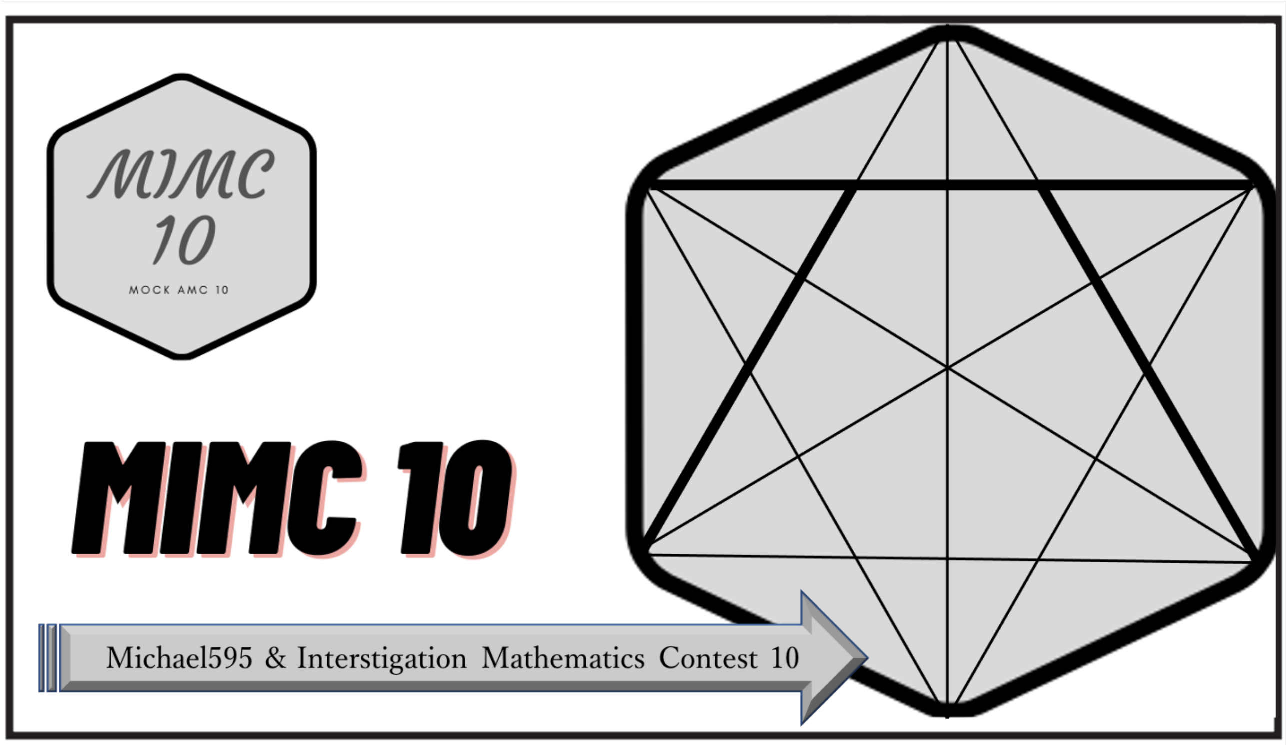


ANSWER KEY

1. **(E)** 288	2. **(A)** 35	3. **(B)** $-\frac{1}{15}$	4. **(C)** 18	5. **(D)** d^2
6. **(A)** 0	7. **(C)** 11	8. **(D)** 8	9. **(D)** 2520	10. **(D)** $B\ 3:2\ D$
11. **(C)** $\frac{1}{2}$	12. **(D)** 25	13. **(A)** 0	14. **(D)** 60	15. **(E)** $\frac{1}{5}$
16. **(B)** 372	17. **(B)** $\frac{18198}{27335}$	18. **(E)** 48	19. **(C)** 12	20. **(B)** 2022
21. **(D)** 80	22. **(E)** 2	23. **(A)** 340	24. **(B)** $\frac{3}{253}$	25. **(C)** 30

Try clicking on each cell!

The MIMC Committee reserves the right to disqualify scores from an individual if it determines that the required security procedures were not followed.

The Committee will publish a projected AIME floor, Distinction and Distinguished Honor Roll, however, there will not be a mock AIME hosted by MIMC Committee.

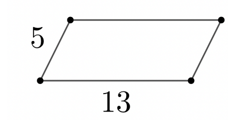
1. Calculate $1^1 + 2^2 + 3^3 + 4^4$.

   (A) 10 **(B)** 38 **(C)** 96 **(D)** 286 **(E)** 288

Solution. Directly Calculating, we have

$$1^1 + 2^2 + 3^3 + 4^4 = 1 + 4 + 27 + 256 = \boxed{\textbf{(E) } 288}.$$

2. If one of the height of the parallelogram below is 7, find its area.



   (A) 35 **(B)** 60 **(C)** 65 **(D)** 91 **(E)** This parallelogram cannot exist.

Solution. Notice that the altitude given must be corresponding to the base with length 5, since otherwise there would be a right triangle with leg length 7 and hypotenuse length 5. Therefore, the area is $7 \cdot 5 = \boxed{\textbf{(A) } 35}$.

