Reinforcement Learning

Lecture 4: First RL algorithms

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Objectives of this lecture

Introduce basic algorithms of reinforcement learning. Solving MDPs without the knowledge of the reward function and the transition probabilities.

- Preliminaries: Stochastic approximation
- Off-policy algorithm: Q-learning and its variants
- On-policy algorithm: SARSA

Lecture 4: outline

1. Q-learning and stochastic approximation
2. SARSA (State-Action-Reward-State-Action) algorithm
3. Improving Q-learning convergence rate

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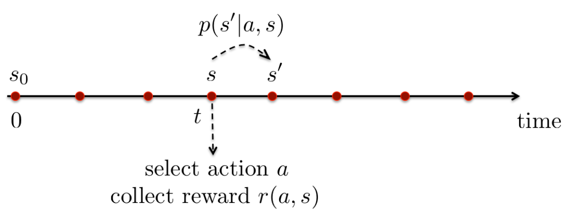
Infinite-horizon discounted MDP



- Stationary transition probabilities: $p(s'|s,a)$
- Stationary reward: $r(s,a)$, uniformly bounded
- Objective: for a given discount factor $\lambda \in [0,1)$, find a policy $\pi \in MD$ maximising (over all possible policies)

$$\lim_{T\to\infty} \mathbb{E}\left[\sum_{u=0}^{T} \lambda^u r(s_u^\pi, a_u^\pi)\right]$$

Discounted RL

- Learn $\pi^\star$ (the optimal policy) from the data
- Off-policy design problems. Data = (a given trajectory $(s_t, a_t, r(s_t, a_t))_{t=0}^{T}$), output at time T = a policy π_T
- On-policy design problems. In each step t:
  - Observe the transition s_{t-1}, a_{t-1}, s_t and the reward $r(s_{t-1}, a_{t-1})$
  - Devise a policy π_t and select the next action $a_t = \pi_t(s_t)$

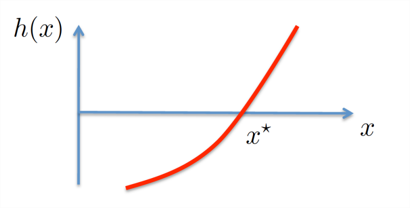
How can we solve Bellman's fixed point equation w/o knowing p and r?

$$V^\star(s) = \max_{a \in A_s}(r(s,a) + \lambda \sum_j p(j|s,a) V^\star(j)$$

$$\pi^\star(s) \in \arg\max_{a \in A_s}(r(s,a) + \lambda \sum_j p(j|s,a) V^\star(j)$$

Stochastic Approximation

Find the root of an increasing function from noisy measurements



Assume that at the n-th iteration, you select x_n

You get a noisy measurement $y_n = h(x_n) + M_n$ with $\mathbb{E}[M_n] = 0$

Stochastic Approximation: Example

$(M_n)_{n\geq 0}$ is i.i.d. Gaussian with mean 0 and variance 1.
Algorithm: $x_{n+1} = x_n - \frac{1}{n}(h(x_n) + M_n)$

Then

$$x_{n+p} = x_n - \sum_{t=n}^{n+p} \frac{h(x_t)}{t} - \sum_{t=n}^{n+p} \frac{M_t}{t}$$

The noise is averaged out after a while: for white Gaussian noise, $Var(\sum_{t\geq n} \frac{M_t}{t})$ tends to 0 as $n \to \infty$. "Hence", $x_n \to x_\infty$ as $n \to \infty$ almost surely where $h(x_\infty) = 0$

Robbins-Monro Algorithm (1951): $x_{n+1} = x_n - \alpha_n(h(x_n) + M_n)$

A generic SA algorithm

Let $X_n = (X_n(1), ..., X_n(d))^\top \in \mathbb{R}^d$ satisfying:

$$X_{n+1} = X_n + \alpha_n[h(X_n) + M_{n+1} + N_{n+1}] ,$$

Assumptions:

(A1) $h : \mathbb{R}^d \to \mathbb{R}^d$ is lipschitz

(A2) (Diminishing step sizes) $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$.

(A3) (Martingale difference) $\forall n$, $\mathbb{E}[M_{n+1}|\mathcal{F}_n] = 0$ where $\mathcal{F}_n = \sigma(X_0, M_1, N_1, \ldots, M_n, N_n, X_n)$ and $\forall n$, $\mathbb{E}[\|M_{n+1}\|^2 \mid \mathcal{F}_n] \leq c_0(1 + \|X_n\|^2)$.

