Chapter 1

Algorithms to Solve Integer Programs

1.1 LP to solve IP

Recall that he linear relaxation of an integer program is the linear programming problem after removing the integrality constraints

Integer Program:	Linear Relaxation:
$\max \quad z_{IP} = c^\top x$ $Ax \leq b$ $x \in \mathbb{Z}^n$	$\max \quad z_{LP} = c^\top x$ $Ax \leq b$ $x \in \mathbb{R}^n$

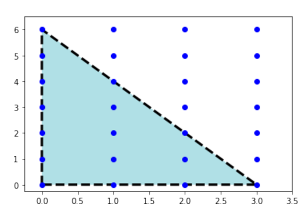
Theorem 1. *It always holds that*

$$z^*_{IP} \leq z^*_{LP}. \tag{1.1.1}$$

Furthermore, if x^*_{LP} is integral (feasible for the integer program), then

$$x^*_{LP} = x^*_{IP} \quad \text{and } z^*_{LP} = z^*_{IP}. \tag{1.1.2}$$

Example 1:



Consider the problem

$$\begin{aligned} \max z =& 3x_1 + 2x_1 \\ & 2x_1 + x_2 \leq 6 \\ & x_1, x_2 \geq 0; x_1, x_2 \text{ integer} \end{aligned}$$

1.1.1 Rounding LP Solution can be bad!

Consider the two variable knapsack problem

$$\max 3x_1 + 100x_2 \tag{1.1.3}$$
$$x_1 + 10x_2 \leq 10 \tag{1.1.4}$$
$$x_i \in \{0,1\} \text{ for } i = 1,2. \tag{1.1.5}$$

Then $x^*_{LP} = [1, 0.99]$ and $z^*_{LP} = 1 \cdot 3 + 0.99 \cdot 100 = 3 + 99 = 102$.

But $x^*_{IP} = [0,1]$ with $z^*_{IP} = 0 \cdot 3 + 1 \cdot 100 = 100$.

Suppose that we rounded the LP solution.

$x^*_{LP-Rounded-Down} = [1,0]$. Then $z^*_{LP-Rounded-Down} = 1 \cdot 3 = 3$. Which is a terrible solution!

How can we avoid this issue?

Cool trick! Using two different strategies gives you at least a 1/2 approximation to the optimal solution.

1.1.2 Rounding LP solution can be infeasible!

Now only could it produce a poor solution, it is not always clear how to round to a feasible solution.

1.1.3 Fractional Knapsack

The fractional knapsack problem has an exact greedy algorothm.

https://www.youtube.com/watch?time_continue=424&v=m1p-eWxrt6g

https://www.geeksforgeeks.org/fractional-knapsack-problem/

1.2 Branch and Bound

See http://web.tecnico.ulisboa.pt/mcasquilho/compute/_linpro/TaylorB_module_c.pdf for some nice notes on branch and bound.

1.2.1 Algorithm

Algorithm 1 Branch and Bound - Maximization

Input: Integer Linear Problem with max objective
Output: Exact Optimal Solution x^*

1: Set $LB = -\infty$.
2: Solve LP relaxation.
 a: If x^* is integer, stop!
 b: Otherwise, choose fractional entry x_i^* and branch onto subproblems: (i) $x_i \leq \lfloor x_i^* \rfloor$ and (ii) $x_i \geq \lceil x_i^* \rceil$.
3: Solve LP relaxation of any subproblem.
 a: If LP relaxation is infeasible, prune this node as "Infeasible"
 b: If $z^* < LB$, prune this node as "Suboptimal"
 c: x^* is integer, prune this nodes as "Integer" and update $LB = \max(LB, z^*)$.
 d: Otherwise, choose fractional entry x_i^* and branch onto subproblems: (i) $x_i \leq \lfloor x_i^* \rfloor$ and (ii) $x_i \geq \lceil x_i^* \rceil$. Return to step 2 until all subproblems are pruned.
4: Return best integer solution found.

1.2.2 General Branching

and 9 square board feet of wood, and a chair requires 1 hour of labor and 5 square board feet of wood. Currently, 6 hours of labor and 45 square board feet of wood are available. Each table contributes \$8 to profit, and each chair contributes \$5 to profit. Formulate and solve an IP to maximize Telfa's profit.

Solution Let

$$x_1 = \text{number of tables manufactured}$$
$$x_2 = \text{number of chairs manufactured}$$

Because x_1 and x_2 must be integers, Telfa wants to solve the following IP:

$$\begin{aligned}
\max z &= 8x_1 + 5x_2 \\
\text{s.t.} \quad x_1 + x_2 &\le 6 \qquad \text{(Labor constraint)} \\
9x_1 + 5x_2 &\le 45 \qquad \text{(Wood constraint)} \\
x_1, x_2 \ge 0&;\ x_1, x_2 \text{ integer}
\end{aligned}$$

The branch-and-bound method begins by solving the LP relaxation of the IP. If all the decision variables assume integer values in the optimal solution to the LP relaxation, then the optimal solution to the LP relaxation will be the optimal solution to the IP. We call the LP relaxation subproblem 1. Unfortunately, the optimal solution to the LP relaxation is $z = \frac{165}{4}$, $x_1 = \frac{15}{4}$, $x_2 = \frac{9}{4}$ (see Figure 11). From Section 9.1, we know that (optimal z-value for IP) $\le$ (optimal z-value for LP relaxation). This implies that the optimal z-value for the IP cannot exceed $\frac{165}{4}$. Thus, the optimal z-value for the LP relaxation is an **upper bound** for Telfa's profit.

Our next step is to partition the feasible region for the LP relaxation in an attempt to find out more about the location of the IP's optimal solution. We arbitrarily choose a variable that is fractional in the optimal solution to the LP relaxation—say, x_1. Now observe that every point in the feasible region for the IP must have either $x_1 \le 3$ or $x_1 \ge 4$. (Why can't a feasible solution to the IP have $3 < x_1 < 4$?) With this in mind, we "branch" on the variable x_1 and create the following two additional subproblems:

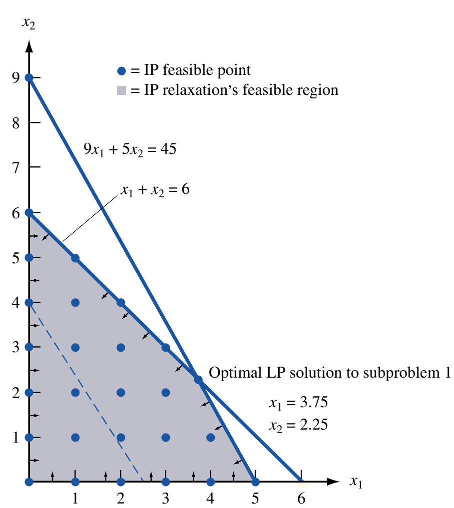


FIGURE 11
Feasible Region for Telfa Problem

Observe that neither subproblem 2 nor subproblem 3 includes any points with $x_1 = \frac{15}{4}$. This means that the optimal solution to the LP relaxation cannot recur when we solve subproblem 2 or subproblem 3.

