W05 Notes

Hydrostatic force

Videos, Organic Chemistry Tutor

- Hydrostatic pressure problems

01 Theory

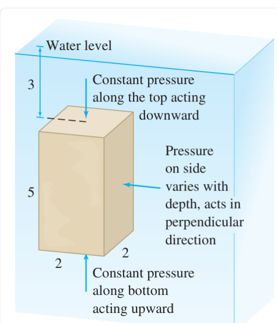
The pressure in a liquid is a function of the depth alone. This is a fundamental fact about liquids.

> **Pressure function**
>
> The fluid pressure in a liquid is a function of depth:
>
> $$p(h) = \rho g h$$
>
> - $\rho =$ fluid density
> - $g =$ gravity constant

In SI units:

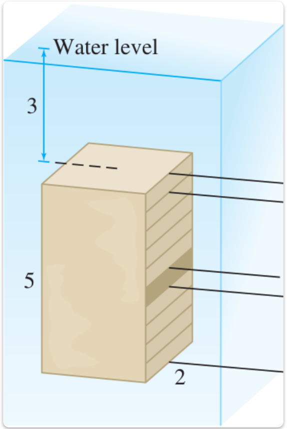
- $\rho = 1000 \mathrm{kg}/\mathrm{m}^3$
- $g = 9.8 \mathrm{m}/\mathrm{s}^2$

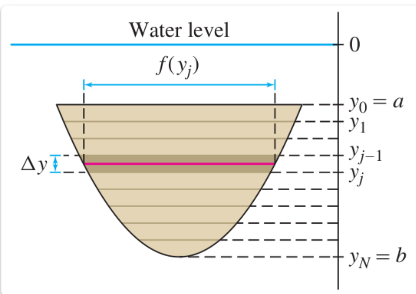
The pressure of a fluid acts upon any surface in the fluid by exerting a force perpendicular to the surface. Force is pressure times area. If the pressure varies across the surface, the total force must be calculated using an integral to add up differing contributions of force on each portion of the surface.



> **Fluid force on submerged plate**
>
> Total fluid force on plate:
>
> $$F = \rho g \int_a^b y\, w(y)\, dy$$
>
> - $w(y) =$ width of horizontal slice
> - $a, b =$ vertical limits of surface
>
> Use $y = a$ for top of plate (shallow) and $y = b$ for bottom of plate (deepest).
> Have $y = 0$ at the surface.

- ⚠ This formula assumes the plate is *oriented straight vertically*, not slanting.
- ⓘ Note that y *increases with depth*, reverse from the usual y-axis.

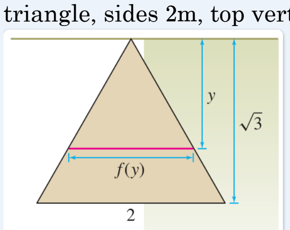




02 Illustration

Fluid force on a triangular plate

Find the total force on the submerged *vertical* plate with the following shape: Equilateral triangle, sides 2m, top vertex at the surface, liquid is oil with density $\rho = 900\text{kg/m}^3$.



Solution

1. Establish coordinate system: height y increases going *down*.
2. Compute width function $w(y)$.
    - Drop a perpendicular from top vertex to the base.
    - Pythagorean Theorem: vertical height is $\sqrt{3}$.
    - Similar triangles: ratio $w(y)/y$ must equal ratio $2/\sqrt{3}$.
    - Solve for $w(y)$:

$$w(y) = 2y/\sqrt{3}$$

3. Write integral using width function.
    - Bounds: shallowest: $y = 0$; deepest: $y = \sqrt{3}$.
    - Integral formula:

$$F = \rho g \int_0^{\sqrt{3}} y\, w(y)\, dy = 900 \cdot 9.8 \int_0^{\sqrt{3}} y \cdot 2y/\sqrt{3}\, dy$$

4. Compute integral.
    - Simplify constants:

$$900 \cdot 9.8 \cdot \frac{2}{\sqrt{3}} \approx 10184.5$$

    - Compute integral without constants:

$$\int_0^{\sqrt{3}} y^2 \, dy = \frac{y^3}{3}\Big|_0^{\sqrt{3}} = \sqrt{3}$$

    - Combine for the final answer: $10184.5 \cdot \sqrt{3} = 17640$

Moments and center of mass

Videos, Math Dr. Bob:

