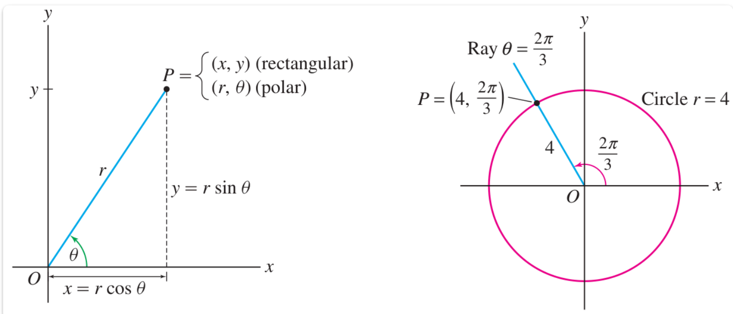
W14 Notes

Polar curves

01 Theory - Polar points, polar curves

Polar coordinates are pairs of numbers (r, θ) which identify points in the plane in terms of *distance to origin* and *angle from $+x$-axis*:



Converting Polar ↔ Cartesian

$$\begin{aligned} &\text{Polar} \to \text{Cartesian} \\ &x = r\cos\theta \\ &y = r\sin\theta \end{aligned} \qquad \begin{aligned} &\text{Cartesian} \to \text{Polar} \\ &r = \sqrt{x^2 + y^2} \\ &\tan\theta = \frac{y}{x} \quad (x \neq 0) \end{aligned}$$

Polar coordinates have *many redundancies*: unlike Cartesian which are unique!

- For example: $(r, \theta) = (r, \theta + 2\pi)$
  - And therefore also $(r, \theta) = (r, \theta - 2\pi)$ (negative θ can happen)
- For example: $(-r, \theta) = (r, \theta + \pi)$ for every r, θ
- For example: $(0, \theta) = (0, 0)$ for any θ

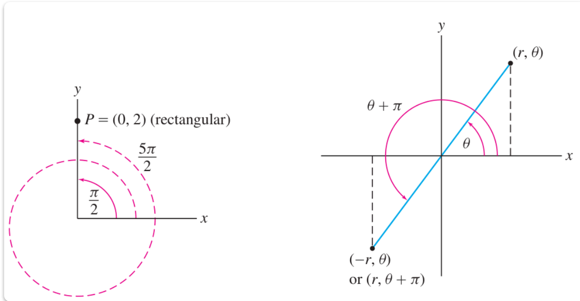
Polar coordinates *cannot be added*: they are not vector components!

- For example $(5, \pi/3) + (2, \pi/6) \neq (7, \pi/2)$
- Whereas Cartesian coordinates can be added: $(1, 4) + (2, -2) = (3, 2)$

⚠️ The transition formulas Cartesian → Polar require careful choice of θ.

- The standard definition of $\tan^{-1}\left(\frac{y}{x}\right)$ sometimes gives *wrong θ*
  - This is because it uses the restricted domain $\theta \in (-\pi/2, \pi/2)$; the polar interpretation is: only points in Quadrant I and Quadrant IV (SAFE QUADRANTS)
- Therefore: *check signs* of x and y to see *which quadrant*, maybe need π-correction!
  - Quadrant I or IV: polar angle is $\tan^{-1}\left(\frac{y}{x}\right)$

- **Quadrant II or III:** polar angle is $\tan^{-1}\left(\frac{y}{x}\right)+\pi$



Equations (as well as points) can also be converted to polar.

For Cartesian $\to$ Polar, look for cancellation from $\cos^2\theta+\sin^2\theta=1$.

For Polar $\to$ Cartesian, try to keep θ inside of trig functions.

- For example:

$$r=\sin^2\theta \qquad \gg\gg \qquad \sqrt{x^2+y^2}=\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2$$

02 Illustration

Converting to polar: π-correction

Compute the polar coordinates of $\left(-\frac{1}{2},+\frac{\sqrt{3}}{2}\right)$ and of $\left(+\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$.