3. Find $\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}} - \frac{1}{1+\frac{1}{1+\frac{1}{1}}}$.

   (A) $-\frac{2}{5}$ **(B)** $-\frac{1}{15}$ **(C)** 0 **(D)** $\frac{1}{15}$ **(E)** $\frac{2}{5}$

Solution. Calculating, we have

$$\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}} - \frac{1}{1+\frac{1}{1+\frac{1}{1}}} = \frac{1}{1+\frac{2}{3}} - \frac{2}{3} = \frac{3}{5} - \frac{2}{3} = \boxed{\textbf{(B) } -\frac{1}{15}}.$$

4. There are 100 locked boxes placed in a 10×10 grid. There are keys for adjacent boxes in each box (diagonally touching does not count as adjacent). There may be more than one key for each box, but you only need one key to unlock the corresponding box. If you only have the key to the bottom left box to start with, what is the minimum number of boxes you have to open in order to obtain the key to the top right box?

   (A) 9 **(B)** 10 **(C)** 18 **(D)** 19 **(E)** 20

Solution. Notice that the minimum number of boxes you have to open is one less than the number of boxes on the exterior border. Therefore, the answer is $10 + 10 - 1 - 1 = \boxed{\textbf{(C) } 18}$.

5. Let $ABCD$ be a square with diagonal length d, and let $EFGH$ be a square with diagonal length d^2. Find the ratio of the area of the square $EFGH$ to the area of the square $ABCD$. Express your answer in terms of d.

   (A) $d\sqrt{2}$ **(B)** d **(C)** $2d$ **(D)** d^2 **(E)** $2d^2$

Solution. Since they are both squares, they are similar to each other. Thus the ratio of their area is the square of the ratio of their diagonals. So the answer is $\left(\frac{d^2}{d}\right)^2 = \boxed{\textbf{(D) } d^2}$.

Remark. It is also possible to fin the exact area of each square, since we can find the side lengths to be $\frac{d}{\sqrt{2}}$ and $\frac{d^2}{\sqrt{2}}$, respectively. Thus the area of each square is $\frac{d^2}{2}$ and $\frac{d^4}{2}$, respectively.

6. How many ordered triples of positive integers (a, b, c) are there satisfying both $a^2 + b^2 = c^2$ and $a^6 + b^6 = c^6$?

   (A) 0 **(B)** 3 **(C)** 18 **(D)** 54 **(E)** infinitely many

Solution. Plugging in the first equation into the second gives $a^6 + b^6 = (a^2 + b^2)^3 = a^6 + b^6 + a^2b^2$, which is a contradiction since a, b, c are all positive integers. Therefore, there is $\boxed{\textbf{(A) } 0}$ ordered triple of positive integers (a, b, c) that satisfy this constraint.

7. Find the smallest possible positive integer k such that $11^{99} - k$ is a multiple of 15.

   (A) 1 **(B)** 6 **(C)** 11 **(D)** 13 **(E)** 14

Solution. That is $11^{99} - k = 15n$ for some integer n. Therefore, $11^{99} \equiv k \pmod{15}$ and we simply need to find the remainder when 11^{99} is divided by 15. We have

$$\begin{aligned} 11^{99} &= (-4)^{99} \pmod{15} \\ &= \left[(-4)^2\right]^{44} \cdot (-4) \pmod{15} \\ &= 1^{44} \cdot (-4) \pmod{15} \\ &= 11 \pmod{15} \end{aligned}$$

Therefore, the smallest possible positive integer k that is also remainder 11 when divided by 15 is $\boxed{\textbf{(C) } 11}$.

8. Define an operation $a \star b = 2ab + a + b$ for any integers a, b. How many ordered pairs of integers (x, y) are there such that $x \star y = 10$?

   (A) 0 **(B)** 2 **(C)** 4 **(D)** 8 **(E)** 12

Solution. $2(a \star b) = 4ab + 2a + 2b$, therefore, we have $a \star b = \frac{(2a+1)(2b+1)-1}{2} = 10 \implies (2a + 1)(2b + 1) = 21$. Notice that there are 8 ordered pairs of integers (x, y) such that $xy = 21$ and the function f where $(x, y) \mapsto (2x + 1, 2y + 1)$ is bijective. Therefore, our answer is $\boxed{\textbf{(D) } 8}$.

9. An odd integer is called *uniform* if the remainder when the integer is divided by $2, 3, 4, 5, 6, 7, 8, 9$ are all the same. For example, 1 is *uniform* because the remainder is always 1, but 3 is not *uniform*. What is the minimum positive difference between two *uniform* integers?

   (A) 1 **(B)** 2 **(C)** 1260 **(D)** 2520 **(E)** 362880

Solution. Notice that an odd integer is *uniform* if the remainder when the integer is divided by $2, 3, 4, 5, 6, 7, 8, 9$ are all the same. The remainder when the integer is divided by 2 can only be 0 or 1, but it must be even if the remainder is 0. Therefore, the common remainder must be 1. If k is uniform, then $k - 1$ must be a multiple of $2, 3, 4, 5, 6, 7, 8$ and 9. So $k - 1$ must be a multiple of $\text{lcm}(2, 3, 4, 5, 6, 7, 8, 9) = 2520$. The minimum positive difference would then be $\boxed{\textbf{(D) } 2520}$, and we can check that 1 and 2521 are both *uniform*, so this difference is achievable.

