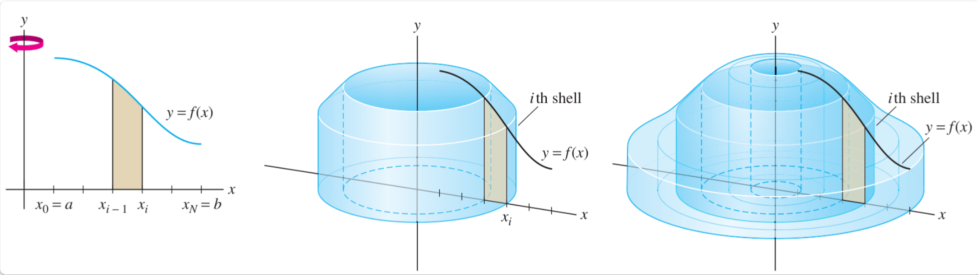
Unit 01 notes

Volume using cylindrical shells

01 Theory

Take a graph $y = f(x)$ in the first quadrant of the xy-plane. Rotate this about the y-axis. The resulting 3D body is symmetric around the axis. We can find the volume of this body by using an integral to add up the volumes of infinitesimal **shells**, where each shell is a *thin cylinder*.



The volume of each cylindrical shell is $2\pi\, R\, h\, \Delta r$:

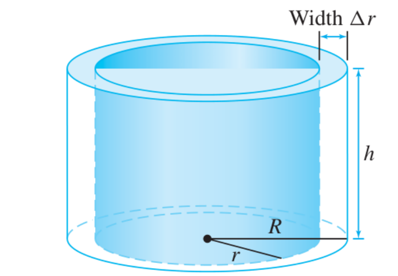


FIGURE 1 The volume of the cylindrical shell is approximately

$$2\pi Rh\Delta r$$

where $\Delta r = R - r$.

In the limit as $\Delta r \to dr$ and the number of shells becomes infinite, their total volume is given by an integral.

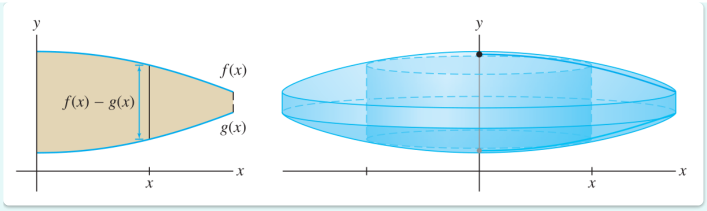
Volume by shells - general formula

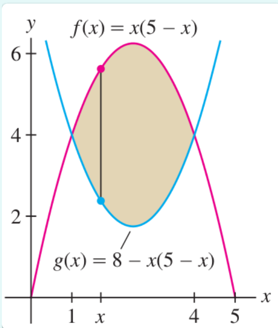
$$V = \int_a^b 2\pi Rh\, dr$$

In any concrete volume calculation, we simply interpret each factor, 'R' and 'h' and 'dr', and determine a and b in terms of the variable of integration that is set for r.

Shells vs. washers

Can you see why shells are sometimes easier to use than washers?





02 Illustration

Example - Revolution of a triangle

Problem: A rotation-symmetric 3D body has cross section given by the region between $y = 3x + 2$, $y = 6 - x$, $x = 0$, and is rotated around the y-axis. Find the volume of this 3D body.

Solution:

1. Define the cross section region.
    - Bounded above-right by $y = 6 - x$.
    - Bounded below-left by $y = 3x + 2$.
    - These intersect at $x = 1$.
    - Bounded at left by $x = 0$.
2. Define range of integration variable.
    - Rotated around y-axis, therefore use x for integration variable (shells!).
    - Integral over $x \in [0, 1]$:

$$V = \int_0^2 2\pi R h \, dr$$

3. Interpret R.
    - Radius of shell-cylinder equals distance along x:

$$R(x) = x$$

4. Interpret h.
    - Height of shell-cylinder equals distance from lower to upper bounding lines:

$$\begin{aligned} h(x) &= (6 - x) - (3x + 2) \\ &= 4 - 4x \end{aligned}$$

5. Interpret dr.
   - dr is limit of Δr which equals Δx here so $dr = dx$.
6. Plug data in volume formula.
   - Insert data and compute integral:

$$V = \int_0^2 2\pi Rh\, dr$$

$$= \int_0^2 2\pi \cdot x(4 - 4x)\, dx$$

$$= 2\pi \left(2x^2 - \frac{4x^3}{3}\right)\Big|_0^1 = \frac{4\pi}{3}$$

Exercise - revolved sinusoid

Consider the region given by revolving the first hump of $y = \sin(x)$ about the y-axis. Find the volume of this region using the method of shells.

