrees Celsius), we can observe the system making quick, sudden transitions between the two states. The flickering happens at about Time = 100, 235, 360, and 370 before the system makes its official transition to the lower state at Time = 500.

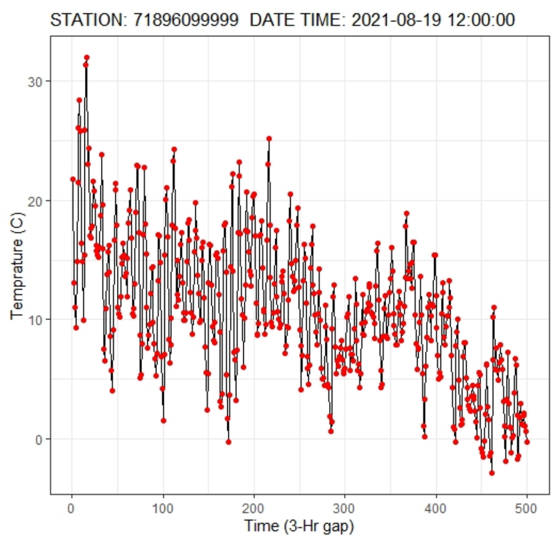
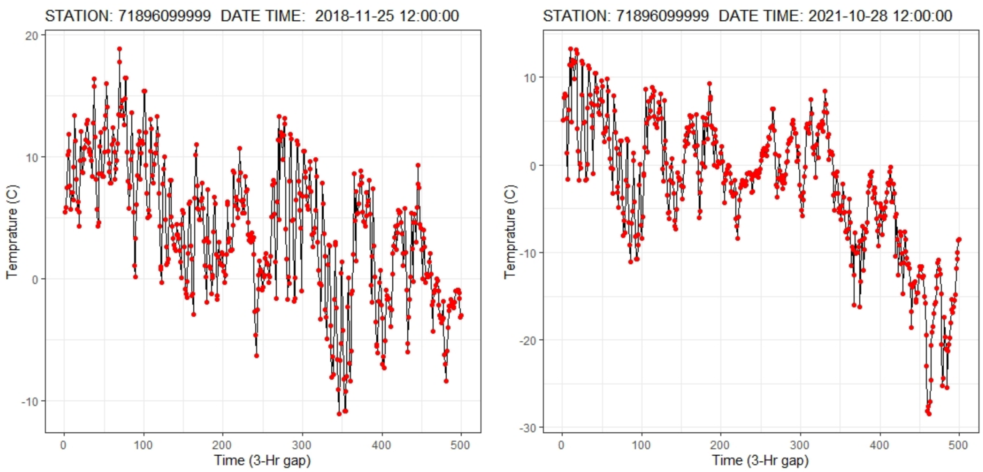


Figure 5: Backwards identification of a Fold Normal Form Bifurcation. The bifurcation is observed to happen around Time = 20 but the model does not predict $P(bifurcation) \geq 90\%$ until Time = 500. The result is that we see the Bifurcation in the historical data if we are measuring at Time= 500.

These results provides evidence that Transcritical Normal Form Bifurcations exist in the temperature data and can represent Cold Snaps, but we need a larger sample size of detected bifurcation to verify this.



(a) First Detection

(b) Second Detection

Figure 6: Detected Transcritical Bifurcations in the temperature Data.

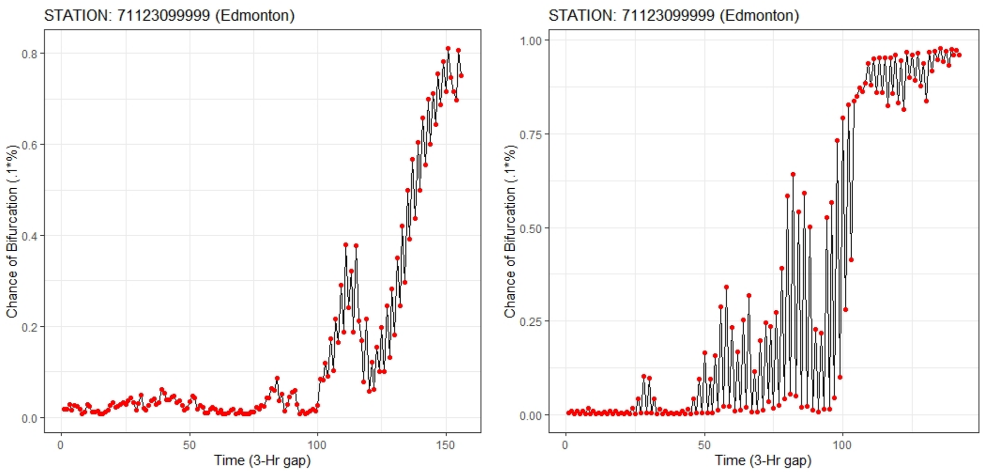
4.3 Early Warning Signals

We were not able to find conclusive evidence of EWS in the temperature data before each bifurcation by inspection. This was due to the noise in the unsmoothed data, which was visually analysed. This noised made it very difficult to identify and observable distinct pattern that could be classified as a common EWS by inspection.

