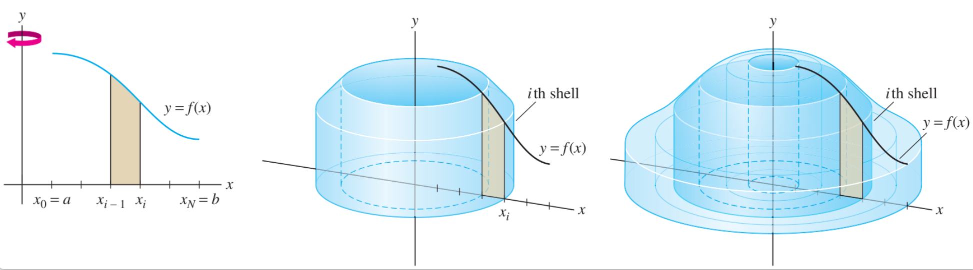
Volume using cylindrical shells

01 Theory

Take a graph $y = f(x)$ in the first quadrant of the xy-plane. Rotate this about the y-axis. The resulting 3D body is symmetric around the axis. We can find the volume of this body by using an integral to add up the volumes of infinitesimal **shells**, where each shell is a *thin cylinder*.



The volume of each cylindrical shell is $2\pi\, R\, h\, \Delta r$:

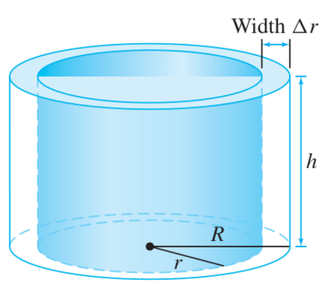


FIGURE 1 The volume of the cylindrical shell is approximately

$$2\pi R h \Delta r$$

where $\Delta r = R - r$.

In the limit as $\Delta r \to dr$ and the number of shells becomes infinite, their total volume is given by an integral.

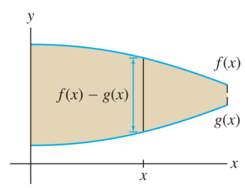
Volume by shells - general formula

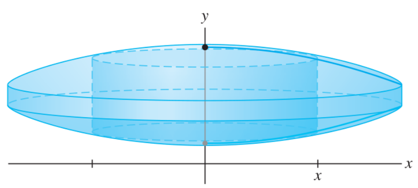
$$V = \int_a^b 2\pi R h \, dr$$

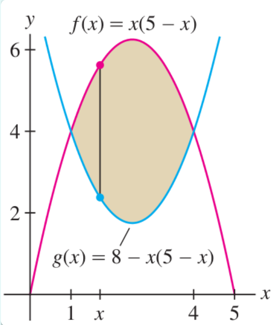
In any concrete volume calculation, we simply interpret each factor, 'R' and 'h' and 'dr', and determine a and b in terms of the variable of integration that is set for r.

Shells vs. washers

Can you see why shells are sometimes easier to use than washers?







02 Illustration

Example - Revolution of a triangle

A rotation-symmetric 3D body has cross section given by the region between $y = 3x + 2$, $y = 6 - x$, $x = 0$, and is rotated around the y-axis. Find the volume of this 3D body.

Solution

1. Define the cross section region.
   - Bounded above-right by $y = 6 - x$.
   - Bounded below-right by $y = 3x + 2$.
   - These intersect at $x = 1$.
   - Bounded at left by $x = 0$.
2. Define range of integration variable.
   - Rotated around y-axis, therefore use x for integration variable (shells!).
   - Integral over $x \in [0, 1]$:

$$V = \int_0^1 2\pi R h \, dr$$

3. Interpret R.
   - Radius of shell-cylinder equals distance along x:

$$R(x) = x$$

4. Interpret h.
   - Height of shell-cylinder equals distance from lower to upper bounding lines:

$$\begin{aligned} h(x) &= (6-x)-(3x+2) \\ &= 4-4x \end{aligned}$$

5. Interpret dr.
   - dr is limit of Δr which equals Δx here so $dr = dx$.
6. Plug data in volume formula.
   - Insert data and compute integral:

$$\begin{aligned} V &= \int_0^1 2\pi R h \, dr \\ &= \int_0^1 2\pi \cdot x(4-4x)\, dx \\ &= 2\pi \left(2x^2 - \frac{4x^3}{3} \right) \Bigg|_0^1 = \frac{4\pi}{3} \end{aligned}$$

Exercise - Revolution of a sinusoid

Consider the region given by revolving the first hump of $y = \sin(x)$ about the y-axis. Find the volume of this region using the method of shells.

Solution

Integration by substitution

[This section is non-examinable. It is included for comparison to IBP.]

03 Theory

The method of ***u*-substitution** is applicable when the integrand is a *product*, with one factor a composite whose *inner function's derivative* is the other factor.

u-substitution

Suppose the integral has this format, for some functions f and u:

$$\int f(u(x)) \cdot u'(x)\, dx$$

Then the rule says we may convert the integral into terms of u considered as a variable, like this:

$$\int f(u(x)) \cdot u'(x)\, dx \quad \gg\!\gg \quad \int f(u)\, du$$

The technique of u-substitution comes from the **chain rule for derivatives**:

$$\frac{d}{dx} F\big(u(x)\big) = f(u(x)) \cdot u'(x)$$

Here we let $F' = f$. Thus $\displaystyle\int f(x)\, dx = F(x) + C$ for some C.

Now, if we *integrate both sides* of this equation, we find:

$$F\big(u(x)\big) = \int f(u(x)) \cdot u'(x)\, dx$$

And of course $F\big(u\big) = \displaystyle\int f(u)\, du - C$.

