W11 - Examples

Power series as functions

Geometric series: algebra meets calculus

Consider the geometric series as a power series functions:

$$\frac{1}{1-x} \quad = \quad 1 + x + x^2 + x^3 + \cdots$$

Take the derivative of both sides of the *function*:

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) \quad \gg\gg \quad \frac{1}{(1-x)^2} \quad \gg\gg \quad \left(\frac{1}{1-x}\right)^2$$

This means f satisfies the identity:

$$f' = f^2$$

Now compute the derivative of the *series*:

$$1 + x + x^2 + x^3 + \cdots \quad \overset{\frac{d}{dx}}{\gg\gg} \quad 1 + 2x + 3x^2 + 4x^3 + \cdots$$

On the other hand, compute the square of the series:

$$\left(1 + x + x^2 + x^3 + \cdots\right)^2 \quad \gg\gg \quad 1 + 2x + 3x^2 + 4x^3 + \cdots$$

So we find that the *same relationship holds*, namely $f' = f^2$, for the closed formula and the series formula for this function.

Manipulating geometric series: algebra

Find power series that represent the following functions:
(a) $\dfrac{1}{1+x}$ (b) $\dfrac{1}{1+x^2}$ (c) $\dfrac{x^3}{x+2}$ (d) $\dfrac{3x}{2-5x}$

Solution

(a) $\dfrac{1}{1+x}$

1. Rewrite in format $\frac{1}{1-u}$.
   - Introduce double negative:

$$\frac{1}{1+x} = \frac{1}{1-(-x)}$$

   - Choose $u = -x$.
2. Plug $u = -x$ into geometric series.
   - Geometric series in u:

$$1 + u + u^2 + u^3 + \cdots$$

   - Plug in $u = -x$:

$$\gg\gg \qquad 1 + (-x) + (-x)^2 + (-x)^3 + \cdots$$

   - Simplify:

$$\gg\gg \qquad 1 - x + x^2 - x^3 + \cdots$$

- Final answer:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

(b) $\dfrac{1}{1+x^2}$

1. ≡ Rewrite in format $\frac{1}{1-u}$.
   - Rewrite:

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

   - Choose $u = -x^2$.
2. Plug $u = -x^2$ into geometric series.
   - Geometric series in u:

$$1 + u + u^2 + u^3 + \cdots$$

   - Plug in $u = -x^2$:

$$\gg\gg \quad 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots \quad \gg\gg \quad 1 - x^2 + x^4 - x^6 + \cdots$$

   - Final answer:

$$\frac{1}{1+x} = 1 - x^2 + x^4 - x^6 + \cdots$$

(c) $\dfrac{x^3}{x+2}$

1. Rewrite in format $Ax^3 \cdot \frac{1}{1-u}$.
   - Rewrite:

$$\frac{x^3}{x+2} \quad \gg\gg \quad x^3 \cdot \frac{1}{2+x} \quad \gg\gg \quad x^3 \cdot \frac{1}{2\left(1+\frac{x}{2}\right)}$$

$$\gg\gg \quad \frac{1}{2}x^3 \cdot \frac{1}{1+\frac{x}{2}} \quad \gg\gg \quad \frac{1}{2}x^3 \cdot \frac{1}{1-\left(-\frac{x}{2}\right)}$$

   - Choose $u = -\frac{x}{2}$. Here $Ax^3 = \frac{1}{2}x^3$.
2. Plug $u = -x^2$ into geometric series.
   - Geometric series in u:

$$1 + u + u^2 + u^3 + \cdots$$

   - Plug in $u = -\frac{x}{2}$:

$$\gg\gg \quad 1 + (-\tfrac{x}{2}) + (-\tfrac{x}{2})^2 + (-\tfrac{x}{2})^3 + \cdots$$

$$\gg\gg \quad 1 - \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^3 + \cdots$$

   - Obtain:

$$\frac{1}{1-\left(-\frac{x}{2}\right)} = 1 - \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^3 + \cdots$$

3. ≡ Multiply by $\frac{1}{2}x^3$.

