Unit 01 notes

Volume using cylindrical shells

Review

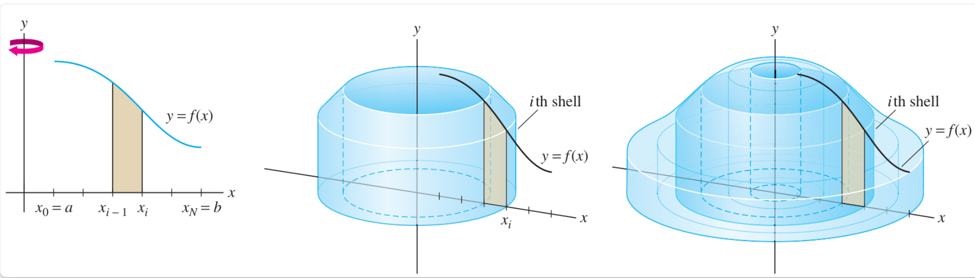
- [Volume using cross-sectional area]
- [Disk/washer method - 01]
- [Disk/washer method - 02]
- [Disk/washer method - 03]

Shells

- [Shell method - 01]
- [Shell method - 02]
- [Shell method - 03]

01 Theory

Take a graph $y = f(x)$ in the first quadrant of the xy-plane. Rotate this about the y-axis. The resulting 3D body is symmetric around the axis. We can find the volume of this body by using an integral to add up the volumes of infinitesimal **shells**, where each shell is a *thin cylinder*.



The volume of each cylindrical shell is $2\pi\, R\, h\, \Delta r$:

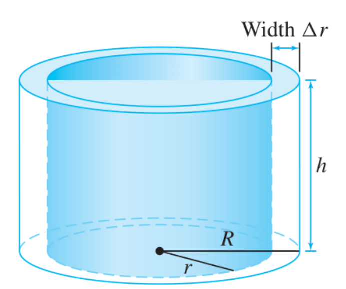


FIGURE 1 The volume of the cylindrical shell is approximately

$$2\pi Rh\Delta r$$

where $\Delta r = R - r$.

In the limit as $\Delta r \to dr$ and the number of shells becomes infinite, their total volume is given by an integral.

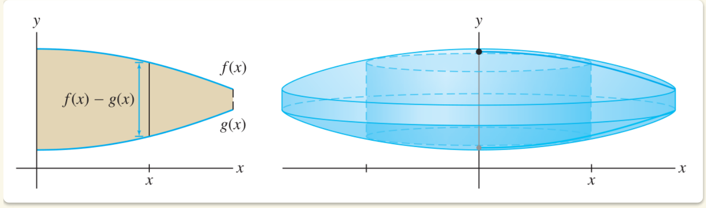
Volume by shells - general formula

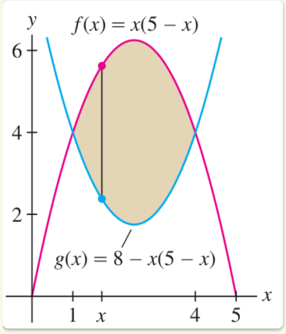
$$V = \int_a^b 2\pi Rh\,dr$$

In any concrete volume calculation, we simply interpret each factor, 'R' and 'h' and 'dr', and determine a and b in terms of the variable of integration that is set for r.

Shells vs. washers

Can you see why shells are sometimes easier to use than washers?





02 Illustration

Example - Revolution of a triangle

A rotation-symmetric 3D body has cross section given by the region between $y = 3x + 2$, $y = 6 - x$, $x = 0$, and is rotated around the y-axis. Find the volume of this 3D body.

Solution

1. Define the cross section region.
   - Bounded above-right by $y = 6 - x$.
   - Bounded below-right by $y = 3x + 2$.
   - These intersect at $x = 1$.
   - Bounded at left by $x = 0$.
2. Define range of integration variable.
   - Rotated around y-axis, therefore use x for integration variable (shells!).
   - Integral over $x \in [0, 1]$:

$$V = \int_0^1 2\pi R h\, dr$$

3. Interpret R.
   - Radius of shell-cylinder equals distance along x:

$$R(x) = x$$

4. Interpret h.

- Height of shell-cylinder equals distance from lower to upper bounding lines:

$$\begin{aligned} h(x) &= (6-x)-(3x+2) \\ &= 4-4x \end{aligned}$$

5. Interpret dr.
   - dr is limit of Δr which equals Δx here so $dr = dx$.
6. Plug data in volume formula.
   - Insert data and compute integral:

$$\begin{aligned} V &= \int_0^1 2\pi Rh\,dr \\ &= \int_0^1 2\pi \cdot x(4-4x)\,dx \\ &= 2\pi\left(2x^2 - \frac{4x^3}{3}\right)\Big|_0^1 = \frac{4\pi}{3} \end{aligned}$$

