Volume using cylindrical shells

01 - Revolution of a triangle

Problem: A rotation-symmetric 3D body has cross section given by the region between $y = 3x + 2$, $y = 6 - x$, $x = 0$, and is rotated around the y-axis. Find the volume of this 3D body.

Solution:

1. Define the cross section region.
   - Bounded above-right by $y = 6 - x$.
   - Bounded below-left by $y = 3x + 2$.
   - These intersect at $x = 1$.
   - Bounded at left by $x = 0$.
2. Define range of integration variable.
   - Rotated around y-axis, therefore use x for integration variable (shells!).
   - Integral over $x \in [0, 1]$:

$$V = \int_0^2 2\pi R h \, dr$$

3. Interpret R.
   - Radius of shell-cylinder equals distance along x:

$$R(x) = x$$

4. Interpret h.
   - Height of shell-cylinder equals distance from lower to upper bounding lines:

$$\begin{aligned} h(x) &= (6 - x) - (3x + 2) \\ &= 4 - 4x \end{aligned}$$

5. Interpret dr.
   - dr is limit of Δr which equals Δx here so $dr = dx$.
6. Plug data in volume formula.
   - Insert data and compute integral:

$$\begin{aligned} V &= \int_0^2 2\pi R h \, dr \\ &= \int_0^2 2\pi \cdot x(4 - 4x) \, dx \\ &= 2\pi \left(2x^2 - \frac{4x^3}{3} \right) \Bigg|_0^1 = \frac{4\pi}{3} \end{aligned}$$

Integration by parts

02 - A and T factors

Problem: Compute the integral: $\displaystyle\int x \cos x \, dx$

Solution:

1. Choose $u = x$.
   - Set $u(x) = x$ because x *simplifies* when differentiated.
     (By the trick: x is *Algebraic*, i.e. more "u", and $\cos x$ is *Trig*, more "v".)
   - Remaining factor must be v':

   $$v'(x) = \cos x$$

2. Compute u' and v.
   - Derive u:

   $$u' = 1$$

   - Antiderive v':

   $$v = \sin x$$

   - Obtain chart:

   $$\begin{array}{c|c} u = x & v' = \cos x \\ \hline u' = 1 & v = \sin x \end{array} \quad \begin{array}{ccc} \longrightarrow & \int u \cdot v' & \text{original} \\ \longrightarrow & \int u' \cdot v & \text{final} \end{array}$$

3. Plug into IBP formula.
   - Plug in all data:

   $$\int x \cos x \, dx = x \sin x - \int 1 \cdot \sin x \, dx$$

   - Compute integral on RHS:

   $$\int x \cos x \, dx = x \sin x + \cos x + C$$

   Note: the *point* of IBP is that this integral is easier than the first one!

4. Final answer is: $x \sin x + \cos x + C$