Full explanation of u-substitution

The substitution method comes from the **chain rule for derivatives**. The rule simply comes from *integrating on both sides* of the chain rule.

1. Setup: functions $F' = f$ and $u(x)$.
   - Let F and f be any functions satisfying $F' = f$, so F is an antiderivative of f.
   - Let u be another *function* and take x for its independent variable, so we can write $u(x)$.
2. The chain rule for derivatives.
   - Using primes notation:

     $$\big(F \circ u\big)' = (F' \circ u) \cdot u'$$

   - Using differentials in variables:

     $$\frac{d}{dx} F\big(u(x)\big) = f(u(x)) \cdot u'(x)$$

3. Integrate both sides of chain rule.
   - Integrate with respect to x:

     $$\frac{d}{dx} F\big(u(x)\big) = f(u(x)) \cdot u'(x) \quad \overset{\int}{\gg\!\gg} \quad \int \frac{d}{dx} F\big(u(x)\big) = \int f(u(x$$

     $$\overset{\text{FTC}}{\gg\!\gg} \quad F(u(x)) = \int f(u(x)) \cdot u'($$

4. Introduce 'variable' u from the u-format of the integral.
   - Treating u as a variable, the definition of F gives:

     $$F(u) = \int f(u)\, du + C$$

- Set the 'variable' u to the 'function' u output:

$$F(u)\Big|_{u=u(x)} = F(u(x))$$

- Combining these:

$$\begin{aligned} F(u(x)) &= F(u)\Big|_{u=u(x)} \\ &= \int f(u)\,du\Big|_{u=u(x)} + C \end{aligned}$$

5. Substitute for $F(u(x))$ in the integrated chain rule.
   - Reverse the equality and plug in:

$$\int f(u(x))\cdot u'(x)\,dx = F(u(x)) = \int f(u)\,du\Big|_{u=u(x)} + C$$

6. This is "u-substitution" in final form.

Integration by parts

04 Theory

The method of **integration by parts** (abbreviated IBP) is applicable when the integrand is a *product* for which one factor is easily integrated while the other *becomes simpler* when differentiated.

Integration by parts

Suppose the integral has this format, for some functions u and v:

$$\int u\cdot v'\,dx$$

Then the rule says we may convert the integral like this:

$$\int u\cdot v'\,dx \quad \gg\gg \quad u\cdot v - \int u'\cdot v\,dx$$

This technique comes from the **product rule for derivatives**:

$$(u\cdot v)' = u'\cdot v + u\cdot v'$$

Now, if we *integrate both sides* of this equation, we find:

$$u\cdot v = \int u'\cdot v\,dx + \int u\cdot v'\,dx$$

and the IBP rule follows by algebra.

Full explanation of integration by parts ›

1. Setup: functions u and v' are established.

- Recognize functions $u(x)$ and $v'(x)$ in the integrand:

$$\int u \cdot v' \, dx$$

2. Product rule for derivatives.
   - Using primes notation:

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

3. Integrate both sides of product rule.
   - Integrate with respect to an input variable labeled 'x':

$$(u \cdot v)' = u' \cdot v + u \cdot v' \quad \overset{\int}{\ggg} \quad \int (u \cdot v)' \, dx = \int u' \cdot v \, dx + \int u$$

$$\overset{\text{FTC}}{\ggg} \quad u \cdot v = \int u' \cdot v \, dx + \int u \cdot v' \, dx$$

   - Rearrange with algebra:

$$\int u \cdot v' \, dx = u \cdot v - \int u' \cdot v$$

4. This is "integration by parts" in final form.

Addendum: *definite* integration by parts

3. Definite version of FTC.
   - Apply FTC to $u \cdot v$:

$$\int_a^b (u \cdot v)' \, dx = u \cdot v \Big|_a^b$$

4. Integrate the derivative product rule using specified bounds.
   - Perform definite integral on both sides, plug in definite FTC, then rearrange:

$$\int_a^b u \cdot v' \, dx = u \cdot v \Big|_a^b - \int_a^b u' \cdot v$$

Choosing factors well

IBP is symmetrical. How do we know which factor to choose for u and which for v?

Here is a trick: the acronym "LIATE" spells out the order of choices – to the left for u and to the right for v:

$$\text{LIATE}:$$

$$u \leftarrow \text{Logarithmic} - \text{Inverse\_trig} - \text{Algebraic} - \text{Trig} - \text{Exponential} \rightarrow v$$

05 Illustration

Example - A and T factors

Compute the integral: $\int x \cos x \, dx$

Solution

1. Choose $u = x$.
    - Set $u(x) = x$ because x *simplifies* when differentiated. (By the trick: x is *Algebraic*, i.e. more "u", and $\cos x$ is *Trig*, more "v".)
    - Remaining factor must be v':

    $$v'(x) = \cos x$$

2. Compute u' and v.
    - Derive u:

    $$u' = 1$$

    - Antiderive v':

    $$v = \sin x$$

    - Obtain chart:

    $$\begin{array}{c|c} u = x & v' = \cos x \\ \hline u' = 1 & v = \sin x \end{array} \quad \begin{array}{ccl} \longrightarrow & \int u \cdot v' & \quad \text{original} \\ \longrightarrow & \int u' \cdot v & \quad \text{final} \end{array}$$

3. Plug into IBP formula.
    - Plug in all data:

    $$\int x \cos x \, dx = x \sin x - \int 1 \cdot \sin x \, dx$$

    - Compute integral on RHS:

    $$\int x \cos x \, dx = x \sin x + \cos x + C$$

    Note: the *point* of IBP is that this integral is easier than the first one!

4. Final answer is: $x \sin x + \cos x + C$

Exercise - Hidden A

Compute the integral:

$$\int \ln x \, dx$$

Solution