Exercise - Revolution of a sinusoid

Consider the region given by revolving the first hump of $y = \sin(x)$ about the y-axis. Set up an integral that gives the volume of this region using the method of shells.

Solution

Integration by substitution

[Note: this section is non-examinable. It is included for comparison to IBP.]

- Integration by Substitution 1: $\int \frac{-x}{(x+1)-\sqrt{x+1}}\,dx$
- Integration by Substitution 2: $\int \frac{x^5}{(1-x^3)^3}\,dx$
- Integration by Substitution 3: $\int_0^1 x^2(1+x)^4\,dx$
- Integration by Substitution 4: $\int \frac{2x+3}{\sqrt{2x+1}}\,dx$
- Integration by Substitution 5: $\int \frac{\sin x}{\cos^3 x}\,dx$
- Integration by Substitution: Definite integrals, various examples

03 Theory

The method of ***u*-substitution** is applicable when the integrand is a *product*, with one factor a composite whose *inner function's derivative* is the other factor.

Substitution

Suppose the integral has this format, for some functions f and u:

$$\int f(u(x)) \cdot u'(x)\,dx$$

Then the rule says we may convert the integral into terms of u considered as a variable, like this:

$$\int f(u(x)) \cdot u'(x)\, dx \quad \gg\gg \quad \int f(u)\, du$$

The technique of u-substitution comes from the **chain rule for derivatives**:

$$\frac{d}{dx} F\big(u(x)\big) = f(u(x)) \cdot u'(x)$$

Here we let $F' = f$. Thus $\int f(x)\, dx = F(x) + C$ for some C.

Now, if we *integrate both sides* of this equation, we find:

$$F\big(u(x)\big) = \int f(u(x)) \cdot u'(x)\, dx$$

And of course $F(u) = \int f(u)\, du - C$.

Full explanation of u-substitution

The substitution method comes from the **chain rule for derivatives**. The rule simply comes from *integrating on both sides* of the chain rule.

1. Setup: functions $F' = f$ and $u(x)$.
   - Let F and f be any functions satisfying $F' = f$, so F is an antiderivative of f.
   - Let u be another *function* and take x for its independent variable, so we can write $u(x)$.
2. The chain rule for derivatives.
   - Using primes notation:

   $$\big(F \circ u\big)' = (F' \circ u) \cdot u'$$

   - Using differentials in variables:

   $$\frac{d}{dx} F\big(u(x)\big) = f(u(x)) \cdot u'(x)$$

3. Integrate both sides of chain rule.
   - Integrate with respect to x:

   $$\frac{d}{dx} F\big(u(x)\big) = f(u(x)) \cdot u'(x) \quad \overset{\int}{\gg\gg} \quad \int \frac{d}{dx} F\big(u(x)\big) = \int f(u(x)) \cdot u'(x)$$

   $$\overset{\text{FTC}}{\gg\gg} \quad F(u(x)) = \int f(u(x)) \cdot u'(x)$$

4. Introduce 'variable' u from the u-format of the integral.
   - Treating u as a variable, the definition of F gives:

   $$F(u) = \int f(u)\, du + C$$

   - Set the 'variable' u to the 'function' u output:

   $$F(u)\Big|_{u=u(x)} = F(u(x))$$

- Combining these:

$$F(u(x)) = F(u)\Big|_{u=u(x)}$$

$$= \int f(u)\,du\,\Big|_{u=u(x)} + C$$

5. Substitute for $F(u(x))$ in the integrated chain rule.
   - Reverse the equality and plug in:

$$\int f(u(x))\cdot u'(x)\,dx = F(u(x)) = \int f(u)\,du\,\Big|_{u=u(x)} + C$$

6. This is “u-substitution” in final form.

Integration by parts

Videos:

- Integration by Parts 1: $\int e^x\,dx$ and $\int \ln x\,dx$
- Integration by Parts 2: $\int \tan^{-1} x\,dx$ and $\int x\sec x\,dx$
- Integration by Parts 3: Definite integrals
- Example: $\int e^{3x}\cos 4x\,dx$, two methods:
  - Double IBP
  - Fast Solution
- Integration by Parts 6: $\int \sec^5 x\,dx$

04 Theory

The method of **integration by parts** (abbreviated IBP) is applicable when the integrand is a *product* for which one factor is easily integrated while the other *becomes simpler* when differentiated.

