W12 - Examples

Parametric curves

Parametric circles

The standard equation of a circle of radius R centered at the point (h,k):

$$(x-h)^2+(y-k)^2=R^2$$

This equation says that the *distance* from a point (x,y) on the circle to the center point (h,k) equals R. This fact defines the circle.

Parametric coordinates for the circle:

$$x=h+R\cos t,\qquad y=k+R\sin t,\qquad t\in[0,2\pi)$$

For example, the unit circle $x^2+y^2=1$ is parametrized by $x=\cos t$ and $y=\sin t$.

Parametric lines

A line is the set of points satisfying:

$$y=mx+b\qquad\text{some } a,b$$

Vertical lines cannot be described in this way, we must use equations like $x=a$.

Parametric coordinates for a line:

$$x=a+rt,\qquad y=b+st,\qquad t\in(-\infty,+\infty)$$

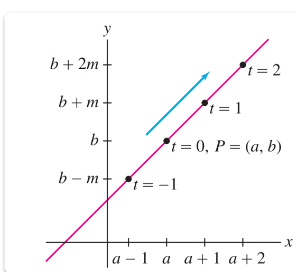
By choosing a,b,c,d appropriately, any line may be described.

For example, a vertical line $x=a$ is given by setting $a=a$ and $b,r,s=0$.

A non-vertical line $y=mx+b$ is given by setting $b=b,\ s=m$ and $a=0,\ r=1$.

For another example, the line $y-a=m(x-b)$ which passes through $P=(a,b)$ with slope m is given by:

$$(x,y)=(a+t,b+mt)$$



Parametric ellipses

The general equation of an ellipse centered at (h,k) with half-axes a and b is:

$$\left(\frac{x-h}{a}\right)^2+\left(\frac{y-k}{b}\right)^2=1$$

This equation represents a *stretched unit circle*:

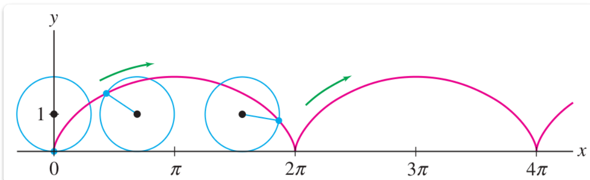
- by a in the x-axis
- by b in the y-axis

Parametric coordinate functions for the general ellipse:

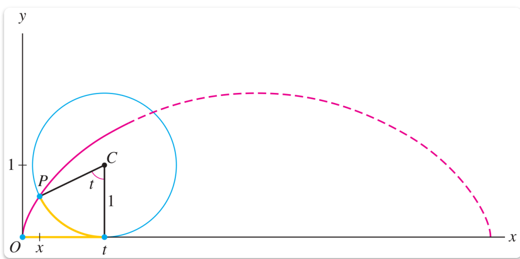
$$x = h + a\cos t, \qquad y = k + b\sin t, \qquad t \in [0, 2\pi)$$

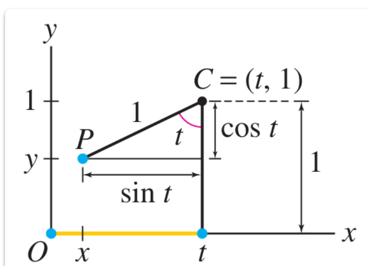
Parametric cycloids

The cycloid is the curve traced by a pen attached to the rim of a wheel as it rolls.



It is easy to describe the cycloid parametrically. Consider the geometry of the situation:





The center C of the wheel is moving rightwards at a constant speed of 1, so its position is $(t, 1)$. The angle is revolving at the same constant rate of 1 (in *radians*) because the *radius* is 1.

The triangle shown has base $\sin t$, so the x coordinate is $t - \sin t$. The y coordinate is $1 - \cos t$.

