Chapter 1

Algorithms to Solve Integer Programs

1.1 LP to solve IP

Recall that he linear relaxation of an integer program is the linear programming problem after removing the integrality constraints

Integer Program:

$$\begin{aligned} \max \quad & z_{IP} = c^\top x \\ & Ax \le b \\ & x \in \mathbb{Z}^n \end{aligned}$$

Linear Relaxation:

$$\begin{aligned} \max \quad & z_{LP} = c^\top x \\ & Ax \le b \\ & x \in \mathbb{R}^n \end{aligned}$$

Theorem 1. *It always holds that*

$$z^*_{IP} \le z^*_{LP}. \tag{1.1.1}$$

Furthermore, if x^*_{LP} is integral (feasible for the integer program), then

$$x^*_{LP} = x^*_{IP} \quad \text{and } z^*_{LP} = z^*_{IP}. \tag{1.1.2}$$

Example 1:
Consider the problem

$$\begin{aligned} \max z = & 3x_1 + 2x_1 \\ & 2x_1 + x_2 \le 6 \\ & x_1, x_2 \ge 0; x_1, x_2 \text{ integer} \end{aligned}$$

1.1.1 Rounding LP Solution can be bad!

Consider the two variable knapsack problem

$$\max 3x_1 + 100x_2 \tag{1.1.3}$$

$$x_1 + 10x_2 \le 10 \tag{1.1.4}$$

$$x_i \in \{0, 1\} \text{ for } i = 1, 2. \tag{1.1.5}$$

Then $x^*_{LP} = [1, 0.99]$ and $z^*_{LP} = 1 \cdot 3 + 0.99 \cdot 100 = 3 + 99 = 102$.
But $x^*_{IP} = [0, 1]$ with $z^*_{IP} = 0 \cdot 3 + 1 \cdot 100 = 100$.
Suppose that we rounded the LP solution.
$x^*_{LP-Rounded-Down} = [1, 0]$. Then $z^*_{LP-Rounded-Down} = 1 \cdot 3 = 3$. Which is a terrible solution!
How can we avoid this issue?

Cool trick! Using two different strategies gives you at least a 1/2 approximation to the optimal solution.

1.1.2 Rounding LP solution can be infeasible!

Now only could it produce a poor solution, it is not always clear how to round to a feasible solution.

1.1.3 Fractional Knapsack

The fractional knapsack problem has an exact greedy algorothm.

https://www.youtube.com/watch?time_continue=424&v=m1p-eWxrt6g

https://www.geeksforgeeks.org/fractional-knapsack-problem/

1.2 Branch and Bound

See http://web.tecnico.ulisboa.pt/mcasquilho/compute/_linpro/TaylorB_module_c.pdf for some nice notes on branch and bound.