Integration by parts

Suppose the integral has this format, for some functions u and v:

$$\int u\cdot v'\,dx$$

Then the rule says we may convert the integral like this:

$$\int u\cdot v'\,dx \quad \gg\gg \quad u\cdot v - \int u'\cdot v\,dx$$

This technique comes from the **product rule for derivatives**:

$$(u\cdot v)' = u'\cdot v + u\cdot v'$$

Now, if we *integrate both sides* of this equation, we find:

$$u\cdot v = \int u'\cdot v\,dx + \int u\cdot v'\,dx$$

and the IBP rule follows by algebra.

Full explanation of integration by parts

1. Setup: functions u and v' are established.
   - Recognize functions $u(x)$ and $v'(x)$ in the integrand:

$$\int u \cdot v'\, dx$$

2. Product rule for derivatives.
   - Using primes notation:

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

3. Integrate both sides of product rule.
   - Integrate with respect to an input variable labeled 'x':

$$(u \cdot v)' = u' \cdot v + u \cdot v' \quad \overset{\int}{\ggg} \quad \int (u \cdot v)'\, dx = \int u' \cdot v\, dx + \int u \cdot v'\, dx$$

$$\overset{\text{FTC}}{\ggg} \quad u \cdot v = \int u' \cdot v\, dx + \int u \cdot v'\, dx$$

   - Rearrange with algebra:

$$\int u \cdot v'\, dx = u \cdot v - \int u' \cdot v$$

4. This is "integration by parts" in final form.

Addendum: *definite* integration by parts

3. Definite version of FTC.
   - Apply FTC to $u \cdot v$:

$$\int_a^b (u \cdot v)'\, dx = u \cdot v \Big|_a^b$$

4. Integrate the derivative product rule using specified bounds.
   - Perform definite integral on both sides, plug in definite FTC, then rearrange:

$$\int_a^b u \cdot v'\, dx = u \cdot v \Big|_a^b - \int_a^b u' \cdot v$$

Choosing factors well

IBP is symmetrical. How do we know which factor to choose for u and which for v?

Here is a trick: the acronym "LIATE" spells out the order of choices – to the left for u and to the right for v:

$$\text{LIATE}:$$

$$u \leftarrow \text{Logarithmic} - \text{Inverse\_trig} - \text{Algebraic} - \text{Trig} - \text{Exponential} \rightarrow v$$

05 Illustration

Example - A and T factors

Compute the integral: $\int x \cos x \, dx$

Solution

1. Choose $u = x$.
   - Set $u(x) = x$ because x *simplifies* when differentiated.
     (By the trick: x is *Algebraic*, i.e. more "u", and $\cos x$ is *Trig*, more "v".)
   - Remaining factor must be v':

$$v'(x) = \cos x$$

2. Compute u' and v.
   - Derive u:

$$u' = 1$$

   - Antiderive v':

$$v = \sin x$$

   - Obtain chart:

$$\begin{array}{c|c} u = x & v' = \cos x \\ \hline u' = 1 & v = \sin x \end{array} \quad \begin{array}{lll} \longrightarrow & \int u \cdot v' & \text{original} \\ \longrightarrow & \int u' \cdot v & \text{final} \end{array}$$

3. Plug into IBP formula.
   - Plug in all data:

$$\int x \cos x \, dx = x \sin x - \int 1 \cdot \sin x \, dx$$

   - Compute integral on RHS:

$$\int x \cos x \, dx = x \sin x + \cos x + C$$

   Note: the *point* of IBP is that this integral is easier than the first one!

