

# Events and outcomes

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## 01 - Coin flipping

Flip a fair coin two times and record both results.

- *Outcomes*: sequences, like  $HH$  or  $TH$ .
- *Sample space*: all possible sequences, i.e. the set  $S = \{HH, HT, TH, TT\}$ .
- *Events*: for example:
  - $A = \{HH, HT\} = \text{"first was heads"}$
  - $B = \{HT, TH\} = \text{"exactly one heads"}$
  - $C = \{HT, TH, HH\} = \text{"at least one heads"}$

With this setup, we may combine events in various ways to generate other events:

- *Complex events*: for example:
  - $A \cap B = \{HT\}$ , or in words:  
"first was heads" AND "exactly one heads" = "heads-then-tails"

Notice that the last one is a *complete description*, namely the *outcome*  $HT$ .

- $A \cup B = \{HH, HT, TH\}$ , or in words:  
"first was heads" OR "exactly one heads"  
= "starts with heads, else it's tails-then-heads"

## 02 - Coin flipping: counting subsets

Flip a fair coin five times and record the results.

How many elements are in the sample space? (How big is  $S$ ?)

How many events are there? (How big is  $\mathcal{F}$ ?)

[Solution](#)

# Probability models

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## 03 - Lucia is Host or Player

The professor chooses three students at random for a game in a class of 40, one to be Host, one to be Player, one to be Judge. What is the probability that Lucia is either Host or Player?

≡ Solution ▾

1. ≡ Set up the probability model.

- Label the students 1 to 40. Write  $L$  for Lucia's number.
- *Outcomes*: assignments such as  $(H, P, J) = (2, 5, 8)$   
These are ordered triples with *distinct* entries in  $1, 2, \dots, 40$ .

- **Sample space:**  $S$  is the collection of all such distinct triples
- **Events:** any subset of  $S$
- **Probability measure:** assume all outcomes are equally likely, so  $P[(i, j, k)] = P[(r, l, p)]$  for all  $i, j, k, r, l, p$
- In total there are  $40 \cdot 39 \cdot 38$  triples of distinct numbers.
- Therefore  $P[(i, j, k)] = \frac{1}{40 \cdot 39 \cdot 38}$  for any **specific** outcome  $(i, j, k)$ .
- Therefore  $P[A] = \frac{|A|}{40 \cdot 39 \cdot 38}$  for any event  $A$ . (Recall  $|A|$  is the **number** of outcomes in  $A$ .)

2. ➡ Define the desired event.

- Want to find  $P[\text{"Lucia is Host or Player"}]$
- Define  $A = \text{"Lucia is Host"}$  and  $B = \text{"Lucia is Player"}$ . Thus:

$$A = \{(L, j, k) \mid \text{any } j, k\}, \quad B = \{(i, L, k) \mid \text{any } i, k\}$$

- So we seek  $P[A \cup B]$ .

3. ➡ Compute the desired probability.

- Importantly,  $A \cap B = \emptyset$  (mutually exclusive).  
There are no outcomes in  $S$  in which Lucia is **both** Host and Player.
- By **additivity**, we infer  $P[A \cup B] = P[A] + P[B]$ .
- Now compute  $P[A]$ .
  - There are  $39 \cdot 38$  ways to choose  $j$  and  $k$  from the students besides Lucia.
  - Therefore  $|A| = 39 \cdot 38$ .
  - Therefore:

$$P[A] \gg \gg \frac{|A|}{40 \cdot 39 \cdot 38} \gg \gg \frac{39 \cdot 38}{40 \cdot 39 \cdot 38} \gg \gg \frac{1}{40}$$

- Now compute  $P[B]$ . It is similar:  $P[B] = \frac{1}{40}$ .
- Finally compute that  $P[A] + P[B] = \frac{1}{20}$ , so the answer is:

$$P[A \cup B] \gg \gg P[A] + P[B] \gg \gg \frac{1}{20}$$

## 04 - iPhones and iPads

At Mr. Jefferson's University, 25% of students have an iPhone, 30% have an iPad, and 60% have neither.

What is the probability that a randomly chosen student has **some** iProduct?  
(Q1)

What about **both**? (Q2)

☰ Solution ▾

1. ➡ Set up the probability model.

- A student is chosen at random: an *outcome* is the chosen student.
- *Sample space*  $S$  is the set of all students.
- Write  $O$  = “has iPhone” and  $A$  = “has iPad” concerning the chosen student.
- All students are equally likely to be chosen: therefore  $P[E] = \frac{|E|}{|S|}$  for any event  $E$ .
- Therefore  $P[O] = 0.25$  and  $P[A] = 0.30$ .
- Furthermore,  $P[O^c A^c] = 0.60$ . This means 60% have “not iPhone AND not iPad”.

