Fuzzy C-Means Clustering

The Algorithm

Fuzzy c-means (FCM) is a method of clustering which allows one piece of data to belong to two or more clusters. This method (developed by Dunn in 1973 and improved by Bezdek in 1981) is frequently used in pattern recognition. It is based on minimization of the following objective function:

$$J_m = \sum_{i=1}^{N}\sum_{j=1}^{C} u_{ij}^{m}\left\|x_i - c_j\right\|^2 \quad , \quad 1 \le m < \infty$$

where m is any real number greater than 1, u_{ij} is the degree of membership of x_i in the cluster j, x_i is the ith of d-dimensional measured data, c_j is the d-dimension center of the cluster, and ||*|| is any norm expressing the similarity between any measured data and the center.

Fuzzy partitioning is carried out through an iterative optimization of the objective function shown above, with the update of membership u_{ij} and the cluster centers c_j by:

$$u_{ij} = \frac{1}{\sum_{k=1}^{C}\left(\frac{\left\|x_i - c_j\right\|}{\left\|x_i - c_k\right\|}\right)^{\frac{2}{m-1}}} \quad , \quad c_j = \frac{\sum_{i=1}^{N} u_{ij}^{m} \cdot x_i}{\sum_{i=1}^{N} u_{ij}^{m}}$$

This iteration will stop when $\max_{ij}\left\{\left|u_{ij}^{(k+1)} - u_{ij}^{(k)}\right|\right\} < \varepsilon$, where ε is a termination criterion between 0 and 1, whereas k are the iteration steps. This procedure converges to a local minimum or a saddle point of J_m.

The algorithm is composed of the following steps:

1. *Initialize $U=[u_{ij}]$ matrix, $U^{(0)}$*
2. *At k-step: calculate the centers vectors $C^{(k)}=[c_j]$ with $U^{(k)}$*

$$c_j = \frac{\sum_{i=1}^{N} u_{ij}^{m} \cdot x_i}{\sum_{i=1}^{N} u_{ij}^{m}}$$

3. *Update* $U^{(k)}$, $U^{(k+1)}$

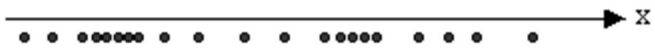
$$u_{ij} = \frac{1}{\sum_{k=1}^{c} \left(\frac{\|x_i - c_j\|}{\|x_i - c_k\|} \right)^{\frac{2}{m-1}}}$$

4. *If* $|| U^{(k+1)} - U^{(k)}|| < \varepsilon$ *then STOP; otherwise return to step 2.*

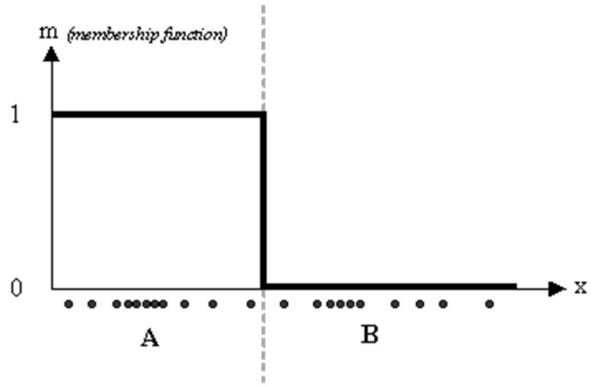
Remarks

As already told, data are bound to each cluster by means of a Membership Function, which represents the fuzzy behaviour of this algorithm. To do that, we simply have to build an appropriate matrix named U whose factors are numbers between 0 and 1, and represent the degree of membership between data and centers of clusters.

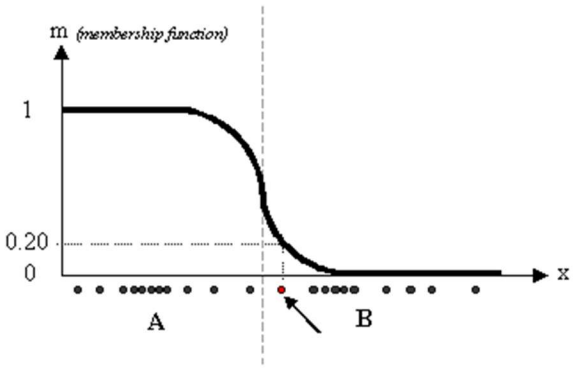
For a better understanding, we may consider this simple mono-dimensional example. Given a certain data set, suppose to represent it as distributed on an axis. The figure below shows this:



Looking at the picture, we may identify two clusters in proximity of the two data concentrations. We will refer to them using ‘A’ and ‘B’. In the first approach shown in this tutorial - the k-means algorithm - we associated each datum to a specific centroid; therefore, this membership function looked like this:



In the FCM approach, instead, the same given datum does not belong exclusively to a well defined cluster, but it can be placed in a middle way. In this case, the membership function follows a smoother line to indicate that every datum may belong to several clusters with different values of the membership coefficient.



In the figure above, the datum shown as a red marked spot belongs more to the B cluster rather than the A cluster. The value 0.2 of 'm' indicates the degree of membership to A for such datum. Now, instead of using a graphical representation, we introduce a matrix U whose factors are the ones taken from the membership functions:

$$U_{MxC} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ .. & .. \\ 0 & 1 \end{bmatrix} \qquad U_{MxC} = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \\ 0.6 & 0.4 \\ .. & .. \\ 0.9 & 0.1 \end{bmatrix}$$

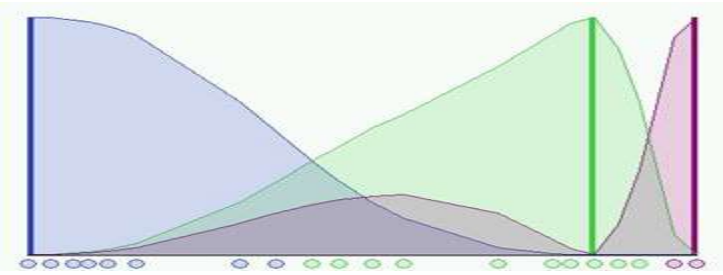
(a) (b)

The number of rows and columns depends on how many data and clusters we are considering. More exactly we have C = 2 columns (C = 2 clusters) and N rows, where C is the total number of clusters and N is the total number of data. The generic element is so indicated: u_{ij}.
In the examples above we have considered the k-means (a) and FCM (b) cases. We can notice that in the first case (a) the coefficients are always unitary. It is so to indicate the fact that each datum can belong only to one cluster. Other properties are shown below:

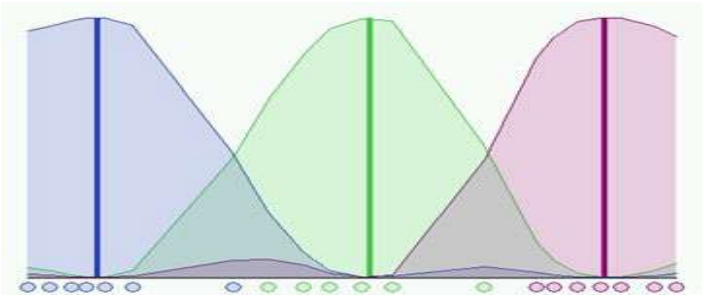
- $u_{ij} \in [0,1] \qquad \forall i, j$
- $\sum_{j=1}^{C} u_{ik} = 1 \qquad \forall i$
- $0 < \sum_{i=1}^{N} u_{ij} < N \quad \forall N$

An Example

Here, we consider the simple case of a mono-dimensional application of the FCM. Twenty data and three clusters are used to initialize the algorithm and to compute the U matrix. Figures below (taken from our interactive demo) show the membership value for each datum and for each cluster. The color of the data is that of the nearest cluster according to the membership function.



In the simulation shown in the figure above we have used a fuzzyness coefficient m = 2 and we have also imposed to terminate the algorithm when $\max_{ij}\left\{\left|u_{ij}^{(k+1)}-u_{ij}^{(k)}\right|\right\}<0.3$. The picture shows the initial condition where the fuzzy distribution depends on the particular position of the clusters. No step is performed yet so that clusters are not identified very well. Now we can run the algorithm until the stop condition is verified. The figure below shows the final condition reached at the 8th step with m=2 and ε =0.3:



Is it possible to do better? Certainly, we could use an higher accuracy but we would have also to pay for a bigger computational effort. In the next fig. we can see a better result having used the same initial conditions and ε=0.01, but we needed 37 steps!