10. In a group of soccer tournament, four teams, A, B, C, and D play in a round-robin style where each team plays every other team exactly once. For example, if the final score between A and B is $x : y$, then A wins if $x > y$, B wins if $x < y$, and they draw otherwise. x goals are scored for A and conceded for B, and y goals are scored for B and conceded for A. In a score table, all of the statistics from all

games for one team is added up. Sometime in the middle of the tournament, the score table looks like this:

Team	Games Played	Wins	Draws	Losses	Goals Scored	Goals Conceded
A	3	1	0	2	1	3
B	3	2	1	0	5	2
C	2	1	1	0	1	0
D	2	0	0	2	2	4

Then what is the final score of the match between B and D?

(A) $B\ 1:0\ D$ **(B)** $B\ 2:1\ D$ **(C)** $B\ 3:0\ D$ **(D)** $B\ 3:2\ D$
(E) The match has not happened yet.

Solution. A has played all three games and won one, having only scored one goal. Therefore, A must have won D with a score of $1:0$.

Then B has also played all three games, so the game between B and D must have happened. D must have scored 2 goals and conceded 3 goals to make up the rest of the goals scored and conceded since it has only played 2 games. Therefore, the score is $\boxed{\textbf{(D) } B\ 3:2\ D}$.

Remark. If we continue, we can deduce the score of every match:

A	$0:2$	B	B	$0:0$	C
A	$0:1$	C	B	$3:2$	D
A	$1:0$	D	C	TBD*	D

*This game has not happened yet.

11. Two real numbers x, y such that $-4 \leq x \leq y \leq 4$ are chosen at random. What is the probability that $|x+y| = |x| + |y|$?

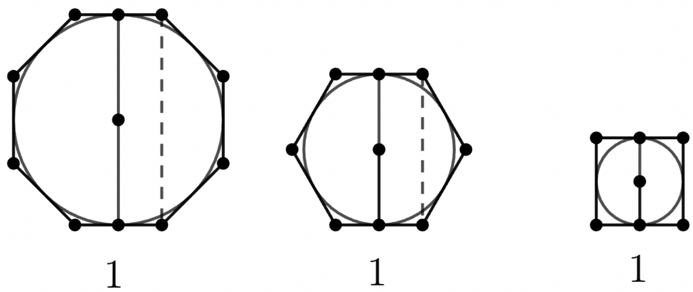
(A) $\frac{1}{4}$ **(B)** $\frac{25}{64}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{9}{16}$ **(E)** $\frac{25}{32}$

Solution. Only when $|x| = x, |y| = y$ or $|x| = -x, |y| = -y$the equation can be satisfied. Now, consider a square with side length 8 centered at origin, we can take all ordered pairs of real numbers in the first quadrant or the third quadrant, which by symmetry, gives $\boxed{\textbf{(C) } \frac{1}{2}}$.

12. There are three circles. ω_1 is inscribed in a regular octagon with side length 1, ω_2 is inscribed in a regular hexagon with side length 1, ω_3 is inscribed in a square with side length 1. Let $[\omega]$ denote the area of the circle ω. If $\frac{[\omega_1]}{[\omega_2][\omega_3]} = \frac{a+b\sqrt{c}}{d\pi}$ in the simplest form, find $a+b+c+d$ (some of a, b, c, d may be 0).

(A) 4 **(B)** 7 **(C)** 15 **(D)** 25 **(E)** 30

Solution. Draw a segment from an intersection of the circle and the polygon to the point diametrically opposite. Note that this is a diameter of the circle and the endpoints are midpoints of the edges. Repeat for each of the three circles.



Then note that the dashed lines have the same length as the diameter, so the radius r_1, r_2, r_3 for each of the circles $\omega_1, \omega_2, \omega_3$ are

$$r_1 = \frac{1}{2}\left(\frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}}\right) = \frac{\sqrt{2}}{2} + \frac{1}{2},$$
$$r_2 = \frac{1}{2}\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2},$$
$$r_3 = \frac{1}{2}(1) = \frac{1}{2}.$$

Therefore, the areas are

$$[\omega_1] = \pi r_1^2 = \frac{1}{4}(3+2\sqrt{2})\pi,$$
$$[\omega_2] = \pi r_2^2 = \frac{3}{4}\pi,$$
$$[\omega_3] = \pi r_3^2 = \frac{1}{4}\pi.$$

Thus, $\frac{[\omega_1]}{[\omega_2][\omega_3]} = \frac{\frac{1}{4}(3+2\sqrt{2})\pi}{\frac{3}{4}\pi\cdot\frac{1}{4}\pi} = \frac{12+8\sqrt{2}}{3\pi}$, so the answer is $12+8+2+3 = \boxed{\textbf{(D) } 25}$.

13. Given real numbers x, m, find the minimum value of

$$\sqrt{\sqrt{x^2 - 2mx + 2x + m^2 - 2m + 1} + m - 1}$$

given that it is defined in the real numbers.

(A) 0 **(B)** $\sqrt{\sqrt{2}-1}$ **(C)** $\sqrt{2}$ **(D)** $\sqrt{\sqrt{2}+1}$ **(E)** $\sqrt{3}$

Solution. Swap the order of terms $2x$ and m^2 gives

$$\sqrt{\sqrt{x^2 - 2mx + m^2 + 2x - 2m + 1} + m - 1} = \sqrt{\sqrt{(x-m)^2 + 2(x-m) + 1} + m - 1}$$
$$= \sqrt{\pm(x - m + 1) + m - 1}$$

Which when $x \geq m-1$, $\sqrt{\sqrt{x-m+1} + m - 1} = \sqrt{x}$ which the minimum is obviously 0. Otherwise when $x < m-1$, $\sqrt{\sqrt{x-m+1} + m - 1} = \sqrt{2m - x - 2}$ which the minimum is still 0 when $x = 2m-2$. Thus, the minimum value achievable is $\boxed{\textbf{(A) } 0}$.

