# *Specimen Geometriae Luciferae*

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G.W. Leibniz, 1695

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It has often been observed by men endowed with keen judgement that Geometers, though they deliver the truest and most certain things, and confirm them so that one cannot withhold assent, yet they neither sufficiently enlighten the mind, nor open the fountains of discovery, while the reader feels themselves captured and bound, but not sufficiently able to grasp how they have fallen into this trap. This issue makes people admire more than understand the demonstrations of Geometers, and not perceive enough fruit for the improvement of the intellect, also profitable for other disciplines, and which seems to me in fact to be the most powerful use of Mathematical demonstrations. Then, as I often pondered these matters, very many things occurred to me that seemed to help restore the causes and reopen the fountains, so I decided to write down a sample of them with an informal style and freer structure, just as it comes now to mind, saving a more rigorous method of explaining them for another time.

Geometers use, or can use, various concepts taken from elsewhere, namely about what is same and what distinct, or i.e. coincident and non-coincident, about what is-in[1] or not is-in, about determined and undetermined, about congruent and incongruent, about similar and dissimilar, about whole and part, about equal, greater, and lesser, about continuous and interrupted, about change, and finally, what is properly their own, about situs and extension.

The doctrine about coincident and non-coincident is itself the doctrine of logic about the forms of syllogisms. Hence, we take it that things which coincide with the same third thing coincide with each other; if one of two coincidents did not coincide with the third, neither would the other coincide with it. A Geometer shows thus that the point where two diameters of a circle (that is, straight lines dividing the circle into two congruent parts) intersect coincides with the point where another two diameters of the same circle intersect. See fig. 36.

Some part of the doctrine about what is-in something else was even involved in demonstrations by Aristotle in his Prior Analytics, for he observed that the predicate is-in the subject, that is, the notion of the predicate [is-in] the notion of the subject, even though on the other hand individuals of the subject are-in individuals of the predicate. And at this point more universal things could be demonstrated about that containing and that contained, or being-in, which would be as useful in matters of Logic as in Geometry. I gave a sample of these when I demonstrated in fig. 37 that if $A$ is in $B$ and $B$ is in $C$, then $A$ also is in $C$; in fig. 38 that if $A$ is in $L$ and $B$ is in $L$, then the composite of $A$ and $B$ also is in $L$; in fig. 39 that if $A$ is in $B$ and $B$ is in $A$, then $A$ and $B$ coincide. I also solved the problem of finding arbitrarily many things such that nothing new can be composed from them, which happens if they are-in each other mutually, successively [continue]; as when $A$ is in $B$ and $B$ in $C$ and $C$ in $D$ etc., then nothing new can be composed from these. This

[1] Here and throughout, we use the hyphenated expression to represent Leibniz's term of art 'inesse'.

can also be exhibited in another way, as when there are five things $A$, $B$, $C$, $D$, $E$, and $A \oplus B$ coincides with $C$, and $A$ is in $D$, and finally $B \oplus D$ coincides with $E$, then nothing new can be composed from them however they may be combined. From this I also show how more things of a given number should relate with respect to coincidence and being-in, so that useful combinations could be arranged from this for composing something new. And part of the general Combinatorial Science of universally accepted formulas is involved in these things, to which not only Geometry, but also Logistics or the universal Mathematics treating of Magnitudes and Ratios in general, is elsewhere shown to be subordinate.

Next is the doctrine of the determined and the undetermined, when of course, from certain givens a requirement is so circumscribed that only a unique thing can be found which satisfies these conditions. There is also semidetermined, when indeed not a unique thing but multiple, of fixed number, or i.e. finite in number, can be exhibited that satisfy them. Thus, given two points $A$, $B$, the line $AB$ or i.e. the minimal path from one to the other is determined (fig. 40); but if a point $C$ is sought in the plane whose distances from the given points $A$ and $B$ are of a given magnitude, the problem is semidetermined, since two points in the same plane can be found, say $C$ and $(C)$, that satisfy the requirement. But only a unique circle can be found whose circumference passes through three given points $A$, $B$, $C$. And hence if two circles are proposed, and it is found in the course of argument that each of them passes through three proposed points, it is certain that those circles, which are two in name, are really one and the same or coincide. Whether I hold the given conditions to be determining can be recognized from them themselves, when they are such that they contain the generation or production of the thing sought, or at least that they demonstrate its possibility, and in the course of generating or demonstrating one always proceeds in a determinate manner, so that nothing is left up to decision or choice. If, indeed, one does arrive at the generation of the thing or the demonstration of its possibility by proceeding in this way, then certainly the problem is thoroughly determined.

From here I deduced many remarkable and very useful Axioms, which still seem to me inadequately observed. The most powerful of these is that a determiner can be substituted for a determined in a new determination in which the determined determines something in turn, while preserving this determination. Thus, if we say that the indefinite line passing through $A$ and $B$ (fig. 41) is the locus of all points relating in a determined way to $A$ and $B$, or i.e. unique with their situses to $A$ and $B$, I will demonstrate from there that, for two other points taken in the same line such as $C$ and $A$ (taking now one of the earlier points for the sake of ease and brevity), the same line is determined also by these two points, or i.e., that any point in the same line is unique with its situs to $A$ and $C$. The demonstration is like this: Let there be a line through $A$ and $B$, each point of which, say $L$, is unique with its situs to $A$ and $B$, so that no other point can be found relating in the same way to $A$ and $B$ (which is a property of the line), or i.e. $A.B.L.$ un. (this is how I will write determination), and take another point $C$ on the same line; I claim that any point of the line, like $L$, is also unique with its situs to $A$ and $C$, or i.e. $A.C.L$ un. Indeed, $A.B.L.$ un. (by hypothesis) and $A.B.C.$ un. (since $C$ is on the line through $A$, $B$); now remove $B$ in the latter determination by means of the prior determination, substituting $A, L$ for $B$ (by the present axiom, because $B$ is determined by $A.L$); and so in the latter determination, instead of $A.B.C.$ we will have $A.A.L.C.$ un. But the repetition of $A$ here is useless, that is, if $A.A.L.C.$ is un., then $A.L.C$ also is un., or i.e. $L$ is unique with its situs to $A$ and $C$, which is what we set out to demonstrate.

We see from this example that a new kind of calculus is born, used by no mortal until now, in which magnitudes do not enter, but rather points, and where calculation is not done by equations, but through determinations, or i.e. congruences and coincidences. Determination can in fact be resolved into coincidence by means of congruence in this way: $A.B.L.$ un. means: if the situs $A.B.L$ is congruent to the situs $A.B.Y$, [then] $L$ coincides with $Y$. I normally denote coincidence by such a sign: $\infty$, and congruence alone by such a sign: $\propto$. And hence $A.B.L.$ un. means the same thing as the following conditional proposition: If $A.B.L \propto A.B.Y$, then $L \infty Y$, where I use the letter $Y$ for an indefinite point, in imitation of the Algebraists, for whom the last letters, such as $x$, $y$, usually signify indefinite magnitudes. For, whatever point you take, say $Y$, which relates in the same way to the points $A$ and $B$ as $L$ relates to the points $A$ and $B$, it necessarily coincides with $L$, supposing of course that the situs of $L$ to $A$ and $B$ is unique, or i.e. that $L$ is on the line passing through $A$ and $B$.

Let us pass on, therefore, to explain congruences. Things are *congruent* that cannot be distinguished in any way if they are observed by themselves, like the two triangles $ABC$ and $AB(C)$ in fig. 40, for which nothing prevents us placing the one on the other so that they coincide. So now they are only distinguished by position, or i.e. the relation to something else already given in position, such as can happen, given another point $L$, when $ABC$ relates in a different way to $L$ than $AB(C)$ relates to $L$, for example if $L$ is closer to $C$ than to $(C)$. It is necessary, though, that another $L$ could be found that relates in the same way to $AB(C)$ as $L$ relates to $ABC$, so that $ABCL$ and $AB(C)(L)$ are congruent; otherwise, if such as could not be done for $AB(C)$ could be done for $ABC$ (so that $(L)$ could not be found for the former as $L$ for the latter), eo ipso $ABC$ and $AB(C)$ could be distinguished, or i.e. would not be congruent. And this itself is an axiom of the greatest moment, that if two things $ABC$ and $AB(C)$ are congruent and some $L$ is found relating in a certain way to the one $ABC$, then also another $(L)$ exists, or i.e. is possible, that relates in the same way to the other $AB(C)$. Now I denote thus: $A.B.C \propto L.M.N$ what signifies that the three points $A$, $B$, $C$ are situated among themselves in the same way as the three points $L$, $M$, $N$. But this is to be understood respectively according to the prescribed order, so of course when $A.B.C$ and $L.M.N$ are understood to be congruent, or to coincide, or to be able to be placed onto each other, $A$ coincides with $L$, and $B$ with $M$, and $C$ with $N$. Hence if $A.B.C \propto L.M.N$, it follows that also $A.B \propto L.M$, and likewise for the others. But in order to obtain $A.B.C \propto L.M.N$, we must first prove $A.B \propto L.M$ and $A.C \propto L.N$ and $B.C \propto M.N$, and then finally indeed by composing we may safely say that $A.B.C \propto L.M.N$. So we see (fig. 43) that triangles $ABC$ and $LMN$ may have two equal sides, $AB$ [equal] to $LM$ and $AC$ [equal] to $LN$, but nevertheless not be congruent because they do not have equal third sides, $BC$ and $MN$. Now the way in general in which a congruence of combinations of higher degree could be obtained from congruences of combinations of lower degree, and that one does not need all triples to find the congruence of a quadruple, but only three, and for obtaining a congruence of quintuples, five triples, and of sextuples, seven triples, and so forth to infinity, will appear below when we talk about similarities.

Now it is also clear in general that from the respective congruence of all combinations of one degree, one could always conclude that all combinations of another degree are congruent, for example all triples from all doubles, since from all combinations of one degree, for example from all the doubles of four congruent things, one can conclude that the whole combination of the four things itself, or i.e. the quadruple $A.B.C.D$, is congruent with $L.M.N.P$. Now from the congruence of the whole combination it follows that any lower combination, or i.e. any triple, is congruent to the corresponding one; therefore from all doubles, all triples.

From these we learn the manifest distinction of congruences from coincidences and existences-in or i.e. enclosings. For (fig. 44) if the line $AB$ coincides with the line $LM$, and at the same time the line $AC$ coincides with $LN$, then the line $BC$ also coincides with the line $MN$. When $AB$ and $LM$ coincide, eo ipso the point $A$ also coincides with $L$ and $B$ with $M$; and when $AC$ and $LN$ coincide, eo ipso the point $C$ also coincides with the point $N$; when, therefore, the points $A$, $B$, $C$ coincide with $L$, $M$, $N$ respectively, and hence $B$, $C$ with $L$, $M$, then the lines $BC$ and $MN$ also coincide. From the nature of the line as far as existences-in, I showed elsewhere that, if $A$ is-in $L$ and $B$ [is-in] $M$, then $A \oplus B$ will also be-in $L \oplus M$, and if $A \oplus B$ is-in $L \oplus M$, and $A \oplus C$ is-in $L \oplus N$, then $A \oplus B \oplus C$ will also be-in $L \oplus M \oplus N$, which mode of arguing cannot be imitated with congruences and similarities.

Now from these things that we just said about the distinction between coincidences and congruences, flows again a reason why triangles $ABC$ and $(L)(M)(N)$ (fig. 44) are congruent if the sides $AB$ and $(L)(M)$ as well as $AC$ and $(L)(N)$ are congruent, possibly not mentioning the third ones $AC$ and $(M)(N)$, provided that the angles at $A$ and $(L)$ are congruent. For if the line $(L)(M)$ is congruent to the line $AB$ and the line $(L)(N)$ to the line $AC$, and also the angle at $(L)$ to the angle at $A$, then the lines $(L)(M)$ and $(L)(N)$ can be transferred onto $AB$ and $AC$, with their situs preserved, and so $(L)(M)(N)$ can be placed onto $ABC$, such that $AB$ and $LM$ as well as $AC$ and $LN$ coincide; therefore, by the nature of coincidence, $BC$ and $MN$ also coincide; and so if the enclosing lines as well as their angles are congruent, then the bases will also be congruent, and so the whole triangle [congruent] to the triangle.

