Mathematical Foundations for Robotics: From Groups and Vector Spaces to Lie Groups, Screw Theory, and Kinematics

Summary of the lectures of ***Mechanics of Mechanisms and Machines***
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(Super incomplete draft; Check correctness.)

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Contents

Intro

These notes are written as a self-contained, graduate-level introduction to the mathematical language of modern robotics kinematics.

1 The Rigid Body and Kinematics

Definition 1.1 (Kinematics vs. Dynamics)**.** ***Kinematics*** *is the study of motion (position, velocity, acceleration) without considering the forces or torques that cause it.* ***Dynamics*** *is the study of motion* with *consideration of forces and torques (e.g., using Newton's laws).*

These notes, and much of the main notes, focus on kinematics.

The Rigid Body Assumption

A **rigid body** is an idealized collection of particles where the distance between any two particles remains constant, regardless of any forces applied.

This is the fundamental assumption of these notes. While no real object is perfectly rigid (materials deform), it is an excellent model for robot links, tools, and most solid objects in mechanics.

Because the body is rigid, we don't need to track every particle. We only need to describe the overall **pose** of the body in space.

2 Configuration: Describing the Pose

To describe the pose of a rigid body, we need two pieces of information: where it is (position) and how it's oriented (orientation).

2.1 Reference Frames

The most important concept is the **reference frame**. All positions and orientations are described *relative* to a frame.

- **Space Frame** $\{s\}$: This is a fixed, inertial frame. You can think of it as the "world" or the "room" the robot is in. We'll denote vectors in this frame with a subscript s, like p_s.
- **Body Frame** $\{b\}$: This frame is rigidly attached to the rigid body and moves with it. We'll denote vectors in this frame with a subscript b, like p_b.

The **pose** of the rigid body is a complete description of the position and orientation of its body frame $\{b\}$ relative to the space frame $\{s\}$.

2.2 Position

Position is the easy part. We track the location of the **origin** of the body frame $\{b\}$ relative to the origin of the space frame $\{s\}$. This is just a vector $p \in \mathbb{R}^3$.

2.3 Orientation and SO(3)

Orientation describes how the body frame is "rotated." We can represent this by writing the axes of the body frame $\{b\}$ (call them $\hat{x}_b, \hat{y}_b, \hat{z}_b$) in the coordinates of the space frame $\{s\}$.

We collect these three vectors as the columns of a 3×3 matrix R:

$$R = \begin{bmatrix} \hat{x}_b & \hat{y}_b & \hat{z}_b \end{bmatrix}$$

This matrix R has special properties because the axes $\{\hat{x}_b, \hat{y}_b, \hat{z}_b\}$ form an orthonormal basis:

1. **Orthogonal**: The columns are mutually orthogonal and have unit length. This means $R^\top R = \mathrm{I}$. (This also means $R^{-1} = R^\top$).
2. **Right-handed**: To preserve the "handedness" of the coordinate system (i.e., not flip it into a mirror image), we require $\det(R) = +1$.

Definition 2.1 (Special Orthogonal Group SO(3))**.** *The set of all* 3×3 *matrices* R *such that* $R^\top R = \mathrm{I}$ *and* $\det(R) = 1$ *is called the **Special Orthogonal Group**, denoted* SO(3)*.*

This is the *mathematical object* for representing rotations, and it is a central topic in the main notes (Section 6). A matrix $R \in \mathrm{SO}(3)$ can be used to "rotate" vectors. If v_b is a vector in the body frame, its representation in the space frame is $v_s = Rv_b$.

2.4 Pose and SE(3)

We now combine position p and orientation R to describe the full pose. Let's find the coordinates of a point q (which is fixed to the rigid body) in the space frame. If the point has coordinates q_b in the body frame $\{b\}$, its space-frame coordinates q_s are:

$$q_s = p + Rq_b$$

This is a **rigid body transformation**. It's a linear rotation (Rq_b) followed by a linear translation ($+p$).

Homogeneous Transformation Matrices

The equation $q_s = p + Rq_b$ is an *affine* transformation, not a purely linear one (due to the $+p$). This is awkward for composition. We can make it linear by adding a "1" to our vectors (called *homogeneous coordinates*):

$$\bar{q}_s = \begin{bmatrix} q_s \\ 1 \end{bmatrix}, \qquad \bar{q}_b = \begin{bmatrix} q_b \\ 1 \end{bmatrix}$$

Now, we can write the transformation as a single 4×4 matrix multiplication:

$$\begin{bmatrix} q_s \\ 1 \end{bmatrix} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} q_b \\ 1 \end{bmatrix}$$

This 4×4 matrix H is called a **homogeneous transformation matrix**.

Definition 2.2 (Special Euclidean Group SE(3))**.** *The set of all* 4×4 *matrices* H *of the form*

$$H = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}, \quad \text{where } R \in \mathrm{SO}(3), p \in \mathbb{R}^3$$

is called the ***Special Euclidean Group****, denoted* SE(3)*.*

This is the *mathematical object* for representing rigid body poses, and it is a central topic in the main notes (Section 6).

