W15 Notes

Complex algebra

01 Theory - Complex arithmetic

The complex numbers $\mathbb{C}$ are sums of real and imaginary numbers. Every complex number can be written uniquely in 'Cartesian' form:

$$z = a + bi, \qquad a,\, b \in \mathbb{R}$$

To add, subtract, scale, and multiply complex numbers, treat 'i' like a constant.

Simplify the result using $i^2 = -1$.

For example:

$$(1+3i)(2-2i) \quad \gg\gg \quad 2 - 2i + 6i - 6i^2$$

$$\gg\gg \quad 2 + 4i - 6(-1) \quad \gg\gg \quad 8 + 4i$$

Complex conjugate

Every complex number has a **complex conjugate**:

$$z = a + bi \qquad \gg\gg \qquad \bar{z} = a - bi$$

For example:

$$\overline{2+5i} \quad = \quad 2 - 5i$$

$$\overline{2-5i} \quad = \quad 2 + 5i$$

In general, $\bar{\bar{z}} = z$.

Conjugates are useful mainly because they eliminate imaginary parts:

$$(2+5i)(2-5i) \quad \gg\gg \quad 4 + 25 \quad \gg\gg \quad 29$$

In general:

$$(a+bi)(a-bi) \quad \gg\gg \quad a^2 - abi + bia - b^2 i^2 \quad \gg\gg \quad a^2 + b^2 \in \mathbb{R}$$

Complex division

To divide complex numbers, use the conjugate to eliminate the imaginary part in the denominator.

For example, reciprocals:

$$\frac{1}{a+bi} \quad \gg\gg \quad \frac{1}{a+bi}\cdot\frac{a-bi}{a-bi}$$

$$\gg\gg \quad \frac{a-bi}{a^2+b^2} \quad \gg\gg \quad \left(\frac{a}{a^2+b^2}\right)+\left(\frac{-b}{a^2+b^2}\right)i$$

More general fractions:

$$\frac{a+bi}{c+di} \quad \gg\gg \quad \frac{a+bi}{c+di}\cdot\frac{c-di}{c-di}$$

$$\gg\gg \quad \frac{ac+bd+(bc-ad)i}{c^2+d^2} \quad \gg\gg \quad \frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i$$

Multiplication preserves conjugation

For any z, $w \in \mathbb{C}$:

$$\overline{zw} = \bar{z}\bar{w}$$

Therefore, one can take products or conjugates in either order.

02 Illustration

Example - Complex multiplication

Compute the products:

(a) $(1-i)(4-7i)$ (b) $(2+5i)(2-5i)$

Solution

(a) $(1-i)(4-7i)$

Expand:

$$(1-i)(4-7i) \quad \gg\gg \quad 4-7i-4i+7i^2$$

Simplify i^2:

$$\gg\gg \quad 4-7i-4i+7(-1)$$

$$\gg\gg \quad -3-11i$$

(b) $(2+5i)(2-5i)$

Expand:

$$(2+5i)(2-5i) \quad \gg\gg \quad 4-10i+10i-25i^2$$

Simplify i^2:

$$\gg\gg \quad 4-10i+10i-25(-1) \quad \gg\gg \quad 29$$

☷ Example - Complex division

Compute the following divisions of complex numbers:

(a) $\dfrac{1}{-3+i}$ (b) $\dfrac{1}{i}$ (c) $\dfrac{1}{7i}$ (d) $\dfrac{2+5i}{2-5i}$

Solution

(a) $\dfrac{1}{-3+i}$

Conjugate is $-3-i$:

$$\frac{1}{-3+i} \quad \gg\gg \quad \frac{1}{-3+i}\cdot\frac{-3-i}{-3-i}$$

Simplify:

$$\gg\gg \quad \frac{-3-i}{9+1} \quad \gg\gg \quad \frac{-3}{10}+\frac{-1}{10}i$$