And from this very example we can illustrate this remarkable and very useful Axiom: Things determined in the same way from congruent things are congruent. Thus, since, in general, from two lines given in magnitude and their angle given in magnitude and position, a triangle is determined or i.e. given in position, hence if two triangles $ABC$ and $(L)(M)(N)$ are given, having legs $AB$ congruent with $(LM)$ and $AC$ with $(L)(N)$, as well as a congruent angle that they enclose, angle $A$ with angle $(L)$, the triangles themselves will be congruent. Similarly, since from three lines given in magnitude the angles of a triangle are also given in magnitude, and so everything is determined which would prevent congruence by being different, hence if two triangles have three respectively equal lines, and hence congruent lines (since equal lines are congruent), the triangles themselves will be congruent. And this, on closer consideration, is seen to coincide with the method of superposition of Euclid.

Other axioms are also relevant here, such as: things congruent to the same thing are congruent to each other; and: [with] things congruent to each other, if a third is incongruent to one of them, then it will be also incongruent to the other; these are nonetheless just corollaries of the axioms about 'the same' and 'different'. In things that are congruent, in fact, everything is the same, except position, so that they differ only in number. And in general, whatever can be done or be said of one of the congruents can also be done and be said of the other, with this one exception, that things applying in the one differ in number or position from those applying in the other. Thus we will understand not only two elbows or two feet to be congruent, but also two pounds, taken abstractly, two hours, two equal degrees of speed. It is also noteworthy that if the peripheries of two bodies are congruent, then also the bodies themselves are congruent, because if the boundaries are congruent in actuality or i.e. coincide, the bodies also coincide. But it is not necessary for surfaces and curves to coincide or be congruent whose extremes coincide or are congruent. It can nonetheless be said in general, that two extensions coincide or are congruent if the things in itself that can be touched from outside, or i.e. that can be common to itself and the outside, coincide or are congruent. Hence, because surfaces and curves (but not solids) can be touched everywhere from outside, it is not sufficient for their boundaries to be congruent or coincident. But in general it is the nature of space, of extension (and so also of body, inasmuch as we conceive nothing other than space to be present in it), that in the inside it is everywhere congruent and indistinguishable (as if I stir in the midst of water, or feel in the midst of darkness, and do not hit anything) and it could only be distinguished through those things that can be touched from the outside, or i.e. are common to it and another (with which it may not have any common part). Hence also if two surfaces or curves are found to be uniform, with their extremes congruent or even actually coinciding, then they themselves will be congruent or actually coincident.

Equals arise from congruents. Certainly what are congruent, or can be rendered congruent, if needed, by transformation, are called equal. Thus in fig. 45, triangles $BAD$, $BCD$, $BCE$, $BFE$ are congruent and therefore equal; since triangle $EBD$ is also equal to the square $ABCD$, though indeed the triangle and square are not congruent, nevertheless in this case a square congruent to the former can be made from the triangle by a transposition of its parts, for if you transfer the one part $BCD$ of triangle $EBD$ onto the congruent $BFE$, with the other part $ECB$ remaining, then from $BFE$ and $ECB$ the square $BCEF$ becomes congruent to the square $ABCD$. Usually, moreover, we designate equality with the sign =, that is, $A = B$ signifies that $A$ and $B$ are equal.

Things can also be called 'equal' whose magnitude is the same. And magnitude is a certain attribute of things, a fixed species of which cannot be determined by any definition or by any fixed concepts, but a certain fixed measure is needed which one may consult, and hence if God made the entire world larger with all its parts preserving the same proportion, there would be no grounds for noticing it. Nonetheless, with one fixed thing taken, as it were a measure, the magnitude of other things can also be recognized by its application to them and applying numbers of repetitions. And so magnitude is determined by the number of parts that are equal to each other, or unequal by some fixed rule. And some thing may be incommensurable with respect to a measure or with respect to things to which the repeated measure is exactly congruent, yet with infinitely continued subtraction of the thing from the measure or the measure from the thing as many times as possible, and of the remainder from what is subtracted, then, from a progression of numbers expressing repetitions, the quantity of the thing is recognized with respect to the measure. And hence those things are equal that relate in the same way to the measure with respect to repetition, and it is clear by that fact that they can be made congruent, since they are resolved in the same way into parts individually congruent to one another.

From this one also understands what Mathematicians call ratio or i.e. proportion. For if $A$ and $B$ are two things, and the one $A$ is accepted as the measure, then the *magnitude* of the other $B$ is expressed by some number (or series of numbers proceeding according to a fixed law), setting $A$ to be expressed by unity. But if neither is the measure, then the number expressing $B$ by $A$, as if $A$ were the measure or unity, expresses the ratio or proportion of $A$ to $B$. And in general the expression of one thing by another homogeneous one (or i.e. one resolvable into congruent things) expresses the ratio of one to the other, and hence ratio is the simplest relation of the two with respect to magnitude, in which no third thing homogeneous to them is assumed for expressing the magnitude of the one from the magnitude of the other by its value. For example, let $A$ and $B$ be two magnitudes (fig. 46), and we want to determine their ratio to each other. Suppose $A$ is large and $B$ small, and therefore subtract $B$ from $A$ as many times as possible, for example 2 times, and let $C$ remain. This $C$ is necessarily smaller than $B$, and so let $C$ be subtracted again from $B$ as many times as possible. Now suppose it can be subtracted 1 time and the remainder is $D$, and $D$ can be subtracted from $C$ again 1 time and the remainder is $E$, and finally $D$ can be subtracted from $E$ 2 times and the remainder is Nothing. Clearly, $A = 2B + C$ (1) and $B = 1C + D$ (2); therefore by substituting for $B$ in eqn. 1 the value expressed in eqn. 2, $A = 2C + 2D + 1C$ (3), or i.e. $A = 3C + 2D$ (4). Again $C = 1D + E$ (5); therefore (from eqns. 4 and 5), $A = 5D + 3E$ (6), and (from eqns. 2 and 5), $B = 2D + E$ (7). Finally $D = 2E$ (8). Therefore (from eqns. 6 and 8) comes $A = 13E$ (9) and (from eqns. 7 and 8), $B = 5E$ (10). From this we see that $E$ is the greatest measure common to all, and setting $E$ as unity, we have $A = 13$ and $B = 5$. But whatever unity is assumed, $A$ and $B$ will still be to each other as the numbers 13 and 5, and $A$ will be thirteen fifths of $B$ or $A = \frac{13}{5}B$ (that is $A = \frac{13}{5}$ if $B$ were unity), namely $A$ is $13E$, while $E$ is a fifth of $B$; on the other hand, $B$ will be five thirteenths of $A$ or $B = \frac{5}{13}A$, for $B = 5E$ whereas $E$ is one thirteenth of $A$. Now it is clear that the quantities homogeneous to $A$ and $B$ arising here in order are

$$\begin{gathered} \mathrm{A} \\ B \\ C \\ D \\ E \\ 13E \\ 5E \\ 3E \\ 2E \\ 1E, \end{gathered}$$

and the numbers of subtractions or *quotients* are 2, 1, 1, 2. And if we cannot arrive at some final thing (like $E$ here) that measures all the others by exact repetitions of itself, so that $A$ and $B$ cannot be resolved into parts congruent to this measure itself, and thus also to each other, then we will not arrive at values expressed by numbers of this kind, produced merely by the repetition of a unity; however, from the progression of quotients itself we can recognize and determine a kind of ratio; as here the ratio of $A$ and $B$ is given by the sequence of quotients 2, 1, 1, 2, when by doing the subtractions such a sequence arises, then even if the sequence proceeds to infinity, which happens for those magnitudes which are said to be incommensurable to each other, yet if only the progression of the sequence is given, by that itself the ratio of the magnitudes will be given, and the longer we continue the sequence, the closer we approach to it.

There are, however, infinitely many other ways of expressing magnitudes either through a series or through certain operations or certain motions. In this way I discovered that with the square of the diameter being $\frac{1}{1}$, the circle is $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11}$, etc. That is, if the square of the diameter is set to be a square foot (the diameter being a foot), the Circle is the square of the diameter one time, minus (because we have taken too much) a third part, plus (since we removed too much) its fifth part, minus (because we readded too much) a seventh part, and so on according to a series of odd numbers understood to continue; this series differs from the magnitude of the circle less than any given quantity, and hence it coincides with it. For if we say $1 - \frac{1}{3}$, the error is less than $\frac{1}{5}$, since otherwise we would not have added too much by adding $\frac{1}{5}$; and again if we say $1 - \frac{1}{3} + \frac{1}{5}$, the error is less than $\frac{1}{7}$, since otherwise we would not have subtracted too much by subtracting $\frac{1}{7}$, and so on. Therefore, by continuing to some [point] the error is always less than the fraction following next; and if any quantity is given, no matter how small, a fraction can be found expressing something even smaller.

But it is chiefly for the common use of calculating in numbers, and in practice, that the expression of magnitudes is conveyed by numbers of parts of a Geometric progression, for instance of decimal. But since that cannot itself be expressed well in an exact figure, we will use the Binary [Bimal], which, naturally, is also the first and simplest. Namely let us divide the line $AB$ in fig. 47 into two equal parts or i.e. two halves, and each half again into two equal parts, so we will have four quarters, and bisecting the quarters again, eight eighths, and so on, sixteen sixteenths, etc. In the same way we could divide the line into 10, 100, 1000, 10000, etc. parts. Now let $CD$ be a quantity to be estimated by the scale of equal parts descending according to the geometric progression that we made. Let us place $CD$ onto the scale $AB$ and $C$ of course onto $A$, and let us see where in our scale the other endpoint $D$ falls. And we will compare $D$ first with the points of the larger division, and proceeding step by step from there to the smaller ones. And since $CD$ is less than the scale $AB$ (for if it were greater, then we would have first subtracted the scale as many times as possible), $D$ will fall between $A$ and $B$; now we see that $CD = \frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{1}{32}$ and something still further but smaller than $\frac{1}{32}$; and so if the scale is not further subdivided, that expression is sufficient at least for this, that the error is less than $\frac{1}{32}$. And if we subdivide once more, we can have such an expression of $CD$ by the scale $AB$ that the error is less than $\frac{1}{64}$. And so on. Thus similarly, if the scale is divided into 10, 100, 1000, 10000 parts and so on, we can arrange that the error is less than $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, $\frac{1}{10000}$ etc.