- Distribute:

$$\frac{1}{2}x^3 \cdot \frac{1}{1-\left(-\frac{x}{2}\right)} \qquad \gg\gg \qquad \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 - \frac{1}{16}x^6 + \cdots$$

- Final answer:

$$\frac{x^3}{x+2} = \frac{1}{2}x^3 - \frac{1}{4}x^4 + \frac{1}{8}x^5 - \frac{1}{16}x^6 + \cdots$$

(d) $\dfrac{3x}{2-5x}$

1. Rewrite in format $Ax \cdot \frac{1}{1-u}$.
    - Rewrite:

$$\frac{3x}{2-5x} \qquad \gg\gg \qquad 3x \cdot \frac{1}{2-5x}$$

$$\gg\gg \qquad 3x \cdot \frac{1}{2\left(1-\frac{5x}{2}\right)} \qquad \gg\gg \qquad \frac{3}{2}x \cdot \frac{1}{1-\frac{5x}{2}}$$

    - Choose $u = \frac{5x}{2}$. Here $Ax = \frac{3}{2}x$.
2. Plug $u = \frac{5x}{2}$ into geometric series.
    - Geometric series in u:

$$1 + u + u^2 + u^3 + \cdots$$

    - Plug in $u = \frac{5x}{2}$:

$$\gg\gg \qquad 1 + \left(\frac{5x}{2}\right) + \left(\frac{5x}{2}\right)^2 + \left(\frac{5x}{2}\right)^3 + \cdots$$

$$\gg\gg \qquad 1 + \frac{5}{2}x + \frac{25}{4}x^2 + \frac{125}{8}x^3 + \cdots$$

    - Obtain:

$$\frac{1}{1-\frac{5x}{2}} = 1 + \frac{5}{2}x + \frac{25}{4}x^2 + \frac{125}{8}x^3 + \cdots$$

3. Multiply by $\frac{3}{2}x$.
    - Distribute:

$$\frac{3}{2}x \cdot \frac{1}{1-\frac{5x}{2}} \qquad \gg\gg \qquad \frac{3}{2}x + \frac{15}{4}x^2 + \frac{75}{8}x^3 + \frac{375}{16}x^4 + \cdots$$

    - Final answer:

$$\frac{3x}{2-5x} = \frac{3}{2}x + \frac{15}{4}x^2 + \frac{75}{8}x^3 + \frac{375}{16}x^4 + \cdots$$

Manipulating geometric series: calculus

Find power series that represent the following functions:
(a) $\ln(1+x)$ (b) $\tan^{-1}(x)$

Solution

(a) $\ln(1+x)$

1. Differentiate to obtain similarity to geometric sum formula.

- Differentiate $\ln(1+x)$:

$$\frac{d}{dx}\ln(1+x) = \frac{1}{1+x} \qquad \gg\gg \qquad \frac{1}{1-(-x)}$$

2. Find power series of differentiated function.
   - Power series by modifying $\frac{1}{1-u}$ with $u = -x$:

$$\frac{1}{1-(-x)} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

3. Integrate series to find original function.
   - Integrate both sides:

$$\int \frac{1}{1-(-x)}\,dx = \int 1 - x + x^2 - x^3 + x^4 - \cdots\,dx$$

$$\ln(1+x) = D + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

   - Use known point to solve for D:

$$\ln(1+0) = D + 0 + 0 + \cdots \qquad \gg\gg \qquad 0 = D$$

   - Final answer:

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

(b) $\tan^{-1} x$

1. Differentiate to obtain similarity to geometric sum formula.
   - Differentiate $\tan^{-1} x$:

$$\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2} \qquad \gg\gg \qquad \frac{1}{1-(-x^2)}$$

2. Find power series of differentiated function.
   - Power series by modifying $\frac{1}{1-u}$ with $u = -x^2$:

$$\frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

3. Integrate series to find original function.
   - Integrate both sides:

$$\int \frac{1}{1-(-x^2)}\,dx = \int 1 - x^2 + x^4 - x^6 + x^8 - \cdots\,dx$$

$$\tan^{-1}(x) = D + x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

   - Use known point to solve for D:

$$\tan^{-1}(0) = D + 0 - 0 + \cdots \qquad \gg\gg \qquad 0 = D$$

   - Final answer:

$$\tan^{-1}(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots$$

- Notice: by evaluating at $x = 1$ we get the Leibniz formula:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Recognizing and manipulating geometric series: Part I

(a) Evaluate $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$.
(Hint: consider the series of $\ln(1-x)$.)

(b) Find a series approximation for $\ln(2/3)$.

Solution

(a) Evaluate $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$. (Hint: consider the series of $\ln(1-x)$.)

