Chapter 1

Integral polyhedra, TU matrices, TDI systems

1.1 Integral polyhedra

1.1.1 Basics

Definition 1 (Integral polyhedron). *A polyhedron P is called integral if every minimal face of P contains an integral vector.*

Remark 2. *If P has vertices, then P is integral if and only if every vertex is an integral vector.*

1.1.2 Properties

Theorem 3. *Let $P \subseteq \mathbb{R}^n$ be a polyhedron. Then the following are equivalent:*

1. $P = \text{conv}(P \cap \mathbb{Z}^n)$
2. *P is integral*
3. $\max\{c^T x : x \in P\}$ *has an integral optimal solution for all $c \in \mathbb{R}^n$ such that the optimal value is finite.*
4. $\max\{c^T x : x \in P\}$ *has an integral optimal solution for all $c \in \mathbb{Z}^n$ such that the optimal value is finite.*
5. $\max\{c^T x : x \in P\}$ *is an integer for all $c \in \mathbb{Z}^n$ such that the optimal value is finite.*

1.2 Unimodular and totally unimodular matrices

1.2.1 Unimodular matrices

Definition 4 (Unimodular matrix). *A matrix $A \in \mathbb{R}^{m \times n}$ is called unimodular if: (1) All entries are integers. (2) A has full rank. (3) Every $m \times m$ square submatrix of A has determinant $-1, 0, 1$.*

Theorem 5. *Let $A \in \mathbb{Z}^{m \times n}$ be a full row rank matrix. Then the polyhedron $P = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$ is integral for all $b \in \mathbb{Z}^n$ if and only if A is unimodular.*

1.2.2 Totally unimodular matrices

Definition 6 (Totally unimodular matrix). *The matrix $A \in \mathbb{R}^{m\times n}$ is called totally unimodular if every square submatrix of A has determinant $-1, 0, 1$.*

Theorem 7. *Let $A \in \mathbb{Z}^{m\times n}$. Then the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b,\ x \geq 0\}$ is integral for all $b \in \mathbb{Z}^n$ if and only if A is totally unimodular.*

Theorem 8. *Let $A \in \mathbb{Z}^{m\times n}$ be a totally unimodular matrix. Then the polyhedron $P = \{x \in \mathbb{R}^n : Ax \leq b,\ x \geq 0\}$ is integral for all $b \in \mathbb{Z}^n$.*

1.2.3 How to detect unimodularity and totally unimodularity

Theorem 9 (Basic properties). *Let $A \in \mathbb{Z}^{n\times m}$. Then the following are equivalent:*

1. *A is totally unimodular*
2. *A^T is totally unimodular*
3. *$[A\ I]$ is totally unimodular (where $I \in \mathbb{R}^n$ denotes the identity matrix)*
4. *$[A\ I]$ is unimodular*

Theorem 10. *Let $A \in \mathbb{Z}^{m\times n}$. Then A is totally unimodular if and only if for all $J \subseteq \{1, \dots, m\}$ there exists J_1, J_2 such that*

1. *$J_1 \cap J_2 = \emptyset$ and $J = J_1 \cup J_2$*
2. *For all $i = 1, \dots, n$ we have*
$$\left| \sum_{j\in J_1} a_{ji} - \sum_{j\in J_2} a_{ji} \right| \leq 1$$

Remark 11. *A analogous result can be written in terms of the columns instead of the rows of A.*

1.2.4 Examples of totally unimodular matrices

Classical examples of matrices that are totally unimodular are: network flow matrices, the node-incidence matrix for a bipartite graph, interval matrices.

1.3 Totally dual integral systems

1.3.1 Basics

Definition 12 (Totally dual integral system). *Let $A \in \mathbb{Q}^{m\times n}$ and $b \in \mathbb{Q}^m$. The system $Ax \leq b$ is totally dual integral system (TDI) if for each integral vector $c \in \mathbb{Z}^n$ such that*

$$\max\{c^T x : Ax \leq b\}$$

is finite, then the dual

$$\min\{b^T y : A^T y = c, y \geq 0\}$$

has an integral optimal solution.

1.3.2 Properties

Theorem 13. *Let* $A \in \mathbb{Q}^{m\times n}$ *and* $b \in \mathbb{Z}^m$*. If* $Ax \leq b$ *is TDI then* $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ *is an integral polyhedron.*

Remark 14. *The condition* $b \in \mathbb{Z}^n$ *is crucial in the proof of the theorem above.*

1.3.3 Totally unimodularity and TDI systems

Theorem 15. *Let* $A \in \mathbb{Q}^{m\times n}$ *be a totally unimodular matrix. Then the system* $Ax \leq b$ *is TDI for all* $b \in \mathbb{R}^n$*.*

1.3.4 Examples of TDI systems

Classical examples of TDI systems are: the independent set formulation for matroids, matchings.