By this method arises the remarkable convenience that all quantities which would be expressed by fractions can be expressed in integers as precisely as desired. Indeed, let there be a seventh part of a foot, or whatever other portion or fraction. Let us take 100000 etc. and divide it by 7 continuing as far as desired, it will yield 1428571428571428 etc. or $\frac{1}{7} = \frac{1}{10} + \frac{4}{100} + \frac{2}{1000} + \frac{8}{10000}$ etc. or i.e. $1x + 4x^2 + 2x^3 + 8x^4$ etc. setting $x = \frac{1}{10}$ and $x^2$ to be $\frac{1}{100}$ or i.e. the square of $\frac{1}{10}$, and $x^3$ to be the cube of $\frac{1}{10}$, and so on. And the error is always smaller than one of the last portions, where we stopped, in this case less than $\frac{1}{10000}$. Here, moreover, it is consummately noteworthy that a period always arises when the quantity is commensurable to the proposed unity, as in this case 142847 repeats to infinity. Hence the nature of the progression is perfectly recognized. It is clear that this takes place whether we estimate the magnitude by calculation or by actual application to the proposed scale. But the Binary [Bimal] progression has this remarkable [property], that the coefficients, or i.e. the numbers by which the powers $x$, $x^2$, $x^3$ etc. are multiplied, are just 1 or 0. There are still other ways of expressing magnitudes, allowing them indeed to be incommensurable with the unity, where it can nevertheless happen that certain powers of them or i.e. something arising out of them could be co-measured with the unity or scale. That this may appear by an example, consider fig. 48, where $AB$ is a line, for instance a foot, and its square, or i.e. a square foot, is $ABCD$. Let the other line $BD$ be equal to $AB$, so that the angle $ABD$ at $B$ is right, and let the line $AD$ be drawn. And above the line $BD$ (= $AB$) let there be a square $BEFD$, equal to the square $ABCD$ (or i.e. $AC$), and finally above the line $AD$ let there be a square $ADGH$. Now it is well established, not only from Euclid's Elements, but also by inspection itself of the figure, that the square $ADGH$ is twice the square $ABCD$, or i.e. is equal to the square of $AC$ and $BF$ taken together. Indeed, drawing the diagonals $AG$, $DH$, intersecting at $L$, the square $ADGH$ will be resolved into four triangles $ALD$, $DLG$, $GLH$, and $HLA$, equal and congruent to each other, and the square of $AC$ drawn with the diagonal $DB$ is resolved into two such triangles; therefore, the square $ADGH$ is twice the square of $AC$, and hence the square or i.e. power of the line $AB$ (namely the square of $AC$) can be co-measured with the square or i.e. power of the line $AD$ (namely with the square $ADGH$). But let us see now whether the lines $AB$ and $AD$ themselves could be co-measured, or both expressed by numbers, meaning of course rational numbers, which can be expressed by repetition of unity, or i.e. of whatever fixed portion of that unity (which exhausts the unity by its own repetition). Let us set $AB$ to be 1 (namely one foot), and ask what is $AD$; it should be a number which multiplied by itself (or squared) produces 2, namely twice what $AB$ squared produces. Certainly such a number cannot be an integer. For it must be less than 2 (because 2, 3, and other larger numbers squared, or drawn on themselves, produce more than 2, for 2 on 2 gives 4, and 3 on 3 gives 9 etc.), but still it must be larger than 1 (because 1 on 1 gives 1, not 2); therefore it falls between 1 and 2, so it cannot be an integer, but rather a fraction. Actually no fractional number is better. For the square of every fractional number is a fractional number, whereas 2 is indeed an integer that should be the square of $AD$, and so $AD$ is neither an integer nor a fractional number, and thus not rational, but surd. And thus either it is expressed geometrically by drawing lines, as in the figure, or by calculation—and that either mechanically by approximation, or exactly, as if I said that it was $\frac{1414}{1000}$ or 1414 thousandths of a foot, or more accurately $\frac{14142136}{10000000}$ (or 14142136 ten-thousand-thousandths), for this fraction multiplied by itself will yield $\frac{20000001}{10000000}$ and a little more, so that its difference from 2 is less than one thousand-thousandth. $AD$ is expressed exactly either in common numbers by an infinite series, or by surds. It would be too drawn-out to explain here how $AD$ is expressed from $AB$ by an infinite series. Algebraically or with surds $AD$ is expressed by the notation for extracting a square root from 2, or i.e. putting $AB = 1$, then $AD$ will be $\sqrt[2]{2}$, that is the square root of 2, or the number whose square is 2. This surd notation is useful in calculation, since it vanishes by multiplication [of the number] by itself, which cannot equally be said of the Notation of Trisection of an Angle or anything else that has nothing in common with calculation. It will be worthwhile to uncover here the real source of incommensurable quantities, namely, from where they arise in the nature of things. Now their cause is *ambiguity*, or i.e. when the thing sought is semidetermined by the givens (on this [see] above) so that several (but finitely many) things satisfy them, and no method could be applied to the givens for distinguishing one from the other. Let us show that this happens in the very example of the preceding paragraph, where we were seeking a number that multiplied by itself makes 2. It should be known, moreover, that such numbers always come in pairs. Indeed, 4 can be produced from +2 times +2 as well as from −2 times −2. And so $\sqrt[2]{4}$ is an ambiguous number, and it signifies +2 as well as −2; similarly, $\sqrt[2]{9}$ is an ambiguous number and signifies +3 as well as −3. Therefore also $\sqrt[2]{2}$ is an ambiguous number, and $\frac{1414}{1000}$ satisfies it as well as $-\frac{1414}{1000}$. Thus, by its nature, or i.e. in general, $\sqrt[2]{a}$ cannot be reduced to something rational since every rational is determined; nevertheless the extraction proceeds by chance, that is, of course, in those numbers that arose through such an operation. Ambiguity can be shown also with lines. Let there be (fig. 49) a circle whose diameter $BM$ is 3 and a portion $AB$ of that is 1. Let $AD$ be drawn at right angles from the point $A$ meeting the circle at $D$; then $AD = \sqrt[2]{2}$

will hold, or i.e. the square of $AD$ will be 2. For from the nature of a circle the square of $AD$ is equal to the rectangle under $BA$ or i.e. 1 and [times?] under $AM$ or 2, which rectangle is 2. But this very construction shows that by the same law that we found the point $D$, we could have also found a point $(D)$ by drawing the line from $A$ in the opposite direction, and so if $AD$ is $\frac{1414}{1000}$, then $A(D)$ will be $-\frac{1414}{1000}$. This is also the reason why such problems cannot be solved by lines alone, since a line intersects a straight line in just one point, but a circle is crossed by a line in two points, and hence solves ambiguous problems of this kind.

Actually these surd expressions also provide us with a way of expressing impossible or i.e. imaginary quantities through calculation. For every line crosses another line of the same plane (unless they are parallel); but a circle does not cross a line whose distance from the center is greater than the radius, and a problem that should be solved by such an intersection is imaginary or i.e. impossible. Indeed $\sqrt[2]{-aa}$ (or something similar) occurs in the value of the quantity we seek, whose square is $-aa$, which is then impossible because such a number $\sqrt[2]{-aa}$ is neither positive nor privative, or i.e. the line sought cannot be exhibited by motion either forward or backward. If it were either positive or privative, the square would nonetheless be positive, as we already remarked before; whereas its square nevertheless comes out negative. These imaginary quantities are nonetheless of service for expressing real quantities, so that some real quantities could not be expressed by calculus except by the intervention of the imaginary ones, as is shown elsewhere, but then the imaginary ones are virtually annihilated.

Having explained the nature of magnitude and measure well enough, however, let us return to the consideration of equality, where it should be noted that two things can be shown to be equals if it is shown that one is neither lesser nor greater than the other, and yet they are homogeneous, or i.e. one can be transformed into the other. Thus Archimedes exhibits a certain cylinder equal to a sphere, a triangle equal to a parabola; now it is clear that a sphere can be transformed into a cylinder if a liquid filling the sphere is poured into the cylinder. That a parabola can be transformed into a triangle, or that a triangle and a parabola are homogeneous, can be shown, since their ratio can be found to be the same as that of a line to a line.

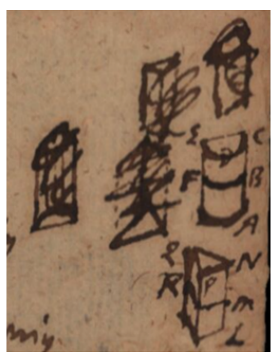

Figure 1: [no caption]

I prove it as follows: Let there be (fig. 15) two prisms or cylindrical bodies $AE$ and $LQ$, let the base or horizontal section of the one be the parabola, say $CDE$ (or others congruent to it), and let the base of the other $LQ$ be the triangle $NPQ$. Suppose first that $AE$ is full of liquid up to the altitude $AB$, which, if it is poured from there into $LQ$, we suppose it to be filled to the altitude $LM$, and that the filled portion $LMR$ of $LQ$ is equal to the portion of $AE$ filled at first with the same liquid, namely $ABF$. Now the quantities of such cylindrical portions come from the altitude multiplied by the base, or are in a composite ratio of the altitudes and bases, therefore when the portions are equal, the bases will be reciprocally as the altitudes, or the parabola $CDE$ will be to the triangle $NPQ$ as the line $LM$ to the line $AB$; thus if another triangle is made which also is to $NPQ$ as the line $AB$

to the line $LM$, which surely can be done by common Geometry [communem Geometriam] (and also, it is understood at first glance from the nature of similar triangles, of which more soon), it is clear that a triangle equal to this parabola is given, or i.e. that the parabola can be transformed into a triangle.

We also identify magnitudes from generation or motion, as in this case a method is given of estimating such a body from the motion of the base along the altitude by which the cylindrical body is generated; thus the rectangle subtended by two lines is estimated from the product [ex ductu] of the line by the line. By this method a surface and also solids generated by rotation are estimated, and to this pertains that spectacular theorem that the thing generated by the motion of some extension is equal to the thing generated by that extension drawn along [ducto in] the path of the center of gravity, certain rather remarkable generalizations of which I gave elsewhere. However, these truths can be demonstrated by reductio ad absurdum, or by applying the preceding method, when it is shown that something cannot be greater or lesser than is asserted.

Also the method through indivisibles and infinites, or rather through the infinitely small or infinitely large, or through the infinitesimal and infinituple, is extraordinarily useful. For it contains a certain resolution as it were into a common measure, though smaller than any given quantity; or a means by which it is shown, by neglecting some things which make an error smaller than anything given and thus nothing, that of two things which are to be compared, one is transformable into the other by transposing. But one should realize that a curve is not composed of points, nor a surface of curves, nor a body of surfaces, but a curve of little curves, a surface of little surfaces, and a body of little bodies indefinitely small; that is, it is shown that two extensions can be compared by resolving them into little parts equal or congruent to each other, however small, just as into a common measure, and the error is always smaller than one of such little parts, or at least of a constant or decreasing finite ratio to one, whence it is clear that the error of such a comparison is smaller than anything given. The Method of Exhaustion is also pertinent here, somewhat differently than before though they come to the same thing at the source. Whereby it is shown how there is a certain infinite sequence of magnitudes, of which a first can be obtained, and the last ones, which continuously approach some proposed [magnitude], such that the difference eventually becomes less than [anything] given, and so in the end nothing, or rather it is exhausted. And thus the final magnitude of this sequence (which we had said was obtained) is equal to the proposed Magnitude; but here it is apparent we have only touched on these things.

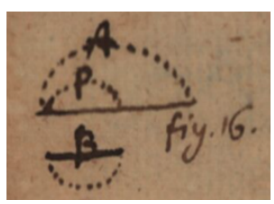


Figure 2: [no caption]

We have not yet defined what is greater and lesser, which by all means must be done. Therefore I say, Less than something is what is equal to a part of it, or (fig. 16) if there are two things $A$ and $B$, and $p$ is a part of $A$ equal to $B$, then $A$ we call *Greater* and $B$ *Lesser*. From here that celebrated Axiom is immediately demonstrated, that the whole is greater than its part, assuming only the other axiom true in itself [per se] or identical, that certainly each and every thing endowed with quantity is as great as it is, or is equal to itself, or that every triple-foot is triple-foot, etc. The demonstration, comprising a single syllogism, is thus: *Whatever is equal to a part p of the whole A, that is less than the whole A* (from the definition of lesser); *now the part p of the whole A is equal to the part p of the whole A, that is, to itself,* (identical or true in itself [per se] by the Axiom), therefore *the* *part $p$ of the whole $A$ is less than the whole $A$*, or the whole is greater than the part.

