

# Unit 01 notes

## Trig power products

Videos, Math Dr. Bob:

- [Trig power products](#):  $\int \cos^m x \sin^n x dx$
- [Trig differing frequencies](#):  $\int \cos mx \sin nx dx$
- [Trig tan and sec](#):  $\int \tan^m x \sec^n x dx$
- [Secant power](#):  $\int \sec^5 x dx$

Videos, Organic Chemistry Tutor:

- [Trig power product techniques](#)
- [Trig substitution](#)

## 06 Theory

### Review: trig identities

- $\sin^2 x + \cos^2 x = 1$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

#### 📌 Trig power product: sin / cos

A sin / cos power product has this form:

$$\int \cos^m x \cdot \sin^n x dx$$

for some integers  $m$  and  $n$  (even negative!).

To compute these integrals, use a sequence of these techniques:

- **Swap an even bunch.**
- **$u$ -sub for power-one.**
- **Power-to-frequency conversion.**

- 🕒 Memorize these three techniques!

Examples of trig power products:

- $\int \sin x \cdot \cos^7 x dx$
- $\int \sin^3 x dx$
- $\int \sin^2 x \cdot \cos^2 x dx$

#### 📌 Swap an even bunch

If *either*  $\cos^m x$  or  $\sin^n x$  is an *odd* power, use

$$\sin^2 x \gg \gg 1 - \cos^2 x$$

$$\text{OR } \cos^2 x \gg \gg 1 - \sin^2 x$$

(maybe repeatedly) to convert an **even bunch** to the opposite trig type.

An **even bunch** is *all but one* from the odd power.

For example:

$$\begin{aligned} \sin^5 x \cdot \cos^8 x &\gg \gg \sin x (\sin^2 x)^2 \cdot \cos^8 x \\ &\gg \gg \sin x (1 - \cos^2 x)^2 \cdot \cos^8 x \\ &\gg \gg \sin x (1 - 2\cos^2 x + \cos^4 x) \cdot \cos^8 x \\ &\gg \gg \sin x (\cos^8 x - 2\cos^{10} x + \cos^{12} x) \\ &\gg \gg \sin x \cos^8 x - 2\sin x \cos^{10} x + \sin x \cos^{12} x \end{aligned}$$



#### *u*-sub for power-one

If  $m = 1$  or  $n = 1$ , *perform u-substitution* to do the integral.

The *other* trig power becomes a  $u$  power; the power-one becomes  $du$ .

For example, using  $u = \cos x$  and thus  $du = -\sin x dx$  we can do:

$$\int \sin x \cos^8 x dx \gg \gg \int -\cos^8 x (-\sin x dx) \gg \gg -\int u^8 du$$

-  By combining these tricks you can do any power product with at least one odd power!
  - Leave a power-one from the odd power when swapping an even bunch.
-  Notice:  $1 = \sin^0 x = \cos^0 x$ , even powers. So the method works for  $\int \sin^3 x dx$  and similar.

#### Power-to-frequency conversion

Using these ‘power-to-frequency’ identities (maybe repeatedly):

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x), \quad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

change an even power (either type) into an odd power of cosine.

For example, consider the power product:

$$\sin^4 x \cdot \cos^6 x$$

You can substitute appropriate powers of  $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$  and  $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ :

$$\begin{aligned} \sin^4 x \cdot \cos^6 x &\gg \gg (\sin^2 x)^2 \cdot (\cos^2 x)^3 \\ &\gg \gg \left(\frac{1}{2}(1 - \cos 2x)\right)^2 \cdot \left(\frac{1}{2}(1 + \cos 2x)\right)^3 \end{aligned}$$

By doing some annoying algebra, this expression can be expanded as a sum of *smaller* powers of  $\cos 2x$ :

$$\left(\frac{1}{2}(1 - \cos 2x)\right)^2 \cdot \left(\frac{1}{2}(1 + \cos 2x)\right)^3$$

$$\gg \gg \frac{1}{32} \left(1 + \cos(2x) - 2 \cos^2(2x) - 2 \cos^3(2x) + \cos^4(2x) + \cos^5(2x)\right)$$

Each of these terms can be integrated by repeating the same techniques.