Solution

For $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ we observe first that it lies in Quadrant II.

Next compute:

$$\tan^{-1}\left(\frac{\sqrt{3}/2}{-1/2}\right) \quad \gg\gg \quad \tan^{-1}\left(-\sqrt{3}\right) \quad \gg\gg \quad -\pi/3$$

This angle is in Quadrant IV. We *add π* to get the polar angle in Quadrant II:

$$\theta=\pi-\pi/3 \quad \gg\gg \quad 2\pi/3$$

The radius is of course 1 since this point lies on the unit circle. Therefore polar coordinates are $(r,\theta)=(1,2\pi/3)$.

For $\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$ we observe first that it lies in Quadrant IV.

Next compute:

$$\tan^{-1}\left(\frac{+\sqrt{2}/2}{-\sqrt{2}/2}\right) \quad \gg\gg \quad \tan^{-1}(-1) \quad \gg\gg \quad -\pi/4$$

This is the correct angle because Quadrant IV is SAFE. So the point in polar is $(1, -\pi/4)$.

☰ Shifted circle in polar

For example, let's convert a shifted circle to polar. Say we have the Cartesian equation:

$$x^2 + (y-3)^2 = 9$$

Then to find the polar we substitute $x = r\cos\theta$ and $y = r\sin\theta$ and simplify:

$$x^2 + (y-3)^2 = 9$$

$$\gg\gg \qquad r^2\cos^2\theta + (r\sin\theta - 3)^2 = 9$$

$$\gg\gg \qquad r^2\cos^2\theta + r^2\sin^2\theta - 6r\sin\theta + 9 = 9$$

$$\gg\gg \qquad r^2(\sin^2\theta + \cos^2\theta) - 6r\sin\theta = 0$$

$$\gg\gg \qquad r^2 - 6r\sin\theta = 0 \qquad \gg\gg \qquad r = 6\sin\theta$$

So this shifted circle *is the polar graph of the polar function* $r(\theta) = 6\sin\theta$.

03 Theory - Polar limaçons

To draw the polar graph of some function, it can help to first draw the Cartesian graph of the function. (In other words, set $y = r$ and $x = \theta$, and draw the usual graph.) By tracing through the points on the Cartesian graph, one can visualize the trajectory of the polar graph.

This Cartesian graph may be called a **graphing tool** for the polar graph.

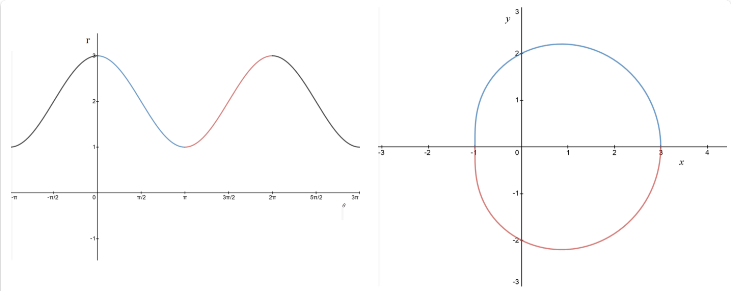
A limaçon is the polar graph of $r = a + b\cos\theta$.

Any limaçon shape can be obtained by adjusting c in this function (and rescaling):

$$r = 1 + c\cos\theta$$

Limaçon satisfying $r(\theta) = 1$: unit circle.

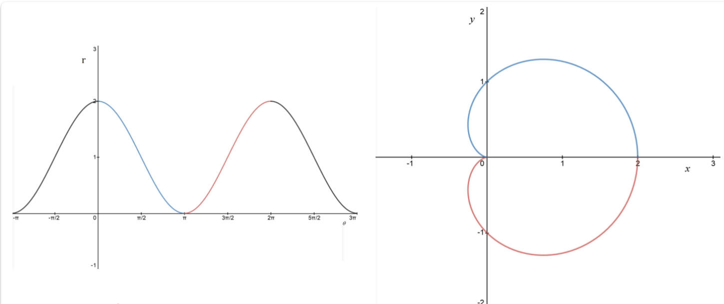
Limaçon satisfying $r(\theta) = 2 + \cos\theta$: 'outer loop' circle with 'dimple':



