Calculus II - Lecture notes - W04

Videos, Math Dr. Bob:

- Formula for Arc Length 1 (good basic understanding and examples)
- Formula for Arc Length 2

Videos, Khan Academy:

- Arc length integration example

Arc length

01 Theory

The **total arc length** of a curve is just the length of the curve.

The 'arc length' (not "total") is a quantity measuring the length "as you go along," usually given as a function of the points on the curve. It measures the length from some starting point 'up to' the given point.

We can use calculus to calculate the arc length of many curves. If the curve is the graph of a function, and we know the function and its derivative (whether from a formula or a data table), we can use integration to find the arc length.

Arc-length formula

The arc length s of the graph of $f(x)$ over $x \in [a,b]$ is:

$$s = \int_a^b \sqrt{1 + \big(f'(x)\big)^2}\, dx$$

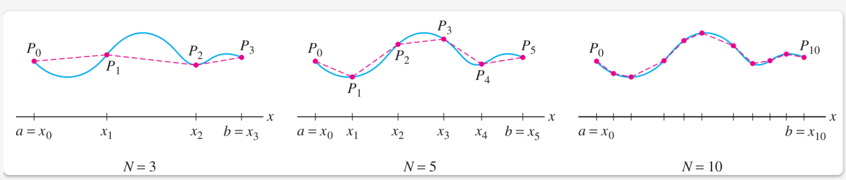
(This formula applies when $f'(x)$ exists and is continuous on $[a,b]$.)

The arc length *function* $s(x)$ of the graph of $f(x)$, starting from $x = a$, is:

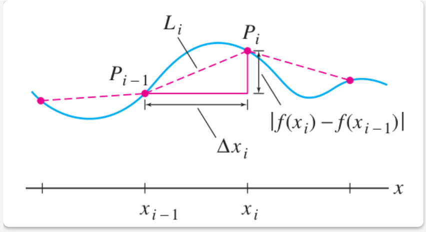
$$s(x) = \int_a^x \sqrt{1 + \big(f'(t)\big)^2}\, dt$$

Arc-length formula - explanation

The arc-length integral is the limit of Riemann sums that add the lengths of straight line segments whose endpoints lie on the curve, and which together approximate the curve.



Each tiny line segment is the hypotenuse of a triangle with horizontal Δx and vertical Δy.



We can approximate the vertical Δy using the *derivative*:

$$\Delta y \approx \frac{dy}{dx}\Delta x = f'(x)\Delta x$$

Considering infinitesimals in the limit, we have $\Delta x \to dx$ (horizontal side) and $\Delta y \to dy = f'\,dx$ (vertical side). The Pythagorean Theorem gives:

$$ds = \sqrt{dx^2 + dy^2}$$

which we can simplify using $dy = f'\,dx$:

$$\gg\gg \quad \sqrt{dx^2 + (f'dx)^2} \quad \gg\gg \quad \sqrt{1 + (f')^2}\,dx$$

The integral of these infinitesimals gives the arc length:

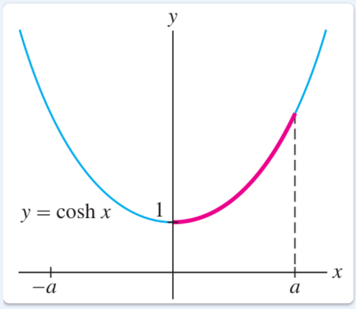
$$s(a) = \int_0^a ds = \int \sqrt{1 + (f')^2}\,dx$$

02 Illustration

Example - Arc length of chain in terms of position

A hanging chain describes a **catenary** shape. ('Catenary' is to hyperbolic trig as 'sinusoid' is to normal trig.) The graph of the hyperbolic cosine is a catenary:

$$y = f(x) = \cosh x$$



Let us compute the arc length of this catenary on the portion from $x = 0$ to $x = a$.

Solution

1. Arc-length formula.
    - Give arc length $s(a)$, a function of $a \geq 0$:

$$s(a) = \int_0^a \sqrt{1 + (f')^2}\,dx$$

2. Compute $f'(x)$.

- Hyperbolic trig derivative:

$$\frac{d}{dx}\cosh(x) = \sinh(x)$$

3. Plug into formula.
   - Arc length:

$$s(a) = \int_0^a \sqrt{1+\sinh^2(x)}\, dx$$

4. Hyperbolic trig identity.
   - Fundamental identity:

$$\cosh^2 x - \sinh^2 x = 1$$

   - Rearrange:

$$1 + \sinh^2 x = \cosh^2 x$$

5. Plug into formula and compute.
   - Arc length:

$$\int_0^a \sqrt{1+\sinh^2(x)}\, dx \quad \gg\gg \quad \int_0^a \sqrt{\cosh^2 x}\, dx \quad \gg\gg \quad \int_0^a \cosh x\, dx$$

   - Compute integral:

$$\int_0^a \cosh x\, dx = \sinh a$$

   - The arc length of a catenary curve matches the area under the catenary curve!

