W01 - Examples

Events and outcomes

> ### ≡ Example - Coin flipping
>
> Flip a fair coin two times and record both results.
>
> - *Outcomes:* sequences, like HH or TH.
> - *Sample space:* all possible sequences, i.e. the set
> $S = \{HH, HT, TH, TT\}$.
> - *Events:* for example:
>   - $A = \{HH, HT\} =$ "first was heads"
>   - $B = \{HT, TH\} =$ "exactly one heads"
>   - $C = \{HT, TH, HH\} =$ "at least one heads"
>
> With this setup, we may combine events in various ways to generate other events:
>
> - *Complex events:* for example:
>   - $A \cap B = \{HT\}$, or in words:
>
>   "first was heads" AND "exactly one heads" = "heads-then-tails"
>
>   Notice that the last one is a *complete description*, namely the *outcome HT*.
>   - $A \cup B = \{HH, HT, TH\}$, or in words:
>
>   "first was heads" OR "exactly one heads"
>   = "starts with heads, else it's tails-then-heads"

> ### ✎ Exercise - Coin flipping: counting subsets
>
> Flip a fair coin five times and record the results.
>
> How many elements are in the sample space? (How big is S?)
> How many events are there? (How big is $\mathcal{F}$?)
>
> Solution

Probability models

> ### ≡ Example - Lucia is Host or Player

The professor chooses three students at random for a game in a class of 40, one to be Host, one to be Player, one to be Judge. What is the probability that Lucia is either Host or Player?

Solution ⌄

1. ⧉ Set up the probability model.
   - Label the students 1 to 40. Write L for Lucia's number.
   - *Outcomes:* assignments such as $(H, P, J) = (2, 5, 8)$
     These are ordered triples with *distinct* entries in 1, 2, …, 40.
   - *Sample space:* S is the collection of all such distinct triples
   - *Events:* any subset of S
   - *Probability measure*: assume all outcomes are equally likely, so $P[(i, j, k)] = P[(r, l, p)]$ for all i, j, k, r, l, p
   - In total there are $40 \cdot 39 \cdot 38$ triples of distinct numbers.
   - Therefore $P[(i, j, k)] = \frac{1}{40 \cdot 39 \cdot 38}$ for any *specific* outcome (i, j, k) .
   - Therefore $P[A] = \frac{|A|}{40 \cdot 39 \cdot 38}$ for any event A. (Recall $|A|$ is the *number* of outcomes in A.)

2. ⇉ Define the desired event.
   - Want to find $P[\text{"Lucia is Host or Player"}]$
   - Define $A = $ "Lucia is Host" and $B = $ "Lucia is Player". Thus:
     $$A = \{(L, j, k) \mid \text{any } j, k\}, \qquad B = \{(i, L, k) \mid \text{any } i, k\}$$
   - So we seek $P[A \cup B]$.

3. ⧉ Compute the desired probability.
   - Importantly, $A \cap B = \emptyset$ (mutually exclusive).
     There are no outcomes in S in which Lucia is *both* Host and Player.
   - By *additivity*, we infer $P[A \cup B] = P[A] + P[B]$.
   - Now compute $P[A]$.
     - There are $39 \cdot 38$ ways to choose j and k from the students besides Lucia.
     - Therefore $|A| = 39 \cdot 38$.
     - Therefore:
       $$P[A] \quad \gg\gg \quad \frac{|A|}{40 \cdot 39 \cdot 38} \quad \gg\gg \quad \frac{39 \cdot 38}{40 \cdot 39 \cdot 38} \quad \gg\gg \quad \frac{1}{40}$$
   - Now compute $P[B]$. It is similar: $P[B] = \frac{1}{40}$.
   - Finally compute that $P[A] + P[B] = \frac{1}{20}$, so the answer is:
     $$P[A \cup B] \quad \gg\gg \quad P[A] + P[B] \quad \gg\gg \quad \frac{1}{20}$$

≡ **Example - iPhones and iPads**

At Mr. Jefferson's University, 25% of students have an iPhone, 30% have an iPad, and 60% have neither.

What is the probability that a randomly chosen student has *some* iProduct? (Q1)

What about *both*? (Q2)

≡ **Solution** ∨

1. ⌷ Set up the probability model.
   - A student is chosen at random: an *outcome* is the chosen student.
   - *Sample space* S is the set of all students.
   - Write $O =$ "has iPhone" and $A =$ "has iPad" concerning the chosen student.
   - All students are equally likely to be chosen: therefore $P[E] = \frac{|E|}{|S|}$ for any event E.
   - Therefore $P[O] = 0.25$ and $P[A] = 0.30$.
   - Furthermore, $P[O^c A^c] = 0.60$. This means 60% have "not iPhone AND not iPad".

2. ≡ Define the desired event.
   - Q1: desired event $= O \cup A$
   - Q2: desired event $= OA$

3. ⌷ Compute the probabilities.
   - We do not believe O and A are exclusive.
   - Try: apply inclusion-exclusion:

   $$P[O \cup A] = P[O] + P[A] - P[OA]$$

   - We know $P[O] = 0.25$ and $P[A] = 0.30$. So this formula, with given data, RELATES Q1 and Q2.
   - Notice the complements in $O^c A^c$ and try *Negation*.
   - *Negation:*

   $$P[(OA)^c] = 1 - P[OA]$$

   DOESN'T HELP.
   - Try again: Negation:*

   $$P[(O^c A^c)^c] = 1 - P[O^c A^c]$$

   - And De Morgan (or a Venn diagram!):

   $$(O^c A^c)^c \quad \gg\gg \quad O \cup A$$

- Therefore:

$$P[O \cup A] \quad \gg \gg \quad P[(O^c A^c)^c]$$

$$\gg \gg \quad 1 - P[O^c A^c] \quad \gg \gg \quad 1 - 0.6 = 0.4$$

- We have found Q1: $P[O \cup A] = 0.40$.
- Applying the RELATION from inclusion-exclusion, we get Q2:

$$P[O \cup A] = P[O] + P[A] - P[OA]$$

$$\gg \gg \quad 0.40 = 0.25 + 0.30 - P[OA]$$

$$\gg \gg \quad P[OA] = 0.15$$