- Moments and CoM 01: Points masses on a line
- Moments and CoM 02: Points masses in the plane
- Moments and CoM 03: Planar lamina of uniform density
- Moments and CoM 04: Integral formula for planar lamina
- Moments and CoM 05: Rod of non-uniform density

03 Theory

Moment

The **moment** of a region to an axis is the total (integral) of mass times distance to that axis:

"Moment to x:"

$$M_x = \int \rho \, y \, dA \qquad \text{(general formula)}$$

$$M_x = \int_c^d \rho \, y \left(g_2(y) - g_1(y)\right) dy \qquad \text{(region between functions } g_2 \text{ and } g_1\text{)}$$

"Moment to y:"

$$M_y = \int \rho \, x \, dA \qquad \text{(general formula)}$$

$$M_y = \int_a^b \rho \, x \left(f_2(x) - f_1(x)\right) dx \qquad \text{(region between functions } f_2 \text{ and } f_1\text{)}$$

- Notice the *swap* in letters! M_y has integral with x while M_x has integral with y.
    - Because M_y needs distance to *x-axis*.
- Notice that if you remove x or y factors from the integrands, the integrals give **total mass M**.

These formulas are obtained by slicing the region into rectangular strips that are parallel to the axis in question.

The area *per strip* is then:

- $f(x)\,dx$ — region under $y = f(x)$
- $(f_2 - f_1)\,dx$ — region between f_1 and f_2
- $g(y)\,dy$ — region ‘under’ $x = g(y)$
- $(g_2 - g_1)\,dy$ — region between g_1 and g_2

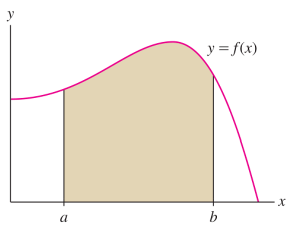


FIGURE 5 Lamina occupying the region under the graph of $f(x)$ over $[a, b]$.

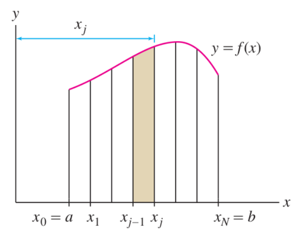
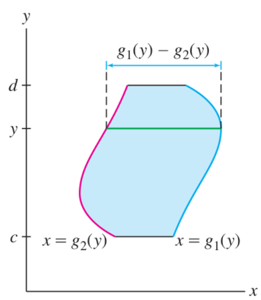


FIGURE 6 The shaded strip is nearly rectangular of area with $f(x_j)\Delta x$.



The idea of moment is related to:

- Torque balance and angular inertia
- Center of mass

The **center of mass (CoM)** of a solid body is a single point with two important properties:

1. CoM = “average position” of the body
   - The average position determines an *effective center* of dynamics. For example, gravity acting on every bit of mass of a rigid body acts the same as a force on the CoM alone.
2. CoM = “balance point” of the body
   - The net *torque* (rotational force) about the CoM, generated by a force distributed over the body’s mass, equivalently a force on the CoM, is zero.

Note:

- When the body has *uniform density*, then the CoM is also called the **centroid**.

Center of mass from moments

Coordinates of the CoM:

$$\bar{x} = \frac{M_y}{M}, \qquad \bar{y} = \frac{M_x}{M}$$

Here M_x and M_y are the moments and M is the total mass of the body.

Center of mass from moments - explanation

Notice how these formulas work. The total mass is always $M = \int \rho \, dA$. The moment to y (for example) is $M_y = \int \rho \, x \, dA$. Dividing these:

$$\bar{x} = \frac{M_y}{M} \quad \gg\gg \quad \frac{\int \rho \, x \, dA}{\int \rho \, dA} \quad \gg\gg \quad \frac{\int x \, dA}{\int dA} \quad \gg\gg \quad \frac{\int x \, dA}{A}$$

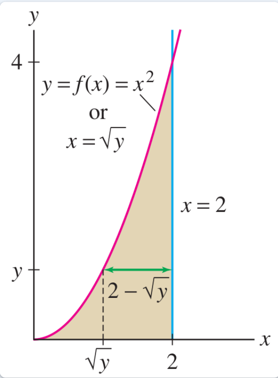
where $A =$ total area.

In other words, through the formula $\bar{x} = \frac{M_y}{M}$, we find that $\bar{x}$ is the *average value of x* over the region with area A.