Worked Example (2D Pose). Imagine a 2D body frame $\{b\}$ rotated by $90°$ counter-clockwise (about z) relative to $\{s\}$, with its origin at $p = (5, 2, 0)$. The rotation matrix is $R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. The position is $p = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}$. The SE(3) matrix is:

$$H = \begin{bmatrix} 0 & -1 & 0 & 5 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A point $q_b = (1, 0, 0)$ in the body frame is at what space-frame location?

$$\begin{bmatrix} q_s \\ 1 \end{bmatrix} = H \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0(1) + (-1)(0) + 0(0) + 5(1) \\ 1(1) + 0(0) + 0(0) + 2(1) \\ 0(1) + 0(0) + 1(0) + 0(1) \\ 0(1) + 0(0) + 0(0) + 1(1) \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

So, $q_s = (5, 3, 0)$. This makes sense: the point is "one unit in the body's x-direction," which is the space-frame's y-direction, starting from $(5, 2)$.

3 Velocity: Twists

Now we describe the *motion* of a rigid body. The pose $H(t)$ is now a function of time. We want to find its velocity.

3.1 Linear and Angular Velocity

The velocity of the body frame's origin is simply the time derivative of its position:

$$v_s(t) = \dot{p}(t)$$

The angular velocity is more complex. It's the time derivative of the orientation, $\dot{R}(t)$. Let's analyze $\dot{R}$. We know $R(t)^\top R(t) = \mathrm{I}$. Taking the time derivative (using the product rule):

$$\dot{R}^\top R + R^\top \dot{R} = 0$$

Let $\hat{\omega}_b = R^\top \dot{R}$. The equation shows $\hat{\omega}_b^\top = -\hat{\omega}_b$. This means $\hat{\omega}_b$ is a 3×3 **skew-symmetric matrix**.

Definition 3.1 (Skew-Symmetric Matrices $\mathfrak{so}(3)$)**.** *A matrix A is skew-symmetric if $A^\top = -A$. The set of all 3×3 skew-symmetric matrices is denoted $\mathfrak{so}(3)$. Any $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$ can be mapped to a matrix in $\mathfrak{so}(3)$ via the* hat *map:*

$$\hat{\omega} = \omega \wedge = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

The vee *map ($\vee$) is the inverse: $\hat{\omega} \vee = \omega$. A key property is that $\hat{a}b = a \times b$ (the cross product).*

The matrix $\hat{\omega}_b = R^\top \dot{R}$ corresponds to a vector ω_b, which we call the **body angular velocity**. We can also define the **spatial angular velocity** ω_s from the matrix $\hat{\omega}_s = \dot{R}R^\top$. (Note: $\hat{\omega}_s = R\hat{\omega}_b R^\top$, or $\omega_s = R\omega_b$.)

3.2 Twists and $\mathfrak{se}(3)$

We can now combine the linear velocity $v_s = \dot{p}$ and angular velocity ω_s of the body.

Definition 3.2 (Twist)**.** *A **spatial twist**, denoted $\boldsymbol{\xi}_s$, is a 6-dimensional vector that combines the angular velocity ω_s and the linear velocity v_s of the body frame's origin:*

$$\boldsymbol{\xi}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} \in \mathbb{R}^6$$

This $\boldsymbol{\xi}_s$ describes the complete, instantaneous velocity of the rigid body. This physical concept maps directly to the *Lie algebra* $\mathfrak{se}(3)$. Let's take the derivative of the homogeneous matrix H:

$$\dot{H} = \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix}$$

We can factor this in the space frame:

$$\dot{H} = \begin{bmatrix} \dot{R}R^\top R & v_s \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega}_s R & v_s \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega}_s & v_s \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

Let $\hat{\boldsymbol{\xi}}_s = \begin{bmatrix} \hat{\omega}_s & v_s \\ 0 & 0 \end{bmatrix}$. Then we have the fundamental equation:

$$\dot{H} = \hat{\boldsymbol{\xi}}_s H$$

Definition 3.3 (The Lie Algebra $\mathfrak{se}(3)$)**.** *The set of all* 4×4 *matrices* $\hat{\boldsymbol{\xi}}$ *of the form*

$$\hat{\boldsymbol{\xi}} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}, \quad \textit{where } \hat{\omega} \in \mathfrak{so}(3), v \in \mathbb{R}^3$$

is called the ***Lie algebra*** $\mathfrak{se}(3)$*. It is the set of all possible instantaneous velocities (twists) of a rigid body.*

This is the *mathematical object* for representing twists, and it is a central topic in the main notes (Section 6 & 7).

Worked Example (Types of Twists).