From Figure 12, we see that every point in the feasible region for the Telfa IP is included in the feasible region for subproblem 2 or subproblem 3. Also, the feasible regions for subproblems 2 and 3 have no points in common. Because subproblems 2 and 3 were created by adding constraints involving x_1, we say that subproblems 2 and 3 were created by **branching** on x_1.

We now choose any subproblem that has not yet been solved as an LP. We arbitrarily choose to solve subproblem 2. From Figure 12, we see that the optimal solution to subproblem 2 is $z = 41$, $x_1 = 4$, $x_2 = \frac{9}{5}$ (point C). Our accomplishments to date are summarized in Figure 13.

A display of all subproblems that have been created is called a **tree.** Each subproblem is referred to as a **node** of the tree, and each line connecting two nodes of the tree is called an **arc.** The constraints associated with any node of the tree are the constraints for the LP relaxation plus the constraints associated with the arcs leading from subproblem 1 to the node. The label t indicates the chronological order in which the subproblems are solved.

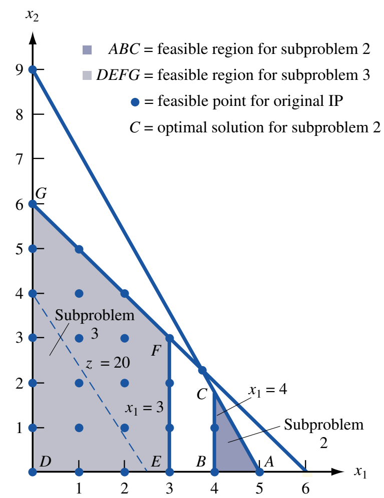


FIGURE 12
Feasible Region for Subproblems 2 and 3 of Telfa Problem

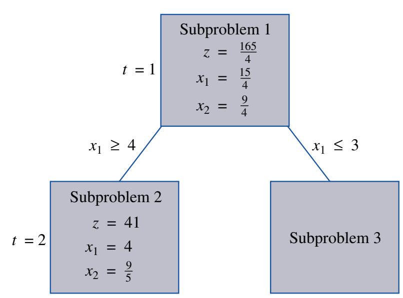


FIGURE 13
Telfa Subproblems 1 and 2 Solved

Because x_2 is the only fractional variable in the optimal solution to subproblem 2, we branch on x_2. We partition the feasible region for subproblem 2 into those points having $x_2 \geq 2$ and $x_2 \leq 1$. This creates the following two subproblems:

Subproblem 4 Subproblem 1 + Constraints $x_1 \geq 4$ and $x_2 \geq 2$ = subproblem 2 + Constraint $x_2 \geq 2$.

Subproblem 5 Subproblem 1 + Constraints $x_1 \geq 4$ and $x_2 \leq 1$ = subproblem 2 + Constraint $x_2 \leq 1$.

The feasible regions for subproblems 4 and 5 are displayed in Figure 14. The set of unsolved subproblems consists of subproblems 3, 4, and 5. We now choose a subproblem to

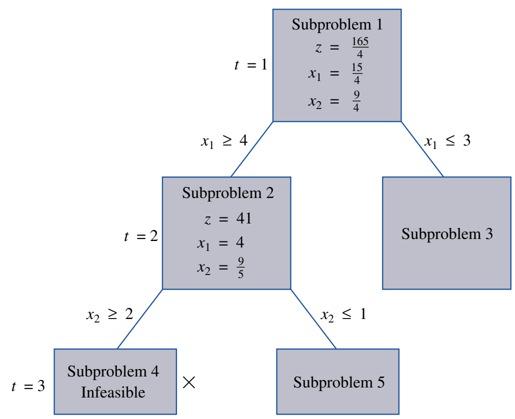


FIGURE 15
Telfa Subproblems 1, 2, and 4 Solved

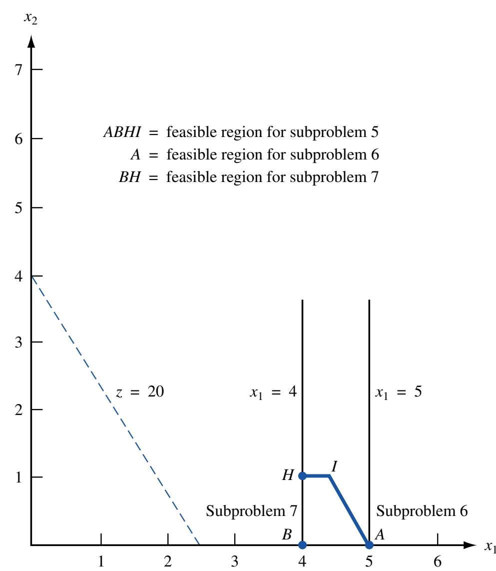


FIGURE 16
Feasible Regions for Subproblems 6 and 7 of Telfa Problem

Together, subproblems 6 and 7 include all integer points that were included in the feasible region for subproblem 5. Also, no point having $x_1 = \frac{40}{9}$ can be in the feasible region for subproblem 6 or subproblem 7. Thus, the optimal solution to subproblem 5 will not recur when we solve subproblems 6 and 7. Our tree now looks as shown in Figure 17.

Subproblems 3, 6, and 7 are now unsolved. The LIFO rule implies that we next solve subproblem 6 or subproblem 7. We arbitrarily choose to solve subproblem 7. From Figure 16, we see that the optimal solution to subproblem 7 is point H: $z = 37$, $x_1 = 4$, $x_2 = 1$. Both x_1 and x_2 assume integer values, so this solution is feasible for the original IP. We now know that subproblem 7 yields a feasible integer solution with $z = 37$. We also know that subproblem 7 cannot yield a feasible integer solution having $z > 37$. Thus, further branching on subproblem 7 will yield no new information about the optimal solution to the IP, and subproblem has been fathomed. The tree to date is pictured in Figure 18.

