W14 - Examples

Statistical testing cont'd

ML test: Smoke detector

Suppose that a smoke detector sensor is configured to produce $8\,\mathrm{V}$ when there is smoke, and $0\,\mathrm{V}$ otherwise. But there is background noise with distribution $\mathcal{N}(0, 3^2\,\mathrm{V})$.

Design an ML test for the detector electronics to decide whether to activate the alarm.

What are the three error probabilities? (Type I, Type II, Total.)

Solution

First, establish the conditional distributions:

$$X \mid H_0 \;\sim \mathcal{N}(0, 3^2) \qquad X \mid H_1 \;\sim \mathcal{N}(8, 3^2)$$

Density functions:

$$f_{X|H_0} \;=\; \frac{1}{\sqrt{2\pi 9}} e^{-\frac{1}{2}\left(\frac{x-0}{3}\right)^2} \qquad f_{X|H_1} \;=\; \frac{1}{\sqrt{2\pi 9}} e^{-\frac{1}{2}\left(\frac{x-8}{3}\right)^2}$$

The ML condition becomes:

$$\frac{1}{\sqrt{2\pi 9}} e^{-\frac{1}{2}\left(\frac{x-0}{3}\right)^2} \quad \overset{?}{\geq} \quad \frac{1}{\sqrt{2\pi 9}} e^{-\frac{1}{2}\left(\frac{x-8}{3}\right)^2}$$

$$\gg\gg \quad -\frac{1}{2}\left(\frac{x-0}{3}\right)^2 \quad \overset{?}{\geq} \quad -\frac{1}{2}\left(\frac{x-8}{3}\right)^2$$

$$\gg\gg \quad x^2 \quad \overset{?}{\leq} \quad (x-8)^2$$

$$\gg\gg \quad x \leq 4$$

Therefore, A_0 is $x \leq 4$, while A_1 is $x > 4$.

The decision rule is: activate alarm when $x > 4$.

Type I error:

$$P_{FA} \;=\; P[A_1 \mid H_0] \quad \gg\gg \quad P[X > 4 \mid H_0]$$

$$\gg\gg \quad 1 - P\left[\frac{X-0}{3} \leq \frac{4}{3} \,\middle|\, H_0\right]$$

$$\gg\gg \quad 1 - P[Z \leq 1.3333] \quad \gg\gg \quad \approx 0.0912$$

Type II error:

$$P_{\text{Miss}} = P[A_0 \mid H_1] \gg\gg P[X \leq 4 \mid H_1]$$

$$\gg\gg \quad P\left[\frac{X-8}{3} \leq \frac{4-8}{3} \,\middle|\, H_1\right]$$

$$\gg\gg \quad P[Z \leq -1.3333] \quad \gg\gg \quad \approx 0.0912$$

Total error:

$$P_{\text{ERR}} = P_{FA} \cdot 0.5 + P_{\text{Miss}} \cdot 0.5 \quad \approx \quad 0.0912$$

MAP test: Smoke detector

Suppose that a smoke detector sensor is configured to produce 8 V when there is smoke, and 0 V otherwise. But there is background noise with distribution $\mathcal{N}(0, 3^2\,\text{V})$.

Suppose that the background chance of smoke is 5%. Design a MAP test for the alarm.

What are the three error probabilities? (Type I, Type II, Total.)

Solution

First, establish priors:

$$P[H_0] = 0.95 \qquad P[H_1] = 0.05$$

The MAP condition becomes:

$$\frac{1}{\sqrt{2\pi 9}} e^{-\frac{1}{2}\left(\frac{x-0}{3}\right)^2} \cdot 0.95 \quad \overset{?}{\geq} \quad \frac{1}{\sqrt{2\pi 9}} e^{-\frac{1}{2}\left(\frac{x-8}{3}\right)^2} \cdot 0.05$$

$$\gg\gg \quad e^{-\frac{1}{2}\left(\frac{x-0}{3}\right)^2} \quad \overset{?}{\geq} \quad e^{-\frac{1}{2}\left(\frac{x-8}{3}\right)^2} \cdot \frac{0.05}{0.95}$$

$$\gg\gg \quad -\frac{1}{2}\left(\frac{x-0}{3}\right)^2 \quad \overset{?}{\geq} \quad -\frac{1}{2}\left(\frac{x-8}{3}\right)^2 + \ln\left(\frac{0.05}{0.95}\right)$$

$$\gg\gg \quad x^2 \quad \overset{?}{\leq} \quad (x-8)^2 - 18\ln\left(\frac{0.05}{0.95}\right)$$

$$\gg\gg \quad x \leq 7.31$$

Therefore, A_0 is $x \leq 7.31$, while A_1 is $x > 7.31$.

The decision rule is: activate alarm when $x > 7.31$.

