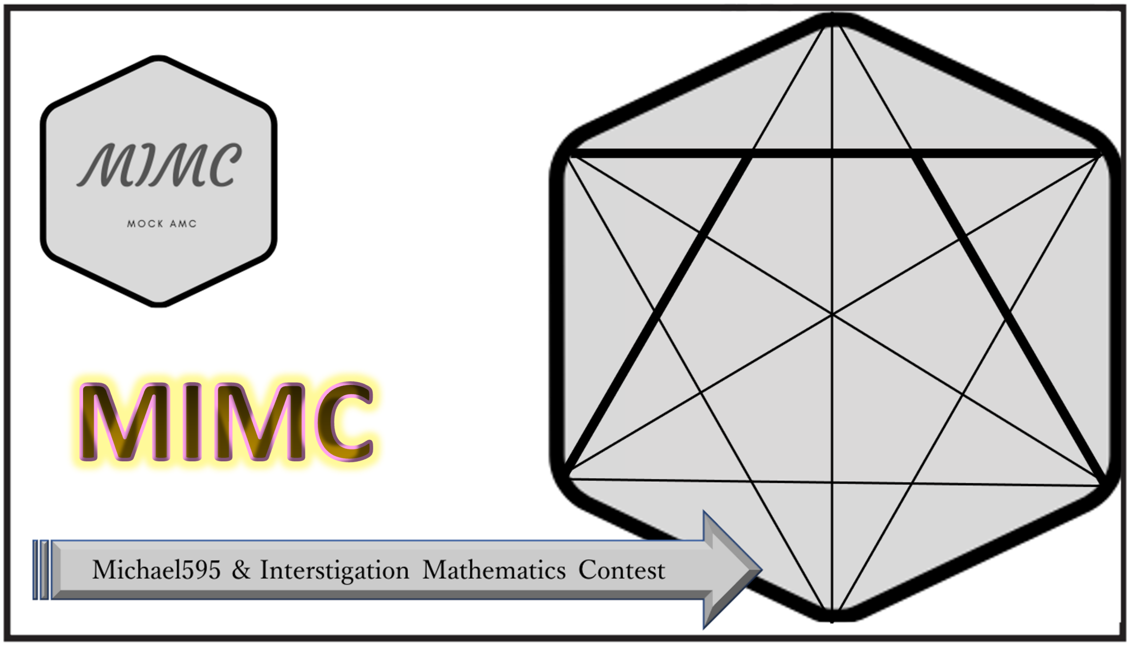
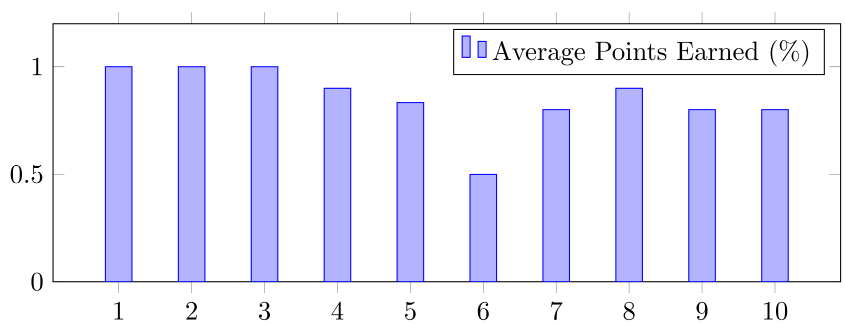


STATISTICS

Number of Responses:	**11**	Minimum:	**52**
Mean:	**85.55**	First Quartile:	**75**
Standard Deviation:	**15.81**	Median:	**97**
Mode:	**100**	Third Quartile:	**100**
Range:	**48**	Maximum:	**100**



The MIMC Committee reserves the right to disqualify scores from an individual if it determines that the required security procedures were not followed.