Graphing Tool

Polar Graph

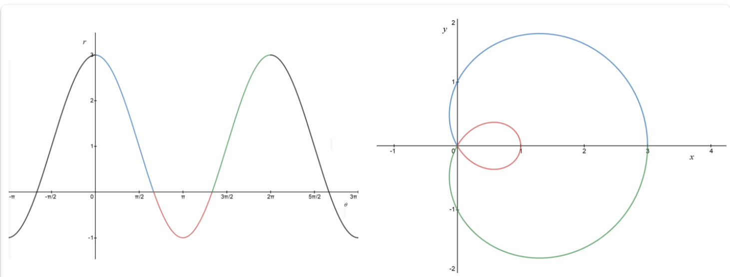
Limaçon satisfying $r(\theta) = 1 + \cos\theta$: ‘cardioid’ = ‘outer loop’ circle with ‘dimple’ that creates a cusp:



Graphing Tool

Polar Graph

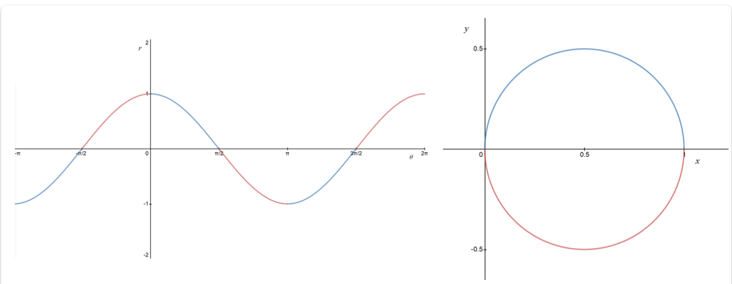
Limaçon satisfying $r(\theta) = 1 + 2\cos\theta$: ‘dimple’ pushes past cusp to create ‘inner loop’:



Graphing Tool

Polar Graph

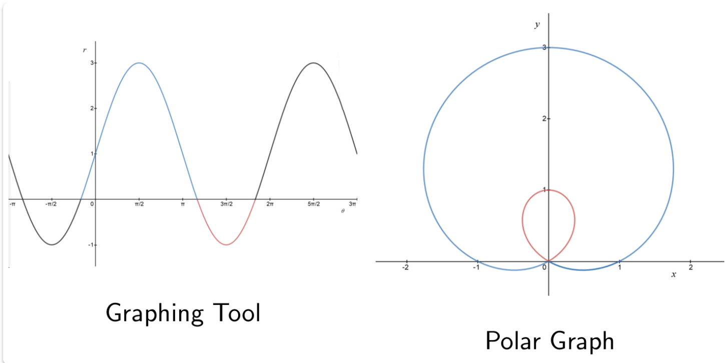
Limaçon satisfying $r(\theta) = \cos\theta$: ‘inner loop’ only, no outer loop exists:



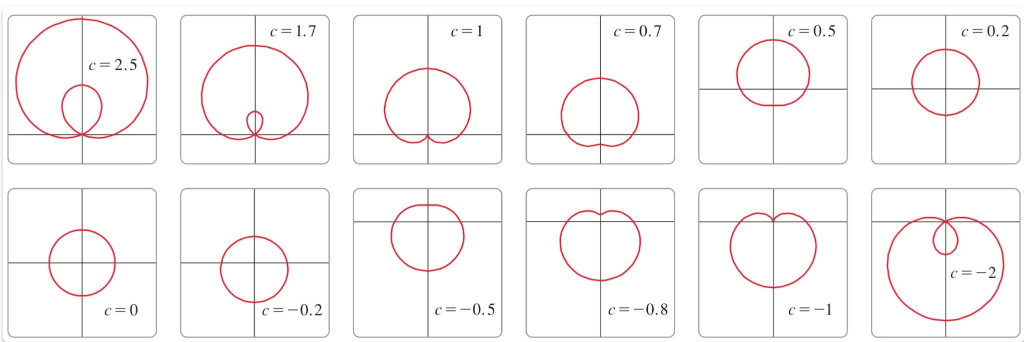
Graphing Tool

Polar Graph

Limaçon satisfying $r(\theta) = 1 + 2\sin\theta$: ‘inner loop’ and ‘outer loop’ and rotated $\circlearrowleft$ 90°:



Transitions between limaçon types, $y = 1 + c\sin\theta$:



Notice the transition points at $|c| = 0.5$ and $|c| = 1$:

The *flat spot* occurs when $c = \pm 0.5$

- Smaller c gives *convex shape*

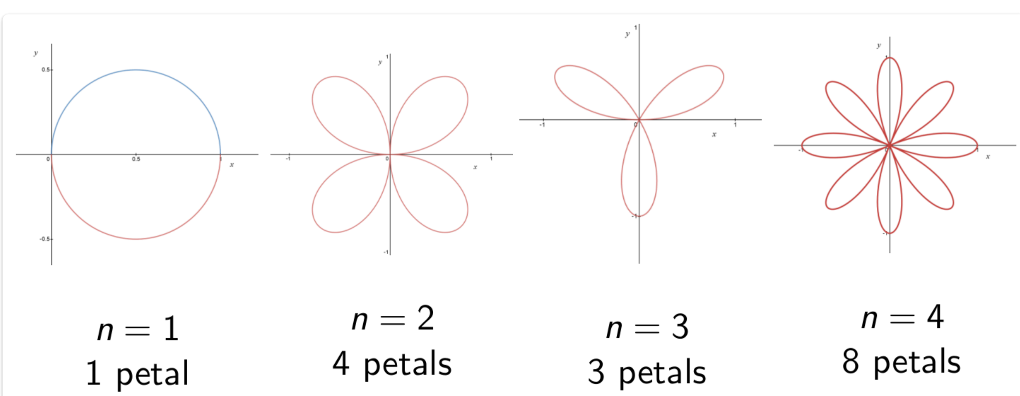
The *cusp* occurs when $c = \pm 1$

- Smaller c gives *dimple* (assuming $|c| > 0.5$)
- Larger c gives *inner loop*

04 Theory - Polar roses

Roses are polar graphs of this form:

$$r(\theta) = \sin(n\theta) \qquad n = 1,\, 2,\, 3,\, \ldots$$



The pattern of petals:

- $n = 2k$ (even): obtain $2n$ petals
  - These petals traversed *once*
- $n = 2k+1$ (odd): obtain n petals
  - These petals traversed *twice*
- Either way: total-petal-traversals: always $2n$

Calculus with polar curves

05 Theory - Polar tangent lines, arclength

Polar arclength formula

The arclength of the polar graph of $r(\theta)$, for $\theta \in [\theta_0, \theta_1]$:

$$L \quad = \quad \int_{\theta_0}^{\theta_1} \sqrt{r'(u)^2 + r(u)^2}\, du$$

To derive this formula, *convert to Cartesian* with parameter θ:

$$r = r(\theta) \quad \gg\gg \quad (x, y) = (r\cos\theta,\, r\sin\theta)$$

From here you can apply the familiar arclength formula with θ in the place of t.

Extra - Derivation of polar arclength formula

Let $r = r(\theta)$ and convert to parametric Cartesian, so $x = r\cos\theta$ and $y = r\sin\theta$.