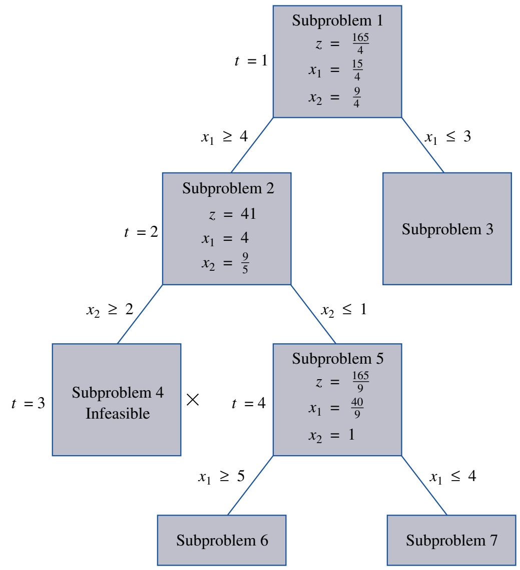


FIGURE 17
Telfa Subproblems 1, 2, 4, and 5 Solved

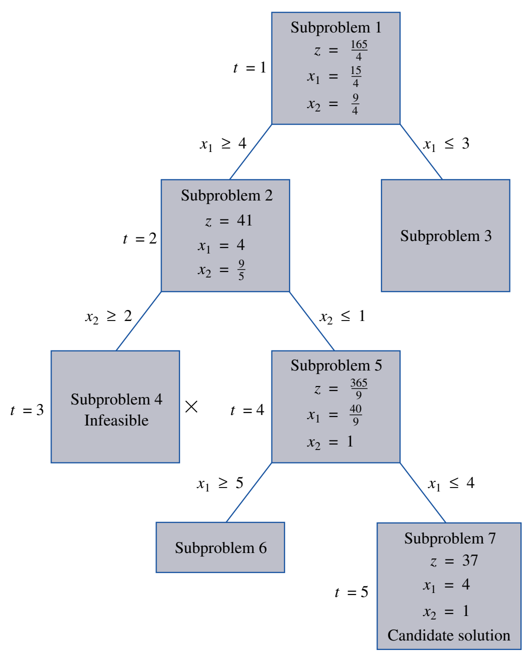
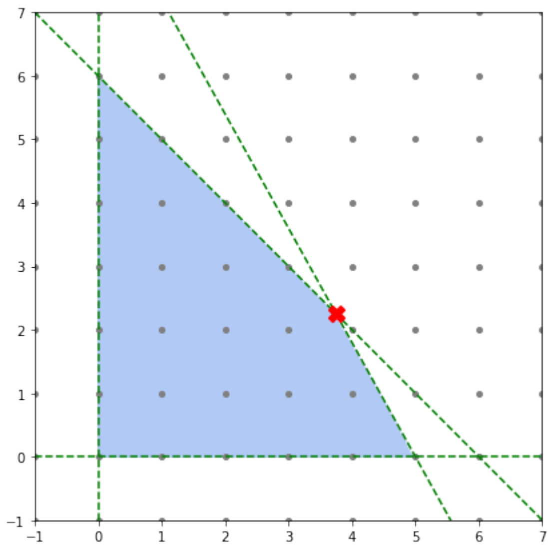


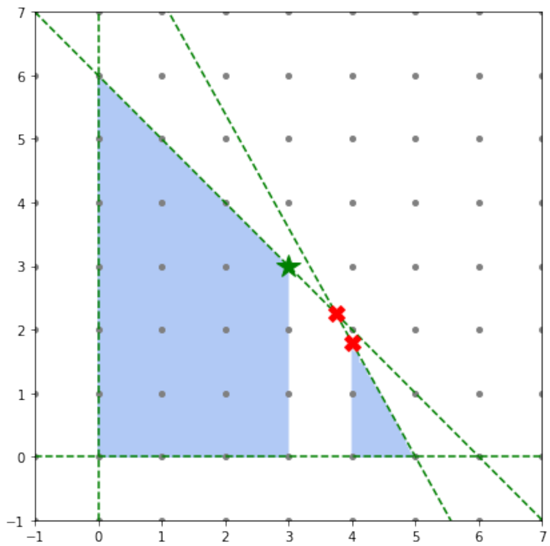
FIGURE 18
Branch-and-Bound Tree After Five Subproblems Have Been Solved

Example 2: See Example 9 in Chapter 9 of the textbook (Winston - Operations Research Applications and Algorithms).

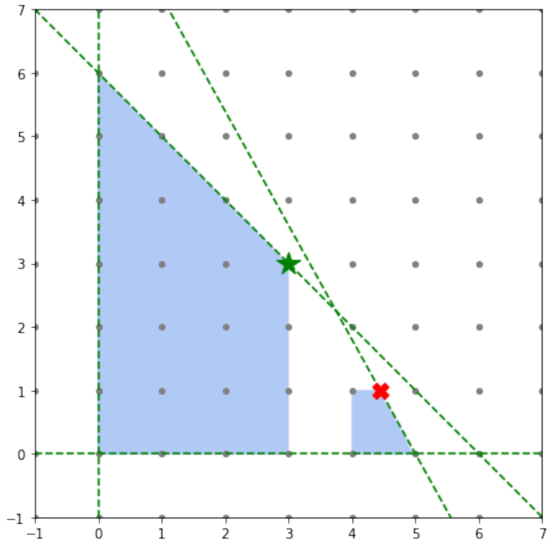
$x = [3.75, 2.25]$, $obj = 41.25$



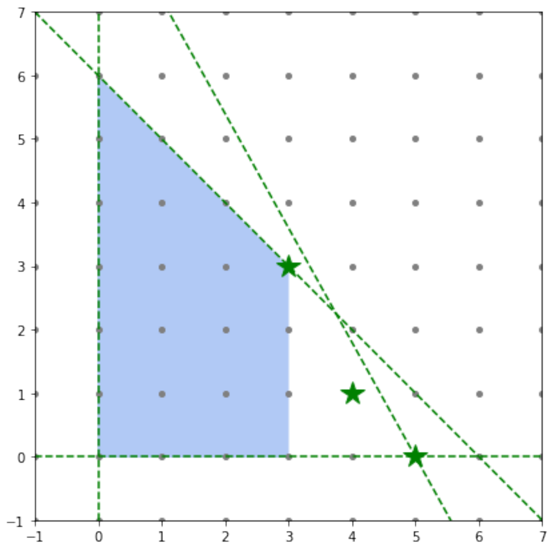
$x = [3, 3]$ $obj = 39.0$
$x = [4, 1.8]$,, $obj = 41.0$



$x = [3, 3]$, $obj = 39.0$
$x = [4.44, 1]$ $obj = 40.55$.
Infeasible Region



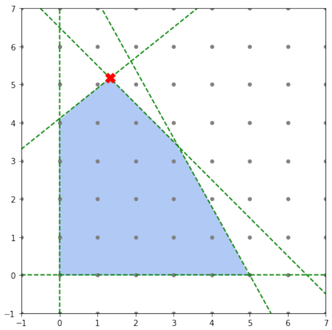
$x = [3, 3]$, $obj = 39.0$
$x = [4, 1]$, $obj = 37.0$
$x = [5, 0]$, $obj = 40.0$
Infeasible Region