(A4) (Additional noise) $\forall n$, .$\|N_n\|^2 \leq c_n(1 + \|X_n\|^2)$ a.s., where $\lim_{n\to\infty} c_n = 0$ a.s..

(A5) (Stability) $\dot{x} = h(x)$ has a unique globally stable equilibrium $x^\star$. $\forall x$, $h_\infty(x) = \lim_{c\to\infty} \frac{h(cx)}{c}$ exists and 0 is the only globally stable point of $\dot{x} = h_\infty(x)$.

Convergence

Let $X_n = (X_n(1), ..., X_n(d))^\top \in \mathbb{R}^d$ satisfying for all n:

$$X_{n+1} = X_n + \alpha_n[h(X_n) + M_{n+1} + N_{n+1}] \ ,$$

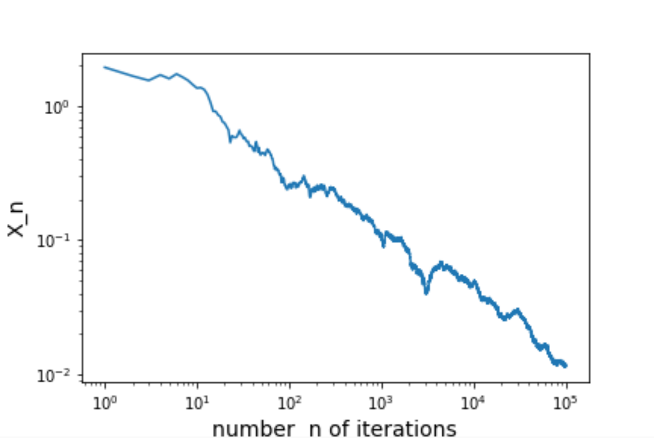
Theorem. If (A1)-(A5) hold, for any initial condition X_0,

$$\lim_{n\to\infty} X_n = x^\star, \quad \text{almost surely,}$$

where $x^\star$ is the only globally stable point of $\dot{x} = h(x)$.

Example

$h(x) = \arctan(x)$, $\alpha_n = 1/n$, $M_n \sim \mathcal{N}(0,1)$, $N_n = 0$



Asynchornous SA algorithm

Let $X_n = (X_n(1), ..., X_n(d))^\top \in \mathbb{R}^d$. At each iteration n, only a random set of coordinates $I_n \subset \{1, \dots, d\}$ of X_n are updated: for $1 \le i \le d$,

$$X_{n+1}(i) = \begin{cases} X_n(i) + \alpha_{\mathcal{I}_n(i)}[h(X_n; i) + M_{n+1}(i) + N_{n+1}(i)] & \text{if } i \in I_n \\ X_n(i) & \text{otherwise} \end{cases}$$

where $\mathcal{I}_n(i)$ is the number of updates of the i-th coordinate up to time n, i.e., $\mathcal{I}_n(i) := \sum_{m=0}^{n} 1[i \in I_m]$ and $h(x; i)$ is the i-th entry of $h(x)$.

Asynchornous SA algorithm

Assumptions:

(B1) (Linearly growing $\mathcal{I}_n(i)$) There exists a deterministic $\Delta > 0$ such that for all $1 \le i \le d$, $\liminf_{n\to\infty} \mathcal{I}_n(i)/n \ge \Delta$ a.s. Furthermore, for $c > 0$ and all $1 \le i, j \le d$, the limit of $\big(\sum_{m=\mathcal{I}_n(i)}^{\bar{\mathcal{I}}_n(c,i)} \alpha_m\big)/\big(\sum_{m=\mathcal{I}_n(j)}^{\bar{\mathcal{I}}_n(c,j)} \alpha_m\big)$ as $n \to \infty$ exists a.s. where $\bar{\mathcal{I}}_n(c,i) := \mathcal{I}_{N_n(c)}(i)$ with $N_n(c) := \min\big\{N > n : \sum_{m=n+1}^{N} \alpha_m > c\big\}$.

