Summary of Research
March 2021

Introduction

My research is interdisciplinary, drawing on topology, algebra, and physics. The main motivation is a 35 year-old program to create useful Topological Quantum Field Theories. TQFTs are intricate algebro-topological machines that promise to shed light in a systematic way on the "wild world" phenomena in the smooth structures of 4-manifolds that was originally discovered using gauge theory. (Gauge theory studies certain physically naturally PDEs on manifolds that *a priori* have infinite dimensional solution spaces. The "wild world" results of gauge theory are unfortunately *ad hoc.*) Roughly, TQFTs give systematic algebraic presentations of the structures of "cutting and gluing" of manifolds. I am working on two projects currently, and both can be understood ultimately as contributions to the development of prototypes of $4d$ TQFTs.

A second motivational aspect has emerged for the first project, along with the possibility of another kind of application, and it is the program in representation theory seeking to extend or transfer the structures and relationships in Lie theory that are determined by Weyl groups to the more general class of complex reflection groups, and to do this by characterizing those structures in terms of the Cherednik algebras associated to reflection groups.

Background

Quantum topology

The module category of quantum groups for $q = e^{\frac{2\pi i}{k}}$ a root of unity, $U_q(\mathfrak{g})$-mod, is a modular tensor category from which one constructs the Witten-Reshetikhin-Turaev invariant of 3-manifolds, which in turn recovers the Jones polynomial on links. (Recall that any 3-manifold is a surgery on a framed link, and that link invariants which are further invariant under 'Fenn-Rourke' moves are 3-manifold invariants.) The Jones polynomial has integer coefficients, and these are 'categorified' in Khovanov's knot homology, whose graded dimensions give those coefficients. The WRT invariant doesn't have integral coefficients, and nobody knows how to categorify it yet. A categorification of WRT could be a powerful knew source of information about 3-manifolds. Their development could mirror the already successful program that produced knot Floer homology (categorifying the Alexander polynomial) and Heegaard-Floer homology of 3-manifolds. It is possible to think of the former as quantum invariants associated to Lie algebras $\mathfrak{g}$, and the latter as a quantum invariant associated to the Lie superalgebra $\mathfrak{gl}(1|1)$. It is, furthermore, noteworthy that Khovanov homology was used recently to show that the Conway knot is not 'smoothly slice' despite being 'topologically slice'; this result is a concrete indicator that categorified $\mathfrak{g}$ quantum invariants are capable of detecting smooth structure phenomena. (No parallel result is known on the Heegaard-Floer side, to my knowledge.)

Quantum algebra

Quantum groups $U_q(\mathfrak{g})$ are certain algebras deforming the enveloping algebra of a Lie algebra $U(\mathfrak{g})$, over $\mathbb{C}(q)$ instead of $\mathbb{C}$. The q powers store additional data of 'grading', one could say, with combinatorial integers distributed over q powers. The relations of $U_q(\mathfrak{g})$ use a quantum number $[n]_q = q^{-n+1}+q^{-n+3}+\cdots+q^{n-1}$ where the relations of $U(\mathfrak{g})$ use n. (Define $[n]_q!$ and $\begin{bmatrix} n \\ m \end{bmatrix}_q$ similarly. So $[n]_q \to n$ as $q \to 1$, etc.) The Lie generators h_α are replaced by q^{h_α} and the commutators $[e_\alpha, f_\alpha]$ return 'weight' eigenvalues that are also q-numbers. The algebra $U_q(\mathfrak{g})$ is a Hopf algebra. Its coproduct is not symmetric. There is a quantum 'R-matrix' that yields a highly nontrivial commutor on the tensor product of $U_q(\mathfrak{g})$-modules U and V:

$$\mathrm{flip} \circ R : U \otimes V \to V \otimes U.$$

This operation satisfies the braid relations on triples $U \otimes V \otimes W$.