But here there we must already explain somewhat what a whole and a part are. Of course it is clear that a part is in the whole, or by supposing the whole the part is immediately supposed, or by supposing the part along with certain other parts by that very fact the whole is supposed, so that the parts taken with their position differ from the whole only in name, or the name of the whole is merely put in place of them in arguments for the sake of brevity. However, there are others things that are in it, even though they are not parts, such as points that can be taken on a curve, the diameter which can be taken in a circle; and so the part ought to be Homogeneous with the whole; and hence if two things $A$ and $B$ are homogeneous and $A$ is in $B$, then $A$ will be the whole and $B$ the part, and so the demonstrations I gave elsewhere about the containing and the contained or existing-in can be transferred to the whole and part. But what Homogeneous is, we partly have touched and partly will explain further. From these definitions of equal, greater, lesser, whole, and part very many axioms can be demonstrated, which were assumed by Euclid. That the whole is greater than its part as we showed already. That the whole can be composed in a certain way out of its parts, or that parts can be assigned which taken together coincide with it, is clear from what was said in the previous paragraph, that is, from the nature of things existing-in. What is less than the lesser is less than the greater, or if $A$ is less than $B$, and $B$ less than $C$, then $A$ will be less than $C$, or $A + L = B$ and $B + M = C$, therefore $A + L + M = C$. Now these axioms, that from adding or subtracting equals from equals, equals result, and other things of this kind are immediately demonstrated from this, that Equals are those which are the same in magnitude, or which can be substituted for each other with the magnitude preserved, and if things are handled in the same way with respect to magnitude (according to all determinate methods of handling, by which only a unique thing is produced), then equals will result. From here it immediately appears that by addition, subtraction, and multiplication of equals to equals, equals will result; but if roots of the same denomination are extracted from equals, either pure or stricken, then it is not necessary for equals to immediately result, because the problem of extracting roots is ambiguous by its nature and absolutely speaking. And so it is not allowed to say that those things must be equal that produce equals when multiplied by themselves or with the same thing in the same way. Thus, there can be two unequal numbers (namely 1 and 2) for which the remainder taken from 3 (2 or 1) multiplied by the number itself (1 or 2) produces an equal namely 2.

Now, after talking about magnitude and equals, it is time for us to talk about shape or form and similars; indeed, the usefulness of similarity in Geometry is very great, but its nature has not been given a good enough explanation, hence many things that are immediately clear at first glance to the one considering them are demonstrated in a roundabout way. It is well known from Euclid's book of Givens, what things are given in position, what things in magnitude, and finally what things in shape. If something is given in *position* from certain given things, then something else which is given out of the same things in the same (determinate) way must be coincident with the first, or the same in number; if something is given in *magnitude* from certain given things, and something else is given from the same or equal things in the same (determinate) way, then it will be equal to the first; if something is given in *shape* from certain things, and something else is given from the same or similar things in the same determinate way, it will be of the same shape as the first or similar. Finally those things that are similar and equal are congruent. And those things which are given both in magnitude and shape, they can be said to be given in *example* or *type*, so that those things that are of the same type or example, that is both in quality or form and quantity, are called congruent. Further, those which cannot be distinguished in any way, neither by themselves nor through other things, are certainly the same or coincident, and such things, in matters where nothing besides extension is considered, are those that have the same position or that actually agree with the same locus. But there are others which come together through all things or are of the same type or example, yet still differ in number, as right angles, two eggs similar through everything, two characters of the same type pressed on uniform wax. It is clear that these things, if looked at in themselves, cannot be distinguished in any way, even if they are brought together. They are only distinguished with respect to situs toward external things. Thus if two eggs are perfectly similar and equal, and are placed next to each other, it can only be noted that one is farther east or west than the other, farther north or south, farther up or down, or closer to some other body outside of them. And these things are called congruent, which are such that nothing more can be affirmed about the one that is not possible to be understood also regarding the other with only a distinction in number or individual, or of the position which one has a some certain time, since multiple things are never in the same place at the same time, nor one thing in multiple places. But those things are similar whose shape of definition is the same, or which are of the same lowest shape, as all circles are of the same shape, or the same definition fits each, nor can a circle be divided into distinct shapes that differ by some definition. Indeed, although one circle can be one foot, another half a foot, etc., however no definition can be given of a foot, but there is need of some fixed and permanent type; hence, measures of things tend to be made out of durable material, and so one person proposed that the pyramids of Egypt be used, which have already endured so many centuries and likely will endure a long time yet. Thus, as long as we suppose that neither the globe of the earth, nor the motions of the stars will notably change, the same quantity of the tilt of the earth that we investigated can be investigated by posterity. And if some shapes keep the same magnitude over the whole world and many centuries, like the cells of bees seem to some people to do, a constant measure could also be taken from this. Finally, as long as we suppose that nothing will notably change in the cause of gravity, nor in the motion of the stars, posterity can learn our measures with the aid of a pendulum. But if, as I already said elsewhere, God changed everything, preserving the same proportion, every measure would have perished for us, and we would not be able to see how much things had been changed, because a measure cannot be comprehended in a certain definition and so cannot be retained in memory either, but we need a real conservation of it. From all these things I judge the difference between magnitude and shape, between quantity and quality to be clear.

So if two things are similar, they cannot be distinguished individually in themselves. For example, two unequal circles will never be distinguished as long as each of them is looked at individually. All theorems, all constructions, all properties, proportions, respects that can be noted in one circle can also be noted in the other. As the diameter relates to the side of a certain regular polygon inscribed or circumscribed in the one, so also it will relate in the other; as the one circle relates to its circumscribed square, so also will the other to its own; hence it is clear at once by permutation that circles are as the square of the diameters, for because $A$ is to $B$ as $L$ is to $M$ (fig. 52), by permutation $A$ will be to $L$ as $B$ to $M$. And generally from this it is clear that similar surfaces are as the squares of homologous lines, and similar bodies as the cubes of homologous lines. From this also Archimedes took that the centers of gravity of similar figures are similarly located. And so to distinguish two similars, for instance two circles, we need not only to look at them individually and do the task by memory, but we need to look at them simultaneously and actually move them to each other, or apply some common real measure to them bringing it from one to the other, or something already measured or to be measured by the application of a real measure. And so at last it will appear whether they are congruent or not. Indeed, if some homologous things from the two similars, e.g. the diameters of the two circles, or the parameters of two parabolas, it is necessary for the similars themselves clearly also to be congruent and so also equal. It is not true that if similars are added or subtracted from similars, then similars will come out, unless they are added or subtracted in the same way in both cases. And generally whatever is determined from similars in a similar manner or in the same way, those are similar; and if they are semidetermined, where the problem is ambiguous, at least to each one of the semidetermined things on one side will correspond one of the semidetermined things on the other side, which will be similar to it. This can also be said of equals, congruents, and coincidents. If two homologous things from two similar coincide, the two similars will only be congruent, since whatever things coincide are congruent, and when there exist congruent homologous things from similars, they are congruent.

Furthermore, I usually denote similarity in this way ~, and $A \sim B$ signifies $A$ sim. $B$. From things individually similar, as I said, it is not allowed to conclude that the composites are similar, and it is possible that $AB \sim LM$ and $AC \sim LN$ and $BC \sim MN$, but it is not allowed to conclude $ABC \sim LMN$; otherwise since every line is similar to any other line, one could conclude that any figure you want was similar to any other; however, such a line of reasoning goes forward for congruences. But in groups of three or more such argumentation goes forward, which is remarkable. Namely, if all triples on the one side are similar to the combinations of triples on the other side, also quadruples, quintuples, etc., the fusions will therefore be similar, or if (fig. 53) $ABC \sim LMN$ and $ABD \sim LMP$ and $ACD \sim LNP$ and $BCD \sim MNP$, $ABCD$ will be $\sim LMNP$. But whether one of the triples can be omitted or can be concluded from the others, let us see, for instance whether $BCD \sim MNP$ can be omitted. Take a $LMN$ similar to triangle $ABC$ and $LMP$ similar to $ABD$, it is clear that given $ABCD$ and assuming $LMN$ (which is given in shape) is given in magnitude and position arbitrarily and $LMP$ is given in shape and magnitude, and since we also have the position of $LM$ (because $LM$ is assumed in $LMN$) it is clear that $P$ falls on the circle described by the motion of triangle $LMP$ around $LM$ as the axis. In the plane however $P$ can only be taken in two [places] when $L$ and $M$ remain, namely in $P$ or in $\pi$ (because the circumferences of this circle punctures the plane in two points). The third similarity, namely $ACD \sim LNP$ shows that out of these $P$ ought to be chosen, excluding $\pi$, since $ACD$ is not $\sim LN\pi$. And so in the plane in this way everything is determined, or out of only three similarities of corresponding triples one concludes the similarity of the fourth triple and so also of the whole quadruple, and when in the written figure $A$, $B$, $C$, $D$ are in the same plane, also $L$, $M$, $N$, $P$ will be in the same plane. But absolutely, in space if $A$, $B$, $C$, $D$ are understood to be placed anywhere, let us see what will happen to the similarities of triples for concluding the similarity of the whole quadruple. And so since we have from the first two similarities two things, $LMN$ (assumed in position and magnitude, given in shape) and the circle described by the point $P$ rotated about the axis $LM$, attached rigidly to the axis, hence from $ACD \sim LNP$, since $LP$ and $NP$ are given as we have already $LN$, the circle described by rotating the point $P$ about the axis $LN$, attached to the axis, is also given. These two circles are not in the same plane, but they are both in right planes to the plane $LMN$, or both are orthogonal to the plane $LMN$. They must necessarily meet each other, else the thing sought would be impossible, however it is clear from elsewhere that it is possible (from general postulates, since it is possible to have something similar to anything anywhere), and so these two circles meet each other. But two circles orthogonal to a plane in which they have their centers relate in the same way with respect to the plane, above this plane just as below this plane, therefore when they meet each other, they meet each other as much above the plane as below the plane, and so in two points. Now there remains $BCD \sim MNP$, where since $MN$ is given in position and $MNP$ in shape, certainly $MNP$ will be given in type or magnitude and shape, or again a circle described by the point $P$ with the axis $MN$ will be given. And because it intersects each of [the others] in two points, and at least one intersection coincides with both, or it is incident on a point where the two previous circles intersect each other, else the problem would be impossible, it is necessary that both intersections coincide with the previous intersections. Hence, the third circle exhibits nothing new, and therefore the three triples suffice to conclude the fourth; but the problem is semidetermined, and the matter reduces to the same as if it was proposed, given the distances of one point from three points, to find that fourth point, a problem which is semidetermined. But the way that we demonstrated it in this place is remarkable and mental, and the very method by which we established arguments for similarity from there is also remarkable, when partly assumed three points, partly obtained that they were such as they should be, whence the problem for fourth is determined, so that a quadruple is similar to a quadruple. For finding a quintuple to be similar to another, let there first be found one similar quadruple, which happens with three triangles or triples. There remains one point besides this, and it is clearly determined from the givens, namely its given distances from the four points; and so there is need of only two more triples or triangles that the new point enters into. More precisely, as we have shown,

Let $LMN$, $LMP$, $LNP$ be similar to $ABC$, $ABD$, $ACD$;
$ABCD$ will be similar to $LMNP$, And so also $BCD$ similar to $MNP$.

We ask, from what additional [assumptions] would we conclude that $ABCDE$ is similar to $LMNPQ$. We found a little earlier that $LMNP$ is similar to $ABCD$, hence because $LMNP$ is given in position, and hence in magnitude even more, and $LMNPQ$ is given in shape (because it is given that it is similar to $ABCDE$), it is necessary that $LMNPQ$ is also given in magnitude, or the lines $LQ$, $MQ$, $NQ$, $PQ$ are given in magnitude; therefore, the point $Q$ is given in position, for it has been shown elsewhere that a point with a given situs to four points not located in the same plane is determined or unique. But to return to our triples, it suffices to add the following to the three earlier similarities of triples:

That $ABE$ and $CDE$ be similar to $LMQ$ and $NPQ$, So that $ABCDE$ will be similar to $LMNPQ$.

