W15 - Examples

Complex algebra

Complex multiplication

Compute the products:

(a) $(1-i)(4-7i)$ (b) $(2+5i)(2-5i)$

Solution

(a) $(1-i)(4-7i)$

Expand:

$$(1-i)(4-7i) \quad \gg\gg \quad 4-7i-4i+7i^2$$

Simplify i^2:

$$\gg\gg \quad 4-7i-4i+7(-1)$$

$$\gg\gg \quad -3-11i$$

(b) $(2+5i)(2-5i)$

Expand:

$$(2+5i)(2-5i) \quad \gg\gg \quad 4-10i+10i-25i^2$$

Simplify i^2:

$$\gg\gg \quad 4-10i+10i-25(-1) \quad \gg\gg \quad 29$$

Complex division

Compute the following divisions of complex numbers:

(a) $\dfrac{1}{-3+i}$ (b) $\dfrac{1}{i}$ (c) $\dfrac{1}{7i}$ (d) $\dfrac{2+5i}{2-5i}$

Solution

(a) $\dfrac{1}{-3+i}$

Conjugate is $-3-i$:

$$\frac{1}{-3+i} \quad \gg\gg \quad \frac{1}{-3+i}\cdot\frac{-3-i}{-3-i}$$

Simplify:

$$\gg\gg \quad \frac{-3-i}{9+1} \quad \gg\gg \quad \frac{-3}{10}+\frac{-1}{10}i$$

(b) $\frac{1}{i}$

Conjugate is $-i$:

$$\frac{1}{i} \quad \gg\gg \quad \frac{1}{i} \cdot \frac{-i}{-i} \quad \gg\gg \quad -i$$

(c) $\frac{1}{7i}$

Factor out the $1/7$:

$$\frac{1}{7i} \quad \gg\gg \quad \frac{1}{7} \cdot \frac{1}{i}$$

Use $\frac{1}{i} = -i$:

$$\gg\gg \quad \frac{1}{7} \cdot (-i) \quad \gg\gg \quad \frac{-1}{7} i$$

(d) $\frac{2+5i}{2-5i}$

Denominator conjugate is $2+5i$:

$$\frac{2+5i}{2-5i} \quad \gg\gg \quad \frac{2+5i}{2-5i} \cdot \frac{2+5i}{2+5i}$$

Simplify:

$$\gg\gg \quad \frac{4+20i+25i^2}{4+25} \quad \gg\gg \quad \frac{-21}{29} + \frac{20}{29} i$$

Complex product, quotient, power using Euler

Start with two complex numbers:

$$z = 2e^{i\frac{\pi}{2}} \qquad w = 5e^{i\frac{\pi}{3}}$$

Product zw:

$$zw \quad \gg\gg \quad \left(2e^{i\frac{\pi}{2}}\right) \cdot \left(5e^{i\frac{\pi}{3}}\right)$$

$$\gg\gg \quad (2 \cdot 5)\left(e^{i\frac{\pi}{2}}\right)\left(e^{i\frac{\pi}{3}}\right) \quad \gg\gg \quad 10e^{i\frac{\pi}{2} + i\frac{\pi}{3}} \quad \gg\gg \quad 10e^{i\frac{5\pi}{6}}$$

Quotient z/w:

$$z/w \quad \gg\gg \quad \left(2e^{i\frac{\pi}{2}}\right) \Big/ \left(5e^{i\frac{\pi}{3}}\right)$$

$$\gg\gg \frac{2e^{i\frac{\pi}{2}}}{5e^{i\frac{\pi}{3}}} \quad \gg\gg \quad \frac{2}{5} e^{i\frac{\pi}{2}} e^{-i\frac{\pi}{3}} \quad \gg\gg \quad \frac{2}{5} e^{i\frac{\pi}{6}}$$

Power z^8:

$$z^8 \quad \gg\gg \quad \left(2e^{i\frac{\pi}{2}}\right)^8$$

$$\gg\gg \quad 2^8\left(e^{i\frac{\pi}{2}}\right)^8 \quad \gg\gg \quad 512e^{i\cdot 4\pi}$$

Notice:

$$e^{i\cdot 4\pi} \quad \gg\gg \quad \left(e^{2\pi i}\right)^2 \quad \gg\gg \quad 1^2 \quad \gg\gg \quad 1$$

Simplify:

$$512e^{i\cdot 4\pi} \quad \gg\gg \quad 512$$

Thus: $z^8 = 512$.

Complex power from Cartesian

Compute $(3+3i)^4$.

Solution

First convert to exponential form:

$$3+3i \quad \gg\gg \quad 3\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)$$

$$\gg\gg \quad 3\sqrt{2}e^{i\frac{\pi}{4}}$$

Compute the power:

$$(3+3i)^4 \quad \gg\gg \quad \left(3\sqrt{2}e^{i\frac{\pi}{4}}\right)^4$$

$$\gg\gg \quad 324e^{i\pi} \quad \gg\gg \quad -324$$

Finding all 4^{th} roots of 16

Compute all the 4^{th} roots of 16.

Solution

Write $16 = 16e^{0i}$.