1. Find the series representation of $\ln(1-x)$ following the hint.
    - Notice that $\frac{d}{dx}\ln(1-x) = \frac{-1}{1-x}$.
    - We know the series of $\frac{-1}{1-x}$:

    $$\frac{-1}{1-x} = -(1+x+x^2+\cdots) = -1-x-x^2-\cdots$$

    - Notice that $\int \frac{-1}{1-x}\,dx = \ln(1-x)+C$; this is the desired function when $C=0$.
    - Integrate the series term-by-term:

    $$\int \frac{-1}{1-x}\,dx = \int -1-x-x^2-\cdots\,dx \qquad \gg\gg \qquad \ln(1-x) = D - x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$

    - Solve for D using $\ln(1-0)=0$, so $0 = D-0-0-\cdots$ and thus $D=0$. So:

    $$\ln(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots \quad = \quad \sum_{n=1}^{\infty} -\frac{x^n}{n!}$$

2. Notice the similar formula.
    - The series formula $\sum_{n=1}^{\infty} -\frac{x^n}{n!}$ looks similar to the formula $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$.
3. Choose $x=-1$ to recreate the desired series.
    - We obtain equality by setting $x=-1$ because $-(-1)^n = (-1)^{n+1} = (-1)^{n-1}$.
4. Final answer is $\ln(1--1) = \ln 2$.

(b) Find a series approximation for $\ln(2/3)$.

1. Observe that $\ln(2/3) = \ln(1-1/3)$.
    - Therefore we can use the series $\ln(1-x) = -x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots$
2. Plug $x=1/3$ into the series for $\ln(1-x)$.
    - Plug in and simplify:

$$\begin{aligned}\ln(2/3) = \ln(1-1/3) &= -1/3 - \frac{(1/3)^2}{2} - \frac{(1/3)^3}{3} - \cdots \\ &= -\frac{1}{3} - \frac{1}{3^2\cdot 2} - \frac{1}{3^3\cdot 3} - \cdots\end{aligned}$$

Recognizing and manipulating geometric series: Part 2

(a) Find a series representing $\tan^{-1}(x)$.

(b) Find a series representing $\int \frac{dx}{1+x^4}$.

Solution

(a) Find a series representing $\tan^{-1}(x)$.

1. ⚠ Notice that $\frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$.
2. Obtain the series for $\frac{1}{1+x^2}$.
   - Let $u = -x^2$:

$$\frac{1}{1+x^2} \quad \gg\gg \quad \frac{1}{1-u} = 1 + u + u^2 + \cdots$$
$$\gg\gg \quad 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

3. Integrate the series for $\frac{1}{1+x^2}$ by terms.
   - Set up the strategy. We know:

$$\int \frac{1}{1+x^2}\,dx = \tan^{-1}(x) + C$$

   and:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

   - Integrate term-by-term:

$$= \int 1 - x^2 + x^4 - x^6 + x^8 - \cdots\,dx = D + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

   - Conclude that:

$$\tan^{-1}(x) + C = D + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

4. Solve for $D - C$ by testing at $\tan^{-1}(0) = 0$.
   - Plugging in, obtain:

$$\tan^{-1}(0) = D - C + 0 + \cdots + 0$$

   so $D - C = 0$.
5. Final answer is $\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$.

(b) Find a series representing $\displaystyle\int \frac{dx}{1+x^4}$.

1. Find a series representing the integrand.
   - Integrand is $\frac{1}{1+x^4}$.
   - Rewrite integrand in format of geometric series sum:

$$\frac{1}{1+x^4} \quad \gg\gg \quad \frac{1}{1-(-x^4)} \quad \gg\gg \quad \frac{1}{1-u}, \quad u = -x^4$$

   - Write the series:

$$\frac{1}{1-u} = 1 + u + u^2 + u^3 + \cdots \quad \gg\gg \quad 1 - x^4 + x^8 - x^{12} + x^{16} - \cdots \quad = \sum_{n=0}^{\infty} (-1)^n x^{4n}$$

2. Integrate the integrand series by terms.
   - Integrate term-by-term:

$$\int 1 - x^4 + x^8 - x^{12} + x^{16} - \cdots\,dx \quad \gg\gg \quad C + x - \frac{x^5}{5} + \frac{x^9}{9} - \frac{x^{13}}{13} + \frac{x^{17}}{17} - \cdots$$

   - This is our final answer.

Taylor and Maclaurin series

Maclaurin series of e^x

What is the Maclaurin series of $f(x) = e^x$?

Solution

Because $\dfrac{d}{dx}e^x = e^x$, we find that $f^{(n)}(x) = e^x$ for all n.

So $f^{(n)}(0) = e^0 = 1$ for all n.

So $a_n = \dfrac{1}{n!}$ for all n. Thus:

$$e^x = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots$$

Maclaurin series of $\cos x$

Find the Maclaurin series representation of $\cos x$.

