W10 Notes

Ratio test and Root test

01 Theory

Ratio Test (RaT)

Applicability: Any series with nonzero terms.

Test Statement:

Suppose that $\left|\frac{a_{n+1}}{a_n}\right| \longrightarrow L$ as $n \to \infty$.

Then:

$$\begin{aligned} L < 1: &\qquad \sum_{n=1}^{\infty} a_n \quad \text{converges absolutely} \\ L > 1: &\qquad \sum_{n=1}^{\infty} a_n \quad \text{diverges} \\ L = 1 \text{ or DNE}: &\qquad \text{test inconclusive} \end{aligned}$$

Extra - Ratio test: explanation

To understand the ratio test, consider this series:

$$\sum_{n=0}^{\infty} \frac{2^n}{n!} \quad = \quad 1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \cdots$$

- The term $\frac{2^3}{3!}$ is given by multiplying the prior term by $\frac{2}{3}$.
- The term $\frac{2^4}{4!}$ is given by multiplying the prior term by $\frac{2}{4}$.
- The term a_n is created by multiplying the prior term by $\frac{2}{n}$.

When $n > 3$, the multiplication factor giving the next term is necessarily less than $\frac{2}{3}$. Therefore, when $n > 3$, the terms shrink *faster than those of a geometric series* having $r = \frac{2}{3}$. Therefore this series converges.

Similarly, consider this series:

$$\sum_{n=0}^{\infty} \frac{10^n}{n!} \quad = \quad 1 + \frac{10}{1!} + \frac{10^2}{2!} + \frac{10^3}{3!} + \cdots$$

Write $R_n = \frac{a_n}{a_{n-1}}$ for the ratio from the prior term a_{n-1} to the current term a_n. For this series, $R_n = \frac{10}{n}$.

This ratio falls below $\frac{10}{11}$ when $n > 11$, after which the terms necessarily shrink faster than those of a geometric series with $r = \frac{10}{11}$. Therefore this series converges.

The main point of the discussion can be stated like this:

$$R_n \to L < 1 \quad \text{as} \quad n \to \infty$$

Whenever this is the case, then *eventually* the ratios are bounded below some $r < 1$, and the series terms are smaller than those of a converging geometric series.

Extra - Ratio test: proof

Let us write $R_n = \left|\frac{a_{n+1}}{a_n}\right|$ for the ratio to the next term from term n.

Suppose that $R_n \to L$ as $n \to \infty$, and that $L < 1$. This means: eventually the ratio of terms is close to L; so eventually it is less than 1.

More specifically, let us define $r = \frac{L+1}{2}$. This is the point halfway between L and 1. Since $R_n \to L$, we know that eventually $R_n < r$.

Any geometric series with ratio r converges. Set $c = a_N$ for N big enough that $R_N < r$. Then the terms of our series satisfy $|a_{N+n}| \leq cr^n$, and the series starting from a_N is absolutely convergent by comparison to this geometric series.

(Note that the terms $a_1, \ldots, a_{N-1}$ do not affect convergence.)

02 Illustration

Example - Ratio test

(a) Observe that $\displaystyle\sum_{n=0}^{\infty} \frac{10^n}{n!}$ has ratio $R_n = \frac{10}{n}$ and thus $R_n \to 0 < 1$. Therefore the RaT implies that this series converges.

⚠ Notice this technique!

Simplify the ratio:

$$\frac{\frac{10^{n+1}}{(n+1)!}}{\frac{n!}{10^n}} \quad \gg\gg \quad \frac{10^{n+1}}{(n+1)!} \cdot \frac{n!}{10^n}$$

$$\gg\gg \quad \frac{10 \cdot 10^n}{(n+1)n!} \cdot \frac{n!}{10^n} \quad \gg\gg \quad \frac{10}{n}$$

We *frequently* use these rules:

$$10^{n+1} = 10^n \cdot 10, \qquad (n+1)! = (n+1)n!$$

to simplify ratios having exponents and factorials.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ has ratio $R_n = \frac{(n+1)^2}{2^{n+1}} \Big/ \frac{n^2}{2^n}$.

