

Number Systems

Chapter 9

Dr. Ronald H.Y. Chung

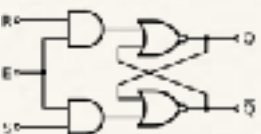


THE UNIVERSITY OF HONG KONG

DEPARTMENT OF
COMPUTER SCIENCE

The Decimal System

- ❖ System based on decimal digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) to represent numbers
- ❖ For example the number 83 means eight tens plus three:
 - ❖ $83 = (8 * 10) + 3$
- ❖ The number 4728 means four thousands, seven hundreds, two tens, plus eight:
 - ❖ $4728 = (4 * 1000) + (7 * 100) + (2 * 10) + 8$
- ❖ The decimal system is said to have a base, or radix, of 10. This means that each digit in the number is multiplied by 10 raised to a power corresponding to that digit's position:
 - ❖ $83 = (8 * 10^1) + (3 * 10^0)$
 - ❖ $4728 = (4 * 10^3) + (7 * 10^2) + (2 * 10^1) + (8 * 10^0)$



Decimal Fractions

- ❖ The same principle holds for decimal fractions, but negative powers of 10 are used. Thus, the decimal fraction 0.256 stands for 2 tenths plus 5 hundredths plus 6 thousandths:

- ❖ $0.256 = (2 * 10^{-1}) + (5 * 10^{-2}) + (6 * 10^{-3})$

- ❖ A number with both an integer and fractional part has digits raised to both positive and negative powers of 10:

- ❖ $472.256 = (4 * 10^2) + (7 * 10^1) + (2 * 10^0) + (2 * 10^{-1}) + (5 * 10^{-2}) + (6 * 10^{-3})$

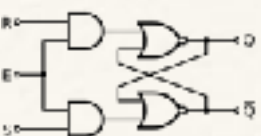
4	7	2	2	5	6
100s	10s	1s	tenths	hundredths	thousandths
10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}
position 2	position 1	position 0	position -1	position -2	position -3

- ❖ *Most significant digit*

- ❖ The leftmost digit (carries the highest value)

- ❖ *Least significant digit*

- ❖ The rightmost digit



Positional Number Systems

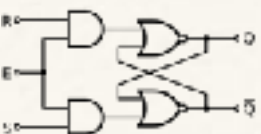
- ❖ Each number is represented by a string of digits in which each digit position i has an associated weight r^i , where r is the radix, or base, of the number system.
- ❖ The general form of a number in such a system with radix r is
 - ❖ $(\dots a_3a_2a_1a_0.a_{-1}a_{-2}a_{-3}\dots)_r$
 - ❖ where the value of any digit a_i is an integer in the range $0 \leq a_i < r$. The dot between a_0 and a_{-1} is called the radix point
- ❖ Example positional number systems with base 7

Position	4	3	2	1	0	-1
Value in exponential form	7^4	7^3	7^2	7^1	7^0	7^{-1}
Decimal value	2401	343	49	7	1	1/7



The Binary System

- ❖ Only two digits, 1 and 0
- ❖ Represented to the base 2
- ❖ The digits 1 and 0 in binary notation have the same meaning as in decimal notation:
 - ❖ $0_2 = 0_{10}$
 - ❖ $1_2 = 1_{10}$
- ❖ To represent larger numbers each digit in a binary number has a value depending on its position:
 - ❖ $10_2 = (1 * 2^1) + (0 * 2^0) = 2_{10}$
 - ❖ $11_2 = (1 * 2^1) + (1 * 2^0) = 3_{10}$
 - ❖ $100_2 = (1 * 2^2) + (0 * 2^1) + (0 * 2^0) = 4_{10}$
- ❖ Again, fractional values are represented with negative powers of the radix:
 - ❖ $1001.101 = 2^3 + 2^0 + 2^{-1} + 2^{-3} = 9.625_{10}$