- **Pure Translation**: The body slides without rotating. $\omega_s = 0$, $v_s = (1, 0, 0)$ (moving along x-axis). $\boldsymbol{\xi}_s = (0, 0, 0, 1, 0, 0)^\top$. $\hat{\boldsymbol{\xi}}_s = \begin{bmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$. This is a *prismatic joint.*
- **Pure Rotation**: The body rotates about the space z-axis (which passes through its origin). $\omega_s = (0, 0, 1)$, $v_s = (0, 0, 0)$. $\boldsymbol{\xi}_s = (0, 0, 1, 0, 0, 0)^\top$. $\hat{\boldsymbol{\xi}}_s = \begin{bmatrix} 0&-1&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$. This is a *revolute joint.*

The main notes (Section 7.1, 8.1) build on this to show that *any* rigid motion (a "screw") is just a linear combination of these basic twists.

4 Forces: Wrenches

Kinematics describes motion; dynamics relates motion to **forces** and **torques** (also called **moments**).

4.1 Force and Torque

A **force** $f \in \mathbb{R}^3$ is a "push" or "pull" applied to the body. A **torque** (or **moment**) $m \in \mathbb{R}^3$ is a "twist" or "rotation" applied to the body. If a force f is applied at a point q, it creates a torque m_o about a reference point o (like the origin) given by the cross product:

$$m_o = (q - o) \times f$$

Just as we can combine all instantaneous velocities into a single $\boldsymbol{\xi}$, we can combine all forces and torques acting on a body into a single object.

Definition 4.1 (Wrench)**.** *A* ***wrench****, denoted $\boldsymbol{\mathcal{F}}$, is a* 6*-dimensional vector that combines the total torque m and total force f acting on a body, referenced to a specific frame (e.g., the space frame).*

$$\boldsymbol{\mathcal{F}}_s = \begin{bmatrix} m_s \\ f_s \end{bmatrix} \in \mathbb{R}^6$$

4.2 Duality: Power and the Reciprocal Product

Twists and wrenches are "dual" to each other. Their relationship is **power**.

- The power P_{lin} generated by a force f moving at velocity v is $P_{\text{lin}} = f^\top v$.
- The power P_{ang} generated by a torque m rotating at angular velocity ω is $P_{\text{ang}} = m^\top \omega$.

The total instantaneous power P generated by a wrench $\boldsymbol{\mathcal{F}}$ on a body moving with twist $\boldsymbol{\xi}$ is the sum of these two:

$$P = m_s^\top \omega_s + f_s^\top v_s$$

The Reciprocal Product

This power calculation $P = m^\top \omega + f^\top v$ is a key operation. It is the dot product of the $6D$ wrench and twist vectors.

$$P = \boldsymbol{\mathcal{F}} \circ \boldsymbol{\xi} = \begin{bmatrix} m \\ f \end{bmatrix} \cdot \begin{bmatrix} \omega \\ v \end{bmatrix}$$

This is called the **reciprocal product** in the main notes (Section 7.2).

If $\boldsymbol{\mathcal{F}} \circ \boldsymbol{\xi} = 0$, the wrench generates no power on the twist. This means the force/torque is "reciprocal" (or orthogonal) to the motion. This concept is *critical* for understanding constraints in parallel robots (Section 9).

5 Connection to Robot Mechanisms

These concepts are the building blocks for robotics:

1. A robot is a chain of rigid bodies (**links**).
2. Links are connected by **joints**, which *constrain* the relative motion.
3. A joint's allowed motion is described by a **twist**.
   - A **revolute (R) joint** (like an elbow) allows a pure rotation twist $\boldsymbol{\xi} = (\omega, r \times \omega)$.
   - A **prismatic (P) joint** (like a slider) allows a pure translation twist $\boldsymbol{\xi} = (0, v)$.
4. The total motion of the robot's end-effector is found by "adding up" the twists from each joint. This is the **Product of Exponentials (POE)** formula (Section 8.1).
5. The velocity of the end-effector is related to the joint velocities ($\dot{\theta}$) by the **Jacobian** (Section 8.2).
6. The forces at the end-effector are related to the torques at the joint motors by the **Jacobian** (using wrench duality).

You now have the physical motivation for *why* SO(3), SE(3), $\mathfrak{so}(3)$, $\mathfrak{se}(3)$, $\boldsymbol{\xi}$, and $\boldsymbol{\mathcal{F}}$ are the fundamental objects used in the main notes.

Exercises (with short solutions).

M.1 Pose Calculation: A body frame $\{b\}$ is at $p_s = (1, 2, 3)$ and rotated 90° *clockwise* about the space x-axis. What is its pose matrix $H \in \mathrm{SE}(3)$? *Solution:* Clockwise about x is $R_x(-\pi/2)$. $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-\pi/2) & -\sin(-\pi/2) \\ 0 & \sin(-\pi/2) & \cos(-\pi/2) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$. $H = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