We are interested in what becomes of the module category $U_q(\mathfrak{g})$-mod "around $q = 0$", for reasons discussed with Project 1 below. In the early 90's, Kashiwara found a category $\mathsf{Crys}(\mathfrak{g})$ of certain directed graphs he called 'crystals' that are defined on the data of a $U_q(\mathfrak{g})$-module M. He took a lattice $\mathcal{L} \subset M$ over the ring of functions regular at $q = 0$, and considered the quotient $\mathcal{L}/q\mathcal{L}$. Crystal 'vertices' form a basis B_M of this quotient, and certain 'Kashiwara operators' $\widetilde{e}_i$, $\widetilde{f}_i$ are inverse (partial) bijections moving vertices along 'i-strings', raising and lowering their weights (or falling off the end of a string, giving 0). This category has a monoidal structure as well as a 'cactus' commutor $\sigma : B_{M_1} \otimes B_{M_2} \xrightarrow{\sim} B_{M_2} \otimes B_{M_1}$ satisfying 'cactus relations' that are related to but distinct from the braid relations.

The category $U_q(\mathfrak{g})$-mod can be considered a deformation of a (trivial) linearization of the crystal category. In fact, the cactus commutor does arise from a commutor on $U_q(\mathfrak{g})$-mod, the Drinfel'd unitarized R-matrix, written $\bar{R}$. But important q-power data is lost by unitarizing. The original R-matrix and its concomitant braiding structure, and actually the very factor lost by unitarizing, degenerates in a naive $q \to 0$ limit. So, one hopes and expects to find an algebraic structure giving a new concept of 'deformation of $\mathsf{Crys}(\mathfrak{g})$', but which is not (quite) the module category of $U_q(\mathfrak{g})$. For applications to quantum topology, one hopes for a braiding structure on this new deformation.

Quantum geometry

One approach to the study of module categories of quantum groups uses the geometry of configuration spaces:

$$\mathsf{Conf}_n(\mathbb{C}) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for } i \neq j\}.$$

There are natural bundles $\pi_{n,k} : \mathsf{Conf}_{n+k}(\mathbb{C}) \to \mathsf{Conf}_n(\mathbb{C})$ whose fibers are called 'discriminantal arrangements'. The geometry of these arrangements relates the module categories of Lie algebras to those of quantum groups. The fundamental ingredient is a function of the form $\Pi(z_i - z_j)^{a_{ij}/\kappa}$ with some parameters a_{ij} related to a Cartan matrix, with κ from the q parameter: $q = e^{2\pi i/\kappa}$. Such a function determines a local system, and chains in homology of the arrangement with coefficients in that system can be described by Hochschild homology of a quantum group. On the other hand, such a function determines a 'Knizhnik-Zamolodchikov' differential equation defining a flat connection for functions over $\mathsf{Conf}_n(\mathbb{C})$ taking values in the corresponding classical representation of the Lie algebra. The relation between $U(\mathfrak{g})$-mod (or a trivial deformation thereof) and $U_q(\mathfrak{g})$-mod that one finally obtains is an equivalence of braided tensor categories. While the equivalence was first shown by Drinfel'd-Kohno, and proven in stronger form by Kazhdan-Lusztig, the geometric approach described here was worked out by Varchenko, collaborating somewhat with Schechtman. The ideas can be traced to Varchenko's early work with Arnol'd in singularity theory that led to his study of integrals of holomorphic forms over families of vanishing cycles.

This geometric picture is attractive from our point of view because Varchenko's hypergeometric KZ solutions can be given in explicit integral forms having asymptotic expansions in the limit $\kappa \to 0i$. The first terms of these asymptotics are related to the crystal base, and the cactus commutor can be computed from the monodromy of first terms as the base point is moved within a compactification $\overline{\mathcal{M}_{0,n+1}}(\mathbb{R})$ of the real locus $\mathsf{Conf}_n(\mathbb{R})$. (This compactification agrees with the Deligne-Mumford space of genus 0 curves with n marked points.) While the fundamental group of $\mathsf{Conf}_n(\mathbb{C})$ is the braid group, that of $\overline{\mathcal{M}_{0,n+1}}(\mathbb{R})$ is called the 'cactus group'. This relation between the cactus commutor and asymptotics of KZ solutions was conjectured by Etingof and proven in the $\mathfrak{sl}_2$ case by Rybnikov. In higher rank, the cactus commutor has been determined as a monodromy using other structures. (Gaudin algebras, Bethe Ansatz, shift-of-argument algebras — not quantum groups and KZ solutions.)

