Mixed Integer Nonlinear Programming

Theory and Applications

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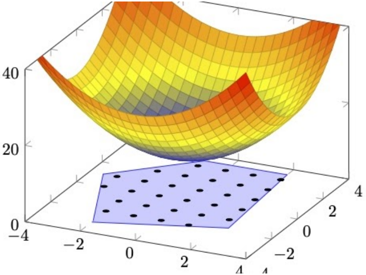


Outline

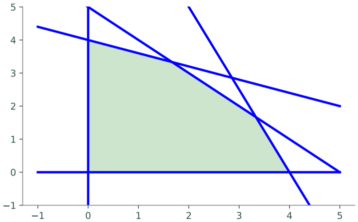
1 Linear Programming

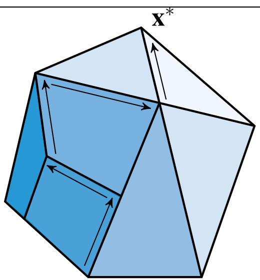
Linear Programming (LP):

Polynomial time (P)

Given a matrix $A \in \mathbb{R}^{m\times n}$, vector $b \in \mathbb{R}^m$ and vector $c \in \mathbb{R}^n$, the *linear programming* problem is

$$\begin{aligned} \max \quad & c^\top x \\ s.t. \quad & Ax \le b \\ & x \ge 0 \end{aligned} \tag{1}$$





Example: Toy Maker Excel PuLP Gurobipy

Consider the problem of a toy company that produces toy planes and toy boats. The toy company can sell its planes for \$10 and its boats for \$8 dollars. It costs \$3 in raw materials to make a plane and \$2 in raw materials to make a boat. A plane requires 3 hours to make and 1 hour to finish while a boat requires 1 hour to make and 2 hours to finish. The toy company knows it will not sell anymore than 35 planes per week. Further, given the number of workers, the company cannot spend anymore than 160 hours per week finishing toys and 120 hours per week making toys. The company wishes to maximize the profit it makes by choosing how much of each toy to produce.

$$\begin{cases} \max & 7x_1 + 6x_2 \\ s.t. & 3x_1 + x_2 \leq 120 \\ & x_1 + 2x_2 \leq 160 \\ & x_1 \leq 35 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{cases} \tag{2}$$

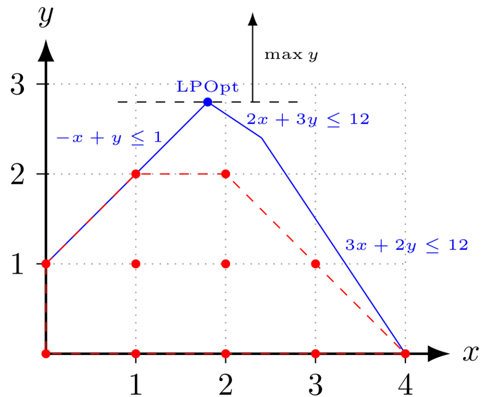
1.1. Amazing Results in Linear Programming

1. There are fast (polynomial time!) algorithms to solve Linear Programming.
2. If objective is bounded, then there exists an optimal solution.
3. If the objective is unbounded, there exists a ray on which the objective is unbiounded.
4. Strong duality theorem

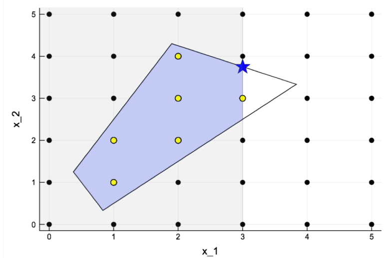
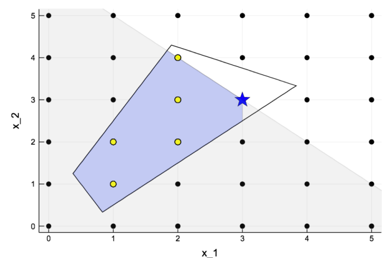
2 Integer Programming

Integer Linear Programming (ILP):
NP-Complete
Given a matrix $A \in \mathbb{R}^{m \times n}$, vector $b \in \mathbb{R}^m$ and vector $c \in \mathbb{R}^n$,
the *integer linear programming* problem is

$$\begin{aligned} \max \quad & c^\top x \\ s.t. \quad & Ax \leq b \\ & x \in \mathbb{Z}^n \end{aligned} \tag{3}$$



