How to view math: as a language

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It is occasionally worthwhile to consider what kind of a thing mathematics *is*. This kind of reflection can help us place our mathematical endeavors in the context of our larger purposes in life.

Thinking about the nature of math can have consequences for downstream questions, such as questions about the best ways to learn or train in mathematics, and questions about the best norms to maintain in a community of professionals.

The prior question is a philosophical one, and the typical method of philosophers is to draw distinctions. Good distinctions serve in definitions, definitions are collected into theories, theories are expressed in special cases, and sometimes unexpected consequences are discovered. More importantly, the process of working through tentative distinctions and definitions can bring understanding, i.e. clarity about our concepts and insight into what is really going on. Downstream consequences of this kind of insight may be less explicit but no less impactful.

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A philosopher might consider the form of a good answer to the question to consist of three steps. Find a classification of natural kinds of human activity, make a claim that mathematics falls into one of these kinds, and give an argument for the claim.

It will be convenient to take a shortcut. This essay will be concerned with the following claims:

- Mathematics is not a sport, it is not a science, and it is not a technology.
- Mathematics is a language, i.e. it is a symbolic art.

Our main effort will involve cutting apart these four predicates, describing essential features of language, and showing how readily mathematics falls into the class of symbolic arts. While math shares aspects and features with (sports, sciences, technologies), it is not quite right to say that it *is* any of those things.

In a substantial second part of the essay we will draw some pedagogical consequences from the linguistic perspective on the nature of mathematics, and take the liberty of making a few remarks about higher criticism. In other words, we will ask how and why to learn math, and what makes for good math.

(A reader with inordinate interest in classification schemas, disappointed with our shortcut, should look at a list of university offices to find a familiar starting point on the long road to the kind of answer they prefer.)

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It is in no way essential to our main claims that the four predicates should identify similar types of things, i.e. adjacent categories within a higher category. On the contrary, it would be entirely compatible with those claims if every science were to be contained within a linguistic art.

The kind of explanation of these claims that we will be giving is compatible with the facts that mathematics has features and aspects of a science, a sport, and a technology. In fact, other symbolic arts do too. But perhaps it has these aspects *because* it is a symbolic art, and it has them in the peculiar way in which any sufficiently rich symbolic art *must* have them.

A container ship is not identical with its containers, nor with the union of a boat and some containers, even though it is often convenient to speak metonymically as if it were. The activity of mathematics should not be confused with one of its typical features or aspects.

Mathematics and science §

Mathematicians and natural scientists both pursue knowledge and understanding in their respective research domains. Their methods involve distinctive mixtures of action, observation, and reflection. In this they are distinguished from scholars in the humanities, who tend not so much to observe reactions to their actions, and particularly the philosophers and historians, whose enterprise does not essentially depend upon action at all.

Experiments are intended to elicit novel but repeatable presentations of nature. These presentations often take the form of images, either literally or in an extended sense. Repeatability is necessary for the kind of universality that we believe nature, or our understanding of nature, should possess. In any case, the presentations we elicit are studied, with attentive patience, and through a conceptual-observational framework that gives meaning to our observation.

An experiment exposes nature. It is designed to bring something new into view, something new which answers to an expectation produced by a burgeoning conceptual framework. Sometimes our theory-determined expectation may be vague and thin — the more vague and thin, the more exploratory the experiment is deemed. Still, in every case an experiment brings something to light, something distinct from the experimental act itself. After the experimental act, the new thing brought to light is observed and considered.

The distinction between the experimental act and the new image elicited by that act is essential to the concept of experimental natural science. It is an abstract expression of a familiar modern criticism of the older 'natural philosophy' practiced by Greeks and Scholastics, namely that their way of knowing was founded upon observation and reflection alone. The modern mind takes it that it is not enough to mix reflection and observation: to understand the way nature really works we have to send her messages and see how she replies.

This relation between action, response, and observation is the essential point, and mathematics does not follow this pattern. In math there is not a correlate of the response of nature, that image produced in experiment and distinct from the action. Rather, in math, the action is identical with the object of observation. Or, more precisely, a record or residue of the action as it were compresses and expresses the action so it becomes easier to consider, and new insight is sought by considering this residue.

