W11 Notes

Power series as functions

01 Theory

Given a numerical value for x within the interval of convergence of a power series, the series sum may be considered as the output $f(x)$ of a function f.

Many techniques from algebra and calculus can be applied to such power series functions.

Addition and Subtraction:

$$\begin{array}{rcl} f & = & a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\ g & = & b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots \\ \hline \\ f + g & = & (a_0 + b_0) + (a_1 + b_1)\,x + (a_2 + b_2)\,x^2 + \cdots \end{array}$$

Summation notation:

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n \quad = \quad \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

Scaling:

$$cf \;=\; ca_0 + (ca_1)\,x + (ca_2)\,x^2 + \cdots$$

Summation notation:

$$c\sum_{n=0}^{\infty} a_n x^n \quad = \quad \sum_{n=0}^{\infty} (ca_n)\,x^n$$

Extra - Multiplication and composition

Multiplication:

$$f \cdot g = \left(a_0 + a_1 x + a_2 x^2 + \cdots\right) \cdot \left(b_0 + b_1 x + b_2 x^2 + \cdots\right)$$

$$= a_0 b_0 + (a_0 b_1 + a_1 b_0)\,x + (a_0 b_2 + a_1 b_1 + a_2 b_0)\,x^2 + \cdots$$

For example, suppose that the geometric power series $f(x) = 1 + x + x^2 + x^3 + \cdots$ converges, so $|x| < 1$. Then we have for its square:

$$\begin{aligned} f \cdot f \;=\; f(x)^2 \;&=\; (1 + x + x^2 + \cdots) \cdot (1 + x + x^2 + \cdots) \\ &=\; 1 + (1+1)x + (1+1+1)x^2 + \cdots \\ &=\; 1 + 2x + 3x^2 + 4x^3 + \cdots \\ &=\; \sum_{n=0}^{\infty} (n+1) x^n \end{aligned}$$

Composition:

$$f(-x) \;=\; 1 - x + x^2 - x^3 + x^4 - \cdots$$

$$\begin{aligned} f(2x^3) \;&=\; 1 + 2x^3 + (2x^3)^2 + \cdots \\ &=\; 1 + 2x^3 + 4x^6 + 8x^9 + \cdots \end{aligned}$$

Assume:

$$f \;=\; a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \;=\; \sum_{n=0}^{\infty} a_n x^n$$

Then:

Differentiation:

$$\frac{df}{dx} \;=\; a_1 + (2a_2)\,x + (3a_3)\,x^2 + \cdots \quad = \quad \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Antidifferentiation:

$$\int f(x)\,dx \;=\; C + a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots \quad = \quad C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

For example, for the geometric series we have:

$$f = \; 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$\frac{df}{dx} = \; 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

$$\int f\,dx = \; C + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots$$

Do the series created with sums, products, derivatives etc., all converge? On what interval?

For the algebraic operations, the resulting power series will converge wherever both of the original series converge.

For calculus operations, the *radius* is preserved, but the *endpoints* are not necessarily:

Power series calculus - Radius preserved

If the power series $f(x)$ has radius of convergence R, then the power series $f'(x)$ and $\int f\,dx$ also have the same radius of convergence R.

Power series calculus - Endpoints not preserved

It is possible that a power series $f(x)$ converges at and endpoint a of its interval of convergence, yet f' and $\int f\,dx$ do *not* converge at a.

Extra - Proof of radius for derivative and integral series

Suppose $f(x)$ has radius of convergence $R = L^{-1}$:

$$\left|\frac{a_{n+1}}{a_n}\right| \cdot |x| \longrightarrow L \cdot |x| \qquad \text{as } \; n \to \infty$$

Consider now the derivative f' and its ratios of successive terms:

$$\left|\frac{(n+1)a_{n+1}x^n}{n a_n x^{n-1}}\right| = \left(\frac{n+1}{n}\right) \cdot \left|\frac{a_{n+1}}{a_n}\right| \cdot |x| \quad \xrightarrow{n\to\infty} \quad 1 \cdot L \cdot |x| = L \cdot |x|$$

Consider instead the antiderivative $\int f\,dx$ and its ratios of successive terms:

$$\left|\frac{\left(\frac{1}{n+1}\right)a_n x^{n+1}}{\left(\frac{1}{n}\right)a_{n-1}x^n}\right| = \left(\frac{n}{n+1}\right)\cdot\left|\frac{a_n}{a_{n-1}}\right|\cdot|x| \quad \xrightarrow{n\to\infty} \quad 1\cdot L\cdot|x| = L\cdot|x|$$

In both these cases the ratio test provides that the series converges when $|x| < L^{-1}$.