Solution

Use the Derivative-Coefficient Identity to solve for the coefficients:

$$a_n \quad = \quad \frac{f^{(n)}(0)}{n!}$$

n	$f^{(n)}(x)$	$f^{(n)}(0)$	a_n
0	$\cos x$	1	1
1	$-\sin x$	0	0
2	$-\cos x$	-1	$-1/2$
4	$\sin x$	0	0
5	$\cos x$	1	$1/24$
6	$-\sin x$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$

By studying the generating pattern of the coefficients, we find for the series:

$$\cos x \quad = \quad 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Maclaurin series from other Maclaurin series

(a) Find the Maclaurin series of $\sin x$ using the Maclaurin series of $\cos x$.

(b) Find the Maclaurin series of $f(x) = x^2 e^{-5x}$ using the Maclaurin series of e^x.

(c) Using (b), find the *value* of $f^{(22)}(0)$.

Solution

(a)

1. Remember that $\frac{d}{dx}\cos x = -\sin x$
2. Differentiate $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
   - Differentiate term-by-term:

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \gg\gg \qquad 0 - 2\frac{x^1}{2!} + 4\frac{x^3}{4!} - 6\frac{x^5}{6!} + \cdots$$

$$= \quad -\frac{x^1}{1!} + \frac{x^3}{3!} - \frac{x^5}{5!} - \cdots$$

- Take negative because $\sin x = -\frac{d}{dx}\cos x$:

$$\gg\gg \qquad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

3. Final answer is $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$

(b)

1. Recall the series $e^u = 1 + \frac{u^1}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots$
2. Compute the series for e^{-5x}
   - Set $u = -5x$:

$$1 + \frac{u^1}{1!} + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots \qquad \gg\gg \qquad 1 + \frac{(-5x)}{1!} + \frac{(-5x)^2}{2!} + \frac{(-5x)^3}{3!} + \cdots$$

3. Compute the product.
   - Product of series:

$$x^2 e^{-5x} \qquad \gg\gg \qquad x^2\left(1 + \frac{(-5x)}{1!} + \frac{(-5x)^2}{2!} + \frac{(-5x)^3}{3!} + \cdots\right)$$

$$= x^2 - 5x + \frac{25}{2}x^2 - \frac{125}{3!}x^3 + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{5^n x^{n+2}}{n!}$$

(c)

1. Derivatives at $x = 0$ are calculable from series coefficients.
   - Suppose we know the series $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$
   - Then $f^{(n)}(0) = n! \cdot a_n$.
   - It may be easier to compute a_n for a given $f(x)$ than to compute the derivative *functions* $f^{(n)}(x)$ and then evaluate them.
2. Compute a_{22}.
   - Write the series such that it reveals the coefficients:

$$\sum_{n=0}^{\infty} (-1)^n \frac{5^n x^{n+2}}{n!} \qquad \gg\gg \qquad \sum_{n=0}^{\infty} \left((-1)^n \frac{5^n}{n!}\right) x^{n+2}, \qquad \Longrightarrow \qquad a_{n+2} = (-1)^n \frac{5^n}{n!}$$

   - Always have a_{22} is the coefficient of x^{22}.
   - Compute a_{22}:

$$a_{22} = (-1)^{20} \frac{5^{20}}{20!} \qquad \gg\gg \qquad 5^{20} \frac{1}{20!}$$

3. Compute $f^{(22)}(0)$.
   - Use formula $f^{(22)}(0) = n! \cdot a_n$:

$$f^{(22)}(0) = 22! \cdot a_{22}$$

$$= 5^{20} \cdot \frac{22!}{20!}$$

Computing a Taylor series

Find the Taylor series of $f(x) = \sqrt{x+1}$ centered at $c = 3$.

Solution

A Taylor series is just a Maclaurin series that isn't centered at $c = 0$.

The general format looks like this:

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

The coefficients satisfy $a_n = \frac{f^{(n)}(c)}{n!}$. (Notice the c.)

We find the coefficients by computing the derivatives and evaluating at $x = 3$:

$$\begin{aligned} f(x) &= (x+1)^{1/2}, & f(3) &= 2 \\ f'(x) &= \frac{1}{2}(x+1)^{-1/2}, & f'(3) &= \frac{1}{4} \\ f''(x) &= -\frac{1}{4}(x+1)^{-3/2}, & f''(3) &= -\frac{1}{32} \\ f'''(x) &= \frac{3}{8}(x+1)^{-5/2}, & f'''(3) &= \frac{3}{256} \\ f^{(4)}(x) &= -\frac{15}{16}(x+1)^{-7/2}, & f^{(4)}(3) &= -\frac{15}{2048} \end{aligned}$$

By dividing by $n!$ we can write out the first terms of the series:

$$f(x) \;=\; \sqrt{x+1} \quad = \quad 2 + \frac{1}{4}(x-3) - \frac{1}{64}(x-3)^2 + \frac{1}{512}(x-3)^3 - \frac{5}{16,384}(x-3)^4 + \cdots$$

Applications of Taylor series

Taylor polynomial approximations

Let $f(x) = \sin x$ and let $T_n(x)$ be the Taylor polynomials expanded around $c = 0$.