Then:

$$ds = \sqrt{(x')^2 + (y')^2}\, d\theta$$

$$\begin{aligned} x' = (r\cos\theta)' \quad &\gg\gg \quad r'\cos\theta - r\sin\theta \\ y' = (r\sin\theta)' \quad &\gg\gg \quad r'\sin\theta + r\cos\theta \end{aligned}$$

Therefore:

$$\begin{aligned} (x')^2 + (y')^2 \quad \gg\gg \quad & r'^2\cos^2\theta - 2rr'\cos\theta\sin\theta + r^2\sin^2\theta \\ & + r'^2\sin^2\theta + 2rr'\sin\theta\cos\theta + r^2\cos^2\theta \\ & = r'^2 + r^2 \end{aligned}$$

Therefore:

$$ds = \sqrt{(x')^2 + (y')^2}\, d\theta \qquad \gg\gg \qquad \sqrt{r'^2 + r^2}\, d\theta$$

Therefore:

$$L \quad = \quad \int_{\theta_0}^{\theta_1} \sqrt{r'(u)^2 + r(u)^2}\, du$$

06 Illustration

Finding vertical tangents to a limaçon

Let us find the vertical tangents to the limaçon (the cardioid) given by $r = 1 + \sin\theta$.

1. Convert to Cartesian parametric.
   - Plug $r(\theta)$ into $x = r\cos\theta$ and $y = r\sin\theta$:

$$r(\theta) = 1 + \sin\theta \quad \gg\gg \quad (x, y) = \Big((1+\sin\theta)\cos\theta,\ (1+\sin\theta)\sin\theta\Big)$$

2. Compute x' and y'.
   - Derivatives of both coordinates:

$$(x',\, y') \quad \gg\gg$$

$$\Big(\cos\theta\cos\theta + (1+\sin\theta)(-\sin\theta),\ \cos\theta\sin\theta + (1+\sin\theta)\cos\theta\Big)$$

   - Simplify:

$$\gg\gg \qquad \Big(\cos^2\theta - \sin^2\theta - \sin\theta,\ \cos\theta\,(1 + 2\sin\theta)\Big)$$

3. The vertical tangents occur when $x'(\theta) = 0$.
   - Set equation: $x' = 0$:

$$x'(\theta) = 0 \quad \gg\gg \quad \cos^2\theta - \sin^2\theta - \sin\theta = 0$$

   - ⚠ Solve equation.
     - Convert to *only* $\sin\theta$:

$$\gg\gg \quad (1 - \sin^2\theta) - \sin^2\theta - \sin\theta = 0$$

     - Substitute $A = \sin\theta$ and simplify:

$$\gg\gg \quad 1 - 2A^2 - A = 0 \quad \gg\gg \quad 2A^2 + A - 1 = 0$$

     - Solve:

$$A = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \gg\gg$$

$$\frac{-1 \pm \sqrt{1 - 4\cdot 2\cdot(-1)}}{2\cdot 2} \quad \gg\gg \quad \frac{1}{2},\ -1$$

     - Solve for θ:

$$A = \sin\theta \quad \gg\gg \quad \sin\theta = \frac{1}{2},\ -1$$

$$\gg\gg \quad \theta = \frac{\pi}{6},\ \frac{5\pi}{6}\ (\text{for } 1/2) \quad \text{and} \quad \theta = \frac{3\pi}{2}\ (\text{for } -1)$$

4. Compute final points.

- In polar coordinates, the final points are:

$$(r, \theta) = (1 + \sin\theta,\ \theta)\Big|_{\theta=\frac{\pi}{6},\ \frac{5\pi}{6},\ \frac{3\pi}{2}}$$

$$\gg\gg \quad \left(\frac{3}{2}, \frac{\pi}{6}\right),\ \left(\frac{3}{2}, \frac{5\pi}{6}\right),\ \left(0, \frac{3\pi}{2}\right)$$

- In Cartesian coordinates:
  - For $\theta = \frac{\pi}{6}$:

$$(x, y)\Big|_{\theta=\frac{\pi}{6}} \quad \gg\gg \quad \Big((1 + \sin\theta)\cos\theta,\ (1 + \sin\theta)\sin\theta\Big)\Big|_{\theta=\frac{\pi}{6}}$$

$$\gg\gg \quad \left(\left(1 + \frac{1}{2}\right)\frac{\sqrt{3}}{2},\ \left(1 + \frac{1}{2}\right)\frac{1}{2}\right) \quad \gg\gg \quad \left(\frac{3\sqrt{3}}{4},\ \frac{3}{4}\right)$$