02 Illustration

Example - Geometric series: algebra meets calculus

Consider the geometric series as a power series functions:

$$\frac{1}{1-x} \quad = \quad 1 + x + x^2 + x^3 + \cdots$$

Take the derivative of both sides of the *function*:

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) \quad \gg\gg \quad \frac{1}{(1-x)^2} \quad \gg\gg \quad \left(\frac{1}{1-x}\right)^2$$

This means f satisfies the identity:

$$f' = f^2$$

Now compute the derivative of the *series*:

$$1 + x + x^2 + x^3 + \cdots \quad \overset{\frac{d}{dx}}{\gg\gg} \quad 1 + 2x + 3x^2 + 4x^3 + \cdots$$

On the other hand, compute the square of the series:

$$\left(1 + x + x^2 + x^3 + \cdots\right)^2 \quad \gg\gg \quad 1 + 2x + 3x^2 + 4x^3 + \cdots$$

So we find that the *same relationship holds*, namely $f' = f^2$, for the closed formula and the series formula for this function.

Example - Manipulating geometric series: algebra

Find power series that represent the following functions:

(a) $\dfrac{1}{1+x}$ (b) $\dfrac{1}{1+x^2}$ (c) $\dfrac{x^3}{x+2}$ (d) $\dfrac{3x}{2-5x}$

Solution

(a) $\dfrac{1}{1+x}$

1. Rewrite in format $\frac{1}{1-u}$.
   - Introduce double negative:

$$\frac{1}{1+x} = \frac{1}{1-(-x)}$$

   - Choose $u = -x$.
2. Plug $u = -x$ into geometric series.
   - Geometric series in u:

$$1 + u + u^2 + u^3 + \cdots$$

- Plug in $u = -x$:

$$\gg\gg \qquad 1 + (-x) + (-x)^2 + (-x)^3 + \cdots$$

- Simplify:

$$\gg\gg \qquad 1 - x + x^2 - x^3 + \cdots$$

- Final answer:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

(b) $\dfrac{1}{1+x^2}$

1. Rewrite in format $\frac{1}{1-u}$.

- Rewrite:

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

- Choose $u = -x^2$.

2. Plug $u = -x^2$ into geometric series.

- Geometric series in u:

$$1 + u + u^2 + u^3 + \cdots$$

- Plug in $u = -x^2$:

$$\gg\gg \qquad 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots \qquad \gg\gg \qquad 1 - x^2 + x^4 - x^6 + \cdots$$

- Final answer:

$$\frac{1}{1+x} = 1 - x^2 + x^4 - x^6 + \cdots$$

(c) $\dfrac{x^3}{x+2}$

1. Rewrite in format $Ax^3 \cdot \frac{1}{1-u}$.

- Rewrite:

$$\frac{x^3}{x+2} \qquad \gg\gg \qquad x^3 \cdot \frac{1}{2+x} \qquad \gg\gg \qquad x^3 \cdot \frac{1}{2\left(1+\frac{x}{2}\right)}$$

$$\gg\gg \qquad \frac{1}{2}x^3 \cdot \frac{1}{1+\frac{x}{2}} \qquad \gg\gg \qquad \frac{1}{2}x^3 \cdot \frac{1}{1-\left(-\frac{x}{2}\right)}$$

- Choose $u = -\frac{x}{2}$. Here $Ax^3 = \frac{1}{2}x^3$.

2. Plug $u = -x^2$ into geometric series.

- Geometric series in u:

$$1 + u + u^2 + u^3 + \cdots$$

- Plug in $u = -\frac{x}{2}$:

$$\ggg \qquad 1 + (-\tfrac{x}{2}) + (-\tfrac{x}{2})^2 + (-\tfrac{x}{2})^3 + \cdots$$

$$\ggg \qquad 1 - \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^3 + \cdots$$

- Obtain:

$$\frac{1}{1-\left(-\frac{x}{2}\right)} = 1 - \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^3 + \cdots$$