(B2) (Slowly decreasing α_n) The sequence $\{\alpha_n\}$ satisfies that $\alpha_{n+1} \le \alpha_n$ eventually and that for $c \in (0,1)$, $\sup_n \alpha_{\lfloor cn \rfloor}/\alpha_n < \infty$ and $\big(\sum_{m=0}^{\lfloor cn \rfloor} \alpha_m\big)/\big(\sum_{m=0}^{n} \alpha_m\big) \to 1$, where $\lfloor cn \rfloor$ is the integer part of cn.

Convergence

Let $X_n = (X_n(1), ..., X_n(d))^\top \in \mathbb{R}^d$ satisfying for all n and for $1 \leq i \leq d$,

$$X_{n+1}(i) = \begin{cases} X_n(i) + \alpha_{\mathcal{I}_n(i)}[h(X_n; i) + M_{n+1}(i) + N_{n+1}(i)] & \text{if } i \in I_n \\ X_n(i) & \text{otherwise} \end{cases}$$

Theorem. If (A1)-(A5) and (B1)-(B2) hold, for any initial condition X_0,

$$\lim_{n \to \infty} X_n = x^\star, \quad \text{almost surely,}$$

where $x^\star$ is the only globally stable point of $\dot{x} = h(x)$.

Application to discounted RL problems

Can we apply the R-M algorithm to evaluate the value function in an online manner?

- Remember that $V^\star$ is a fixed point of L, i.e., $L(V^\star) - V^\star = 0$

$$\forall s,\ V^\star(s) = \max_{a \in A_s}(r(s,a) + \lambda \sum_j p(j|s,a)V^\star(j))$$

- At time n, assume we have an estimate V_n of $V^\star$. Assume that we can try all actions a and observe the new state $S_{n+1}(s_n, a)$ a r.v. such that $\mathbb{P}[S_{n+1} = j | s_n, a] = p(j|s_n, a)$. We can compute

$$y_n = \max_{a \in A_{s_n}} \left(r(s_n, a) + \lambda V_n(S_{n+1}(s_n, a)) \right) - V_n(s_n)$$

Application to discounted RL problems

To apply the R-M algorithm to evaluate $V^\star$, we would need that $\mathbb{E}[y_n] = L(V_n)(s_n) - V_n(s_n)$. However:

$$\begin{aligned} L(V_n)(s_n) - V_n(s_n) &= \max_{a \in A_{s_n}} (r(s_n, a) + \lambda \sum_j p(j|s_n, a) V_n(j)) - V_n(s_n) \\ &= \max_{a \in A_{s_n}} (r(s_n, a) + \lambda \mathbb{E}(V_n(S_{n+1}(s_n, a)))) - V_n(s_n) \end{aligned}$$

whereas

$$\mathbb{E}[y_n] = \mathbb{E}[\max_{a \in A_{s_n}} (r(s_n, a) + \lambda V_n(S_{n+1}(s_n, a)))] - V_n(s_n)$$

and $\mathbb{E}[\max(X, Y)] \neq \max(\mathbb{E}[X], \mathbb{E}[Y])$

Q-function

We apply the R-M algorithm to the estimate the Q-function instead.

$Q(s, a)$ is the maximum expected reward starting from state s and taking action a:

$$Q(s, a) = r(s, a) + \lambda \sum_j p(j|s, a) V^\star(j)$$

Note that $V^\star(s) = \max_{a \in A_s} Q(s, a)$, and hence

$$Q(s, a) = r(s, a) + \lambda \sum_j p(j|s, a) \max_b Q(j, b)$$

Q **is the fixed point of an operator** H (defined on $\mathbb{R}^{S \times A}$)

$$(HQ)(s, a) = r(s, a) + \lambda \sum_j p(j|s, a) \max_b Q(j, b)$$

Q-function and Stochastic Approximation

- At time n, assume we have an estimate Q_n of Q. Assume that we can try all actions a and observe the new state $S_{n+1}(s_n, a)$ a r.v. such that $\mathbb{P}[S_{n+1} = j | s_n, a] = p(j|s_n, a)$. We can compute for all a

$$y_n = r(s_n, a) + \lambda \max_b Q_n(S_{n+1}(s_n, a), b)) - Q_n(s_n, a)$$

- y_n can be used in the R-M algorithm converging to Q, because

$$\begin{aligned}
\mathbb{E}[y_n] &= r(s_n, a) + \lambda \mathbb{E}[\max_b Q_n(S_{n+1}(s_n, a), b))] - Q_n(s_n, a) \\
&= r(s_n, a) + \lambda \sum_j p(j|s_n, a) \max_b Q_n(j, b) - Q_n(s_n, a) \\
&= H(Q_n)(s_n, a) - Q_n(s_n, a)
\end{aligned}$$

Q-learning

We observe the trajectory of the system under some behaviour policy π_b: $(s_n, a_n, r_n)_{n\geq 0}$, where r_n is the reward collected in the n-th step.