Calogero-Moser geometry

Associated to each complex reflection group $(W, \mathfrak{h})$ is a Cherednik algebra $\mathsf{CA}_{t,c}$ defined with parameters $t \in \mathbb{C}$ and $c : S_W \to \mathbb{C}$ for the simple reflections S_W. The center of this algebra at $t = 0$ has a large spectrum $\mathfrak{F} := \mathsf{Spec}(Z(\mathsf{CA}_{0,c}))$. The group W acts diagonally on $\mathfrak{h} \times \mathfrak{h}^*$ and the quotient is a (singular) variety covering $\mathfrak{h}/W \times \mathfrak{h}^*/W$. Etingof and Ginzburg showed that $\mathfrak{F}$ is actually a deformation of the covering:

$$\Upsilon : \mathfrak{F} \to \mathcal{C} \times \mathfrak{h}/W \times \mathfrak{h}^*/W.$$

They also found (the amazing fact) that $\mathfrak{F}$ agrees with a well-known 'Calogero-Moser' space admitting a quantum integrable system and giving its name to this geometry.

The relevance of this space comes from an analogy with the quantum group representation theory. Crystal bases are defined in quantum groups arising from Weyl groups. An analogous structure exists in the Hecke algebras arising from Coxeter groups. (Coxeter groups comprise the reflection groups over $\mathbb{R}$, including Weyl groups.) This structure is called 'Kazhdan-Lusztig cells'. There is not a known generalization of KL cells from Coxeter to complex reflection groups, but Bonnafé-Rouquier have conjectured that so-called 'Calogero-Moser cells' defined using the geometry (the Galois theory) of the covering Υ are a generalization of KL cells. Their conjectural extension has been proved for W of rank 2, and recently for all of type A. The point is that the family of algebras $\mathsf{CA}_{t,c}$ controls these cells and relates to crystals at $t = 0$, while for $t = 1$ it relates to KZ equations. Let us explain these two aspects somewhat further.

Kazhdan-Lusztig cells are certain collections of Kazhdan-Lusztig basis elements of the Hecke algebra $\mathcal{H}_W$. These basis elements are uniquely defined by their image in a $q = 0$ quotient and the condition of symmetry under a natural involution. This much has precise parallels for $U_q(\mathfrak{g})$ using its canonical basis and $q \leftrightarrow q^{-1}$ involution, and where the $q = 0$ projection gives crystal vertices. Indeed Lusztig has created a crystalline algebra out of $U_q(\mathfrak{g})$ under $q \to 0$ following this analogy. But, we do not know quite how to deform his algebra.

Now for the relation to KZ equations. First, $\mathsf{CA}_{t,c} \cong \mathsf{CA}_{1,c}$ for $t \neq 0$, so we focus on the latter. Then, $\mathsf{CA}_{1,c}$ is a $\mathbb{C}[\mathfrak{h}]$-bimodule. Consider $\mathfrak{h}_{reg}$ the regular locus of $\mathfrak{h}$, off all hyperplanes. Using certain differential 'Dunkl' operators defined on $\mathfrak{h}_{reg}$ one has an isomorphism of the localization with equivariant D-modules:

$$\mathsf{CA}_{1,c} \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}[\mathfrak{h}_{reg}] \xrightarrow{\sim} D(\mathfrak{h}_{reg}) \rtimes W.$$

Let M be a sufficiently nice $\mathsf{CA}_{1,c}$-module. (In a version of Category $\mathcal{O}$.) Then the localization M_{reg} is a W-equivariant D-module, i.e. an equivariant vector bundle on $\mathfrak{h}_{reg}$ with flat connection, i.e. (Riemann-Hilbert taking sections, then the quotient) a local system on $\mathfrak{h}_{reg}/W$. This gives 'KZ equations of type W'. Of course, the flat connection gives rise to a monodromy representation of the braid group B_W of type W. It has been shown that these representations of B_W factor through the Hecke algebra $\mathcal{H}_W(q)$ of type W. (Recall $\mathcal{H}_W(q) = \mathbb{C}[B_W]/I$ for $I = \left\langle (t_H - 1)\prod_j (t_H - \theta_{H,j} q_{H,j}) \right\rangle_{H \text{ a hyperplane}}$.) These KZ equations agree with the classical ones for $W = S_n$ and $M = \mathbb{C}[\mathfrak{h}]$.

