Polynomial Optimization

$$\begin{aligned} \max \quad & h(x) \\ \text{s.t.} \quad & f_1(x) = 0, \ \ldots, \ f_m(x) = 0, \\ & g_1(x) \geq 0, \ \ldots, \ g_k(x) \geq 0 \\ & x \in \mathbb{R}^n \end{aligned}$$

Almost every combinatorial optimization can be modelled like this

Vertex Cover

Find a minimum size vertex set that covers every edge.

- $$\begin{aligned} \min \quad & \Sigma x_i \\ \text{s.t.} \quad & x_i + x_j \geq 1 \qquad (i,j) \in E \\ & x_i^2 = x_i \qquad (x_i \in \{0,1\}) \end{aligned}$$ $\quad x_i = \begin{cases} 1 & \text{if } i \in V \\ 0 & \text{if } i \notin V \end{cases}$

OR.

- $$\begin{aligned} \max \quad & \Sigma x_i \\ \text{s.t.} \quad & x_i \cdot x_j = 0 \end{aligned}$$ $\quad x_i = \begin{cases} 0 & \text{if } i \in V \\ 1 & \text{if } i \notin V. \end{cases}$

Binary Search

We instead solve the feasibility problem

$$\begin{aligned} & h(x) - \gamma \geq 0 \\ & f_i(x) = 0 \qquad i = 1, \ldots, m \\ & g_i(x) \geq 0 \qquad i = 1, \ldots, k \end{aligned}$$

When you know a range on values for objective function, we can do binary search with γ.

Goal: To understand the feasibility question with polynomial constraints.

Problem Setup: $\mathbb{K}$ = field, $\mathbb{K}[x_1, \dots, x_n] = \mathbb{K}[x] = R$

(ring of polynomials with coefficients in $\mathbb{K}$)

$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ (monomial)

Solve $\begin{cases} f_i(x) = 0 & i = 1, \dots, m \\ g_i(x) \geq 0 & i = 1, \dots, k \end{cases}$

$p(x) = \sum_\alpha u_\alpha x^\alpha$, degree$(p(x)) = \max_{\alpha : u_\alpha \neq 0} \left(\sum_{i=1}^n \alpha_i \right)$

$F := \{ f_1, \dots, f_m \}$

$\langle F \rangle_R = \{ \sum_{i=1}^m \beta_i f_i \mid \beta_i \in R \}$ "Ideal generated by F"

Defn: A polynomial $s(x)$ is a Sum of Squares (SOS) if

$$s(x) = \sum_{i \in I} [q_i(x)]^2$$

Defn: Let $g_1, \dots, g_k \in R$

$$\text{cone}(G) = \{ s_0 + \sum_i s_i g_i + \sum_{\{i,j\}} s_{ij} g_i g_j + \dots + s_{ijk\dots} g_i g_j g_k \dots \mid$$

s_α is a SOS polynomial $\}$

Lemma: Fredholm's Alternative

$\nexists\, x$ such that $Ax + b = 0 \iff \exists\, \mu$ such that $\mu^T A = 0$, $\mu^T b = 1$

Theorem Hilbert's (weak) Nullstellensatz

$\nexists\, x$ such that $f_i(x) = 0$, $i = 1, \dots, m$

$\iff \exists\, \beta_1, \dots, \beta_m \in R$ s.t. $\sum \beta_i f_i = 1$

$\iff 1 \in \langle F \rangle_R$

(Proof by Cox, Little, & O'Shea)

General Farkas' Lemma:

$$\nexists x \text{ s.t. } Ax+b=0,\ Cx+d \geq 0$$
$$\Leftrightarrow \exists \mu, \lambda \text{ with } \lambda \geq 0 \text{ s.t.}$$
$$\mu^T A + \lambda^T C = 0$$
$$\mu^T b + \lambda^T d = -1$$

Proof uses: Hahn Banach Theorem = Separation Hyperplane Theorem

Theorem: Positivestellensatz (Stengle 1973)

$$\nexists x \text{ such that } f_i(x)=0,\ i=1,\dots,m$$
$$\text{\& } g_i(x) \geq 0,\ i=1,\dots,k$$
$$\Leftrightarrow$$
$$\exists\ f \in \langle F \rangle_R,\ g \in \text{Cone}(G) \text{ such that } f+g=1$$

Example: $f_1(x) = x_1^2 - 1$, $f_2(x) = 2x_1x_2 + x_3$, $f_3(x) = x_1 + x_2$, $f_4(x) = x_1 + x_3$

$\{x \mid f_i(x) = 0\}$??

Algorithm to find infeasibility certificate:

$$\mu_1(x^2-1) + \mu_2(2x_1x_2+x_3) + \mu_3(x_1+x_2) + \mu_4(x_1+x_3) = 1$$

- Assuming μ_i's are constant

$$\Rightarrow \left.\begin{aligned} -\mu_1 &= 1 \\ \mu_2+\mu_4 &= 0 \\ \mu_2+\mu_4 &= 0 \\ \mu_3 &= 0 \\ \mu_1 &= 0 \end{aligned}\right\} \text{infeasible} \Rightarrow \text{try with } \mu_i\text{'s as linear functions, then quadratics, etc.}$$

Theorem: $\exists$ an exponential bound on the degrees of β_i's in the Hilbert infeasibility certificate.

Algorithm: NulLA (uses linear algebra to solve system of polynomial equations)

for $d = 1, \dots, D$ (exponentially bounded)

1. try to find β_i of degree d s.t. $\sum \beta_i f_i = 1$
2. If yes, stop → report infeasible
3. If no, continue with $d = d+1$.

If END → report feasible.

SOS $\quad p(x) = \sum_{i \in I} \{g_i(x)\}^2$

Theorem 1: $p(x) \in \mathbb{R}$ is SOS $\Leftrightarrow$ $p(x) = z^T Q z$ where Q is PSD, z is vector of monomials

Example:
$$p(x_1, x_2) = x_1^2 - x_1 x_2^2 + x_2^4 + 1$$
$$= \frac{3}{4}(x_1 - x_2)^2 + \frac{1}{4}(x_1 + x_2^2) + 1$$
$$= \frac{1}{6}\begin{bmatrix} 1 & x_2 & x_2^2 & x_1 \end{bmatrix} \begin{bmatrix} 6 & 0 & -2 & 0 \\ 0 & 4 & 0 & 0 \\ -2 & 0 & 6 & -3 \\ 0 & 0 & -3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ x_2 \\ x_2^2 \\ x_1 \end{bmatrix}$$

Defn: A symmetric matrix is PSD if $x^T A x \geq 0 \quad \forall x \in \mathbb{R}^n$

Theorem: Let A be a symmetric matrix.

A is PSD $\Leftrightarrow$ all eigenvalues of A are real & ≥ 0

$\Leftrightarrow$ $A = C^T C$ where C is a $k \times n$ matrix.

Proof Theorem 1: ($\Leftarrow$) $p(x) = z^T Q z = z^T C^T C z = \sum_{i}^{\text{rows of } C} (c_i z)^2 = \text{SOS}$

($\Rightarrow$) $p(x) = \sum [g_i(x)]^2 = x^T Q^T Q x$ where $Q = \begin{bmatrix} g_1 \\ \vdots \end{bmatrix}$ coefficients, $x = \begin{bmatrix} 1 \\ x_1 \\ x_2 \\ \vdots \end{bmatrix}$ all monomials.

$\square$