4. Final answer is: $x \sin x + \cos x + C$

Exercise - Hidden A

Compute the integral:

$$\int \ln x \, dx$$

Solution

Trig power products

Videos, Math Dr. Bob:

- Trig power products: $\int \cos^m x \sin^n x \, dx$
- Trig differing frequencies: $\int \cos mx \sin nx \, dx$

- [Trig tan and sec]: $\int \tan^m x \sec^n x\, dx$
- [Secant power]: $\int \sec^5 x\, dx$

Videos, Organic Chemistry Tutor:

- [Trig power product techniques]
- [Trig substitution]

06 Theory

Review: trig identities

- $\sin^2 x + \cos^2 x = 1$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Trig power product: sin / cos

A sin / cos power product has this form:

$$\int \cos^m x \cdot \sin^n x\, dx$$

for some integers m and n (even negative!).

To compute these integrals, use a sequence of these techniques:

- **Swap an even bunch.**
- **u-sub for power-one.**
- **Power-to-frequency conversion.**

- Memorize these three techniques!

Examples of trig power products:

- $\displaystyle\int \sin x \cdot \cos^7 x\, dx$
- $\displaystyle\int \sin^3 x\, dx$
- $\displaystyle\int \sin^2 x \cdot \cos^2 x\, dx$

Swap an even bunch

If *either* $\cos^m x$ or $\sin^n x$ is an *odd* power, use

$$\sin^2 x \quad \gg\gg \quad 1 - \cos^2 x$$

$$\text{OR} \quad \cos^2 x \quad \gg\gg \quad 1 - \sin^2 x$$

(maybe repeatedly) to convert an **even bunch** to the opposite trig type.

An **even bunch** is *all but one* from the odd power.

For example:

$$\begin{aligned}
\sin^5 x \cdot \cos^8 x \quad &\gg\gg \quad \sin x\,(\sin^2 x)^2 \cdot \cos^8 x \\
&\gg\gg \quad \sin x\,(1-\cos^2 x)^2 \cdot \cos^8 x \\
&\gg\gg \quad \sin x\,\left(1 - 2\cos^2 x + \cos^4 x\right) \cdot \cos^8 x \\
&\gg\gg \quad \sin x\,\left(\cos^8 x - 2\cos^{10} x + \cos^{12} x\right) \\
&\gg\gg \quad \sin x \cos^8 x - 2\sin x \cos^{10} x + \sin x \cos^{12} x
\end{aligned}$$

u-sub for power-one

If $m = 1$ or $n = 1$, *perform u-substitution* to do the integral.

The *other* trig power becomes a u power; the power-one becomes du.

For example, using $u = \cos x$ and thus $du = -\sin x\,dx$ we can do:

$$\int \sin x \cos^8 x\,dx \quad \gg\gg \quad \int -\cos^8 x(-\sin x\,dx) \quad \gg\gg \quad -\int u^8\,du$$

- By combining these tricks you can do any power product with at least one odd power!
    - Leave a power-one from the odd power when swapping an even bunch.
- Notice: $1 = \sin^0 x = \cos^0 x$, even powers. So the method works for $\int \sin^3 x\,dx$ and similar.

Power-to-frequency conversion

Using these 'power-to-frequency' identities (maybe repeatedly):

$$\sin^2 x = \tfrac{1}{2}(1-\cos 2x), \qquad \cos^2 x = \tfrac{1}{2}(1+\cos 2x)$$

change an even power (either type) into an odd power of cosine.

For example, consider the power product:

$$\sin^4 x \cdot \cos^6 x$$

You can substitute appropriate powers of $\sin^2 x = \frac{1}{2}(1-\cos 2x)$ and $\cos^2 x = \frac{1}{2}(1+\cos 2x)$:

$$\begin{aligned}
\sin^4 x \cdot \cos^6 x \quad &\gg\gg \quad \left(\sin^2 x\right)^2 \cdot \left(\cos^2 x\right)^3 \\
&\gg\gg \quad \left(\tfrac{1}{2}(1-\cos 2x)\right)^2 \cdot \left(\tfrac{1}{2}(1+\cos 2x)\right)^3
\end{aligned}$$

By doing some annoying algebra, this expression can be expanded as a sum of *smaller* powers of $\cos 2x$:

$$\left(\tfrac{1}{2}(1-\cos 2x)\right)^2 \cdot \left(\tfrac{1}{2}(1+\cos 2x)\right)^3$$

$$\gg\gg \quad \frac{1}{32}\Big(1 + \cos(2x) - 2\cos^2(2x) - 2\cos^3(2x) + \cos^4(2x) + \cos^5(2x)\Big)$$

Each of these terms can be integrated by repeating the same techniques.

