W12 - Homework

Summations

01

Summation of three: Rolling mixed dice

You have three dice. One has 4, one has 6, and one has 12 sides.

How many 4s do you expect to see if you roll these dice together?

02

Jumble of coins

In my pocket I have a jumble of coins: 5 dimes, 4 quarters, 3 nickels, 3 pennies, and one big 50¢-piece. I draw three at random. What is the expected value of the three?

03

Counting flip flops

A bag contains 50 marbles, 30 blue and 20 red. A sequence of zeros and ones is created by pulling the marbles out one at a time (without replacement) and writing a 1 if the marble drawn is blue and a zero if it is red.

How many pairs of adjacent digits in the sequence are expected to differ from each other?

Hint: Use a sum of 49 indicators.

Central Limit Theorem

04

Normal approximation - Eating hot dogs

Frank is a competitive hot dog eater. He eats $1\,\mathrm{hd}$ in $15\,\mathrm{sec}$ with $\sigma = 4\,\mathrm{sec}$.

What is the probability that Frank manages to consume $64\,\mathrm{hd}$ in $15\,\mathrm{min}$ or less, in an upcoming competition? Use a normal approximation from the CLT to estimate this probability.

State the reason that the normal approximation is applicable.

05

Continuity correction: De Moivre-Laplace

A fair die is rolled 300 times.

Use a normal approximation to estimate the probability that exactly 100 outcomes are either 3 or 6.

Do this with and without the continuity correction.

06

Normal approximation - Ventilator filters

A mechanical ventilator model uses air filters that last 100 hours on average with a standard deviation of 30 hours.

How many filters should be stocked so that the supply lasts 2,000 hours with probability at least 95%? Use a normal approximation to estimate the answer.

State the reason that the normal approximation is applicable.

07

Normal approximation - Grading many exams

An instructor has 50 exams to grade. The grading time for each exam follows a distribution with an average of 20 minutes and variance of 16 minutes. Assume the grading times per exam are independent.

Roughly what are the odds that after 450 minutes of grading, at least half the exams will be graded? Use a normal approximation to estimate the answer.

State the reason that the normal approximation is applicable.

08

Indicator method, exchangeability, summation rules

A class has 40 students: 24 women and 16 men. Each period the teacher selects a random student to present an exercise on the board from among those who have not presented already.

Let X count the number of times a man was chosen after 15 class periods.

(a) ⋆ Find $E[X]$.
(b) ★ Find $\mathrm{Var}[X]$.

Hint: Is X_j independent of X_i? Do you know $E[X_j]$ anyway?

09

★ Graphing convergence to a bell curve

Let X_i be independent RVs each having the following PMF:

$$P_{X_i}(x) \;=\; \begin{cases} 0.5 & x = +1 \\ 0.5 & x = -1 \end{cases}$$

Notice that $E[X_i] = 0$ and $\mathrm{Var}(X_i) = 1$ for each X_i.

Define $S_n = X_1 + \cdots + X_n$. So $E[S_n] = 0$ and $\text{Var}(S_n) = n$, and therefore $\text{Var}\left(\frac{S_n}{\sqrt{n}}\right) = 1$.

By the CLT, $\frac{S_n}{\sqrt{n}}$ should converge to the standard normal distribution as $n \to \infty$. In this problem you explore the limit process by direct computation of the cases $n = 1, 2, 3, 4, 5$.

(a) Compute the PMF of S_n in terms of n.

Hint steps:

1. $P_{S_n}(w)$ is the number of sequences of n outcomes of ± 1 having a total sum of w, times the probability of any particular such sequence.
2. Find the probability of any particular sequence of n outcomes of ± 1.
3. There are $\binom{n}{\ell}$ ways to get ℓ outcomes of $+1$ and $n - \ell$ outcomes of -1. Solve for ℓ in terms of w.
4. Put 2. and 3. together in 1. to get your formula.

(b) Compute the PMF of $\frac{S_n}{\sqrt{n}}$ for $n = 1, 2, 3, 4, 5$ using your formula from (a) together with a scaling (derived variable calculation).

(c) Draw the graphs of the PMF of S_n and the PMF of $S_n/\sqrt{n}$ for $n = 1, 2, 3, 4, 5$.