## 07 Illustration

### Example - Trig power product with an odd power

Compute the integral:

$$\int \cos^2 x \cdot \sin^5 x \, dx$$

#### Solution

1. Swap over the even bunch.

- Max even bunch leaving power-one is  $\sin^4 x$ :

$$\sin^5 x \gg \gg \sin x (\sin^2 x)^2 \gg \gg \sin x (1 - \cos^2 x)^2$$

- Apply to  $\sin^5 x$  in the integrand:

$$\int \cos^2 x \cdot \sin^5 x \, dx \gg \gg \int \cos^2 x \cdot \sin x (1 - \cos^2 x)^2 \, dx$$

2. Perform  $u$ -substitution on the power-one integrand.

- Set  $u = \cos x$ .
- Hence  $du = -\sin x \, dx$ . Recognize this in the integrand.
- Convert the integrand:

$$\begin{aligned} \int \cos^2 x \cdot \sin x (1 - \cos^2 x)^2 \, dx &\gg \gg \int \cos^2 x \cdot (1 - \cos^2 x)^2 (\sin x \, dx) \\ &\gg \gg \int u^2 \cdot (1 - u^2)^2 \, du \end{aligned}$$

3. Perform the integral.

- Expand integrand and use power rule to obtain:

$$\int u^2 \cdot (1 - u^2)^2 \, du = \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$$

- Insert definition  $u = \cos x$ :

$$\begin{aligned} \int \cos^2 x \cdot \sin^5 x \, dx &\gg \gg \int u^2 \cdot (1 - u^2)^2 \, du \\ &\gg \gg \frac{1}{3}\cos^3 x - \frac{2}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C \end{aligned}$$

4. This is our final answer.

## 08 Theory

### 📌 Trig power product: tan / sec or cot / csc

A tan / sec power product has this form:

$$\int \tan^m x \cdot \sec^n x \, dx$$

A cot / csc power product has this form:

$$\int \cot^m x \cdot \csc^n x \, dx$$

To integrate these, **swap an even bunch** using:

- $\tan^2 x + 1 = \sec^2 x$

OR:

- $\cot^2 x + 1 = \csc^2 x$

Or do **u-substitution** using:

- $u = \tan x \rightsquigarrow du = \sec^2 x \, dx$
- $u = \sec x \rightsquigarrow du = \sec x \tan x \, dx$

OR:

- $u = \cot x \rightsquigarrow du = -\csc^2 x \, dx$
- $u = \csc x \rightsquigarrow du = -\csc u \cot u \, dx$

Note:

- ⚠️ There is no simple “power-to-frequency conversion” for tan / sec !

We can modify the power-one technique to solve some of these. We need to swap over an even bunch *from the odd power* so that exactly the  $du$  factor is left behind.

Considering all the possibilities, one sees that this method works when:

- $\tan^m x$  is an *odd* power
- $\sec^n x$  is an *even* power

Quite a few cases escape this method:

- Any  $\int \tan^m x \, dx$
- Any  $\int \tan^m x \cdot \sec^n x \, dx$  for  $m$  even and  $n$  odd

These tricks don't work for  $\int \tan x \, dx$  or  $\int \sec x \, dx$  or  $\int \tan^4 x \sec^5 x \, dx$ , among others.

### 📌 Special integrals: tan and sec

We have:

$$\int \tan x \, dx = \ln |\sec x| + C$$

$$\int \sec x \, dx = \ln |\sec x + \tan x| + C$$

- ⚠️ These integrals should be memorized individually.

### Deriving special integrals - tan and sec

The first formula can be found by  $u$ -substitution, considering that  $\tan x = \frac{\sin x}{\cos x}$ .

The second formula can be derived by multiplying  $\sec x$  by a special “1”, computing instead  $\int \frac{\sec x (\sec x + \tan x)}{\sec x + \tan x} dx$  by expanding the numerator and doing  $u$ -sub on the denominator.

## 09 Illustration

### Example - Trig power product with tan and sec

Compute the integral:

$$\int \tan^5 x \cdot \sec^3 x \, dx$$

#### Solution

1.  $\Rightarrow$  Try  $du = \sec^2 x \, dx$ .

- Factor  $du$  out of the integrand:

$$\int \tan^5 x \cdot \sec^3 x \, dx \quad \gg \gg \quad \int \tan^5 x \cdot \sec x (\sec^2 x \, dx)$$

- We then must swap over remaining  $\sec x$  into the  $\tan x$  type.
- Cannot do this because  $\sec x$  has odd power. Need even to swap.

2.  $\Rightarrow$  Try  $du = \sec x \tan x \, dx$ .