M.2 Finding a Twist: A body is spinning at 2 rad/s about an axis passing through the point $r = (0, 5, 0)$ in the space frame, with direction $\omega_{\mathrm{dir}} = (0, 0, 1)$. What is the spatial twist $\boldsymbol{\xi}_s$? *Solution:* $\omega_s = 2 \cdot (0, 0, 1) = (0, 0, 2)$. The origin of the frame is at $p = 0$. The velocity of a point r on the axis is 0. But the velocity v_s is the velocity of the origin ($p = 0$) of a frame that is *on* the rotation axis. The formula for the linear velocity of a frame at p whose motion is defined by rotation ω_s about a point r is $v_s = -\omega_s \times (r - p) = -\omega_s \times r$. Wait, the formula in the main notes (Section 7) is simpler: for a rotation about an axis ω through r, the linear component is $v = r \times \omega = (0, 5, 0) \times (0, 0, 2) = (10, 0, 0)$. So $\boldsymbol{\xi}_s = (\omega_s, v_s) = (0, 0, 2, 10, 0, 0)^\top$. *Self-correction:* The formula $v = r \times \omega$ (or $-\omega \times r$) is correct. Let's use the cross product $\hat{\omega}_s r$: $\hat{\omega}_s r = \begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ 0 \end{bmatrix}$. So $v_s = -\omega_s \times r = (0, 0, 2) \times (0, 5, 0) = (-10, 0, 0)$. $\boldsymbol{\xi}_s = (\omega_s, v_s) = (0, 0, 2, -10, 0, 0)^\top$. *Note:* The main notes use $r \times \omega$ (Example 7), which gives $(10, 0, 0)$. Let's stick to that convention. $\boldsymbol{\xi}_s = (\omega, r \times \omega) = (0, 0, 2, 10, 0, 0)^\top$.

M.3 Power Calculation: A wrench $\boldsymbol{\mathcal{F}}_s = (m, f) = (0, 10, 0, 0, 0, 5)$ is applied to a body moving with twist $\boldsymbol{\xi}_s = (\omega, v) = (2, 0, 0, 1, 1, 0)$. What is the power? *Solution:* $P = \boldsymbol{\mathcal{F}} \circ \boldsymbol{\xi} = m^\top \omega + f^\top v$ $P = (0 \cdot 2 + 10 \cdot 0 + 0 \cdot 0) + (0 \cdot 1 + 0 \cdot 1 + 5 \cdot 0) = 0 + 0 = 0$. The wrench is reciprocal to the twist; no power is generated.

6 Summary and Refresher

6.1 Sets, functions, fields

Definition 6.1 (Set)**.** *A* set *is a collection of elements. We write $x \in S$ to denote that x is an element of S.*

A *function* $f: A \to B$ maps each $a \in A$ to a unique $b \in B$. The *image* $f(A) = \{f(a) \mid a \in A\}$; the *preimage* of $Y \subseteq B$ is $f^{-1}(Y) = \{a \in A \mid f(a) \in Y\}$.

A *field* (e.g. $\mathbb{R}, \mathbb{C}$) is a set with addition and multiplication obeying the usual laws (associativity, commutativity, distributivity, identities, inverses). We will work over $\mathbb{R}$ unless stated otherwise.

6.2 Real vector spaces

Definition 6.2 (Vector space)**.** *A real vector space V is a set equipped with addition $+$ and scalar multiplication λv for $\lambda \in \mathbb{R}$, satisfying closure, associativity, commutativity of addition, identities, inverses, and distributivity.*

Typical examples: $\mathbb{R}^n$, spaces of polynomials P_n, spaces of matrices $\mathbb{R}^{m\times n}$. A *subspace* $U \subseteq V$ is closed under addition and scalar multiplication. A set $\{v_i\}$ *spans* V if every $v \in V$ is a linear combination of $\{v_i\}$. It is *independent* if $\sum \alpha_i v_i = 0$ implies all $\alpha_i = 0$. A *basis* is spanning and independent; its size is the *dimension* $\dim V$.

6.3 Linear maps and matrices

A map $T: V \to W$ is *linear* if $T(\alpha v + \beta w) = \alpha T(v) + \beta T(w)$. With chosen bases, linear maps correspond to matrices. The *kernel* (null space) $\operatorname{null}(T) = \{v \mid T(v) = 0\}$ and the *image* $\operatorname{im}(T) = \{T(v) \mid v \in V\}$.

Theorem 6.3 (Rank–nullity)**.** *For finite-dimensional V,*

$$\dim V = \operatorname{rank}(T) + \dim \operatorname{null}(T). \tag{1}$$

6.4 Inner products and norms

An *inner product* $\langle \cdot, \cdot \rangle$ on V is symmetric, bilinear, and positive-definite. It induces a norm $\|v\| = \sqrt{\langle v, v\rangle}$. In $\mathbb{R}^n$ with the standard inner product $\langle x, y\rangle = x^\top y$, orthogonality means $\langle x, y\rangle = 0$.

6.5 Matrix exponential and logarithm

For a square matrix A,

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}, \qquad \log(\mathrm{I} + A) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} A^k \quad (\|A\| < 1). \tag{2}$$

If A and B commute ($AB = BA$), then $e^{A+B} = e^A e^B$. For noncommuting matrices, the Baker–Campbell–Hausdorff (BCH) series quantifies the deviation.

6.6 Multivariable calculus

Gradients, Jacobians, and differentials generalize directional rates of change. If $f\colon \mathbb{R}^n \to \mathbb{R}^m$, its Jacobian $J_f(x) \in \mathbb{R}^{m\times n}$ satisfies $f(x+\Delta x) \approx f(x) + J_f(x)\Delta x$.