04 Illustration

CoM of a parabolic plate

Find the CoM of the region depicted:



Solution

1. Compute the total mass.
   - Area under the curve with density factor ρ:

$$M = \int_0^2 \rho \, x^2 \, dx \quad \gg\gg \quad \rho \left. \frac{x^3}{3} \right|_0^2 \quad \gg\gg \quad \frac{8\rho}{3}$$

2. Compute M_y.
   - Formula:

$$M_y = \int_a^b \rho \, x \, dA$$

   - Interpret and calculate:

$$M_y = \int_0^2 \rho \, x \, f(x) \, dx \quad \gg\gg \quad \rho \int_0^2 x^3 \, dx$$

$$\gg\gg \quad 4\rho = M_y$$

3. Compute M_x.
   - Formula:

$$M_x = \int_c^d \rho \, y \, dA$$

- Width of horizontal strips between the curves:

$$w(y) = 2 - \sqrt{y}$$

- Interpret dA:

$$dA = (2 - \sqrt{y})\, dy$$

- Plug data into integral:

$$M_x = \int_c^d \rho\, y\, dA \quad \gg\gg \quad \int_0^4 \rho\, y(2 - \sqrt{y})\, dy$$

- Calculate integral:

$$\int_0^4 \rho\, y(2 - \sqrt{y})\, dy \quad \gg\gg \quad \int_0^4 \rho\, 2y\, dy - \int_0^4 \rho\, y^{3/2}\, dy$$

$$\gg\gg \quad \frac{16\rho}{5} = M_x$$

4. Compute CoM coordinates from moments.
   - CoM formulas:

$$\bar{x} = \frac{M_y}{M}, \qquad \bar{y} = \frac{M_x}{M}$$

   - Insert data:

$$\bar{x} = \frac{4\rho}{8\rho/3} \quad \gg\gg \quad \frac{3}{2}$$

$$\bar{y} = \frac{16\rho/5}{8\rho/3} \quad \gg\gg \quad \frac{6}{5}$$

   - Therefore CoM is located at $(\bar{x}, \bar{y}) = (\frac{3}{2}, \frac{6}{5})$.

05 Theory

A downside of the technique above is that to find M_x we needed to convert the region into functions in y. This would be hard to do if the region was given as the area under a curve $y = f(x)$ but $f^{-1}(y)$ cannot be found analytically. An alternative formula that works in this situation.

Midpoint of strips for opposite variables

When the region lies between $f_1(x)$ and $f_2(x)$, we can find M_x with an x-integral:

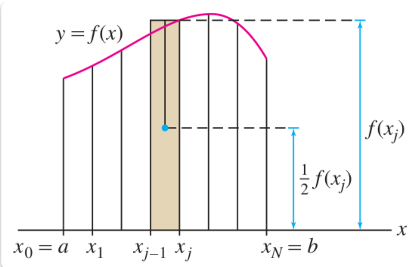
$$M_x = \int_c^d \rho\, \tfrac{1}{2}\left(f_2^2 - f_1^2\right) dx \qquad \text{(region between } f_1 \text{ and } f_2\text{)}$$

When the region lies between $g_1(y)$ and $g_2(y)$, we can find M_y with a y-integral:

$$M_y = \int_a^b \rho\, \tfrac{1}{2}\left(g_2^2 - g_1^2\right) dx \qquad \text{(region between } g_1 \text{ and } g_2\text{)}$$

- Use $f_1 = 0$ or $g_1 = 0$ for regions “under a curve” $y = f_2(x)$ or $x = g_2(y)$.

The idea for these formulas is to treat each vertical strip as a point concentrated at the CoM of the vertical strip itself.



The height to this midpoint is $\frac{1}{2}f(x)$, and the area of the strip is $f(x)\,dx$, so the integral becomes $\int \rho\,\frac{1}{2}f(x)^2\,dx$.