3. Multiply by $\frac{1}{2}x^3$.

- Distribute:

$$\frac{1}{2}x^3 \cdot \frac{1}{1-\left(-\frac{x}{2}\right)} \quad \gg\gg \quad \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 - \frac{1}{16}x^6 + \cdots$$

- Final answer:

$$\frac{x^3}{x+2} = \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 - \frac{1}{16}x^6 + \cdots$$

(d) $\dfrac{3x}{2-5x}$

1. Rewrite in format $Ax \cdot \frac{1}{1-u}$.

- Rewrite:

$$\frac{3x}{2-5x} \quad \gg\gg \quad 3x \cdot \frac{1}{2-5x}$$

$$\gg\gg \quad 3x \cdot \frac{1}{2\left(1-\frac{5x}{2}\right)} \quad \gg\gg \quad \frac{3}{2}x \cdot \frac{1}{1-\frac{5x}{2}}$$

- Choose $u = \frac{5x}{2}$. Here $Ax = \frac{3}{2}x$.

2. Plug $u = \frac{5x}{2}$ into geometric series.

- Geometric series in u:

$$1 + u + u^2 + u^3 + \cdots$$

- Plug in $u = \frac{5x}{2}$:

$$\gg\gg \quad 1 + \left(\tfrac{5x}{2}\right) + \left(\tfrac{5x}{2}\right)^2 + \left(\tfrac{5x}{2}\right)^3 + \cdots$$

$$\gg\gg \quad 1 + \frac{5}{2}x + \frac{25}{4}x^2 + \frac{125}{8}x^3 + \cdots$$

- Obtain:

$$\frac{1}{1-\frac{5x}{2}} = 1 + \frac{5}{2}x + \frac{25}{4}x^2 + \frac{125}{8}x^3 + \cdots$$

3. Multiply by $\frac{3}{2}x$.

- Distribute:

$$\frac{3}{2}x \cdot \frac{1}{1-\frac{5x}{2}} \quad \gg\gg \quad \frac{3}{2}x + \frac{15}{4}x^2 + \frac{75}{8}x^3 + \frac{375}{16}x^4 + \cdots$$

- Final answer:

$$\frac{3x}{2-5x} = \frac{3}{2}x + \frac{15}{4}x^2 + \frac{75}{8}x^3 + \frac{375}{16}x^4 + \cdots$$

Example - Manipulating geometric series: calculus

Find a power series that represents $\ln(1+x)$.

Solution

1. Differentiate to obtain similarity to geometric sum formula.
   - Differentiate $\ln(1+x)$:

$$\frac{d}{dx}\ln(1+x) = \frac{1}{1+x} \qquad \gg\gg \qquad \frac{1}{1-(-x)}$$

2. Find power series of differentiated function.
   - Power series by modifying $\frac{1}{1-u}$ with $u = -x$:

$$\frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

3. Integrate series to find original function.
   - Integrate both sides:

$$\int \frac{1}{1-(-x)}\, dx = \int 1 - x + x^2 - x^3 + x^4 - \cdots\, dx$$

$$\ln(1+x) = D + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

   - Use known point to solve for D:

$$\ln(1+0) = D + 0 + 0 + \cdots \qquad \gg\gg \qquad 0 = D$$

   - Final answer:

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

Example - Recognizing and manipulating geometric series: Part I

(a) Evaluate $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$.

(Hint: consider the series of $\ln(1-x)$.)

(b) Find a series approximation for $\ln(2/3)$.

Solution

(a) Evaluate $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$. (Hint: consider the series of $\ln(1-x)$.)

1. Find the series representation of $\ln(1-x)$ following the hint.
   - Notice that $\frac{d}{dx}\ln(1-x) = \frac{-1}{1-x}$.
   - We know the series of $\frac{-1}{1-x}$:

$$\frac{-1}{1-x} = -(1 + x + x^2 + \cdots) = -1 - x - x^2 - \cdots$$

   - Notice that $\int \frac{-1}{1-x}\, dx = \ln(1-x) + C$; this is the desired function when $C = 0$.
   - Integrate the series term-by-term:

$$\int \frac{-1}{1-x}\, dx = \int -1 - x - x^2 - \cdots\, dx \qquad \gg\gg \qquad \ln(1-x) = D - x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$

- Solve for D using $\ln(1-0)=0$, so $0=D-0-0-\cdots$ and thus $D=0$. So:

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots \quad = \quad \sum_{n=1}^{\infty} -\frac{x^n}{n!}$$

2. Notice the similar formula.
   - The series formula $\sum_{n=1}^{\infty} -\frac{x^n}{n!}$ looks similar to the formula $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$.
3. Choose $x=-1$ to recreate the desired series.
   - We obtain equality by setting $x=-1$ because $-(-1)^n = (-1)^{n+1} = (-1)^{n-1}$.
4. Final answer is $\ln(1--1) = \ln 2$.

