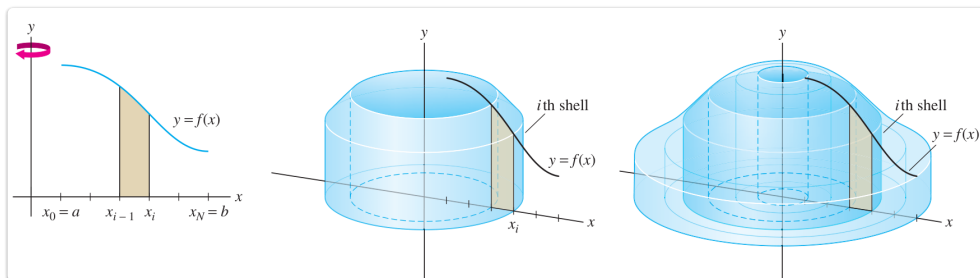


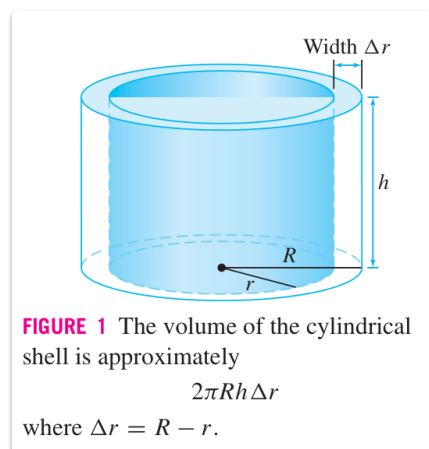
# Volume using cylindrical shells

## 01 Theory

Take a graph  $y = f(x)$  in the first quadrant of the  $xy$ -plane. Rotate this about the  $y$ -axis. The resulting 3D body is symmetric around the axis. We can find the volume of this body by using an integral to add up the volumes of infinitesimal **shells**, where each shell is a *thin cylinder*.



The volume of each cylindrical shell is  $2\pi R h \Delta r$ :



In the limit as  $\Delta r \rightarrow dr$  and the number of shells becomes infinite, their total volume is given by an integral.

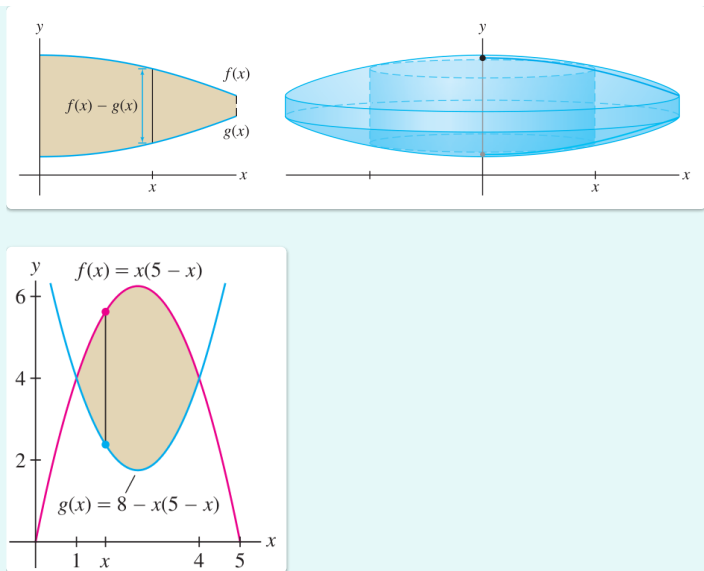
### Volume by shells - general formula

$$V = \int_a^b 2\pi R h \, dr$$

In any concrete volume calculation, we simply interpret each factor, ' $R$ ' and ' $h$ ' and ' $dr$ ', and determine  $a$  and  $b$  in terms of the variable of integration that is set for  $r$ .

### Shells vs. washers

Can you see why shells are sometimes easier to use than washers?



## 02 Illustration

### ≡ Example - Revolution of a triangle

A rotation-symmetric 3D body has cross section given by the region between  $y = 3x + 2$ ,  $y = 6 - x$ ,  $x = 0$ , and is rotated around the  $y$ -axis. Find the volume of this 3D body.

#### ≡ Solution ▾

##### 1. ≡ Define the cross section region.

- Bounded above-right by  $y = 6 - x$ .
- Bounded below-right by  $y = 3x + 2$ .
- ⓘ These intersect at  $x = 1$ .
- Bounded at left by  $x = 0$ .