1. We can just plug the recursive formula:

$$\begin{aligned} a_1 &= 2a_0 + 2^{1-1} = \boxed{1}, \\ a_2 &= 2a_1 + 2^{2-1} = \boxed{4}, \\ a_3 &= 2a_2 + 2^{3-1} = \boxed{12}, \\ a_4 &= 2a_3 + 2^{4-1} = \boxed{32}, \\ a_5 &= 2a_4 + 2^{5-1} = \boxed{80}. \end{aligned}$$

2. We can list out the factorization of all terms from question 1:

$$\begin{aligned} a_1 &= 1 = 1 \cdot 2^0 \\ a_2 &= 4 = 2^2 \cdot 1 = 2 \cdot 2^1 \\ a_3 &= 12 = 2^2 \cdot 3 = 3 \cdot 2^2 \\ a_4 &= 32 = 2^5 = 4 \cdot 2^3 \\ a_5 &= 80 = 2^4 \cdot 5 \cdot 2^4 \end{aligned}$$

From this pattern, $a_n = n \cdot 2^{n-1}$, which $a_{2023} = \boxed{2023 \cdot 2^{2022}}$.

3. Now we have to prove the pattern that we have seen in problem 2. The easiest way to do so is by induction.

Claim: $a_n = n \cdot 2^{n-1}$ for all nonnegative integer n.

Base Case: $a_0 = 0 = 0 \cdot 2^{-1}$.

Inductive Hypothesis: $a_i = i2^{i-1}$ for some i.

Inductive Step: We now have to show that $a_{i+1} = (i+1)2^i$. By our recursive formula, we know that

$$a_{i+1} = 2a_i + 2^i = 2(i2^{i-1}) + 2^i = i2^i + 2^i = (i+1)2^i,$$

completing our induction. Thus by the Principle of Mathematical Induction, $a_n = n \cdot 2^{n-1}$ for all nonnegative integer n. $\square$

4. From our recursive formula,

$$
\begin{aligned}
a_n &= 2a_{n-1} + 2^{n-1} \\
a_n &= 2\left(2a_{n-2} + 2^{n-2}\right) + 2^{n-1} = 4a_{n-2} + 2 \cdot 2^{n-1} = 4a_{n-2} + 2^n \\
a_n &= 4\left(2a_{n-3} + 2^{n-3}\right) + 2^n = 8a_{n-3} + 2^{n-1} + 2^n = 8a_{n-3} + 3 \cdot 2^{n-1}
\end{aligned}
$$

Now we observe that $a_n = 2^{n-k}a_k + (n-k)2^{n-1}$. We prove by induction.

Claim: $a_n = 2^{n-k}a_k + (n-k)2^{n-1}$ for all $0 \le k < n$

Base Case: $a_n = 2^{n-(n-1)}a_{n-1} + (n-(n-1))2^{n-1} = 2a_{n-1} + 2^{n-1}$, which is the recursive formula according to the problem statement.

Inductive Hypothesis: $a_n = 2^{n-k}a_k + (n-k)2^{n-1}$.

Inductive Step:

$$
\begin{aligned}
a_n &= 2^{n-k}a_k + (n-k)2^{n-1} \\
a_n &= 2^{n-k}\left(2a_{k-1} + 2^{k-1}\right) + (n-k)2^{n-1} \\
a_n &= 2^{n-k} \cdot 2a_{k-1} + 2^{n-k+k-1} + (n-k)2^{n-1} \\
a_n &= 2^{n-k+1}a_{k-1} + (n-k+1)2^{n-1}
\end{aligned}
$$

□

Note: This formula is not necessarily unique. For one, we have seen in multiple submissions with the formula $a_n = \frac{n2^{n-k}a_k}{k}$, which is true trivially by the explicit formula. This formula, and any other correct formulas relating a_k and a_n, is also marked correct.

5. If you list a couple of these sequences, you may find a similar pattern as a_n.

Claim: $b_{n,c} = n \cdot c^{n-1}$ for all nonnegative integer n.

Base Case: $b_{0,c} = 0 = 0 \cdot c^{-1}$.

Inductive Hypothesis: $b_{i,c} = ic^{i-1}$ for some i.