(b) Find a series approximation for $\ln(2/3)$.

1. Observe that $\ln(2/3) = \ln(1-1/3)$.
   - Therefore we can use the series $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$
2. Plug $x=1/3$ into the series for $\ln(1-x)$.
   - Plug in and simplify:

$$\begin{aligned}\ln(2/3) = \ln(1-1/3) &= -1/3 - \frac{(1/3)^2}{2} - \frac{(1/3)^3}{3} - \cdots \\ &= -\frac{1}{3} - \frac{1}{3^2\cdot 2} - \frac{1}{3^3\cdot 3} - \cdots\end{aligned}$$

Example - Recognizing and manipulating geometric series: Part II

(a) Find a series representing $\tan^{-1}(x)$ using differentiation.

(b) Find a series representing $\int \frac{dx}{1+x^4}$.

Solution

(a) Find a series representing $\tan^{-1}(x)$.

1. Notice that $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$.
2. Obtain the series for $\frac{1}{1+x^2}$.
   - Let $u=-x^2$:

$$\frac{1}{1+x^2} \quad \gg\gg \quad \frac{1}{1-u} = 1+u+u^2+\cdots$$

$$\gg\gg \quad 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

3. Integrate the series for $\frac{1}{1+x^2}$ by terms.
   - Set up the strategy. We know:

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$$

and:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

- Integrate term-by-term:

$$\gg\gg \quad \int 1 - x^2 + x^4 - x^6 + x^8 - \cdots \, dx$$

$$\gg\gg \quad D + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

- Conclude that:

$$\tan^{-1}(x) + C = D + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

4. Solve for $D - C$ by testing at $\tan^{-1}(0) = 0$.
   - Plugging in, obtain:

$$\tan^{-1}(0) = D - C + 0 + \cdots + 0$$

   so $D - C = 0$.
5. Final answer is $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$.

(b) Find a series representing $\displaystyle\int \frac{dx}{1+x^4}$.

1. Find a series representing the integrand.
   - Integrand is $\frac{1}{1+x^4}$.
   - Rewrite integrand in format of geometric series sum:

$$\frac{1}{1+x^4} \quad \gg\gg \quad \frac{1}{1-(-x^4)} \quad \gg\gg \quad \frac{1}{1-u}, \quad u = -x^4$$

   - Write the series:

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots \quad \gg\gg \quad 1 - x^4 + x^8 - x^{12} + x^{16} - \cdots \quad = \sum_{n=0}^{\infty} (-1)^n x^{4n}$$

2. Integrate the integrand series by terms.
   - Integrate term-by-term:

$$\int 1 - x^4 + x^8 - x^{12} + x^{16} - \cdots \, dx \quad \gg\gg \quad C + x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13} + \frac{x^{17}}{17} - \cdots$$

   - This is our final answer.

Taylor and Maclaurin series

03 Theory

Suppose that we have a power series function:

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Consider the *successive derivatives* of f:

$$\begin{array}{rclclclclcl}
f(x) & = & a_0 & + & a_1 x & + & a_2 x^2 & + & a_3 x^3 & + & a_4 x^4 & + & \cdots \\
f'(x) & = & 0 & + & a_1 & + & 2 \cdot a_2 x^1 & + & 3 \cdot a_3 x^2 & + & 4 \cdot a_4 x^3 & + & \cdots \\
f''(x) & = & 0 & + & 0 & + & 2 \cdot a_2 & + & 3 \cdot 2 \cdot a_3 x^1 & + & 4 \cdot 3 \cdot a_4 x^2 & + & \cdots \\
f'''(x) & = & 0 & + & 0 & + & 0 & + & 3 \cdot 2 \cdot 1 \cdot a_3 & + & 4 \cdot 3 \cdot 2 \cdot a_4 x^1 & + & \ldots \\
\vdots & & \vdots & & & & \vdots & & & & \vdots \\
f^{(n)}(x) & = & 0 & + & 0 & + & 0 & + & 0 & + & \cdots + n! \cdot a_n & + & \cdots
\end{array}$$