Simplify this:

$$\frac{(n+1)^2}{2^{n+1}} \Big/ \frac{n^2}{2^n} \qquad \gg\gg \qquad \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}$$

$$\gg\gg \quad \frac{(n+1)^2 \cdot 2^n}{n^2 \cdot 2 \cdot 2^n} \quad \gg\gg \quad \frac{n^2+2n+1}{2n^2} \longrightarrow \frac{1}{2} \text{ as } n \to \infty$$

So the series *converges absolutely* by the ratio test.

(c) Observe that $\sum_{n=1}^{\infty} n^2$ has ratio $R_n = \frac{n^2+2n+1}{n^2} \to 1$ as $n \to \infty$.

So the ratio test is *inconclusive*, even though this series fails the SDT and obviously diverges.

(d) Observe that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ has ratio $R_n = \frac{n^2}{n^2+2n+1} \to 1$ as $n \to \infty$.

So the ratio test is *inconclusive*, even though the series converges as a p-series with $p = 2 > 1$.

(e) More generally, the ratio test is usually *inconclusive for rational functions*; it is more effective to use LCT with a p-series.

03 Theory

Root Test (RooT)

Applicability: Any series.

Test Statement:

Suppose that $\sqrt[n]{|a_n|} \longrightarrow L$ as $n \to \infty$.

Then:

$$\begin{aligned} L < 1: &\quad \sum_{n=1}^{\infty} a_n \quad \text{converges absolutely} \\ L > 1: &\quad \sum_{n=1}^{\infty} a_n \quad \text{diverges} \\ L = 1 \text{ or DNE}: &\quad \text{test inconclusive} \end{aligned}$$

Extra - Root test: explanation

The fact that $\sqrt[n]{|a_n|} \to L$ and $L < 1$ implies that eventually $\sqrt[n]{|a_n|} < r$ for all high enough n, where $r = \frac{L+1}{2}$ is the midpoint between L and 1.

Now, the equation $\sqrt[n]{|a_n|} < r$ is equivalent to the equation $|a_n| < r^n$.

Therefore, eventually the terms $|a_n|$ are each less than the corresponding terms of this convergent geometric series:

$$\sum_{n=1}^{\infty} r^n \; = \; 1 + r + r^2 + r^3 + \cdots$$

04 Illustration

Root test examples

(a) Observe that $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^n$ has roots of terms:

$$|a_n|^{1/n} = \left(\left(\frac{1}{n}\right)^n\right)^{1/n} = \frac{1}{n}$$

Because $\frac{1}{n} \to 0 < 1$ as $n \to \infty$, the RooT shows that the series converges.

(b) Observe that $\sum_{n=1}^{\infty}(-1)^n\left(\frac{n}{2n+1}\right)^n$ has roots of terms:

$$\sqrt[n]{|a_n|} = \frac{n}{2n+1} \to \frac{1}{2} < 1$$

Because $\frac{n}{2n+1} \to \frac{1}{2}$ as $n \to \infty$, the RooT shows that the series converges.

(c) Observe that $\sum_{n=1}^{\infty}\left(\frac{3}{n}\right)^n$ converges because $\sqrt[n]{|a_n|} = \frac{3}{n} \to 0$ as $n \to \infty$.

Ratio test versus root test

Determine whether the series $\sum_{n=1}^{\infty}\frac{n^2 4^n}{5^{n+2}}$ converges absolutely or conditionally or diverges.

Solution

Before proceeding, rewrite somewhat the general term as $\left(\frac{n}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^n$.

Now we solve the problem first using the ratio test. By plugging in $n+1$ we see that

$$a_{n+1} = \left(\frac{n+1}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^{n+1}$$

So for the ratio R_n we have:

$$\left(\frac{n+1}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^{n+1} \cdot \left(\frac{5}{n}\right)^2 \cdot \left(\frac{5}{4}\right)^n \qquad \gg\gg \qquad \frac{n^2+2n+1}{n^2} \cdot \frac{4}{5} \longrightarrow \frac{4}{5} < 1 \text{ as } n \to \infty$$

Therefore the series converges absolutely by the ratio test.

