Unit 01 notes

Volume using cylindrical shells

Review

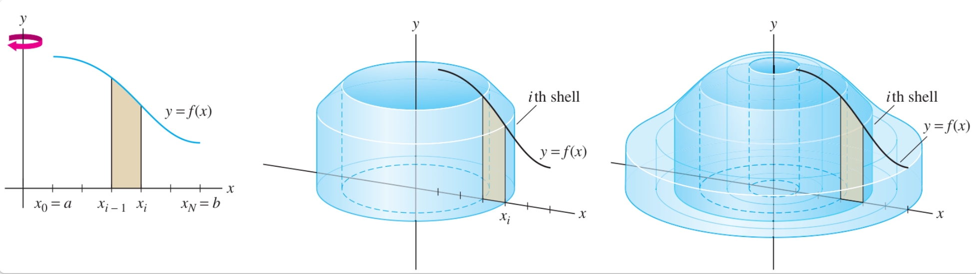
- Volume using cross-sectional area
- Disk/washer method - 01
- Disk/washer method - 02
- Disk/washer method - 03

Shells

- Shell method - 01
- Shell method - 02
- Shell method - 03

01 Theory

Take a graph $y = f(x)$ in the first quadrant of the xy-plane. Rotate this about the y-axis. The resulting 3D body is symmetric around the axis. We can find the volume of this body by using an integral to add up the volumes of infinitesimal **shells**, where each shell is a *thin cylinder*.



The volume of each cylindrical shell is $2\pi\, R\, h\, \Delta r$:

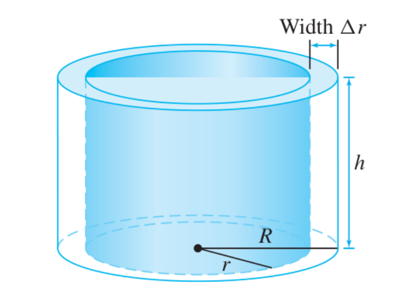


FIGURE 1 The volume of the cylindrical shell is approximately

$$2\pi Rh\Delta r$$

where $\Delta r = R - r$.

In the limit as $\Delta r \to dr$ and the number of shells becomes infinite, their total volume is given by an integral.

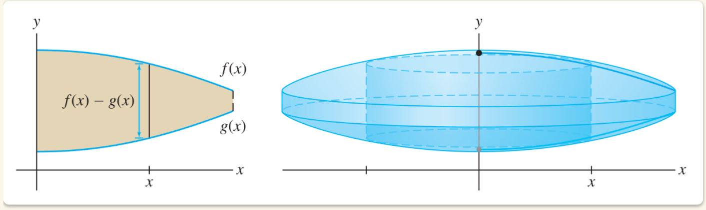
Volume by shells - general formula

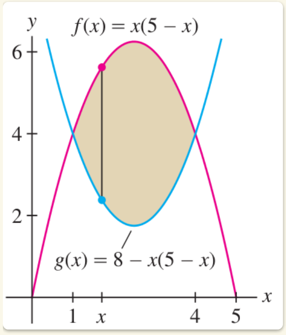
$$V = \int_a^b 2\pi Rh\, dr$$

In any concrete volume calculation, we simply interpret each factor, ‘R’ and ‘h’ and ‘dr’, and determine a and b in terms of the variable of integration that is set for r.

Shells vs. washers

Can you see why shells are sometimes easier to use than washers?





02 Illustration

Example - Revolution of a triangle

A rotation-symmetric 3D body has cross section given by the region between $y = 3x + 2$, $y = 6 - x$, $x = 0$, and is rotated around the y-axis. Find the volume of this 3D body.