Exercise - Arc length of a line segment

Find the arc length of the *straight line* given by the formula $y = 3x + 1$ over $x \in [0, 3]$.

Check your answer using the Pythagorean Theorem.

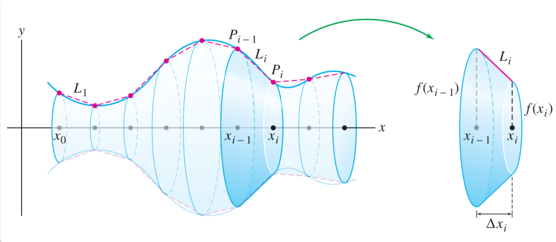
Surface areas of revolutions - thin bands

Videos, Math Dr. Bob

- [Area of a Surface of Revolution]

03 Theory

The infinitesimal of arc length along a curve, ds, can be used to find the **surface area** of a surface of revolution. The circumference of an **infinitesimal band** is $2\pi R$ and the width of such a band is ds.



The general formula for the surface area is:

$$S = \int_a^b 2\pi R \, ds$$

In any given problem we need to find the appropriate expressions for R and ds in terms of the variable of integration. For regions rotated around the x-axis, the variable will be x; for regions rotated about the y-axis it will be y.

Assuming the region is rotated around the x-axis, and the cross section in the xy-plane is the graph of f and so $R = f(x)$, the formula above becomes:

Area of revolution formula - thin bands

The surface area S of the surface of revolution given by $R = f(x)$ is given by the formula:

$$S = \int_a^b 2\pi f(x) \sqrt{1 + \left(f'\right)^2} \, dx$$

In this formula, we assume $f(x) \geq 0$ and f' is continuous. The surface is the revolution of $f(x)$ on $x \in [a, b]$ around the x-axis.

04 Illustration

Example - Surface area of a sphere

Using the fact that a sphere is given by revolving a semicircle, verify the formula $S = 4\pi r^2$ for the surface area of a sphere.

Solution

1. Sphere as surface of revolution.
   - Sphere of radius r given by revolving upper semicircle.
   - Upper semicircle:

   $$x^2 + y^2 = r^2, \qquad y \geq 0$$

   - Upper semicircle as function of x:

   $$y = f(x) = \sqrt{r^2 - x^2}, \qquad x \in [-r, r]$$

2. Surface area formula.
   - Bounds are $x = -r$ and $x = +r$.
   - Function is $f(x) = \sqrt{r^2 - x^2}$
   - Plug data into formula:

   $$S = \int_{-r}^{+r} 2\pi \sqrt{r^2 - x^2} \sqrt{1 + \left(f'\right)^2} \, dx$$

3. Compute $\left(f'\right)^2$.

- Power rule and chain rule:

$$f'(x) = \frac{1}{2}(r^2 - x^2)^{-1/2}(-2x)$$

- Algebra:

$$\gg\gg \qquad -x(r^2 - x^2)^{-1/2}$$

- Squaring:

$$\left(f'\right)^2 = \frac{x^2}{r^2 - x^2}$$

4. Compute integrand.
   - Compute $1 + \left(f'\right)^2$:

$$1 = \frac{r^2 - x^2}{r^2 - x^2}$$

$$1 + \left(f'\right)^2 \qquad \gg\gg \qquad \frac{r^2 - x^2}{r^2 - x^2} + \frac{x^2}{r^2 - x^2} \qquad \gg\gg \qquad \frac{r^2}{r^2 - x^2}$$

   - Integrand factors become:

$$\sqrt{r^2 - x^2}\sqrt{1 + \left(f'\right)^2} \qquad \gg\gg \qquad \sqrt{r^2 - x^2}\sqrt{\frac{r^2}{r^2 - x^2}} \qquad \gg\gg \qquad r$$

5. Compute integral.
   - Surface area again:

$$S = \int_{-r}^{+r} 2\pi r \, dx$$

$$\begin{aligned} &= 2\pi r x \Big|_{-r}^{+r} = 2\pi r r - 2\pi r(-r) \\ &= 4\pi r^2 \end{aligned}$$

   - This is the desired surface area formula $S = 4\pi r^2$.