14. For all positive integers $n > 3$, what is the minimum number of positive divisors of

$$n(n^2-1)(n^2-4)(n^2-9)?$$

(A) 20 **(B)** 30 **(C)** 40 **(D)** 60 **(E)** 120

Solution. $n(n^2-1)(n^2-4)(n^2-9) = (n+3)(n+2)(n+1)n(n-1)(n-2)(n-3) = 7!\binom{n+3}{7}$. Since $n > 3$, and the minimum number of factor occurs when $\binom{n+3}{7} = 1$. This is equivalent to asking the number of positive divisors of $7!$, and $7! = 2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1$, which there are $(4+1)(2+1)(1+1)(1+1) = \boxed{\textbf{(D) } 60}$.

15. Koal is playing a game of buttons! Koal starts with 1, and there are eight buttons in front of them that takes an the number k that Koal has and updates the number to $2k, 3k, 4k, 5k, 6k, 7k, 8k, k+1$, respectively. Given that Koal presses each button once, what is the probability that Koal will get an odd number?

(A) $\frac{1}{20}$ **(B)** $\frac{3}{40}$ **(C)** $\frac{1}{10}$ **(D)** $\frac{1}{8}$ **(E)** $\frac{1}{5}$

Solution. Consider that the final number will be expressed as $8! + a$ which a is the product of all the numbers on the buttons that are pressed after the $k+1$ button. Therefore, this implies that all the odd buttons must be pressed after $k+1$, which is equivalent to the statement that all the even buttons must be pressed before the $k+1$ button, and the positions of the odd buttons are independent of this arrangement. By symmetry, the probability is $\boxed{\textbf{(E) } \dfrac{1}{5}}$.

16. Let A be a sequence of positive integers in increasing order such that all elements in A can be expressed as

$$3^{2^{a_1}} + 3^{2^{a_2}} + \cdots + 3^{2^{a_n}}$$

for which $a_1, a_2, \ldots, a_n$ are distinct nonnegative integers. Given that all integers a that can be written in that form are in the sequence A, find the remainder when A_{30} is divided by 1000.

(A) 213 **(B)** 372 **(C)** 375 **(D)** 492 **(E)** 495

Solution. Assume without loss of generality that $a_1 > a_2 > \cdots > a_n$. Let $t_i = 3^{2^i}$. Note that t_i increases as i increases, and $t_i > t_{i-1} + t_{i-2} + \cdots + t_0$. We just have to choose which i to use. Then consider a binary number b_k for A_k, the kth element in A, such that the ith digit from the right is 1 if t_i is included in A and 0 otherwise. For example, if the binary number is $b_k = 1011_2$, then

$$A_k = t_3 + t_1 + t_0 = 3^{2^3} + 3^{2^1} + 3^{2^0} = 3^8 + 3^1 + 3^0.$$

Since A is a sequence sorted in increasing order, $\{b_k\}$ is also a sequence sorted in increasing order. That is, $b_k < b_{k'} \iff A_k < A_{k'}$. Also $b_1 = 1_2 = 1$, and each nonnegative integer k corresponds to a valid b_k, so $b_k = k$. Therefore, $b_{30} = 30 = 11110$ and

$$A_{30} = 3^{2^4} + 3^{2^3} + 3^{2^2} + 3^{2^1}$$
$$A_{30} = 3^{16} + 3^8 + 3^4 + 3^2$$
$$A_{30} \equiv 721 + 561 + 81 + 9 \pmod{1000}$$
$$A_{30} \equiv \boxed{\textbf{(B) } 372} \pmod{1000}$$

Remark: It is possible to do the last step with Binomial Theorem, if you convert $3^2 = 9 = 10 - 1$. It may be faster.

17. Kidderminster has five 4-sided dice, and he rolls the dice as follows. He starts by rolling all five dice at once. Whenever he rolls a 1, he will stop rolling that die. Then he throws the remaining dice again until there are none left. Let P_n be the probability that Kidderminster rolls exactly n dice at once at some point. Find P_3.

(A) $\frac{351}{781}$ **(B)** $\frac{18198}{27335}$ **(C)** $\frac{2}{3}$ **(D)** $\frac{1856}{2783}$ **(E)** $\frac{3}{4}$

Solution. For P_n, we only concern $P(P_n \text{ and } P_k)$ for all $n < k \leq 5$. Let A be the probability that Kidderminster achieves P_3 from P_5, and B be the probability that Kidderminster achieves P_3 from P_4. Then, if Kidderminster currently has 5 dice, then there is a probability of $\frac{243}{1024}$ for him to repeat P_5, a probability of $\frac{405}{1024}$ probability to proceed with P_4, and $\frac{270}{1024}$ to achieve P_3. Therefore, we have

$$P_5 = \frac{243}{1024}P_5 + \frac{405}{1024}P_4 + \frac{270}{1024}$$
$$781P_5 = 405P_4 + 270$$

Now for P_4, we have

$$P_4 = \frac{81}{256}P_4 + \frac{27}{64}$$
$$175P_4 = 108 \implies P_4 = \frac{108}{175}$$