07 Illustration

Example - Trig power product with an odd power

Compute the integral:

$$\int \cos^2 x \cdot \sin^5 x \, dx$$

Solution

1. Swap over the even bunch.
   - Max even bunch leaving power-one is $\sin^4 x$:

$$\sin^5 x \quad \gg\gg \quad \sin x \left(\sin^2 x\right)^2 \quad \gg\gg \quad \sin x \left(1 - \cos^2 x\right)^2$$

   - Apply to $\sin^5 x$ in the integrand:

$$\int \cos^2 x \cdot \sin^5 x \, dx \quad \gg\gg \quad \int \cos^2 x \cdot \sin x \left(1 - \cos^2 x\right)^2 dx$$

2. Perform u-substitution on the power-one integrand.
   - Set $u = \cos x$.
   - Hence $du = \sin x \, dx$. Recognize this in the integrand.
   - Convert the integrand:

$$\int \cos^2 x \cdot \sin x \left(1 - \cos^2 x\right)^2 dx \quad \gg\gg \quad \int \cos^2 x \cdot \left(1 - \cos^2 x\right)^2 \left(\sin x \, dx\right)$$

$$\gg\gg \quad \int u^2 \cdot (1 - u^2)^2 \, du$$

3. Perform the integral.
   - Expand integrand and use power rule to obtain:

$$\int u^2 \cdot (1 - u^2)^2 \, du = \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$$

   - Insert definition $u = \cos x$:

$$\int \cos^2 x \cdot \sin^5 x \, dx \quad \gg\gg \quad \int u^2 \cdot (1 - u^2)^2 \, du$$

$$\gg\gg \quad \frac{1}{3}\cos^3 x - \frac{2}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$$

4. This is our final answer.

08 Theory

Trig power product: $\tan / \sec$ **or** $\cot / \csc$

A $\tan / \sec$ power product has this form:

$$\int \tan^m x \cdot \sec^n x \, dx$$

A $\cot / \csc$ power product has this form:

$$\int \cot^m x \cdot \csc^n x \, dx$$

To integrate these, **swap an even bunch** using:

- $\tan^2 x + 1 = \sec^2 x$

OR:

- $\cot^2 x + 1 = \csc^2 x$

Or do **u-substitution** using:

- $u = \tan x \ \rightsquigarrow \ du = \sec^2 x\, dx$
- $u = \sec x \ \rightsquigarrow \ du = \sec x \ \tan x\, dx$

OR:

- $u = \cot x \ \rightsquigarrow \ du = -\csc^2 x\, dx$
- $u = \csc x \ \rightsquigarrow \ du = -\csc u \ \cot u\, dx$

Note:

- ⓘ There is no simple "power-to-frequency conversion" for tan / sec !

We can modify the power-one technique to solve some of these. We need to swap over an even bunch *from the odd power* so that exactly the du factor is left behind.

Considering all the possibilities, one sees that this method works when:

- $\tan^m x$ is an *odd* power
- $\sec^n x$ is an *even* power

Quite a few cases escape this method:

- Any $\int \tan^m x\, dx$
- Any $\int \tan^m x \cdot \sec^n x\, dx$ for m even and n odd

These tricks don't work for $\int \tan x\, dx$ or $\int \sec x\, dx$ or $\int \tan^4 x \ \sec^5 x\, dx$, among others.

Special integrals: tan and sec

We have:

$$\int \tan x\, dx = \ln|\sec x| + C$$

$$\int \sec x\, dx = \ln|\sec x + \tan x| + C$$

- ⓘ These integrals should be memorized individually.

Deriving special integrals - tan and sec

The first formula can be found by u-substitution, considering that $\tan x = \frac{\sin x}{\cos x}$.

The second formula can be derived by multiplying $\sec x$ by a special "1", computing instead $\int \frac{\sec x(\sec x+\tan x)}{\sec x+\tan x}\, dx$ by expanding the numerator and doing u-sub on the denominator.