Indeed, from $ABE \sim LMQ$, because $ABE$ and $LM$ are given, $LQ$ and $MQ$ will be given, and from $CDE \sim NPQ$, because $CDE$ and $NP$ are given, $NQ$ and $PQ$ will be given. For the two $ABE \sim LMQ$ and $CDE \sim NPQ$, we could have also used $ACE \sim LNQ$ and $BDE \sim MPQ$, or $ADE \sim LPQ$ and $BCE \sim MNQ$, observing that in the two similarities we have conjoined nothing is in common except $E$ and $Q$. From here it is also clear that the similarity of a quintuple is given from the similarity of three quadruples. For from these five similarities of triples thus gather three quadruples:

- From $\underline{ABC}$, $\underline{ABD}$, $\underline{ACD}$ similar to $LMN$, $LMP$, $LNP$, we conclude $ABCD \sim LMNP$.
- From $\underline{ABE}$, $ACE$, $BCE$ similar to $LMQ$, $LNQ$, $MNQ$, we conclude $ABCE \sim LMNQ$.
- From $ACE$, $\underline{ADE}$, $CDE$ similar to $LNQ$, $LPQ$, $NPQ$, we conclude $ACDE \sim LNPQ$.

Indeed, there is need of at least three quadruples to obtain the five triples (which we have underlined) sufficient for the quintuple. For a similarity of a sextuple if we want that $ABCDEF \sim LMNPQR$, let us make $ABCDE \sim LMNPQ$, for which there is need of the five triples stated above. Then because every point is determined enough from its situs to four other points being given, we only need to find $LR$, $MR$, $NR$, $PR$, which will happen the same way as above by assuming only two similarities of triples having nothing other than $F$ and $R$ in common, namely that $LMR$, $NPR$ would be similar to $\underline{ABF}$, $\underline{CDF}$, whence together with the five similarities above we conclude the sextuple $ABCDEF \sim LMNPQR$. And thus from three similar triples or triangles (or from three similar pyramids) we conclude the similarity of two quintuples or the five-cornered solids filled up from them; from seven similar triples or triangles we conclude the similarity of two sextuple or the six-cornered solids filled up from them, and so forth to infinity, supposing that more than three of the points are not in one plane. From one, three, five, seven, nine, etc. similar triangles, we conclude the similarity of two triples, quadruples, quintuples, sextuples, septuples, etc. formed by them or the four-cornered (or pyramidal), five-cornered, six-cornered, seven-cornered solids. Here note that the number of faces of a solid is not immediately defined from the number of corners. But it will be worthwhile to track down the progression by which higher combinations are sufficiently concluded from quadruples or pyramids, and from quintuples or five-cornered solids, and so forth which it is now downhill to establish with the aid of the sufficient triple already found.

But here most importantly this should be noted, that the same things that we said about similarities, regarding concluding similarities of higher combinations from triples, quadruples, quintuples, etc., those can furthermore be applied to congruences. Indeed, $LMPN$ is found to be congruent to $ABCD$ (fig. 53) in the same way as $LMPN$ is found to be similar to $ABCD$, the only difference being that, while for finding similarity one could assume the first line $LM$ arbitrarily, for finding congruence one must assume that $LM$ is equal to $AB$; having $LM$ already, whence one has the triangle $LMN$ in type (being similar to the given $ABC$) which then can be assumed in position, and placed wherever one likes. Now from here since the distances of the point $P$ from the points $L$, $M$, $N$ are given, the point $P$ can be obtained, and $LMNP$ (a pyramidal solid) becomes similar or also congruent to $ABCD$. And this method should be noted: whatever things suffice for constructing something according to a prescribed condition, in this case similarity or congruence, they also suffice for concluding that very condition from them. Congruences have this privilege at least, that they can be concluded from congruences of pairs or lines, but for new similarities nothing can be concluded from similarities of pairs or lines, indeed all lines are similar to each other; but from the similarities of triangles or triples one can conclude similarities of other polygons as well as solids. And because for concluding similarity of a four-cornered [figure] in the plane or in a solid, perhaps also the same number of triangles is needed for concluding similarity of higher polygons in the plane or in a solid, which we do not have space to discuss now.

Finally, for two figures to be similar, it is necessary for their angles to be congruent, which I show as follows: otherwise, if they did not have corresponding or homologous angles that were equal and so also congruent, then they could be distinguished in themselves individually. Indeed if (fig. 54) angle $A$ is not congruent to angle $(A)$, hence in $AC$ taking $AD = AB$ and adjoining $DB$, and similarly in $(A)(C)$ taking $(A)(D) = (A)(B)$ and adjoining $(D)(B)$, the ration of $DB$ to $AB$ will not be the same as that of $(D)(B)$ to $(A)(B)$, therefore or hence $ABC$ and $(A)(B)(C)$ can be distinguished. On the contrary, if all the angles are the same, one can show that the triangles themselves are similar as follows: from one side and all angles being given the triangle is given; now the side is similar to the side (of course, any straight line is similar to any straight line) and the angle is congruent to the angle, therefore the triangles are determined from similar and congruent things in the same way, and so they are similar. For making quadrilterals, pentagons, etc. similar (either in the plane or in a solid), it is not only necessary for all the angles to be equal, because a polygon higher than a triangle is not immediately given from one side and all angles being given, and so however many sides are needed for determining a quadrilateral, pentagon, etc. with all angles being given, the ratio of those sides can be assumed to be the same as in the quadrilateral and other given polygon, and from there with the angles being the same the figure is similar, because from these sides and angles the figure can also be constructed; and in general, whether all sides or all angles, or only some sides and some angles, provided that the givens are sufficient for constructing the figure, and the problem is completely determined from them (or semidetermined such that the several things satisfying it are congruent or similar to each other), then it is sufficient that no dissimilarity can be noted in these givens, and so the angles in both are equal but the corresponding given sides in both are proportional, for similar figures to be understood to arise from both. And it was already remarked above that if some (or one) homologous things in two similar figures are congruent, then all the rest are congruent. But by all mean coincidence cannot be concluded from one coincidence of homologous things, but according to the nature of a figure more or fewer coincidences of homologous things are needed to conclude coincidence of everything. By this method since the corresponding angles of similar figures are necessarily equal and thus congruent, Geometers have made it so that they have no need for special rules about similarity and in fact so that everything in Geometry that can be asserted about similarity can be demonstrated through congruences. This is admittedly helpful for demonstrations that force the intellect, but thus there is often need for large detours, while on the other hand, through the consideration of similarity itself, one can pre-know the same things by a short hand and a simple intuition of the mind, a certain mental analysis depending less on the inspection of figures and on images.

Furthermore, in about the same way as equals are born out of congruents, Homogeneous things are born out of similars, which is worthwhile to note, for just as equals are those things that either are congruent or can be rendered congruent by transformation, so Homogeneous things are those that are either similar (whose homogeneity is self-evident, like that of two squares to each other or two circles to each other) or at least can be rendered similar by transformation. Now this transformation occurs if nothing is taken away or added but nonetheless it becomes something else, when a certain transformation occurs with certain parts preserved, as when we cut the square $ABCD$ (in fig. 10) into two triangles $ABD$ and $BCD$, and by reconjoining them differently (for instance by transferring $ABD$ into $BCE$) we thence form triangle $DBE$; but certain transformations do not preserve any parts, as when a straight line is to be transformed into a curve, a rounded surfaces into a plane, and a completely rectilinear thing into a curvilinear one or vice versa; then therefore only the smallest things are preserved, and the transformation is when from one thing another arises, with at least the smallest things remaining the same, and it is thus preserved in perfect real transformation through a flexible or liquid thing. And in mental transformation we can use quasi-smallest things, that is indefinitely small, in place of the smallest things, so that a quasi-transformation occurs, because a quasi-curvilinear thing is used instead of a curvilinear thing, namely a rectilinear polygon; arbitrarily large numbers of sides if the quasi-transformation that we seek proceeds in this way; or the error or difference between the quasi-transformation and the true one becomes ever smaller and smaller, so that it eventually becomes smaller than any given thing, the true transformation can be concluded. And because those things, one of which arises from the other by transformation, are equal, it is clear also that those things are Homogeneous to each other which are themselves similar, or for which there are equal things that are similar [to each other].

It also clear that Homogeneous things are those that are generated by continuous increment or decrement of the same thing, at least excepting the minima and maxima or extrema. Thus if we suppose that a path or curve grows continuously by the motion of a point, the curves described by one point are homogeneous to each other, and indeed curves generated by distinct points, since they are allowed to be dissimilar, and it is clear that that dissimilarity arises from certain particular obstacles that cannot change homogeneity. And the same holds for those things that are described by the motion of a curve or surface. But one should understand a motion in which a describing point does not march through the tracks of another describing point. And indeed we can imagine homogeneous things arising from each other continuously, as a circle transformed continuously into various ellipses can pass through infinitely many ellipses of all possible shapes. And in general in Homogeneous things this axiom has a place, that whatever passes continuously from one extreme to another passes through all intermediates; however it does not pertain to the angle of contact, which is really not a middle, but is of another and clearly heterogeneous nature.

Euclid defines Homogeneous things differently as those things, by the subtraction of one of which from the other, and of the remainder from the one subtracted, and so continuing forever, either nothing remains, or less than any given quantity. But that given quantity, which the remainder must be smaller than, must also be of an ascertained homogeneity, and it will be of an ascertained homogeneity if it is similar to one of them or measures one of them by repetition. And so if for two given quantities a common quasi-measure can be found smaller than a true measure of one of them assumed arbitrarily small, then it can be said the two are homogeneous to each other. This definition is indeed correct and useful for putting together cogent demonstrations, but it does not equally illuminate the mind as the one that is taken from the consideration of similarity. And truly one follows from the other, since from such a resolution into a quasi-common measure it is shown that one can be transformed into the other, or at least into something similar to it such that the error is smaller than any given. Indeed, it is evident that whatever things have a common measure can also be transformed such that the one becomes similar to the other.

Besides something must be said about the Continuum and about Change, before we come to explaining Extension and Motion (which are species of them). A Continuum is a whole, of which any two cointegrating parts (or those that coincide with the whole when taken together) have something in common, and even if they are not redundant or if they have no part in common, or if the aggregate of their magnitudes is equal to the aggregate of the whole, then at least they have some boundary in common. And further in order to pass from one to the other continuously, not by a jump, it is necessary to pass through that common boundary. Hence is demonstrated what Euclid tacitly assumed in his first part, that two circles in the same plane, one of which is partly inside and partly outside the other, intersect each other somewhere, as when one circle (fig. 55) is described by radius $AC$, the other by radius $BC$, and $AC$ and $BC$ are equal to each other and to $AB$, it is evident that some $B$ which is in one circumference $DCB$ falls inside the other circle $ACE$, because $B$ is its center, but in turn it is clear that $D$, where the extended line $BA$ meets the circumference $DCB$, falls outside the circle $ACE$, and so the circumference $DCB$, since it is continuous and is found partly inside and partly outside the circle $ACE$, intersects its circumference somewhere. And in general, if a some continuous curve is on some surface, and it is partly inside and partly outside a part of that surface, it intersects the periphery of this part somewhere. And if some continuous surface is partly inside some solid and partly outside, it necessarily intersects the periphery of the solid somewhere. If it was only inside or only outside, and yet meets the periphery or boundary of the one, then it is said to be tangent to it, that is the intersections coincide with each other.

We can even express this by some kind of calculus, as when a part of an extension is $\overline{Y}$ (fig. 56) and each point falling in this part $\overline{Y}$ is called by one general name $Y$, while every point of that extension falling outside this part is called by one general name $Z$, and so the whole extension taken outside that part $\overline{Y}$ is called $\overline{Z}$; it is clear that points falling on the periphery of the part $\overline{Y}$ are common to $\overline{Y}$ and $\overline{Z}$ or can be called partially $Y$ and $Z$, that is, it can be said that some $Y$ are $Z$ and some $Z$ are $Y$. Now the whole extension is composed of $\overline{Y}$ and $\overline{Z}$ together, or is $\overline{Y} \oplus \overline{Z}$, as every point of it is either $Y$ or $Z$, allowing that some are both $Y$ and $Z$. Let us suppose some other new extension is given, say $AXB$, existing in the proposed extension $\overline{Y} \oplus \overline{Z}$., and this new extension we will call generally $\overline{X}$, so that any point of it will be $X$; it is clear first of all that every $X$ is either $Y$ or $Z$. But if it is known from the givens that some $X$ is $Y$ (for instance $A$ that falls inside $\overline{Y}$) and again that some $X$ is $Z$ (for instance $B$ that falls outside $\overline{Y}$ and so in $\overline{Z}$), it follows that some $X$ is both $Y$ and $Z$ at the same time. Hence, although otherwise in general nothing follows in this way from particulars, yet in a continuum such a thing can be concluded from them because of the peculiar nature of continuity. So to sum up the consequence in a few words: If there are three continua $\overline{X}$, $\overline{Y}$, and $\overline{Z}$ and every $X$ is either $Y$ or $Z$, and some $X$ is $Y$ and some $X$ is $Z$, then some $X$ will be $Y$ and $Z$ at the same time. Hence one also concludes, $\overline{X} \oplus \overline{Y}$ comprises a new continuum, because some $Y$ is $Z$ or some $Z$ is $Y$.

