# W01 - Examples

### Events and outcomes

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≡ Example - Coin flipping
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Flip a fair coin two times and record both results.

- Outcomes: sequences, like HH or TH.
- *Sample space*: all possible sequences, i.e. the set  $S = \{HH, HT, TH, TT\}$ .
- *Events:* for example:
  - $A = \{HH, HT\} =$  "first was heads"
  - $B = \{HT, TH\} =$  "exactly one heads"
  - $C = \{HT, TH, HH\} =$  "at least one heads"

With this setup, we may combine events in various ways to generate other events:

- *Complex events:* for example:
  - $A \cap B = \{HT\}$ , or in words:

"first was heads" AND "exactly one heads" = "heads-then-tails"

Notice that the last one is a *complete description*, namely the *outcome HT*.

•  $A \cup B = \{HH, HT, TH\}$ , or in words:

"first was heads" OR "exactly one heads"

 $= \hbox{``starts with heads}, \hbox{else it's tails-then-heads''}$ 

#### Exercise - Coin flipping: counting subsets

Flip a fair coin five times and record the results.

How many elements are in the sample space? (How big is S?)

How many events are there? (How big is  $\mathcal{F}$ ?)

Solution

### Probability models

#### ≡ Example - Lucia is Host or Player

The professor chooses three students at random for a game in a class of 40, one to be Host, one to be Player, one to be Judge. What is the probability that Lucia is either Host or Player?

#### **Solution Solution**

#### 1. \ Set up the probability model.

- Label the students 1 to 40. Write *L* for Lucia's number.
- *Outcomes:* assignments such as (H, P, J) = (2, 5, 8)These are ordered triples with *distinct* entries in 1, 2, ..., 40.
- Sample space: S is the collection of all such distinct triples
- *Events:* any subset of *S*
- Probability measure: assume all outcomes are equally likely, so P[(i,j,k)] = P[(r,l,p)] for all i,j,k,r,l,p
- In total there are  $40 \cdot 39 \cdot 38$  triples of distinct numbers.
- Therefore  $P[(i, j, k)] = \frac{1}{40 \cdot 39 \cdot 38}$  for any *specific* outcome (i, j, k).
- Therefore  $P[A]=\frac{|A|}{40\cdot 39\cdot 38}$  for any event A. (Recall |A| is the number of outcomes in A.)

#### 2. ⇒ Define the desired event.

- Want to find *P*["Lucia is Host or Player"]
- Define A = "Lucia is Host" and B = "Lucia is Player". Thus:

$$A = \big\{ (L,j,k) \mid \text{any } j,k \big\}, \qquad B = \big\{ (i,L,k) \mid \text{any } i,k \big\}$$

• So we seek  $P[A \cup B]$ .

#### 3. ₩ Compute the desired probability.

- Importantly,  $A \cap B = \emptyset$  (mutually exclusive). There are no outcomes in S in which Lucia is both Host and Player.
- By *additivity*, we infer  $P[A \cup B] = P[A] + P[B]$ .
- Now compute P[A].
  - There are  $39 \cdot 38$  ways to choose j and k from the students besides Lucia.
  - Therefore  $|A| = 39 \cdot 38$ .
  - Therefore:

$$P[A]$$
  $\gg \gg$   $\frac{|A|}{40 \cdot 39 \cdot 38}$   $\gg \gg$   $\frac{39 \cdot 38}{40 \cdot 39 \cdot 38}$   $\gg \gg$   $\frac{1}{40}$ 

- Now compute P[B]. It is similar:  $P[B] = \frac{1}{40}$ .
- Finally compute that  $P[A] + P[B] = \frac{1}{20}$ , so the answer is:

$$P[A \cup B] \gg P[A] + P[B] \gg \frac{1}{20}$$

#### **≡** Example - iPhones and iPads

At Mr. Jefferson's University, 25% of students have an iPhone, 30% have an iPad, and 60% have neither.

What is the probability that a randomly chosen student has some iProduct? (Q1)

What about both? (Q2)

#### **=**Solution

#### 1. **□** Set up the probability model.

- A student is chosen at random: an *outcome* is the chosen student.
- *Sample space S* is the set of all students.
- Write O = "has iPhone" and A = "has iPad" concerning the chosen student.
- All students are equally likely to be chosen: therefore  $P[E] = \frac{|E|}{|S|}$  for any event E.
- Therefore P[O] = 0.25 and P[A] = 0.30.
- Furthermore,  $P[O^cA^c]=0.60$ . This means 60% have "not iPhone AND not iPad".

#### $2. \equiv$ Define the desired event.

- Q1: desired event =  $O \cup A$
- Q2: desired event = OA

#### 3. **□** Compute the probabilities.

- We do not believe *O* and *A* are exclusive.
- Try: apply inclusion-exclusion:

$$P[O \cup A] = P[O] + P[A] - P[OA]$$

- We know P[O]=0.25 and P[A]=0.30. So this formula, with given data, RELATES Q1 and Q2.
- Notice the complements in  $O^cA^c$  and try *Negation*.
- Negation:

$$P[(OA)^c] = 1 - P[OA]$$

#### DOESN'T HELP.

• Try again: *Negation:* 

$$P[(O^c A^c)^c] = 1 - P[O^c A^c]$$

• And De Morgan (or a Venn diagram!):

$$(O^cA^c)^c \gg \gg O \cup A$$

• Therefore:

$$P[O \cup A] \gg \gg P[(O^c A^c)^c]$$

$$\gg \gg 1 - P[O^c A^c] \gg \gg 1 - 0.6 = 0.4$$

- We have found Q1:  $P[O \cup A] = 0.40$ .
- Applying the RELATION from inclusion-exclusion, we get Q2:

$$P[O \cup A] = P[O] + P[A] - P[OA]$$

$$\gg \gg 0.40 = 0.25 + 0.30 - P[OA]$$

$$\gg \gg P[OA] = 0.15$$

## Conditional probability

#### Exercise - Simplifying conditionals

Let  $A \subset B$ . Simplify the following values:

$$P[A \mid B], \quad P[A \mid B^c], \quad P[B \mid A], \quad P[B \mid A^c]$$

Solution

#### **≡** Coin flipping: at least 2 heads

Flip a fair coin 4 times and record the outcomes as sequences, like HHTH.