Example 3: Consider the two variable example with

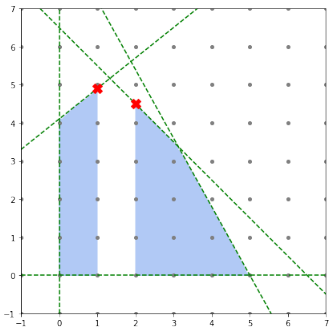
$$
\begin{aligned}
\max -3x_1 + 4x_2 \\
2x_1 + 2x_2 \le 13 \\
-8x_1 + 10x_2 \le 41 \\
9x_1 + 5x_2 \le 45 \\
0 \le x_1 \le 10, \text{ integer} \\
0 \le x_2 \le 10, \text{ integer}
\end{aligned}
$$

$x = [1.33, 5.167] obj = 16.664$



$x = [1, 4.9] obj = 16.5998$

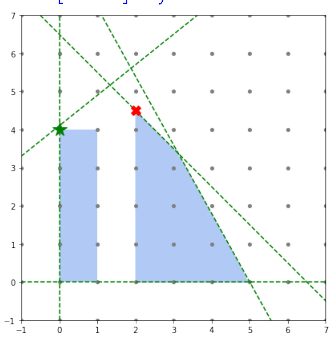
$x = [2, 4.5] obj = 12.0$



Infeasible Region

$x = [0.4.] obj = 16.0$

$x = [2.4.5] obj = 12.0$



1.2.3 Knapsack Problem and 0/1 branching

Consider the problem

$$\begin{aligned} \max \quad & 16x_1 + 22x_2 + 12x_3 + 8x_4 \\ \text{s.t.} \quad & 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\ & 0 \leq x_i \leq 1 \quad i = 1, 2, 3, 4 \\ & x_i \in \{0, 1\} \quad i = 1, 2, 3, 4 \end{aligned}$$

What is the optimal solution if we remove the binary constraints?

$$
\begin{aligned}
\max \quad & c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \\
\text{s.t.} \quad & a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 \leq b \\
& 0 \leq x_i \leq 1 \quad i = 1, 2, 3, 4
\end{aligned}
$$

How do I find the solution to this problem?

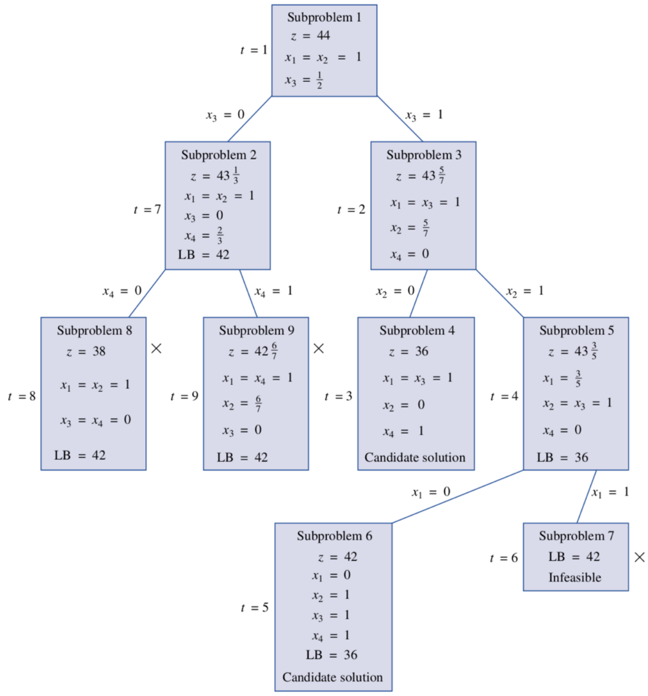
$$
\begin{aligned}
\max \quad & c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \\
\text{s.t.} \quad & (a_1 - A)x_1 + (a_2 - A)x_2 + (a_3 - A)x_3 + (a_4 - A)x_4 \le 0 \\
& 0 \le x_i \le m_i \quad i = 1, 2, 3, 4
\end{aligned}
$$

How do I find the solution to this problem?

Consider the problem

$$
\begin{aligned}
\max \quad & 16x_1 + 22x_2 + 12x_3 + 8x_4 \\
\text{s.t.} \quad & 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\
& 0 \leq x_i \leq 1 \quad i = 1, 2, 3, 4 \\
& x_i \in \{0, 1\} \quad i = 1, 2, 3, 4
\end{aligned}
$$

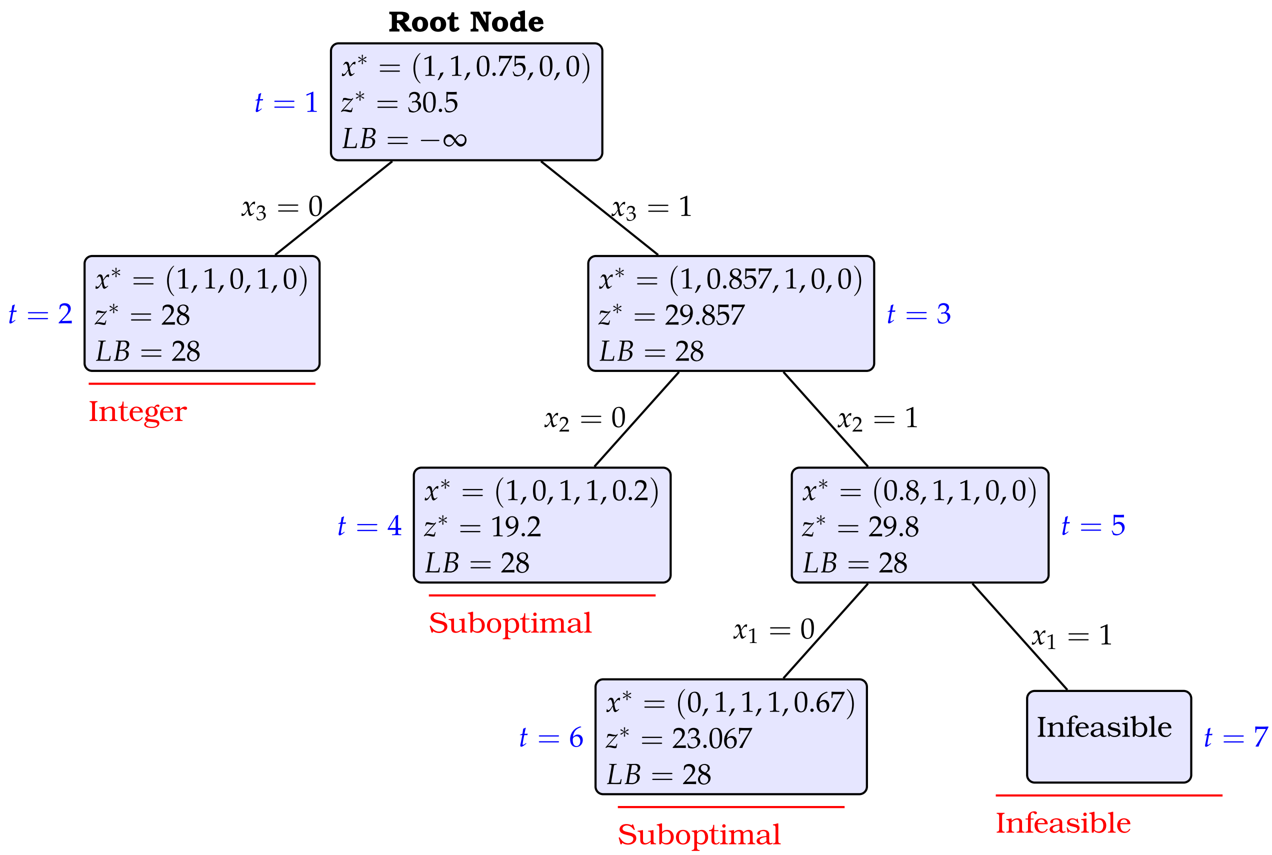
We can solve this problem with branch and bound.