Solution

1. Set up the integral for shells.
   - Integration variable: $r = x$, the distance of a shell to the y-axis.
   - Then $dr = dx$ and $h = \sin x$, the height of a shell.
   - Bounds: one hump is given by $x \in [0, \pi]$.
   - Thus:

$$V = \int_0^\pi 2\pi x \sin x\, dx$$

2. Perform the integral using IBP.
   - Choose $u = 2\pi x$ and $v' = \sin x$ since x is A and $\sin x$ is T.
   - Then $u' = 2\pi$ and $v = -\cos x$.
   - Recall IBP formula:

$$\int uv'\, dx = uv - \int u'v\, dx$$

   - Insert data in IBP formula:

$$\int_0^\pi 2\pi x \sin x\, dx \quad \gg\gg \quad (2\pi x)(-\cos x)\Big|_0^\pi - \int_0^\pi 2\pi(-\cos x)\, dx$$

   - Compute first term:

$$-2\pi x \cos x\Big|_0^\pi \quad \gg\gg \quad -2\pi(\pi)(-1) - -2\pi(0)(+1) \quad \gg\gg \quad 2\pi^2$$

- Compute integral term:

$$-\int_0^\pi 2\pi(-\cos x)\,dx \quad \gg\gg \quad 2\pi \sin\Big|_0^\pi \quad \gg\gg \quad 0$$

- So the answer is $2\pi^2$.

Integration by substitution

[This section is non-examinable. It is included for comparison to IBP.]

03 Theory

The method of **u-substitution** is applicable when the integrand is a *product*, with one factor a composite whose *inner function's derivative* is the other factor.

u-substitution

Suppose the integral has this format, for some functions f and u:

$$\int f(u(x)) \cdot u'(x)\,dx$$

Then the rule says we may convert the integral into terms of u considered as a variable, like this:

$$\int f(u(x)) \cdot u'(x)\,dx \quad \gg\gg \quad \int f(u)\,du$$

The technique of u-substitution comes from the **chain rule for derivatives**:

$$\frac{d}{dx}F\big(u(x)\big) = f(u(x)) \cdot u'(x)$$

Here we let $F' = f$. Thus $\displaystyle\int f(x)\,dx = F(x) + C$ for some C.

Now, if we *integrate both sides* of this equation, we find:

$$F\big(u(x)\big) = \int f(u(x)) \cdot u'(x)\,dx$$

And of course $F\big(u\big) = \displaystyle\int f(u)\,du - C$.

Full explanation of u-substitution ›

The substitution method comes from the **chain rule for derivatives**. The rule simply comes from *integrating on both sides* of the chain rule.

1. Setup: functions $F' = f$ and $u(x)$.
    - Let F and f be any functions satisfying $F' = f$, so F is an antiderivative of f.
    - Let u be another *function* and take x for its independent variable, so we can write $u(x)$.

2. The chain rule for derivatives.
    - Using primes notation:

$$(F \circ u)' = (F' \circ u) \cdot u'$$

    - Using differentials in variables:

$$\frac{d}{dx} F\big(u(x)\big) = f(u(x)) \cdot u'(x)$$

3. Integrate both sides of chain rule.
    - Integrate with respect to x:

$$\frac{d}{dx} F\big(u(x)\big) = f(u(x)) \cdot u'(x) \quad \overset{\int}{\ggg} \quad \int \frac{d}{dx} F\big(u(x)\big) = \int f(u(x)) \cdot u'(x)$$

$$\overset{\text{FTC}}{\ggg} \quad F(u(x)) = \int f(u(x)) \cdot u'(x)$$

4. Introduce 'variable' u from the u-format of the integral.
    - Treating u as a variable, the definition of F gives:

$$F(u) = \int f(u)\, du + C$$

    - Set the 'variable' u to the 'function' u output:

$$F(u)\,\Big|_{u=u(x)} = F(u(x))$$

    - Combining these:

$$F(u(x)) = F(u)\,\Big|_{u=u(x)}$$

$$= \int f(u)\, du\,\Big|_{u=u(x)} + C$$

5. Substitute for $F(u(x))$ in the integrated chain rule.
    - Reverse the equality and plug in:

$$\int f(u(x)) \cdot u'(x)\, dx = F(u(x)) = \int f(u)\, du\,\Big|_{u=u(x)} + C$$

6. This is "u-substitution" in final form.

Integration by parts

04 Theory

The method of **integration by parts** (abbreviated IBP) is applicable when the integrand is a *product* for which one factor is easily integrated while the other *becomes simpler* when differentiated.