Inductive Step: We now have to show that $b_{i+1,c} = (i+1)c^i$. By our recursive formula, we know that

$$b_{i+1,c} = cb_{i,c} + c^i = c(ic^{i-1}) + c^i = ic^i + c^i = (i+1)c^i,$$

completing our induction. Thus by the Principle of Mathematical Induction, $b_{n,c} = n \cdot c^{n-1}$ for all nonnegative integer n. □

6. **Claim:** There are infinite such functions f.

   Proof: We know that $f(n) = n \cdot 2^{n-1}$ would work as it is indeed continuous and differentiable over $\mathbb{R}$. Now we can add something such that the function would still hit all of the nonnegative integer points the same. For example, $\sin x$ is periodic, so

   $$f(n) = n \cdot 2^{n-1} + k\sin(nx)$$

   would work for all $k \in \mathbb{R}$, giving us infinite options. (Note that is just an example of what we could have chosen. It would work as long as the final function is continuous and differentiable.) $\square$

7. **Claim:** There is no polynomial f satisfying the conditions.

   Proof: Consider a polynomial $f(x)$ such that $f(n) = a_n$ for all nonnegative integer n. Then since $a_n = n \cdot 2^{n-1}$, we know that $f(n) = n \cdot 2^{n-1}$. This means that

   $$\frac{f(n)}{n \cdot 2^{n-1}} = 1$$

   for all nonnegative integer n. However, exponential function 2^{n-1} grows much faster than a polynomial. More formally,

   $$\lim_{n\to\infty} \frac{f(n)}{n \cdot 2^{n-1}} = 0$$

   if we repeatedly apply L'Hôpital's Rule. (If you just stated that exponential grows much faster than a polynomial, you will only be deducted one point as this is aimed for people without knowledge of calculus.) Thus the quotient cannot stay at 1, which means that there does not exist a polynomial f such that $f(n) = a_n$ for all nonegative integer n. $\square$

8. We use a table to keep track of S_n and a_n, respectively.

n	a_n	S_n
0	0	0
1	1	1
2	4	5
3	12	17
4	32	49
5	80	129
6	192	321

Now consider the values of $S_n - 1$ for all $n > 0$, then we get $0, 4, 16, 48, 128, 320$, which are $(1-1)2^1, (2-1)2^2, (3-1)2^3, (4-1)2^4, (5-1)2^5, (6-1)2^6$. Thus, we seek to prove that $S_n = (n-1)2^n + 1$. To accomplish this goal, we would use our best friend — induction.

Base Case: $S_1 = 1 = (1-1)2^1 + 1$.

Inductive Hypothesis: $S_n = (n-1)2^n + 1$.

Inductive Step: If $S_n = (n-1)2^n + 1$ holds for $n \geq 1$ and $S_{n+1} = S_n + a_{n+1}$, then

$$\begin{aligned}
S_{n+1} &= (n-1)2^n + 1 + (n+1) \cdot 2^n \\
S_{n+1} &= (n-1)2^n + (n+1) \cdot 2^n + 1 \\
S_{n+1} &= (n-1+n+1)2^n + 1 \\
S_{n+1} &= 2n \cdot 2^n + 1 \\
S_{n+1} &= n \cdot 2^{n+1} + 1
\end{aligned}$$

□

9. We can just plug in the formulas for a_n and S_n. We would get

$$\frac{a_n}{S_n} = \boxed{\frac{n2^{n-1}}{(n-1)2^n + 1}}.$$

10. From the answer of problem 9,

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{S_n} &= \lim_{n\to\infty} \frac{n \cdot 2^{n-1}}{(n-1)2^n + 1} \\
&= \lim_{n\to\infty} \frac{2^{n-1}}{\frac{n-1}{n} \cdot 2^n + \frac{1}{n}} \\
&= \lim_{n\to\infty} \frac{n}{n-1} \cdot \lim_{n\to\infty} \frac{2^{n-1}}{2^n} \\
&= 1 \cdot \frac{1}{2} \\
&= \boxed{\frac{1}{2}}.
\end{aligned}$$