1.2.1 Algorithm

Algorithm 1 Branch and Bound - Maximization

Input: Integer Linear Problem with max objective

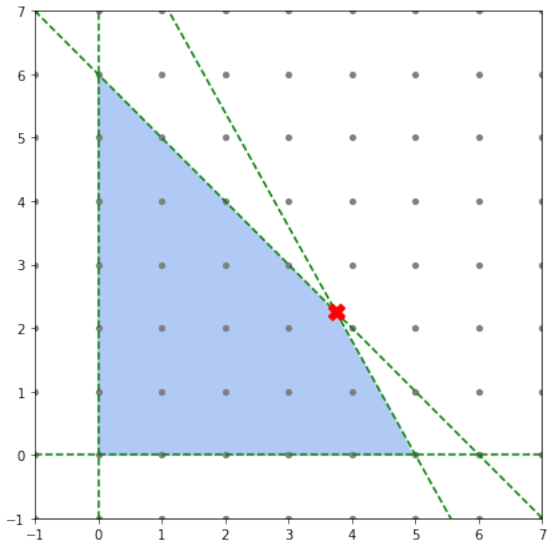
Output: Exact Optimal Solution x^*

1: Set $LB = -\infty$.
2: Solve LP relaxation.
   a: If x^* is integer, stop!
   b: Otherwise, choose fractional entry x^*_i and branch onto subproblems: (i) $x_i \leq \lfloor x^*_i \rfloor$ and (ii) $x_i \geq \lceil x^*_i \rceil$.
3: Solve LP relaxation of any subproblem.
   a: If LP relaxation is infeasible, prune this node as "Infeasible"
   b: If $z^* < LB$, prune this node as "Suboptimal"
   c: x^* is integer, prune this nodes as "Integer" and update $LB = \max(LB, z^*)$.
   d: Otherwise, choose fractional entry x^*_i and branch onto subproblems: (i) $x_i \leq \lfloor x^*_i \rfloor$ and (ii) $x_i \geq \lceil x^*_i \rceil$. Return to step 2 until all subproblems are pruned.
4: Return best integer solution found.

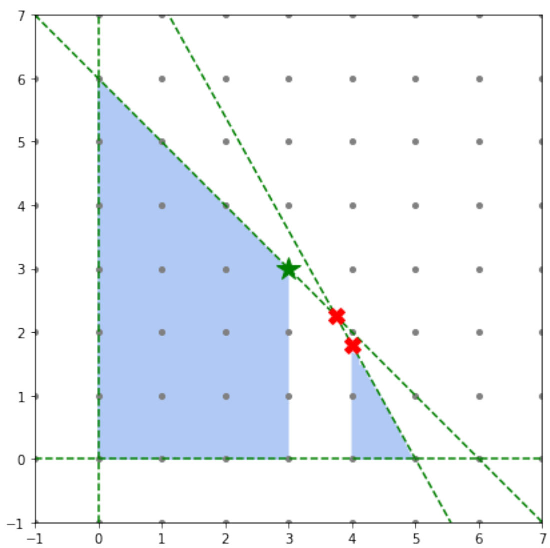
1.2.2 General Branching

Example 2: See Example 9 in Chapter 9 of the textbook (Winston - Operations Research Applications and Algorithms).

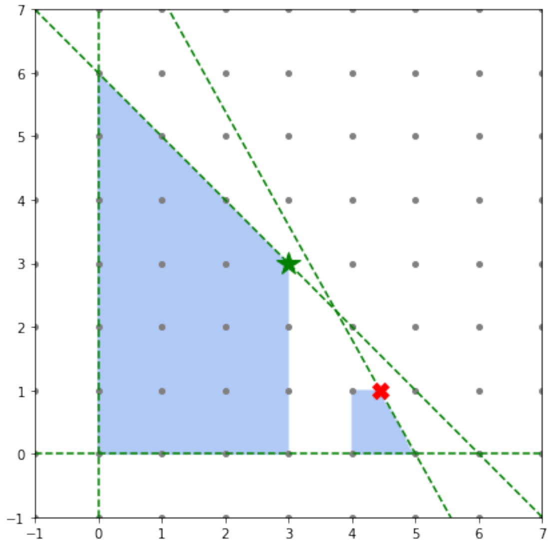
$x = [3.75, 2.25]$, $obj = 41.25$



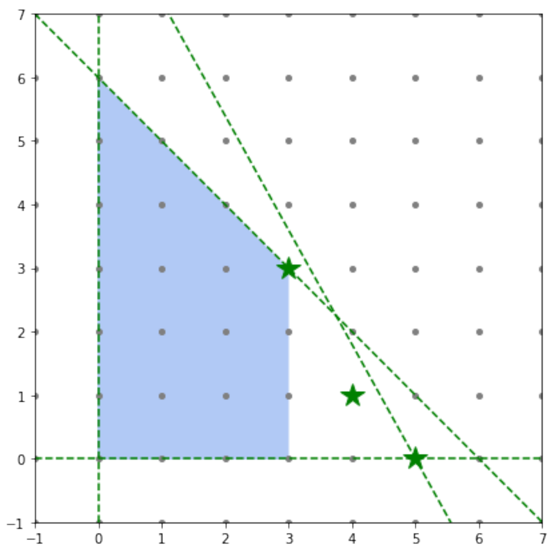
$x = [3, 3]$ $obj = 39.0$
$x = [4, 1.8]$,, $obj = 41.0$