By considering the alternating series error bound, find the first n for which $T_n(0.02)$ must have error less than 10^{-6}.

Solution

1. ≡ Write the Maclaurin series of $\sin x$ because we are expanding around $c = 0$.
    - Alternating sign, odd function:

$$\sin x \quad = \quad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

2. ⚠ Notice this series is alternating, so AST error bound formula applies.
    - AST error bound formula is:

$$|E_n| \leq a_{n+1}$$

    - Here the series is $S = a_0 - a_1 + a_2 - a_3 + \cdots$ and $E_n = S - S_n$ is the error.
    - Notice that $x = 0.02$ is part of the terms a_i in this formula.
3. Implement error bound to set up equation for n.
    - Find n such that $a_{n+1} \leq 10^{-6}$, and therefore by the AST error bound formula:

$$|E_n| \leq a_{n+1} \leq 10^{-6}$$

    - Plug in $x = 0.02$.
    - From the series of $\sin x$ we obtain for a_{2n+1}:

$$a_{2n+1} = \frac{0.02^{2n+1}}{(2n+1)!}$$

    - We seek the first time it happens that $a_{2n+1} \leq 10^{-6}$.
4. Solve for the first time $a_{2n+1} \leq 10^{-6}$.

- Equations to solve:

$$\frac{0.02^{2n+1}}{(2n+1)!} \leq 10^{-6} \qquad \text{but:} \quad \frac{0.02^{2(n-1)+1}}{(2(n-1)+1)!} \not\leq 10^{-6}$$

- Method: list the values:

$$\frac{0.02^1}{1!} = 0.02, \qquad \frac{0.02^3}{3!} \approx 1.33 \times 10^{-6}, \qquad \frac{0.02^5}{5!} \approx 2.67 \times 10^{-11}, \qquad \dots$$

- The first time a_{2n+1} is below 10^{-6} happens when $2n+1 = 5$.

5. Interpret result and state the answer.
   - When $2n+1 = 5$, the term $\dfrac{x^{2n+1}}{(2n+1)!}$ at $x = 0.02$ is less than 10^{-6}.
   - Therefore the sum of prior terms is accurate to an error of less than 10^{-6}.
   - The sum of prior terms equals $T_4(0.02)$.
   - Since $T_4(x) = T_3(x)$ because there is no x^4 term, the same sum is $T_3(0.02)$.
   - The final answer is $n = 3$.
   - We do not immediately infer that the answer is 5, nor solve $2n+1 = 5$ to get $n = 2$. Those are wrong!

Taylor polynomials to approximate a definite integral

Approximate $\displaystyle\int_0^{0.3} e^{-x^2}\,dx$ using a Taylor polynomial with an error no greater than 10^{-5}.

Solution

1. Write the series of the integrand.
   - Plug $u = -x^2$ into the series of e^u:

$$e^u = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \cdots \qquad \gg\gg \qquad e^{-x^2} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$$

2. Compute definite integral by terms.
   - Antiderivative by terms:

$$\int 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\,dx \qquad \gg\gg \qquad x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$$

   - Plug in bounds for definite integral:

$$\int_0^{0.3} e^{-x^2}\,dx \qquad \gg\gg \qquad x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots \Big|_0^{0.3}$$

$$\gg\gg \qquad 0.3 - \frac{0.3^3}{3!} + \frac{0.3^5}{5!} - \frac{0.3^7}{7!} + \cdots$$

3. Notice AST, apply error formula.
   - Compute some terms:

$$\frac{0.3^3}{3!} \approx 0.0045, \qquad \frac{0.3^5}{5!} \approx 2.0 \times 10^{-5}, \qquad \frac{0.3^7}{7!} \approx 4.34 \times 10^{-8}$$

   - So we can guarantee an error less than 4.34×10^{-5} by summing the first terms through $\frac{0.3^5}{5!}$.

4. Final answer is $0.3 - \dfrac{0.3^3}{3!} + \dfrac{0.3^5}{5!} \approx 0.291243$.