W04 - Examples

Bernoulli process

24 - Binomial variable counting ones in repeated die rolls

A standard die is rolled 6 times. Use a binomial variable to find the probability of rolling at least 4 ones.

Solution

1. Labels
   - Let $S_6 \sim \text{Bin}(6, \frac{1}{6})$.
   - Interpret: S_6 counts the ones appearing over 6 rolls.
   - We seek $P[S_6 \geq 4]$.
2. Calculation
   - Exclusive events:

$$P[S_6 \geq 4] \quad \gg\gg \quad P[S_6 = 4] + P[S_6 = 5] + P[S_6 = 6]$$

$$\gg\gg \quad \binom{6}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^2 + \binom{6}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^1 + \binom{6}{6}\left(\frac{1}{6}\right)^6\left(\frac{5}{6}\right)^0$$

$$\gg\gg \quad \frac{203}{23328}$$

25 - Roll die until

Roll a fair die repeatedly. Find the probabilities that:

- (a) At most 2 threes occur in the first 5 rolls
- (b) There is no three in the first 4 rolls
- (c) There is no three in the first 4 rolls, but there is at least one in the first 20 rolls

Solution

(a)

1. Labels.
   - Use $S_5 \sim \text{Bin}(6, 1/6)$ to count the number of threes among the first six rolls.
   - Seek $P[S_5 \leq 2]$ as the answer.
2. Calculations.
   - Divide into exclusive events:

$$P[S_5 \leq 2] \quad \gg\gg \quad P[S_5 = 0] + P[S_5 = 1] + P[S_5 = 2]$$

$$\gg\gg \quad \binom{5}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^5 + \binom{5}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^4 + \binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3$$

$$\gg\gg \quad \frac{625}{648}$$

(b)

1. ≡ Labels.
   - Use $N \sim \text{Geom}(1/6)$ to give the roll number of the first time a three is rolled.
   - Seek $P[N > 4]$ as the answer.
2. Sum the PMF formula for $\text{Geom}(1/6)$.
   - Compute:

$$P[N > 4] \quad \gg\gg \quad \sum_{k=5}^{\infty} \left(\frac{5}{6}\right)^{k-1} \left(\frac{1}{6}\right)$$

3. ⚠ Geometric series formula.
   - For any geometric series:

$$a + ar + ar^2 + ar^3 + \cdots \quad \gg\gg \quad \frac{a}{1-r}$$

   - Apply formula:

$$\sum_{k=5}^{\infty} \left(\frac{5}{6}\right)^{k-1} \left(\frac{1}{6}\right) \quad \gg\gg \quad \left(\frac{5}{6}\right)^4$$

4. ≡ Final answer is $P[N > 4] = (5/6)^4$.

(c)

1. ≡ Labels.
   - Event A means "no three in the first 4 rolls."
   - Event B means "no three in rolls 5 through 20."
   - ! These events are independent!
   - Seek $P(AB^c)$ as answer.
2. Known probabilities.
   - Know: $P[A] = (5/6)^4$
   - Know: $P[B] = (5/6)^{16}$
3. Apply product rule for independent events.
   - Product rule and negation rule:

$$P(AB^c) \quad \gg\gg \quad P(A)(1 - P(B))$$

   - Insert data:

$$\gg\gg \quad \left(\frac{5}{6}\right)^4 \left(1 - \left(\frac{5}{6}\right)^{16}\right) \quad \gg\gg \quad \approx 0.456$$

26 - Cubs winning the World Series

Suppose the Cubs are playing the Yankees for the World Series. The first team to 4 wins in 7 games wins the series. What is the probability that the Cubs win the series?

Assume that for any given game the probability of the Cubs winning is $p = 45\%$ and losing is $q = 55\%$.

Solution

(a) Using a binomial distribution

1. ≡ Label.
   - Let $X \sim \text{Bin}(7, p)$.

- Thus $P_X(4)$ is the probability that the Cubs win exactly 4 games over 7 played.
- Seek $P_X(4) + P_X(5) + P_X(6) + P_X(7)$ as the answer.