Let  $A_{\geq 2}$  be the event that there are at least two heads, and  $A_{\geq 1}$  the event that there is at least one heads.

First let's calculate  $P[A_{\geq 2}]$ .

Define  $A_2$ , the event that there were exactly 2 heads, and  $A_3$ , the event of exactly 3, and  $A_4$  the event of exactly 4. These events are exclusive, so:

$$P[A_{\geq 2}] = P[A_2 \cup A_3 \cup A_4] \quad \gg \gg \quad P[A_2] + P[A_3] + P[A_4]$$

Each term on the right can be calculated by counting:

$$P[A_2] = rac{|A_2|}{2^4} \quad \gg \gg \quad rac{{4 \choose 2}}{16} \quad \gg \gg \quad rac{6}{16}$$

$$P[A_3] = rac{|A_3|}{2^4} \quad \gg \gg \quad rac{{4 \choose 1}}{16} \quad \gg \gg \quad rac{4}{16}$$

$$P[A_4] = rac{|A_4|}{2^4} \quad \gg \gg \quad rac{inom{4}{0}}{16} \quad \gg \gg \quad rac{1}{16}$$

Therefore,  $P[A_{\geq 2}] = \frac{11}{16}$ .

Now suppose we find out that "at least one heads definitely came up". (Meaning that we know  $A_{\geq 1}$ .) For example, our friend is running the experiment and tells us this fact about the outcome.

Now what is our estimate of likelihood of  $A_{\geq 2}$ ?

The formula for conditioning gives:

$$P[A_{\geq 2} \mid A_{\geq 1}] = \frac{P[A_{\geq 2} \cap A_{\geq 1}]}{P[A_{\geq 1}]}$$

Now  $A_{\geq 2} \cap A_{\geq 1} = A_{\geq 2}$ . (Any outcome with at least two heads automatically has at least one heads.) We already found that  $P[A_{\geq 2}] = \frac{11}{16}$ . To compute  $P[A_{\geq 1}]$  we simply add the probability  $P[A_1]$ , which is  $\frac{4}{16}$ , to get  $P[A_{\geq 1}] = \frac{15}{16}$ .

Therefore:

$$P[A_{\geq 2} \mid A_{\geq 1}] = rac{11/16}{15/16} \gg \gg rac{11}{15}$$

Flip a coin. If the outcome is heads, roll two dice and add the numbers. If the outcome is tails, roll a single die and take that number. What is the probability of getting a tails AND a number at least 3?

#### **=**Solution

This "two-stage" experiment lends itself to a solution using the multiplication rule for conditional probability.

- $1. \equiv$  Label the events of interest.
  - Let *H* and *T* be the events that the coin showed heads and tails, respectively.
  - Let  $A_1, \ldots, A_{12}$  be the events that the final number is  $1, \ldots, 12$ , respectively.
  - The value we seek is  $P[TA>_3]$ .
- 2. 
  Observe known (conditional) probabilities.
  - We know that  $P[H] = \frac{1}{2}$  and  $P[T] = \frac{1}{2}$ .
  - We know that  $P[A_5 \mid T] = \frac{1}{6}$ , for example, or that  $P[A_2 \mid H] = \frac{1}{36}$ .
- 3. 

  ⇒ Apply "multiplication" rule.
  - This rule gives:

$$P[TA_{\geq 3}] = P[T] \cdot P[A_{\geq 3} \mid T]$$

- We know  $P[T] = \frac{1}{2}$  and can see by counting that  $P[A_{\geq 3} \mid T] = \frac{2}{3}$ .
- Therefore  $P[TA_{\geq 3}] = \frac{1}{3}$ .

#### ≡ Multiplication: draw two cards

Two cards are drawn from a standard deck (without replacement).

What is the probability that the first is a 3, and the second is a 4?

#### **=**Solution

This "two-stage" experiment lends itself to a solution using the multiplication rule for conditional probability.

- $1. \equiv$  Label events.
  - Write *T* for the event that the first card is a 3
  - Write *F* for the event that the second card is a 4.
  - We seek P[TF].
- $2. \equiv$  Write down knowns.
  - We know  $P[T] = \frac{4}{52}$ . (It does not depend on the second draw.)
  - Easily find  $P[F \mid T]$ .
    - If the first is a 3, then there are four 4s remaining and 51 cards.
    - So  $P[F \mid T] = \frac{4}{51}$ .
- $3. \equiv$  Apply multiplication rule.

• Multiplication rule:

$$P[TF] = P[T] \cdot P[F \mid T]$$

$$P[TF] = \frac{4}{52} \cdot \frac{4}{51} \quad \gg \gg \quad \frac{4}{13 \cdot 51}$$

• Therefore  $P[TF] = \frac{4}{663}$