# *De Magnitudine et Mensura*

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(1) *Magnitude* is what is expressed in a thing by the number of parts congruent to a given thing which is called the Measure.

*Note*. For example, the magnitude of a line is expressed by the number of feet or inches, that is, of parts any of which are congruent to a foot or an inch in some actually given material (as if brass or wood). Just so the magnitude of a fathom (the amount that human arms can be extended), needing to be designated by something certain (as if a bulk measure [per aversionem]), is reckoned to be expressed by number as six feet, or seventy-two inches, since a foot is twelve inches. The magnitude of a cubit is one and a half feet, or one foot and six inches, or eighteen inches. We set, moreover, the magnitude of a foot or an inch to be given in a real device. Hence it is also clear that the magnitude of one and the same thing can be expressed by various numbers, as the measure is varied; indeed sometimes various measures are connected to each other, such as when the cubit is designated simultaneously by feet and inches.

(2) *Homogeneous* things are those whose magnitudes can be expressed by numbers while taking the same measure as the unity for all of them.

*Note*. Thus if a foot is the unity, then an inch will be $\frac{1}{12}$, a cubit $\frac{3}{2}$, a fathom 6. But if the inch is set as unity, then a foot will be 12, a cubit 18, and a fathom 72. And in this way the length of any straight line can be expressed by a whole number if, when the measure is subtracted some number of times (for example when three feet are subtracted), then nothing remains (so then it would be a three foot line). But if, when the measure (or foot) is subtracted as many times as possible, something remains, then we can take a certain part of the foot (for example a tenth) in order to measure it also, which can be subtracted again as many times as possible from this residue (for example seven times in succession, and with the foot taken as the unity, the number of the subtracted quantity will be the fraction 3 and $\frac{7}{10}$, or $\frac{37}{10}$). And if the thing to be measured is exhausted in this manner and nothing remains, that number will correspond to the thing and express its magnitude. But if instead something survives, then indeed we can take a new part of the measure again, perhaps a hundredth, and subtract it as many times as possible; and if the *error* smaller than the hundredth part does not seem sufficiently important to us, then we can be content with this *approximation* by the measure and tenths or hundredths of the measure, otherwise progressing to thousandths and beyond. In practice we are accustomed, however, to apply a *scale*, that is, a certain constant division of the measure made in brass or another durable material, indeed by tenths and tenths of tenths, or hundredths, and thousandths, and so on, since fractions expressed in decimals in this way can be treated in the form of integers, which are customarily exhibited to us by a decadic progression, that is by units, tens, hundreds, thousands, ten thousands etc.; thus Ludolphus of Colonia discovered, by prolonged calculation, that with the diameter of a circle being 1, the circumference is $3+\frac{1}{10}+\frac{4}{100}+\frac{1}{1000}+\frac{5}{10000}+\frac{9}{100000}+\frac{2}{1000000}$, or (what is allowed in decimals) joining them up in one fraction $\frac{3141592}{1000000}$ etc. up until .... it is settled. But since approximations of this kind, even if they suffice for common practice, never give exact knowledge of the magnitude we seek, therefore we proceed by a scientific progression long enough until a series of progression to infinity appears; and to this end we do not employ decimals indiscriminately, or any other constant divisions of the scale, but rather adjust the fractions to the nature of the thing, that we may arrive more easily, of course, at the law of progression. And thus I discovered, if the diameter is $\frac{1}{8}$, the circumference will be $\frac{1}{3} + \frac{1}{35} + \frac{1}{99} + \frac{1}{195}$ and so on to infinity, setting the numerator of the fraction to be the unit, but the denominators to be what results from two odds, 1 and 3, 5 and 7, 9 and 11, 13 and 15, 17 and 19, and so on, multiplied together. And by this method, not only can all the approximations obtainable by continuing be expressed simultaneously, but the error can even be made smaller than whatever given; namely it can be shown that if the circumference is called $\frac{1}{3}$, then the error will come out smaller than $\frac{1}{5}$; if it is called $\frac{1}{3} + \frac{1}{35}$, then the error will come out smaller than $\frac{1}{9}$; if it is called $\frac{1}{3} + \frac{1}{35} + \frac{1}{99}$, then the error will come out smaller than $\frac{1}{13}$, and so on, always by taking the first from the pair of consecutive odds. The whole infinite series exactly expresses the nature of the circle. But we have derived this *rationally* from the inner nature of the circle; on the other hand, the magnitude of the circumference of the circle or another curved path can be obtained *mechanically* using a thread, fitted to the rigid curved path and afterwards extended in a straight line and applied to a scale, or rolling the rigid curved path in the plane, provided that the rolling is guided by a thread or chain for security, lest dragging be mixed with it. The fact is also obtained by motion, when two mobile things traverse a line and a curve with uniform velocity, for the absolute paths in the same times will be as the velocities of the mobile things. And if the thing to be measured is a surface, another surface can be taken as the measure, for example a square foot which (or the determinate parts of which) can be subtracted as many times as possible from a planar surface; and if the surface is not planar, one should see whether it is easily transformed into a plane. For the measure of a solid, another solid would be taken, such as a cubic foot, and one would proceed in the same way. The magnitudes of solids could also be compared by immersing them in liquid, and measuring the amount it is raised in the vessel; but also by weighing, if both are developed from the same material; and the same thing can be carried over to curves and surfaces in some appropriate way [suo quodam modo]. And thus Galileo investigated the dimension of a Cycloid by weights, although by this method he did not obtain the true dimension discovered later by others by scientific reasoning. A surface and a solid are also sometimes conveniently measured by motion, as the traces, as it were, of a curve or a surface. In general, however, all estimation from our definition of magnitude reduces to the repetition of a certain measure expressed by numbers, or to the number assignable to a thing supposing that unity is assigned to another given thing. And furthermore, by this reckoning, not only extensions and diffusions of parts outside a part [difusiones partium extra parte], as in space and time, but also intensions or degrees of qualities and actions, indeed also laws, values, likelihoods, perfections, and other inextensible things are called back to numbers, a measure being found to which, or to the several parts of which, those that are found in the thing to be measured are congruent, but by repetition of which, or of the several parts of which, the magnitude to be estimated is formed. Of what great moment this consideration is, and how the strength of true Universal Mathesis, or the art of estimation in general, consists in it, is shown in our *Specimen Dynamicum*.