Midpoint of strips formula - full explanation

- If the strip is located at some x, with y values from 0 up to $f(x)$, then:

$$\text{CoM of strip} = \left(x,\ \frac{1}{2}f(x)\right)$$

- The area of the strip is $dA = f(x)\,dx$. So the integral formula for M_x can be recast:

$$M_x = \int y\,dA \quad \gg\gg \quad \int_a^b \tfrac{1}{2}f(x)\cdot f(x)\,dx \quad \gg\gg \quad \int_a^b \tfrac{1}{2}f^2\,dx$$

- If the vertical strips are between $f_1(x)$ and $f_2(x)$, then the *midpoints* of the strips are given by the 'average' function:

$$\frac{1}{2}\Big(f_1(x) + f_2(x)\Big)$$

- The *height* of each strip is $f_2(x) - f_1(x)$, so $dA = (f_2 - f_1)\,dx$.
- Putting this together:

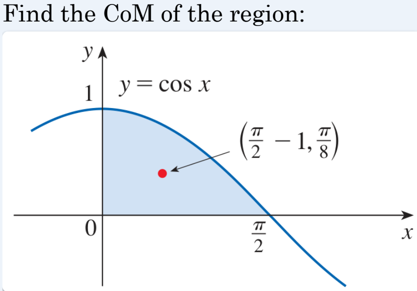
$$M_x = \int y\,dA \quad \gg\gg \quad \int_a^b \tfrac{1}{2}\big(f_1(x) + f_2(x)\big)\cdot(f_2 - f_1)\,dx$$

$$\gg\gg \quad \int_a^b \tfrac{1}{2}\left(f_2^2 - f_1^2\right)dx$$

06 Illustration

Computing CoM using only vertical strips

Find the CoM of the region:



Solution

1. Compute the total mass M.

- Area under the curve times density ρ:

$$\int_0^{\pi/2} \rho \cos x \, dx = \rho \sin x \Big|_0^{\pi/2} = \rho$$

2. Compute M_y using vertical strips.

- Plug $f(x) = \cos x$ into formula:

$$M_y = \int_a^b \rho \, x \, f(x) \, dx \quad \gg\gg \quad \int_0^{\pi/2} \rho \, x \cos x \, dx$$

- Integration by parts.
  - Set $u = x$, $v' = \cos x$; then $u' = 1$, $v = \sin x$.
  - IBP formula:

$$\int_a^b uv' \, dx = uv\Big|_a^b - \int_a^b u'v \, dx$$

  - Plug in data:

$$\int_0^{\pi/2} \rho \, x \cos x \, dx = \rho \, x \sin x \Big|_0^{\pi/2} - \rho \int_0^{\pi/2} \sin x \, dx$$

  - Evaluate:

$$\gg\gg \qquad \frac{\pi\rho}{2} \cdot 1 - \rho \left(-\cos\frac{\pi}{2} - -\cos 0\right) = \rho\left(\frac{\pi}{2} - 1\right)$$

3. Compute M_x *also* using vertical strips.

- Plug $f_2(x) = f(x) = \cos x$ and $f_1(x) = 0$ into formula:

$$M_x = \int_0^{\pi/2} \rho \, \tfrac{1}{2} f_2^2 \, dx \quad \gg\gg \quad \int_0^{\pi/2} \rho \, \tfrac{1}{2}\cos^2 x \, dx$$

- Integration by 'power to frequency conversion':
  - Use $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$:

$$\int_0^{\pi/2} \rho \, \tfrac{1}{2}\cos^2 x \, dx = \frac{\rho}{4} \int_0^{\pi/2} (1 + \cos 2x) \, dx$$

  - Integrate:

$$\gg\gg \qquad \frac{\rho}{4} x\Big|_0^{\pi/2} + \frac{\rho \sin 2x}{8}\Big|_0^{\pi/2} = \frac{\pi\rho}{8}$$

4. Compute CoM.

- CoM via moment formulas:

$$\bar{x} = \frac{M_y}{M}, \qquad \bar{y} = \frac{M_x}{M}$$

- Plug in data:

$$\bar{x} = \frac{\rho(\pi/2 - 1)}{\rho} \quad \gg\gg \quad \frac{\pi}{2} - 1$$

- Plug in data:

$$\bar{y} = \frac{\pi\rho/8}{\rho} \quad \gg\gg \quad \frac{\pi}{8}$$

- CoM is given by $(\bar{x}, \bar{y}) = \left(\frac{\pi}{2} - 1, \frac{\pi}{8}\right)$.

☰ CoM of region between line and parabola

Compute the CoM of the region below $y = x$ and above $y = x^2$ with $x \in [0, 1]$.