09 Illustration

Example - Trig power product with tan and sec

Compute the integral:

$$\int \tan^5 x \cdot \sec^3 x \, dx$$

Solution

1. Try $du = \sec^2 x \, dx$.
    - Factor du out of the integrand:

    $$\int \tan^5 x \cdot \sec^3 x \, dx \qquad \gg\gg \qquad \int \tan^5 x \cdot \sec x \left(\sec^2 x \, dx\right)$$

    - We then must swap over remaining $\sec x$ into the $\tan x$ type.
    - Cannot do this because $\sec x$ has odd power. Need even to swap.
2. Try $du = \sec x \tan x \, dx$.
    - Factor du out of the integrand:

    $$\int \tan^5 x \cdot \sec^3 x \, dx \qquad \gg\gg \qquad \int \tan^4 x \cdot \sec^2 x \left(\sec x \tan x \, dx\right)$$

    - Swap remaining $\tan x$ into $\sec x$ type:

    $$\int (\tan^2 x)^2 \cdot \sec^2 x \left(\sec x \tan x \, dx\right)$$

    $$\gg\gg \quad \int \left(\sec^2 x - 1\right)^2 \cdot \sec^2 x \left(\sec x \tan x \, dx\right)$$

    - Substitute $u = \sec x$ and $du = \sec x \tan x \, dx$:

    $$\gg\gg \qquad \int (u^2 - 1)^2 \cdot u^2 \, du$$

3. Compute the integral in u and convert back to x.
    - Expand the integrand:

    $$\gg\gg \qquad \int u^6 - 2u^4 + u^2 \, du$$

    - Apply power rule:

    $$\gg\gg \qquad \frac{u^7}{7} - 2\frac{u^5}{5} + \frac{u^3}{3} + C$$

    - Plug back in, $u = \sec x$:

    $$\gg\gg \qquad \frac{\sec^7 x}{7} - 2\frac{\sec^5 x}{5} + \frac{\sec^3 x}{3} + C$$

4. This is our final answer.

Trig substitution

Videos, Math Dr. Bob:

- Trig sub 1: Basics and $\int \frac{1}{\sqrt{36-x^2}}\,dx$ and $\int \frac{x}{36+x^2}\,dx$ and $\int \frac{1}{\sqrt{x^2-36}}\,dx$
- Trig sub 2: $\int \frac{dx}{(1+x^2)^{5/2}}$
- Trig sub 3: $\int \frac{x^2}{\sqrt{1-4x^2}}\,dx$
- Trig sub 4: $\int \sqrt{e^{2x}-1}\,dx$
- Trig sub 5: $\int \frac{\sqrt{4-36x^2}}{x^2}\,dx$

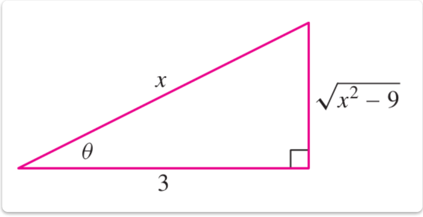
10 Theory

Certain algebraic expressions have a secret meaning that comes from the Pythagorean Theorem. This meaning has a very simple expression in terms of trig functions of a certain angle.

For example, consider the integral:

$$\int \frac{1}{x^2\sqrt{x^2-9}}\,dx$$

Now consider this triangle:



The triangle determines the relation $x = 3\sec\theta$, and it implies $\sqrt{x^2-9} = 3\tan\theta$.

Now plug these into the integrand above:

$$\frac{1}{x^2\sqrt{x^2-9^2}} \quad \gg\gg \quad \frac{1}{9\sec^2\theta\cdot 3\tan\theta}$$

Considering that $dx = 3\sec\theta\,\tan\theta\,d\theta$, we obtain a very reasonable trig integral:

$$\int \frac{1}{x^2\sqrt{x^2-9^2}}\,dx \quad \gg\gg \quad \int \frac{3\sec\theta\,\tan\theta}{27\sec^2\theta\,\tan\theta}\,d\theta$$

$$\gg\gg \quad \frac{1}{9}\int \cos\theta\,d\theta \quad \gg\gg \quad \frac{1}{9}\sin\theta + C$$

We must rewrite this in terms of x using $x = 3\sec\theta$ to finish the problem. We need to find $\sin\theta$ assuming that $\sec\theta = \frac{x}{3}$. To do this, refer back to the triangle to see that $\sin\theta = \frac{\sqrt{x^2-9}}{x}$. Plug this in for our final value of the integral:

$$\frac{1}{9}\sin\theta + C \quad \gg\gg \quad \frac{\sqrt{x^2-9}}{9x} + C$$

Here is the moral of the story:

- Re-express the Pythagorean expression *using a triangle and a trig substitution*.
  - In this way, square roots of quadratic polynomials can be eliminated.

