Unit 03 - Essential problems

Sequences

L'Hopital practice - converting indeterminate form

By imitating the technique in from the L'Hopital's Rule example, find the limit of the sequence:

$$a_n = \sqrt{n}\,\ln\left(1+\frac{1}{n}\right)$$

Series basics

Geometric series

Compute the following summation values using the sum formula for geometric series.

(a) $\displaystyle\sum_{n=0}^{\infty} 5^{-n}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{2+3^n}{5^n}$ (c) $\displaystyle\sum_{n=-4}^{\infty} \left(-\frac{4}{9}\right)^n$ (d) $\displaystyle\sum_{n=0}^{\infty} e^{3-2n}$

Limits and convergence

For each sequence, either write the limit value (if it converges), or write 'diverges'.

(a) $b_n = \dfrac{5n-1}{12n+9}$ (b) $b_n = (-1)^n\left(\dfrac{5n-1}{12n+9}\right)$ (c) $a_n = \sqrt{4+\dfrac{1}{n}}$ (d) $a_n = \cos^{-1}\left(\frac{n^3}{n^3+1}\right)$

(d) $a_n = 10 + \left(-\dfrac{1}{9}\right)^n$ (e) $a_n = 10 + \left(-\dfrac{1}{9}\right)^n$ (f) $c_n = 1.01^n$ (g) $a_n = 2^{1/n}$

(h) $c_n = \dfrac{n!}{9^n}$ (i) $a_n = \dfrac{3n^2+n+2}{2n^2-3}$ (j) $a_n = \dfrac{\cos n}{n}$ (k) $d_n = \ln 5^n - \ln n!$

(l) $a_n = \left(2+\dfrac{4}{n^2}\right)^{1/3}$ (m) $c_n = \ln\left(\dfrac{2n+1}{3n+4}\right)$ (n) $y_n = \dfrac{e^n}{2^n}$ (o) $a_n = \dfrac{(\ln n)^2}{n}$

(p) $a_n = \dfrac{(-1)^n(\ln n)^2}{n}$ (q) $b_n = \dfrac{3-4^n}{2+7\cdot 4^n}$ (r) $a_n = \left(1+\dfrac{1}{n}\right)^n$ (s) $a_n = \dfrac{1}{\ln\left(1+\frac{1}{n}\right)}$

(t) $a_n = n\sin\dfrac{\pi}{n}$

Positive series

Integral Test, Direct Comparison Test, Limit Comparison Test

Determine whether the series converges by checking applicability and then applying the designated convergence test.

(a) Integral Test: $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

(b) Direct Comparison Test: $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^5 + 4n + 1}$

(c) Limit Comparison Test: $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{n^4 - 1}$

Limit Comparison Test (LCT)

Use the Limit Comparison Test to determine whether the series converges:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + \ln n}$$

Show your work. You must check that the test is applicable.

Alternating series

Absolute and conditional convergence

Apply the Alternating Series Test (AST) to determine whether the series are absolutely convergent, conditionally convergent, or divergent.

Show your work. You must check that the test is applicable.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1/3}}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^4}{n^3 + 1}$

Alternating series: error estimation

Find the approximate value of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!}$ such that the error E_n satisfies $|E_n| < 0.005$.

How many terms do you really need?

Ratio test and Root test

Ratio and root tests

Apply the ratio test or the root test to determine whether each of the following series is absolutely convergent, conditionally convergent, or divergent.

(a) $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^{100}}$

(b) $\sum_{n=0}^{\infty} \left(\frac{5n}{10n+4}\right)^n$

(c) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{3^n}$

Power series: Radius and Interval

Power series - radius and interval

Find the radius and interval of convergence for these power series:

(a) $\sum_{n=0}^{\infty} (-1)^n \frac{(x+3)^n}{n!}$

(b) $\sum_{n=1}^{\infty} (-1)^n \frac{(x-7)^n}{n}$

(c) $\sum_{n=12}^{\infty} n^n (x-2)^n$

Series tests: strategy tips

Various limits, Part II

Find the limits. You may use $+\infty$ or $-\infty$ or DNE as appropriate. Braces indicate sequences.

- C = Convergent
- AC = Absolutely Convergent
- CC = Conditionally Convergent
- D = Divergent

a_n	$\lim_{n\to\infty} a_n$	$\{a_n\}$ **C or D**	$\lim_{n\to\infty} (-1)^n a_n$	$\{(-1)^n a_n\}$ **C or D**	$\sum a_n$ **AC, CC, or D**	$\sum (-1)^n a_n$ **AC, CC, or D**
$\frac{4n!}{2^n}$						
$\frac{(n+2)3^n}{n!}$						
$\frac{4^n}{(3n)^n}$						
$\frac{1}{(2n+1)!}$						

Power series as functions

Modifying geometric power series

Consider the geometric power series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$.

For this problem, you should modify the series for $\frac{1}{1-x}$.

(a) Write $\frac{1}{5-x}$ as a power series centered at $c = 0$ and determine its radius of convergence.

(b) Write $\frac{1}{16+2x^3}$ as a power series centered at $c = 0$ and determine its radius of convergence.

Finding a power series

Find a power series representation for these functions:

(a) $f(x) = \dfrac{x^2}{x^4 + 81}$ (b) $g(x) = x^2 \ln(1 + x)$

Tayler and Maclaurin series

Maclaurin series I

For each of these functions, find the Maclaurin series.

(a) $x \ln(1 - 5x)$ (b) $x^2 \cos(x^3)$

Discovering the function from its Maclaurin series II

For each of these series, identify the function of which it is the Maclaurin series, and evaluate the function at an appropriate choice of x to find the total sum for the series.

(a) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+1}}{4^{2n+1}(2n+1)!}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{2n}}{n!}$ (c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n+2}}{3^{2n+1}(2n)!}$

Summing a Maclaurin series by guessing its function

For each of these series, identify the function of which it is the Maclaurin series:

(a) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{5x^{4n+2}}{(2n+1)!}$ (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-5x)^{n+1}}{n+1}$

Now find the total sums for these series:

(c) $\sum_{n=0}^{\infty} \frac{(-5)^n}{n!}$ (d) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{9^n (2n)!}$

(Hint: for (c)-(d), do the process in (a)-(b), then evaluate the resulting function somewhere.)

Large derivative at $x = 0$ using pattern of Maclaurin series

Consider the function $f(x) = x^2 \sin\left(5x^3\right)$. Find the value of $f^{(35)}(0)$.

(Hint: find the rule for coefficients of the Maclaurin series of $f(x)$ and then plug in 0.)

Applications of Taylor series

Approximating $1/e$

Using the series representation of e^x, show that:

$$\frac{1}{e} = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$$

Now use the alternating series error bound to approximate $\frac{1}{e}$ to an error within 10^{-3}.

Some estimates using series

For each of these estimates, use the error bound formula for alternating series.

Find an infinite series representation of $\int_0^1 \sin(x^2)\, dx$ and then use your series to estimate this integral to within an error of 10^{-3}.