##### 2. ⇌ Define range of integration variable.

- Rotated around  $y$ -axis, therefore use  $x$  for integration variable (shells!).
- Integral over  $x \in [0, 1]$ :

$$V = \int_0^1 2\pi R h \, dx$$

##### 3. ≡ Interpret $R$ .


- Radius of shell-cylinder equals distance along  $x$ :

$$R(x) = x$$


##### 4. ≡ Interpret $h$ .

- Height of shell-cylinder equals distance from lower to upper bounding lines:

$$\begin{aligned}h(x) &= (6 - x) - (3x + 2) \\&= 4 - 4x\end{aligned}$$

5.  Interpret  $dr$ .

- $dr$  is limit of  $\Delta r$  which equals  $\Delta x$  here so  $dr = dx$ .

6.  Plug data in volume formula.

- Insert data and compute integral:

$$\begin{aligned}V &= \int_0^1 2\pi R h \, dr \\&= \int_0^1 2\pi \cdot x(4 - 4x) \, dx \\&= 2\pi \left( 2x^2 - \frac{4x^3}{3} \right) \Big|_0^1 = \frac{4\pi}{3}\end{aligned}$$

### Exercise - Revolution of a sinusoid

Consider the region given by revolving the first hump of  $y = \sin(x)$  about the  $y$ -axis. Find the volume of this region using the method of shells.

[Solution](#)

## Integration by substitution

[This section is non-examinable. It is included for comparison to IBP.]

### 03 Theory

The method of ***u*-substitution** is applicable when the integrand is a *product*, with one factor a composite whose *inner function's derivative* is the other factor.

#### *u*-substitution

Suppose the integral has this format, for some functions  $f$  and  $u$ :

$$\int f(u(x)) \cdot u'(x) \, dx$$

Then the rule says we may convert the integral into terms of  $u$  considered as a variable, like this:

$$\int f(u(x)) \cdot u'(x) dx \gg \gg \int f(u) du$$

The technique of  $u$ -substitution comes from the **chain rule for derivatives**:

$$\frac{d}{dx} F(u(x)) = f(u(x)) \cdot u'(x)$$

Here we let  $F' = f$ . Thus  $\int f(x) dx = F(x) + C$  for some  $C$ .

Now, if we *integrate both sides* of this equation, we find:

$$F(u(x)) = \int f(u(x)) \cdot u'(x) dx$$

And of course  $F(u) = \int f(u) du - C$ .

### Full explanation of $u$ -substitution >

The substitution method comes from the **chain rule for derivatives**.

The rule simply comes from *integrating on both sides* of the chain rule.

1.  $\Rightarrow$  Setup: functions  $F' = f$  and  $u(x)$ .

- Let  $F$  and  $f$  be any functions satisfying  $F' = f$ , so  $F$  is an antiderivative of  $f$ .
- Let  $u$  be another *function* and take  $x$  for its independent variable, so we can write  $u(x)$ .

2.  $!$  The chain rule for derivatives.

- Using primes notation:

$$(F \circ u)' = (F' \circ u) \cdot u'$$

- Using differentials in variables:

$$\frac{d}{dx} F(u(x)) = f(u(x)) \cdot u'(x)$$

3.  $!$  Integrate both sides of chain rule.

- Integrate with respect to  $x$ :

$$\frac{d}{dx} F(u(x)) = f(u(x)) \cdot u'(x) \gg \gg \int \frac{d}{dx} F(u(x)) = \int f(u(x))$$

$$\stackrel{\text{FTC}}{\gg \gg} F(u(x)) = \int f(u(x)) \cdot u'(x) dx$$

4.  $\Leftarrow$  Introduce 'variable'  $u$  from the  $u$ -format of the integral.

- Treating  $u$  as a variable, the definition of  $F$  gives:

$$F(u) = \int f(u) du + C$$

- Set the ‘variable’  $u$  to the ‘function’  $u$  output:

$$F(u) \Big|_{u=u(x)} = F(u(x))$$

- Combining these:

$$\begin{aligned} F(u(x)) &= F(u) \Big|_{u=u(x)} \\ &= \int f(u) du \Big|_{u=u(x)} + C \end{aligned}$$

5.  $\Rightarrow$  Substitute for  $F(u(x))$  in the integrated chain rule.