When these functions are evaluated at $x = 0$, all terms with a positive x-power become zero:

$$\begin{aligned} f(0) &= & a_0 &= & a_0 \\ f'(0) &= & a_1 &= & a_1 \\ f''(0) &= & 2\cdot a_2 &= & 2!\cdot a_2 \\ f'''(0) &= & 3\cdot 2\cdot a_3 &= & 3!\cdot a_3 \\ \vdots &= & \vdots &= & \vdots \\ f^{(n)}(0) &= & n\cdot(n-1)\cdots 2\cdot 1\cdot a_n &= & n!\cdot a_n \end{aligned}$$

This last formula is the basis for Taylor and Maclaurin series:

Power series: Derivative-Coefficient Identity

$$f^{(n)}(0) \quad = \quad n!\cdot a_n$$

This identity holds for a power series function $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ which has a nonzero radius of convergence.

We can apply the identity in both directions:

- Know $f(x)$? $\quad\rightsquigarrow\quad$ Calculate a_n for any n.
- Know a_n? $\quad\rightsquigarrow\quad$ Calculate $f^{(n)}(0)$ for any n.

Many functions can be ‘expressed’ or ‘represented’ near $x = c$ (i.e. for small enough $|x-c|$) as convergent power series. (This is true for almost all the functions encountered in pre-calculus and calculus.)

Such a power series representation is called a **Taylor series**.
When $c = 0$, the Taylor series is also called the **Maclaurin series**.

One power series representation we have already studied:

$$\frac{1}{1-x} \quad = \quad 1 + x + x^2 + x^3 + \cdots$$

Whenever a function has a power series (Taylor or Maclaurin), the Derivative-Coefficient Identity may be applied to *calculate the coefficients* of that series.

Conversely, sometimes a series can be interpreted as an *evaluated power series* coming from $x = c$ for some c. If the closed form function format can be obtained for this power series, the *total sum of the original series may be discovered* by putting $x = c$ in the argument of the function.

04 Illustration

Example - Maclaurin series of e^x

What is the Maclaurin series of $f(x) = e^x$?

Solution

Because $\frac{d}{dx}e^x = e^x$, we find that $f^{(n)}(x) = e^x$ for all n.

So $f^{(n)}(0) = e^0 = 1$ for all n. Therefore $a_n = \frac{1}{n!}$ for all n by the Derivative-Coefficient identity.

Thus:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Example - Maclaurin series of $\cos x$

Find the Maclaurin series representation of $\cos x$.

Solution

Use the Derivative-Coefficient Identity to solve for the coefficients:

$$a_n = \frac{f^{(n)}(0)}{n!}$$

n	$f^{(n)}(x)$	$f^{(n)}(0)$	a_n
0	$\cos x$	1	1
1	$-\sin x$	0	0
2	$-\cos x$	-1	$-1/2$
3	$\sin x$	0	0
4	$\cos x$	1	$1/24$
5	$-\sin x$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$

By studying the generating pattern of the coefficients, we find for the series:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

Maclaurin series from other Maclaurin series

(a) Find the Maclaurin series of $\sin x$ using the Maclaurin series of $\cos x$.

(b) Find the Maclaurin series of $f(x) = x^2 e^{-5x}$ using the Maclaurin series of e^x.

(c) Using (b), find the *value* of $f^{(22)}(0)$.

Solution

(a)

1. Remember that $\frac{d}{dx}\cos x = -\sin x$
2. Differentiate $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
   - Differentiate term-by-term:

     $$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \gg\gg \quad 0 - 2\frac{x^1}{2!} + 4\frac{x^3}{4!} - 6\frac{x^5}{6!} + \cdots$$

     $$= -\frac{x^1}{1!} + \frac{x^3}{3!} - \frac{x^5}{5!} - \cdots$$

   - Take negative because $\sin x = -\frac{d}{dx}\cos x$:

     $$\gg\gg \quad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

3. Final answer is $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

(b)

1. Recall the series $e^u = 1 + \frac{u^1}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots$
2. Compute the series for e^{-5x}.
   - Set $u = -5x$:

$$1 + \frac{u^1}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots \quad \gg\gg \quad 1 + \frac{(-5x)}{1!} + \frac{(-5x)^2}{2!} + \frac{(-5x)^3}{3!} + \cdots$$