Solution

1. ☰ Name the functions: $f_1(x) = x^2$ (lower) and $f_2(x) = x$ (upper) over $x \in [0, 1]$.
2. ☰ Compute the mass M.
   - Area between curves times density:

$$M = \int_0^1 \rho\,(f_2 - f_1)\,dx \quad \gg\gg \quad \rho \int_0^1 x - x^2\,dx = \frac{\rho}{6}$$

3. Compute M_y using vertical strips.
   - Plug into formula:

$$M_y = \int_0^1 \rho\,x\,(f_2 - f_1)\,dx = \rho \int_0^1 x(x - x^2)\,dx = \frac{\rho}{12}$$

4. Compute M_x also using vertical strips.
   - Plug into formula:

$$M_x = \int_0^1 \rho\,\tfrac{1}{2}\left(f_2^2 - f_1^2\right)dx \quad \gg\gg \quad \rho \int_0^1 \tfrac{1}{2}(x^2 - x^4)\,dx = \frac{2\rho}{30}$$

5. ☰ Compute CoM.
   - Using CoM via moment formulas:

$$\bar{x} = \frac{\rho/12}{\rho/6} \quad \gg\gg \quad \frac{1}{2}$$

$$\bar{y} = \frac{2\rho/30}{\rho/6} \quad \gg\gg \quad \frac{2}{5}$$

   - CoM is given by $(\bar{x}, \bar{y}) = \left(\frac{1}{2}, \frac{2}{5}\right)$.

07 Theory

Two useful techniques for calculating moments and (thereby) CoMs:

- Additivity principle
- Symmetry

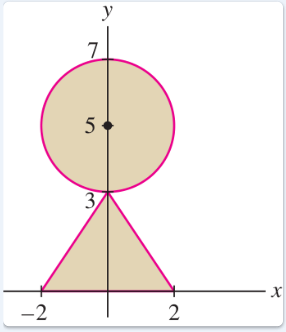
Additivity says that you can *add moments of parts* of a region to get the total moment of the region (to a given axis).

A symmetry principle is that if a region is *mirror symmetric across some line*, then the CoM must lie on that line.

08 Illustration

☰ Center of mass using moments and symmetry

Find the center of mass of the region below.



Solution

1. Symmetry: CoM on y-axis.
   - Because the region is symmetric in the y-axis, the CoM must lie on that axis.
   - Therefore $\bar{x} = 0$.
2. Additivity of moments.
   - Write M_x for the total x-moment (distance measured to the x-axis from above).
   - Write M_x^{tri} and M_x^{circ} for the x-moments of the triangle and circle.
   - *Additivity of moments* equation:

$$M_x = M_x^{\text{tri}} + M_x^{\text{circ}}$$

3. Find moment of the circle M_x^{circ}.
   - By symmetry we know $\bar{x}^{\text{circ}} = 0$.
   - By symmetry we know $\bar{y}^{\text{circ}} = 5$.
   - Area of circle with $r = 2$ is $A = 4\pi$, so total mass is $M = 4\pi\rho$.
   - Centroid-from-moments equation:

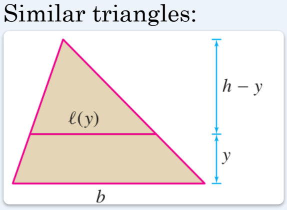
$$\bar{y}^{\text{circ}} = \frac{M_x^{\text{circ}}}{M}$$

   - Solve the equation for M_x^{circ}.
     - Solve:

$$\bar{y}^{\text{circ}} = \frac{M_x^{\text{circ}}}{M} \quad \gg\gg \quad 5 = \frac{M_x^{\text{circ}}}{4\pi\rho}$$

$$\gg\gg \quad M_x^{\text{circ}} = 20\pi\rho$$

4. Find moment of the triangle M_x^{tri} using integral formula
   - Similar triangles:



   - Similarity equation:

$$\frac{\ell(y)}{h-y} = \frac{b}{h} \quad \gg\gg \quad \ell(y) = b - \frac{b}{h}y$$

   - Integral formula:

$$M_x^{\text{tri}} = \rho \int_0^h y\ell(y)\,dy = \rho \int_0^h y\left(b - \frac{b}{h}y\right)dy$$

- Perform integral:

$$\rho \int_0^h y \left(b - \frac{b}{h} y\right) dy \quad \gg\gg \quad \rho \left(\frac{by^2}{2} - \frac{by^3}{3h}\right)\Big|_0^h = \frac{\rho b h^2}{6}$$