Conversion Between Binary and Decimal

- ❖ Binary notation to decimal notation:
 - ❖ Multiply each binary digit by the appropriate power of 2 and add the results
- ❖ Decimal notation to binary notation:
 - ❖ Integer and fractional parts are handled separately



Converting Integer Part

- For the integer part, recall that in binary notation, an integer represented by

$$b_{m-1}b_{m-2}\dots b_2b_1b_0 \quad b_i = 0 \text{ or } 1$$

- This integer has the value

$$(b_{m-1} * 2^{m-1}) + (b_{m-2} * 2^{m-2}) + \dots + (b_1 * 2^1) + b_0$$

- Suppose it is required to convert a decimal integer N into binary form. If we divide N by 2, in the decimal system, and obtain a quotient N_1 and a remainder R_0 , we may write

$$N = 2 * N_1 + R_0 \quad R_0 = 0 \text{ or } 1$$

- Next, we divide the quotient N_1 by 2. Assume that the new quotient is N_2 and the new remainder R_1 . Then

$$N_1 = 2 * N_2 + R_1 \quad R_1 = 0 \text{ or } 1$$

- so that

$$N = 2(2N_2 + R_1) + R_0 = (N_2 * 2^2) + (R_1 * 2^1) + R_0$$

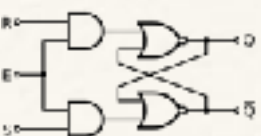
- If next

$$N_2 = 2N_3 + R_2$$

- we have

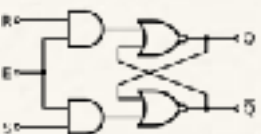
$$N = (N_3 * 2^3) + (R_2 * 2^2) + (R_1 * 2^1) + R_0$$

$$\begin{array}{r} N_{m+1} \\ 2 \overline{) N_m} \\ \underline{- 2N_{m+1}} \\ R_m \end{array}$$