3. Compute the product.
   - Product of series:

$$x^2 e^{-5x} \quad \gg\gg \quad x^2\left(1 + \frac{(-5x)}{1!} + \frac{(-5x)^2}{2!} + \frac{(-5x)^3}{3!} + \cdots\right)$$

$$\gg\gg \quad x^2 - 5x^3 + \frac{25}{2}x^4 - \frac{125}{3!}x^5 + \cdots$$

$$\gg\gg \quad \sum_{n=0}^{\infty}(-1)^n \frac{5^n x^{n+2}}{n!}$$

(c)

1. Derivatives at $x = 0$ are calculable from series coefficients.
   - Suppose we know the series $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$
   - Then $f^{(n)}(0) = n! \cdot a_n$.
   - It may be easier to compute a_n for a given $f(x)$ than to compute the derivative *functions* $f^{(n)}(x)$ and then evaluate them.
2. Compute a_{22}.
   - Write the series such that it reveals the coefficients:

$$\sum_{n=0}^{\infty}(-1)^n \frac{5^n x^{n+2}}{n!} \quad \gg\gg \quad \sum_{n=0}^{\infty}\left((-1)^n \frac{5^n}{n!}\right) x^{n+2}$$

$$\implies \quad a_{n+2} = (-1)^n \frac{5^n}{n!}$$

   - Coefficient with a_{n+2} corresponds to the term with x^{n+2}, *not necessarily* the $(n+2)^{\text{th}}$ term (e.g. if the first term is x^2 as here).
   - Compute a_{22}:

$$a_{22} = (-1)^{20}\frac{5^{20}}{20!} \quad \gg\gg \quad 5^{20}\frac{1}{20!}$$

3. Compute $f^{(22)}(0)$.
   - Use Derivative-Coefficient Identity:

$$f^{(22)}(0) \quad = \quad 22! \cdot a_{22}$$

$$\gg\gg \quad 5^{20} \cdot \frac{22!}{20!} \quad \gg\gg \quad 5^{20} \cdot 22 \cdot 21$$

Computing a Taylor series

Find the first five terms of the Taylor series of $f(x) = \sqrt{x+1}$ centered at $c = 3$.

Solution
A Taylor series is just a Maclaurin series that isn't centered at $c = 0$.

The general format looks like this:

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

The coefficients satisfy $a_n = \frac{f^{(n)}(c)}{n!}$. (Notice the c.)

We find the coefficients by computing the derivatives and evaluating at $x = 3$:

$$\begin{aligned}
f(x) &= (x+1)^{1/2}, & f(3) &= 2 \\
f'(x) &= \frac{1}{2}(x+1)^{-1/2}, & f'(3) &= \frac{1}{4} \\
f''(x) &= -\frac{1}{4}(x+1)^{-3/2}, & f''(3) &= -\frac{1}{32} \\
f'''(x) &= \frac{3}{8}(x+1)^{-5/2}, & f'''(3) &= \frac{3}{256} \\
f^{(4)}(x) &= -\frac{15}{16}(x+1)^{-7/2}, & f^{(4)}(3) &= -\frac{15}{2048}
\end{aligned}$$

By dividing by $n!$ we can write out the first terms of the series:

$$f(x) \;=\; \sqrt{x+1} \;=\; 2 + \frac{1}{4}(x-3) - \frac{1}{64}(x-3)^2 + \frac{1}{512}(x-3)^3 - \frac{5}{16,384}(x-3)^4 + \cdots$$

05 Theory

⚠ Study these!

- Memorize all of these series!
- Recognize all of these series!
- Recognize all of these summation formulas!

$$\begin{aligned}
\frac{1}{1-x} &= 1 + x + x^2 + \cdots &&= \sum_{n=0}^{\infty} x^n, \quad R = 1, \quad \text{interval: } (-1,1) \\
\ln(1-x) &= -\frac{x}{1} - \frac{x^2}{2} - \frac{x^3}{3} - \cdots &&= \sum_{n=0}^{\infty} -\frac{x^{n+1}}{n+1}, \quad R = 1, \quad \text{interval: } [-1,1) \\
\tan^{-1} x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots &&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad R = 1, \quad \text{interval: } [-1,1] \\
e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots &&= \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad R = \infty \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots &&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad R = \infty \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots &&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad R = \infty
\end{aligned}$$