$x = [3, 3]$, $obj = 39.0$
$x = [4.44, 1]$ $obj = 40.55$.
Infeasible Region



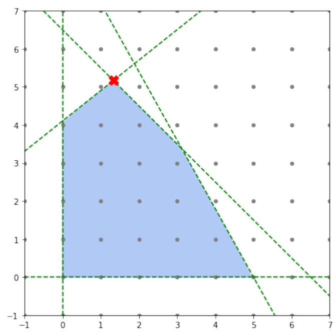
$x = [3, 3]$, $obj = 39.0$
$x = [4, 1]$, $obj = 37.0$
$x = [5, 0]$, $obj = 40.0$
Infeasible Region



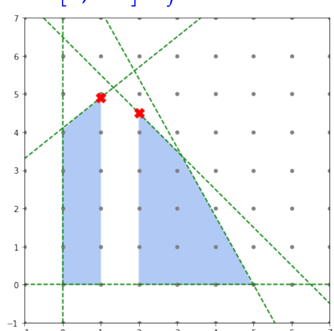
Example 3: Consider the two variable example with

$$
\begin{aligned}
\max -3x_1 + 4x_2 \\
2x_1 + 2x_2 \leq 13 \\
-8x_1 + 10x_2 \leq 41 \\
9x_1 + 5x_2 \leq 45 \\
0 \leq x_1 \leq 10, \text{ integer} \\
0 \leq x_2 \leq 10, \text{ integer}
\end{aligned}
$$

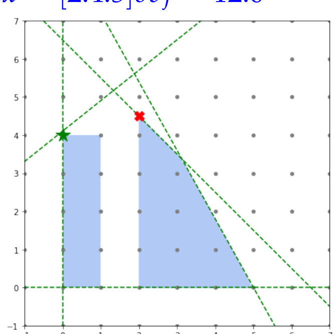
$x = [1.33, 5.167] obj = 16.664$



$x = [1, 4.9] obj = 16.5998$
$x = [2, 4.5] obj = 12.0$



Infeasible Region
$x = [0.4.] obj = 16.0$
$x = [2.4.5] obj = 12.0$



1.2.3 Knapsack Problem and 0/1 branching

Consider the problem

$$
\begin{aligned}
\max \quad & 16x_1 + 22x_2 + 12x_3 + 8x_4 \\
\text{s.t.} \quad & 5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14 \\
& 0 \leq x_i \leq 1 \quad i = 1,2,3,4 \\
& x_i \in \{0,1\} \quad i = 1,2,3,4
\end{aligned}
$$

What is the optimal solution if we remove the binary constraints?

$$
\begin{aligned}
\max \quad & c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \\
\text{s.t.} \quad & a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 \le b \\
& 0 \le x_i \le 1 \quad i = 1, 2, 3, 4
\end{aligned}
$$

How do I find the solution to this problem?

$$\begin{aligned} \max \quad & c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \\ \text{s.t.} \quad & (a_1 - A)x_1 + (a_2 - A)x_2 + (a_3 - A)x_3 + (a_4 - A)x_4 \leq 0 \\ & 0 \leq x_i \leq m_i \quad i = 1, 2, 3, 4 \end{aligned}$$

