

W10 - Examples

Ratio test and Root test

Ratio test examples

(a) Observe that $\sum_{n=0}^{\infty} \frac{10^n}{n!}$ has ratio $R_n = \frac{10}{n}$ and thus $R_n \rightarrow 0 < 1$. Therefore the RaT implies that this series converges.

⚠ Notice this technique!

Simplify the ratio:

$$\begin{aligned} \frac{\frac{10^{n+1}}{(n+1)!}}{\frac{n!}{10^n}} &\gg \gg \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n} \\ &\gg \gg \frac{10 \cdot 10^n}{(n+1)n!} \cdot \frac{n!}{10^n} \gg \gg \frac{10}{n} \end{aligned}$$

We *frequently* use these rules:

$$10^{n+1} = 10^n \cdot 10, \quad (n+1)! = (n+1)n!$$

to simplify ratios having exponents and factorials.

(b) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ has ratio $R_n = \frac{(n+1)^2}{2^{n+1}} \bigg/ \frac{n^2}{2^n}$.

Simplify this:

$$\begin{aligned} \frac{(n+1)^2}{2^{n+1}} \bigg/ \frac{n^2}{2^n} &\gg \gg \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} \\ &\gg \gg \frac{(n+1)^2 \cdot 2^n}{n^2 \cdot 2 \cdot 2^n} \gg \gg \frac{n^2 + 2n + 1}{2n^2} \rightarrow \frac{1}{2} \text{ as } n \rightarrow \infty \end{aligned}$$

So the series *converges absolutely* by the ratio test.

(c) Observe that $\sum_{n=1}^{\infty} n^2$ has ratio $R_n = \frac{n^2 + 2n + 1}{n^2} \rightarrow 1$ as $n \rightarrow \infty$.

So the ratio test is *inconclusive*, even though this series fails the SDT and obviously diverges.

(d) Observe that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ has ratio $R_n = \frac{n^2}{n^2 + 2n + 1} \rightarrow 1$ as $n \rightarrow \infty$.

So the ratio test is *inconclusive*, even though the series converges as a p -series with $p = 2 > 1$.

(e) More generally, the ratio test is usually *inconclusive for rational functions*; it is more effective to use LCT with a p -series.

Root test examples

(a) Observe that $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^n$ has roots of terms:

$$|a_n|^{1/n} = \left(\left(\frac{1}{n}\right)^n\right)^{1/n} = \frac{1}{n}$$

Because $\frac{1}{n} \rightarrow 0 < 1$ as $n \rightarrow \infty$, the Root Test shows that the series converges.

(b) Observe that $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{2n+1}\right)^n$ has roots of terms:

$$\sqrt[n]{|a_n|} = \frac{n}{2n+1} \rightarrow \frac{1}{2} < 1$$

Because $\frac{n}{2n+1} \rightarrow \frac{1}{2}$ as $n \rightarrow \infty$, the Root Test shows that the series converges.

(c) Observe that $\sum_{n=1}^{\infty} \left(\frac{3}{n}\right)^n$ converges because $\sqrt[n]{|a_n|} = \frac{3}{n} \rightarrow 0$ as $n \rightarrow \infty$.

Ratio test versus root test

Determine whether the series $\sum_{n=1}^{\infty} \frac{n^2 4^n}{5^{n+2}}$ converges absolutely or conditionally or diverges.

Solution

Before proceeding, rewrite somewhat the general term as $\left(\frac{n}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^n$.

Now we solve the problem first using the ratio test. By plugging in $n+1$ we see that

$$a_{n+1} = \left(\frac{n+1}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^{n+1}$$

So for the ratio R_n we have:

$$\left(\frac{n+1}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^{n+1} \cdot \left(\frac{5}{n}\right)^2 \cdot \left(\frac{5}{4}\right)^n \gg \gg \frac{n^2 + 2n + 1}{n^2} \cdot \frac{4}{5} \rightarrow \frac{4}{5} < 1 \text{ as } n \rightarrow \infty$$

Therefore the series converges absolutely by the ratio test.

Now solve the problem again using the root test. We have for $\sqrt[n]{|a_n|}$:

$$\left(\left(\frac{n}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^n\right)^{1/n} = \left(\frac{n}{5}\right)^{2/n} \cdot \frac{4}{5}$$

To compute the limit as $n \rightarrow \infty$ we must use logarithmic limits and L'Hopital's Rule. So, first take the log:

$$\ln \left(\left(\frac{n}{5}\right)^{2/n} \cdot \frac{4}{5} \right) = \frac{2}{n} \ln \frac{n}{5} + \ln \frac{4}{5}$$

Then for the first term apply L'Hopital's Rule:

$$\frac{\ln \frac{n}{5} \xrightarrow{d/dx} \frac{1}{n/5} \cdot \frac{1}{5}}{n/2 \xrightarrow{d/dx} 1/2} \gg \gg \frac{1/n}{1/2} \gg \gg \frac{2}{n} \rightarrow 0 \text{ as } n \rightarrow \infty$$

So the first term goes to zero, and the second (constant) term is the value of the limit. So the log limit is $\ln \frac{4}{5}$, and the limit (before taking logs) must be $e^{\ln \frac{4}{5}}$ (inverting the log using e^x) and this is $\frac{4}{5}$. Since $\frac{4}{5} < 1$, the root test also shows that the series converges absolutely.