Evaluate roots formula:

$$\left(16e^{0i}\right)^{\frac{1}{4}} \quad \gg\gg \quad w_k = 16^{\frac{1}{4}}e^{i\left(\frac{0}{4}+k\frac{2\pi}{4}\right)}$$

Simplify:

$$\gg\gg \quad 2e^{i\cdot k\frac{\pi}{2}} \quad \gg\gg \quad 2,\ 2i,\ -2,\ -2i$$

Finding 2^{nd} roots of $2i$

Find both 2^{nd} roots of $2i$.

Solution

Write $2i = 2e^{i\frac{\pi}{2}}$.
Evaluate roots formula:

$$\left(2e^{i\frac{\pi}{2}}\right)^{\frac{1}{2}} \quad \gg\gg \quad w_k = \sqrt{2}e^{i\left(\frac{\pi/2}{2} + k\frac{2\pi}{2}\right)}$$

$$\gg\gg \quad \sqrt{2}e^{i\left(\frac{\pi}{4} + k\pi\right)}$$

Compute the options: $k = 0,\, 1$:

$$\gg\gg \quad \sqrt{2}e^{i\frac{\pi}{4}},\ \sqrt{2}e^{i\frac{5\pi}{4}}$$

Convert to rectangular:

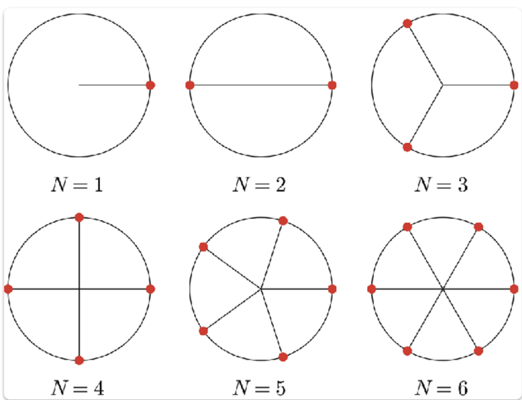
$$\gg\gg \quad \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right),\ \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$$

$$\gg\gg \quad 1+i,\ 1-i$$

Some roots of unity

Find the 1^{st} and 2^{nd} and 3^{rd} and 4^{th} and 5^{th} and 6^{th} roots of the number 1.

Solution



1^{st}

Write $1 = e^{0i}$. Evaluate roots formula. There is no possible k:

$$\left(e^{0i}\right)^{\frac{1}{1}} \quad \gg\gg \quad e^{0i} \quad \gg\gg \quad 1$$

2^{nd}

Write $1 = e^{0i}$. Evaluate roots formula in terms of k:

$$\left(e^{0i}\right)^{\frac{1}{2}} \quad \gg\gg \quad w_k = e^{i\left(\frac{0}{2} + k\frac{2\pi}{2}\right)} \qquad k = 0,\, 1$$

Compute the two options, $k = 0,\, 1$:

$$\gg\gg \quad 1,\ e^{\pi i} \quad \gg\gg \quad 1,\ -1$$

3^{rd}

Evaluate roots formula in terms of k:

$$\left(e^{0i}\right)^{\frac{1}{3}} \quad \gg\gg \quad w_k = e^{i\left(\frac{0}{3}+k\frac{2\pi}{3}\right)}$$

Compute the options: $k = 0,\ 1,\ 2$:

$$\gg\gg \quad 1,\ e^{i\frac{2\pi}{3}},\ e^{i\frac{4\pi}{3}} \quad \gg\gg \quad 1, \quad -\frac{1}{2}+\frac{\sqrt{3}}{2}i, \quad -\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

4^{th}

Evaluate roots formula:

$$\left(e^{0i}\right)^{\frac{1}{4}} \quad \gg\gg \quad w_k = e^{i\left(\frac{0}{4}+k\frac{2\pi}{4}\right)}$$

Compute the options: $k = 0,\ 1,\ 2,\ 3$:

$$1,\ e^{i\frac{\pi}{2}},\ e^{i\pi},\ e^{i\frac{3\pi}{2}} \quad \gg\gg \quad 1,\ i,\ -1,\ -i$$

5^{th}

Evaluate roots formula:

$$\left(e^{0i}\right)^{\frac{1}{5}} \quad \gg\gg \quad w_k = e^{i\left(\frac{0}{5}+k\frac{2\pi}{5}\right)}$$

Compute the options: $k = 0,\ 1,\ 2,\ 3,\ 4$:

$$1,\ e^{i\frac{2\pi}{5}},\ e^{i\frac{4\pi}{5}},\ e^{i\frac{6\pi}{5}},\ e^{i\frac{8\pi}{5}}$$

Don't simplify, it's not feasible.

6^{th}

Evaluate roots formula:

$$\left(e^{0i}\right)^{\frac{1}{6}} \quad \gg\gg \quad w_k = e^{i\left(\frac{0}{6}+k\frac{2\pi}{6}\right)}$$

Compute the options: $k = 0,\ 1,\ 2,\ 3,\ 4,\ 5$:

$$1,\ e^{i\frac{2\pi}{6}},\ e^{i\frac{4\pi}{6}},\ e^{i\frac{6\pi}{6}},\ e^{i\frac{8\pi}{6}},\ e^{i\frac{10\pi}{6}}$$

Simplify:

$$\gg\gg \quad 1,\ \frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -1,\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i,\ \frac{1}{2}-\frac{\sqrt{3}}{2}i$$