Plug in gives

$$P_5 = \frac{405P_4 + 270}{781} = \frac{81 \cdot \frac{108}{35} + 270}{781} = \frac{8748 + 270 \cdot 35}{35 \cdot 781} = \boxed{\textbf{(B)}\ \frac{18198}{27335}}.$$

18. It's well known that $1 + 3 + \cdots + (2k - 1) = k^2$. What is the number of positive divisors of

$$1^2 + 3^2 + 5^2 + \cdots + 99^2?$$

(A) 12 **(B)** 18 **(C)** 24 **(D)** 36 **(E)** 48

Solution. We know that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$. Thus

$$\begin{aligned}
1^2 + 3^2 + 5^2 + \cdots + 99^2 &= 1^2 + 2^2 + 3^2 + \cdots + 99^2 - (2^2 + 4^2 + 6^2 + \cdots + 98^2) \\
&= 1^2 + 2^2 + 3^2 + \cdots + 99^2 - 2^2(1^2 + 2^2 + 3^2 + \cdots + 49^2) \\
&= \frac{99 \cdot 100 \cdot 199}{6} - 4 \cdot \frac{49 \cdot 50 \cdot 99}{6} \\
&= 33 \cdot (50 \cdot 199 - 2 \cdot 50 \cdot 49) \\
&= 33 \cdot 50 \cdot 101 \\
&= 2 \cdot 3 \cdot 5^2 \cdot 11 \cdot 101.
\end{aligned}$$

Therefore it has $2 \cdot 2 \cdot 3 \cdot 2 \cdot 2 = \boxed{\textbf{(E)}\ 48}$ positive divisors.

19. Find the number of ordered triples (x, y, z) such that $x, y, z \in \{1, 2, 3, 4, 5, 6\}$ and that 5 divides the expression

$$x^2 - xy + y^2 - yz + z^2 - zx.$$

(A) 0 **(B)** 6 **(C)** 12 **(D)** 15 **(E)** 18

Solution. Multiply and divide by 2 gives

$$\frac{2x^2 - 2xy + 2y^2 - 2yz + 2z^2 - 2xz}{2} = \frac{(x-y)^2 + (y-z)^2 + (z-x)^2}{2}$$

Notice that the factor of $\frac{1}{2}$ does not affect the divisibility of 5. Listing out all possible remainders when a perfect square is divided by 5, we get that $0^2 \equiv 0 \pmod 5, 1^2 \equiv 1 \pmod 5, 2^2 \equiv -1 \pmod 5, 3^2 \equiv -1 \pmod 5, 4^2 \equiv 1 \pmod 5$. This implies that n^2 can only leave a remainder of $0, 1, -1$ when divided by 5.

Now, we consider all possible ordered triples (a_1, a_2, a_3) such that $a_i \in \{-1, 0, 1\}$ for all $1 \leq i \leq 3$ and $a_1 + a_2 + a_3 = 0$, which there are only 7 possibilities: the 6 permutations of $(-1, 0, 1)$ and the single case $(0, 0, 0)$. The latter implies that $a_1 - a_2, a_2 - a_3, a_1 - a_3$ are all multiples of 5, which can be obtained when $a_1 = a_2 = a_3$ with 6 cases. Furthermore, we can also have $a_1 - a_2 = 5, a_2 = a_3$, and their permutations. In this case, we have $\binom{3}{2}$ to choose which two elements are equal, and notice that the element must be either 1 to 6 because otherwise, the value obtained when adding or subtracting a 5 will not be in the set. Therefore, we have $\binom{3}{2} \cdot 2$ to choose the two elements that are equal, and the third one is fixed: when the two elements are both 1, then the third must be 6, and vice versa.

Now, we consider the second case when (a_1, a_2, a_3) is a permutation of $(-1, 0, 1)$. Because of symmetry, we can assume that $(a_1 - a_2)^2, (a_2 - a_3)^2, (a_1 - a_3)^2$ are $-1, 0, 1 \pmod 5$, respectively. $a_1 - a_3 = a_1 - a_2 + a_2 - a_3 \equiv a_1 - a_2 \pmod 5$, but this creates a contradiction because each value corresponds to exactly one value in modulo 5, so an integer cannot be both 1 and $-1 \pmod 5$ at the same time.

Therefore, the answer is $\boxed{\textbf{(C) } 12}$.

20. Let a_n be a sequence defined as $a_n = 2a_{n-1} + 4a_{n-2}$. Given that $a_0 = 0$ and $a_1 = 1$, find the number of factors of 2 in a_{2023}.

(A) 0 **(B)** 2022 **(C)** 2023 **(D)** 2024 **(E)** 4046

Solution. We can list out a couple of terms to see if there are any patterns, and we hypothesize that $a_n = 2^{n-1} \cdot F_n$ where F_n is the Fibonacci sequence with $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. We can prove the pattern with induction.

Claim: $a_n = 2^{n-1} \cdot F_n$ for $n \geq 0$.

Base Case: $a_0 = 2^{-1} \cdot F_0 = 0$ and $a_1 = 2^0 \cdot F_1 = 1$.

Inductive Hypothesis: For some $i \in \mathbb{N}$ with $i \geq 2$, $a_{i-1} = 2^{i-2} \cdot F_{i-1}$ and $a_{i-2} = 2^{i-3} \cdot F_{i-2}$.