Worked Example (Jacobian linearization). For $f(x) = \begin{bmatrix} \sin x_1 \\ x_1 x_2 \end{bmatrix}$, the Jacobian is $J_f(x) = \begin{bmatrix} \cos x_1 & 0 \\ x_2 & x_1 \end{bmatrix}$.

Exercises (with short solutions).

P.1 Show that the set of polynomials of degree $\leq n$ is a vector space and find a basis. *Solution:* Closure holds; a basis is $\{1, t, t^2, \dots, t^n\}$.

P.2 Prove rank–nullity for a 3×5 matrix by dimension counting. *Solution:* Columns span a subspace of $\mathbb{R}^3$ of dimension rank; nullity equals $5 - \text{rank}$.

7 Group Theory Essentials

Definition 7.1 (Group). *A group $(G, \cdot)$ is a set with a binary operation $\cdot$ satisfying closure, associativity, identity e, and inverses g^{-1} for all $g \in G$.*

7.1 Motivation from symmetry

Rotations of a rigid body compose associatively, have an identity (do nothing), and each rotation has an inverse (rotate back). Hence the set of all rotations with composition forms a group.

7.2 Subgroups, homomorphisms

A subset $H \subseteq G$ is a *subgroup* if it is a group under the same operation. A *homomorphism* $\phi\colon G \to H$ preserves the operation: $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$. The kernel $\ker\phi = \{g \mid \phi(g) = e\}$ is a normal subgroup. Cosets partition G and lead to quotient groups G/N.

Examples. $(\mathbb{R}, +)$, $(\mathbb{R} \setminus \{0\}, \times)$, the set of $n \times n$ invertible matrices $\mathrm{GL}(n)$ under multiplication, $\mathrm{SO}(3)$ under multiplication.

Exercises (with short solutions).

- **G.1** Show that $\mathrm{SO}(3)$ is a group. *Solution:* Closure by product of orthogonal matrices with determinant 1; associativity inherited from matrix multiplication; identity I; inverse $R^\top$.
- **G.2** Describe a nontrivial homomorphism from $(\mathbb{R}, +)$ to $(\mathbb{R}^+, \times)$. *Solution:* $\phi(t) = e^t$.

8 Vector Spaces and Linear Maps

8.1 Basics

See prerequisites. In robotics we constantly move between coordinate representations; linear maps and bases make this precise.

8.2 Null spaces, images, rank–nullity

Given $A \in \mathbb{R}^{m\times n}$, the solution set to $Ax = 0$ is $\mathrm{null}(A)$. The possible outputs form $\mathrm{im}(A)$. rank–nullity links these: $\mathrm{rank}(A) + \dim \mathrm{null}(A) = n$.

8.3 Matrix representation and change of basis

Let $\{e_i\}$ and $\{\tilde{e}_i\}$ be bases. The change-of-basis matrix P satisfies $[v]_{\tilde{e}} = P^{-1}[v]_e$. If T is linear, its matrix in the new basis is $\tilde{A} = P^{-1}AP$.

Worked Example (Orthogonal change of basis). If $Q \in \mathrm{O}(n)$, then $\tilde{A} = Q^\top AQ$. Orthogonal changes preserve inner products and lengths.

Exercises (with short solutions).

V.1 For $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, find $\mathrm{null}(A - \mathrm{I})$. *Solution:* $(A - \mathrm{I}) = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$; null space spanned by $(1, 0)^\top$.

V.2 Show that if Q is orthogonal, then $\|Qx\| = \|x\|$. *Solution:* $\|Qx\|^2 = x^\top Q^\top Qx = x^\top x$.

9 Euclidean Vector Spaces

9.1 Inner products, norms, and orthogonality

An *inner product space* $(V, \langle\cdot,\cdot\rangle)$ supports projections and orthogonal decompositions.

9.2 Projection theorem and Gram–Schmidt

Theorem 9.1 (Projection)**.** *Given a subspace $U \subset V$ and $v \in V$, there is a unique decomposition $v = u + w$ with $u \in U$ and $w \in U^{\perp}$. u is the projection of v onto U.*

Gram–Schmidt orthonormalizes any independent set $\{v_i\}$ to an orthonormal basis $\{q_i\}$.

9.3 Cross product and determinant

In $\mathbb{R}^3$, the cross product $a \times b$ is orthogonal to both a and b with magnitude $\|a\|\|b\| \sin\theta$. It satisfies $a \cdot (b \times c) = \det[a\ b\ c]$.

9.4 Coordinate transformations and orthogonal matrices

Orthogonal matrices Q preserve inner products: $\langle Qx, Qy\rangle = \langle x, y\rangle$. They model rotations and reflections.

Worked Example (Orthogonal projection). Project $v = (1, 2, 2)$ onto the line spanned by $u = (1, 1, 0)$. The unit direction is $\hat{u} = u/\|u\| = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, so $\mathrm{proj}_u(v) = \langle v, \hat{u}\rangle\hat{u} = \frac{3}{\sqrt{2}}\hat{u} = (\frac{3}{2}, \frac{3}{2}, 0)$.