Parameter. Step sizes (α_n)

1. Initialization. Select a Q-function $Q_0 \in \mathbb{R}^{S\times A}$

2. Q-function improvement. For $n \geq 0$. Update the Q-function as follows: $\forall s, a$

$$Q_{n+1}(s_n, a_n) = Q_n(s_n, a_n)$$

$$+ 1_{(s,a)=(s_n,a_n)} \alpha_{\nu_n(s_n,a_n)} \left[r_n + \gamma \max_{b\in\mathcal{A}} Q_n(s_{n+1}, b) - Q_n(s_n, a_n) \right]$$

where $\nu_n(s,a) := \sum_{m=0}^{n} 1[(s,a) = (s_m, a_m)]$.

Q-learning convergence

Theorem. Assume that the step sizes (α_n) satisfy (A2), and that the sets (I_n) defined through the behaviour policy π_b satisfy (B1)-(B2). For any discount factor $\lambda \in (0,1)$:

$$\lim_{n\to\infty} Q_n = Q, \quad \text{almost surely.}$$

The conditions required in the above theorem are met if $\alpha_n = \frac{1}{n+1}$ and if the behaviour policy yields an irreducible Markov chain (e.g. unichain model). They are also met for the ϵ-greedy policy: it selects an action uniformly at random w.p. ϵ and $a_n \in \arg\max_{a\in A_{s_n}} Q_n(s_n, a)$ w.p. $1-\epsilon$.

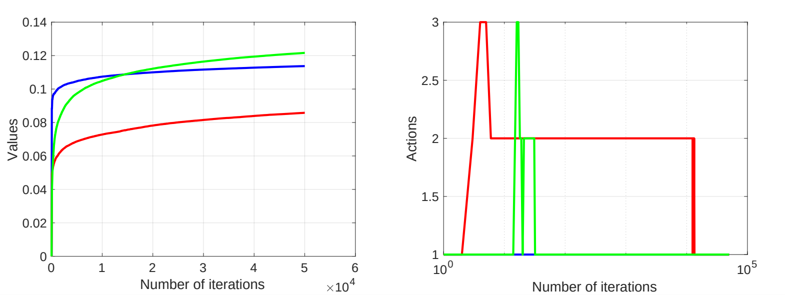
Q-learning: demo

The crawling robot ...

https://www.youtube.com/watch?v=2iNrJx6IDEo

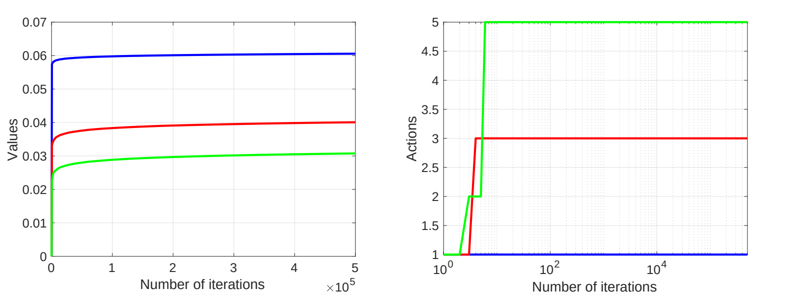
Q-learning: example

Randomly selected MDPs with 3 states and 3 actions.



Q-learning: example

Randomly selected MDPs with 3 states and 20 actions.