(b) $\dfrac{1}{i}$

Conjugate is $-i$:

$$\frac{1}{i} \quad \gg\gg \quad \frac{1}{i}\cdot\frac{-i}{-i} \quad \gg\gg \quad -i$$

(c) $\dfrac{1}{7i}$

Factor out the $1/7$:

$$\frac{1}{7i} \quad \gg\gg \quad \frac{1}{7}\cdot\frac{1}{i}$$

Use $\frac{1}{i}=-i$:

$$\gg\gg \quad \frac{1}{7}\cdot(-i) \quad \gg\gg \quad \frac{-1}{7}i$$

(d) $\dfrac{2+5i}{2-5i}$

Denominator conjugate is $2+5i$:

$$\frac{2+5i}{2-5i} \quad \gg\gg \quad \frac{2+5i}{2-5i}\cdot\frac{2+5i}{2+5i}$$

Simplify:

$$\gg\gg \quad \frac{4+20i+25i^2}{4+25} \quad \gg\gg \quad \frac{-21}{29}+\frac{20}{29}i$$

Complex exponential

03 Theory - cis, Euler, products, powers

Multiplication of complex numbers is much easier to understand when the numbers are written using polar form.

There is a shorthand 'cis' notation:

$$\begin{aligned} a+bi \quad &\gg\gg \quad r\cos\theta + r\sin\theta\, i \\ &\gg\gg \quad r\big(\cos\theta + i\sin\theta\big) \\ &\gg\gg \quad r\,\mathrm{cis}\,\theta \end{aligned}$$

The cis notation stands for $\cos\theta + i\sin\theta$.

For example:

$$\sqrt{2}-\sqrt{2}i \quad \gg\gg \quad \sqrt{2}\,(1-i) \quad \gg\gg \quad 2\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)$$

$$\gg\gg \quad 2\cos\left(-\frac{\pi}{4}\right)+2\sin\left(-\frac{\pi}{4}\right)i$$

$$\gg\gg \quad 2\,\mathrm{cis}\left(-\frac{\pi}{4}\right)$$

Euler Formula

General **Euler Formula**:

$$re^{i\theta} \quad = \quad r\cos\theta + i\,r\sin\theta$$

On the unit circle:

$$e^{i\theta} \quad = \quad \cos\theta + i\sin\theta$$

The form $re^{i\theta}$ expresses the *same data* as the cis form.

The principal advantage of the form $re^{i\theta}$ is that it *reveals the rule for multiplication*:

Complex multiplication - Exponential form

$$r_1e^{i\theta_1}\cdot r_2e^{i\theta_2} \quad = \quad (r_1r_2)\,e^{i(\theta_1+\theta_2)}$$

In words:

- Multiply radii

- Add angles

Notice:

$$\text{multiply by } e^{i\frac{\pi}{2}} \qquad \Longleftrightarrow \qquad \text{rotate by } +90^\circ$$

Notice:

$$e^{i\frac{\pi}{2}} = +i$$

Therefore i ‘acts upon’ other numbers by rotating them 90° counterclockwise!

De Moivre’s Theorem - Complex powers

In exponential notation:

$$\left(re^{i\theta}\right)^n \; = \; r^n e^{i\cdot n\theta}$$

In cis notation:

$$\left(r \operatorname{cis} \theta\right)^n \; = \; r^n \operatorname{cis}(n\theta)$$

Expanded cis notation:

$$\left(r\cos\theta + i\, r\sin\theta\right)^n \; = \; r^n\cos(n\theta) + i\, r^n\sin(n\theta)$$

So the power of n acts like this:

- **Stretch**: r to r^n
- **Rotate** by n increments of θ

Extra - Derivation of Euler Formula

Recall the power series for e^x:

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots \qquad = \sum_{i=0}^{\infty}\frac{1}{i!}x^i$$