Type I error:

$$P_{FA} = P[A_1 \mid H_0] \quad \gg\gg \quad P[X > 7.31 \mid H_0]$$

$$\gg\gg \quad 1 - P[Z \leq 2.4367] \quad \gg\gg \quad \approx 0.007411$$

Type II error:

$$P_{\text{Miss}} = P[A_0 \mid H_1] \quad \gg\gg \quad P[X \leq 7.31 \mid H_1]$$

$$\gg\gg \quad P[Z \leq -0.23] \quad \gg\gg \quad \approx 0.4090$$

Total error:

$$P_{\text{ERR}} \;=\; P_{FA} \cdot 0.95 + P_{\text{Miss}} \cdot 0.05 \quad\approx\quad 0.02749$$

MC Test: Smoke detector

Suppose that a smoke detector sensor is configured to produce 8 V when there is smoke, and 0 V otherwise. But there is background noise with distribution $\mathcal{N}(0, 3\,\mathrm{V})$.

Suppose that the background chance of smoke is 5%. Suppose the cost of a miss is $50\times$ the cost of a false alarm. Design an MC test for the alarm.

Compute the expected cost.

Solution

We have priors:

$$P[H_0] \;=\; 0.95 \qquad P[H_1] \;=\; 0.05$$

And we have costs:

$$C_{10} = 1 \qquad C_{01} = 50$$

(The ratio of these numbers is all that matters in the inequalities of the condition.)

The MC condition becomes:

$$\frac{1}{\sqrt{2\pi 9}} e^{-\frac{1}{2}\left(\frac{x-0}{3}\right)^2} \cdot 0.95 \cdot \boxed{1} \quad \overset{?}{\geq} \quad \frac{1}{\sqrt{2\pi 9}} e^{-\frac{1}{2}\left(\frac{x-8}{3}\right)^2} \cdot 0.05 \cdot \boxed{50}$$

$$\gg\gg \quad e^{-\frac{1}{2}\left(\frac{x-0}{3}\right)^2} \quad \overset{?}{\geq} \quad e^{-\frac{1}{2}\left(\frac{x-8}{3}\right)^2} \cdot \frac{2.5}{0.95}$$

$$\gg\gg \quad -\frac{1}{2}\left(\frac{x-0}{3}\right)^2 \quad \overset{?}{\geq} \quad -\frac{1}{2}\left(\frac{x-8}{3}\right)^2 + \ln\left(\frac{2.5}{0.95}\right)$$

$$\gg\gg \quad x^2 \quad \overset{?}{\leq} \quad (x-8)^2 - 18\ln\left(\frac{2.5}{0.95}\right)$$

$$\gg\gg \quad x \leq 2.91$$

Therefore, A_0 is $x \leq 2.91$, while A_1 is $x > 2.91$.

The decision rule is: activate alarm when $x > 2.91$.

Type I error:

$$P_{FA} \;=\; P[A_1 \mid H_0]$$

$$\gg\gg \quad P[X > 2.91 \mid H_0] \quad \gg\gg \quad \approx 0.1660$$

Type II error:

$$P_{\text{Miss}} \;=\; P[A_0 \mid H_1]$$

$$\gg\gg \quad P[X \leq 2.91] \quad \gg\gg \quad \approx 0.04488$$

Total error:

$$P_{\text{ERR}} \;=\; P_{FA} \cdot 0.95 + P_{\text{Miss}} \cdot 0.05 \quad\approx\quad 0.1599$$

PMF of total cost:

$$P_C(c) \quad = \quad \begin{cases} 0.002244 & c = 50 \\ 0.1577 & c = 1 \\ 0.840056 & c = 0 \end{cases}$$

Therefore $E[C] \; = \; 0.27$.

Mean square error

Minimal MSE estimate given PMF

Suppose X has the following PMF:

k	**1**	**2**	**3**	**4**	**5**
$P_X(k)$	0.15	0.28	0.26	0.19	0.13

Find the minimal MSE estimate of X, given that X is even. What is the error of this estimate?

Solution

The minimal MSE given A is just $E[X \mid A]$ where $A = \{2, 4\}$.

First compute the conditional PMF:

$$P_{X|A}(k) \quad = \quad \begin{cases} 0.19/0.47 & k = 4 \\ 0.28/0.47 & k = 2 \end{cases}$$

Therefore:

$$\hat{x}_A \quad = \quad 2\frac{0.28}{0.47} + 4\frac{0.19}{0.47} \quad \approx \quad 2.80851$$

The error is:

$$e_A \quad = \quad (2 - 2.81)^2 \frac{0.28}{0.47} + (4 - 2.81)^2 \frac{0.19}{0.47}$$

$$\ggg \quad \approx 0.9633$$

Minimal MSE estimate from joint PDF

Here is the joint PDF of X and Y:

$$f_{X,Y} \quad = \quad \begin{cases} 8xy & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the minimal MSE estimate of X in terms of Y.

What is the estimate of X when $Y = 0.2$? When $Y = 0.8$?

Solution

$$\hat{x}_M(y) \quad = \quad \frac{2}{3} \cdot \frac{1 - y^3}{1 - y^2}$$

$\hat{x}_M(0.2) \; = \; 0.6889 \qquad \hat{x}_M(0.8) \; = \; 0.9037$

Estimating on a variable interval

Suppose that $R \sim \text{Unif}((0, 1))$ and suppose $X \sim \text{Unif}(0, R)$.