Lecture 4: outline

1. Q-learning and stochastic approximation
2. **SARSA (State-Action-Reward-State-Action) algorithm**
3. Improving Q-learning convergence rate

SARSA

State-Action-Reward-State-Action

- **On-policy** algorithm: we select actions according to a policy defined through Q_n
- ϵ-greedy policy:

$$\text{w.p. } 1-\epsilon, \text{select } a \in \arg\max_b Q_n(s_n, b)$$
$$\text{w.p. } \epsilon, \text{select } a \text{ uniformly at random}$$

SARSA

Parameter. Step sizes (α_n)

1. **Initialization.** Select a Q-function $Q_0 \in \mathbb{R}^{S \times A}$
2. Q-**function improvement.** For $n \geq 0$, select an action a_n according to a policy $\pi_n(Q_n)$, observe $r(s_n, a_n)$ and the next state s_{n+1}, select a_{n+1} according to $\pi_n(Q_n)$. Update the Q-function as follows: $\forall s, a$,

$$
\begin{aligned}
&Q_{n+1}(s,a) = Q_n(s,a) \\
&+ 1_{(s,a)=(s_n,a_n)} \alpha_n \left(r(s,a) + \lambda Q_n(s_{n+1}, a_{n+1}) - Q_n(s,a) \right)
\end{aligned}
$$

SARSA convergence

Theorem. Assume that the step sizes (α_n) satisfy (A2), and that SARSA is based on the ϵ-greedy policy. For any discount factor $\lambda \in (0, 1)$:

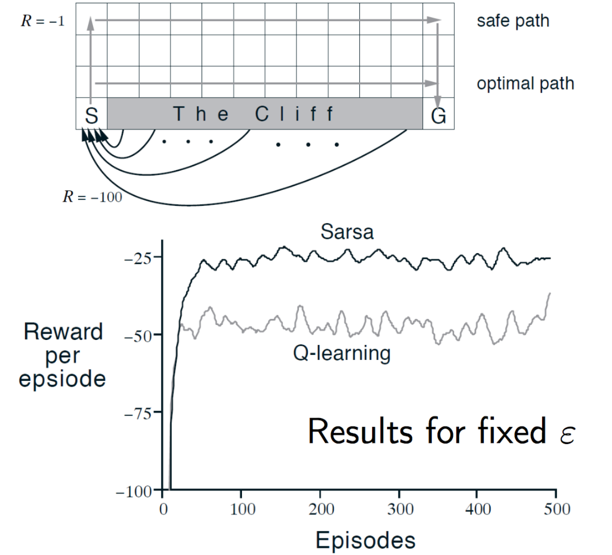
$$\lim_{n\to\infty} Q_n = Q^\epsilon, \quad \text{almost surely}$$

where $Q^\epsilon(s,a) = r(s,a) + \lambda \sum_j p(j|s,a) V^{\star\epsilon}(j)$ and $V^{\star\epsilon}$ is the value function of the policy selecting an optimal action w.p. $1-\epsilon$ and a (uniform) random action w.p. ϵ.

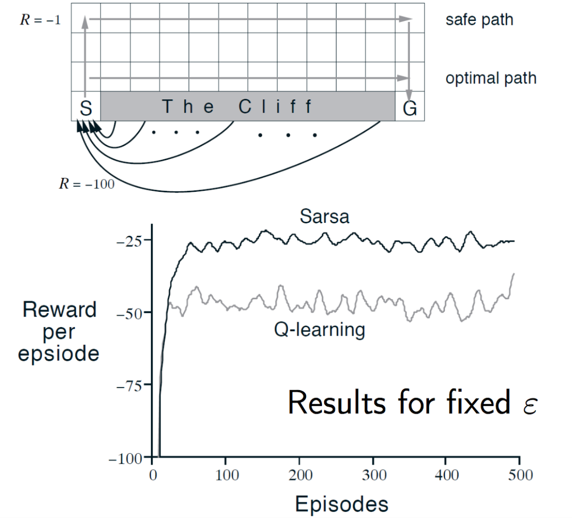
On-policy algorithms are "safer", they do not explore (state, action) pairs yielding very negative rewards (for fixed exploration rate $\epsilon > 0$, SARSA does not converge to the optimal policy)

On vs Off-policy Algorithms

R. Sutton, NIPS 2015



On vs Off-policy Algorithms



https://studywolf.wordpress.com/2013/07/01/reinforcement-learning-sarsa-vs-q-learning/

Lecture 4: outline

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Slow Convergence of Q-learning

A synchronous version of Q-learning with $\alpha_k = \frac{1}{k+1}$, where every $(s, a) \in S \times A$ is observed in each round k, and

$$
\begin{aligned}
Q_{k+1} &= (1 - \alpha_k) Q_k + \alpha_k (H Q_k - \varepsilon_k) \\
&= \frac{1}{k+1} \sum_{j=0}^{k} (H Q_j - \varepsilon_j)
\end{aligned}
$$

Note that the estimation error ε_k is asymptotically averaged out, i.e., $\frac{1}{k+1} \sum_{j=0}^{k} \varepsilon_j \to 0$ a.s. as $k \to \infty$.