Inductive Step: Then we have to show $a_i = 2^{i-1} \cdot F_i$. By the recursive formula given,

$$\begin{aligned} a_i &= 2a_{i-1} + 4a_{i-2} \\ &= 2 \cdot 2^{i-2} \cdot F_{i-1} + 4 \cdot 2^{i-3} \cdot F_{i-2} \\ &= 2^{i-1}(F_{i-1} + F_{i-2}) \\ &= 2^{i-1} \cdot F_i, \end{aligned}$$

as desired. Therefore, by the Principle of Mathematical Induction, $a_n = 2^{n-1} \cdot F_n$. Therefore, $a_{2023} = 2^{2022} \cdot F_{2023}$. Since Fibonacci sequences are alternating even and odd, the 2023rd term is odd. Thus the answer is just $\boxed{\textbf{(B) } 2022}$.

21. How many values of n are there such that $0 \le n < 300$ and $n^4 - 1$ is a multiple of 30?

(A) 8 **(B)** 15 **(C)** 30 **(D)** 80 **(E)** 150

Solution. By the Chinese remainder theorem, each ordered triple $(a_1 \pmod 2, a_2 \pmod 3, a_3 \pmod 5)$ is congruent to a unique value in $\mathbb{Z}_{30}$. Therefore, $n^4 - 1$ is a multiple of 30 implies that $n^4 \equiv 1 \pmod{2, 3, 5}$, and each ordered element in the set

$$\{(a, b, c) : a \in \{0, 1\}, b \in \{0, 1, 2\}, c \in \{0, 1, 2, 3, 4\}, a^4 \equiv 1 \pmod 2, b^4 \equiv 1 \pmod 3, c^4 \equiv 1 \pmod 5\}$$

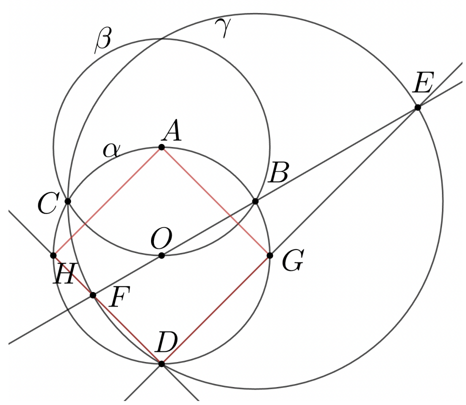
Now, notice that $a = 1$ is the only possibility, and b can be both 1 and 2, while c can be all of $1, 2, 3, 4$, since $1 - 1, 2^4 - 1, 3^4 - 1, 4^4 - 1$ are all multiples of 5. Thus, $c \in \{1, 2, 3, 4\}$. By the Chinese Remainder Theorem, there are a total of $1 \cdot 2 \cdot 4 = 8$ distinct values of n.

Since the remainders reset between every multiples of 30, there are a total of $\frac{300}{30} \cdot 8 = \boxed{\textbf{(D) } 80}$ distinct values of n.

22. There is a point A on a circle α with center O and radius 1. Draw a circle β with center A and radius OA. The let the points of intersection of α and β be B and C. Then draw circle γ with center B and radius BC where γ intersects α again at point D. Let the line OB intersect γ at points E, F. Then let line DE intersect α again at G, line DF intersect α again at H. Find the area of the quadrilateral $AGDH$.

(A) $\frac{\sqrt{6}-\sqrt{2}}{2}$ **(B)** $\sqrt{2}$ **(C)** $\sqrt{3}$ **(D)** $\frac{\sqrt{2}+\sqrt{6}}{2}$ **(E)** 2

Solution. Note that $BC = BD$, so triangle BCD is isosceles. Furthermore, since α and β have the same radius, $AO = AB = BO = AC = CO$, so $\triangle ABO$ and $\triangle ACO$ are equilateral triangles. Thus $\angle COB = 120°$. Furthermore, $BO = OD = 1$ and $BD = BC = \sqrt{3}$, so $\angle BOD = 120°$. Thus $\angle COD = 120°$, implying that $\triangle BCD$ is an equilateral triangle.



Therefore, $\angle ABC + \angle CBD = 30^\circ + 60^\circ = 90^\circ$, so AD is a diameter. Furthermore, EF is the diameter of γ, so $\angle GDH = \angle EDF = 90^\circ$. Thus HG is also a diameter. Therefore, $AGDH$ is a rectangle.

Now note that $\angle ADG = \angle ADB + \angle BDG$, and since $\triangle BDE$ is an isosceles triangle,

$$\angle ADB = \frac{1}{2}\angle AOB = 30^\circ,$$
$$\angle BDE = \frac{180^\circ - \angle DBE}{2} = \frac{\angle FBD}{2} = \frac{\angle FOD}{4} = \frac{\angle AOB}{4} = \frac{60^\circ}{4} = 15^\circ.$$

Thus $\angle ADG = 30^\circ + 15^\circ = 45^\circ$, meaning that $\triangle ADG$ must be a right isosceles triangle, so $AG = GD$. Therefore, quadrilateral $AGDH$ is a square, so its area is $(\sqrt{2})^2 = \boxed{\textbf{(E) } 2}$.

23. If I choose a positive integer n less than 1000, the probability that the decimal expansion of $\frac{1}{n}$ is a purely repeating decimal with a period of 6 can be written as $\frac{a}{b}$, find $a + b$.