Solution

1. Define the cross section region.
    - Bounded above-right by $y = 6 - x$.
    - Bounded below-right by $y = 3x + 2$.
    - These intersect at $x = 1$.
    - Bounded at left by $x = 0$.
2. Define range of integration variable.
    - Rotated around y-axis, therefore use x for integration variable (shells!).
    - Integral over $x \in [0, 1]$:

$$V = \int_0^1 2\pi R h \, dr$$

3. Interpret R.
    - Radius of shell-cylinder equals distance along x:

$$R(x) = x$$

4. Interpret h.

- Height of shell-cylinder equals distance from lower to upper bounding lines:

$$\begin{aligned} h(x) &= (6-x)-(3x+2) \\ &= 4-4x \end{aligned}$$

5. Interpret dr.
   - dr is limit of Δr which equals Δx here so $dr = dx$.
6. Plug data in volume formula.
   - Insert data and compute integral:

$$\begin{aligned} V &= \int_0^1 2\pi Rh\,dr \\ &= \int_0^1 2\pi \cdot x(4-4x)\,dx \\ &= 2\pi\left(2x^2 - \frac{4x^3}{3}\right)\bigg|_0^1 = \frac{4\pi}{3} \end{aligned}$$

Exercise - Revolution of a sinusoid

Consider the region given by revolving the first hump of $y = \sin(x)$ about the y-axis. Set up an integral that gives the volume of this region using the method of shells.

Solution

Integration by substitution

[Note: this section is non-examinable. It is included for comparison to IBP.]

- Integration by Substitution 1
- Integration by Substitution 2
- Integration by Substitution 3
- Integration by Substitution 4
- Integration by Substitution 5
- Integration by Substitution: Definite Integrals

03 Theory

The method of **u-substitution** is applicable when the integrand is a *product*, with one factor a composite whose *inner function's derivative* is the other factor.

u-substitution

Suppose the integral has this format, for some functions f and u:

$$\int f(u(x)) \cdot u'(x)\,dx$$

Then the rule says we may convert the integral into terms of u considered as a variable, like this:

$$\int f(u(x)) \cdot u'(x)\, dx \quad \gg\gg \quad \int f(u)\, du$$

The technique of u-substitution comes from the **chain rule for derivatives**:

$$\frac{d}{dx}F\big(u(x)\big) = f(u(x)) \cdot u'(x)$$

Here we let $F' = f$. Thus $\int f(x)\, dx = F(x) + C$ for some C.

Now, if we *integrate both sides* of this equation, we find:

$$F\big(u(x)\big) = \int f(u(x)) \cdot u'(x)\, dx$$

And of course $F\big(u\big) = \int f(u)\, du - C$.

Full explanation of u-substitution

The substitution method comes from the **chain rule for derivatives**. The rule simply comes from *integrating on both sides* of the chain rule.

1. Setup: functions $F' = f$ and $u(x)$.
   - Let F and f be any functions satisfying $F' = f$, so F is an antiderivative of f.
   - Let u be another *function* and take x for its independent variable, so we can write $u(x)$.
2. The chain rule for derivatives.
   - Using primes notation:

   $$\big(F \circ u\big)' = (F' \circ u) \cdot u'$$

   - Using differentials in variables:

   $$\frac{d}{dx}F\big(u(x)\big) = f(u(x)) \cdot u'(x)$$

3. Integrate both sides of chain rule.
   - Integrate with respect to x:

   $$\frac{d}{dx}F\big(u(x)\big) = f(u(x)) \cdot u'(x) \quad \overset{\int}{\gg\gg} \quad \int \frac{d}{dx}F\big(u(x)\big) = \int f(u(x)) \cdot u'(x)$$

   $$\overset{\text{FTC}}{\gg\gg} \quad F(u(x)) = \int f(u(x)) \cdot u'(x)$$

4. Introduce 'variable' u from the u-format of the integral.
   - Treating u as a variable, the definition of F gives:

   $$F(u) = \int f(u)\, du + C$$

   - Set the 'variable' u to the 'function' u output:

   $$F(u)\Big|_{u=u(x)} = F(u(x))$$

- Combining these:

$$F(u(x)) = F(u)\Big|_{u=u(x)}$$

$$= \int f(u)\,du\,\Big|_{u=u(x)} + C$$

5. Substitute for $F(u(x))$ in the integrated chain rule.
    - Reverse the equality and plug in:

$$\int f(u(x)) \cdot u'(x)\,dx = F(u(x)) = \int f(u)\,du\,\Big|_{u=u(x)} + C$$

6. This is "u-substitution" in final form.

Integration by parts

Videos:

- Integration by Parts 1
- Integration by Parts 2
- Integration by Parts 3
- Example: $\int e^{3x} \cos 4x\,dx$, two methods:
    - Double IBP
    - Fast Solution

04 Theory

The method of **integration by parts** (abbreviated IBP) is applicable when the integrand is a *product* for which one factor is easily integrated while the other *becomes simpler* when differentiated.

Integration by parts

Suppose the integral has this format, for some functions u and v:

$$\int u \cdot v'\,dx$$

Then the rule says we may convert the integral like this:

$$\int u \cdot v'\,dx \quad \gg\gg \quad u \cdot v - \int u' \cdot v\,dx$$

This technique comes from the **product rule for derivatives**:

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

Now, if we *integrate both sides* of this equation, we find:

$$u \cdot v = \int u' \cdot v\,dx + \int u \cdot v'\,dx$$

and the IBP rule follows by algebra.

Full explanation of integration by parts

1. Setup: functions u and v' are established.

- Recognize functions $u(x)$ and $v'(x)$ in the integrand:

$$\int u \cdot v' \, dx$$

2. Product rule for derivatives.
   - Using primes notation:

$$(u \cdot v)' = u' \cdot v + u \cdot v'$$

3. Integrate both sides of product rule.
   - Integrate with respect to an input variable labeled 'x':

$$(u \cdot v)' = u' \cdot v + u \cdot v' \quad \overset{\int}{\ggg} \quad \int (u \cdot v)' \, dx = \int u' \cdot v \, dx + \int u \cdot v' \, dx$$

$$\overset{\text{FTC}}{\ggg} \quad u \cdot v = \int u' \cdot v \, dx + \int u \cdot v' \, dx$$

   - Rearrange with algebra:

$$\int u \cdot v' \, dx = u \cdot v - \int u' \cdot v$$

4. This is "integration by parts" in final form.

Addendum: *definite* integration by parts

3. Definite version of FTC.
   - Apply FTC to $u \cdot v$:

$$\int_a^b (u \cdot v)' \, dx = u \cdot v \Big|_a^b$$

4. Integrate the derivative product rule using specified bounds.
   - Perform definite integral on both sides, plug in definite FTC, then rearrange:

$$\int_a^b u \cdot v' \, dx = u \cdot v \Big|_a^b - \int_a^b u' \cdot v$$

Choosing factors well

IBP is symmetrical. How do we know which factor to choose for u and which for v?

Here is a trick: the acronym "LIATE" spells out the order of choices – to the left for u and to the right for v:

$$\text{LIATE}:$$

$$u \; \leftarrow \text{Logarithmic} - \text{Inverse\_trig} - \text{Algebraic} - \text{Trig} - \text{Exponential} \rightarrow v$$

05 Illustration

Example - A and T factors

Compute the integral: $\displaystyle\int x \cos x \, dx$

Solution

1. Choose $u = x$.
   - Set $u(x) = x$ because x *simplifies* when differentiated.
     (By the trick: x is *Algebraic*, i.e. more "u", and $\cos x$ is *Trig*, more "v".)
   - Remaining factor must be v':
   $$v'(x) = \cos x$$
2. Compute u' and v.
   - Derive u:
   $$u' = 1$$
   - Antiderive v':
   $$v = \sin x$$
   - Obtain chart:
   $$\begin{array}{c|c} u = x & v' = \cos x \\ \hline u' = 1 & v = \sin x \end{array} \quad \begin{array}{ccc} \longrightarrow & \int u \cdot v' & \text{original} \\ \longrightarrow & \int u' \cdot v & \text{final} \end{array}$$
3. Plug into IBP formula.
   - Plug in all data:
   $$\int x \cos x \, dx = x \sin x - \int 1 \cdot \sin x \, dx$$
   - Compute integral on RHS:
   $$\int x \cos x \, dx = x \sin x + \cos x + C$$
   Note: the *point* of IBP is that this integral is easier than the first one!
4. Final answer is: $x \sin x + \cos x + C$