The convergence of recursion $Q_{k+1} = \frac{1}{k+1} \sum_{j=0}^{k} H Q_j$ is slower than $Q_{k+1} = H Q_k$ since the former is dragged down by immature $\{Q_j\}_{j<k}$.

Speedy Q-learning

Synchronous version of speedy Q-learning [Azar et al., NIPS 2011] is

$$\begin{aligned} Q_{k+1} &= \alpha_k Q_k + (1-\alpha_k)[kHQ_k - (k-1)HQ_{k-1} - \varepsilon_k] \\ &= HQ_k + \frac{1}{k+1}(HQ_{-1} - HQ_k) - \frac{1}{k+1}\sum_{j=0}^{k} \varepsilon_j \end{aligned}$$

which is asymptotically $Q_{k+1} = HQ_k$.

Asynchronous Speedy Q-learning

We observe the trajectory under behaviour policy π_b: $(s_n, a_n, r_n)_{n\geq 0}$.

Parameter. Initial Q-function $Q_0 \in \mathbb{R}^{S\times A}$

Initialization. Set $Q_{-1} = Q_0$, $k = 0$

Main loop. Repeat 1 & 2 along $(s_n, a_n, r_n)_{n\geq 0}$.

1. **Speedy Q update.** Let $\alpha_k = \frac{1}{k+1}$ and define $T_k Q(s,a) = \frac{1}{\nu_{k,n}(s,a)} \sum_{i=1}^{\nu_{k,n}(s,a)} [r_{k,i}(s,a) + \lambda \max_{b\in A} Q(s_{k,i}(s,a), b)]$ where $r_{k,i}$ and $s_{k,i}$ are the reward and next state at i-th visit of (s,a) in round k, and $\nu_{k,n}(s,a)$ is the number of visits at (s,a) up to time n in round k.

$$
\begin{aligned}
Q_{k+1}(s_n, a_n) &= (1-\alpha_k) Q_k(s_n, a_n) \\
&+ \alpha_k \left(k T_k Q_k(s_n, a_n) - (k-1) T_k Q_{k-1}(s_n, a_n)\right) .
\end{aligned}
$$

2. **Move to next round.** If all possible state-action pairs have been visited, increase k by 1.

Zap Q-learning [Devraj & Meyn NIPS 2017]

Achieving the fastest convergence among Q-learning like algorithms
Designing the optimal gain matrix G_n so that the recursion $X_{n+1} = X_n + G_n f(X_n)$ converges the fastest

Asymptotic covariance: a CTL states that $\sqrt{n}(X_n - X^\star) \sim \mathcal{N}(0, \Sigma)$.

Newton-Raphson method: How can we design G_n so as to minimise the positive semidefinite asymptotic covariance matrix Σ.

The asymptotic covariance is related to the sample complexity in some sense...

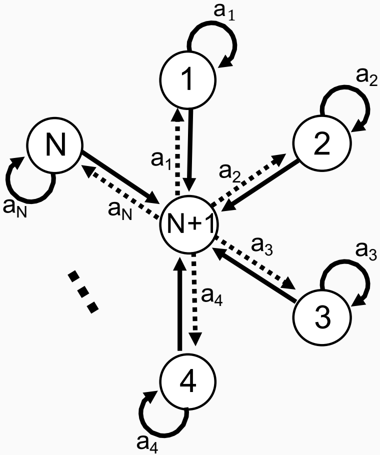
Zap Q-learning [Devraj & Meyn NIPS 2017]

Parameter. Initial Q-function $Q_0 \in \mathbb{R}^{S \times A}$ and step size α_n, β_n

1. **Initialization.** Let Q_0 be a column vector in $\mathbb{R}^{S \times A}$
2. **Zap Q update.** Let $\hat{A}_{n+1} = \hat{A}_n + \beta_{n+1}[A_{n+1} - \hat{A}_n]$ and $A_{n+1} = 1_{s_n,a_n}[\lambda 1_{s_{n+1},a^*_{n+1}} - 1_{s_n,a_n}]^\top$ where $a^*_{n+1} = \arg\max_{a \in A} Q_n(s_{n+1}, a)$ and $1_{s_n,a_n}$ is a column vector whose entry is 1 at (s_n, a_n) and 0 elsewhere. Find $\hat{A}_n^\dagger$ which is pseudo inverse of $\hat{A}_n$. Then,