The optimal solution was found at $t = 5$ at subproblem 6 to be $x^* = (0, 1, 1, 1)$, $z^* = 42$.

Example: Binary Knapsack Solve the following problem with branch and bound.

$$\begin{aligned} \max \quad & z = 11x_1 + 15x_2 + 6x_3 + 2x_4 + x_5 \\ \text{Subject to:} \quad & 5x_1 + 7x_2 + 4x_3 + 3x_4 + 15x_5 \leq 15 \\ & x_i \text{ binary}, i = 1, \ldots, 4 \end{aligned}$$



1.2.4 Traveling Salesman Problem solution via Branching

1.3 Cutting Planes

Cutting planes are inequalities $\pi^\top x \leq \pi_0$ that are valid for the feasible integer solutions that the cut off part of the LP relaxation. Cutting planes can create a tighter description of the feasible region that allows for the optimal solution to be obtained by simply solving a strengthened linear relaxation.

The cutting plane procedure, as demonstrated in Figure **??**. The procedure is as follows:

1. Solve the current LP relaxation.
2. If solution is integral, then return that solution. STOP
3. Add a cutting plane (or many cutting planes) that cut off the LP-optimal solution.
4. Return to Step 1.

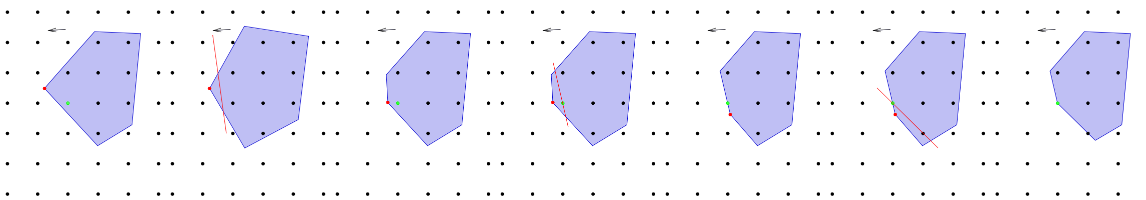


Figure 1.1: *The cutting plane procedure.*

In practice, this procedure is integrated in some with with branch and bound and also other primal heuristics.

1.3.1 Chvátal Cuts

Chvátal Cuts are a general technique to produce new inequalities that are valid for feasible integer points.

Chvátal Cuts:
Suppose

$$a_1 x_1 + \cdots + a_n x_n \leq d \tag{1.3.1}$$

is a valid inequality for the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$, then

$$\lfloor a_1 \rfloor x_1 + \cdots + \lfloor a_n \rfloor x_n \leq \lfloor d \rfloor \tag{1.3.2}$$

is valid for the integer points in P, that is, it is valid for the set $P \cap \mathbb{Z}^n$. Equation (**??**) is called a Chvátal Cut.

We will illustrate this idea with an example.

Example 4: Recall example **??**. The model was
Model

min	$p + n + d + q$	total number of coins used
s.t.	$p + 5n + 10d + 25q = 83$	sums to 83¢
	$p, d, n, q \in \mathbb{Z}_+$	each is a non-negative integer

From the equality constraint we can derive several inequalities.

1. Divide by 25 and round down both sides:

$$\frac{p + 5n + 10d + 25q}{25} = 83/25 \quad \Rightarrow \quad q \leq 3$$

2. Divide by 10 and round down both sides:

$$\frac{p + 5n + 10d + 25q}{10} = 83/10 \quad \Rightarrow \quad d + 2q \leq 8$$

3. Divide by 5 and round down both sides:

$$\frac{p+5n+10d+25q}{10} = 83/5 \quad \Rightarrow \quad n+2d+5q \leq 16$$

4. Multiply by 0.12 and round down both sides:

$$0.12(p+5n+10d+25q = 0.12(83) \quad \Rightarrow \quad d+3q \leq 9$$

These new inequalities are all valid for the integer solutions. Consider the new model:

New Model

$$\begin{array}{lll} \min & p+n+d+q & \text{total number of coins used} \\ \text{s.t.} & p+5n+10d+25q = 83 & \text{sums to } 83¢ \\ & q \leq 3 & \\ & d+2q \leq 8 & \\ & n+2d+5q \leq 16 & \\ & d+3q \leq 9 & \\ & p,d,n,q \in \mathbb{Z}_+ & \text{each is a non-negative integer} \end{array}$$

The solution to the LP relaxation is exactly $q = 3, d = 0, n = 1, p = 3$, which is an integral feasible solution, and hence it is an optimal solution.

1.3.2 Gomory Cuts

Gomory cuts are a type of Chvátal cut that is derived from the simplex tableau. Specifically, suppose that

$$x_i + \sum_{i \in N} \tilde{a}_i x_i = \tilde{b}_i \tag{1.3.3}$$

is an equation in the optimal simplex tableau.