(A) 340 **(B)** 1018 **(C)** 1019 **(D)** 1020 **(E)** 1021

Solution. Notice that if $\frac{1}{n} = 0.\overline{a_1a_2a_3a_4a_5a_6}$, then $10^6\left(\frac{1}{n}\right) = \overline{a_1a_2a_3a_4a_5a_6}.\overline{a_1a_2a_3a_4a_5a_6}$. Therefore

$$(10^6 - 1)\frac{1}{n} = \overline{a_1a_2a_3a_4a_5a_6}.$$

Thus, we need $10^6 - 1$ to be a multiple of n. However, we also cannot have $10^k - 1$ to be a multiple of n such that $k \mid 6$.

$$10^6 - 1 = (10^3-1)(10^3+1) = (10-1)(10+1)(10^2+10+1)(10^2-10+1) = 9\cdot 11\cdot 111\cdot 91 = 3^3\cdot 7\cdot 11\cdot 13\cdot 37$$

While $10-1 = 3^2, 10^2-1 = 3^2\cdot 11, 10^3-1 = 3^3\cdot 37$. This implies that n is not divisible by $3^2, 3^2\cdot 11, 3^3\cdot 37$. Now, we simply need to find the number of divisors of 999999 such that it is not a divisor of at least one of $9, 99, 999$ and it is less than 1000.

Now, we consider a and $\frac{999999}{a}$, which since 999999 is slightly less than 1000^2, we can approximately say that at least one of elements in the set $\{a, \frac{999999}{a}\}$ is less than 1000 if a is a divisor of 999999.

Claim. For all ordered pairs of positive integers a, b such that $ab = 999999$, exactly one of a and b will be less than 1000.

Proof. Assume contradiction that both $a, \frac{999999}{a}$ are greater than 1000, then let $b = \frac{999999}{a}$, we have $ab = 999999$. Since $a, b > 1000$, assume that $a = 1000 + a_1, b = 1000 + b_1$ such that $a_1, b_1 \in \mathbb{N}$, then we have

$$(1000 + a_1)(1000 + b_1) = 999999$$

$$1000000 + 1000a_1 + 1000b_1 + a_1b_1 = 999999$$

$$a_1b_1 + 1000(a_1 + b_1) = -1$$

Which is impossible because $a_1, b_1 \in \mathbb{N}$, and the sum of positive integers cannot be -1. □

Now, assume that both are greater than or equal to 1000, then their product will always be greater than $1000^2 > 999999$. □

Now, since 999999 is not a perfect square, the two elements in the ordered pairs are always distinct and there will always be an even number of divisors. Therefore, we can partition $\mathbb{D}_{999999}$ into groups of 2, and exactly one of the elements in those two will satisfy. Therefore, we know that there are $\frac{\tau(999999)}{2} = 32$ divisors of 999999 that are less than 1000 by the previous claim.

From the set $S = \{a \mid a \in \mathbb{N}, a < 1000, a \mid 999999\}$, we need to subtract the number of elements in S that is a divisor of at least one of $9, 99, 999$, and notice that when a positive integer is divisible by 9, it will always be divisible by $99, 999$ since $9 \mid 99, 999$.

Use the Principle of Inclusion and Exclusion gives that there are $\tau(99)+\tau(999)-\tau(9) = 3\cdot2+4\cdot2-3 = 11$ such elements, which implies that there are $32 - 11 = 21$ such positive integers n such that $\frac{1}{n}$ has a period of 6, and the desired probability is $\frac{21}{999} = \frac{7}{333}$, giving us an answer of $\boxed{\textbf{(A) } 340}$.

24. Let $a_1, a_2, \ldots, a_{2023}$ be the roots of the polynomial

$$x^{2023} - 2023 \cdot 2022x^{2021} + c_{2021}x^{2020} + c_{2020}x^{2019} + \cdots + c_2x + c_1$$

where each c_i is a real constant. Given that $a_1, a_2, \ldots, a_{2023}$ form an arithmetic sequence, what is the square of the common difference of the arithmetic sequence?

(A) 0 **(B)** $\frac{3}{253}$ **(C)** $\frac{4}{337}$ **(D)** $\frac{6}{253}$ **(E)** $\frac{8}{337}$

Solution. By Vieta's formula, we know that

$$\sum_{1\le i\le 2023} a_i = a_1 + a_2 + \cdots + a_{2023} = 0,$$
$$\sum_{1\le i<j\le 2023} a_ia_j = a_1a_2 + a_1a_3 + \cdots + a_{2022}a_{2023} = -2023 \cdot 2022.$$

Since the roots form an arithmatic sequence with sum 0, the middle term must be 0. Therefore, assuming without loss of generality the roots are sorted in increasing order, we can let the common difference be d, so $a_i = (i - 1012)d$. Therefore,

$$\begin{aligned}\sum_{1\le i<j\le 2023} a_ia_j &= \sum_{1\le i<j\le 2023} (i - 1012d)(j - 1012d) \\ &= (-1011d)(-1010d) + (-1011d)(-1009d) + \cdots + (1010d)(1011d).\end{aligned}$$