There is a gap between the action – the experiment – of the scientist and the response of nature. I do not mean a logical or temporal gap, but a gap of identity. The experimental scientist does not *do* the experimental result. She does the experiment, then *collects* the resulting data, and attends to this collection to understand it in the framework of meaningfulness within which she operates.

There is no such gap for the mathematician. His action is to produce a string of mathematical thoughts, and to express them in a symbolic residue. Such action may be part of mathematical experimentation or play, or it may draw from reflection on such play and attempt to move further, to move towards the production of a piece of a much larger meaningful whole, i.e. towards a new definition or theorem. In either case, his action results in an object of further consideration, but that object is a record or residue of the action itself, an externalized trace of the action, and not a response from nature distinct from the action. In the sciences, the experimental result, as a generalized image, is an image *of* something that is provoked or called forth by the experimental procedure; in mathematics, the procedure *itself* is externalized and considered and judged.

It is possible to bring out this distinction also in terms of criteria of success. For the mathematician, criteria of success are applied to judge his act with the assistance of its symbolic residue. An appropriate accumulation of successful acts already *constitutes* a result. For the scientist, the criteria of success are applied to an experiment to judge whether her procedure in the experiment indeed elicited a result, an image, that is meaningful in the conceptual-observational framework in question. Whether the result confirms or discredits some hypothesis is another matter. No quantity of experimental procedures, combined with an assessment of success as experiments, constitutes a result. Indeed, the data of a result must be *added* to the data of some successful procedures, and the combination is published as a product of science. This is to say: in experimental science one must *actually follow* the procedure specified and *then* report on the way nature behaved.

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"Science is inductive, while math is deductive."

This appears to be a truism, and one may wonder why we shouldn't start here. But it is not clear what it really means.

Certainly, in science, a *conceptual framework* is typically supported in a diffuse manner by a diversity of experimental images, but it may be reasonable to doubt the whole framework while accepting the images. Whereas, in math, a theorem is apparently tied in to the framework by a deductive argument starting from definitions (which are not justified deductively), and it is impossible to doubt the claim of the theorem while accepting the definitions (if you follow the proof). This situation doesn't have the symmetry suggested by the truism.

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The dimension of *play* gives another axis on which math and science are separable. Play arises in math in at least three forms. There is the metaphor of a game for the activity of math, there is the aspect of competitive play against other researchers, and there is the activity of playing around with some ideas on the chalkboard to see if they lead somewhere.

In science, apparently the metaphor of a game does not fit so comfortably to the discipline as a whole; and the rules of the competitive game, as it were, tend to feel less well defined. But these comparisons are extrinsic and vague. The third form is more important and permits of an interesting contrast.

Play in the third sense involves, at least, partial attempts at the real activity of a discipline, but in a context where the full requirements of rigor, precision, and completeness are set aside, with the intention of restoring them later when there is a better sense of what actions are worthwhile. This much is common to math and science.

One aspect of play is specifically involved in what children do, namely 'make-believe' or pretense. Mathematicians routinely engage in this kind of play. When a mathematician makes use of expressions of 'mathematical acts' (perhaps written a chalkboard) that are false (strictly, literally), but which in some way point towards a truth, he is engaged in a bit of make-believe. By waving his hands at the unspecified conditions and qualifications, and focusing attention on a 'claim' to see what may be derived from it were it true, he is pretending in a similar way to that in which a child pretends, whether he pretends that a toy train is a train, or a mud pie is a pie. Perhaps this is a way of bracketing reality at a certain layer, 'screening off' we might say, the background requirements, in order to focus attention on something one would like to say in a foreground layer, assuming the context to be right for it. This is, possibly, a description of the working conditions of poetic creativity.