Now solve the problem again using the root test. We have for $\sqrt[n]{|a_n|}$:

$$\left(\left(\frac{n}{5}\right)^2 \cdot \left(\frac{4}{5}\right)^n\right)^{1/n} = \left(\frac{n}{5}\right)^{2/n} \cdot \frac{4}{5}$$

To compute the limit as $n \to \infty$ we must use logarithmic limits and L'Hopital's Rule. So, first take the log:

$$\ln\left(\left(\frac{n}{5}\right)^{2/n} \cdot \frac{4}{5}\right) = \frac{2}{n}\ln\frac{n}{5} + \ln\frac{4}{5}$$

Then for the first term apply L'Hopital's Rule:

$$\frac{\ln \frac{n}{5} \xrightarrow{d/dx} \frac{1}{n/5} \cdot \frac{1}{5}}{n/2 \xrightarrow{d/dx} 1/2} \quad \gg\gg \quad \frac{1/n}{1/2} \quad \gg\gg \quad \frac{2}{n} \longrightarrow 0 \text{ as } n \to \infty$$

So the first term goes to zero, and the second (constant) term is the value of the limit. So the log limit is $\ln \frac{4}{5}$, and the limit (before taking logs) must be $e^{\ln \frac{4}{5}}$ (inverting the log using e^x) and this is $\frac{4}{5}$. Since $\frac{4}{5} < 1$, the root test also shows that the series converges absolutely.

Series tests - strategy tips

05 Theory

It can help to associate certain “strategy tips” to find convergence tests based on certain patterns.

Matching powers → Simple Divergence Test

$$\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$$

Use the SDT because we see the highest power is the *same* ($= 1$) in numerator and denominator.

Rational or Algebraic → Limit Comparison Test

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$$

Use the LCT because we have a *rational or algebraic* function (positive terms).

Not rational, not factorials → Integral Test

$$\sum_{n=1}^{\infty} n e^{-n^2}$$

Use the IT because we do *not* have a rational/algebraic function, and we do *not* see factorials.

Rational, alternating → AST and LCT

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^4+1}$$

Use the AST because it’s alternating. Then use the LCT (to find absolute convergence) because its a rational function.

Factorials → Ratio Test

$$\sum_{n=1}^{\infty} \frac{2^k}{k!}$$

Use the RaT because we see a factorial. (In case of alternating + factorial, use RaT first.)

Recognize geometric → LCT or DCT

$$\sum_{n=1}^{\infty} \frac{1}{2+3^n}$$

Use the LCT or DCT comparing to $\frac{1}{3^n}$ because we see similarity to $\frac{1}{3^n}$ (recognize geometric).

Power series: Radius and Interval

06 Theory

A power series looks like this:

$$f(x) \; = \; a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

Power series are used to *build and study functions*. They allow a uniform "modeling framework" in which many functions can be described and compared. Power series are also convenient for *computers* because they provide a way to store and evaluate *differentiable* functions.

Small x needed for power series

The most important fact about power series is that they work for *small values of x*.

Many power series diverge for $|x|$ too big; but even when they converge, for big $|x|$ they converge more slowly, and partial sum approximations are less accurate.

The idea of a power series is a modification of the idea of a geometric series in which the common ratio r becomes a variable x, and each term has an additional *coefficient parameter* a_n controlling the relative contribution of different orders.

07 Theory

Every power series has a **radius of convergence** and an **interval of convergence**.

Radius of convergence

Consider a power series centered at $x = 0$:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

Define L as the limit of coefficient ratios:

$$L \; = \; \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

Then reciprocal, $R = 1/L$, is the **radius of convergence**; it can be anything in $[0, \infty]$ including either extreme.