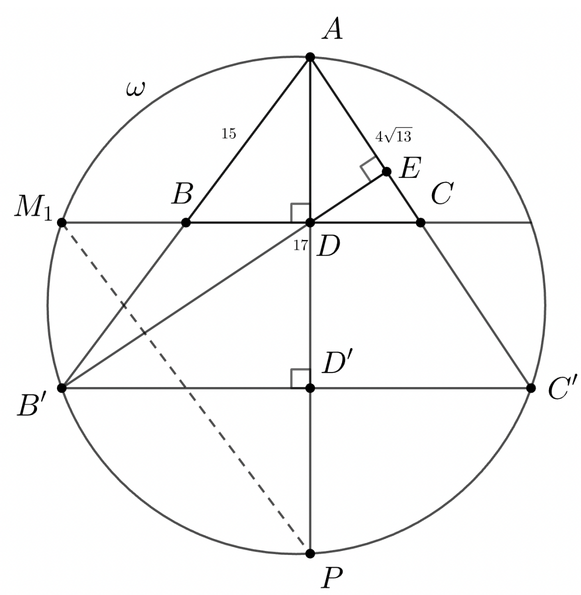
This is almost symmetrical. That is, for a term $k_1k_2d^2$, there is a term $-k_1k_2d^2$ to cancel it out. Except there is no positive term to cancel out $(kd)(-kd)$. Therefore, this sum is simplified to

$$\sum_{k=1}^{1011} -k^2d^2 = -2023 \cdot 2022$$
$$-\frac{(1011)(1012)(2023)}{6}d^2 = -2023 \cdot 2022$$
$$d^2 = \frac{12}{1012} = \boxed{\textbf{(B) } \frac{3}{253}}.$$

25. Let ABC be a triangle such that $AB = 15, AC = 4\sqrt{13}$ and $BC = 17$. Let D be a point on BC such that $AD \perp BC$, and let E a point on AC such that $DE \perp AC$. Let the intersection of lines AB and DE be B', and let C' be the point such that $B'C' \| BC$ and $\triangle AB'C' \sim \triangle ABC$. Let ω be the circumcircle of $\triangle AB'C'$, and define P as the intersection point of line AD and ω, and $P \neq A$. Furthermore, let M_1 be the intersection closer to B of line BC with ω. Find M_1P.

(A) 20 **(B)** $10 + 12\sqrt{5}$ **(C)** 30 **(D)** $2\sqrt{235}$ **(E)** 32

Solution. We first draw the diagram.



Let the intersection of lines AP and $B'C'$ be the D', then AD' and $B'E$ are both altitudes of $\triangle AB'C'$. Then D is the intersection of two altitudes, meaning that it is the orthocenter of $\triangle AB'C'$.

Let $BD = x$, then $DC = 17 - x$. Then by Pythagorean Theorem we have

$$15^2 - x^2 = (4\sqrt{13})^2 - (17 - x)^2,$$
$$225 = 208 - 289 + 34x,$$
$$x = \frac{306}{34} = 9.$$

Let $B = (0, 0)$, then $D = (9, 0)$ and $A = (9, 12)$. The equation of line AB is $y = \frac{4}{3}x$. Now, $C = (17, 0)$, so the equation of line AC is $y = -\frac{3}{2}x + 12$. Since $\overline{DE}$ and $\overline{AC}$ are perpendicular, the product of their slope is -1. Therefore, the slope of line DE is $\frac{2}{3}$. Let the equation of $\overline{DE}$ be $y = \frac{2}{3}x + b$. Plug D into the equation gives $0 = 9 \cdot \frac{2}{3} + b \implies b = -6$. $y = \frac{2}{3}x - 6$. B' is defined as the intersection of $\overline{DE}$ and $\overline{AB}$. Equating gives $\frac{4}{3}x = \frac{2}{3}x - 6 \implies \frac{2}{3}x = -6 \implies x = -9. y = -12$. Since both BB' and AB are 15, we can conclude that B is the midpoint of AB'. Similarly, D is the midpoint of AD'.

Then let M_1' be the reflection of D over point B. That is, let M_1' be the point on line BD such that $M_1' \neq D$ and $M_1'B = BD$. Then since $\angle ABD = \angle B'BM_1'$, $BD = BM_1'$, $AB = BB'$, we have $\triangle ABD \cong \triangle B'BM_1'$. Similarly, $\triangle B'BD \cong \triangle ABM_1$. Also since C, C', D', D are concyclic,

$$\begin{aligned}\angle AM_1'B' + \angle AC'B' &= \angle B'M_1'B + \angle AM_1'B + (180^\circ - \angle D'DE),\\ &= \angle ADB + \angle B'DB + (180^\circ - \angle ADB'),\\ &= 180^\circ.\end{aligned}$$

Thus A, B', C', M_1' are concyclic, so $M_1' = M_1$.

Let P' be the reflection of D over point D'. That is, let P' be the point on line DD' such that $P' \neq D$ and $PD' = D'D$. Then since $D'C' = D'C'$, $DD' = D'P'$, $\angle DD'C' = \angle P'D'C' = 90^\circ$, we have $\triangle DD'C' \cong \triangle P'D'C'$. Thus

$$\begin{aligned}\angle P'C'B' = \angle PC'D' &= \angle DC'D', && \\ &= \angle DED', && (C', D', D, E \text{ cyclic})\\ &= \angle D'EB', && \\ &= \angle D'AB', && (A, B', D', E \text{ cyclic})\end{aligned}$$

so A, B', P', C' concyclic, meaning that $P = P'$.

Therefore, we have that $\angle BDA = \angle M_1DP$, $\frac{BD}{M_1D} = \frac{DA}{DP} = \frac{DD'}{DP} = \frac{1}{2}$, so $\triangle ABD \sim \triangle PM_1D \implies M_1P = 2AB = \boxed{\textbf{(C) } 30}$.