Categorical Lie theory

A 'weak' categorification of a finite dimensional $\mathfrak{sl}_2$-module is a string of additive categories $\mathcal{C}_r$ and additive 'raising and lowering' functors $E : \mathcal{C}_r \to \mathcal{C}_{r+2}$, $F : \mathcal{C}_r \to \mathcal{C}_{r-2}$, with isomorphisms of functors

$$\begin{aligned} EF|_{\mathcal{C}_r} &\cong FE|_{\mathcal{C}_r} \oplus I^r_{\mathcal{C}_r} \quad \text{if } r \geq 0 \\ FE|_{\mathcal{C}_r} &\cong EF|_{\mathcal{C}_r} \oplus I^r_{\mathcal{C}_r} \quad \text{if } r \leq 0. \end{aligned}$$

So if, for example, $M \in \mathcal{C}_3, E(M) \cong 0$ 'highest weight', then $EF(M) \cong M \oplus M \oplus M$. Each weight space of the $\mathfrak{sl}_2$-module has been replaced by a weight category, and the commutator of E and F, instead of multiplying a weight vector by its weight, takes a weight object and returns weight-many copies of that object.

The starting point of categorical representation theory is the discovery of algebras H_n of natural transformations acting on compositions F^n. Several naturally occurring categorifications have H_n actions, and by use of H_n actions one can establish uniqueness theorems, prove highly nontrivial category equivalences that are of independent interest, and (hopefully) define a monoidal product on categorical representations. So, one speaks of 'strong categorical $\mathfrak{sl}_2$ actions' or simply '2-reps of $\mathfrak{sl}_2$' when there is such an action. In the case of $\mathfrak{sl}_2$ one can use the 'nil-affine Hecke algebra', and for more general Kac-Moody algebras, the 'quiver Hecke algebras' act on compositions of functors F_α.

There is a different description of this setup of a 2-rep of $\mathfrak{sl}_2$. A 1-rep is a module over the (associative) enveloping algebra $U(\mathfrak{sl}_2)$; we can add a system of idempotents to make a category $\dot{U}(\mathfrak{sl}_2)$, and our 1-rep can be described as a linear functor from $\dot{U}(\mathfrak{sl}_2)$ to Vect. Then, there is an upgrade of $\dot{U}(\mathfrak{sl}_2)$ to a 2-category, called $\mathcal{U}(\mathfrak{sl}_2)$, with rules for the H_n actions built in. The objects are indexed by $r \in \mathbb{Z}$. A 2-rep of $\mathfrak{sl}_2$ can be described in this language as a 2-functor from $\mathcal{U}(\mathfrak{sl}_2)$ to Cat, the 2-category of categories. In terms of this description, we could say that the discovery of H_n actions can be packaged as a discovery of the appropriate Lie 2-category defined by some Cartan data.

In the context of categorification, the q-numbers of quantum groups do become graded dimensions. There is a graded version of $\mathcal{U}(\mathfrak{sl}_2)$ that categorifies $\dot{U}_q(\mathfrak{sl}_2)$, and an idempotent completion of $\mathcal{U}(\mathfrak{sl}_2)$ categorifies a $\mathbb{Z}[q,q^{-1}]$-form of $\dot{U}_q(\mathfrak{sl}_2)$.

Project 1

Recently, Gukov et al. found a 3-manifold invariant $\hat{Z}_a(q)$ that is a q-series with integral coefficients. They discovered that the WRT invariants can be recovered from $\hat{Z}_a(q)$ using some arguments involving $q \to e^{2\pi i/k}$. There are reasons from physics to believe these coefficients are the dimensions of spaces of 'BPS states', and thus that the series can be categorified into a new 3-manifold homology. We look forward ultimately to the emergence of a $4d$ TQFT.