We can understand some continuum not only in things that exist at the same time, indeed not only in time and space, but also in some change and the aggregate of all states of some continuous change, e.g. if we suppose that a circle is continuously transformed and passes through all shapes of Ellipses while preserving its magnitude, the aggregate of all these states or all these Ellipses can be conceived in the likeness of a continuum, even though all these Ellipses are not placed next to each other, and they do not all exist at any one time, but one becomes another. However, we can take ones congruent to them instead of them, or compose some solid consisting of all those ellipsis, or whose sections parallel to the base are all those ellipses taken in order. But if we conceive of a sphere being transformed into equal Spheroids in order, then we cannot exhibit some real continuum fused in this way out of those spheroids, because in extension alone we do not have more than three dimensions. But if we want to use some new consideration, for instance weight, we can exhibit a fourth dimension, and thus exhibit a real solid of parts heterogeneous or of distinct weights, which by its sections parallel to the same base represents all the spheroids. But there is not even need to ascend to a fourth dimension or use weights besides extension, for, instead of the spheroids, let us only take the right-angled figures proportional to them, which is certainly possible, and a plane can be fused from them, whose sections parallel to the base be proportional to the spheroids corresponding in order and so will represent the continuous transmutation of the sphere into spheroids. Indeed, it suffices for us to be able to assume some line $AX$ (fig. 57) that runs from some moving point $X$ that starts from $A$, and let us suppose that, corresponding to each portion of the line or abscissa as $AX$, we can exhibit a state of the sphere continuously transmuted into spheroids with the magnitude preserved, represented through the line $XY$ or such that the right-angled coordinates $XY$ correspond to the spheroids in order, or so that $XY$ is to $AB$ as the ratios of the conjugate axes (by which the spheroid is determined when the magnitude is given, which is always the same here) are to unity (for in the sphere the ratio is of equality). In this way it is clear how the continuous change is represented through the line $AX$ and the curve $BY$ or through the plane figure $BAXYB$, but if the shape had not been changed with the magnitude retained, but the magnitude had been changed with the shape retained, those $XY$ would have been proportional to the magnitudes or states. But now when the shape is changed, they are at least proportional to something determining the shape. However, pondering the matter, the line $AX$ alone is sufficient, so that we may conceive that a portion of the line can be taken corresponding to each logarithm of the ratio of the conjugate axes, which vanishes at $A$ or in the case of equality. But if we want to take abscissas corresponding not to logarithms but to the ratios, then for the case of the sphere or the circle an abscissa $CA$ should be taken, representing the unity, which will continuously grow as the ratio of the axes grows. On the other hand, it continuously shrinks when the ratios shrink, and it vanishes at $C$, when the circle is transformed into an Ellipse, or the sphere into a spheroid, of infinitely small longitude. And in transmuting, if this change occurs according to only one consideration, as here, only the ratio of the axes changes, since the Ellipses can only change in one way while preserving their magnitude; but if we ordered the circle to vary in infinitely many times infinitely many ways, namely according to magnitude just as according to shape, so that it must pass through all types of Ellipses, then that change would have to be represented not through a line or a curve, but through some surface; it is the same the magnitude of the circle had needed to be preserved, but it had had to be transformed into Ellipses of the second degree, of which there are not only infinitely many shapes [or species], but also infinitely many genera, and under each genus infinitely many shapes, and so infinitely many times infinitely many shapes. And if you ordered the circle to be transmuted not only through all shapes of Ellipses of the second degree but also to vary its magnitude, and so to pass through all types of Ellipses of the second degree, then the states of the circle would be infinitely many times infinitely many times infinitely many, and all the changes would have to be represented by a solid. And if the circle had to pass through all types of Ellipses or Ovals of the third degree, all the variations could not be exhibited in one continuum except through a fourth dimension, using for instance weight, or some heterogeneity of extension. And so forth. In this way it is necessary that at one moment infinitely many, indeed sometimes infinitely many times infinitely many, changes occur, else one eternity would not suffice to run through all the variations.

And so from these things the nature of continuous change is also understood, and truly it does not suffice for it that between any states an intermediate one can be found; indeed, certain progressions can be thought up in which such interpolation proceeds perpetually, however so that one cannot fuse together a continuum from it; rather it is necessary that one understands a continuous cause that operates at each moment, or that for each point of some indefinite line some corresponding state can be assigned, as we said. And such changes can be understood in respect of position, shape, magnitude, velocity, and even other qualities that are not of this consideration, such as heat and light. Thus also the Angle of contact is in no way homogeneous to a common angle, indeed it is not even syngenes, as a point to a curve, but it relates to it in some way as an angle to a line; for neither can a continuous generation of a fixed law be thought up which equally passes through angles of contact and angles of straight lines. It is the same regarding the angle of osculation invented by me, and other older things. Of course the angle of intersection of two curves that intersect each other is the same as that of their tangent lines; the angle of contact of two curves tangent to each other is the same as the angle of contact of the two osculating circles to the curves, as I showed elsewhere.

Before we depart from here, something else must be said about Relation or the habitude of things toward each other, which differs greatly from ratio or proportion, which indeed is only one certain simpler species of it. Now there are perfect or determining relations, through which one thing can be found from others; there are indeterminate relations, when something relates to another, but so that the knowledge of its habitude does not suffice for determining one from the other being given, unless new things or new conditions are added. Sometimes new conditions are added, but sometimes also new things. In relations one can also look at homeoptosis and hetereoptosis. Namely, if there is a certain relation between homogeneous things $A$, $B$, $C$, and each one of these three things relates in the same way, so that by permuting their places in the formula nothing other than the previous relation would arise, then the relation will be some absolute Homeoptosis; however it can also happen that only some of the homogeneous things falling into the relation relate by homeoptosis, for instance $A$ and $B$, allowing $C$ to relate differently than $A$ and $B$. And this Homeoptosis is of great importance in reasoning. It can also happen that there is a certain relation between $A$ and $B$ (where however one still requires that other things homogeneous to them enter the relation) where $A$ is determined from $B$ being given, but $B$ is only semi-determined from $A$, indeed it may even be undetermined. I would like to illustrate these things by example. Let $ABCYA$ be the quadrant of a circle (fig. 58), the magnitude of whose radius $AC$ or $CB$ or $CY$ will be called $a$, and the magnitude of the right sine $YX$ will be called $y$, and the magnitude of the sine of the complement $CX$ will be called $x$. It is clear that the square of $CY$ equals the squares of $CX$ and of $YX$ together, or the equation $xx+yy=aa$ holds, which expresses the relation among these three homogeneous things $x$, $y$, and $a$, by means of which $y$ or the right sine can be obtained from the given $a$ and $x$ or from the radius and the sine of the complement. In this relation, it is clear that $x$ and $y$ relate by homeoptosis, and $a$ relates in a different way from them. It is also clear that the relation is semidetermining with respect to position, even though it is absolutely determining with respect to mass. Indeed, $y=\sqrt{aa-xx}$, which is ambiguous and signifies $y=+\sqrt{aa-xx}$ as much as $y=-\sqrt{aa-xx}$, of which the former signifies $XY$ and the latter $X(Y)$; however, $XY$ and $X(Y)$ are congruent or equal in mass. It is also clear that the $a$ or the magnitude of the radius is constant, or it relates in the same way, and any $x$ and $y$ are indefinite, for just as from the givens $CX$ and $XY$ the radius is obtained (by extracting a square root of the sum of the squares), so from $C_2X$ and ${}_2X_2Y$ the radius is obtained in the same way. Such constant magnitudes relating in the same way to other indefinite ones are usually called parameters. But just as here we have explained the relation of points of a quadrant, called $Y$, to the right points $X$, or the way by which from the magnitude of the radius being given and the points $A$, $B$, $C$ being given in position, from the point $X$ of the line a corresponding point $Y$ of the circle can be found (albeit in a dual way or semi-determinately), so we will also be able to give another simpler relation, by means of which from the points of one line that is given in position the corresponding points of another line, also given in position, in the same plane can be determined in order, which relation will be found to be much simpler. In fig. 59, let there be lines $\overline{X}$ and $\overline{Y}$ of the same plane intersecting each other at the point $A$, so that some $X$ is $A$ and some $Y$ is also $A$, and in that case $X \infty Y$. The lines $\overline{X}$ and $\overline{Y}$ being given in position and with a common point $A$, the angle which they form will also be given, and so also the ratio of the lines $AX$ and $XY$, assuming that $XY$ is the normal ordinate to $AX$; let this ratio be expressed by some number $n$ and the equation of $AX$ to $XY$ (or $x$ to $y$) will be as 1 to $n$ or as unity to that number, and $y=nx$ will hold. Hence, it is clear that this relation between $x$ and $y$ is so simple that there is no need to assume some third thing homogeneous to them, or some other line, much less a higher extension; indeed the $n$ that we assumed is a number only or a magnitude not needing any position, but is determined by shape or concept alone, nor is it homogeneous to those lines. And this simple relation of two Homogeneous magnitudes is nothing other than a ratio, that is, the relation is given between these two lines, $\overline{X}$ and $\overline{Y}$ existing in the same given plane, since if one of them is given in position, and their common point $A$ is given, and finally the ration between $XY$ and $AX$ or between the ordinate $y$ and the abscissa $x$ is the same as between the number $n$ and the unity 1, then the other line will also be given in position.

Next, I will show by example that every relation between two homogeneous things or between two things merely endowed with magnitude, so that nothing else besides numbers enters in, is a ratio or proportion, even if it is sometimes involved so that the nature of each one appears. Let the equation $x^2 + 2xy = yy$ (1) hold, which no other real magnitude enters than these two $x$ and $y$ homogeneous to each other, and let us suppose them to be lines, hence let us write $\frac{y}{x} = n$ (2) so that $n$ is the ratio of $x$ to $y$, or at least the quotient or number expressing that relation. Now equation (1) divided by $xx$ yields: $1 + \frac{2y}{x} = \frac{yy}{xx}$ (3) that is (by equation (2)) $1 + 2n = nn$ (4); the matter is thus reduced to only a ratio, or a number to be found that expresses it; and so from equation (1) nothing else is given than the ratio between $y$ and $x$, although that is given here surdly or ambiguously; indeed it becomes $nn - 2n + 1 = 2$ (5) or, extracting a root, $-n + 1 = \sqrt[2]{2}$ or $n = 1 \pm \sqrt[2]{2}$ (6). Hence, this method can be deduced, from the given $x$ or magnitude of $CX$ (fig. 60) to find $y$ or the magnitude of $CY$ or of $C(Y)$. Let there be a right isosceles triangle $CXA$ whose base is $CX = x$, and let a circle $XY(Y)$ be described with center $A$ and radius $AX$, bisecting the extended line $CY$ or $C(Y)$, namely in $Y$ and in $(Y)$; I say that the line $CY$ or $C(Y)$ is what we seek or that it expresses the magnitude $y$ in the equation $xx + 2xy = yy$. If $CX$ is $x$, then $CY$ or $C(Y)$ will be $y$; indeed, $CY$ to $CX$ is as $\sqrt{2} + 1$ to 1 and $C(Y)$ to $CX$ is as $\sqrt{2}$–1 to 1, or putting $CX$ as the unity or 1, $CY$ will equal $CA(\sqrt{2}) + AY$ (or 1) $= \sqrt{2} + 1$ and $C(Y) = CA(\sqrt{2}) - A(Y)$ (or $-1$) $= \sqrt{2} - 1$. And so putting $x$ as unity, $y$ will be the sum or difference of these two $\sqrt{2}$ and 1, where however it must be noted that one root must be understood as negative or false, that is the mass of $C(Y)$ will be $\sqrt{2}$–1, however the minus sign must be prefixed to it so it becomes $-\sqrt{2} + 1$. Hence, $y$ is either $1 + \sqrt{2}$ or $1 - \sqrt{2}$. Furthermore, it is clear from this that the locus of all points $Y$ is the line $CY$, if the locus of all points $X$ is the line $CX$, provided that the angle of the lines is such that every parallel line to the first $XY$ already found, say ${}_2X_2Y$, is always $C_2X$ to $C_2Y$, as we said, or according to the ratio which equation (1) or the ratio found in equation (6) expresses. But the relations between distinct curves among themselves can be expressed not only by parallel lines drawn from one to the other, but also through lines converging on one point, and often one relation is simpler than the other. Thus, if (fig. 61) there is an Ellipse, whose two foci are $A$ and $B$, and any point $Y$ in the Ellipse is taken, then it is a property of the Ellipse, that $AY + BY$ is always equal to a constant line, namely the major axis $CD$ of the Ellipse, and hence that $AY + BY$ and $A(Y) + B(Y)$ are equal to each other.