- Conclude:

$$M_x^{\text{tri}} = \frac{\rho b h^2}{6} = \frac{\rho 4 \cdot 3^2}{6} = 6\rho$$

5. Apply additivity.
   - Additivity formula:

$$M_x = M_x^{\text{tri}} + M_x^{\text{circ}} = \rho(20\pi + 6)$$

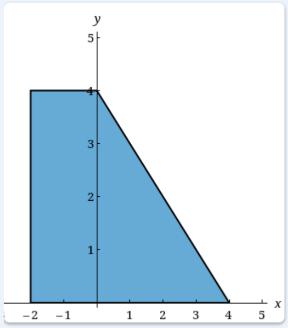
6. Total mass of region.
   - Area of circle is 4π.
   - Area of triangle is $\frac{1}{2} \cdot 4 \cdot 3 = 6$.
   - Thus $M = \rho A = \rho(4\pi + 6)$.
7. Compute center of mass $\bar{y}$ from total M_x and total M.
   - We have $M_x = \rho(20\pi + 6)$ and $M = \rho(4\pi + 6)$.
   - Plug into formula:

$$\bar{y} = \frac{M_x}{M} = \frac{\rho(20\pi + 6)}{\rho(4\pi + 6)} \approx 3.71$$

8. Final answer is $(\bar{x}, \bar{y}) = (0, 3.71)$.

Center of mass - two part region

Find the center of mass of the region which combines a rectangle and triangle (as in the figure) *by computing separate moments*. What are those separate moments? Assume the mass density is $\rho = 1$.



Solution

1. By symmetry, the center of mass of the rectangle is located at $(-1, 2)$.
   - Thus $\bar{x}^{\text{rect}} = -1$ and $\bar{y}^{\text{rect}} = 2$.
2. Find moments of the rectangle.
   - Total mass of rectangle $= M^{\text{rect}} = \rho \times \text{area} = 1 \cdot 8 = 8$.

- Apply moment relation:

$$\bar{x}^{\text{rect}} = \frac{M_y^{\text{rect}}}{M^{\text{rect}}} \qquad \gg\gg \qquad M_y^{\text{rect}} = -8$$

$$\bar{y}^{\text{rect}} = \frac{M_x^{\text{rect}}}{M^{\text{rect}}} \qquad \gg\gg \qquad M_x^{\text{rect}} = 16$$

3. Find moments of the triangle.
   - Area of vertical slice $= \left(4 - \frac{4}{4}x\right) dx$.
   - Distance from y-axis $= x$.
   - Total M_y^{tri} integral:

$$M_y^{\text{tri}} = \int_0^4 \rho x \left(4 - \frac{4}{4}x\right) dx$$

$$= \int_0^4 1 \cdot (4-x)x \, dx = \frac{32}{3}$$

   - Total M_x^{tri} integral:

$$M_x^{\text{tri}} = \int_0^4 \rho \frac{1}{2} f(x)^2 \, dx = \int_0^4 \rho \frac{1}{2} \left(4 - \frac{4}{4}x\right)^2 dx$$

$$= 1 \cdot \frac{1}{2} \int_0^4 (16 - 8x + x^2) \, dx = \frac{32}{3}$$

4. Add up total moments.
   - General formulas: $M_x = M_x^{\text{tri}} + M_x^{\text{rect}}$ and $M_y = M_y^{\text{tri}} + M_y^{\text{rect}}$
   - Plug in data: $M_x = \frac{32}{3} + 16 = \frac{80}{3}$ and $M_y = \frac{32}{3} - 8 = \frac{8}{3}$
5. Find center of mass from moments.
   - Total mass of triangle $= M^{\text{tri}} = \rho \times \text{area} = 1 \cdot \frac{1}{2} \cdot 4 \cdot 4 = 8$.
   - Total combined mass $= M = M^{\text{tri}} + M^{\text{rect}} = 8 + 8 = 16$.
   - Apply moment relation:

$$\bar{x} = \frac{M_y}{M} = \frac{8/3}{16} = \frac{1}{6}$$

$$\bar{y} = \frac{M_x}{M} = \frac{80/3}{16} = \frac{5}{3}$$

   - Therefore, center of mass is $(\bar{x}, \bar{y}) = \left(\frac{1}{6}, \frac{5}{3}\right)$.