The condition of playing has a close relative in musical composition in the act of *improvisation*. A composer brackets the preceding and succeeding parts of a piece, assuming that a context is appropriate (and perhaps could be completed later) and jumps into a micro-moment where the pretended meaning outstrips the literal detail by some margin. There may be another cousin in engineering, namely in the activity of *tinkering* that creative inventors are known for. Here the conditions of a practical implementation are bracketed, and some kind of mechanism demonstrating a principle or an idea is displayed in the foreground, as a kind of demonstration. In neither case, however, is use made specifically of false (strictly, literally) expressions in the service of deeper truth; but perhaps this is only because the judgements of truth and falsehood are not as directly applicable in these domains.

In experimental science, on the other hand, the make-believe form of play does not seem to arise. Instead, as a kind of counterpart, one finds the activity of 'trying things out'. When a scientist runs micro-experiments without all the rigor and apparatus needed to qualify as compelling evidence for a critical observer, but rather in order to 'see what happens', in turn so that she can detect some trace of a direction to move towards a new phenomenon, she is doing something analogous to playing. It is exploratory experimentation. Rather than honing a method of eliciting some kind of image, this type of experiment looks for a type of thing that has never been seen before. This kind of play would seem to be closer to tinkering and improvisation.

The main difference between exploratory experimentation and mathematical playing around is that the former does not involve *acting as if* something false is the case. In math you might really say: "let's *just imagine* that A is like B having relation R to C". And you might write some badly defined morphism, or rely on a property that only holds for a subcategory that you didn't precisely circumscribe. In science you might hear instead: "let's *just try* X, without asking what it means in our framework, and see what happens". Something is suspended in exploratory experimentation, but it is the requirement of meaningfulness of the elicited image in the conceptual-observational framework at hand. For this reason an exploratory experiment is not normally qualifiable as a success or a failure. One might say that a quantity of 'meaning' is suspended when a scientist "tries things", whereas a quantity of 'truth' is suspended when a mathematician "imagines things".

It is worth pointing out that the response of nature we encountered above makes an appearance here as well. After "trying something", the experimenter receives a 'something that happened', and considers this to learn whether it is a result she can make sense of. It is a sort of poking around at nature that is less determined by rational reflection in a conceptual framework, and it provides a fertile soil for new awareness of another reality. This poking around can surely be playful, but it is play in a different sense. The mathematician is engaged in a game of imitation, since that is what make-believe consists in. The scientist is engaged in a game of "let's see what happens", since that is what an experiment consists in. The result of an imitation game is a correspondence dictionary, and the result of an exploratory experiment is an uncovered phenomenon.

(The reader may wonder whether Einstein's analogies qualify as games of imitation. Perhaps, but maybe theory is a vague boundary between experiment and mathematics.)

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A few other dichotomies that distinguish math and science are well-worn in popular discussion, but they can be mentioned anyway to satisfy your expectations.

It is said that science seeks knowledge, while mathematics, like philosophy, seeks understanding. Knowledge consists of true beliefs, while understanding is a matter of an ability to navigate a conceptual framework, and when applied to a particular case it typically means a kind of assimilation. Sometimes the distinction of 'what' and 'why' is used, but this can be unhelpful because answers to 'why' are typically *reasons*, and assimilation in a conceptual framework might not involve rational argumentation *per se*.

Concerning this dichotomy let me give just one brief remark. It is natural to assume an internal logical relation between knowledge, propositions, and objects. That is, simply, knowledge has to do with a selection of the propositions that are true, and propositions refer to objects and ascribe properties to them or groupings of them. Understanding in internally logically related to language and meaning. One understands another expression, whether propositional or poetical, by receiving the meaning it conveys, which may consist in triggering parts of certain abilities that one has (whether physical or mental). To the extent these are reasonable connections to maintain, it is natural that science, which generates descriptions of objects in the world, should produce knowledge, while mathematics has a more subtle relation to knowledge, given how many experts seriously argue that mathematical abstractions do not have a real existence. One need not be a mathematical formalist to agree that there are far fewer serious formalists when it comes to microbiota or neutrons. (And even these are easier to find than formalists about zebras.)

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In mathematics, explanations are typically the same thing as (or abbreviated versions of) justifications of theory, namely the proofs.