Gomory Cut:
The Gomory cut corresponding to the tableau row (**??**) is

$$\sum_{i \in N} (\tilde{a}_i - \lfloor \tilde{a}_i \rfloor) x_i \geq \tilde{b}_i - \lfloor \tilde{b}_i \rfloor \tag{1.3.4}$$

We will solve the following problem using only Gomory Cuts.

$$\begin{array}{rrcl} \min & x_1 - 2x_2 & & \\ \text{s.t.} & -4x_1 + 6x_2 & \leq & 9 \\ & x_1 + x_2 & \leq & 4 \\ & x \geq 0 & , & x_1, x_2 \in \mathbb{Z} \end{array}$$

Step 1: The first thing to do is to put this into standard from by appending slack variables.

$$\begin{array}{rrcl} \min & x_1 - 2x_2 & & \\ \text{s.t.} & -4x_1 + 6x_2 + s_1 & = & 9 \\ & x_1 + x_2 + s_2 & = & 4 \\ & x \geq 0 & , & x_1, x_2 \in \mathbb{Z} \end{array} \tag{1.3.5}$$

We can apply the simplex method to solve the LP relaxation.

Initial Basis

Basis	RHS	x_1	x_2	s_1	s_2
z	0.0	1.0	-2.0	0.0	0.0
s_1	9.0	-4.0	6.0	1.0	0.0
s_2	4.0	1.0	1.0	0.0	1.0

$\vdots$ $\vdots$

Optimal Basis

Basis	RHS	x_1	x_2	s_1	s_2
z	-3.5	0.0	0.0	0.3	0.2
x_1	1.5	1.0	0.0	-0.1	0.6
x_2	2.5	0.0	1.0	0.1	0.4

This LP relaxation produces the fractional basic solution $x_{LP} = (1.5, 2.5)$.

Example 5: (Gomory cut removes LP solution)

We now identify an integer variable x_i that has a fractional basic solution. Since both variables have fractional values, we can choose either row to make a cut. Let's focus on the row corresponding to x_1.

The row from the tableau expresses the equation

$$x_1 - 0.1s_1 + -0.6s_2 = 1.5. \tag{1.3.6}$$

Applying the Gomory Cut (**??**), we have the inequality

$$0.9s_1 + 0.4s_2 \geq 0.5. \tag{1.3.7}$$

The current LP solution is $(x_{LP}, s_{LP}) = (1.5, 2.5, 0, 0)$. Trivially, since $s_1, s_2 = 0$, the inequality is violated.

Example 6: (Gomory Cut in Original Space)

The Gomory Cut (**??**) can be rewritten in the original variables using the equations from (**??**). That is, we can use the equations

$$\begin{array}{rcl} s_1 & = & 9 + 4x_1 - 6x_2 \\ s_2 & = & 4 - x_1 - x_2, \end{array} \tag{1.3.8}$$

which transforms the Gomory cut into the original variables to create the inequality

$$0.9(9 + 4x_1 - 6x_2) + 0.4(4 - x_1 - x_2) \geq 0.5.$$

or equivalently

$$-3.2x_1 + 5.8x_2 \leq 9.2. \tag{1.3.9}$$

As you can see, this inequality does cut off the current LP relaxation.

Example 7: (Gomory cuts plus new tableau) Now we add the slack variable $s_3 \geq 0$ to make the equation

$$0.9s_1 + 0.4s_2 - s_3 = 0.5. \tag{1.3.10}$$

Next, we need to solve the linear programming relaxation (where we assume the variables are continuous).

This leads us to the tableau Notes from Leo Liberti

8.2.3 Gomory Cuts

Gomory cuts are a special kind of Chvátal cuts. Their fundamental property is that they can be inserted in the simplex tableau very easily. This makes them a favorite choice in cutting plane algorithms, which generate cuts iteratively in function of the current incumbent.

Suppose x^* is the optimal solution found by the simplex algorithm deployed on a continuous relaxation of a given MILP. Assume the component x_h^* is fractional. Since $x_h^* \neq 0$, column h is a basic column; thus there corresponds a row t in the simplex tableau:

$$x_h + \sum_{j\in\nu} \bar{a}_{tj} x_j = \bar{b}_t, \tag{8.2}$$

where ν are the nonbasic variable indices, $\bar{a}_{tj}$ is a component of $B^{-1}A$ (B is the nonsingular square matrix of the current basic columns of A) and $\bar{b}_t$ is a component of $B^{-1}b$. Since $\lfloor \bar{a}_{tj} \rfloor \leq \bar{a}_{tj}$ for each row index t and column index j,

$$x_h + \sum_{j\in\nu} \lfloor \bar{a}_{tj} \rfloor x_j \leq \bar{b}_t.$$

Furthermore, since the LHS must be integer, we can restrict the RHS to be integer too:

$$x_h + \sum_{j\in\nu} \lfloor \bar{a}_{tj} \rfloor x_j \leq \lfloor \bar{b}_t \rfloor. \tag{8.3}$$

We now subtract Eq. (8.3) from Eq. (8.2) to obtain the *Gomory cut*:

$$\sum_{j\in\nu} (\lfloor \bar{a}_{tj} \rfloor - \bar{a}_{tj}) x_j \leq (\lfloor \bar{b}_t \rfloor - b_t). \tag{8.4}$$

We can subsequently add a slack variable to the Gomory cut, transform it to an equation, and easily add it back to the current dual simplex tableau as the last row with the slack variable in the current basis.

8.2.3.1 Cutting plane algorithm

In this section we illustrate the application of Gomory cut in an iterative fashion in a cutting plane algorithm. This by solving a continuous relaxation at each step. If the continuous relaxation solution fails to be integral, a separating cutting plane (a valid Gomory cut) is generated and added to the formulation, and the process is repeated. The algorithm terminates when the continuous relaxation solution is integral.

Let us solve the following MILP in standard form:

$$\left.\begin{array}{lrcl} \min & x_1 - 2x_2 & & \\ t.c. & -4x_1 + 6x_2 + x_3 & = & 9 \\ & x_1 + x_2 + x_4 & = & 4 \\ & x \geq 0 & , & x_1, x_2 \in \mathbb{Z} \end{array}\right\}$$

In this example, x_3, x_4 can be seen as slack variables added to an original formulation in canonical form with inequality constraints expressed in x_1, x_2 only.