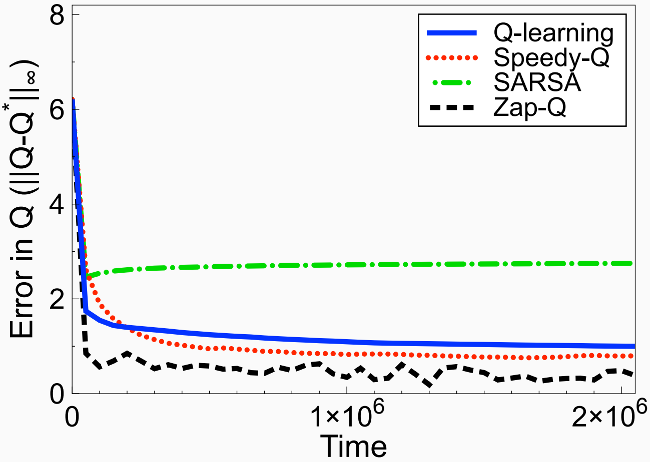
$$Q_{n+1} = Q_n - \alpha_{n+1} \hat{A}_{n+1}^\dagger \left(r_n + \lambda \max_{b \in A} Q_n(s_{n+1}, b) - Q_n(s_n, a_n) \right) 1_{s_n,a_n}$$

Numerical experiments



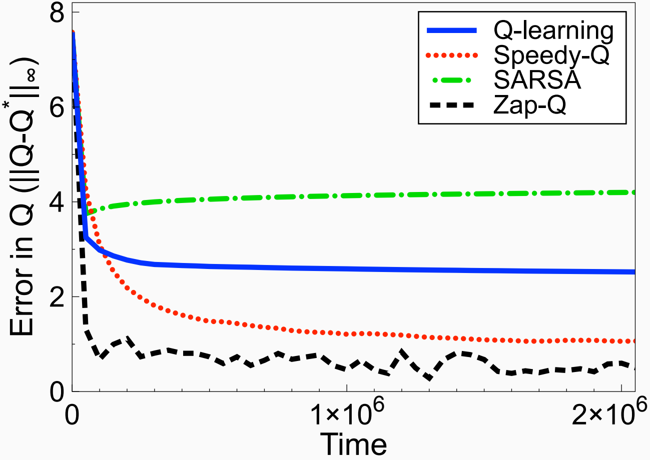
Numerical experiments

$\lambda = 0.8$



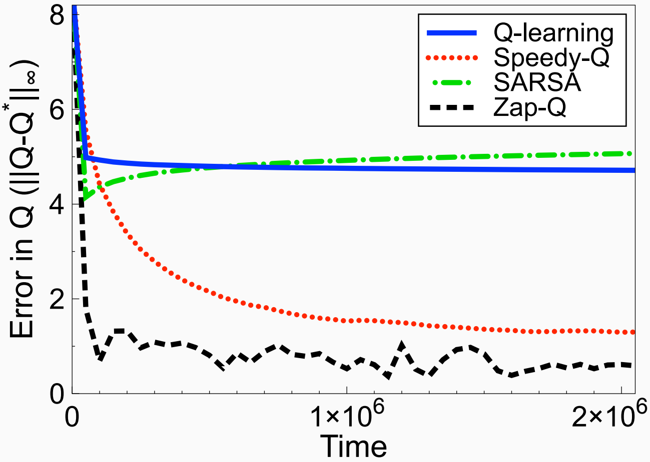
Numerical experiments

$\lambda = 0.9$



Numerical experiments

$\lambda = 0.95$



References

Stochastic approximation

- A Stochastic Approximation method. H. Robbins, S. Monro. 1951, Annals of Math. Stat., 22.
- Random Iterative Models. M. Duflo. 1997, Springer.
- Stochastic Approximation and Recursive Algorithms. H. Kushner, H. Yin. 2003, Springer.
- Stochastic Approximation: A Dynamical Systems viewpoint. V. Borkar. 2008, Cambridge Univ.

Improved Q-learning

- Speedy Q-learning. M. Azar et al., proc. of NIPS 2011.
- Zap Q-learning. A. Devraj, S. Meyn, proc. of NIPS 2017.