In science, theory is justified by experimental results. Theories give explanations of phenomena. But the explanations that theory provides do not themselves justify the theory and its claims. (They may justify funding for science.) Here explanation and justification diverge.

Caveats should be made, of course. There are 'good proofs' in math that are 'more explanatory'. And, in science, there are explanations that seem too beautiful to be false but have no experimental support. (Perhaps supersymmetry is an example.) But there is a reason these feel like caveats.

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An important aspect of the character of the evolution of both math and science is the centrality of communal posing and resolving of problems. This also seems to distinguish them from, say, history, literature, or the arts.

Yet, even here is there is a subtle but significant difference between math and science when it comes to the role of solutions.

You notice a pattern. A pattern without explanation determines a problem. In math and in science you want an explanation for the pattern, to put it in the context of a bigger, simpler, deeper theoretical framework that can explain many other phenomena.

Well, in math an explanation for a pattern may well consist in the knowledge that this pattern is not a coincidence of the small case, but in fact is a true pattern. That is, you notice a fact about the cases to which your intuition and tools are quickly applied within some domain of objects that you can define. The problem is whether this apparent pattern is an actual pattern: whether the situation for small or easy cases in fact actually does have a common description with all the cases.

Problem solutions in experimental science are not like this! The explanation sought (1) needs to add understanding of the very phenomenon that is observed, i.e. some particular image. In math, the facts for the small cases are already transparently clear. In science, the small case, well, the particular image (which may be graphical rendering of a lot of data from many events), needs reveals the entire phenomenon, and needs explanation. There is no *internal* or *necessary* requirement that the explanation extend to additional data *as its subject matter*.

This is about the motion from examples to theories. In math and science one takes theory to explain examples. Problems, in both cases, typically ask for the theoretical background that underlies some phenomenon observed in examples. But in math, the problem must ask whether something that apparently extends beyond the given examples does indeed extend; whereas in science the problem must ask for some principles of foundation and principles of derivation according to which the examples are necessary. So, in science, the solution amounts to a concentration of theoretical energy into a focal point of the example, while in math the theoretical energy flows out of the example into a large class of similar cases.

Mathematics and technology Technology is a neighbor of science and it is natural to ask whether math could be a technology.

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Mathematics has apparent technological qualities in several respects.

All mathematicians develop tools. Some mathematicians join forces to develop big machines for the purpose of attacking a problem or imaging a class of structures. It also happens that some procedure is known, at least in some cases, to be mechanical. These tools, machines, and mechanical procedures are reasonably described as technological. They can be seen as technological from a point of view within mathematics; that is, they appear to be technologies to the practicing mathematician. In this respect, the tools, machines, and procedures developed in the course of the practice of mathematics should be called the 'technological *portion* of mathematics'. Part of what a mathematician does, at work, is to develop mathematical technology. The concept of technology at play here is that of externalization, specifically the externalization of complex reasoning from within consciousness to a written expression outside of consciousness, which the mind engages by way of semi-conscious automated behavior.

Another technological quality is also visible in the course of the practice of math, and we may call it the 'technological *aspect* of mathematics'. This aspect is the way in which mathematics is oriented towards known problems. There is a philosophy according to which mathematics is about problem solving, and we can give this philosophy its due by acknowledging that any bit of math can be considered under the aspect of problem solving. The relation of problem solving to the idea of technology is simple. Technology can be defined as the art and lore of solving problems. Most problems most people care about are perceived shortages of goods. For example, the light bulb solves the problem of the shortage of light and time in the day. It is hard to say anything this general about the character of mathematical problems, but it is probably true that most problems in math are about whether a phenomenon observed in special cases in fact holds in the general case. (And sometimes a solution includes a more rigorous definition of the general case.)

