

# W01 - Examples

## Events and outcomes

01

### ≡ Example - Coin flipping

Flip a fair coin two times and record both results.

- *Outcomes*: sequences, like  $HH$  or  $TH$ .
- *Sample space*: all possible sequences, i.e. the set  $S = \{HH, HT, TH, TT\}$ .
- *Events*: for example:
  - $A = \{HH, HT\}$  = “first was heads”
  - $B = \{HT, TH\}$  = “exactly one heads”
  - $C = \{HT, TH, HH\}$  = “at least one heads”

With this setup, we may combine events in various ways to generate other events:

- *Complex events*: for example:
  - $A \cap B = \{HT\}$ , or in words:  
“first was heads” AND “exactly one heads” = “heads-then-tails”  
  
Notice that the last one is a *complete description*, namely the *outcome*  $HT$ .
  - $A \cup B = \{HH, HT, TH\}$ , or in words:  
“first was heads” OR “exactly one heads”  
= “starts with heads, else it’s tails-then-heads”

02

### 🔗 Exercise - Coin flipping: counting subsets

Flip a fair coin five times and record the results.

How many elements are in the sample space? (How big is  $S$ ?)

How many events are there? (How big is  $\mathcal{F}$ ?)

### ≡ Solution >

There are  $2^5 = 32$  possible sequences, so  $|S| = 32$ .

To count the number of possible subsets, consider that we have 32 distinct items, and a subset is uniquely determined by the binary information – for each item – of whether it is

in or out. Thus there are  $2^{32}$  possibilities. So  $|\mathcal{F}| = 2^{32}$ .

## Probability models

03

### ≡ Example - Lucia is Host or Player

**Problem:** The professor chooses three students at random for a game in a class of 40, one to be Host, one to be Player, one to be Judge. What is the probability that Lucia is either Host or Player?

**Solution:**

#### 1. ⚡ Set up the probability model.

- Label the students 1 to 40. Write  $L$  for Lucia's number.
- **Outcomes:** assignments such as  $(H, P, J) = (2, 5, 8)$   
These are ordered triples with *distinct* entries in  $1, 2, \dots, 40$ .
- **Sample space:**  $S$  is the collection of all such distinct triples
- **Events:** any subset of  $S$
- **Probability measure:** assume all outcomes are equally likely, so  $P[(i, j, k)] = P[(r, l, p)]$  for all  $i, j, k, r, l, p$
- In total there are  $40 \cdot 39 \cdot 38$  triples of distinct numbers.
- Therefore  $P[(i, j, k)] = \frac{1}{40 \cdot 39 \cdot 38}$  for any *specific* outcome  $(i, j, k)$ .
- Therefore  $P[A] = \frac{|A|}{40 \cdot 39 \cdot 38}$  for any event  $A$ . (Recall  $|A|$  is the *number* of outcomes in  $A$ .)

#### 2. ➡ Define the desired event.

- Want to find  $P[\text{"Lucia is Host or Player"}]$
- Define  $A = \text{"Lucia is Host"}$  and  $B = \text{"Lucia is Player"}$ . Thus:

$$A = \{(L, j, k) \mid \text{any } j, k\}, \quad B = \{(i, L, k) \mid \text{any } i, k\}$$

- So we seek  $P[A \cup B]$ .

#### 3. ⚡ Compute the desired probability.

- Importantly,  $A \cap B = \emptyset$  (mutually exclusive).  
There are no outcomes in  $S$  in which Lucia is *both* Host and Player.
- By *additivity*, we infer  $P[A \cup B] = P[A] + P[B]$ .
- Now compute  $P[A]$ .
  - There are  $39 \cdot 38$  ways to choose  $j$  and  $k$  from the students besides Lucia.
  - Therefore  $|A| = 39 \cdot 38$ .
  - Therefore:

$$P[A] \gg \gg \frac{|A|}{40 \cdot 39 \cdot 38} \gg \gg \frac{39 \cdot 38}{40 \cdot 39 \cdot 38} \gg \gg \frac{1}{40}$$

- Now compute  $P[B]$ . It is similar:  $P[B] = \frac{1}{40}$ .
- Finally compute that  $P[A] + P[B] = \frac{1}{20}$ , so the answer is:

$$P[A \cup B] \gg \gg P[A] + P[B] \gg \gg \frac{1}{20}$$

04

### ≡ Example - iPhones and iPads

#### Problem:

At Mr. Jefferson's University, 25% of students have an iPhone, 30% have an iPad, and 60% have neither.

What is the probability that a randomly chosen student has either iProduct? (Q1)

What about both? (Q2)

#### Solution:

##### 1. ≡ Set up the probability model.

- A student is chosen at random: an *outcome* is the chosen student.
- *Sample space*  $S$  is the set of all students.
- Write  $O$  = "has iPhone" and  $A$  = "has iPad" concerning the chosen student.
- All students are equally likely to be chosen: therefore  $P[E] = \frac{|E|}{|S|}$  for any event  $E$ .
- Therefore  $P[O] = 0.25$  and  $P[A] = 0.30$ .
- Furthermore,  $P[O^c A^c] = 0.60$ . This means 60% have "not iPhone AND not iPad".

##### 2. ≡ Define the desired event.

- Q1: desired event =  $O \cup A$
- Q2: desired event =  $OA$

##### 3. ≡ Compute the probabilities.

- We do not believe  $O$  and  $A$  are exclusive.
- Try: apply inclusion-exclusion:

$$P[O \cup A] = P[O] + P[A] - P[OA]$$

- We know  $P[O] = 0.25$  and  $P[A] = 0.30$ . So this formula, with given data, RELATES Q1 and Q2.
- Notice the complements in  $O^c A^c$  and try *Negation*.
- *Negation*:

$$P[(OA)^c] = 1 - P[OA]$$

DOESN'T HELP.

- Try again: Negation.\*

$$P[(O^c A^c)^c] = 1 - P[O^c A^c]$$

- And De Morgan (or a Venn diagram!):

$$(O^c A^c)^c \gg \gg O \cup A$$

- Therefore:

$$P[O \cup A] \gg \gg P[(O^c A^c)^c]$$

$$\gg \gg 1 - P[O^c A^c] \gg \gg 1 - 0.6 = 0.4$$

- We have found Q1:  $P[O \cup A] = 0.40$ .
- Applying the RELATION from inclusion-exclusion, we get Q2:

$$P[O \cup A] = P[O] + P[A] - P[OA]$$

$$\gg \gg 0.40 = 0.25 + 0.30 - P[OA]$$

$$\gg \gg P[OA] = 0.15$$