Furthermore, just as the nature of the curve $AYB$ is conveniently expressed by two normal lines $YX$ and $YX$ sent out from one point $Y$ to a certain two lines given in position, normal to each other, $CA$ and $CB$, so the nature of the curve $Y(Y)$ (fig. 62), which does not remain in any fixed plane, can be expressed, if from any point of it placed at the top, say $Y$, three normal lines $YX$, $YZ$, and $YV$ (which we will call $x$, $z$, $v$) are drawn into three planes $CXA$, $CZB$, $CVD$ normal to each other. And if two equations are given, one for instance between $x$ and $z$, the other between $x$ and $v$, the nature of the curve $Y(Y)$ will be sufficiently determined. The first equation will express the nature of the curve $Z(Z)$ projected onto the plane $CZB$ from the curve $Y(Y)$, the latter the nature of the curve $V(V)$ projected onto the plane $CVD$ from the same curve $Y(Y)$. However, the three planes can be not only normal to each other, but of any sort of a given angle, hence if at least two normal planes and the third one $CVD$ of an indefinite angle are assumed, we can find whether the whole curve $Y(Y)$ does not fall in some plane, which will happen if the plane $CVD$ can be taken arbitrarily, so that the curve $V(V)$ and the curve $Y(Y)$ coincide, or so that the lines $v$ become infinitely small or vanish. Hence also the nature of loci is clear, namely if the point $V$ is placed on a plane (fig. 59) and its distances $YX$ and $YZ$ from two indefinite lines $CX$ and $CZ$ in the same plane given in position, the problem is determined, albeit ambiguously, that is fixed points in the same plane, four in number, can be given which satisfy it. But if the distances themselves are not given, but only their relation to each other, by means of which one is determined from the other given, then the problem is undetermined, or it becomes a locus, for instance in fig. 59 a circle, and we say that all the points $Y$ are toward the circle, if they are of such a nature that when normal conjugate ordinates $YX$ and $YZ$ are drawn from each of them to two lines $CX$ and $CZ$ normal to each other, the squares of the conjugate ordinates taken together are always the same amount or equal the same constant square, for the locus of such points will be toward a circle whose center is $C$ and whose radius is the side of a constant power or square. Similarly in a solid (fig. 62) if the distances $YX$, $YZ$, $YV$ of a point $Y$ from three planes $CXA$, $CZB$, $CVD$ are given, the problem is determined, albeit ambiguously, for certain points finite in number (namely four) satisfy it. But it must be understood that the magnitudes are given when some unity is assumed, if there are as many equations as there are unknowns; and so if to find the three lines $x$, $z$, $v$ three equations are also given (independent from each other), the givens themselves will be understood, and the problem will be determined; but if only two equations are given, the problem will be undetermined in the first degree, or an unknown point $Y$ will not be obtained determinately, but rather $\overline{Y}$ or the locus of all $Y$ or the curve $Y(Y)$ of which every point satisfies these conditions. But if for finding these three magnitudes or lines only one equation is given to us which these three lines enter, then the problem is infinitely many times undetermined, or it is undetermined in the second degree, and the locus is toward a surface or some determined surface is obtained (or semidetermined or ambiguous, namely twin or triplet or quadruplet etc.) of which every point satisfies this condition or the relation expressed by this equation. Hence we now understand what are the loci for a point, curve, and surface, and how points, lines, and surfaces are determined by given equations or relations expressed by equations.

These same things can be explained also through the compositions of rectilinear motions. Indeed (fig. 63) if a regula $RX$ proceeds through a line $\overline{X}$, always in the same plane and always with the same angle preserved and meanwhile some point $Y$ moves on the regula itself, such that if the motion of either one begins at a point $A$ or $X$ or $Y$, and then when the regula arrives at ${}_2X$, ${}_3X$ etc. the point arrives at ${}_2Y$, ${}_3Y$, ${}_4Y$ (that is if the regula has stopped at in the first site $A_1R$, in ${}_2Z$, ${}_3Z$, ${}_4Z$) some curve $\overline{Y}$ or ${}_1Y_2Y_3Y$ etc. will be described by this composite motion, whose nature is given from the given relation between the corresponding $AX$ and $AZ$; for example, if the $AZ$'s are proportional to the $AX$'s or if $A_2X$ is to $A_2Z$ (or to ${}_2X_2Y$) as $A_3X$ is to $A_3Z$, and so forth, or if $A_2X$, $A_3X$, $A_4X$ are as $A_2Z$, $A_3Z$, $A_4Z$, the curve $AYY$ or $\overline{Y}$ will be a line; if the $AZ$'s are in duplicated ratio of the $AX$'s or as their squares, the curve $\overline{Y}$ will be a quadratic parabola; if in a tripled one, it will be a cubic parabola etc. If the $AZ$'s are as the reciprocal of $AX$ or $A_2X$ to $A_3X$ is as $A_3Z$ to $A_2Z$, and the same everywhere, then the curve $\overline{Y}$ will be a Hyperbola, whose asymptotes are $\overline{X}$ and $\overline{Z}$. And so further various curves can arise, which it is not of this place to pursue.

It is important to note in general how one understands from this motion in what parts the curve turns to convexity or concavity, whether it has an inflection point, a vertex or point of reversal, maximal and minimal abscissae or ordinates of its period. First let us suppose in fig. 64 that the velocities of the regula or instantaneous increments ${}_2X_3X$, ${}_3X_4X$, etc. of the $AX$'s (which are indefinitely small) are proportional to the corresponding velocities or instantaneous increments ${}_2Z_3Z$, ${}_3Z_4Z$, etc. of the point or of the conjugate abscissae $AZ$ (or ordinates $XY$); then $AYY$ is a straight line; otherwise, it will be curved. But now (fig. 63) if we suppose that, while the velocity of the regula remains uniform or the instantaneous increments ${}_2X_3X$, ${}_3X_4X$, etc. of the abscissae remain equal, the velocity of the point crows or the instantaneous increments of the conjugate abscissae or of the ordinates $AZ$ grow; or if, while the velocity of the regula grows, the velocity of the point, which earlier was doing the same thing as the velocity of the regula, grows more; or, while the instantaneous increments of the abscissae grow, the instantaneous increments of the ordinates grow even more; then the curve $AYY$ (fig. 63) turns its convexity toward the directrix $AX$, if both grow at the same time, namely those of the abscissae as well as those of the conjugate abscissae or the recession from the fixed point $A$ and those of the regula as well as those of the of moving point on the regula; which must be supposed from the beginning, if indeed in the beginning the regula as well as the moving point on it are understood to recede from $A$. And it is the same if on the other hand both the regula and the point on the regula are understood to continuously approach $A$, and the velocity of that regula or the instantaneous approaches to $A$ remain the same, or grow less than the velocities of instantaneous increments of that point on the regula. But since in this way the point only goes back through the previous path, this remark will not obtain any further consequences. But if it happens that, with the velocities of the regula or the instantaneous increments of the abscissae (namely ${}_2X_3X$ etc.) decreasing, the velocities of the point on the regula or the instantaneous moments of the ordinates ${}_2Z_3Z$ etc. remain uniform, or grow, or at least decrease less than those ${}_2X_3X$ etc., then also the curve $AYY$ turns its convexity toward the directrix $AX$.

On the other hand, from these things it is immediately clear that if the instantaneous increments of the abscissae increase more, or decrease less, than the instantaneous increments of the conjugate abscissae or ordinates, then the curve turns it concavity toward the directrix (or the lines in which the abscissae are taken) if we only suppose that the curve recedes both from the directrix $AX$ and the conjugate directrix $AZ$, or approaches to them, that is in the one directrix as well as the other it recedes from a common point $A$ or approaches it. I say this is clear from the foregoing, if we only change the directrix and its abscissae into the conjugate directrix and the conjugate abscissae in fig. 63 or fig. 65 or conversesly; indeed, it is evident that if a curve turns its concavity to one directrix, it turns its convexity to the conjugate and conversely, when it is receding from both at once of course. Further it is clear from here how an inflection point of a curve arises. Indeed, in fig. 66, if, as the points $X$ of the directrix recede from $A$, the corresponding points $Z$ of the conjugate directrix also recede from $A$, and while before the increments ${}_2Z_3Z$ etc. of the conjugate abscissae increased more or decreased less than the increments ${}_2X_3X$ etc. of the principal abscissae from $A$ up until ${}_3Y$, but at ${}_3Y$ the opposite begins to happen, there the curve has a contrary bend and from concave becomes convex, toward the same parts. That is, if we suppose the rectangle ${}_4X_4Z$ is cut by the curve $A_2Y_4Y$ into two parts $A_4X_4Y_3YA$ and $A_4Z_4Y_3YA$, then while the part $A_3Y$ of the cutting curve turns its concavity to the latter part of the space, the other part ${}_{34}Y$ turns its convexity to the former part of the space, that is, while each straight line or chord in the part $A_3Y$ of the curve, such as $A_2Y$, ${}_2Y_3Y$, fell in the latter part of the space, now each chord in the part ${}_3Y_4Y$ of the curve falls in the former part of the space.

But if we further suppose that the increments of both abscissae, namely the principal and conjugate, or at least [the increments] of one, decrease continuously, and we take the one which decreases, or at least decreases more, and we suppose that its velocity vanishes in the end, and thus further by continued change changes into the opposite, that is the curved line does not recede from $A$ anymore with respect to its abscissa, but rather approaches $A$, then we have a point of reversal. For example in fig. 67 the velocity of $A$ decreases until ${}_4X$, where it vanishes, namely ${}_1X_2X$, ${}_2X_3X$, ${}_3X_4X$, which represent velocities, continuously decrease until they vanish at ${}_4X$, where the velocity of going forward changes to going backward, and $X$ from ${}_4X$ tends to ${}_5X$, ${}_6X$ and again approaches $A$, as the velocity of going backward grows again (at least for some time), but meanwhile $Z$ proceeds with a uniform velocity; now the ordinate ${}_4X_4Y$ when drawn to the curve from the place of reversal of the point $X$, namely from ${}_4X$, is tangent to the curve at ${}_4Y$. It can happen that the points $X$ and $Z$ reverse toward $A$ at the same time, but this is quite singular, and in that case the curve has infinitely many tangents at the point of reversal, as is clear in fig. 68 that the two lines $XY$ and $ZY$ perpendicular to each other are tangent to the curve $AYH$ at the same time; from which it is clear that the whole curve falls inside the rectangle $XZ$, and so every line drawn through $Y$ falling outside the triangle is tangent to the curve, and it seems one can doubt whether there is one curve or rather two $AY$ and $HY$ intersecting each other at $H$; but since such generations can be imagined for one curve, and we have an example in secondary cycloids, nothing prevents the whole $AYH$ from being considered one curve. But if the curve does not have infinitely many tangents, or if $X$ and $Z$ do not reverse at the same time, or if in fig. 68 the curve $AY$ does not tend to $H$, but to $L$, then it is clear that since the one ordinate from $X$, namely $XY$, is tangent to the curve at $Y$, the other $ZY$, which of course is perpendicular to $XY$ and so to the tangent, is also perpendicular to the curve $AYL$, and so it is the maximum or minimum of the ordinates of this period, indeed the maximum when the curve at $Y$ turns its concavity to the directrix $AZ$, and the minimum when it turns its convexity to it.