Applications of Taylor series

06 Theory reminder

Linear approximation is the technique of approximating a specific value of a function, say $f(x_1)$, at a point x_1 that is close to another point x_0 where we *know* the exact value $f(x_0)$. We write Δx for $x_1 - x_0$, and $y_0 = f(x_0)$, and $y_1 = f(x_1)$. Then we write $dy = f'(x_0) \cdot \Delta x$ and use the fact that:

$$y_1 \approx y_0 + dy = y_0 + f'(x_0) \cdot \Delta x$$

Computing a linear approximation

For example, to approximate the value of $\sqrt{4.01}$, set $f(x) = \sqrt{x}$, set $x_0 = 4$ and $y_0 = 2$, and set $x_1 = 4.01$ so $\Delta x = 0.01$.

Then compute: $f'(x) = \frac{1}{2\sqrt{x}}$
So $f'(x_0) = 1/4$.

Finally:

$$y_1 \approx y_0 + f'(x_0) \cdot \Delta x \qquad \gg\gg \qquad y_1 \approx 2 + \frac{1}{4} \cdot 0.01 = 2.0025$$

Now recall the **linearization** of a function, which is itself another function:

Given a function $f(x)$, the linearization $L(x)$ at the basepoint $x = c$ is:

$$L(x) = f(c) + f'(c)(x - c)$$

The graph of this linearization $L(x)$ is the tangent line to the curve $y = f(x)$ at the point $(c, f(c))$.

The linearization $L(x)$ may be used as a replacement for $f(x)$ for values of x near c. The closer x is to c, the more accurate the approximation $L(x)$ is for $f(x)$.

Computing a linearization

We set $f(x) = \sqrt{x}$, and we let $c = 4$.

We compute $f(c) = 2$, and $f'(x) = \frac{1}{2\sqrt{x}}$ so $f'(c) = \frac{1}{4}$.

Plug everything in to find $L(x)$:

$$L(x) = f(c) + f'(c)(x - c) \qquad \gg\gg \qquad L(x) = 2 + \frac{1}{4}(x - 4)$$

Now approximate $f(4.01) \approx L(4.01)$:

$$L(4.01) = 2 + \frac{1}{4}(4.01 - 4) = 2.0025$$

07 Theory

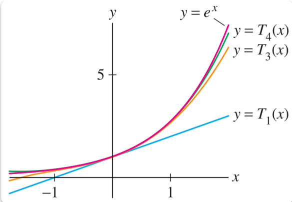
Taylor polynomials

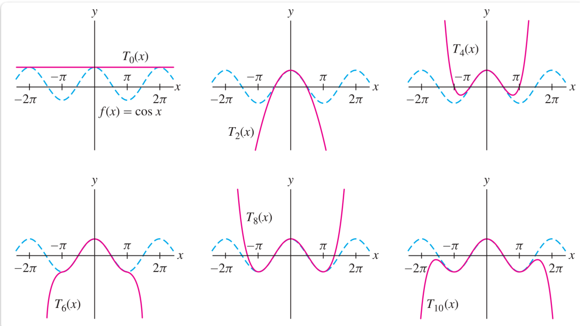
The **Taylor polynomials $T_n(x)$** of a function $f(x)$ are the partial sums of the Taylor series of $f(x)$:

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(c)}{n!}(x - c)^n = f(c) + \frac{f'(c)}{1!}(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \cdots$$

These polynomials are *generalizations of linearization*.
Specifically, $f(c) = T_0(x)$, and $L(x) = T_1(x)$.

The Taylor series $T_n(x)$ is a better approximation of $f(x)$ than $T_i(x)$ for any $i < n$.





Facts about Taylor series

The series $T_n(x)$ has the same derivatives as $f(x)$ at the point $x = c$. This fact can be verified by visual inspection of the series: apply the power rule and chain rule, then plug in $x = c$ and all factors left with $(x - c)$ will become zero.

The difference $f(x) - T_n(x)$ vanishes to order n at $x = c$:

$$\begin{aligned} f(x) - T_n(x) &= \frac{f^{(n)}(c)}{n!}(x-c)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-c)^{n+1} + \cdots \\ &= (x-c)^n \left(\frac{f^{(n)}(c)}{n!} + \frac{f^{(n+1)}(c)}{(n+1)!}(x-c) + \cdots \right) \end{aligned}$$

The factor $(x - c)^n$ drives the whole function to zero with order n as $x \to c$.