2.1. Cutting Planes

Model	LP Solution
$\begin{array}{lrl} \max & x_1 + x_2 & \\ \text{subject to} & -2x_1 + x_2 & \leq 0.5 \\ & x_1 + 2x_2 & \leq 10.5 \\ & x_1 - x_2 & \leq 0.5 \\ & -2x_1 - x_2 & \leq -2 \\ & \color{red}{x_1} & \color{red}{\leq 3} \end{array}$	
$\begin{array}{lrl} \max & x_1 + x_2 & \\ \text{subject to} & -2x_1 + x_2 & \leq 0.5 \\ & x_1 + 2x_2 & \leq 10.5 \\ & x_1 - x_2 & \leq 0.5 \\ & -2x_1 - x_2 & \leq -2 \\ & \color{red}{x_1} & \color{red}{\leq 3} \\ & \color{red}{x_1 + x_2} & \color{red}{\leq 6} \end{array}$	

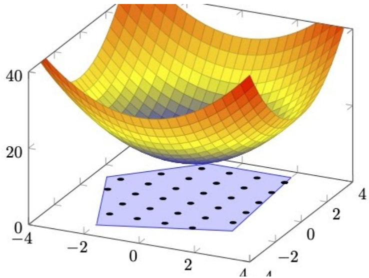
2.2. Amazing results in Integer Programming

1. There exists a finite cutting plane algorithm to solve the problem
2. If the objective is bounded, then there exists an optimal solution (provided that the data is rational)
3. If the objective is unbounded, then there exists a ray on which the objetive is unbounded (provided that the data is rational)
4. Under certain special structural conditions, the problem can be solved in polynomial time.
5. In practice, we can frequently solve general problems fast.

3 Mixed Integer Nonlinear Programming

MINLP:
very *NP-Hard*!
Given a function $f(x)\colon \mathbb{R}^d \to \mathbb{R}$ and other functions $f_i(x)\colon \mathbb{R}^d \to \mathbb{R}$ for $i = 1,\ldots,m$, the *mixed integer nonlinear programming* problem is

$$\begin{aligned} \min \quad & f(x) \\ s.t. \quad & f_i(x) \le 0 \quad \text{for } i = 1,\ldots,m \\ & x \in \mathbb{Z}^n \times \mathbb{R}^{d-n} \end{aligned} \tag{4}$$



3.1. Example: Polynomial optimization

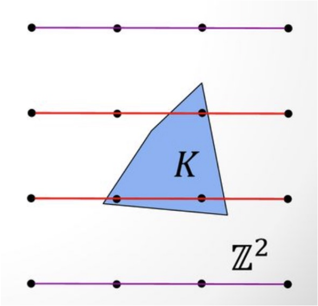
$$
\begin{aligned}
\min \quad & 3x_1 + 4x_2 \\
s.t. \quad & x_1^2 x_2 + 2x_2 + x_3 = 5 \\
& x_1^2 x_2^2 + x_2 + x_4 = 3 \\
& x_1 x_2 - x_1 x_2^2 \leq 1 \\
& 0 \leq x_1 x_2^2 \leq 2 \\
& x_1 \in \mathbb{Z}
\end{aligned}
\tag{5}
$$

3.2. Difficult things about MINLP

1. There may not exist any finite algorithm!
2. Even if the objective is bounded, there may not exist an optimal solution!
3. If the objective is unbounded, there may not be a ray on which the objective is unbounded!
4. Nearly feasible solutions may be super optimal!
5. It is difficult to determine even if there exists a rational solution!
6. Feasible regions are not contiguous!

3.3. Very cool things about MINLP

Flatness Theorem: For any convex body K in $\mathbb{R}^n$, if the *lattice width* $\omega(K) \geq 2n^2$, then K contains an integer point.



Corollary: Suppose f and $f_1, \ldots, f_m$ are convex functions. Then there exists an algorithm to solve the mixed integer program

$$\min f(x) : f_i(x) \leq 0, x \in \mathbb{Z}^n \times \mathbb{R}^{d-n}$$

that runs in time $n^{O(n^2)}\text{poly}(d, m, \langle f_i \rangle)$.

***S*-Lemma:** For quadratic functions f, g with $g(\bar{x}) < 0$ for some $\bar{x} \in \mathbb{R}^n$, the following are equivalent:

(1) The set $\{x \in \mathbb{R}^n : f(x) < 0, g(x) \leq 0\}$ is empty.
(2) There is a $\lambda \geq 0$ such that

$$f(x) + \lambda g(x) \geq 0 \quad \text{for all } x \in \mathbb{R}^n$$

Corollary: The problem

$$\begin{aligned} \min \quad & x^\top Q x + c^\top x \\ s.t. \quad & \|x\|_2 \leq 1 \end{aligned} \tag{6}$$

can be solved in polynomial time.