Exercises (with short solutions).

E.1 Prove $a \cdot (b \times c) = \det[a\ b\ c]$. *Solution:* Expand by components; both sides are the scalar triple product.

E.2 Show $Q \in \mathrm{O}(n) \Rightarrow (Qx) \cdot (Qy) = x \cdot y$. *Solution:* As above via $Q^{\top}Q = \mathrm{I}$.

10 Affine Spaces and Frames

An *affine space* is like a vector space without a distinguished origin. Points p, q differ by a vector $v = \overrightarrow{pq}$. Reference frames in robotics are affine: a frame is an origin o plus orthonormal axes.

10.1 Homogeneous coordinates

Augmenting $x \in \mathbb{R}^3$ with a 1 yields $\bar{x} = \begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathbb{R}^4$. Rigid motions become linear maps in $\mathbb{R}^4$:

$$H = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}, \qquad \bar{x}' = H\bar{x}. \tag{3}$$

Worked Example (Composing motions). Two motions $H_1 = [R_1, p_1], H_2 = [R_2, p_2]$ compose to $H_2 H_1 = [R_2 R_1,\ R_2 p_1 + p_2]$.

Exercises (with short solutions).

A.1 Show that the set $\{[R, p] \mid R \in \mathrm{SO}(3),\, p \in \mathbb{R}^3\}$ is closed under multiplication. *Solution:* See composition formula above.

A.2 Explain why frames are affine objects. *Solution:* Moving the origin changes point coordinates by translation without altering vector differences.

11 Lie Groups (SO(3), SE(3)) and Lie Algebras

11.1 Manifolds in brief

A manifold is a space that locally looks like $\mathbb{R}^n$. Matrix Lie groups (like SO(3) and SE(3)) are smooth manifolds closed under matrix multiplication and inversion.

11.2 SO(3) and its algebra $\mathfrak{so}(3)$

Definition 11.1. $\mathrm{SO}(3) = \{R \in \mathbb{R}^{3\times 3} \mid R^\top R = \mathrm{I},\ \det R = 1\}$ *is a 3D manifold and a group. Its Lie algebra is*

$$\mathfrak{so}(3) = \{\hat{\omega} \in \mathbb{R}^{3\times 3} \mid \hat{\omega}^\top = -\hat{\omega}\} = \left\{ \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} : \omega \in \mathbb{R}^3 \right\}. \tag{4}$$

The *hat* map $\omega \mapsto \hat{\omega}$ embeds vectors into skew-symmetric matrices; the *vee* map is its inverse: $\hat{\omega}\vee = \omega$.

11.3 Exponential map and Rodrigues' formula

The matrix exponential maps $\mathfrak{so}(3)$ to SO(3): $R = \exp(\hat{\omega}\theta)$. For $\|\omega\| = 1$,

$$R = \mathrm{I} + \sin\theta\,\hat{\omega} + (1 - \cos\theta)\,\hat{\omega}^2. \tag{5}$$

11.4 SE(3) and its algebra $\mathfrak{se}(3)$

Definition 11.2. $\mathrm{SE}(3) = \left\{ \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} : R \in \mathrm{SO}(3),\ p \in \mathbb{R}^3 \right\}$ *is a 6D manifold. Its Lie algebra is*

$$\mathfrak{se}(3) = \left\{ \hat{\boldsymbol{\xi}} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} : \omega, v \in \mathbb{R}^3 \right\}. \tag{6}$$

The exponential map $\exp : \mathfrak{se}(3) \to \mathrm{SE}(3)$ is

$$\exp\left(\begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} t \right) = \begin{bmatrix} \exp(\hat{\omega}t) & \mathbf{J}(\omega t)\, v \\ 0 & 1 \end{bmatrix}, \tag{7}$$

where the left Jacobian of SO(3) is

$$\mathbf{J}(\phi) = \mathrm{I} + \frac{1 - \cos\phi}{\phi^2}\hat{\omega} + \frac{\phi - \sin\phi}{\phi^3}\hat{\omega}^2, \quad \phi = \|\omega\| t. \tag{8}$$

If $\omega = 0$ (pure translation), then $\exp\left(\hat{\boldsymbol{\xi}} t\right) = [\mathrm{I}, vt]$.

11.5 Adjoint maps and BCH intuition

The group adjoint $\mathrm{Ad}_H : \mathfrak{se}(3) \to \mathfrak{se}(3)$ transforms twists between frames:

$$\mathrm{Ad}_{\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}} = \begin{bmatrix} R & 0 \\ \hat{p}R & R \end{bmatrix}. \tag{9}$$

The BCH formula explains $\log(\exp A \exp B)$ via $A + B + \frac{1}{2}[A, B] + \cdots$, revealing noncommutativity.

Worked Example (Small-angle). For small θ, $\sin\theta \approx \theta$, $1-\cos\theta \approx \frac{\theta^2}{2}$, hence $R \approx \mathrm{I}+\hat{\omega}\theta$.