Chapter 2

Cutting Planes

2.1 Introduction

2.1.1 Cutting planes

Definition 16 (Cutting plane for IP)**.** *Let $P \subseteq \mathbb{R}^n$ be a polyhedron. An inequality $a^T x \leq b$ is called a cutting plane if*

$$P \cap \mathbb{Z}^n \subseteq \{x \in \mathbb{R}^n : a^T x \leq b\}.$$

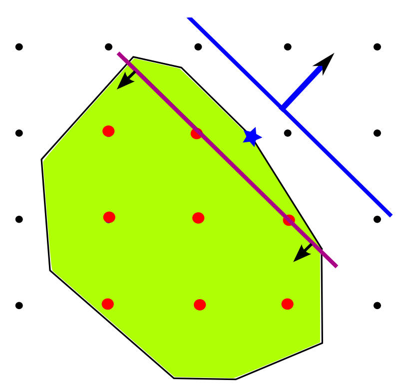
Definition 17 (Cutting plane for MIP)**.** *Let $P \subseteq \mathbb{R}^n$ be a polyhedron. An inequality $a^T x \leq b$ is called a cutting plane if*

$$P \cap (\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}) \subseteq \{x \in \mathbb{R}^n : a^T x \leq b\},$$

where we are assuming that in the MIP only the first n_1 variables must be integers ($n = n_1 + n_2$).

2.1.2 Cutting plane algorithm

Generic cutting plane algorithm



1. **Solve** LP (continuous relaxation of MILP).
2. If solution of LP is **fractional**: add cutting plane and go to (1.)
3. If solution of LP is **integral**: **STOP**.

2.1.3 How to compute cutting planes

Two approaches:

1. Computing cutting planes for general IPs.
   - From “Algebraic” properties: CG cuts, MIR inequalities, functional cuts, etc.

- From "Geometric" properties: lattice-free cuts, etc.

2. Computing cutting planes for specific IPs.

   - Knapsack problem, Node packing, etc. (many many other examples...)

2.2 Computing cutting planes for general IPs

2.2.1 Chvátal-Gomory cuts (for pure integer programs)

Definition 18 (Chvátal-Gomory cut for P)**.** *Let $P \subseteq \mathbb{R}^n$ be a polyhedron. Let $a \in \mathbb{Z}^n$, $b \in \mathbb{R}$ and let $a^T x \leq b$ be a valid inequality for P. Then the inequality*

$$a^T x \leq \lfloor b \rfloor$$

is called a Chvátal-Gomory cut.

Remark 19. *Some examples of CG cuts are: blossom inequalities for the matching problem, clique inequalities for the independent set problem, Gomory's fractional cut.*

A nice property of CG cuts

Definition 20 (Chvátal-Gomory closure of P)**.** *Let $P \subseteq \mathbb{R}^n$ be a polyhedron. Then the set*

$$P' = P \cap \bigcap_{\substack{\alpha^T x \leq \beta \\ \text{is a CG cut for } P}} \{x \in \mathbb{R}^n : \alpha^T x \leq \beta\}$$

is called a Chvátal-Gomory closure.

Theorem 21 (Finiteness of the CG cuts procedure)**.** *Let P_0 be a rational polyhedron and for $k \in \mathbb{Z}_+$ define $P^{k+1} = \left(P^k\right)'$. Then*

1. *For all $k \in \mathbb{Z}_+$, P^k is again a rational polyhedron.*
2. *There exists $t \in \mathbb{Z}_+$ such that $P^t = P_I$.*

2.2.2 Cutting planes from the Simplex tableau

Assume $P = \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$ where $A \in \mathbb{R}^{m \times n}$ is a full-row rank matrix. Let B,N denote the basic and nonbasic variables defining a vertex $(\hat{x}_B, \hat{x}_N)$ of P (where $\hat{x}_N = 0$). You can write the constraints defining P in terms of the basis B:

$$\begin{gathered} x_B = \bar{b} - \bar{A}_N x_N \\ x_B, x_N \geq 0, \end{gathered}$$

where $\bar{b} = A_B^{-1} b$ and $\bar{A}_N = A_B^{-1} A_N$.

Denote $\bar{b} = (\bar{b}_i)_{i \in B}$ and $\bar{A}_N = (\bar{a}_{ij})_{i \in B, j \in N}$. Assume that $\bar{b}_i \notin \mathbb{Z}$, so the vertex is fractional (that is, $(\hat{x}_B, \hat{x}_N) \notin\in \mathbb{Z}^n$), and therefore, we would want to cut off that LP solution.