- Factor  $du$  out of the integrand:

$$\int \tan^5 x \cdot \sec^3 x \, dx \quad \gg \gg \quad \int \tan^4 x \cdot \sec^2 x (\sec x \tan x \, dx)$$

- Swap remaining  $\tan x$  into  $\sec x$  type:

$$\begin{aligned} & \int (\tan^2 x)^2 \cdot \sec^2 x (\sec x \tan x \, dx) \\ & \gg \gg \int (\sec^2 x - 1)^2 \cdot \sec^2 x (\sec x \tan x \, dx) \end{aligned}$$

- Substitute  $u = \sec x$  and  $du = \sec x \tan x \, dx$ :

$$\gg \gg \int (u^2 - 1)^2 \cdot u^2 \, du$$

3.  $\Rightarrow$  Compute the integral in  $u$  and convert back to  $x$ .

- Expand the integrand:


$$\gg \gg \int u^6 - 2u^4 + u^2 \, du$$

- Apply power rule:

$$\gg \gg \frac{u^7}{7} - 2\frac{u^5}{5} + \frac{u^3}{3} + C$$

- Plug back in,  $u = \sec x$ :

$$\gg \gg \frac{\sec^7 x}{7} - 2\frac{\sec^5 x}{5} + \frac{\sec^3 x}{3} + C$$

4.  This is our final answer.

## Trig substitution

Videos, Math Dr. Bob:

- [Trig sub 1](#): Basics and  $\int \frac{1}{\sqrt{36-x^2}} dx$  and  $\int \frac{x}{36+x^2} dx$  and  $\int \frac{1}{\sqrt{x^2-36}} dx$
- [Trig sub 2](#):  $\int \frac{dx}{(1+x^2)^{5/2}}$
- [Trig sub 3](#):  $\int \frac{x^2}{\sqrt{1-4x^2}} dx$
- [Trig sub 4](#):  $\int \sqrt{e^{2x}-1} dx$
- [Trig sub 5](#):  $\int \frac{\sqrt{4-36x^2}}{x^2} dx$

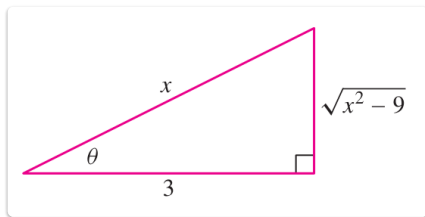
### 10 Theory

Certain algebraic expressions have a secret meaning that comes from the Pythagorean Theorem. This meaning has a very simple expression in terms of trig functions of a certain angle.

For example, consider the integral:

$$\int \frac{1}{x^2 \sqrt{x^2 - 9}} dx$$

Now consider this triangle:



The triangle determines the relation  $x = 3 \sec \theta$ , and it implies  $\sqrt{x^2 - 9} = 3 \tan \theta$ .

Now plug these into the integrand above:

$$\frac{1}{x^2 \sqrt{x^2 - 9^2}} \gg \gg \frac{1}{9 \sec^2 \theta \cdot 3 \tan \theta}$$


Considering that  $dx = 3 \sec \theta \tan \theta d\theta$ , we obtain a very reasonable trig integral:

$$\begin{aligned} \int \frac{1}{x^2 \sqrt{x^2 - 9^2}} dx &\gg \gg \int \frac{3 \sec \theta \tan \theta}{27 \sec^2 \theta \tan \theta} d\theta \\ &\gg \gg \frac{1}{9} \int \cos \theta d\theta \gg \gg \frac{1}{9} \sin \theta + C \end{aligned}$$

We must rewrite this in terms of  $x$  using  $x = 3 \sec \theta$  to finish the problem. We need to find  $\sin \theta$  assuming that  $\sec \theta = \frac{x}{3}$ . To do this, refer back to the triangle to see that  $\sin \theta = \frac{\sqrt{x^2 - 9}}{x}$ . Plug this in for our final value of the integral:

$$\frac{1}{9} \sin \theta + C \gg \gg \frac{\sqrt{x^2 - 9}}{9x} + C$$

Here is the moral of the story:

-  Re-express the Pythagorean expression *using a triangle and a trig substitution*.

- In this way, square roots of quadratic polynomials can be eliminated.

There are always three steps for these trig sub problems:

- (1) Identify the trig sub: find the sides of a triangle and relevant angle  $\theta$ .
- (2) Solve a trig integral (often a power product).
- (3) Refer back to the triangle to convert the answer back to  $x$ .