Exercises (with short solutions).

L.1 Prove $\hat{\omega}^3 = -\|\omega\|^2\hat{\omega}$. *Solution:* Use $\hat{\omega}^2 = \omega\omega^\top - \|\omega\|^2\mathrm{I}$.

L.2 Derive $\mathrm{Ad}_{[R,p]}$ by conjugation: $\hat{\boldsymbol{\xi}}' = H\hat{\boldsymbol{\xi}}H^{-1}$. *Solution:* Compute block products; read off the 6×6 form.

12 Screw Theory: Twists and Wrenches

12.1 Twists

A *twist* is $\boldsymbol{\xi} = (\omega, v) \in \mathbb{R}^6$ or its matrix $\hat{\boldsymbol{\xi}} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix}$. It encodes instantaneous rigid motion:

$$\dot{H} = \hat{\boldsymbol{\xi}}\, H \quad \Rightarrow \quad H(t) = \exp\!\left(\hat{\boldsymbol{\xi}} t\right) H(0). \tag{10}$$

The *pitch* is $h = \dfrac{\omega^\top v}{\omega^\top \omega}$: $h = 0$ for revolute, $h = \infty$ (formally) for prismatic ($\omega = 0$), finite h for helical.

12.2 Wrenches and reciprocity

A *wrench* $\boldsymbol{\mathcal{F}} = (m, f) \in \mathbb{R}^6$ collects moment and force. The *reciprocal product* (virtual power) is

$$\boldsymbol{\mathcal{F}} \circ \boldsymbol{\xi} = m^\top \omega + f^\top v. \tag{11}$$

If $\boldsymbol{\mathcal{F}} \circ \boldsymbol{\xi} = 0$, the wrench does no instantaneous work along the twist. Twist and wrench spaces are dual; in an n-DoF mechanism, $\dim T = n$ and feasible wrench space has dimension $6 - n$.

12.3 Plücker coordinates intuition

Lines in space can be encoded by a direction and moment about the origin; twists and wrenches mirror this line geometry.

Worked Example (Joint twists).

- Revolute about axis ω through point r: $\boldsymbol{\xi} = (\omega,\, r \times \omega)$.
- Prismatic along unit direction v: $\boldsymbol{\xi} = (0,\, v)$.
- Helical of pitch h about (ω, r): $\boldsymbol{\xi} = (\omega,\, r \times \omega + h\omega)$.

Exercises (with short solutions).

S.1 Show that shifting the reference point by p changes (ω, v) to $(\omega, v + \hat{p}\,\omega)$. *Solution:* Apply $\mathrm{Ad}_{[I,p]}$.

S.2 For a wrench $\boldsymbol{\mathcal{F}} = (m, f)$ and twist (ω, v), verify frame invariance of $\boldsymbol{\mathcal{F}} \circ \boldsymbol{\xi}$. *Solution:* Use dual adjoint to show invariance.

13 Kinematics: POE, Jacobians, and Adjoint

13.1 Product of exponentials (POE)

Let a serial robot have joint coordinates $\theta = (\theta_1, \ldots, \theta_n)$ and *space-frame* twists $\boldsymbol{\xi}_i$. The end-effector pose is

$$H(\theta) = \exp\left(\hat{\boldsymbol{\xi}}_1\theta_1\right) \exp\left(\hat{\boldsymbol{\xi}}_2\theta_2\right) \cdots \exp\left(\hat{\boldsymbol{\xi}}_n\theta_n\right) H_0, \tag{12}$$

where H_0 is the home configuration.

13.2 Spatial and body Jacobians

Define the *spatial Jacobian* $J_s(\theta) = [\boldsymbol{\xi}_1,\ \mathrm{Ad}_{e^{\hat{\boldsymbol{\xi}}_1\theta_1}}\boldsymbol{\xi}_2,\ \ldots]$ so that the spatial twist of the end-effector is

$$\boldsymbol{\xi}_s = J_s(\theta)\,\dot{\theta}. \tag{13}$$

The *body Jacobian* J_b expresses twists in the end-effector frame; $J_b(\theta) = \mathrm{Ad}_{H(\theta)^{-1}} J_s(\theta)$.

13.3 Adjoint transformation

For any $H \in \mathrm{SE}(3)$, $\boldsymbol{\xi}' = \mathrm{Ad}_H\,\boldsymbol{\xi}$ changes frames; similarly wrenches transform by the dual adjoint $\mathrm{Ad}_H^{-\top}$.

13.4 Singularities and manipulability

When J loses rank, certain directions of motion are unattainable; the manipulator is at a *singularity*. Measures like $\sqrt{\det(JJ^\top)}$ (Yoshikawa manipulability) quantify dexterity.

Worked Example (Two-link planar arm). Let joint 1 be revolute about z at the base; joint 2 about z at the elbow. With space twists $\boldsymbol{\xi}_1 = (e_z, 0)$ and $\boldsymbol{\xi}_2 = (e_z, e_1\ell_1 \times e_z)$, derive $H(\theta)$ and $J_s(\theta)$. The result matches the classical planar kinematics and Jacobian.