There are always three steps for these trig sub problems:

- (1) Identify the trig sub: find the sides of a triangle and relevant angle θ.
- (2) Solve a trig integral (often a power product).
- (3) Refer back to the triangle to convert the answer back to x.

To speed up your solution process for these problems, *memorize* these three transformations:

(1)

$$\sqrt{a^2 - x^2} \quad \overset{x=a\sin\theta}{\gg\gg} \quad \sqrt{a^2 - a^2\sin^2\theta} = a\cos\theta \qquad \text{from} \quad 1 - \sin^2\theta = \cos^2\theta$$

(2)

$$\sqrt{a^2 + x^2} \quad \overset{x=a\tan\theta}{\gg\gg} \quad \sqrt{a^2 + a^2\tan^2\theta} = a\sec\theta \qquad \text{from} \quad 1 + \tan^2\theta = \sec^2\theta$$

(3)

$$\sqrt{x^2 - a^2} \quad \overset{x=a\sec\theta}{\gg\gg} \quad \sqrt{a^2\sec^2\theta - a^2} = a\tan\theta \qquad \text{from} \quad \sec^2\theta - 1 = \tan^2\theta$$

For a more complex quadratic with linear and constant terms, you will need to first *complete the square* for the quadratic and then do the trig substitution.

11 Illustration

Example - Trig sub in quadratic: completing the square

Compute the integral:

$$\int \frac{dx}{\sqrt{x^2 - 6x + 11}}$$

Solution

1. Notice square root of a quadratic.
2. Complete the square to obtain Pythagorean form.
   - Find constant term for a complete square:

$$x^2 - 6x + \left(\frac{-6}{2}\right)^2 = x^2 - 6x + 9 = (x-3)^2$$

   - Add and subtract desired constant term:

$$x^2 - 6x + 11 \qquad \gg\gg \qquad x^2 - 6x + 9 - 9 + 11$$

   - Simplify:

$$x^2 - 6x + 9 - 9 + 11 \qquad \gg\gg \qquad (x-3)^2 + 2$$

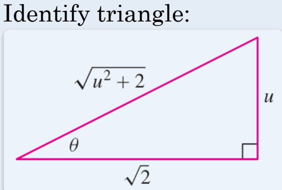
3. Perform shift substitution.
   - Set $u = x - 3$ as inside the square:

$$(x-3)^2 + 2 = u^2 + 2$$

   - Infer $du = dx$.
   - Plug into integrand:

$$\int \frac{dx}{\sqrt{x^2 - 6x + 11}} \qquad \gg\gg \qquad \int \frac{du}{\sqrt{u^2 + 2}}$$

4. Trig sub with $\tan\theta$.
   - Identify triangle:



- Use substitution $u = \sqrt{2}\tan\theta$. (From triangle or memorized tip.)
- Infer $du = \sqrt{2}\sec^2\theta\,d\theta$.
- Plug in data:

$$\int \frac{du}{\sqrt{u^2+2}} \qquad \gg\gg \qquad \int \frac{\sec^2\theta}{\sec\theta}\,d\theta = \int \sec\theta\,d\theta$$

5. Compute trig integral.
   - Use ad hoc formula:

$$\int \sec\theta\,d\theta = \ln|\tan\theta + \sec\theta| + C$$

6. Convert trig back to x.
   - First in terms of u, referring to the triangle:

$$\tan\theta = \frac{u}{\sqrt{2}}, \qquad \sec\theta = \frac{\sqrt{u^2+2}}{\sqrt{2}}$$

   - Then in terms of x using $u = x - 3$.
   - Plug everything in:

$$\ln|\tan\theta + \sec\theta| + C \qquad \gg\gg \qquad \ln\left|\frac{x-3}{\sqrt{2}} + \frac{\sqrt{(x-3)^2+2}}{\sqrt{2}}\right| + C$$

7. Simplify using log rules.
   - Log rule for division gives us:

$$\ln\frac{f(x)}{a} = \ln f(x) - \ln a$$

   - The common denominator $\frac{1}{\sqrt{2}}$ can be pulled outside as $-\ln\sqrt{2}$.
   - The new term $-\ln\sqrt{2}$ can be "absorbed into the constant" (redefine C).
   - So we write our final answer thus:

$$\ln\left|x - 3 + \sqrt{(x-3)^2+2}\right| + C$$