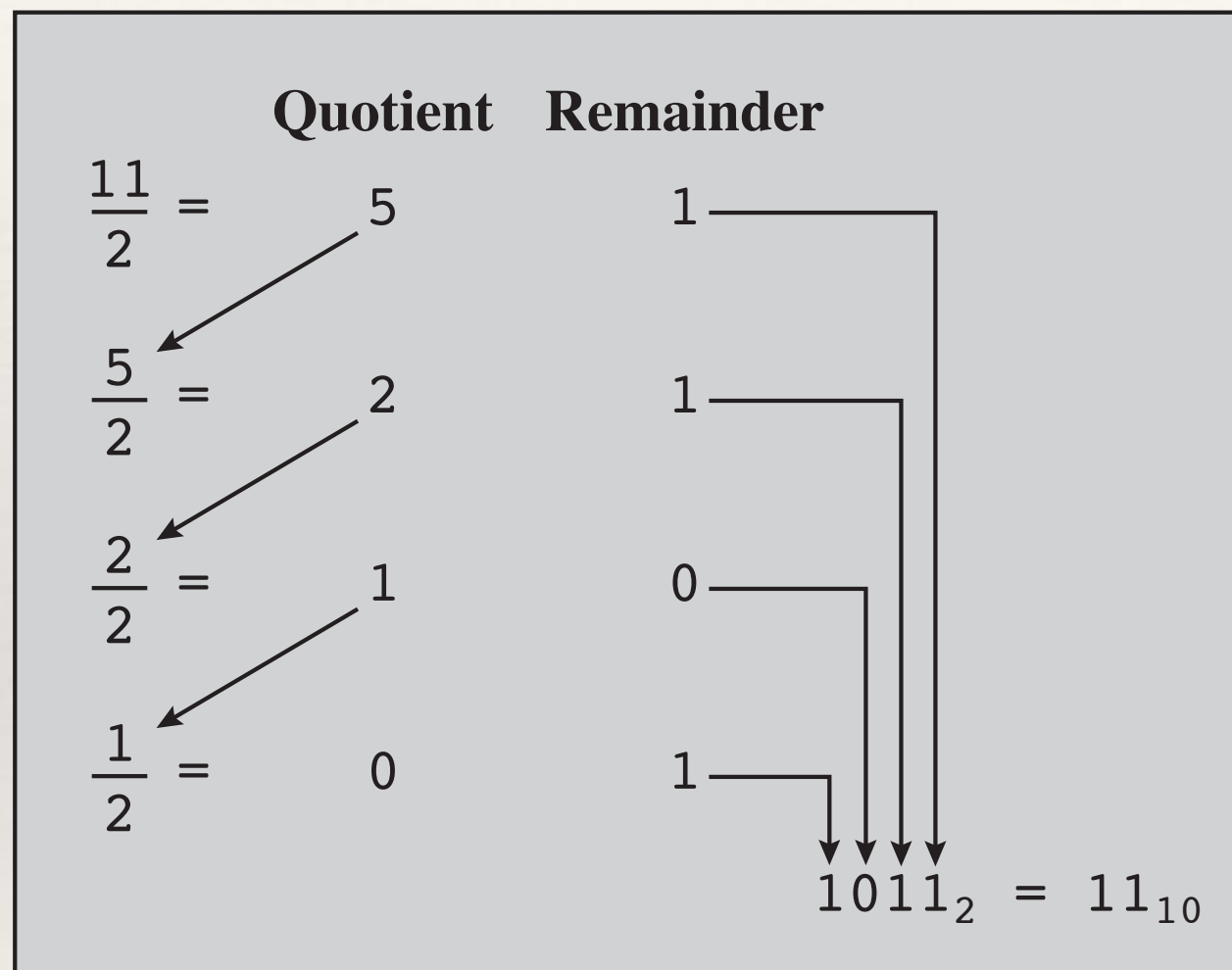


Converting Integer Part (Cont'd)