[1]

[1]Material from notes by Leo Liberti. https://www.lix.polytechnique.fr/~liberti/teaching/ieor/advmathprog.pdf

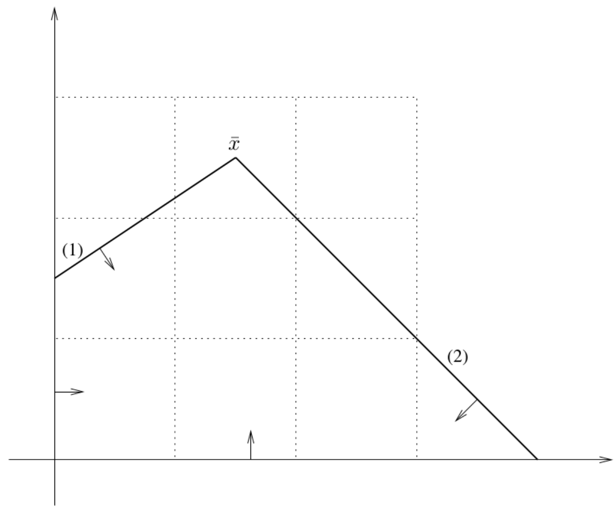
following tableau sequence, where the pivot element is $\boxed{\text{framed}}$.

	x_1	x_2	x_3	x_4
0	1	-2	0	0
9	-4	$\boxed{6}$	1	0
4	1	1	0	1

	x_1	x_2	x_3	x_4
3	$-\frac{1}{3}$	0	$\frac{1}{3}$	0
$\frac{3}{2}$	$-\frac{2}{3}$	1	$\frac{1}{6}$	0
$\frac{5}{2}$	$\boxed{\frac{5}{3}}$	0	$-\frac{1}{6}$	1

	x_1	x_2	x_3	x_4
$\frac{7}{2}$	0	0	$\frac{3}{10}$	$\frac{1}{5}$
$\frac{5}{2}$	0	1	$\frac{1}{10}$	$\frac{2}{5}$
$\frac{3}{2}$	1	0	$-\frac{1}{10}$	$\frac{3}{5}$

The solution of the continuous relaxation is $\bar{x} = (\frac{3}{2}, \frac{5}{2})$, where $x_3 = x_4 = 0$.



We derive a Gomory cut from the first row of the optimal tableau: $x_2 + \frac{1}{10}x_3 + \frac{2}{5}x_4 = \frac{5}{2}$. The cut is formulated as follows:

$$x_i + \sum_{j\in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{b}_i \rfloor, \tag{8.5}$$

where N is the set of nonbasic variable indices and i is the index of the chosen row. In this case we obtain the constraint $x_2 \leq 2$.

We introduce this Gomory cut in the current tableau. Note that if we insert a valid cut in a simplex tableau, the current basis becomes primal infeasible, thus a dual simplex iteration is needed. First of all, we express $x_2 \leq 2$ in terms of the current nonbasic variables x_3, x_4. We subtract the i-th optimal tableau row from Eq. (8.5), obtaining:

$$\begin{aligned} x_i + \sum_{j\in N} \bar{a}_{ij} x_j &\leq \bar{b}_i \\ \Rightarrow \sum_{j\in N} (\lfloor \bar{a}_{ij} \rfloor - \bar{a}_{ij}) x_j &\leq (\lfloor \bar{b}_i \rfloor - \bar{b}_i) \\ \Rightarrow -\frac{1}{10}x_3 - \frac{2}{5}x_4 &\leq -\frac{1}{2}. \end{aligned}$$

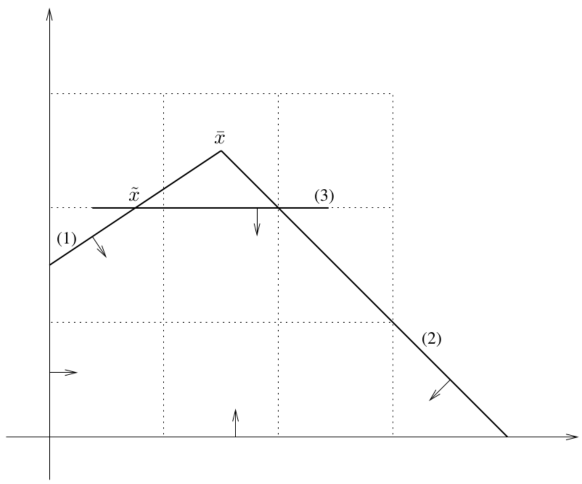
Recall that the simplex algorithm requires the constraints in equation (rather than inequality) form, so

$$-\frac{1}{10}x_3 - \frac{2}{5}x_4 + x_5 = -\frac{1}{2}.$$

We ajoin this constraint as the bottom row of the optimal tableau. We now have a current tableau with an additional row and column, corresponding to the new cut and the new slack variable (which is in the basis):

	x_1	x_2	x_3	x_4	x_5
$\frac{7}{2}$	0	0	$\frac{3}{10}$	$\frac{1}{5}$	0
$\frac{5}{2}$	0	1	$\frac{1}{10}$	$\frac{2}{5}$	0
$\frac{3}{2}$	1	0	$-\frac{1}{10}$	$\frac{3}{5}$	0
$-\frac{1}{2}$	0	0	$-\frac{1}{10}$	$\boxed{-\frac{2}{5}}$	1

The new row corresponds to the Gomory cut $x_2 \leq 2$ (labelled "constraint (3)" in the figure below).



We carry out an iteration of the dual simplex algorithm using this modified tableau. The reduced costs are all non-negative, but $\bar{b}_3 = -\frac{1}{2} < 0$ implies that $x_5 = \bar{b}_3$ has negative value, so it is not primal feasible (as $x_5 \geq 0$ is now a valid constraint). We pick x_5 to exit the basis. The variable j entering the basis is given by:

$$j = \text{argmin}\{\frac{\bar{c}_j}{|\bar{a}_{ij}|} \mid j \leq n \wedge \bar{a}_{ij} < 0\}.$$

In this case, $j = \text{argmin}\{3, \frac{1}{2}\}$, corresponding to $j = 4$. Thus x_4 enters the basis replacing x_5 (the pivot element is indicated in the above tableau). The new tableau is:

	x_1	x_2	x_3	x_4	x_5
$\frac{13}{4}$	0	0	$\frac{1}{4}$	0	$\frac{1}{2}$
2	0	1	0	0	1
$\frac{3}{4}$	1	0	$-\frac{1}{4}$	0	$\frac{3}{2}$
$\frac{5}{4}$	0	0	$\frac{1}{4}$	1	$-\frac{5}{2}$

The optimal solution is $\tilde{x} = (\frac{3}{4}, 2)$. Since this solution is not integral, we continue. We pick the second

$$x_1 - \frac{1}{4}x_3 + \frac{3}{2}x_5 = \frac{3}{4},$$

to generate a Gomory cut

$$x_1 - x_3 + x_5 \leq 0,$$

which, written in terms of the variables x_1, x_2 is

$$-3x_1 + 5x_2 \leq 7.$$

This cut can be written as:

$$-\frac{3}{4}x_3 - \frac{1}{2}x_5 \leq -\frac{3}{4}.$$

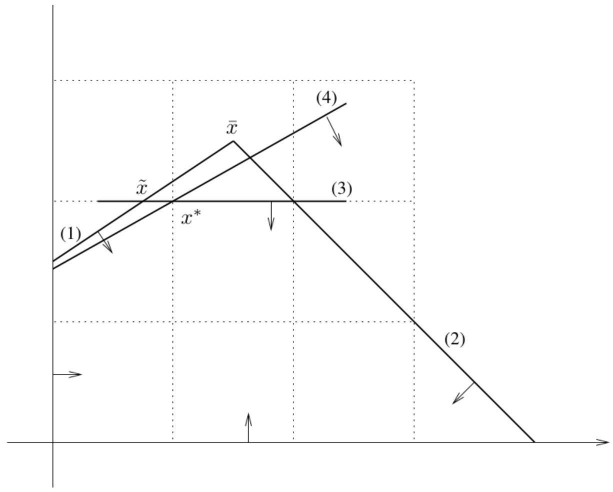
The new tableau is:

	x_1	x_2	x_3	x_4	x_5	x_6
$\frac{13}{4}$	0	0	$\frac{1}{4}$	0	$\frac{1}{2}$	0
2	0	1	0	0	1	0
$\frac{3}{4}$	1	0	$-\frac{1}{4}$	0	$\frac{3}{2}$	0
$\frac{5}{4}$	0	0	$\frac{1}{4}$	1	$-\frac{5}{2}$	0
$-\frac{3}{4}$	0	0	$\boxed{-\frac{3}{4}}$	0	$-\frac{1}{2}$	1

The pivot is framed; the exiting row 4 was chosen because $\bar{b}_4 < 0$, the entering column 5 because $\frac{\bar{c}_3}{|\bar{a}_{43}|} = \frac{1}{3} < 1 = \frac{\bar{c}_5}{|\bar{a}_{45}|}$). Pivoting, we obtain:

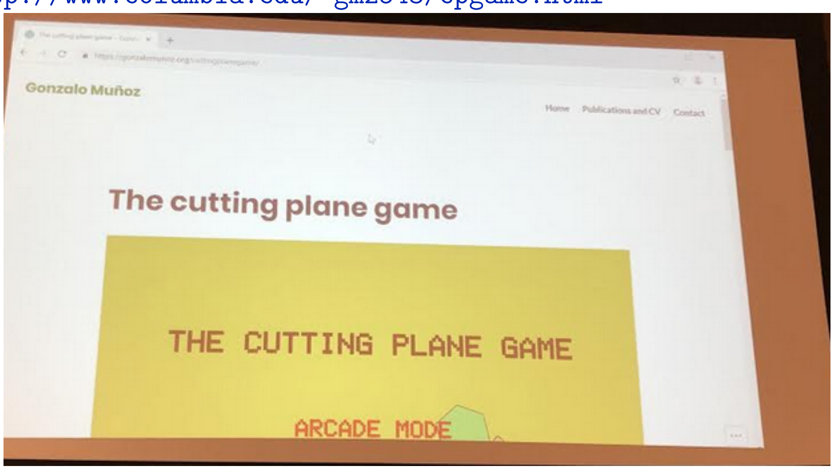
	x_1	x_2	x_3	x_4	x_5	x_6
3	0	0	0	0	$\frac{1}{3}$	$\frac{1}{3}$
2	0	1	0	0	1	0
1	1	0	0	0	$\frac{5}{3}$	$-\frac{1}{3}$
1	0	0	0	1	$-\frac{8}{3}$	$\frac{1}{3}$
1	0	0	1	0	$\frac{2}{3}$	$-\frac{4}{3}$

This tableau has optimal solution $x^* = (1, 2)$, which is integral (and hence optimal). The figure below shows the optimal solution and the last Gomory cut to be generated.



1.3.3 Fun with cutting planes

The Cutting Plane Game:
http://www.columbia.edu/~gm2543/cpgame.html



1.4 Branching Rules

There is a few clever ideas out there on how to choose which variables to branch on. We will not go into this here. But for the interested reader, look into

- Strong Branching
- Pseudo-cost Branching

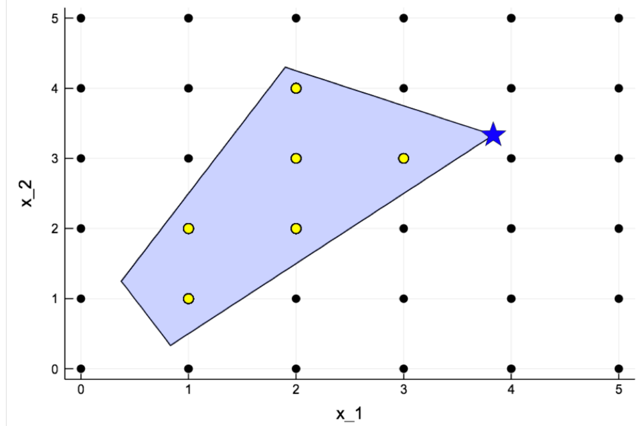
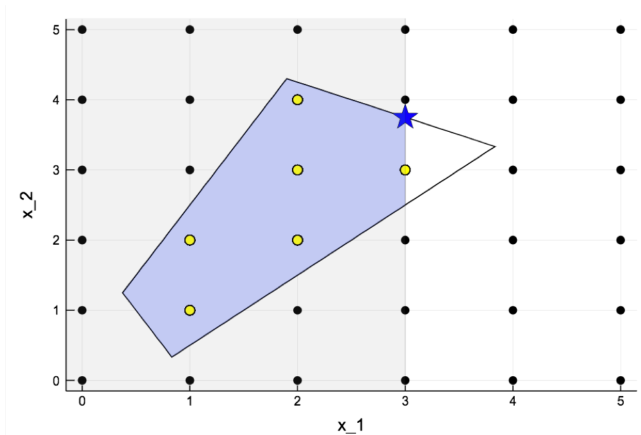
1.5 Lagrangian Relaxation for Branch and Bound

At each note in the branch and bound tree, we want to bound the objective value. One way to get a a good bound can be using the Lagrangian.

See [?] for a description of this.

For a great tutorial, see this: https://my.eng.utah.edu/~kalla/phy_des/lagrange-relax-tutorial-fisher.pdf

1.6 Literature

Model	**LP Solution**
max $x_1 + x_2$ Subject to $-2x_1 + x_2 \leq 0.5$ $x_1 + 2x_2 \leq 10.5$ $x_1 - x_2 \leq 0.5$ $-2x_1 - x_2 \leq -2$ $x_i \quad \forall i \in \{1,2\}$	
max $x_1 + x_2$ Subject to $-2x_1 + x_2 \leq 0.5$ $x_1 + 2x_2 \leq 10.5$ $x_1 - x_2 \leq 0.5$ $-2x_1 - x_2 \leq -2$ $x_i \quad \forall i \in \{1,2\}$	
max $x_1 + x_2$ Subject to $-2x_1 + x_2 \leq 0.5$ $x_1 + 2x_2 \leq 10.5$ $x_1 - x_2 \leq 0.5$ $-2x_1 - x_2 \leq -2$ $x_i \quad \forall i \in \{1,2\}$