This definition of technology as the art of problem solving carries two consequences for the nature of technology. The first is that the conditions of success of a technology precede and are independent of the nature of the technology. (They are, namely, determined by the problem.) One implication of this fact is that different technologies that solve the same problem may differ radically from one another. Consider the radical difference between candles and light bulbs. The second is that, inasmuch as a problem is well-defined, a solution, once discovered, tends to preserve its status indefinitely. The candle or lightbulb, now discovered, will always give possible solutions to the problem of light shortage. These two consequences, when combined, have the further consequence that technology tends to accrete over time, as a growing heap or aggregate. The heap grows as new and improved solutions are added to human knowledge; it is a (mere) aggregate because the solutions are often radically different, do not have relations to each other (in general), and lack structure at the level of the whole collection. In the changing landscape of needs and resources, some technologies will fall out of favor and even become buried and forgotten in the heap. But they can always be uncovered and put back in service to solve the problem they were created to solve. This is not the way of basic science, which involves shifts of paradigm and conceptual framework, and some explanations belonging to an overturned theory may not be possible or even comprehensible in the conceptual framework of the modern day.

Mathematics has a third technological quality which is not visible from within its practice. This we may call the 'technological *product* of mathematics'. It is the conceptual infrastructure that mathematicians develop on their way to proving theorems, and which can be used in application to the real world. (It must be admitted that the infrastructure that is actually used is often developed in connection with such use.) This technological quality is not visible from within math because the acts (whether individual or communal) of application and fitting to application of parts of mathematics are conducted with quite different goals, programs, and methods than those of pure math research. (It is not typically theorems *per se* that are applied, but, rather, the infrastructure byproduct). Further, the appliers of math draw upon the infrastructure of math in a diffuse way, rarely invoking the precise details of mathematical definitions, and rarely making use of rigorous proofs. This is contrary to the main mode of operation for most mathematicians, for whom most of their labor (perhaps not to say creativity) involves working out precise definitions that support rigorous proofs, and indeed these proofs are the felt product of their effort.

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What would it mean to claim that math is a technology?

Considering in turn the technological qualities of math, it is clear that the technological portion and the technological aspect could not be intended. Both of these are visible from within math, which of course entails a limitation on them from the point of view of math, and this makes it impossible to identify them with math.

Arguing also from the other direction, it is possible in both cases to describe some complementary qualities that seem no less essential to the discipline. Concerning the technological portion, the tools and machines of math are merely instruments in the hands of mathematicians to achieve their true goal, which is to build proofs of deep and beautiful theorems. Mathematical industry relies on these instruments, but does not consist merely in their development. The most 'industrial' research topics involve very large and complex mathematical machines, requiring for their development and application the distributed labor of many hands. But it is no less desirable to produce beautiful theorems, so that even here the machines are mere instruments.

Concerning the technological aspect of math, the argument that math cannot be identified with this aspect is, of course, that math is not entirely about solving problems, because it is also about defining good structures, proving structural theorems, and discovering systematic relationships. The activities of defining, proving, and systematizing cannot be adequately explained in terms of problem solving alone. The expression 'theory building' is sometimes heard in math departments to mean either the construction of machines and tools, or any of the triad above, namely defining, proving, and systematizing. Most mathematicians believe that theory building and problem solving characterize complementary lifestyles or personalities with equal claim to validity as approaches to mathematical practice. If that is true, it follows that mathematics cannot be identified with problem solving, and thus it cannot be identified with its technological aspect. But is it true?

At this stage it is important to clarify that the problems guiding mathematics are not to be understood so broadly as just any questions. "Question answering" arguably describes an aspect of all rational thought, and is in no way peculiar to mathematics. The problems of mathematics are supposed to be posed precisely enough that the type of consensus arising around proofs of theorems can also arise around solutions to problems, and even determine the distribution of monetary prizes attached to those problems. Indeed, oftentimes a solution to a longstanding problem is expressed in the form of a theorem. Now, given that the problems in question must be precisely posed, it is obvious that they depend, both formally and logically, on the precise definitions that are given already for the structures in question. That dependence can be applied, by composing aboutnessess, in the following syllogism. Math is about problems, and problems are about mathematically defined structures, so math is about mathematically defined structures. Now, unless it could be shown (which it cannot) that mathematical definitions always arise (in a somewhat direct way) out of attempts to solve problems, it must finally be granted that math is not entirely about problems. It is also about structures, and the theorems (with proofs) that reveal those structures.