One wants to think of the series $\hat{Z}_a(q)$ as an expansion of something around $q = 0$, and hopes for some algebraic structure with q parameter whose representation category around $q = 0$ controls $\hat{Z}_a(q)$. Now the WRT invariants are determined by $U_q(\mathfrak{g})$-mod as a ribbon Hopf algebra, so we expect the structure we seek to relate to $U_q(\mathfrak{g})$. So, finally, our preliminary aim is to understand $U_q(\mathfrak{g})$ *around* $q = 0$, not just *at* $q = 0$ where we expect the structure we seek to give $\mathsf{Crys}(\mathfrak{g})$. As we have seen, the category $U_q(\mathfrak{g})$-mod itself is not the right concept of deformation of $\mathsf{Crys}(\mathfrak{g})$.

There are two basic ingredients to the approach we take for this project. First we have the idea of using $\mathsf{Conf}_n(\mathbb{C})$ geometry and asymptotic solutions of KZ equations. The calculation of asymptotics of KZ solutions gives a link between the crystal base and the $U_q(\mathfrak{g})$ modules away from $q = 0$, indeed even at $q = 1$. The second idea is that of looking to $\mathsf{CA}_{t,c}$ for an analogy. As we described, modules of the algebra $\mathsf{CA}_{t\neq 1,c}$ give rise to KZ equations on a vector bundle over $\mathfrak{h}_{reg}$. On the other hand, the Kazhdan-Lusztig cells (generalizing from crystals) are controlled by the algebra $\mathsf{CA}_{0,c}$. (Using Calogero-Moser cells of $Z(\mathsf{CA}_{0,c})$.) One hopes to construct something on the quantum group side playing a role analogous to that of $\mathsf{CA}_{0,c}$. At present we do not have a specific proposal for such an algebra.

The key contribution of Varchenko's work, with regard to our goals, is to look at KZ solutions *very near the hyperplanes* before taking the $\kappa \to 0i$ limit. Concretely, this means considering a compactification $\mathcal{M}_n$ of $\mathsf{Conf}_n(\mathbb{R})$, obtained by taking iterated blow-ups along the intersections of hyperplanes. (This compactification is related to the de Concini-Procesi wonderful compactification of a hyperplane arrangement.) The space $\mathcal{M}_n$ can be covered by charts called 'asymptotic zones', or just 'zones', that are indexed by binary rooted trees T together with permutations $\sigma \in S_n$ enumerating the leaves. (More precisely, each zone is covered by 2^n distinct charts which are related by flipping the signs of some of the n coordinates.) Then, inside one of these charts (T, σ), with coordinates $u_1, \dots, u_n$, say, we have that $u_i = 0$ corresponds to a specific collection of $\eta_{T,\sigma}(i)$ hyperplanes $z_l = z_k$, depending on the chart. (We think of this as an $\eta_{T,\sigma}(i)$-fold collision of particles at positions z_k.) We can write a general asymptotic form of a KZ solution in a neighborhood of the point $u_i = 0\,\forall i$, and compute the W-monodromy of this expression for the solution on a path crossing a 'wall' $u_i = 0$.

This calculation leads precisely to the Varchenko-Rybnikov method of showing that the action of the cactus group on the tensor product of crystals agrees with the action of the cactus group on the eigenbasis of Gaudin Hamiltonians. We have not described the Gaudin Hamiltonians and their simple spectrum over $\mathcal{M}_n$. Suffice it to say that they serve as a partial analogue of $\mathsf{CA}_{0,c}$ on the quantum side in type A. (Rybnikov wanted to understand the monodromy of the spectrum for other reasons.) In addition to the limitation to type A, the Gaudin Hamiltonians and the 'simple spectrum' result one requires for their use holds on the $\mathbb{R}$ locus only, in general. This $\mathbb{R}$ locus restriction applies also for 'shift of argument' methods that have been used recently to find the monodromy of Gaudin eigenlines very generally. To find a proper analogue of $\mathsf{CA}_{t,c}$ that could deform $\mathsf{Crys}(\mathfrak{g})$, we need to move outside the $\mathbb{R}$-locus.