- Reverse the equality and plug in:

$$\int f(u(x)) \cdot u'(x) dx = F(u(x)) = \int f(u) du \Big|_{u=u(x)} + C$$

6.  $\equiv$  This is “ $u$ -substitution” in final form.

## Integration by parts

### 04 Theory

The method of **integration by parts** (abbreviated IBP) is applicable when the integrand is a *product* for which one factor is easily integrated while the other *becomes simpler* when differentiated.

#### $\boxplus$ Integration by parts

Suppose the integral has this format, for some functions  $u$  and  $v$ :

$$\int u \cdot v' dx$$

Then the rule says we may convert the integral like this:

$$\int u \cdot v' dx \gg \gg u \cdot v - \int u' \cdot v dx$$

This technique comes from the **product rule for derivatives**:

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

Now, if we *integrate both sides* of this equation, we find:

$$u \cdot v = \int u' \cdot v dx + \int u \cdot v' dx$$

and the IBP rule follows by algebra.

#### $\boxtimes$ Full explanation of integration by parts >

1.  $\Rightarrow$  Setup: functions  $u$  and  $v'$  are established.

- Recognize functions  $u(x)$  and  $v'(x)$  in the integrand:

$$\int u \cdot v' dx$$

## 2. Product rule for derivatives.

- Using primes notation:

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

## 3. Integrate both sides of product rule.

- Integrate with respect to an input variable labeled ‘ $x$ ’:

$$(u \cdot v)' = u' \cdot v + u \cdot v' \quad \ggg \quad \int (u \cdot v)' dx = \int u' \cdot v dx + \int u$$

$$\stackrel{\text{FTC}}{\ggg} \quad u \cdot v = \int u' \cdot v dx + \int u \cdot v' dx$$

- Rearrange with algebra:

$$\int u \cdot v' dx = u \cdot v - \int u' \cdot v$$

## 4. This is “integration by parts” in final form.

**Addendum:** *definite* integration by parts

## 3. Definite version of FTC.

- Apply FTC to  $u \cdot v$ :

$$\int_a^b (u \cdot v)' dx = u \cdot v \Big|_a^b$$

## 4. Integrate the derivative product rule using specified bounds.

- Perform definite integral on both sides, plug in definite FTC, then rearrange:

$$\int_a^b u \cdot v' dx = u \cdot v \Big|_a^b - \int_a^b u' \cdot v$$

## Choosing factors well

IBP is symmetrical. How do we know which factor to choose for  $u$  and which for  $v$ ?

Here is a trick: the acronym “LIATE” spells out the order of choices – to the left for  $u$  and to the right for  $v$ :

LIATE :

$u \leftarrow \text{Logarithmic} - \text{Inverse\_trig} - \text{Algebraic} - \text{Trig} - \text{Exponential} \rightarrow v$

## 05 Illustration

### ≡ Example - A and T factors

Compute the integral:  $\int x \cos x \, dx$

#### ≡ Solution ▾

1. ≡ Choose  $u = x$ .

- Set  $u(x) = x$  because  $x$  *simplifies* when differentiated.  
(By the trick:  $x$  is *Algebraic*, i.e. more “ $u$ ”, and  $\cos x$  is *Trig*, more “ $v$ ”.)
- Remaining factor must be  $v'$ :

$$v'(x) = \cos x$$

2. ⇒ Compute  $u'$  and  $v$ .

- Derive  $u$ :

$$u' = 1$$

- Antiderive  $v'$ :

$$v = \sin x$$

- Obtain chart:

$u = x$	$v' = \cos x$	$\longrightarrow$	$\int u \cdot v'$	original
$u' = 1$	$v = \sin x$	$\longrightarrow$	$\int u' \cdot v$	final

3. ⇒ Plug into IBP formula.

- Plug in all data:

$$\int x \cos x \, dx = x \sin x - \int 1 \cdot \sin x \, dx$$

- Compute integral on RHS:

$$\int x \cos x \, dx = x \sin x + \cos x + C$$

Note: the *point* of IBP is that this integral is easier than the first one!

4. ≡ Final answer is:  $x \sin x + \cos x + C$

### ✍ Exercise - Hidden A

Compute the integral:

$$\int \ln x \, dx$$

[Solution](#)