Now further let us join together both variations of a curve, the one which is according to convexity and concavity, and the other which is according to approach and recession with respect to the directrix. Truly a curve can approach as well as recede with respect to the directrix, to which turns either its concavity or its convexity, as in fig. 69 at $(H)$ concave receding, at $(B)$ concave approaching, at $(C)$ convex receding, in $(D)$ convex approaching; but if one consults with both directrices at once, then when it recedes from both, it turns is convexity to one and concavity to the other, as at $(H)$ and at $(C)$; but when it approaches one and recedes from the other, then its turns it concavity to both or convexity to both, as in $(B)$ and $(D)$. And so we must now come to the case where the curve recedes from one directrix and approaches the other, or where $X$ recedes from $A$ but $Z$ approaches $A$, when the curve $\overline{Y}$ turns its concavity or convexity to both directrices, its convexity as in fig. 70 if ${}_2X_3X$ to ${}_3X_4X$ receding from $A$ have a smaller ratio than ${}_2Z_3Z$ to ${}_3Z_4Z$ approaching to $A$, or if as the velocities of recession in one directrix either increase or remain [constant] or decrease, the velocities of approach in the other increase less or decrease more. On the other hand, in fig. 71 the curve turns its concavity to each directrix, if ${}_2X_3X$ to ${}_3X_4X$ receding from $A$ have a greater ratio than ${}_2Z_3Z$ to ${}_3Z_4Z$ approaching $A$, or if as the velocities of recession in one directrix either increase or remain [constant] or decrease, the velocities of approach in the other increase less or decrease more.

From this one understands how it can happen that a curve which before turns its convexity to the directrix now turns its concavity to it, or, on the other hand, even if it does not have an inflection point but remains concave to the same sides, when a reversal occurs in that directrix as in fig. 72 if the motion of $X$ is receding to $A$ at ${}_3X_4X$ and approaching $A$ at ${}_4X_5X$, where it is clear from $(H)$, $(B)$, $(C)$, $(D)$, $(E)$, $(F)$, $(G)$, $(K)$ how many various ways the reversal can occur, so that convexity can be turned afterward to the same line $AX$, to which concavity was turned before, or conversely, where it is clear in $(H)$ and $(B)$ that, as the curve recedes from $AX$ and from $AZ$ and in the point of reversal still recedes from $AX$ but now approaches $AZ$, it turns its convexity at first and its concavity afterward to $AX$; it is the same in $(B)$, where the curve earlier approaches $AX$, then always recedes from it, approaches in 1, recedes in $(B)$ and in 2, and recedes from $AZ$ up until $(B)$, then recedes from it. But in $(C)$ to 1 the concavity is turned to $AX$ at first, then at 2 the convexity, and in both places it recedes, which holds by means of a belly, which contains one reversal with respect to $AZ$, but two reversals with respect to $AX$. Moreover, $(E)$ arises from $(C)$, and $(D)$ from $(F)$ when the belly vanishes into a point, and so the reversals both according to $AZ$ and according to $AX$ coincide there, whence in that point there can be infinitely many tangents, such as we have already touched on above. But if the same belly also contains an inflection point, as in $(G)$ and $(K)$, then when that belly vanishes so that $(L)$ or $(M)$ or $(N)$ is born, and thus the inflection point coincides with the point of reversal, it happens that with no reversal standing in the way the curve turns its convexity or concavity to where it was before, since when two causes of changing the concavity come together, they cancel each other out and the concavity remains the same as it was before with respect to the directrix $AX$, namely in $(L)$, $(M)$, $(N)$ they turn their concavity to it after the reversal just as well as before; if they were inverted, they would turn their convexity to it after the reversal just as well as before.

Furthermore, from this one understands that there are two reasons why a curve changes its concavity, and that which turned its concavity to the directrix $AX$ before now turns its convexity [to it]: One, a reversal of the point $X$ moved in that directrix, as in fig. 73 the curve $YY$ from ${}_3Y$ to ${}_4Y$ turns its convexity to $AX$, but after the reversal at ${}_4Y$ it turns its concavity to it at ${}_5Y$, because the point $X$ recedes from $A$ from ${}_3X$ to ${}_4X$, but it approaches

$A$ from ${}_4X$ to ${}_5X$ or goes backward from ${}_4X$ to ${}_5X$; and the second reason is a point of inflection, when the curve itself really goes from convex to concave or conversely, as in fig. 74, where the curve has an inflection point at ${}_4Y$, so that the tangent line which previously fell on one side of the curve falls on the other side after $Y_4$, but at the point $Y_4$ itself there is no tangent, or rather the tangent coincides with one secant, indeed (fig. 75) the tangent line intersects the curve with an inflection point at $L$ again somewhere at $M$, and when $L$ and $M$ can be moved continuously more and more, it happens that they finally coincide at $N$, where there is no tangent, or rather in a certain respect the tangent and secant are the same at once, whence also in the point of inflection three points of the curve coincide which elsewhere are distinct, two from the tangent (for every tangent is understood to intersect the curve in two coinciding points), one from the secant. And it appears that $LN$ and $LM$ coincide at the point of inflection $N$ of the two parts, just as if two distinct curves $LNS$, $MNR$ turning their convexity are tangent to each other at $N$, so that by going across from one to the other the inflected $LNM$ or the inflected $RNS$ can arise.

But from these two ways by which the concavity of a curve to some directrix changes, distinct from each other, we will be able to define a period inside which one understands there to be some maximum and minimum, since when a curve has many inflection points and many points of reversal, it has distinct maxima and minima for each of its periods. Indeed in fig. 76 the curve $Y$ recedes from its directrix $AX$ up until $B$, then approaches it again, hence the ordinate is maximal at $B$ (if there the turns its concavity to the directrix); furthermore, from $B$ the curve approaches the directrix $AX$ and at the same time recedes from the directrix $AZ$ up until $C$, where there is a point of reversal, or where it still approaches $AX$ but no longer recedes from $AZ$; but from $C$ (where the ordinate to $AX$ is tangent to the curve) up until $D$ it approaches the directrix $AX$ and the directrix $AZ$ at the same time, where it again begins to recede from the directrix $AX$ but still continues to approach $AZ$ up until $E$, where it recedes both from $AZ$ and from $AX$ again. Therefore, the points of reversal, which change the concavity, make periods. Thus, the first period is $ABC$, in which the curve turns its concavity to the directrix $AX$, of which period the maximal ordinate is at $B$; another period is $CDE$, where the curve turns its convexity to the directrix $AX$, and of which the minimal ordinate is at $D$. Furthermore, the curve $CDE$ being drawn out could intersect itself at $F$. And if the whole belly is understood to coincide into a point, then the twofold reversal with respect to the directrix $AZ$ coincides with the single reversal with respect to the directrix $AX$. And thus because the two reversals cancel each other out, in this way it can happen that the curve $(Y)(B)(F)(G)$ (in the same fig. 76), which recedes from the directrix $AX$ from $(B)$ up until $(F)$, recedes from it again after $(F)$, without any inflection point and also without any reversal with respect to the other, conjugate directrix; however, elsewhere the alternation of these is necessary so that a curve can recede again from a directrix which it approached. But let us return to the earlier curve $AYBCDEFG$, and after two periods $ABC$ and $CDE$, let's find the third $EGH$ from the most recent point of reversal $E$ to the next inflection point $H$, of which period the maximal ordinate is at $G$. The fourth period is $HJK$ from the point of inflection $H$ to the new point of reversal $K$, of which period the minimum is at the point $J$. Here one must note, even though two periods immediately next to each other, each of which has its own maximum or minimum with respect to the same directrix $AX$, should be distinguished from each other either by some point of reversal with respect to the conjugate directrix $AZ$ or by some point of inflection on the curve itself, still neither a point of reversal of the conjugate directrix nor a point of inflection immediately makes a period that has a maximum or a minimum, indeed not even multiple points of inflection necessarily make a new period, as is clear from the serpentine $KLM$; however, multiple new points of reversal with respect to the conjugate directrix $AZ$ necessarily make a new period or new periods of maxima and minima for this directrix $AX$, if there are no inflection points on the curve. I demonstrate this thus, because points of reversal with respect to the conjugate directrix are maximal and minimal ordinates to the conjugate directrix, hence if there exist multiple points of reversal with respect to the conjugate directrix, there exist multiple such ordinates to the conjugate directrix, therefore also periods of maxima and minima for the conjugate directrix, because each maximum or minimum has its own period; now these periods to the conjugate directrix $AZ$ are necessarily delimited by points of inflection or by points of reversal to the first directrix $AX$, but here there are no points of inflection by hypothesis, therefore there must be points of reversal with respect to the directrix $AX$, and indeed both maxima and minima and so also periods with respect to the directrix $AX$, which was asserted. Finally one should note that periods (with respect to the same directrix) are regularly such that maxima and minima succeed each other alternately; however, there is an exception in certain cases, as on the same curve $(Y)(B)(F)(G)$ in figure 76 two maxima immediately succeed each other, the ordinate from $B$ to $AX$ and the ordinate from $G$ to $AX$ (unless we want to compute the ordinate from $F$ at the same time, which however does not have its own period because it vanished), and the reason is that there two points of reversal are tacit or suppress each other, and if they were expressly understood and counted, then the rule of alternation would remain true. Similarly, it can happen that a point of reversal and point of inflection coincide, and thus the alternation, as if in the same figure $N$ there were a new period $KLMNP$ from the point of reversal $K$ to the point of inflection $P$, and of this period the maximal ordinate is from $N$ to the directrix $AX$, and again a new period $PQR$ from the point of inflection $P$ to the point of reversal $R$, of which period the maximal ordinate is from $Q$ to the directrix $AX$, from there again a new period $RST$ from the point of reversal $R$ to the point $T$ (and what sort it is should be clear from continuation of the curve), of which period the maximal ordinate is from $S$ to the directrix $AX$. And up until now the alternation of maxima and minima is always preserved; but if we suppose that whole belly $VPQRV$ vanishes into a point $V$, then the ordinate from $V$ to $AX$ could not be said to be a maximum or minimum of ordinates, because the curve $NVST$ is not secant but tangent; therefore, the maximal ordinate of the period $MNV$, namely $N$ in the direction of $AX$, is immediately succeeded by the maximal ordinate of the period $VST$, namely from $S$ to the same directrix, because $R$ and $Q$ the points of reversal and inflection that coincide into one compensate for each other and cancel each other out.

And thus we have scattered some seeds here, from which the certain general elements of curves are born, and the curves can be divided into certain classes by their form. Many other things can be demonstrated from these principles, such as that the direction of the point describing a curve is the same as that of the tangent line; one could also explain the elements of curved lines that are described in a solid by the composition of three motions, while (fig. 62) one plane $CD$ proceed in another $CB$ from $CE$ against $BF$, and in the plane $CG$ the regula $CG$ moves, approaches to $ED$ or recedes from there, and in the regula $CG$ a point $C$ moves against $G$ or recedes from $G$. From these things one can also deduce the manner of drawing tangent lines of curves and finding maxima and minima; but in this place we will not pursue that, nor a full treatment, but we are giving a certain taste and introduction. So much for this time.