In the introduction, we also alluded to our hope that this project contributes to the program of extending Lie theory to complex reflection groups W. We explain that now. Our idea is to imitate the Varchenko technique of studying KZ solutions inside asymptotic zones in the more general context of KZ equations on a vector bundle arising as the localization of a (nice) $\mathsf{CA}_{1,c}$-module. Such an extended Varchenko technique could help us compute the monodromy of KZ solutions for W beyond the case of type A. The asymptotic limit of these solutions, on the other hand, would give information about the Calogero-Moser cells. Ultimately one hopes for an approach to extend the conjecture of Bonnafé-Rouquier that Calogero-Moser cells agree with Kazhdan-Lusztig cells for Coxeter groups beyond those of rank 2 or type A.

Project 2

Raphaël has proposed a construction for tensor 2-products of 2-representations of Kac-Moody algebras. Such a construction could feature in a $4d$ TQFT analogously to the way the tensor product of modules of a Hopf algebra features in $3d$ TQFTs. Consider the quantum group $U_q(\mathfrak{g})$ we discussed: it is a Hopf algebra. Set $\mathcal{C} = U_q(\mathfrak{g})$-mod. The Hopf coproduct provides a monoidal structure on $\mathcal{C}$, and the (compatible) Hopf antipode affords Hom objects adjoint to the monoidal products:

$$\mathcal{C}(X, \mathrm{Hom}(Y, Z)) \cong \mathcal{C}(X \otimes Y, Z).$$

So $\mathcal{C}$ is a rigid semisimple tensor category. Taking q a root of unity, there are finitely many simple objects, and we have a 'fusion category' giving rise to a $3d$ TQFT.

Taking this story up a categorical notch, one desires a subtle and interesting *Hopf category* in order to define a 4d TQFT. Let me say only that a Hopf category will involve the structure of an 'internal' tensor product object (category) and Hom object (category) for any two categories from some collection. Our specific hope, of course, is to construct these objects for 2-reps of Kac-Moody algebras. Raphaël's proposal runs along these lines.

Important related work has been done recently by Ben Webster. Instead of defining a *bona fide* monoidal and Hom structure on the category of 2-reps, one can start by considering all 2-reps that categorify the 1-product of 1-reps, and look for natural conditions under which such 2-reps are unique. Webster found, for any complex semisimple $\mathfrak{g}$, a representation category having the structure of 2-rep of $\mathfrak{g}$ and that categorifies a tensor product of irreducibles $V_{\nu_1} \otimes \cdots \otimes V_{\nu_n}$. Furthermore, together with Losev, he found a natural criterion under which 2-reps categorifying $V_{\nu_1} \otimes \cdots \otimes V_{\nu_n}$ must be equivalent as 2-reps to his own. Webster's category is the representation category of a certain algebra that is very naturally defined by diagrammatic generators and relations as a generalization of the universal algebra Raphaël originally defined for irreducibles. Webster found functors in (the derived category of representations of) his algebra that categorify the structure maps of $U_q(\mathfrak{g})$-mod as a ribbon category. He also showed that these functors recover Khovanov's knot homology. These successes are evidence that Webster's 2-reps categorifying $V_{\nu_1} \otimes \cdots \otimes V_{\nu_n}$ are the 'right' categories, meaning that they should be equivalent in some sense to the output of Raphaël's internal tensor product construction for 2-reps.

The precise goal of this project is to find the sense in which Raphaël's tensor product of 2-reps is equivalent (as 2-rep) to Webster's category, and to establish the equivalence. Since Raphaël's full construction is quite complex, and in general returns an $\mathcal{A}_\infty$ category as the 2-product of abelian or dg 2-reps, our first step is to focus on the simplest case of the fundamental 2-rep of $\mathfrak{sl}_2$ and its n-fold product. For this case the relevant $\mathcal{A}_\infty$ structure simplifies and the output can be presented as a dg category. So our goal is to relate this dg category to Webster's category.