(a) Find $\hat{x}_M(r)$ (b) Find $\hat{r}_M(x)$ (c) Find $\hat{R}_{L_{\min}}(X)$

Solution

(a) Find $\hat{x}_M(r)$.

We know $\hat{x}_M(r) = E[X \mid R = r]$.

Given $R = r$, so X is uniform on $(0, r)$, we have $E[X \mid R = r] = \frac{r}{2}$.

(b) Find $\hat{r}_M(x)$.

We know $\hat{r}_M(x) = E[R \mid X = x]$.

To compute this function, we calculate a sequence of densities.

We know f_R and $f_{X|R}$. From these we derive the joint distribution $f_{X,R}$:

$$f_R(r) = \begin{cases} 1 & r \in (0,1) \\ 0 & \text{otherwise} \end{cases} \qquad f_{X|R}(x|r) = \begin{cases} 1/r & x \in (0,r) \\ 0 & \text{otherwise} \end{cases}$$

$$\gg\gg \quad f_{X,R}(x,r) \quad = \quad f_{X|R} \cdot f_R \ = \begin{cases} 1/r & 0 < x < r < 1 \\ 0 & \text{else} \end{cases}$$

Now extract the marginal f_X:

$$\gg\gg \quad f_X(x) \quad = \quad \int_{-\infty}^{\infty} f_{X,R}(x,r)\, dr$$

$$\gg\gg \quad \int_x^1 \frac{1}{r}\, dr \quad \gg\gg \quad -\ln x \qquad (0 < x < 1)$$

Now deduce the conditional $f_{R|X}$:

$$f_{R|X} \quad = \quad \frac{f_{X,R}}{f_X} \ = \begin{cases} \frac{-1}{r \ln x} & 0 < x < r < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then:

$$E[R \mid X = x] \quad \gg\gg \quad \int_x^1 r \frac{-1}{r \ln x}\, dr$$

$$\gg\gg \quad \frac{x-1}{\ln x}$$

So $\hat{r}_M(x) = \frac{x-1}{\ln x}$.

(c) Find $\hat{R}_{L_{\min}}(X)$.

We need all the basic statistics.

$E[R] \ = \ 1/4$ because $R \sim \text{Unif}((0,1))$.

$\sigma_R \ = \ \frac{(b-a)^2}{12} \ = \ 1/12$.

$E[X] \;=\; 1/4$ using the marginal PDF $f_X(x) = -\ln x$ on $x \in (0,1)$. (IBP and L'Hopital are needed.)

$\sigma_X \;=\; \sqrt{7}/12$ also using the marginal $f_X(x) = -\ln x$.

$E[XR] = 1/6$ using $f_{X,R}(x,r)$, namely:

$$E[XR] \;=\; \int_{r=0}^{1}\int_{x=0}^{r} xr\frac{1}{r}\,dx\,dr$$

$$\gg\gg \quad \int_0^1 \frac{x^2}{2}\,dx \quad \gg\gg \quad \frac{1}{6}$$

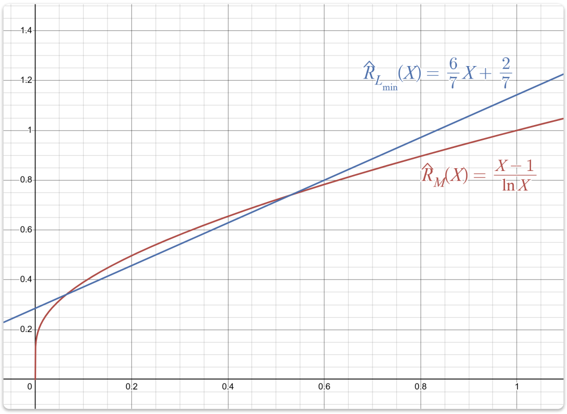
From this we infer $\mathrm{Cov}[X,R] = 1/24$ and $\rho_{X,R} = \sqrt{3/7}$.

Hence:

$$L_{\min}(x) \;=\; \frac{6}{7}x + \frac{2}{7}$$

Thus:

$$\hat{R}_{L_{\min}}(X) \;=\; \frac{6}{7}X + \frac{2}{7}$$



Line of minimal MSE given joint PDF

Here is the joint PDF of X and Y:

$$f_{X,Y} \;=\; \begin{cases} 8xy & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the line giving the linear MSE estimate of X in terms of Y.

What is the expected error of this line, $e_{L_{\min}}$?

What is the estimate of X when $Y = 0.2$? When $Y = 0.8$?

Answers

$\hat{X}_{L_{\min}}(Y) \;=\; 0.3637\,Y + 0.6060$

$e_{L_{\min}} \;=\; 0.02020$

$\hat{x}_{L_{\min}}(0.2) \;=\; 0.67874 \qquad \hat{x}_{L_{\min}}(0.8) \;=\; 0.89696$