If we only considered orders up to n, we might say that $f(x)$ and $T_n(x)$ are the same near c.

08 Illustration

Taylor polynomial approximations

Let $f(x) = \sin x$ and let $T_n(x)$ be the Taylor polynomials expanded around $c = 0$.

By considering the alternating series error bound, find the first n for which $T_n(0.02)$ must have error less than 10^{-6}.

Solution

1. Write the Maclaurin series of $\sin x$ because we are expanding around $c = 0$.
   - Alternating sign, odd function:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

2. ⚠ Notice this series is alternating, so AST error bound formula applies.
   - AST error bound formula is:

$$|E_n| \le a_{n+1}$$

- Here the series is $S = a_0 - a_1 + a_2 - a_3 + \cdots$ and $E_n = S - S_n$ is the error.
- Notice that $x = 0.02$ is part of the terms a_i in this formula.

3. Implement error bound to set up equation for n.
    - Find n such that $a_{n+1} \leq 10^{-6}$, and therefore by the AST error bound formula:

$$|E_n| \leq a_{n+1} \leq 10^{-6}$$

    - Plug in $x = 0.02$.
    - From the series of $\sin x$ we obtain for a_{2n+1}:

$$a_{2n+1} = \frac{0.02^{2n+1}}{(2n+1)!}$$

    - We seek the first time it happens that $a_{2n+1} \leq 10^{-6}$.

4. Solve for the first time $a_{2n+1} \leq 10^{-6}$.
    - Equations to solve:

$$\frac{0.02^{2n+1}}{(2n+1)!} \leq 10^{-6} \qquad \text{but:} \quad \frac{0.02^{2(n-1)+1}}{(2(n-1)+1)!} \not\leq 10^{-6}$$

    - Method: list the values:

$$\frac{0.02^1}{1!} = 0.02, \qquad \frac{0.02^3}{3!} \approx 1.33 \times 10^{-6}, \qquad \frac{0.02^5}{5!} \approx 2.67 \times 10^{-11}, \qquad \ldots$$

    - The first time a_{2n+1} is below 10^{-6} happens when $2n+1 = 5$.

5. Interpret result and state the answer.
    - When $2n+1 = 5$, the term $\dfrac{x^{2n+1}}{(2n+1)!}$ at $x = 0.02$ is less than 10^{-6}.
    - Therefore the sum of prior terms is accurate to an error of less than 10^{-6}.
    - The sum of prior terms equals $T_4(0.02)$.
    - Since $T_4(x) = T_3(x)$ because there is no x^4 term, the same sum is $T_3(0.02)$.
    - The final answer is $n = 3$.
    - It would be wrong to infer at the beginning that the answer is 5, or to solve $2n+1 = 5$ to get $n = 2$.

Taylor polynomials to approximate a definite integral

Approximate $\displaystyle\int_0^{0.3} e^{-x^2}\,dx$ using a Taylor polynomial with an error no greater than 10^{-5}.

Solution

1. Write the series of the integrand.
    - Plug $u = -x^2$ into the series of e^u:

$$e^u = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \cdots \qquad \gg\gg \qquad e^{-x^2} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

2. Compute definite integral by terms.
    - Antiderivative by terms:

$$\int 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\,dx \qquad \gg\gg \qquad x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

- Plug in bounds for definite integral:

$$\int_0^{0.3} e^{-x^2}\,dx \qquad \gg\gg \qquad x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \Big|_0^{0.3}$$

$$\gg\gg \qquad 0.3 - \frac{0.3^3}{3!} + \frac{0.3^5}{5!} - \frac{0.3^7}{7!} + \cdots$$

3. Notice AST, apply error formula.
   - Compute some terms:

$$\frac{0.3^3}{3!} \approx 0.0045, \qquad \frac{0.3^5}{5!} \approx 2.0 \times 10^{-5}, \qquad \frac{0.3^7}{7!} \approx 4.34 \times 10^{-8}$$

   - So we can guarantee an error less than 4.34×10^{-5} by summing the first terms through $\frac{0.3^5}{5!}$.
4. Final answer is $0.3 - \frac{0.3^3}{3!} + \frac{0.3^5}{5!} \approx 0.291243$.