Plug in $x = i\theta$:

$$e^{i\theta} \quad \gg\gg \quad 1 + (i\theta) + \frac{1}{2!}(i\theta)^2 + \frac{1}{3!}(i\theta)^3 + \dots +$$

Simplify terms:

$$\gg\gg \quad 1 + i\theta - \frac{1}{2!}\theta^2 - \frac{1}{3!}i\theta^3 + \frac{1}{4!}\theta^4 + \frac{1}{5!}i\theta^5 - \frac{1}{6!}\theta^6 - \frac{1}{7!}i\theta^7 + \frac{1}{8!}\theta^8 + \cdots$$

Separate by i-factor. Select out the $\overbrace{\text{terms with } i}$:

$$\gg\gg \quad 1 \overbrace{+i\theta} - \frac{1}{2!}\theta^2 \overbrace{-\frac{1}{3!}i\theta^3} + \frac{1}{4!}\theta^4 \overbrace{+\frac{1}{5!}i\theta^5} - \frac{1}{6!}\theta^6 \overbrace{-\frac{1}{7!}i\theta^7} + \frac{1}{8!}\theta^8 + \cdots$$

Separate into a series without i and a series with i:

$$\gg\gg \quad \left(1-\frac{1}{2!}\theta^2+\frac{1}{4!}\theta^4-\cdots\right)+\left(\theta-\frac{1}{3!}\theta^3+\frac{1}{5!}\theta^5-\cdots\right)i$$

Identify $\cos\theta$ and $\sin\theta\, i$. Write trig series:

$$\cos\theta = 1-\frac{1}{2!}\theta^2+\frac{1}{4!}\theta^4-\cdots$$

$$\sin\theta = \theta-\frac{1}{3!}\theta^3+\frac{1}{5!}\theta^5-\cdots$$

Therefore $e^{i\theta}=\cos\theta+i\sin\theta$.

04 Illustration

Example - Complex product, quotient, power using Euler

Start with two complex numbers:

$$z = 2e^{i\frac{\pi}{2}} \qquad w = 5e^{i\frac{\pi}{3}}$$

Product zw:

$$zw \quad \gg\gg \quad \left(2e^{i\frac{\pi}{2}}\right)\cdot\left(5e^{i\frac{\pi}{3}}\right)$$

$$\gg\gg \quad (2\cdot 5)\left(e^{i\frac{\pi}{2}}\right)\left(e^{i\frac{\pi}{3}}\right) \quad \gg\gg \quad 10e^{i\frac{\pi}{2}+i\frac{\pi}{3}} \quad \gg\gg \quad 10e^{i\frac{5\pi}{6}}$$

Quotient z/w:

$$z/w \quad \gg\gg \quad \left(2e^{i\frac{\pi}{2}}\right)\Big/\left(5e^{i\frac{\pi}{3}}\right)$$

$$\gg\gg \frac{2e^{i\frac{\pi}{2}}}{5e^{i\frac{\pi}{3}}} \quad \gg\gg \quad \frac{2}{5}e^{i\frac{\pi}{2}}e^{-i\frac{\pi}{3}} \quad \gg\gg \quad \frac{2}{5}e^{i\frac{\pi}{6}}$$

Power z^8:

$$z^8 \quad \gg\gg \quad \left(2e^{i\frac{\pi}{2}}\right)^8$$

$$\gg\gg \quad 2^8\left(e^{i\frac{\pi}{2}}\right)^8 \quad \gg\gg \quad 512e^{i\cdot 4\pi}$$

Notice:

$$e^{i\cdot 4\pi} \quad \gg\gg \quad \left(e^{2\pi i}\right)^2 \quad \gg\gg \quad 1^2 \quad \gg\gg \quad 1$$

Simplify:

$$512e^{i\cdot 4\pi} \quad \gg\gg \quad 512$$

Thus: $z^8 = 512$.

Example - Complex power from Cartesian

Compute $(3+3i)^4$.