There is another way to see that math is more than problem solving. One can ask the question: “Where do the problems come from?” A direct intuitive answer might look something like this: problems for math arise from vague analogies, or from the phenomena of small cases. Or they come from surveying structures, structural theorems, or structural relationships in an imprecise or conceptual way, and then making conjectures. Now, this sort of answer can be arranged to form an argument that math is not only about solving problems. The argument goes like this. New problems are, apparently, generated from within mathematics. (Indeed, these conceptual surveys, theorems, or small case phenomena, do properly belong to mathematics.) On the other hand, these items frequently arise during mathematical exploration that has no precise relation to any specific problem. Then, if new problems arise from mathematical work that is in no way related to a specific problem, it follows that mathematics cannot be completely explained as problem solving. Something else, whether intuitions or perceptions does not matter, stands above the problems and guides the production and valuation of the problems.

Of course we can admit that problem solving is *essential* for mathematics, in other words that this aspect may be *critical*; we could, if desired, also grant a *leading role* to the activity of identifying and solving problems in determining the course of the evolution of math. It is even possible to believe, in fact, that this leading role advances beyond the role of general concepts, analogies, or the phenomena observable among known structures. All this can be done without denying that problem solving does not, in an important sense, exhaust the proper description of mathematics, and therefore without denying that it is wrong to say that mathematics *is* a technology.

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There is, finally, the quality of mathematics as a technology which is given by its applications outside of mathematics, i.e. as its technological product. An argument that this quality is identifiable with mathematics *per se* looks like this. We grant that the *developers* of mathematics aim to produce theorems, and theorems are not (*per se*) applied outside math. However, the *product* of these developers, i.e. of pure mathematicians, is the precise conceptual infrastructure that can be applied in the ordering and systematizing of the natural world. There are *users* of mathematics, who watch from outside and take and apply this infrastructure. The dynamic here, one finally argues, is the dynamic of a technology. Technology is created by developers and applied by users.

An argument that math cannot be identified with (the art of development of) its technological product cannot be formulated along the same lines as the prior two arguments. This technological quality of math is not visible from a point of view within mathematical practice; it is necessarily attributed from outside; it portrays the whole discipline in a certain light.

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One weakness of this argument can be found in the final step. It is certainly true that mathematicians develop infrastructure that is applied by non-mathematician users. It is not true that the dynamic is quite parallel to the case of any other technology. For other technologies, developers create the technology *for* users. By contrast, pure mathematicians do not create their theorems for anybody else. Certainly this is true in the small. A given mathematician will not typically have a non-mathematical application in mind as the reason to prove a particular theorem. In the large, though, the question is more subtle. It may be possible to make a (somewhat intricate) argument that despite every intention of the individual researcher, the goal of the *discipline* of mathematics, taken as a whole, is indeed to supply a product to its users, namely to the scientists and engineers who want to order and systematize the natural world.

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What is this now? The set of pure mathematicians, taken as a singular entity, is attributed a collective intentionality; or at any rate, like a school of fish, is thought to exhibit the appearance of goal-directed behavior without its substance. This goal is to develop abstract infrastructure for science and engineering.

After considering more carefully this school of fish metaphor, one finds it reasonable to suppose that the apparent intentionality at the level of the collection can be observed. Indeed the collective response can be *seen emerging*, in small collections, from out of individual behavior. Individual fish navigate around obstacles and flee from prey; and individual fish prefer to swim in synchronized patterns around each other. The collective action emerges from the balance of these forces.

In math research communities, one can indeed find a collective intentionality that emerges from the behavior of individual researchers. This reveals itself in large scale research programs, grant applications, motivation sections of conference talks, the agglomeration of project collaborators, and the narrative thread running through a sequence of papers on a given topic. To be sure, these collective behaviors don't seem to introduce specifically global novel phenomena. In any case, it is clear to a professional mathematician that the collective intentionality one *sees emerging* by considering these small or intermediate collections is not any more directed towards infrastructure for users outside mathematics than is the goal-directed behavior of the individual mathematician. The research programs, motivations, and narrative threads would appear to be completely explainable from within mathematics in terms of the ordinary goals of research mathematics, except they are at a larger scale and (accordingly) more vague. The collective goals remain: defining good structures, proving structural theorems about them, and discovering systematic relationships.

