

W01 - Examples

Events and outcomes

≡ Example - Coin flipping

Flip a fair coin two times and record both results.

- *Outcomes*: sequences, like HH or TH .
- *Sample space*: all possible sequences, i.e. the set $S = \{HH, HT, TH, TT\}$.
- *Events*: for example:
 - $A = \{HH, HT\} = \text{"first was heads"}$
 - $B = \{HT, TH\} = \text{"exactly one heads"}$
 - $C = \{HT, TH, HH\} = \text{"at least one heads"}$

With this setup, we may combine events in various ways to generate other events:

- *Complex events*: for example:
 - $A \cap B = \{HT\}$, or in words:
"first was heads" AND "exactly one heads" = "heads-then-tails"

Notice that the last one is a *complete description*, namely the *outcome* HT .
 - $A \cup B = \{HH, HT, TH\}$, or in words:
"first was heads" OR "exactly one heads"
= "starts with heads, else it's tails-then-heads"

✂ Exercise - Coin flipping: counting subsets

Flip a fair coin five times and record the results.

How many elements are in the sample space? (How big is S ?)

How many events are there? (How big is \mathcal{F} ?)

[Solution](#)

Probability models

≡ Example - Lucia is Host or Player

The professor chooses three students at random for a game in a class of 40, one to be Host, one to be Player, one to be Judge. What is the probability that Lucia is either Host or Player?

≡ Solution ∨

1. ≡ Set up the probability model.

- Label the students 1 to 40. Write L for Lucia's number.
- **Outcomes:** assignments such as $(H, P, J) = (2, 5, 8)$
These are ordered triples with *distinct* entries in $1, 2, \dots, 40$.
- **Sample space:** S is the collection of all such distinct triples
- **Events:** any subset of S
- **Probability measure:** assume all outcomes are equally likely, so $P[(i, j, k)] = P[(r, l, p)]$ for all i, j, k, r, l, p
- In total there are $40 \cdot 39 \cdot 38$ triples of distinct numbers.
- Therefore $P[(i, j, k)] = \frac{1}{40 \cdot 39 \cdot 38}$ for any *specific* outcome (i, j, k) .
- Therefore $P[A] = \frac{|A|}{40 \cdot 39 \cdot 38}$ for any event A . (Recall $|A|$ is the *number* of outcomes in A .)

2. ≡ Define the desired event.

- Want to find $P[\text{"Lucia is Host or Player"}]$
- Define $A = \text{"Lucia is Host"}$ and $B = \text{"Lucia is Player"}$. Thus:

$$A = \{(L, j, k) \mid \text{any } j, k\}, \quad B = \{(i, L, k) \mid \text{any } i, k\}$$

- So we seek $P[A \cup B]$.

3. ≡ Compute the desired probability.

- Importantly, $A \cap B = \emptyset$ (mutually exclusive).
There are no outcomes in S in which Lucia is *both* Host and Player.
- By *additivity*, we infer $P[A \cup B] = P[A] + P[B]$.
- Now compute $P[A]$.
 - There are $39 \cdot 38$ ways to choose j and k from the students besides Lucia.
 - Therefore $|A| = 39 \cdot 38$.
 - Therefore:

$$P[A] \gg \gg \frac{|A|}{40 \cdot 39 \cdot 38} \gg \gg \frac{39 \cdot 38}{40 \cdot 39 \cdot 38} \gg \gg \frac{1}{40}$$

- Now compute $P[B]$. It is similar: $P[B] = \frac{1}{40}$.
- Finally compute that $P[A] + P[B] = \frac{1}{20}$, so the answer is:

$$P[A \cup B] \gg \gg P[A] + P[B] \gg \gg \frac{1}{20}$$

≡ Example - iPhones and iPads

At Mr. Jefferson's University, 25% of students have an iPhone, 30% have an iPad, and 60% have neither.

What is the probability that a randomly chosen student has *some* iProduct? (Q1)

What about *both*? (Q2)

≡ Solution ▾

1. ≡ Set up the probability model.

- A student is chosen at random: an *outcome* is the chosen student.
- *Sample space* S is the set of all students.
- Write O = "has iPhone" and A = "has iPad" concerning the chosen student.
- All students are equally likely to be chosen: therefore $P[E] = \frac{|E|}{|S|}$ for any event E .
- Therefore $P[O] = 0.25$ and $P[A] = 0.30$.
- Furthermore, $P[O^c A^c] = 0.60$. This means 60% have "not iPhone AND not iPad".

2. ≡ Define the desired event.

- Q1: desired event = $O \cup A$
- Q2: desired event = OA

3. ≡ Compute the probabilities.