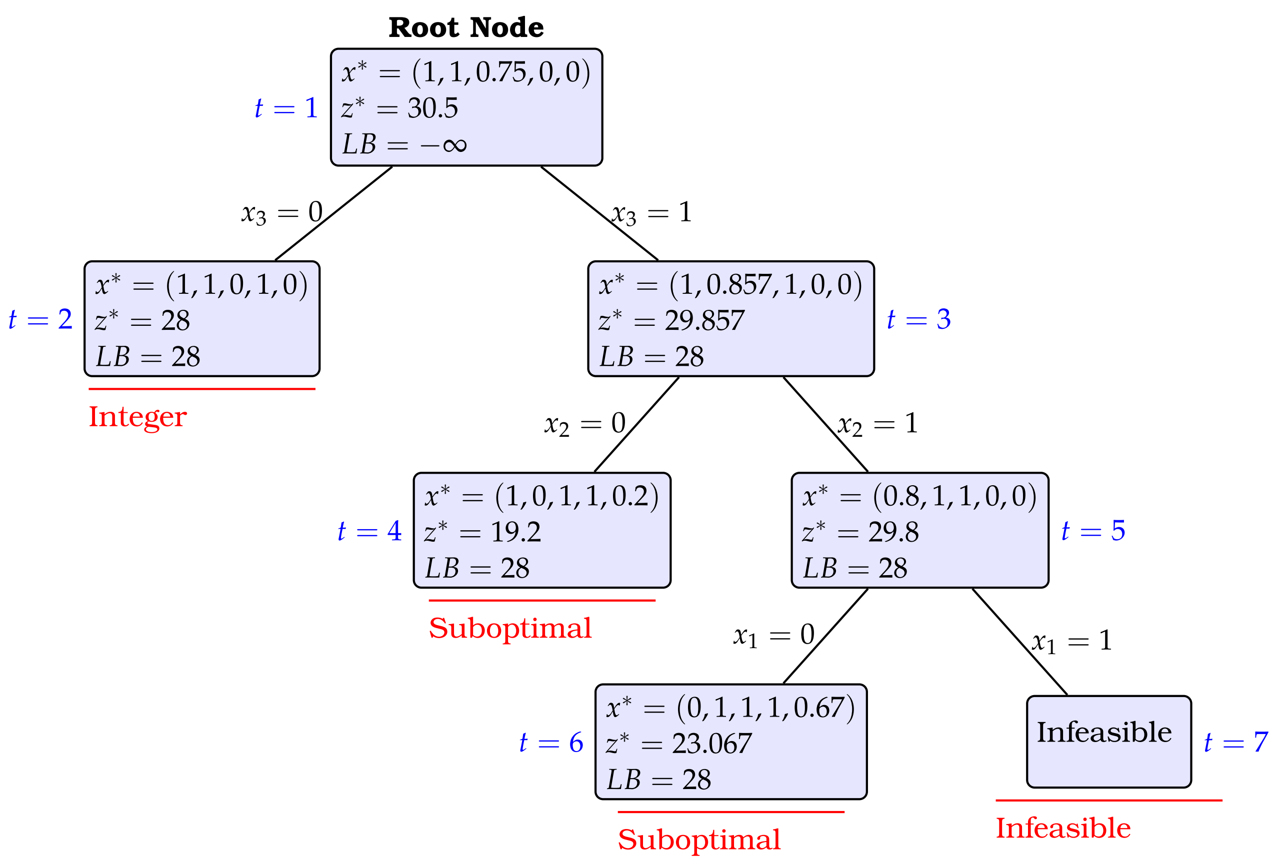
How do I find the solution to this problem?

Consider the problem

$$\begin{aligned} \max \quad & 16x_1 + 22x_2 + 12x_3 + 8x_4 \\ \text{s.t.} \quad & 5x_1 + 7x_2 + 4x_3 + 3x_4 \le 14 \\ & 0 \le x_i \le 1 \quad i = 1,2,3,4 \\ & x_i \in \{0,1\} \quad i = 1,2,3,4 \end{aligned}$$

We can solve this problem with branch and bound.
The optimal solution was found at $t = 5$ at subproblem 6 to be $x^* = (0,1,1,1)$, $z^* = 42$.
Example: Binary Knapsack Solve the following problem with branch and bound.