So the coordinates of the point $P = (x, y)$ are given parametrically by:

$$x = t - \sin t, \qquad y = 1 - \cos t, \qquad t > 0$$

If the circle has another radius, say R, then the parametric formulas change to:

$$x = Rt - R\sin t, \qquad y = R - R\cos t, \qquad t > 0$$

Calculus with parametric curves

Tangent to a cycloid

Find the equation of the tangent line to the cycloid $(4t - 4\sin t,\ 4 - 4\cos t)$ when $t = \frac{\pi}{4}$.

Solution

Compute $x'(\pi/4) = 4 - 2\sqrt{2}$.

- Derivative of $x(t)$:

$$x'(t) = 4 - 4\cos t$$

- Plug in $t = \pi/4$:

$$\begin{aligned} x'(\pi/4) &= 4 - 4\cos(\pi/4) \\ &= 4 - 2\sqrt{2} \end{aligned}$$

Compute $y'(\pi/4) = 4\sin t = 2\sqrt{2}$.

- Derivative of $y(t)$:

$$y'(t) = 4\sin t$$

- Plug in $t = \pi/4$:

$$\begin{aligned} y'(\pi/4) &= 4\sin(\pi/4) \\ &= 2\sqrt{2} \end{aligned}$$

Apply formula $\frac{dy}{dx} = \frac{y'}{x'}$.

- Calculate $\frac{dy}{dx}$ at $t = \pi/4$:

$$\frac{dy}{dx}(\pi/4) = \frac{y'(\pi/4)}{x'(\pi/4)} \quad \gg\gg \quad \frac{2\sqrt{2}}{4 - 2\sqrt{2}}$$

$$\gg\gg \quad \frac{2\sqrt{2}}{4 - 2\sqrt{2}} \cdot \frac{4 + 2\sqrt{2}}{4 + 2\sqrt{2}}$$

$$\gg\gg \quad \frac{8\sqrt{2} + 8}{16 - 8} \quad \gg\gg \quad \sqrt{2} + 1$$

- Slope of tangent line is $m = \sqrt{2} + 1$

A point on the tangent line: $\left(\pi - 2\sqrt{2}, 4 - 2\sqrt{2}\right)$ at $t = \pi/4$.

- Plug $t = \pi/4$ into $\big(x(t), y(t)\big) = (4t - 4\sin t, 4 - 4\cos t)$:

$$\left(4\frac{\pi}{4} - 4\sin(\pi/4), 4 - 4\cos(\pi/4)\right) \quad \gg\gg \quad \left(\pi - 2\sqrt{2}, 4 - 2\sqrt{2}\right)$$

Equation of tangent line: $y = mx + b$.

- Point-slope formulation:

$$y - \left(4 - 2\sqrt{2}\right) = \left(\sqrt{2} + 1\right)\left(x - \left(\pi - 2\sqrt{2}\right)\right)$$

- Simplify:

$$\gg\gg \qquad y = \left(\sqrt{2} + 1\right)\left(x - \pi + 2\sqrt{2}\right) + 4 - 2\sqrt{2}$$

$$\gg\gg \qquad y = \left(\sqrt{2} + 1\right)x + 8 - \left(\sqrt{2} + 1\right)\pi$$

This is our final answer.

Vertical and horizontal tangents of the circle

Consider the circle parametrized by $x = \cos t$ and $y = \sin t$. Find the points where the tangent lines are vertical or horizontal.

Solution

For the points with vertical tangent line, we find where the moving point has $x'(t) = 0$ (purely vertical motion):

$$x'(t) = -\sin t,$$

$$x'(t) = 0 \qquad \gg\gg \qquad -\sin t = 0 \qquad \gg\gg \qquad t = 0, \pi$$

For the points with horizontal tangent line, we find where the moving point has $y'(t) = 0$ (purely horizontal motion):

$$y'(t) = \cos t,$$

$$y'(t) = 0 \qquad \gg\gg \qquad \cos t = 0 \qquad \gg\gg \qquad t = \frac{\pi}{2}, \frac{3\pi}{2}$$