Exercise - Hidden A

Compute the integral:

$$\int \ln x \, dx$$

Solution

Trig power products

06 Theory

Review: trig identities

- $\sin^2 x + \cos^2 x = 1$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Trig power product: $\sin x$ and $\cos x$

A sin / cos power product has this form:

$$\int \sin^m x \cdot \cos^n x \, dx$$

for some integers m and n (even negative!).

To compute these integrals, use a sequence of these techniques:

- **Swap an even bunch.**
- **u-sub for power-one.**
- **Power-to-frequency conversion.**

Memorize these three techniques!

Examples of trig power products:

- $\displaystyle\int \sin x \cdot \cos^7 x \, dx$
- $\displaystyle\int \sin^3 x \, dx$
- $\displaystyle\int \sin^2 x \cdot \cos^2 x \, dx$

Swap an even bunch

If *either* $\sin^m x$ or $\cos^n x$ is an *odd* power, use

$$\begin{aligned} &\sin^2 x \quad \gg\gg \quad 1 - \cos^2 x \\ \text{OR} \quad &\cos^2 x \quad \gg\gg \quad 1 - \sin^2 x \end{aligned}$$

(maybe repeatedly) to convert an **even bunch** to the opposite trig type.

An **even bunch** is *all but one* from the odd power.

For example:

$$\begin{aligned} \sin^5 x \cdot \cos^8 x \quad &\gg\gg \quad \sin x \, (\sin^2 x)^2 \cdot \cos^8 x \\ &\gg\gg \quad \sin x \, (1 - \cos^2 x)^2 \cdot \cos^8 x \\ &\gg\gg \quad \sin x \left(1 - 2\cos^2 x + \cos^4 x\right) \cdot \cos^8 x \\ &\gg\gg \quad \sin x \left(\cos^8 x - 2\cos^{10} x + \cos^{12} x\right) \\ &\gg\gg \quad \sin x \cos^8 x - 2 \sin x \cos^{10} x + \sin x \cos^{12} x \end{aligned}$$

u-sub for power-one

If $m = 1$ or $n = 1$, *perform u-substitution* to do the integral.

The *other* trig power becomes a u power; the power-one becomes du.

For example, using $u = \cos x$ and thus $du = -\sin x \, dx$ we can do:

$$\int \sin x \cos^8 x \, dx \quad \gg\gg \quad \int -\cos^8 x(-\sin x \, dx) \quad \gg\gg \quad -\int u^8 \, du$$

- By combining these tricks you can do any power product with at least one odd power!
- Notice: $1 = \sin^0 x = \cos^0 x$, even powers. So the method works for $\int \sin^3 x \, dx$ and similar.

Power-to-frequency conversion

Using these 'power-to-frequency' identities (maybe repeatedly):

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x), \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

change an even power (either type) into an odd power of cosine.

For example, consider the power product:

$$\sin^4 x \cdot \cos^6 x$$

You can substitute appropriate powers of $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$:

$$\sin^4 x \cdot \cos^6 x \qquad \gg\gg \qquad \left(\sin^2 x\right)^2 \cdot \left(\cos^2 x\right)^3$$

$$\gg\gg \qquad \left(\frac{1}{2}(1 - \cos 2x)\right)^2 \cdot \left(\frac{1}{2}(1 + \cos 2x)\right)^3$$

By doing some annoying algebra, this expression can be expanded as a sum of *smaller* powers of $\cos 2x$:

$$\left(\frac{1}{2}(1 - \cos 2x)\right)^2 \cdot \left(\frac{1}{2}(1 + \cos 2x)\right)^3$$

$$\gg\gg \quad \frac{1}{32}\Big(1 + \cos(2x) - 2\cos^2(2x) - 2\cos^3(2x) + \cos^4(2x) + \cos^5(2x)\Big)$$

Each of these terms can be integrated by repeating the same techniques.