Perhaps we need to consider the possibility of new, global topological, structures in a collective behavior that are difficult to explain in terms of local intentional mechanisms. I am not sure whether schools of fish exhibit such behavior, but there is nothing strange in the idea. (It is part of the idea of emergence of higher-order consciousness.) If this is the kind of product-directedness that is supposed to be attributed to the collective of mathematical researchers, we cannot show it doesn't exist by considering intentionality that we can see emerging, from within the practice, in small collections.

It would be tempting for our opponent to step back here and make the argument very general. (Too general.) For example, she could suggest that all research is (ultimately, in the final analysis, considering the global goals of the human collective) directed towards products for human use. However, precisely the same argument can be brought to the case of music and poetry. It is not my interest here to engage in a symmetric defense of the autonomy of math, music and poetry; in fact it is quite enough for me that the three end up receiving parallel treatment.

The more difficult case is still not quite ruled out. Its possibility can be detected by the following consideration: So much of math seems to be stimulated by problems in the real world. Does this not suggest, uniquely for math, an ultimately external purpose?

Here I can reply only that intentional behavior may be stimulated by a substratum, but this does not mean the behavior is correctly understood as producing something *for* use in relation to that substratum. Here, again, we can return the case to music and poetry, which if anything are surely stimulated (as arts) by experience of emotion, but the claim that their ultimate goal is therefore to produce goods *for* use in relation to experience and emotion is not obviously thereby warranted. And, again, such a claim would be too general to be of any use in understanding these arts. The concept of 'technology' itself would be washed away completely if this kind of claim were admitted in general form.

Math and sport §

The rules of a sport or game are usually precise enough that a third party consensus quickly forms, by a kind of avalanche process, except in very subtle borderline cases, as to whether a sequence of plays or moves is a winning one according to the rules.

The same is true regarding the rules of mathematical inference, so that a consensus forms as to whether a proof is valid and a conjecture becomes a theorem. In this way math resembles a sport or game.

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Math can appear to resemble a sport in another way because of the presence of competition. It is not hard to find in a math research community some race to prove a conjecture or discover a theorem that somewhat resembles the economic contest for a limited resource. In some contexts that limited resource even has a name, i.e. ‘low hanging fruit’. Occasionally mathematicians must be secretive about their progress to avoid motivating others or giving away their key ideas. This competition has a similar feel to the activity of developing intellectual property and filing patents in more overtly technological disciplines.

Sports and games, of course, all involve some element of contest or struggle. One wonders whether this feature is the defining feature of the class of sports and games.

That is not true, since of course *genuine* contest and struggle in the wild is by no means limited to sports, in fact the contrary is apparent, that sports and games involve *artificial* struggle and competition. I.e., they do not involve competition for the resources of *survival*. (A real war is never ‘just a game’.) Still, it is possible to admit this, even grant it as a condition, and suppose, namely, that sports and games are all those activities that involve friendly competition, i.e. competition for a good or resource that does not really matter at the end of the day.

If the presence of friendly competition, in an activity that is perhaps indeed pursued for the recreational pleasure of that competition, is the defining feature of sport, then it would seem that mathematics qualifies. But, again, it is not true — friendly competition is not the defining feature of sport, and its presence is not a reason to think of math as a sport. It is easy to find examples that demonstrate this: drag racing from a stoplight is one example, arguing in a pub is another; in these cases the stakes are low and the competition is artificial (although the adrenaline is not), yet the activities are not really sports or games.

Given that not all friendly contests are sports or games, it does not follow from the presence of friendly competition that mathematics is a sport or game. That conclusion might nonetheless be *suggested* by the presence of friendly competition. We must look closer at the locus of competition in math research communities to see whether it is the same competitiveness as the competitiveness of a sport or game.