Integration by parts

Suppose the integral has this format, for some functions u and v:

$$\int u \cdot v' \, dx$$

Then the rule says we may convert the integral like this:

$$\int u \cdot v' \, dx \quad \gg\gg \quad u \cdot v - \int u' \cdot v \, dx$$

This technique comes from the **product rule for derivatives**:

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

Now, if we *integrate both sides* of this equation, we find:

$$u \cdot v = \int u' \cdot v \, dx + \int u \cdot v' \, dx$$

and the IBP rule follows by algebra.

Full explanation of integration by parts

1. Setup: functions u and v' are established.
   - Recognize functions $u(x)$ and $v'(x)$ in the integrand:

   $$\int u \cdot v' \, dx$$

2. Product rule for derivatives.
   - Using primes notation:

   $$(u \cdot v)' = u' \cdot v + u \cdot v'$$

3. Integrate both sides of product rule.
   - Integrate with respect to an input variable labeled 'x':

   $$(u \cdot v)' = u' \cdot v + u \cdot v' \quad \overset{\int}{\gg\gg} \quad \int (u \cdot v)' \, dx = \int u' \cdot v \, dx + \int u \cdot v' \, dx$$

   $$\overset{\text{FTC}}{\gg\gg} \quad u \cdot v = \int u' \cdot v \, dx + \int u \cdot v' \, dx$$

   - Rearrange with algebra:

   $$\int u \cdot v' \, dx = u \cdot v - \int u' \cdot v$$

4. This is "integration by parts" in final form.

Addendum: *definite* integration by parts

3. Definite version of FTC.
    - Apply FTC to $u \cdot v$:

$$\int_a^b (u \cdot v)' \, dx = u \cdot v \Big|_a^b$$

4. Integrate the derivative product rule using specified bounds.
    - Perform definite integral on both sides, plug in definite FTC, then rearrange:

$$\int_a^b u \cdot v' \, dx = u \cdot v \Big|_a^b - \int_a^b u' \cdot v$$

Choosing factors well

IBP is symmetrical. How do we know which factor to choose for u and which for v?

Here is a trick: the acronym "LIATE" spells out the order of choices – to the left for u and to the right for v:

$$\text{LIATE}: \qquad u \leftarrow \text{Logarithmic} - \text{Inverse\_trig} - \text{Algebraic} - \text{Trig} - \text{Exponential} \rightarrow v$$

05 Illustration

Example - A and T factors

Problem: Compute the integral: $\displaystyle\int x \cos x \, dx$

Solution:

1. Choose $u = x$.
    - Set $u(x) = x$ because x *simplifies* when differentiated.
      (By the trick: x is *Algebraic*, i.e. more "u", and $\cos x$ is *Trig*, more "v".)
    - Remaining factor must be v':

$$v'(x) = \cos x$$

2. Compute u' and v.
    - Derive u:

$$u' = 1$$

    - Antiderive v':

$$v = \sin x$$

    - Obtain chart:

$$\begin{array}{c|c} u = x & v' = \cos x \\ \hline u' = 1 & v = \sin x \end{array} \quad \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} \quad \begin{array}{l} \int u \cdot v' \\ \int u' \cdot v \end{array} \quad \begin{array}{l} \text{original} \\ \text{final} \end{array}$$

3. Plug into IBP formula.
   - Plug in all data:

$$\int x \cos x \, dx = x \sin x - \int 1 \cdot \sin x \, dx$$

   - Compute integral on RHS:

$$\int x \cos x \, dx = x \sin x + \cos x + C$$

   Note: the *point* of IBP is that this integral is easier than the first one!
4. Final answer is: $x \sin x + \cos x + C$

Exercise - hidden A

Compute the integral:

$$\int \ln x \, dx$$

Solution

1. Choose $u = \ln x$.
   - Because Log is farthest right in LIATE.
   - It follows that we must choose $v'(x) = 1$.
2. Compute u' and v.
   - We have $u' = \frac{1}{x}$ and $v = x$.
   - Obtain chart:

$$\begin{array}{c|c} u = \ln x & v' = 1 \\ \hline u' = 1/x & v = x \end{array} \quad \begin{array}{ccc} \longrightarrow & \int u \cdot v' & \text{original} \\ \longrightarrow & \int u' \cdot v & \text{final} \end{array}$$

3. Plug into IBP formula.
   - Plug in all data:

$$\int \ln x \cdot 1 \, dx = x \ln x - \int \frac{1}{x} \cdot x \, dx$$

   - Integrate:

$$-\int \frac{1}{x} \cdot x \, dx \quad \gg\gg \quad -\int 1 \, dx \quad \gg\gg \quad -x + C$$

   - Final answer is: $x \ln x - x + C$