To speed up your solution process for these problems, *memorize* these three transformations:

(1)

$$\sqrt{a^2 - x^2} \quad \begin{matrix} x=a \sin \theta \\ \gg \gg \end{matrix} \quad \sqrt{a^2 - a^2 \sin^2 \theta} = a \cos \theta \quad \text{from} \quad 1 - \sin^2 \theta = \cos^2 \theta$$

(2)

$$\sqrt{a^2 + x^2} \quad \begin{matrix} x=a \tan \theta \\ \gg \gg \end{matrix} \quad \sqrt{a^2 + a^2 \tan^2 \theta} = a \sec \theta \quad \text{from} \quad 1 + \tan^2 \theta = \sec^2 \theta$$

(3)

$$\sqrt{x^2 - a^2} \quad \begin{matrix} x=a \sec \theta \\ \gg \gg \end{matrix} \quad \sqrt{a^2 \sec^2 \theta - a^2} = a \tan \theta \quad \text{from} \quad \sec^2 \theta - 1 = \tan^2 \theta$$

For a more complex quadratic with linear and constant terms, you will need to first *complete the square* for the quadratic and then do the trig substitution.

## 11 Illustration

### ≡ Example - Trig sub in quadratic: completing the square

Compute the integral:

$$\int \frac{dx}{\sqrt{x^2 - 6x + 11}}$$

#### ≡ Solution

1. ☐ Notice square root of a quadratic.

2. ⇌ Complete the square to obtain Pythagorean form.

- Find constant term for a complete square:

$$x^2 - 6x + \left(\frac{-6}{2}\right)^2 = x^2 - 6x + 9 = (x - 3)^2$$

- Add and subtract desired constant term:

$$x^2 - 6x + 11 \quad \gg \gg \quad x^2 - 6x + 9 - 9 + 11$$

- Simplify:

$$x^2 - 6x + 9 - 9 + 11 \quad \gg \gg \quad (x - 3)^2 + 2$$

3. ⇌ Perform shift substitution.

- Set  $u = x - 3$  as inside the square:

$$(x - 3)^2 + 2 = u^2 + 2$$

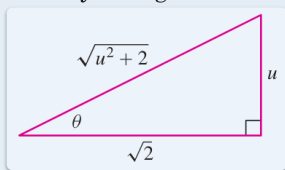
- Infer  $du = dx$ .

- Plug into integrand:

$$\int \frac{dx}{\sqrt{x^2 - 6x + 11}} \gg \gg \int \frac{du}{\sqrt{u^2 + 2}}$$

#### 4. Trig sub with $\tan \theta$ .

- Identify triangle:



- Use substitution  $u = \sqrt{2} \tan \theta$ . (From triangle or memorized tip.)
- Infer  $du = \sqrt{2} \sec^2 \theta d\theta$ .
- Plug in data:

$$\int \frac{du}{\sqrt{u^2 + 2}} \gg \gg \int \frac{\sec^2 \theta}{\sec \theta} d\theta = \int \sec \theta d\theta$$

#### 5. Compute trig integral.

- Use ad hoc formula:

$$\int \sec \theta d\theta = \ln |\tan \theta + \sec \theta| + C$$

#### 6. Convert trig back to $x$ .

- First in terms of  $u$ , referring to the triangle:

$$\tan \theta = \frac{u}{\sqrt{2}}, \quad \sec \theta = \frac{\sqrt{u^2 + 2}}{\sqrt{2}}$$

- Then in terms of  $x$  using  $u = x - 3$ .
- Plug everything in:

$$\ln |\tan \theta + \sec \theta| + C \gg \gg \ln \left| \frac{x-3}{\sqrt{2}} + \frac{\sqrt{(x-3)^2 + 2}}{\sqrt{2}} \right| + C$$

#### 7. Simplify using log rules.

- Log rule for division gives us:

$$\ln \frac{f(x)}{a} = \ln f(x) - \ln a$$

- The common denominator  $\frac{1}{\sqrt{2}}$  can be pulled outside as  $-\ln \sqrt{2}$ .
- The new term  $-\ln \sqrt{2}$  can be “absorbed into the constant” (redefine  $C$ ).
- So we write our final answer thus:

$$\ln \left| x - 3 + \sqrt{(x-3)^2 + 2} \right| + C$$