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Perhaps one reason that math research can feel like a game is that a mathematician’s work *involves directly* the *mathematical moves* of other researchers.

This is in contrast to other sciences. Recall that an act of science (an experiment) elicits some kind of meaningful image, and this image together with its interpretation forms the product of scientific labor. This product is distinct from the experimental act, and the distinction is one in kind and not only in number, since the image produced in an experiment aims to be a type of representation of nature, while the record of the experimental act, in abstract form, is merely a recipe for eliciting that representation. The product of a mathematician’s labor, on the other hand, is identical with his mathematical act, or at least with a record or trace of his act, which is to say, his written proof.

A consequence of the identity in mathematics is that a researcher feels that her work, and the record of her work, constitutes a direct response to the work of her predecessors in the subject area, something like a *move in a game*, with other researchers taking the role of players, whether teammates or opponents, who have made the *prior moves* to which her move responds and upon which it builds. In a sport or game, the *acts* of a player in the game are identical with the 'product' of her work at the game, if that term be admitted here, and these acts are typically made in response to the prior moves of other players.

A consequence of the distinction in the sciences is that a researcher does not feel that his act of science is itself a response to an act of science by another player. For example, his work typically will *not* subsume or draw in the very acts of other researchers (with an exception for experimental technique, which is beside the point). He will, on the contrary, rely on the images of nature elicited by others, and their meaning in the whole system of understanding, to form the background against his new images will have their place. It is up to his own ingenuity to find the technique to elicit new images.

To see another aspect of this same phenomenon, observe that typically a scientist's paper will involve the work of his predecessors mostly in the introduction and literature review sections, and not directly in the section devoted to the description of an experiment (excepting, again, citations of similar techniques) and certainly not in the presentation of the results. A mathematician's paper, on the contrary, will frequently involve the work of predecessors at every point along the way, including in the sections presenting the main theorems.

So we find that mathematical research involves players (researchers) making moves that respond directly to the moves of other players, in the context of a competitive environment where players strive to obtain possession of a limited resource and to achieve certain goals that are reasonably well-defined in advance. This description sounds like that of a sport or game, and clearly it fits mathematics in a way that it does not fit the sciences.

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Music and poetry are like mathematics in that the products of labor are identical with the compositional acts (sometimes real, oftentimes imagined) of the composer or poet. It is also the case that a poet or composer involves directly the works of her colleagues in her work, often implicitly or even explicitly borrowing themes, techniques, or styles from older work (whether that work is created or else borrowed in turn by the earlier artist). All this happens in an environment of friendly competition.

The fact that music and poetry have these features gives a counterexample to the claim that these features define the class of sports or games, and to the argument that math is a sport or game simply because it has these features. This fact suggests that math may be more akin to music and poetry than to sports or games.

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We may observe, though, that music and poetry are not quite like math in at least one regard that we have already discussed. In music and poetry it is not possible to define in advance some problems that the artists are competing to solve, or some limited good or resource that they are competing to obtain. It is not even possible to envision a musical territory that they are competing to explore to discover things yet unseen by humankind. It is true that artists compete with each other: they compete for sponsorship and popularity, and in turn this perhaps amounts to competition for money and glory; but this limited resource for which they compete is quite broad, too broad to distinguish one discipline from another. Any human activity can involve competition for money and glory, not just sports and games or even science and technology. In any reasonable sport or game, as in mathematical research, there is a limited resource of theorems for which researchers compete that is defined internally to the discipline. Moreover, in sports or games, and in mathematics, there are even goods (for example, conjectures) for which researchers compete that are precisely defined in advance. This particular aspect of math resembles a sport or game more than it resembles music or poetry.

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Now let us look even harder at the locus of competition in mathematics. We will start to see the analogy with sports and games come apart at the seam. The seam I mean is the connection between the locus of competition and the defining rules. This seam, or inner relation, is a fundamental aspect of the nature of all sports and games. The disconnection we will find in mathematics has a parallel in music and poetry.