The power series necessarily converges for $|x| < R$ and diverges for $|x| > R$.

Extra - Radius of convergence: explanatory proof

Treat the variable x in the power series $f(x) = a_0 + a_1x + a_2x^2 + \cdots$ as a constant.

Apply the ratio test to this series. The ratio function is:

$$R_n = \left|\frac{a_{n+1}}{a_n}\right| \cdot |x|$$

Since $|x|$ is a constant here, we have:

$$\lim_{n\to\infty} R_n \;=\; L|x|$$

Therefore, the ratio test says that the series converges absolutely when $|x| < 1/L$, and diverges when $|x| > 1/L$.

We can build **shifted power series** for x near another value c. Just replace the variable x with a shifted variable $u = x - c$:

$$a_0 + a_1u + a_2u^2 + a_3u^3 + \ldots$$

$$\gg\gg \quad a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

The radius of convergence of a shifted series is calculated in the same way, using the coefficients:

$$R = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|}$$

However, in the shifted setting, the radius of convergence concerns the *distance from a*: Such a power series converges when $|x-a| < R$ and diverges when $|x-a| > R$.

The **interval of convergence** of a power series is determined by:

- the radius of convergence
- the center point
- special consideration of endpoints

Interval of convergence

The interval of convergence I of a power series $f(x) = a_0 + a_1x + a_2x^2 + \cdots$ is the set of values of x where the series converges.

The interval of convergence I is:

- centered at $x = c$
- extending a distance R to either side of c

- including / excluding the endpoints where $|x-c|=R$ depending on the particular case

To calculate the interval of convergence, follow these steps:

- Observe the center c of the shifted series; $c=0$ corresponds to no shift.
- Take the limit to compute R.
- Write down the *preliminary interval* $(c-R,\, c+R)$.
- Plug each endpoint $c-R$ and $c-R$ into the original series
  - $\to$ check for convergence
- Add in the convergent endpoints. There are 4 total possibilities.

08 Illustration

Example - Radius and interval for a few series

Series	Radius	Interval
$\sum_{n=0}^{\infty} x^n$	$R=1$	$(-1,1)$
$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$	$R=1$	$[1,3)$
$\sum_{n=0}^{\infty} n!\, x^n$	$R=0$	$\{0\}$
$\sum_{n=0}^{\infty} \frac{x^n}{(2n)!}$	$R=\infty$	$(-\infty,\infty)$

Example - Radius of convergence

Find the radius of convergence of the series:

(a) $\sum_{n=0}^{\infty} \frac{x^n}{2^n}$

(b) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

Solution

(a) The ratio of coefficients is $R_n = \left|\frac{a_{n+1}}{a_n}\right| = \frac{1/2^{n+1}}{1/2^n} = 1/2$.

Therefore $R=2$ and the series converges for $|x|<2$.

(b) This power series has $a_{2n+1}=0$, meaning it skips all odd terms.

Instead of the standard ratio function, we take the ratio of successive *even terms*. The series of even terms has coefficients $a_n = \frac{1}{(2n)!}$. So:

$$\left|\frac{a_{n+1}}{a_n}\right| \quad \gg\gg \quad \frac{\frac{1}{(2(n+1))!}}{\frac{1}{(2n)!}}$$

$$\gg\gg \quad \frac{1}{(2n+2)(2n+1)(2n)!} \cdot \frac{(2n)!}{1} \quad \gg\gg \quad \frac{1}{(2n+2)(2n+1)}$$

As $n \to \infty$, this converges to 0, so $L = 0$ and $R = \infty$.