2. Calculate.
   - Use binomial PMF:

$$P_X(k) = \binom{7}{k} p^k q^{6-k}$$

   - Insert data:

$$P_X(4) + \cdots + P_X(7)$$

$$\gg\gg \quad \binom{7}{4} p^4 q^3 + \binom{7}{5} p^5 q^2 + \binom{7}{6} p^6 q^1 + \binom{7}{7} p^7 q^0$$

   - Compute:

$$\gg\gg \quad \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} p^4 q^3 + \frac{7 \cdot 6}{2} p^5 q^2 + \frac{7}{1} p^6 q^1 + 1 \cdot p^7 q^0$$

$$\gg\gg \quad p^4 \left(35q^3 + 21p^1 q^2 + 7p^2 q + p^3\right)$$

   - Convert $q \gg (1-p)$:

$$\gg\gg \quad p^4 \left(35(1-p)^3 + 21p(1-p)^2 + 7p^2(1-p) + p^3\right)$$

$$\gg\gg \quad 35p^4 - 84p^5 + 70p^6 - 20p^7 \quad \gg\gg \quad \approx 0.39$$

(b) Using a Pascal distribution

1. Label.
   - Let $Y \sim \mathrm{Pasc}(4, p)$.
   - Thus $P_Y(k)$ is the probability that the Cubs win their 4^{th} game on game number k.
   - Seek $P_Y(4) + P_Y(5) + P_Y(6) + P_Y(7)$ as the answer.
2. Calculate.
   - Use Pascal PMF:

$$P_Y(k) = \binom{k-1}{3} p^4 q^{k-4}$$

   - Insert data:

$$P_Y(4) + \cdots + P_Y(7)$$

$$\gg\gg \quad \binom{3}{3} p^4 q^0 + \binom{4}{3} p^4 q^1 + \binom{5}{3} p^4 q^2 + \binom{6}{3} p^4 q^3$$

   - Compute:

$$\gg\gg \quad 1 \cdot p^4 + \frac{4}{1} \cdot p^4 q^1 + \frac{5 \cdot 4}{2} p^4 q^2 + \frac{6 \cdot 5 \cdot 4}{3 \cdot 2} p^4 q^3$$

$$\gg\gg \quad p^4 \left(1 + 4q + 10q^2 + 20q^3\right)$$

   - Convert $q \gg (1-p)$:

$$\gg\gg \quad p^4 \left(1 + 4(1-p) + 10(1-p)^2 + 20(1-p)^3\right)$$

$$\gg\gg \quad 35p^4 - 84p^5 + 70p^6 - 20p^7 \quad \gg\gg \quad \approx 0.39$$

- The algebra seems very different, right up to the end!

Expectation and variance

28 - Expected value - rolling dice

Let X be a random variable counting the number of dots given by rolling a single die.

Then:

$$E[X] \quad \gg\gg \quad 1\cdot\frac{1}{6}+2\cdot\frac{1}{6}+\cdots+6\cdot\frac{1}{6} \quad \gg\gg \quad \frac{7}{2}$$

Let S be an RV that counts the dots on a roll of *two* dice.

The PMF of S:

k	2	3	4	5	6	7	8	9	10	11	12
$p_S(k)=P(S=k)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Then:

$$E[S] \quad \gg\gg \quad 2\cdot\frac{1}{36}+3\cdot\frac{2}{36}+4\cdot\frac{3}{36}+\cdots+12\cdot\frac{1}{36} \quad \gg\gg \quad 7$$

- Notice that $\frac{7}{2}+\frac{7}{2}=7$.
    - In general, $E[X+Y]=E[X]+E[Y]$.
    - Let X be a green die and Y a red die.
    - From the earlier calculation, $E[X]=\frac{7}{2}$ and $E[Y]=\frac{7}{2}$.
    - Since $S=X+Y$, we derive $E[S]=7$ by simple addition!

29 - Expectation from PMF of related

Let X have distribution given by this PMF:

x	1	2	3	4	5
$p_X(x)$	$1/7$	$1/14$	$3/14$	$2/7$	$2/7$

Find $E[|X-2|]$.