Sums of Squares Programming: A polynomial $f(x)$ in n-variables is non-negative if we can write it as a *sum of squares*, i.e.,

$$f(x) = \sum_{j=1}^{t} (s_j(x))^2$$

for some polynomials $s_j(x)$.

Equivalently, let X be a vector of monomials. Then when $f(x)$ is a sum of squares, it can be written as

$$f(x) = z^\top Qz$$

where Q is a positive semidefinite matrix.

Corollary: There exists an algorithm for many polynomial optimization problems.

Example.[1] Consider the task proving nonnegativity of the polynomial

$$\begin{aligned} p(x) =& x_1^4 - 6x_1^3x_2 + 2x_1^3x_3 + 6x_1^2x_3^2 + 9x_1^2x_2^2 - 6x_1^2x_2x_3 - 14x_1x_2x_3^2 + 4x_1x_3^3 \\ & + 5x_3^4 - 7x_2^2x_3^2 + 16x_2^4. \end{aligned}$$

Since this is a form (i.e., a homogeneous polynomial), we take

$$z = \left(x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2\right)^T.$$

One feasible solution to the SDP in (5) is given by

$$Q = \begin{pmatrix} 1 & -3 & 0 & 1 & 0 & 2 \\ -3 & 9 & 0 & -3 & 0 & -6 \\ 0 & 0 & 16 & 0 & 0 & -4 \\ 1 & -3 & 0 & 2 & -1 & 2 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 2 & -6 & 4 & 2 & 0 & 5 \end{pmatrix}$$

Upon a decomposition $Q = \sum_{i=1}^3 a_i^T a_i$, with $a_1 = (1,-3,0,1,0,2)^T, a_2 = (0,0,0,1,-1,0)^T, a_3 = (0,0,4,0,0,-1)^T$, one obtains the sos decomposition

$$p(x) = \left(x_1^2 - 3x_1x_2 + x_1x_3 + 2x_3^2\right)^2 + (x_1x_3 - x_2x_3)^2 + \left(4x_2^2 - x_3^2\right)^2.$$

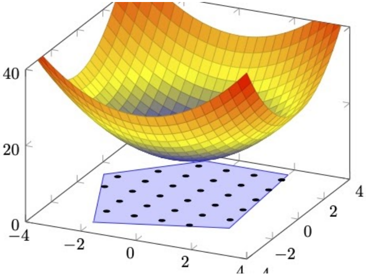
[1] Borrowed from lecture slides of Amir Ali Ahmadi, Princeton ORFE

4 Research interests

4.1. Nonliner Mixed Integer Programming over Polyhedra

Suppose that f is a nonconvex polynomial. Consider

$$\begin{aligned} \max \quad & f(x) \\ s.t. \quad & Ax \leq b \\ & x \in \mathbb{Z}^n \times \mathbb{R}^{d-n} \end{aligned} \tag{7}$$

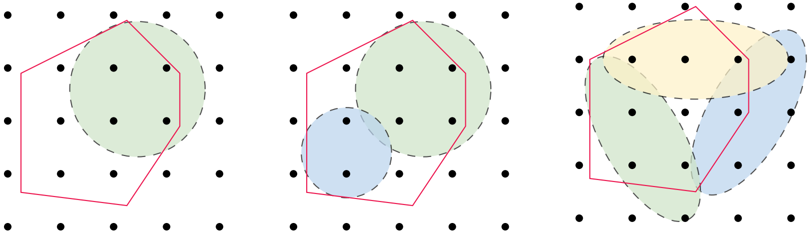


<u>What is known?</u>

1. Some results for 2 integer variables
2. Some results when f is a quadratic
3. Some hardness results in low dimension.

4.2. Reverse convex sets

- Let P be a polyhedron
- Let C_i be convex sets (semi-algebraic?), for $i = 1, \ldots, m$.
- Determine if $P \setminus (C_1 \cup \cdots \cup C_m)$ contains any integer points.



4.3. Combining theories

1. How to make an integer version of the S-Lemma?
2. How to make an integer analog of Sums of Squares Programming?
3. How to derive cutting planes for mixed integer polynomial optimization problems?