Example - Interval of convergence

Find the interval of convergence of the following series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

Solution

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$

1. Apply ratio test.
   - Ratio of successive coefficients:

$$R_n = \left|\frac{1}{n+1} \cdot \frac{n}{1}\right| = \frac{n}{n+1}$$

   - Limit of ratios:

$$R_n = \frac{n}{n+1} \xrightarrow{n\to\infty} 1$$

   - Deduce $L = 1$ and therefore $R = 1$.
   - Therefore:

$$|x-3| < 1 \implies \text{converges}$$

$$|x-3| > 1 \implies \text{diverges}$$

2. Preliminary interval of convergence.
   - Translate to interval notation:

$$|x-3| < 1 \quad \gg\gg \quad x \in (3-1, 3+1)$$

$$\gg\gg \quad x \in (2, 4)$$

3. Final interval of convergence.
   - Check endpoint $x = 2$:

$$\sum_{n=1}^{\infty} \frac{(2-3)^n}{n} \quad \gg\gg \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad \gg\gg \quad \text{converges by AST}$$

- Check endpoint $x = 4$:

$$\sum_{n=1}^{\infty} \frac{(4-3)^n}{n} \quad \gg\gg \quad \sum_{n=1}^{\infty} \frac{1}{n} \quad \gg\gg \quad \text{diverges as } p\text{-series}$$

- Final interval of convergence: $x \in [2, 4)$

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}$

1. Ratio Test.

- Ratio of successive coefficients:

$$R_n = \left|\frac{a_{n+1}}{a_n}\right| \quad \gg\gg \quad \left|\frac{(-3)^{n+1}}{\sqrt{n+2}} \cdot \frac{\sqrt{n+1}}{(-3)^n}\right|$$

$$\gg\gg \quad \frac{3\sqrt{n+2}}{\sqrt{n+1}}$$

- Limit of ratios:

$$\lim_{n\to\infty} R_n \quad \gg\gg \quad \lim_{n\to\infty} \frac{3\sqrt{n+2}}{\sqrt{n+1}} \quad \gg\gg \quad 3$$

- Deduce $L = 3$ and thus $R = 1/3$.
- Therefore:

$$|x| < \frac{1}{3} \implies \text{converges}$$

$$|x| > \frac{1}{3} \implies \text{diverges}$$

- Preliminary interval of convergence: $x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$

2. Check endpoints.

- Check endpoint $x = -1/3$:

$$\sum_{n=0}^{\infty} \frac{\left(-3 \cdot \left(-\frac{1}{3}\right)\right)^n}{\sqrt{n+1}} \quad \gg\gg \quad \sum_{n=0}^{\infty} \frac{1^n}{\sqrt{n+1}} \quad \gg\gg \quad \text{diverges as } p\text{-series}$$

- Check endpoint $x = +1/3$:

$$\sum_{n=0}^{\infty} \frac{\left(-3 \cdot \left(+\frac{1}{3}\right)\right)^n}{\sqrt{n+1}} \quad \gg\gg \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \quad \gg\gg \quad \text{converges by AST}$$

- Final interval of convergence: $x \in (-1/3, 1/3]$

Interval of convergence - further examples

Find the interval of convergence of the following series.

- (a) $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$
- (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n}$

Solution

(a) $\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$

- Ratio of coefficients: $R_n = \frac{n+1}{3n} \longrightarrow \frac{1}{3}$.
- So the $R=3$, center is $x=-2$, and the preliminary interval is $(-2-3, -2+3) = (-5, 1)$.
- Check endpoints: $\sum \frac{n(-3)^n}{3^{n+1}}$ diverges and $\sum \frac{n(3)^n}{3^{n+1}}$ also diverges. Final interval is $(-5, 1)$.

(b) $\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n}$

- Ratio of coefficients: $R_n = \frac{n+1}{n} \longrightarrow 1$.
- So $R=1$, and the series converges when $|4x+1| < 1$.
- ⚠ Extract preliminary interval.
    - Divide by 4:

$$|4x+1| < 1 \quad \overset{\div 4}{\gg\!\gg} \quad |x+1/4| < 1/4 \quad \gg\!\gg \quad x \in (0, 1/2)$$

- Check endpoints: $\sum \frac{(4 \cdot \frac{-1}{2} + 1)^n}{n}$ converges but $\sum \frac{1}{n}$ diverges.
- Final interval of convergence: $[-1/2, 0)$