Solution

1. Compute the PMF.
    - PMF arranged by possible value:

$$\begin{aligned}
P[|X-2|=0] &\quad \gg\gg \quad P[X=2]=\tfrac{1}{14} \\
P[|X-2|=1] &\quad \gg\gg \quad P[X=1]+P[X=3]=\tfrac{1}{7}+\tfrac{3}{14}=\tfrac{5}{14} \\
P[|X-2|=2] &\quad \gg\gg \quad P[X=4]=\tfrac{2}{7} \\
P[|X-2|=3] &\quad \gg\gg \quad P[X=5]=\tfrac{2}{7} \\
P[|X-2|=k] &\quad \gg\gg \quad 0 \quad \text{for} \quad k\neq 0,1,2,3.
\end{aligned}$$

2. Calculate the expectation.
    - Use discrete formula:

$$E[|X-2|]=0\cdot\tfrac{1}{14}+1\cdot\tfrac{5}{14}+2\cdot\tfrac{2}{7}+3\cdot\tfrac{2}{7}=\tfrac{25}{14}$$

Poisson process

31 - Poisson calculation

Suppose $X \sim \text{Pois}(10)$. Find $P[X \leq 13 \mid X \geq 7]$. (Leave the answer in exact form.)

Solution

- Conditioning definition:

$$P[X \leq 13 \mid X \geq 7] \quad \gg\gg \quad \frac{P[7 \leq X \leq 13]}{1 - P[X < 7]}$$

- Expand numerator:

$$P[7 \leq X \leq 13] \quad \gg\gg \quad e^{-10}\frac{10^7}{7!} + e^{-10}\frac{10^8}{8!} + \cdots + e^{-10}\frac{10^{13}}{13!}$$

- Simplify:

$$\gg\gg \quad e^{-10}\frac{10^7}{7!}\left(1 + \frac{10}{8} + \frac{10^2}{8 \cdot 9} + \cdots + \frac{10^6}{8 \cdot 9 \cdots 13}\right)$$

- Compute for denominator:

$$P[X < 7] \quad \gg\gg \quad e^{-10}\frac{10^0}{0!} + e^{-10}\frac{10^1}{1!} + \cdots + e^{-10}\frac{10^6}{6!}$$

32 - Arrivals at a post office

Client arrivals at a post office are modelled well using a Poisson variable.

Each potential client has a very low and independent chance of coming to the post office, but there are many thousands of potential clients, so the arrivals at the office actually come in moderate number.

Suppose the average rate is 5 clients per hour.

- (a) Find the probability that nobody comes in the first 10 minutes of opening. (The cashier is considering being late by 10 minutes to run an errand on the way to work.)
- (b) Find the probability that 5 clients come in the first hour. (I.e. the average is achieved.)
- (c) Find the probability that 9 clients come in the first two hours.

Solution

(a)

1. ≡ Convert rate for desired window.
   - Expect $5/12$ clients every 10 minutes.
   - Let $X \sim \text{Pois}(5/12)$.
   - Seek $P_X(0)$ as the answer.
2. ≡ Compute.
   - Formula:

$$P_X(k) = e^{-5/12}\frac{(5/12)^k}{k!}$$

   - Insert data and compute:

$$P_X(0) \quad \gg\gg \quad e^{-5/12} \quad \gg\gg \quad \approx 0.659$$

(b)

1. Rate is already correct.
    - Let $X \sim \text{Pois}(5)$.
    - Compute the answer:

$$P_X(5) = e^{-5}\frac{5^5}{5!} \quad \gg\gg \quad \approx 0.175$$

(c)

1. Convert rate for desired window
    - Expect 10 clients every 2 hours.
    - Let $X \sim \text{Pois}(10)$.
    - Compute the answer:

$$P_X(9) \quad \gg\gg \quad e^{-10}\frac{10^9}{9!} \quad \gg\gg \quad \approx 0.125$$

- Notice that 0.125 is smaller than 0.175.