Remark 22. *Recall that the vertex $(\hat{x}_B, \hat{x}_N)$ is the only feasible point in P satisfying $x_N = 0$. We will use this fact in order to derive some cutting planes.*

A simple inequality

The following is a valid inequality that cuts off the fractional vertex:

$$\sum_{j \in N} x_j \geq 1.$$

2.2.3 A stronger inequality

Let $N_f = \{j \in N : \bar{a}_{ij} \text{ is fractional}\}$. Then the following is a valid inequality that cuts off the fractional vertex:

$$\sum_{j \in N_f} x_j \geq 1.$$

Gomory's fractional cut

The following inequality can be derived as a CG cut:

$$\sum_{j \in N} (\bar{a}_{ij} - \lfloor \bar{a}_{ij} \rfloor) x_j \geq (\bar{b}_i - \lfloor \bar{b}_i \rfloor).$$

It can be verified that this valid inequality cuts off the fractional vertex.

2.3 Cutting planes from lattice free sets

2.3.1 The general case

Definition 23 (Lattice-free sets)**.** *A set $L \subseteq \mathbb{R}^n$ is a lattice-free set if it does not contain any integral vector in its (topological) interior, that is,* $\text{int}(L) \cap \mathbb{Z}^n = \emptyset$*.*

Let P be a polyhedron and let L be a lattice-free convex set. Then, we can derive cutting planes from L by using the following fact:

$$P \cap \mathbb{Z}^n \subseteq P \setminus \text{int}(L).$$

Such a cutting plane is called a cutting plane derived from a lattice-free set.

Remark 24. *It suffice to consider only the cutting planes defining facets of* $\text{conv}(P \setminus \text{int}(L))$ *as all the cuts not defining these facets are redundant.*

Definition 25 (Maximal lattice-free convex sets)**.** *A maximal lattice-free is a lattice-free convex set that is not strictly contained in any other lattice-free convex set.*

Maximal lattice-free convex sets are important since they give stronger cuts, since $L' \subseteq L$ implies $P \setminus \text{int}(L) \subseteq P \setminus \text{int}(L')$. The following is a nice property of such sets:

Theorem 26. *L is a maximal lattice-free convex set if and only if L is a polyhedron satisfying certain "simple characterization".*

Remark 27. *The fact that all maximal lattice-free convex set are polyhedra is useful because if L a polyhedron, then cutting planes for the set $P \setminus \text{int}(L)$ are likely 'easy' to obtain.*

2.3.2 Split cuts

A special case of a maximal lattice-free convex set is the case of split sets.

Definition 28 (Split set, split cuts)**.** *Let $\pi \in \mathbb{Z}^n$ and let $\pi_0 \in \mathbb{Z}$. Then, a split set is a set of the form*

$$\{x \in \mathbb{R}^n : \pi_0 < \pi^T x < \pi_0 + 1\}.$$

A split cut is a any cutting plane valid for $P \setminus S$, where S is some split set.

Remark 29. *The set $P \setminus S$ can be seen as a disjunction. Let $S = \{x \in \mathbb{R}^n : \pi_0 < \pi^T x < \pi_0 + 1\}$, then*

$$P \setminus S = \{x \in P : \pi^T x \leq \pi_0\} \cup \{x \in P : \pi_0 + 1 \leq \pi^T x\}.$$

In general, disjunctions as the one given by a split set or more general ones are very useful to derived cutting planes for integer programs.

2.4 Mixed-integer rounding cuts (MIR)

2.4.1 Basic MIR inequality

Let $B = \{(u, v) \in \mathbb{Z} \times \mathbb{R} : u + v \geq b,\ v \geq 0\}$. Then the inequality

$$v \geq \big(b - \lfloor b \rfloor\big)\big(\lceil b \rceil - u\big)$$

is valid for the set B.

2.4.2 MIR inequalities from one-row relaxations

Let P be a polyhedron. We want to find valid inequalities for $P \cap (\mathbb{Z}^{|I|} \times \mathbb{R}^{|J|})$.

One-row relaxation

A set of the form

$$Q = \Big\{(x, y) \in \mathbb{R}^{|I|} \times \mathbb{R}^{|J|} : \sum_{i \in I} a_i x_i + \sum_{j \in J} c_j y_j \geq b,\ x, y \geq 0\Big\}$$

is a one-row relaxation of P if $P \subseteq Q$.