07 Illustration

Example - Trig power product with an odd power

Compute the integral:

$$\int \sin^5 x \cdot \cos^2 x \, dx$$

Solution

1. Swap over the even bunch.
    - Max even bunch leaving power-one is $\sin^4 x$:

    $$\sin^5 x \qquad \gg\gg \qquad \sin x \left(\sin^2 x\right)^2 \qquad \gg\gg \qquad \sin x \left(1 - \cos^2 x\right)^2$$

    - Apply to $\sin^5 x$ in the integrand:

    $$\int \sin^5 x \cdot \cos^2 x \, dx \qquad \gg\gg \qquad \int \sin x \left(1 - \cos^2 x\right)^2 \cdot \cos^2 x \, dx$$

2. Perform u-substitution on the power-one integrand.
    - Set $u = \cos x$.
    - Hence $du = \sin x \, dx$. Recognize this in the integrand.

- Convert the integrand:

$$\int \sin x \left(1-\cos^2 x\right)^2 \cdot \cos^2 x \, dx \quad \gg\gg \quad \int \left(1-\cos^2 x\right)^2 \cdot \cos^2 x \left(\sin x \, dx\right)$$

$$\gg\gg \quad \int (1-u^2)^2 \cdot u^2 \, du$$

3. Perform the integral.
   - Expand integrand and use power rule to obtain:

$$\int (1-u^2)^2 \cdot u^2 \, du = \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$$

   - Insert definition $u = \cos x$:

$$\int \sin^5 x \cdot \cos^2 x \, dx \quad \gg\gg \quad \int (1-u^2)^2 \cdot u^2 \, du$$

$$\gg\gg \quad \frac{1}{3}\cos^3 x - \frac{2}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$$

4. This is our final answer.

Example - Trig power product with $\tan x$ and $\sec x$

Compute the integral:

$$\int \tan^5 x \cdot \sec^3 x \, dx$$

Solution

1. Try $du = \sec^2 x \, dx$.
   - Factor du out of the integrand:

$$\int \tan^5 x \cdot \sec^3 x \, dx \quad \gg\gg \quad \int \tan^5 x \cdot \sec x \left(\sec^2 x \, dx\right)$$

   - We then must swap over remaining $\sec x$ into the $\tan x$ type.
   - ⚠ Cannot do this because $\sec x$ has odd power.
     - Swap formula $\tan^2 x + 1 = \sec^2 x$ requires even powers.
2. Try $du = \sec x \, \tan x \, dx$.
   - Factor du out of the integrand:

$$\int \tan^5 x \cdot \sec^3 x \, dx \quad \gg\gg \quad \int \tan^4 x \cdot \sec^2 x \left(\sec x \, \tan x \, dx\right)$$

   - Swap remaining $\tan x$ into $\sec x$ type:

$$\int (\tan^2 x)^2 \cdot \sec^2 x \left(\sec x \, \tan x \, dx\right)$$

$$\gg\gg \quad \int \left(\sec^2 x - 1\right)^2 \cdot \sec^2 x \left(\sec x \, \tan x \, dx\right)$$

- Substitute $u = \sec x$ and $du = \sec x \, \tan x \, dx$:

$$\gg\gg \qquad \int (u^2 - 1)^2 \cdot u^2 \, du$$

3. Compute the integral in u and convert back to x.

- Expand the integrand:

$$\gg\gg \qquad \int u^6 - 2u^4 + u^2 \, du$$

- Apply power rule:

$$\gg\gg \qquad \frac{u^7}{7} - 2\frac{u^5}{5} + \frac{u^3}{3} + C$$

- Plug back in, $u = \sec x$:

$$\gg\gg \qquad \frac{\sec^7 x}{7} - 2\frac{\sec^5 x}{5} + \frac{\sec^3 x}{3} + C$$

4. This is our final answer.