Solution

First convert to exponential form:

$$3+3i \quad \gg\gg \quad 3\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)$$

$$\gg\gg \quad 3\sqrt{2}e^{i\frac{\pi}{4}}$$

Compute the power:

$$(3+3i)^4 \quad \gg\gg \quad \left(3\sqrt{2}e^{i\frac{\pi}{4}}\right)^4$$

$$\gg\gg \quad 324e^{i\pi} \quad \gg\gg \quad -324$$

Complex roots

05 Theory - Roots formula

The exponential notation leads to a formula for a complex n^{th} root of any complex number:

$$\sqrt[n]{re^{i\theta}} \quad = \quad \sqrt[n]{r}\, e^{i\frac{\theta}{n}}$$

⚠️ Every complex number actually has n distinct complex n^{th} roots!

That's two square roots, three cube roots, four 4^{th} roots, etc.

All complex roots

The complex roots of $z = re^{i\theta}$ are given by:

$$w_k \;=\; \sqrt[n]{r}\cdot e^{i\left(\frac{\theta}{n}+k\frac{2\pi}{n}\right)} \qquad \text{for each } k = 0,\,1,\,2,\,\dots,\,n-1$$

In Cartesian notation:

$$w_k \;=\; \sqrt[n]{r}\,\cos\left(\frac{\theta}{n}+k\frac{2\pi}{n}\right)+\sqrt[n]{r}\,\sin\left(\frac{\theta}{n}+k\frac{2\pi}{n}\right)i$$

In words:

- Start with the basic root: $\sqrt[n]{r}\cdot e^{i\frac{\theta}{n}}$
- Rotate by increments of $\frac{2\pi}{n}$ to get all other roots

Extra - Complex roots proof

We must verify that $w_k^n = re^{i\theta}$:

$$\left(\sqrt[n]{r} \cdot e^{i\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right)}\right)^n \quad \gg\gg \quad r^{\frac{n}{n}} \cdot e^{i\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right)n}$$

$$\gg\gg \quad r \cdot e^{i\,(\theta + 2\pi k)} \quad \gg\gg \quad re^{i\theta}e^{2\pi k} \quad \gg\gg \quad re^{i\theta}$$

06 Illustration

Example - Finding all 4^{th} roots of 16

Compute all the 4^{th} roots of 16.

Solution

Write $16 = 16e^{0i}$.

Evaluate roots formula:

$$\left(16e^{0i}\right)^{\frac{1}{4}} \quad \gg\gg \quad w_k = 16^{\frac{1}{4}} e^{i\left(\frac{0}{4} + k\frac{2\pi}{4}\right)}$$

Simplify:

$$\gg\gg \quad 2e^{i \cdot k\frac{\pi}{2}} \quad \gg\gg \quad 2,\ 2i,\ -2,\ -2i$$

Example - Finding 2^{nd} roots of $2i$

Find both 2^{nd} roots of $2i$.

Solution

Write $2i = 2e^{i\frac{\pi}{2}}$.
Evaluate roots formula:

$$\left(2e^{i\frac{\pi}{2}}\right)^{\frac{1}{2}} \quad \gg\gg \quad w_k = \sqrt{2}e^{i\left(\frac{\pi/2}{2} + k\frac{2\pi}{2}\right)}$$

$$\gg\gg \quad \sqrt{2}e^{i\left(\frac{\pi}{4} + k\pi\right)}$$

Compute the options: $k = 0,\ 1$:

$$\gg\gg \quad \sqrt{2}e^{i\frac{\pi}{4}},\ \sqrt{2}e^{i\frac{5\pi}{4}}$$

Convert to rectangular:

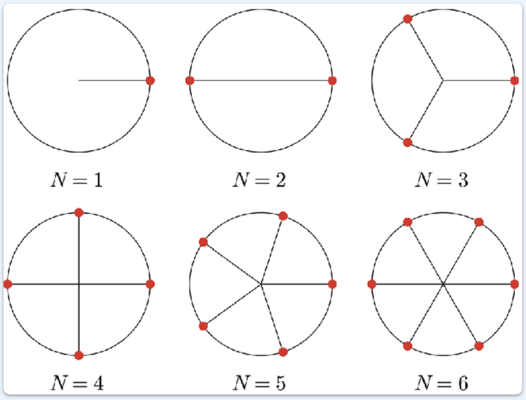
$$\gg\gg \quad \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right),\ \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$$

$$\gg\gg \quad 1 + i,\ 1 - i$$

☰ Example - Some roots of unity

Find the 1^{st} and 2^{nd} and 3^{rd} and 4^{th} and 5^{th} and 6^{th} roots of the number 1.

Solution



1^{st}

Write $1 = e^{0i}$. Evaluate roots formula. There is no possible k:

$$\left(e^{0i}\right)^{\frac{1}{1}} \quad \gg\gg \quad e^{0i} \quad \gg\gg \quad 1$$

2^{nd}

Write $1 = e^{0i}$. Evaluate roots formula in terms of k:

$$\left(e^{0i}\right)^{\frac{1}{2}} \quad \gg\gg \quad w_k = e^{i\left(\frac{0}{2}+k\frac{2\pi}{2}\right)} \qquad k = 0,\, 1$$

Compute the two options, $k = 0,\, 1$:

$$\gg\gg \quad 1,\ e^{\pi i} \quad \gg\gg \quad 1,\ -1$$

3^{rd}

Evaluate roots formula in terms of k:

$$\left(e^{0i}\right)^{\frac{1}{3}} \quad \gg\gg \quad w_k = e^{i\left(\frac{0}{3}+k\frac{2\pi}{3}\right)}$$

Compute the options: $k = 0,\, 1,\, 2$:

$$\gg\gg \quad 1,\ e^{i\frac{2\pi}{3}},\ e^{i\frac{4\pi}{3}} \quad \gg\gg \quad 1, \quad -\frac{1}{2}+\frac{\sqrt{3}}{2}i, \quad -\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

4^{th}

Evaluate roots formula:

$$\left(e^{0i}\right)^{\frac{1}{4}} \quad \gg\gg \quad w_k = e^{i\left(\frac{0}{4}+k\frac{2\pi}{4}\right)}$$

Compute the options: $k = 0,\ 1,\ 2,\ 3$:

$$1,\ e^{i\frac{\pi}{2}},\ e^{i\pi},\ e^{i\frac{3\pi}{2}} \quad \gg\gg \quad 1,\ i,\ -1,\ -i$$

5^{th}

Evaluate roots formula:

$$\left(e^{0i}\right)^{\frac{1}{5}} \quad \gg\gg \quad w_k = e^{i\left(\frac{0}{5}+k\frac{2\pi}{5}\right)}$$

Compute the options: $k = 0,\ 1,\ 2,\ 3,\ 4$:

$$1,\ e^{i\frac{2\pi}{5}},\ e^{i\frac{4\pi}{5}},\ e^{i\frac{6\pi}{5}},\ e^{i\frac{8\pi}{5}}$$

Don’t simplify, it’s not feasible.

6^{th}

Evaluate roots formula:

$$\left(e^{0i}\right)^{\frac{1}{6}} \quad \gg\gg \quad w_k = e^{i\left(\frac{0}{6}+k\frac{2\pi}{6}\right)}$$

Compute the options: $k = 0,\ 1,\ 2,\ 3,\ 4,\ 5$:

$$1,\ e^{i\frac{2\pi}{6}},\ e^{i\frac{4\pi}{6}},\ e^{i\frac{6\pi}{6}},\ e^{i\frac{8\pi}{6}},\ e^{i\frac{10\pi}{6}}$$

Simplify:

$$\gg\gg \quad 1,\ \frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -1,\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i,\ \frac{1}{2}-\frac{\sqrt{3}}{2}i$$