W01 - Examples

Events and outcomes

01

Example - Coin flipping

Flip a fair coin two times and record both results.

- *Outcomes:* sequences, like HH or TH.
- *Sample space:* all possible sequences, i.e. the set $S = \{HH, HT, TH, TT\}$.
- *Events:* for example:
  - $A = \{HH, HT\} =$ "first was heads"
  - $B = \{HT, TH\} =$ "exactly one heads"
  - $C = \{HT, TH, HH\} =$ "at least one heads"

With this setup, we may combine events in various ways to generate other events:

- *Complex events:* for example:
  - $A \cap B = \{HT\}$, or in words:

    $$\text{"first was heads" AND "exactly one heads"} = \text{"heads-then-tails"}$$

    Notice that the last one is a *complete description*, namely the *outcome* HT.
  - $A \cup B = \{HH, HT, TH\}$, or in words:

    $$\begin{aligned} &\text{"first was heads" OR "exactly one heads"} \\ &= \text{"starts with heads, else it's tails-then-heads"} \end{aligned}$$

02

Exercise - Coin flipping: counting subsets

Flip a fair coin five times and record the results.

How many elements are in the sample space? (How big is S?)
How many events are there? (How big is $\mathcal{F}$?)

Solution ›

There are $2^5 = 32$ possible sequences, so $|S| = 32$.

To count the number of possible subsets, consider that we have 32 distinct items, and a subset is uniquely determined by the binary information – for each item – of whether it is in or out. Thus there are 2^{32} possibilities. So $|\mathcal{F}| = 2^{32}$.

Probability models

03

Example - Lucia is Host or Player

Problem: The professor chooses three students at random for a game in a class of 40, one to be Host, one to be Player, one to be Judge. What is the probability that Lucia is either Host or Player?

Solution:

1. Set up the probability model.
    - Label the students 1 to 40. Write L for Lucia's number.
    - *Outcomes:* assignments such as $(H, P, J) = (2, 5, 8)$
      These are ordered triples with *distinct* entries in 1, 2, $\dots$, 40.
    - *Sample space:* S is the collection of all such distinct triples
    - *Events:* any subset of S
    - *Probability measure*: assume all outcomes are equally likely, so $P[(i, j, k)] = P[(r, l, p)]$ for all i, j, k, r, l, p
    - In total there are $40 \cdot 39 \cdot 38$ triples of distinct numbers.
    - Therefore $P[(i, j, k)] = \frac{1}{40 \cdot 39 \cdot 38}$ for any *specific* outcome (i, j, k).
    - Therefore $P[A] = \frac{|A|}{40 \cdot 39 \cdot 38}$ for any event A. (Recall $|A|$ is the *number* of outcomes in A.)
2. Define the desired event.
    - Want to find $P[\text{“Lucia is Host or Player”}]$
    - Define $A =$ “Lucia is Host” and $B =$ “Lucia is Player”. Thus:

$$A = \big\{(L, j, k) \mid \text{any } j, k\big\}, \qquad B = \big\{(i, L, k) \mid \text{any } i, k\big\}$$

    - So we seek $P[A \cup B]$.
3. Compute the desired probability.
    - Importantly, $A \cap B = \emptyset$ (mutually exclusive).
      There are no outcomes in S in which Lucia is *both* Host and Player.
    - By *additivity*, we infer $P[A \cup B] = P[A] + P[B]$.
    - Now compute $P[A]$.

- There are $39 \cdot 38$ ways to choose j and k from the students besides Lucia.
- Therefore $|A| = 39 \cdot 38$.
- Therefore:

$$P[A] \quad \gg\gg \quad \frac{|A|}{40 \cdot 39 \cdot 38} \quad \gg\gg \quad \frac{39 \cdot 38}{40 \cdot 39 \cdot 38} \quad \gg\gg \quad \frac{1}{40}$$

- Now compute $P[B]$. It is similar: $P[B] = \frac{1}{40}$.
- Finally compute that $P[A] + P[B] = \frac{1}{20}$, so the answer is:

$$P[A \cup B] \quad \gg\gg \quad P[A] + P[B] \quad \gg\gg \quad \frac{1}{20}$$

04

Example - iPhones and iPads

Problem:
At Mr. Jefferson's University, 25% of students have an iPhone, 30% have an iPad, and 60% have neither.

What is the probability that a randomly chosen student has either iProduct? (Q1) What about both? (Q2)

Solution:

1. Set up the probability model.
   - A student is chosen at random: an *outcome* is the chosen student.
   - *Sample space* S is the set of all students.
   - Write $O =$ "has iPhone" and $A =$ "has iPad" concerning the chosen student.
   - All students are equally likely to be chosen: therefore $P[E] = \frac{|E|}{|S|}$ for any event E.
   - Therefore $P[O] = 0.25$ and $P[A] = 0.30$.
   - Furthermore, $P[O^c A^c] = 0.60$. This means 60% have "not iPhone AND not iPad".
2. Define the desired event.
   - Q1: desired event $= O \cup A$
   - Q2: desired event $= OA$
3. Compute the probabilities.
   - We do not believe O and A are exclusive.
   - Try: apply inclusion-exclusion:

$$P[O \cup A] = P[O] + P[A] - P[OA]$$

- We know $P[O] = 0.25$ and $P[A] = 0.30$. So this formula, with given data, RELATES Q1 and Q2.
- Notice the complements in $O^c A^c$ and try *Negativity*.
- *Negativity:*

$$P[(OA)^c] = 1 - P[OA]$$

  DOESN'T HELP.
- Try again: *Negativity:*

$$P[(O^c A^c)^c] = 1 - P[O^c A^c]$$

- And De Morgan (or a Venn diagram!):

$$(O^c A^c)^c \quad \gg\gg \quad O \cup A$$

- Therefore:

$$P[O \cup A] \quad \gg\gg \quad P[(O^c A^c)^c]$$

$$\gg\gg \quad 1 - P[O^c A^c] \quad \gg\gg \quad 1 - 0.6 = 0.4$$

- We have found Q1: $P[O \cup A] = 0.40$.
- Applying the RELATION from inclusion-exclusion, we get Q2:

$$P[O \cup A] = P[O] + P[A] - P[OA]$$

$$\gg\gg \quad 0.40 = 0.25 + 0.30 - P[OA]$$

$$\gg\gg \quad P[OA] = 0.15$$