## 2. $\equiv$ Define the desired event.

- Q1: desired event =  $O \cup A$
- Q2: desired event =  $OA$

## 3. $\equiv$ Compute the probabilities.

- We do not believe  $O$  and  $A$  are exclusive.
- Try: apply inclusion-exclusion:

$$P[O \cup A] = P[O] + P[A] - P[OA]$$

- We know  $P[O] = 0.25$  and  $P[A] = 0.30$ . So this formula, with given data, RELATES Q1 and Q2.
- Notice the complements in  $O^c A^c$  and try *Negation*.
- *Negation*:

$$P[(OA)^c] = 1 - P[OA]$$

DOESN'T HELP.

- Try again: *Negation*:

$$P[(O^c A^c)^c] = 1 - P[O^c A^c]$$

- And De Morgan (or a Venn diagram!):

$$(O^c A^c)^c \gg \gg O \cup A$$

- Therefore:

$$P[O \cup A] \gg \gg P[(O^c A^c)^c]$$

$$\gg \gg 1 - P[O^c A^c] \gg \gg 1 - 0.6 = 0.4$$

- We have found Q1:  $P[O \cup A] = 0.40$ .
- Applying the RELATION from inclusion-exclusion, we get Q2:

$$P[O \cup A] = P[O] + P[A] - P[OA]$$

$$\gg \gg 0.40 = 0.25 + 0.30 - P[OA]$$

$$\gg \gg P[OA] = 0.15$$

# Conditional probability

## 05 - Coin flipping: at least 2 heads

Flip a fair coin 4 times and record the outcomes as sequences, like  $HHTH$ .

Let  $A_{\geq 2}$  be the event that there are at least two heads, and  $A_{\geq 1}$  the event that there is at least one heads.

First let's calculate  $P[A_{\geq 2}]$ .

Define  $A_2$ , the event that there were exactly 2 heads, and  $A_3$ , the event of exactly 3, and  $A_4$  the event of exactly 4. These events are exclusive, so:

$$P[A_{\geq 2}] = P[A_2 \cup A_3 \cup A_4] \gg \gg P[A_2] + P[A_3] + P[A_4]$$

Each term on the right can be calculated by counting:

$$P[A_2] = \frac{|A_2|}{2^4} \gg \gg \frac{\binom{4}{2}}{16} \gg \gg \frac{6}{16}$$

$$P[A_3] = \frac{|A_3|}{2^4} \gg \gg \frac{\binom{4}{1}}{16} \gg \gg \frac{4}{16}$$

$$P[A_4] = \frac{|A_4|}{2^4} \gg \gg \frac{\binom{4}{0}}{16} \gg \gg \frac{1}{16}$$

Therefore,  $P[A_{\geq 2}] = \frac{11}{16}$ .

Now suppose we find out that “at least one heads definitely came up”. (Meaning that we know  $A_{\geq 1}$ .) For example, our friend is running the experiment and tells us this fact about the outcome.

Now what is our estimate of likelihood of  $A_{\geq 2}$ ?

The formula for conditioning gives:

$$P[A_{\geq 2} \mid A_{\geq 1}] = \frac{P[A_{\geq 2} \cap A_{\geq 1}]}{P[A_{\geq 1}]}$$

Now  $A_{\geq 2} \cap A_{\geq 1} = A_{\geq 2}$ . (Any outcome with at least two heads automatically has at least one heads.) We already found that  $P[A_{\geq 2}] = \frac{11}{16}$ . To compute  $P[A_{\geq 1}]$  we simply **add** the probability  $P[A_1]$ , which is  $\frac{4}{16}$ , to get  $P[A_{\geq 1}] = \frac{15}{16}$ .

Therefore:

$$P[A_{\geq 2} \mid A_{\geq 1}] = \frac{11/16}{15/16} \gg \gg \frac{11}{15}$$

## 06 - Flip a coin, then roll dice

Flip a coin. If the outcome is heads, roll two dice and add the numbers. If the outcome is tails, roll a single die and take that number. What is the probability of getting a tails AND a number at least 3?

 Solution

This “two-stage” experiment lends itself to a solution using conditional probability.

1.  $\equiv$  Label the events of interest.

- Let  $H$  and  $T$  be the events that the coin showed heads and tails, respectively.
- Let  $A_1, \dots, A_{12}$  be the events that the final number is  $1, \dots, 12$ , respectively.
- The value we seek is  $P[TA_{\geq 3}]$ .

2.  $\equiv$  Observe known (conditional) probabilities.

- We know that  $P[H] = 0.5$  and  $P[T] = 0.5$ .
- We know that  $P[A_5 | T] = \frac{1}{6}$ , for example, or that  $P[A_1 | H] = \frac{1}{12}$ .

3.  $\Rightarrow$  Apply “multiplication” rule.

- This rule gives:

$$P[TA_{\geq 3}] = P[T] \cdot P[A_{\geq 3} | T]$$

- We know  $P[T] = 0.5$  and can see by counting that  $P[A_{\geq 3} | T] = 0.5$ .
- Therefore  $P[TA_{\geq 3}] = 0.25$ .