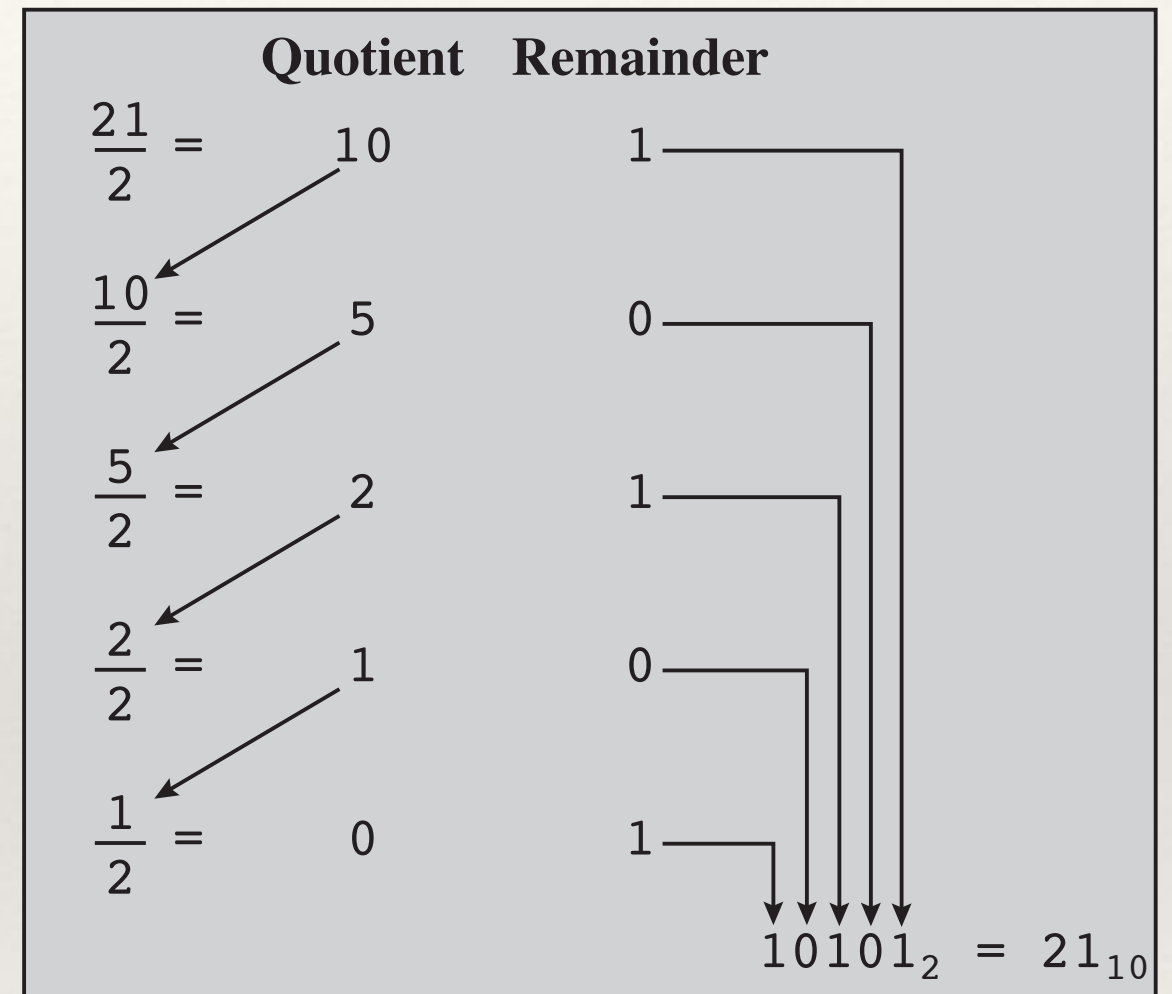
- ❖ Because $N > N_1 > N_2 \dots$,
 - ❖ Continuing this sequence will eventually produce a quotient $N_{m-1} = 1$ (except for the decimal integers 0 and 1, whose binary equivalents are 0 and 1, respectively) ; and
 - ❖ A remainder R_{m-2} , which is 0 or 1
 - ❖ Then
 - ❖ $N = (1 * 2^{m-1}) + (R_{m-2} * 2^{m-2}) + \dots + (R_2 * 2^2) + (R_1 * 2^1) + R_0$
 - ❖ which is the binary form of N . Hence, we convert from base 10 to base 2 by repeated divisions by 2. The remainders and the final quotient, 1, give us, in order of increasing significance, the binary digits of N .



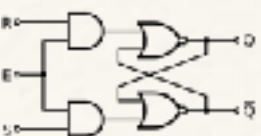
Example Decimal to Binary Conversion for Integers



(a) 11_{10}

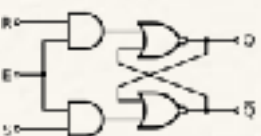


(b) 21_{10}



Converting Fractional Part

- ❖ For the fractional part, recall that in binary notation, a number with a value between 0 and 1 is represented by
 - ❖ $0.b_{-1}b_{-2}b_{-3}\dots$ $b_i = 0$ or 1
- ❖ This fractional part has the value
 - ❖ $(b_{-1} * 2^{-1}) + (b_{-2} * 2^{-2}) + (b_{-3} * 2^{-3}) \dots$
- ❖ This can be rewritten as
 - ❖ $2^{-1} * (b_{-1} + 2^{-1} * (b_{-2} + 2^{-1} * (b_{-3} + \dots) \dots))$
- ❖ Suppose we want to convert the number F ($0 < F < 1$) from decimal to binary notation. We know that F can be expressed in the form
 - ❖ $F = 2^{-1} * (b_{-1} + 2^{-1} * (b_{-2} + 2^{-1} * (b_{-3} + \dots) \dots))$
- ❖ If we multiply F by 2, we obtain,
 - ❖ $2 * F = b_{-1} + 2^{-1} * (b_{-2} + 2^{-1} * (b_{-3} + \dots) \dots)$