- We do not believe O and A are exclusive.
- Try: apply inclusion-exclusion:

$$P[O \cup A] = P[O] + P[A] - P[OA]$$

- We know $P[O] = 0.25$ and $P[A] = 0.30$. So this formula, with given data, RELATES Q1 and Q2.
- Notice the complements in $O^c A^c$ and try *Negation*.
- *Negation*:

$$P[(OA)^c] = 1 - P[OA]$$

DOESN'T HELP.

- Try again: *Negation*:

$$P[(O^c A^c)^c] = 1 - P[O^c A^c]$$

- And De Morgan (or a Venn diagram!):

$$(O^c A^c)^c \gg \gg O \cup A$$

- Therefore:

$$P[O \cup A] \gg \gg P[(O^c A^c)^c]$$

$$\gg \gg 1 - P[O^c A^c] \gg \gg 1 - 0.6 = 0.4$$

- We have found Q1: $P[O \cup A] = 0.40$.
- Applying the RELATION from inclusion-exclusion, we get Q2:

$$P[O \cup A] = P[O] + P[A] - P[OA]$$

$$\gg \gg 0.40 = 0.25 + 0.30 - P[OA]$$

$$\gg \gg P[OA] = 0.15$$

Conditional probability

Exercise - Simplifying conditionals

Let $A \subset B$. Simplify the following values:

$$P[A \mid B], \quad P[A \mid B^c], \quad P[B \mid A], \quad P[B \mid A^c]$$

[Solution](#)

Coin flipping: at least 2 heads

Flip a fair coin 4 times and record the outcomes as sequences, like $HHTH$.

Let $A_{\geq 2}$ be the event that there are at least two heads, and $A_{\geq 1}$ the event that there is at least one heads.

First let's calculate $P[A_{\geq 2}]$.

Define A_2 , the event that there were exactly 2 heads, and A_3 , the event of exactly 3, and A_4 the event of exactly 4. These events are exclusive, so:

$$P[A_{\geq 2}] = P[A_2 \cup A_3 \cup A_4] \gg \gg P[A_2] + P[A_3] + P[A_4]$$

Each term on the right can be calculated by counting:

$$P[A_2] = \frac{|A_2|}{2^4} \gg \gg \frac{\binom{4}{2}}{16} \gg \gg \frac{6}{16}$$

$$P[A_3] = \frac{|A_3|}{2^4} \gg \gg \frac{\binom{4}{1}}{16} \gg \gg \frac{4}{16}$$

$$P[A_4] = \frac{|A_4|}{2^4} \gg \gg \frac{\binom{4}{0}}{16} \gg \gg \frac{1}{16}$$

Therefore, $P[A_{\geq 2}] = \frac{11}{16}$.

Now suppose we find out that “at least one heads definitely came up”. (Meaning that we know $A_{\geq 1}$.) For example, our friend is running the experiment and tells us this fact about the outcome.

Now what is our estimate of likelihood of $A_{\geq 2}$?

The formula for conditioning gives:

$$P[A_{\geq 2} \mid A_{\geq 1}] = \frac{P[A_{\geq 2} \cap A_{\geq 1}]}{P[A_{\geq 1}]}$$

Now $A_{\geq 2} \cap A_{\geq 1} = A_{\geq 2}$. (Any outcome with at least two heads automatically has at least one heads.) We already found that $P[A_{\geq 2}] = \frac{11}{16}$. To compute $P[A_{\geq 1}]$ we simply **add** the probability $P[A_1]$, which is $\frac{4}{16}$, to get $P[A_{\geq 1}] = \frac{15}{16}$.

Therefore:

$$P[A_{\geq 2} \mid A_{\geq 1}] = \frac{11/16}{15/16} \gg \gg \frac{11}{15}$$

≡ Example - Flip a coin, then roll dice

Flip a coin. If the outcome is heads, roll two dice and add the numbers. If the outcome is tails, roll a single die and take that number. What is the probability of getting a tails AND a number at least 3?

≡ Solution

This “two-stage” experiment lends itself to a solution using conditional probability.

1. ≡ Label the events of interest.

- Let H and T be the events that the coin showed heads and tails, respectively.
- Let A_1, \dots, A_{12} be the events that the final number is $1, \dots, 12$, respectively.
- The value we seek is $P[TA_{\geq 3}]$.

2. ≡ Observe known (conditional) probabilities.

- We know that $P[H] = 0.5$ and $P[T] = 0.5$.

- We know that $P[A_5 \mid T] = \frac{1}{6}$, for example, or that $P[A_1 \mid H] = \frac{1}{12}$.

3. \Rightarrow Apply “multiplication” rule.

- This rule gives:

$$P[TA_{\geq 3}] = P[T] \cdot P[A_{\geq 3} \mid T]$$

- We know $P[T] = 0.5$ and can see by counting that $P[A_{\geq 3} \mid T] = 0.5$.
- Therefore $P[TA_{\geq 3}] = 0.25$.