$$\begin{aligned} \max \quad & z = 11x_1 + 15x_2 + 6x_3 + 2x_4 + x_5 \\ \text{Subject to:} \quad & 5x_1 + 7x_2 + 4x_3 + 3x_4 + 15x_5 \le 15 \\ & x_i \text{ binary}, i = 1, \ldots, 4 \end{aligned}$$



1.2.4 Traveling Salesman Problem solution via Branching

1.3 Cutting Planes

Cutting planes are inequalities $\pi^\top x \le \pi_0$ that are valid for the feasible integer solutions that the cut off part of the LP relaxation. Cutting planes can create a tighter description of the feasible region that allows for the optimal solution to be obtained by simply solving a strengthened linear relaxation.

The cutting plane procedure, as demonstrated in Figure **??**. The procedure is as follows:

1. Solve the current LP relaxation.
2. If solution is integral, then return that solution. STOP
3. Add a cutting plane (or many cutting planes) that cut off the LP-optimal solution.
4. Return to Step 1.

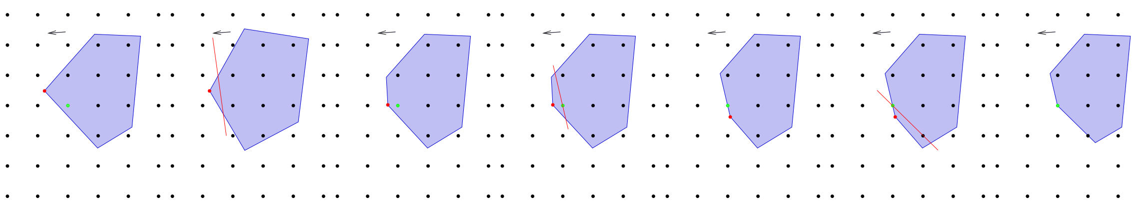


Figure 1.1: *The cutting plane procedure.*

In practice, this procedure is integrated in some with with branch and bound and also other primal heuristics.

1.3.1 Chvátal Cuts

Chvátal Cuts are a general technique to produce new inequalities that are valid for feasible integer points.

Chvátal Cuts:
Suppose

$$a_1 x_1 + \cdots + a_n x_n \leq d \tag{1.3.1}$$

is a valid inequality for the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$, then

$$\lfloor a_1 \rfloor x_1 + \cdots + \lfloor a_n \rfloor x_n \leq \lfloor d \rfloor \tag{1.3.2}$$

is valid for the integer points in P, that is, it is valid for the set $P \cap \mathbb{Z}^n$. Equation (**??**) is called a Chvátal Cut.

We will illustrate this idea with an example.

Example 4: Recall example **??**. The model was
Model

min	$p + n + d + q$	total number of coins used
s.t.	$p + 5n + 10d + 25q = 83$	sums to 83¢
	$p, d, n, q \in \mathbb{Z}_+$	each is a non-negative integer

From the equality constraint we can derive several inequalities.

1. Divide by 25 and round down both sides:

$$\frac{p + 5n + 10d + 25q}{25} = 83/25 \quad \Rightarrow \quad q \leq 3$$

2. Divide by 10 and round down both sides:

$$\frac{p + 5n + 10d + 25q}{10} = 83/10 \quad \Rightarrow \quad d + 2q \leq 8$$

3. Divide by 5 and round down both sides:

$$\frac{p + 5n + 10d + 25q}{10} = 83/5 \quad \Rightarrow \quad n + 2d + 5q \leq 16$$

4. Multiply by 0.12 and round down both sides:

$$0.12(p + 5n + 10d + 25q = 0.12(83) \quad \Rightarrow \quad d + 3q \leq 9$$

These new inequalities are all valid for the integer solutions. Consider the new model:

New Model

$$\begin{array}{lll} \min & p + n + d + q & \text{total number of coins used} \\ \text{s.t.} & p + 5n + 10d + 25q = 83 & \text{sums to } 83¢ \\ & q \leq 3 & \\ & d + 2q \leq 8 & \\ & n + 2d + 5q \leq 16 & \\ & d + 3q \leq 9 & \\ & p, d, n, q \in \mathbb{Z}_+ & \text{each is a non-negative integer} \end{array}$$

The solution to the LP relaxation is exactly $q = 3, d = 0, n = 1, p = 3$, which is an integral feasible solution, and hence it is an optimal solution.