Converting Fractional Part (Cont'd)

- ❖ From this equation, we see that the integer part of $(2 * F)$, which must be either 0 or 1 because $0 < F < 1$, is simply b_{-1} . So we can say $(2 * F) = b_{-1} + F_1$, where $0 < F_1 < 1$ and where
 - ❖ $F_1 = 2^{-1} * (b_{-2} + 2^{-1} * (b_{-3} + 2^{-1} * (b_{-4} + \dots) \dots))$
- ❖ To find b_{-2} , we repeat the process.
- ❖ At each step, the fractional part of the number from the previous step is multiplied by 2.
 - ❖ The digit to the left of the decimal point in the product will be 0 or 1 and contributes to the binary representation, starting with the most significant digit.
 - ❖ The fractional part of the product is used as the multiplicand in the next step.



Example Decimal to Binary Conversion for Fractions

Product	Integer Part	0.110011 ₂
$0.81 \times 2 = 1.62$	1	
$0.62 \times 2 = 1.24$	1	
$0.24 \times 2 = 0.48$	0	
$0.48 \times 2 = 0.96$	0	
$0.96 \times 2 = 1.92$	1	
$0.92 \times 2 = 1.84$	1	

(a) $0.81_{10} = 0.110011_2$ (approximately)

Product	Integer Part	0.01 ₂
$0.25 \times 2 = 0.5$	0	
$0.5 \times 2 = 1.0$	1	

(b) $0.25_{10} = 0.01_2$ (exactly)

Hexadecimal Notation

- ❖ Binary digits are grouped into sets of four bits, called a *nibble*
- ❖ Each possible combination of four binary digits is given a symbol, as follows:

0000 = 0 0100 = 4 1000 = 8 1100 = C

0001 = 1 0101 = 5 1001 = 9 1101 = D

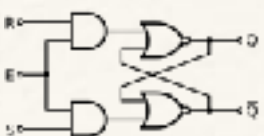
0010 = 2 0110 = 6 1010 = A 1110 = E

0011 = 3 0111 = 7 1011 = B 1111 = F

- ❖ Because 16 symbols are used, the notation is called hexadecimal and the 16 symbols are the hexadecimal digits
- ❖ Thus

$$❖ 2C_{16} = (2_{16} * 16^1) + (C_{16} * 16^0) = (2_{10} * 16^1) + (12_{10} * 16^0) = 44$$

Decimal (base 10)	Binary (base 2)	Hexadecimal (base 16)
0	0000	0
1	0001	1
2	0010	2
3	0011	3
4	0100	4
5	0101	5
6	0110	6
7	0111	7
8	1000	8
9	1001	9
10	1010	A
11	1011	B
12	1100	C
13	1101	D
14	1110	E
15	1111	F
16	0001 0000	10
17	0001 0001	11
18	0001 0010	12
31	0001 1111	1F
100	0110 0100	64
255	1111 1111	FF
256	0001 0000 0000	100



Hexadecimal Notation (Cont'd)

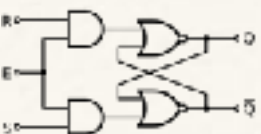
Not only used for representing integers but also as a concise notation for representing any sequence of binary digits

Reasons for using hexadecimal notation are:

It is more compact than binary notation

In most computers, binary data occupy some multiple of 4 bits, and hence some multiple of a single hexadecimal digit

It is extremely easy to convert between binary and hexadecimal notation



Chapter 9 – Summary

- ❖ The decimal system
- ❖ Positional number systems
- ❖ The binary system
- ❖ Converting between binary and decimal
 - ❖ Integers
 - ❖ Fractions
- ❖ Hexadecimal notation