Perimeter of a circle

The perimeter of the circle $(R\cos t, R\sin t)$ is easily found. We have $(x', y') = (-R\sin t, R\cos t)$, and therefore:

$$(x')^2 + (y')^2 = (-R\sin t)^2 + (R\cos t)^2$$

$$\gg\gg \qquad R^2\sin^2 t + R^2\cos^2 t \qquad \gg\gg \qquad R^2$$

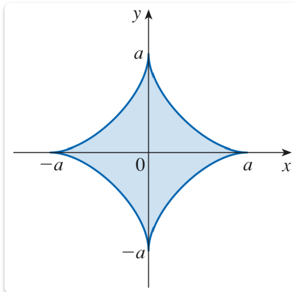
$$ds = \sqrt{(x')^2 + (y')^2}\,dt = R\,dt$$

Integrate around the circle:

$$\text{Perimeter} = \int_0^{2\pi} ds \qquad \gg\gg \qquad \int_0^{2\pi} R\,dt \qquad \gg\gg \qquad Rt\Big|_0^{2\pi} = 2\pi R$$

Perimeter of an asteroid

Find the perimeter length of the 'asteroid' given parametrically by $(x, y) = \left(a\cos^3\theta,\, a\sin^3\theta\right)$ for $a = 2$.



Solution

Notice: Throughout this problem we use the parameter θ instead of t. This does *not* mean we are using polar coordinates!

Compute the derivatives in θ:

$$\left(x', y'\right) = \left(3a\cos^2\theta\sin\theta,\, 3a\sin^2\theta\cos\theta\right)$$

Compute the infinitesimal arc element:

- Compute the sums of squares:

$$(x')^2 + (y')^2 = 9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta$$

$$\gg\gg \qquad 9a^2\sin^2\theta\cos^2\theta\left(\cos^2\theta + \sin^2\theta\right)$$

$$\gg\gg \qquad 9a^2\sin^2\theta\cos^2\theta$$

- Plug into the arc element, simplify:

$$ds = \sqrt{(x')^2 + (y')^2}\,d\theta = \sqrt{9a^2\sin^2\theta\cos^2\theta}\,d\theta$$

$$\gg\gg \qquad ds = 3a\big|\sin\theta\cos\theta\big|\,d\theta$$

Determine the bounds: $\int_0^{\pi/2} ds$ for $1/4$ of the asteroid perimeter.

- The full asteroid requires $4\times$ the length of one edge.
- Notice: The term $\sin\theta\cos\theta$ in the ds formula becomes negative after $\pi/2$!
- Instead we integrate $\int_0^{\pi/2} ds$ and multiply by 4.
- On this interval $[0, \pi/2]$ we have $ds = 3a\sin\theta\cos\theta\,d\theta$.

Integrate the arc element:

$$\int_0^{\pi/2} ds = \int_0^{\pi/2} 3a \sin\theta \cos\theta \, d\theta$$

$$\gg\gg \quad \frac{3a}{2}\int_0^{\pi/2} 2\sin\theta\cos\theta\,d\theta \quad \gg\gg \quad \frac{3a}{2}\int_0^{\pi/2}\sin(2\theta)\,d\theta$$

$$\gg\gg \quad -\frac{3a}{4}\cos(2\theta)\Big|_0^{\pi/2} \quad \gg\gg \quad -\frac{3a}{4}\big(\cos(\pi/2)-\cos(0)\big) \quad \gg\gg \quad \frac{3a}{4}$$

Multiply by 4:

$$\text{arclength} = L = 3a$$

Speed, distance, displacement

The parametric curve $\left(t,\, \frac{2}{3}t^{3/2}\right)$ describes the position of a moving particle (t measuring seconds).

- (a) What is the speed function?

Suppose the particle travels for 8 seconds starting at $t = 0$.

- (b) What is the total distance traveled?
- (c) What is the total displacement?

Solution

(a)
Compute *derivatives*:

$$(x',\, y') = \left(1,\, t^{1/2}\right)$$

Compute the *speed*.

- Find sum of squares:

$$(x')^2 + (y')^2 = 1 + (t^{1/2})^2 = 1 + t$$

- Get the speed function:

$$v(t) = \sqrt{(x')^2 + (y')^2} = \sqrt{1+t}$$

(b)
Distance traveled by using *speed*.