1.3.2 Gomory Cuts

Gomory cuts are a type of Chvátal cut that is derived from the simplex tableau. Specifically, suppose that

$$x_i + \sum_{i \in N} \tilde{a}_i x_i = \tilde{b}_i \tag{1.3.3}$$

is an equation in the optimal simplex tableau.

Gomory Cut:
The Gomory cut corresponding to the tableau row (**??**) is

$$\sum_{i \in N} (\tilde{a}_i - \lfloor \tilde{a}_i \rfloor) x_i \geq \tilde{b}_i - \lfloor \tilde{b}_i \rfloor \tag{1.3.4}$$

We will solve the following problem using only Gomory Cuts.

$$\begin{array}{rrcl} \min & x_1 - 2x_2 & & \\ \text{s.t.} & -4x_1 + 6x_2 & \leq & 9 \\ & x_1 + x_2 & \leq & 4 \\ & x \geq 0 & , & x_1, x_2 \in \mathbb{Z} \end{array}$$

Step 1: The first thing to do is to put this into standard from by appending slack variables.

$$\begin{array}{rrcl} \min & x_1 - 2x_2 & & \\ \text{s.t.} & -4x_1 + 6x_2 + s_1 & = & 9 \\ & x_1 + x_2 + s_2 & = & 4 \\ & x \geq 0 & , & x_1, x_2 \in \mathbb{Z} \end{array} \tag{1.3.5}$$

We can apply the simplex method to solve the LP relaxation.

Initial Basis

Basis	RHS	x_1	x_2	s_1	s_2
z	0.0	1.0	-2.0	0.0	0.0
s_1	9.0	-4.0	6.0	1.0	0.0
s_2	4.0	1.0	1.0	0.0	1.0

⋮ ⋮

Optimal Basis

Basis	RHS	x_1	x_2	s_1	s_2
z	-3.5	0.0	0.0	0.3	0.2
x_1	1.5	1.0	0.0	-0.1	0.6
x_2	2.5	0.0	1.0	0.1	0.4

This LP relaxation produces the fractional basic solution $x_{LP} = (1.5, 2.5)$.

Example 5: (Gomory cut removes LP solution)
We now identify an integer variable x_i that has a fractional basic solution. Since both variables have fractional values, we can choose either row to make a cut. Let's focus on the row corresponding to x_1.

The row from the tableau expresses the equation

$$x_1 - 0.1s_1 + -0.6s_2 = 1.5. \tag{1.3.6}$$

Applying the Gomory Cut (**??**), we have the inequality

$$0.9s_1 + 0.4s_2 \geq 0.5. \tag{1.3.7}$$

The current LP solution is $(x_{LP}, s_{LP}) = (1.5, 2.5, 0, 0)$. Trivially, since $s_1, s_2 = 0$, the inequality is violated.

Example 6: (Gomory Cut in Original Space)

The Gomory Cut (**??**) can be rewritten in the original variables using the equations from (**??**). That is, we can use the equations

$$\begin{aligned} s_1 &= 9 + 4x_1 - 6x_2 \\ s_2 &= 4 - x_1 - x_2, \end{aligned} \tag{1.3.8}$$

which transforms the Gomory cut into the original variables to create the inequality

$$0.9(9 + 4x_1 - 6x_2) + 0.4(4 - x_1 - x_2) \geq 0.5.$$

or equivalently

$$-3.2x_1 + 5.8x_2 \leq 9.2. \tag{1.3.9}$$

As you can see, this inequality does cut off the current LP relaxation.