(3) Things are *Commensurable* with each other when a single common measure can be found exhausting them, by the repetition of which their magnitudes are constituted; but if not, they are called *incommensurable*, and the number to be assigned to that which is incommensurable with the measure taken as unit, is called *surd* or irrational; but if it is commensurable with the unit, then it is termed *rational*.

*Note*. If, of course, by subtracting the measure, or some number of parts constituting the measure by their repetition, as many times as possible, exhaustion is attained, then a common measure, constituting by repetition, can always be obtained. For the whole magnitude to be measured is expressed either by integers or by a composite of integers and fractions. Now any fractions can be reduced to a common divisor, and thus to a common measure. Let the number found to express the desired magnitude of the line be $2 + \frac{2}{3} + \frac{1}{6}$; by reducing to a common denominator it becomes $\frac{17}{6}$; therefore if the foot is 1 or $\frac{6}{6}$, certainly a common measure of the estimated thing and of the foot will be $\frac{1}{6}$, which quantity is contained in the estimated thing seventeen times, and in the foot six times. But if the sixth part of a foot, or a two inch line, is taken as the unit or measure, then the foot will be as 6, while the line to be estimated will be as 17, and thus the foot and the line will be commensurable. But if the fractions continue to infinity, and they cannot be collected by summing into one assignable integral or fractional number, then the magnitude to be estimated will be incommensurable to that to which we assigned the unit, or to a *number* of its *parts* (that is, to those producing it by repetition). For instance if there were a line which consisted of one foot and two tenths of a foot, and three hundredths and four thousandths, and five $10,000^{\text{ths}}$, and likewise to infinity, so that by setting the foot as 1, the line would be $\frac{1}{1} + \frac{2}{10} + \frac{3}{100} + \frac{4}{1000} + \frac{5}{10000} + \frac{6}{100000} + \frac{7}{1000000}$ etc. or $\frac{1234567}{1000000}$ etc. or in the manner of decimals 1.234567 etc. Then indeed the line to be measured would never be exhausted, but nevertheless, its true magnitude is considered to be expressed exactly. Indeed whenever the number is rational, as they call it, or commensurable to the unit, then also, when expressed in decimals, it continues periodically, such that the same characters always recur to infinity, as we will show in its own place (which does not happen here, as the construction itself shows). Furthermore, this method of investigating the common measure has been produced by Mathematicians, as we shall explain in its own place, that the lesser is subtracted from the greater as many times as can be, and then the Residue is subtracted again from the same previously subtracted Lesser, and the second Residue is subtracted from the second (i.e. the first residue), and similarly the third from the second Residue; and so necessarily either we will arrive at exhaustion, and the last subtracted thing exhausting [the rest] is in fact the greatest common measure, contained as many times in the first magnitude (or the greater of those compared) as the unit is-in the product of all quotients multiplied by each other, or else if residues are left over to infinity, the original two quantities will be incommensurable, inasmuch as all the residues are; but this series of quotients that express how many times each lesser could be subtracted from the preceding one, if based on a fixed rule, gives the scientific comparison of the two magnitudes. At the same time, we could imagine by a kind of fiction that all quantities are homogeneous, as though commensurable with each other, namely by contriving some infinitesimal, or infinitely small, elements. The calculus of Logarithms rests on such a fiction, with some fixed Logarithmic Element being established. A similar fiction takes place in Geometry, by imagining the situation as if all curves consisted of infinitely many little line segments infinitely small, and thus as if curved lines were polygons with infinitely many sides; or as if surfaces consisted of infinitely many little planar faces, that is, as if every concave or convex solid were a polyhedron with hedra of infinite smallness. In the same way it can be imagined that all solids consist of equal elementary particles [corpusculis] infinite in number and infinitely small in magnitude. And this fiction cannot introduce error, since (if you proceed duly from the hypothesis), the error never becomes greater than some elementary particles, which has no comparison with the whole; concerning this, see our Lemmas about incomparables. Hence if, for the fictitious or infinitely small elementary particles, we take real assignable things however small, it can be shown that the error which could appear to be admitted to the computation is smaller than any given error; that is, none could be assigned. But although one could conceive, to imitate commensurability, that these infinitesimals or infinitely small elements are equal to each other, nonetheless sometimes it is better to imagine proceeding by some other method that is useful to assist the computation. These things are more apparent from that deeper part of the Doctrine of magnitudes or Universal Mathematics [Mathesis Universalis], in which of course the science of the infinite is contained.