

W02 - Examples

Trig power products

≡ Example - Trig power product with an odd power

Compute the integral:

$$\int \cos^2 x \cdot \sin^5 x \, dx$$

≡ Solution

1. ≡ Swap over the even bunch.

- Max even bunch leaving power-one is $\sin^4 x$:

$$\sin^5 x \quad \gg \gg \quad \sin x (\sin^2 x)^2 \quad \gg \gg \quad \sin x (1 - \cos^2 x)^2$$

- Apply to $\sin^5 x$ in the integrand:

$$\int \cos^2 x \cdot \sin^5 x \, dx \quad \gg \gg \quad \int \cos^2 x \cdot \sin x (1 - \cos^2 x)^2 \, dx$$

2. ≡ Perform u -substitution on the power-one integrand.

- Set $u = \cos x$.
- Hence $du = -\sin x \, dx$. Recognize this in the integrand.
- Convert the integrand:

$$\begin{aligned} \int \cos^2 x \cdot \sin x (1 - \cos^2 x)^2 \, dx &\gg \gg \int \cos^2 x \cdot (1 - \cos^2 x)^2 (-\sin x \, dx) \\ &\gg \gg \int u^2 \cdot (1 - u^2)^2 \, du \end{aligned}$$

3. ≡ Perform the integral.

- Expand integrand and use power rule to obtain:

$$\int u^2 \cdot (1 - u^2)^2 \, du = \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$$

- Insert definition $u = \cos x$:

$$\begin{aligned} \int \cos^2 x \cdot \sin^5 x \, dx &\gg \gg \int u^2 \cdot (1 - u^2)^2 \, du \\ &\gg \gg \frac{1}{3}\cos^3 x - \frac{2}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C \end{aligned}$$

4. ≡ This is our final answer.

≡ Example - Trig power product with tan and sec

Compute the integral:

$$\int \tan^5 x \cdot \sec^3 x \, dx$$

≡ Solution

1. ➞ Try $du = \sec^2 x \, dx$.

- Factor du out of the integrand:

$$\int \tan^5 x \cdot \sec^3 x \, dx \quad \gg \gg \quad \int \tan^5 x \cdot \sec x (\sec^2 x \, dx)$$

- We then must swap over remaining $\sec x$ into the $\tan x$ type.
- Cannot do this because $\sec x$ has odd power. Need even to swap.

2. ➞ Try $du = \sec x \tan x \, dx$.

- Factor du out of the integrand:

$$\int \tan^5 x \cdot \sec^3 x \, dx \quad \gg \gg \quad \int \tan^4 x \cdot \sec^2 x (\sec x \tan x \, dx)$$

- Swap remaining $\tan x$ into $\sec x$ type:

$$\begin{aligned} & \int (\tan^2 x)^2 \cdot \sec^2 x (\sec x \tan x \, dx) \\ & \gg \gg \quad \int (\sec^2 x - 1)^2 \cdot \sec^2 x (\sec x \tan x \, dx) \end{aligned}$$

- Substitute $u = \sec x$ and $du = \sec x \tan x \, dx$:

$$\gg \gg \quad \int (u^2 - 1)^2 \cdot u^2 \, du$$

3. ➞ Compute the integral in u and convert back to x .

- Expand the integrand:

$$\gg \gg \quad \int u^6 - 2u^4 + u^2 \, du$$

- Apply power rule:

$$\gg \gg \quad \frac{u^7}{7} - 2\frac{u^5}{5} + \frac{u^3}{3} + C$$

- Plug back in, $u = \sec x$:

$$\gg \gg \quad \frac{\sec^7 x}{7} - 2\frac{\sec^5 x}{5} + \frac{\sec^3 x}{3} + C$$

4. ➞ This is our final answer.


Trig substitution


≡ Example - Trig sub in quadratic: completing the square

Compute the integral:

$$\int \frac{dx}{\sqrt{x^2 - 6x + 11}}$$

Solution

1.  Notice square root of a quadratic.

2.  Complete the square to obtain Pythagorean form.

- Find constant term for a complete square:


$$x^2 - 6x + \left(\frac{-6}{2}\right)^2 = x^2 - 6x + 9 = (x - 3)^2$$

- Add and subtract desired constant term:

$$x^2 - 6x + 11 \quad \gg \gg \quad x^2 - 6x + 9 - 9 + 11$$

- Simplify:

$$x^2 - 6x + 9 - 9 + 11 \quad \gg \gg \quad (x - 3)^2 + 2$$


3.  Perform shift substitution.

- Set $u = x - 3$ as inside the square:

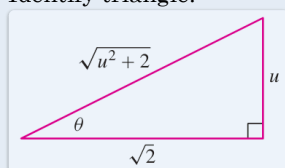
$$(x - 3)^2 + 2 = u^2 + 2$$

- Infer $du = dx$.
- Plug into integrand:

$$\int \frac{dx}{\sqrt{x^2 - 6x + 11}} \quad \gg \gg \quad \int \frac{du}{\sqrt{u^2 + 2}}$$


4.  Trig sub with $\tan \theta$.

- Identify triangle:




- Use substitution $u = \sqrt{2} \tan \theta$. (From triangle or memorized tip.)
- Infer $du = \sqrt{2} \sec^2 \theta d\theta$.
- Plug in data:

$$\int \frac{du}{\sqrt{u^2 + 2}} \quad \gg \gg \quad \int \frac{\sec^2 \theta}{\sec \theta} d\theta = \int \sec \theta d\theta$$

5.  Compute trig integral.

- Use ad hoc formula:

$$\int \sec \theta d\theta = \ln |\tan \theta + \sec \theta| + C$$

6.  Convert trig back to x .

- First in terms of u , referring to the triangle:

$$\tan \theta = \frac{u}{\sqrt{2}}, \quad \sec \theta = \frac{\sqrt{u^2 + 2}}{\sqrt{2}}$$

- Then in terms of x using $u = x - 3$.

- Plug everything in:

$$\ln |\tan \theta + \sec \theta| + C \quad \gg \gg \quad \ln \left| \frac{x-3}{\sqrt{2}} + \frac{\sqrt{(x-3)^2 + 2}}{\sqrt{2}} \right| + C$$

7. ➡ Simplify using log rules.

- Log rule for division gives us:

$$\ln \frac{f(x)}{a} = \ln f(x) - \ln a$$

- The common denominator $\frac{1}{\sqrt{2}}$ can be pulled outside as $-\ln \sqrt{2}$.
- The new term $-\ln \sqrt{2}$ can be “absorbed into the constant” (redefine C).
- So we write our final answer thus:

$$\ln \left| x - 3 + \sqrt{(x-3)^2 + 2} \right| + C$$