- Compute total distance traveled function:

$$s(t) = \int_{u=0}^{t} \sqrt{1+u}\,du$$

Integrate.

- Substitute $w = 1 + u$ and $dw = du$.
- New bounds are 1 and $1 + t$.
- Calculate:

$$\gg\gg \quad \int_1^{1+t} \sqrt{w}\, dw \quad \gg\gg \quad \frac{2}{3} w^{3/2} \Big|_1^{1+t} \quad \gg\gg \quad \frac{2}{3}\left((1+t)^{3/2} - 1\right)$$

Insert $t = 8$ for the answer.

- The distance traveled up to $t = 8$ is:

$$s(8) = \frac{2}{3}\left(9^{3/2} - 1\right) \quad \gg\gg \quad \frac{2}{3}(27 - 1) \quad \gg\gg \quad \frac{52}{3}$$

- This is our final answer.

(c)

Displacement formula: $d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$
- Pythagorean formula for distance between given points.

Compute starting and ending points.
- For starting point, insert $t = 0$:

$$\big(x(t), y(t)\big)\Big|_{t=0} \quad \gg\gg \quad \left(t, \frac{2}{3} t^{3/2}\right)\Big|_{t=0} \quad \gg\gg \quad (0, 0)$$

- For ending point, insert $t = 8$:

$$\big(x(t), y(t)\big)\Big|_{t=8} \quad \gg\gg \quad \left(t, \frac{2}{3} t^{3/2}\right)\Big|_{t=8} \quad \gg\gg \quad \left(8, \frac{2}{3} 8^{3/2}\right) \quad \gg\gg \quad \left(8, \frac{32\sqrt{2}}{3}\right)$$

Plug points into distance formula.

- Insert $(0, 0)$ and $\left(8, 32\sqrt{2}/3\right)$:

$$\sqrt{8^2 + \left(\frac{32\sqrt{2}}{3}\right)^2} \quad \gg\gg \quad \sqrt{64 + \frac{2048}{9}} \quad \gg\gg \quad \frac{\sqrt{2624}}{3}$$

- This is our final answer.

Surface of revolution - parametric circle

By revolving the unit upper semicircle about the x-axis, we can compute the surface area of the unit sphere.

The parametrization of the unit upper semicircle is: $(x, y) = (\cos t,\, \sin t)$.

The derivative is: $(x', y') = (-\sin t,\, \cos t)$.

Therefore, the arc element:

$$ds = \sqrt{(x')^2 + (y')^2}\, dt \quad \gg\gg \quad \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt \quad \gg\gg \quad dt$$

Now for R we choose $R = y(t) = \sin t$ because we are revolving about the x-axis.

Plugging all this into the integral formula and evaluating gives:

$$A = \int_0^\pi 2\pi \sin t\, dt \quad \gg\gg \quad -2\pi \cos t\Big|_0^\pi \quad \gg\gg \quad 4\pi$$

Notice: This method is a little easier than the method using the graph $y = \sqrt{1 - x^2}$.

Surface of revolution - parametric curve

Set up the integral which computes the surface area of the surface generated by revolving about the x-axis the curve $(t^3,\, t^2 - 1)$ for $0 \leq t \leq 1$.

Solution

For revolution about the x-axis, we set $R = y(t) = t^2 - 1$.

Then compute ds:

$$ds = \sqrt{(x')^2 + (y')^2} \quad \gg\gg \quad \sqrt{(3t^2)^2 + (2t)^2} \quad \gg\gg \quad \sqrt{9t^4 + 4t^2}$$

$$\gg\gg \quad \sqrt{t^2(9t^2 + 4)} \quad \gg\gg \quad t\sqrt{9t^2 + 4}$$

Therefore the desired integral is:

$$A = \int_0^1 2\pi R\, ds \quad \gg\gg \quad \int_0^1 2\pi(t^2 - 1)t\sqrt{9t^2 + 4}\, dt$$