Power series: Radius and Interval

Radius of convergence

Find the radius of convergence of the series:

(a) $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$

(b) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

Solution

(a) The ratio of coefficients is $R_n = \left| \frac{a_{n+1}}{a_n} \right| = \frac{1/2^{n+1}}{1/2^n} = 1/2$.

Therefore $R = 2$ and the series converges for $|x| < 2$.

(b) This power series has $a_{2n+1} = 0$, meaning it skips all odd terms.

Instead of the standard ratio function, we take the ratio of successive *even terms*. The series of even terms has coefficients $a_n = \frac{1}{(2n)!}$. So:

$$\left| \frac{a_{n+1}}{a_n} \right| \gg \gg \frac{\frac{1}{(2(n+1))!}}{\frac{1}{(2n)!}} \\ \gg \gg \frac{1}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{1} \gg \gg \frac{1}{(2n+2)(2n+1)}$$

As $n \rightarrow \infty$, this converges to 0, so $L = 0$ and $R = \infty$.

Radius and interval for a few series

Series	Radius	Interval
$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$	$R = 1$	$[1, 3)$
$\sum_{n=0}^{\infty} n! x^n$	$R = 0$	$\{0\}$
$\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$	$R = \infty$	$(-\infty, \infty)$

Interval of convergence

Find the interval of convergence of the following series.

(a) $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$

(b) $\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

Solution

$$(a) \sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$$

1. Apply ratio test.

- Ratio of successive coefficients:

$$R_n = \left| \frac{1}{n+1} \cdot \frac{n}{1} \right| = \frac{n}{n+1}$$

- Limit of ratios:

$$R_n = \frac{n}{n+1} \xrightarrow{n \rightarrow \infty} 1$$

- Deduce $L = 1$ and therefore $R = 1$.
- Therefore:

$$|x-3| < 1 \implies \text{converges}$$

$$|x-3| > 1 \implies \text{diverges}$$

2. Preliminary interval of convergence.

- Translate to interval notation:

$$|x-3| < 1 \implies x \in (3-1, 3+1)$$

$$\implies x \in (2, 4)$$

3. Final interval of convergence.

- Check endpoint $x = 2$:

$$\sum_{n=1}^{\infty} \frac{(2-3)^n}{n} \implies \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \implies \text{converges by AST}$$

- Check endpoint $x = 4$:

$$\sum_{n=1}^{\infty} \frac{(4-3)^n}{n} \implies \sum_{n=1}^{\infty} \frac{1}{n} \implies \text{diverges as } p\text{-series}$$

- Final interval of convergence: $x \in [2, 4)$

$$(b) \sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$$

1. Ratio Test.

- Ratio of successive coefficients:

$$R_n = \left| \frac{a_{n+1}}{a_n} \right| \implies \left| \frac{(-3)^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n} \right|$$

$$\implies \frac{3\sqrt{n+2}}{\sqrt{n+1}}$$

- Limit of ratios:

$$\lim_{n \rightarrow \infty} R_n \implies \lim_{n \rightarrow \infty} \frac{3\sqrt{n+2}}{\sqrt{n+1}} \implies 3$$

- Deduce $L = 3$ and thus $R = 1/3$.

- Therefore:

$$|x| < \frac{1}{3} \implies \text{converges}$$

$$|x| > \frac{1}{3} \implies \text{diverges}$$

- Preliminary interval of convergence: $x \in (-\frac{1}{3}, \frac{1}{3})$

2. Check endpoints.

- Check endpoint $x = -1/3$:

$$\sum_{n=0}^{\infty} \frac{(-3 \cdot (-\frac{1}{3}))^n}{\sqrt{n+1}} \gg \gg \sum_{n=0}^{\infty} \frac{1^n}{\sqrt{n+1}} \gg \gg \text{diverges as } p\text{-series}$$

- Check endpoint $x = +1/3$:

$$\sum_{n=0}^{\infty} \frac{(-3 \cdot (+\frac{1}{3}))^n}{\sqrt{n+1}} \gg \gg \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \gg \gg \text{converges by AST}$$

- Final interval of convergence: $x \in (-1/3, 1/3]$

Interval of convergence - further examples

Find the interval of convergence of the following series.


- (a) $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$
- (b) $\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n}$

Solution

(a) $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

- Ratio of coefficients: $R_n = \frac{n+1}{3n} \rightarrow \frac{1}{3}$.
- So the $R = 3$, center is $x = -2$, and the preliminary interval is $(-2-3, -2+3) = (-5, 1)$.
- Check endpoints: $\sum \frac{n(-3)^n}{3^{n+1}}$ diverges and $\sum \frac{n(3)^n}{3^{n+1}}$ also diverges. Final interval is $(-5, 1)$.

(b) $\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n}$

- Ratio of coefficients: $R_n = \frac{n+1}{n} \rightarrow 1$.
- So $R = 1$, and the series converges when $|4x+1| < 1$.
-  Extract preliminary interval.

- Divide by 4:

$$|4x+1| < 1 \xrightarrow{\div 4} |x+1/4| < 1/4 \gg \gg x \in (0, 1/2)$$

- Check endpoints: $\sum \frac{(4 \cdot \frac{-1}{2} + 1)^n}{n}$ converges but $\sum \frac{1}{n}$ diverges.
- Final interval of convergence: $[-1/2, 0)$