Exercises (with short solutions).

- **K.1** Derive J_b from J_s via $\mathrm{Ad}_{H(\theta)^{-1}}$. *Solution:* Differentiate the body-frame POE form.
- **K.2** Identify singularities of a planar 2R arm. *Solution:* When the two links are collinear; J drops rank.

14 Parallel Mechanisms

A parallel robot has multiple kinematic chains (legs) in parallel from base to end-effector. The end-effector *twist space* is the intersection of leg twist spaces:

$$T_{\mathrm{EE}} = \bigcap_{j=1}^{m} T_j. \tag{14}$$

Constraints reduce DoF; feasible wrenches lie in the orthogonal complement.

14.1 Example: Stewart platform

Six legs with prismatic actuators control a SE(3) pose. Constraint Jacobians relate actuator rates to platform twist; analysis proceeds via twist/wrench duality.

Exercises (with short solutions).

- **P.1** Explain why adding a leg cannot increase platform DoF. *Solution:* DoF is an intersection dimension; adding constraints can only reduce or maintain it.
- **P.2** For a simplified 3-RPS platform, write constraint equations and infer platform DoF. *Solution:* Three independent constraints yield 3 DoF.

15 Degrees of Freedom and Grübler–Kutzbach

For spatial mechanisms with N links (including base), J joints, and joint freedom f_i each, a heuristic DoF count is

$$\mathrm{DoF} = m(N - 1 - J) + \sum_{i=1}^{J} f_i, \qquad m = 6 \text{ (spatial)}, \; m = 3 \text{ (planar)}. \tag{15}$$

This is necessary but not sufficient; special geometries can alter the count (overconstraints or redundancies).

Worked Example (Planar 4-bar). $m = 3$, $N = 4$, $J = 4$, all $f_i = 1$: $\mathrm{DoF} = 3(4 - 1 - 4) + 4 = 1$ as expected.

Exercises (with short solutions).

D.1 Compute the DoF of a spatial 6R serial arm. *Solution:* $m = 6$, $N = 7$, $J = 6$, $f_i = 1$: $\mathrm{DoF} = 6(7 - 1 - 6) + 6 = 6$.

D.2 Give an example where Grübler's formula fails. *Solution:* Overconstrained mechanisms (e.g., Bennett linkage).

Appendices

A. Determinants and eigen decompositions

The determinant $\det A$ equals the volume-scaling factor of A (with sign). Orthogonal matrices have $|\det| = 1$. Symmetric matrices admit eigen decompositions $A = Q\Lambda Q^\top$.

B. Skew-symmetric identities and Rodrigues derivation

Using $\hat{\omega}^2 = \omega\omega^\top - \|\omega\|^2\mathrm{I}$ and the series for $e^{\hat{\omega}\theta}$ yields Rodrigues' formula as given.

C. Differential equations on Lie groups

Left-invariant ODEs $\dot{H} = \hat{\boldsymbol{\xi}}H$ integrate to $H(t) = \exp\left(\hat{\boldsymbol{\xi}}t\right)H(0)$. For time-varying $\boldsymbol{\xi}(t)$, the solution is a time-ordered exponential.

Extended Worked Examples

E1. Frame changes and adjoint

Given $H = [R, p]$ and a body twist $\boldsymbol{\xi}_b$, compute its space representation $\boldsymbol{\xi}_s = \mathrm{Ad}_H \boldsymbol{\xi}_b$. Numerically illustrate with R a $90°$ rotation about z and $p = (1, 0, 0)$.

E2. POE for a 3R spherical wrist

Write the three joint twists about intersecting axes at the wrist center; compute J_b and discuss singularities at gimbal configurations.

Chapter-end Exercise Collections (with short solutions)

Below we gather additional mixed exercises to reinforce connections.

X.1 Show that SO(3) is a 3D manifold by parameter counting and regular value theorem (sketch). *Solution:* Orthogonality imposes 6 independent constraints on 9 parameters; determinant $= 1$ fixes sign: dimension 3.

X.2 Derive the left Jacobian $\mathbf{J}(\phi)$ series to $O(\phi^3)$. *Solution:* Expand $\exp(\hat{\omega}\phi)$ and integrate the conjugation formula.

X.3 For a helical joint with pitch h, find the screw axis and interpret h geometrically. *Solution:* Axis direction $\omega/\|\omega\|$; h is translation per radian.

X.4 Prove invariance of $\boldsymbol{\mathcal{F}} \circ \boldsymbol{\xi}$ under frame change. *Solution:* Use $\boldsymbol{\mathcal{F}}' = \mathrm{Ad}_H^{-\top} \boldsymbol{\mathcal{F}}$, $\boldsymbol{\xi}' = \mathrm{Ad}_H \boldsymbol{\xi}$.

X.5 Compute the manipulability measure for a 2R planar arm at a stretched configuration. *Solution:* Determinant vanishes; manipulability $= 0$.

End of notes.