Remark 30. *One-row relaxations can be constructed by using any valid inequality for P. In particular one could obtain a valid inequality by combining rows of the matrix and vector defining P.*

Applying the basic MIR inequality

We first relax the inequality defining Q. Let $I' \subseteq I$ and consider the following mixed-integer set:

$$B = \Big\{(x, y) \in \mathbb{Z}^{|I|} \times \mathbb{R}^{|J|} : \Big(\sum_{i \in I \setminus I'} x_i + \sum_{i \in I} \lfloor a_i \rfloor x_i\Big) + \Big(\sum_{i \in I'} (a_i - \lfloor a_i \rfloor) x_i + \sum_{j \in J} \max\{0, c_j\} y_j\Big) \geq b\Big\}.$$

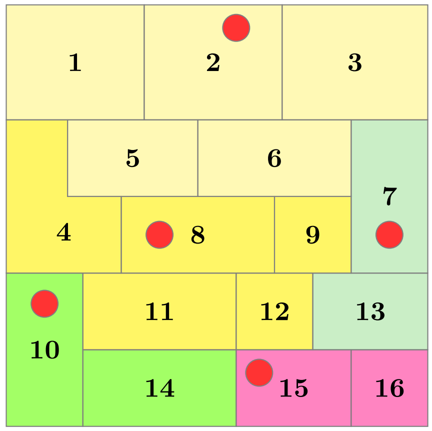


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Figure 2.1: **tikz/Illustration1.pdf**

Sicne the first part of the l.h.s. of the inequality is integral and the second part is non-negative, we can apply the procedure described in Section 2.4.1. We obtain the following valid inequality for B:

$$\Big(\sum_{i\in I'}(a_i-\lfloor a_i\rfloor)x_i+\sum_{j\in J}\max\{0,c_j\}y_j\Big)\geq(b-\lfloor b\rfloor)\bigg(\lceil b\rceil-\Big(\sum_{i\in I\setminus I'}x_i+\sum_{i\in I}\lfloor a_i\rfloor x_i\Big)\bigg).$$

Remark 31. *The above inequality is valid for $P\cap(\mathbb{Z}^{|I|}\times\mathbb{R}^{|J|})$, for all $I'\subseteq I$. The set $I'=\{i\in I:(a_i-\lfloor a_i\rfloor)<(b-\lfloor b\rfloor)\}$ gives the strongest inequality of this form.*

2.5 Gomory Mixed-integer cut (GMI)

Consider the one-row relaxation $Q=\{(x,y)\in\mathbb{Z}^{|I|}\times\mathbb{R}^{|J|}:\sum_{i\in I}a_ix_i+\sum_{j\in J}c_jy_j=b,\ x,y\geq 0\}$. Denote $f_0=b-\lfloor b\rfloor$ and for $i\in I$ denote $f_i=a_i-\lfloor a_i\rfloor$.

We will assume that $0<f_0<1$. In this case, the Gomory mixed-integer cut (GMI) is given by

$$\sum_{\substack{i\in I\\ f_i\leq f_0}}\frac{f_i}{f_0}x_i+\sum_{\substack{i\in I\\ f_i>f_0}}\frac{1-f_i}{1-f_0}x_i+\sum_{\substack{j\in J\\ c_j>0}}\frac{c_j}{f_0}y_j+\sum_{\substack{j\in J\\ c_j<0}}\frac{c_j}{1-f_0}y_j\geq 1.$$

Remark 32.

- *The validity of GMI cuts follows from the fact that they are also split cuts.*
- *In the pure integer programming case (that is, $J=\emptyset$), GMI gives a cut that is stronger than the Gomory's fractional cut.*

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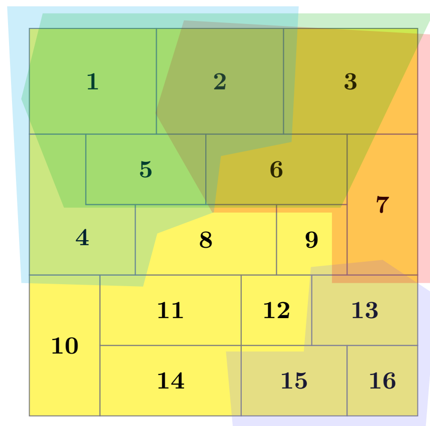

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Figure 2.2: **tikz/Illustration2.pdf**

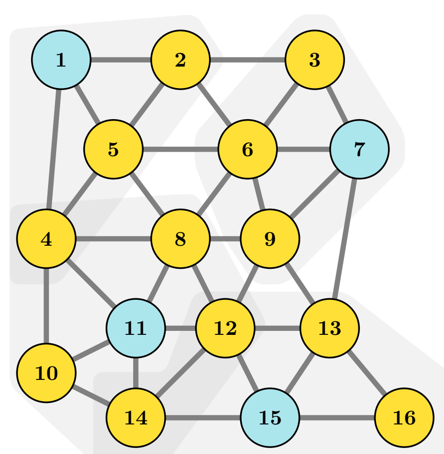

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Figure 2.3: **tikz/Illustration3.pdf**