We were able to find conclusive evidence of the existence of EWS in the probability data before both Fold and Transcritical Normal Form Bifurcation. We were able to identify increases in bifurcation probability percentages as each Normal Form Bifurcation approached (Figure 7(a) and 7(b)). This increase in probability is caused by time series being analysed becoming more and more similar to the training data as the system approached bifurcation. This causes the model to predict a higher and higher bifurcation probability as the bifurcation approached, which creates a distinct EWS. We also identified that if the Bifurcation probability was over 50%, then a Bifurcation would occur. This allowed for the prediction distinct Normal Form Bifurcations on 5 average 5 days in advance with this method.

The second EWS we identified was an increase in bifurcation probability variance, which often happens simultaneously with the increase in bifurcation probability as the bifurcation approaches. When there is no bifurcation approaching, the model always estimates a very low bifurcation probabiliy with little variance between data points. This variance is always less then 5% bifurcation probability. As the Bifurcation approaches, the system starts to predict a higher and higher chance of Bifurcation, which causes a natural increase in the variance (Figure 7(a) and 7(b)). The variance also increases because the model does not identify every time series to look exactly like the approaching bifurcation due to the specifics of the training data. Lastly, the increase in this variance can often be identified before the steady increase in bifurcation probability as far as 2 weeks in advance (Figure 7(a) and 7(b)). This is due to the specifics of the training data as well.

Overall, there is sufficient evidence to conclude that EWS exist before each Normal Form Bifurcation detected, and can be used to predict these bifurcations in advance.



(a) (b)

Figure 7: Clear EWS before Fold Normal Form Bifurcations in Edmonton data. The EWS are the increase in bifurcation probability percentage leading up to the bifurcation as well as the increase in variance of the bifurcation probability percentage. As we can see in both (a) and (b) that both happen simultaneously, with the increase in variance happening before the steady increase in bifurcation probability percentage.

5 Obstacles, Limitations, and Future Exploration

The main obstacles we faced in this project were a lack time and a lack of the computational power to properly complete the project. An extensive amount of background knowledge had to be acquired in coding as well as data acquisition, manipulation, and filtering to complete this project. Along the way, we discovered that we needed high powered computing to filter and organize the data we were able to acquire. this is because running the R code we created to filter and organize our data would take about 2 weeks time to run on a standard computer. We also estimated that learning how to use the Compute Canada clusters would also take about 2 weeks time. With these obstacles, we decided to decrease our original sample size of 300 global cities down to 13 Canadian cities with the goal of obtaining preliminary results for the project before the end of the semester. This lead to around 2.75 months of the 3.5 months of allotted time for the project to be put towards acquiring, cleaning, and organizing our data into an analysable format. The remainder of the time was used to run our Short Long Term Memory Machine Learning Model on the data as well as extracting and analysing the preliminary results. As a result, there was only time to do an analysis by inspection on the Fold and Transcritical probability data. To make this study more sufficient, we need to test our model on a larger sample size and have mathematically sound methods to verify bifurcations discoveries the EWS generated by the model, i.e not inspection.

The limitation of our method is that our model is only as good as our training data. If the training is not good or does not accurately represent what is happening in the data, then our model will detect nothing, even if there is a large drop or spike in temperature.

The preliminary results open up a lot of doors for further research with this project in particular. The proof that Fold and Transcritical Normal Form Bifurcations exist in in temperature data serves as motivation to test our model on the large data set of 300 cities, using the Computer Canada clusters. It also serves as motivation, along with the physics behind heatwaves and cold snaps, to test other climate system covariates such as pressure, dew point, wind speed, and cloud cover for EWS.

6 Relevance

This research is relevant to both applied mathematics and climate modeling. This is because it represents a unique, cutting-edge application of bifurcation theory for gaining insights into extreme temperature transitions, climate trends, and the EWS of these extreme temperature transitions. By proving the existence of Normal Form Bifurcations and EWS in weather data, this project paves the way for further research and eventually a novel, standardized method to be developed to predict temperature transitions in any given city, where local weather data is available. This method would require much less computational power, financial cost, and ideally offer more predictive power than today's most common techniques for predicting temperature transitions. As a result, the output of this project conducted on the full scale of 300 cities and with the consideration of multiple covariates, such as pressure, dew point, wind speed, and cloud cover, opens the door for future research that is both innovative and mathematically sound.

The proof of existence of bifurcations in climate data also opens up the door for research into a low-dimensional dynamical system that can accurately represent climate data from local measurements. This system would be easy to analyze and interpret, due to its low dimensional structure. Today climate models are often systems of partial differential equations which require many measurements across a large area of space to be accurate, and intense numerical simulation to make predictions. Discovery of this model would fill the gap there currently is in local climate modeling and forecasting from a single point in space. If discovered, this low dimensional model would create research opportunities understand how this model fully behaves, and its relevance for atmospheric sciences.

7 Acknowledgements

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