Example 7: (Gomory cuts plus new tableau) Now we add the slack variable $s_3 \geq 0$ to make the equation

$$0.9s_1 + 0.4s_2 - s_3 = 0.5. \tag{1.3.10}$$

Next, we need to solve the linear programming relaxation (where we assume the variables are continuous).

1.4 Branching Rules

There is a few clever ideas out there on how to choose which variables to branch on. We will not go into this here. But for the interested reader, look into

- Strong Branching
- Pseudo-cost Branching

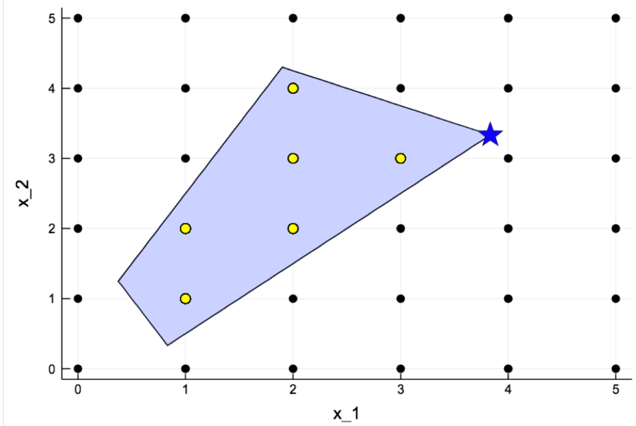
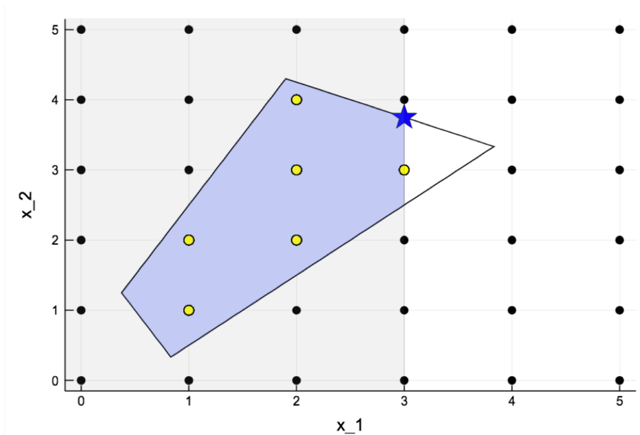
1.5 Lagrangian Relaxation for Branch and Bound

At each note in the branch and bound tree, we want to bound the objective value. One way to get a a good bound can be using the Lagrangian.

See [**?**] for a description of this.

For a great tutorial, see this: https://my.eng.utah.edu/~kalla/phy_des/lagrange-relax-tutorial-fisher.pdf

1.6 Literature

Model	**LP Solution**
$\begin{array}{rl} \max & x_1 + x_2 \\ \text{Subject to} & -2x_1 + x_2 \leq 0.5 \\ & x_1 + 2x_2 \leq 10.5 \\ & x_1 - x_2 \leq 0.5 \\ & -2x_1 - x_2 \leq -2 \\ & x_i \quad \forall i \in \{1,2\} \end{array}$	
$\begin{array}{rl} \max & x_1 + x_2 \\ \text{Subject to} & -2x_1 + x_2 \leq 0.5 \\ & x_1 + 2x_2 \leq 10.5 \\ & x_1 - x_2 \leq 0.5 \\ & -2x_1 - x_2 \leq -2 \\ & x_i \quad \forall i \in \{1,2\} \end{array}$	
$\begin{array}{rl} \max & x_1 + x_2 \\ \text{Subject to} & -2x_1 + x_2 \leq 0.5 \\ & x_1 + 2x_2 \leq 10.5 \\ & x_1 - x_2 \leq 0.5 \\ & -2x_1 - x_2 \leq -2 \\ & x_i \quad \forall i \in \{1,2\} \end{array}$