  - For $\theta = \frac{5\pi}{6}$:

$$(x, y)\Big|_{\theta=\frac{5\pi}{6}} \quad \gg\gg \quad \Big((1 + \sin\theta)\cos\theta,\ (1 + \sin\theta)\sin\theta\Big)\Big|_{\theta=\frac{5\pi}{6}}$$

$$\gg\gg \quad \left(\left(1 + \frac{1}{2}\right)\frac{-\sqrt{3}}{2},\ \left(1 + \frac{1}{2}\right)\frac{1}{2}\right) \quad \gg\gg \quad \left(-\frac{3\sqrt{3}}{4},\ \frac{3}{4}\right)$$

  - For $\theta = \frac{3\pi}{2}$:

$$(x, y)\Big|_{\theta=\frac{3\pi}{2}} \quad \gg\gg \quad \Big((1 + \sin\theta)\cos\theta,\ (1 + \sin\theta)\sin\theta\Big)\Big|_{\theta=\frac{3\pi}{2}}$$

$$\gg\gg \quad \Big((1 - 1)\cdot 0,\ (1 - 1)\cdot(-1)\Big) \quad \gg\gg \quad (0, 0)$$

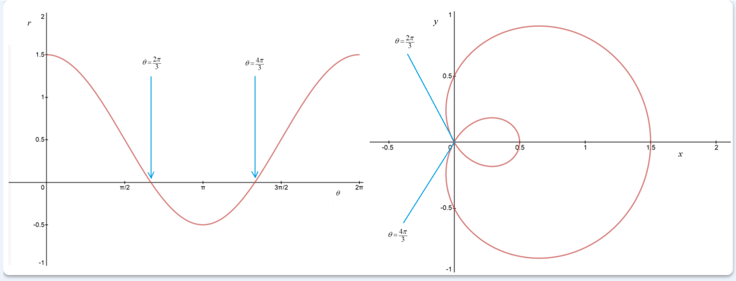
5. ⚠ Correction: $(0, 0)$ is a cusp.
   - The point $(0, 0)$ at $\theta = \frac{3\pi}{2}$ is on the cardioid, but the curve is not smooth there, this is a cusp.
   - Still, the left- and right-sided tangents exists and are equal, so in a sense we can still say the curve has vertical tangent at $\theta = \frac{3\pi}{2}$.

Length of the inner loop

Consider the limaçon given by $r(\theta) = \frac{1}{2} + \cos\theta$. How long is its inner loop? Set up an integral for this quantity.

Solution

The inner loop is traced by the moving point when $\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$. This can be seen from the graph:



Therefore the length of the inner loop is given by this integral:

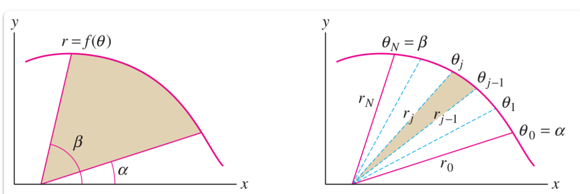
$$L = \int_{2\pi/3}^{4\pi/3} \sqrt{(-\sin\theta)^2 + \left(\frac{1}{2} + \cos\theta\right)^2}\, d\theta \quad \gg\gg \quad \int_{2\pi/3}^{4\pi/3} \sqrt{5/4 + \cos\theta}\, d\theta$$

07 Theory - Polar area

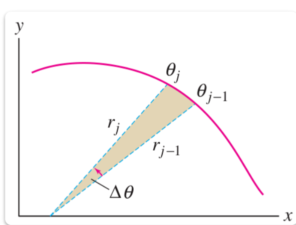
Sectorial area from polar curve

$$A \quad = \quad \int_\alpha^\beta \frac{1}{2} r(\theta)^2\, d\theta$$

The “area under the curve” concept for graphs of functions in Cartesian coordinates translates to a “sectorial area” concept for polar graphs. To compute this area using an integral, we divide the region into Riemann sums of small sector slices.



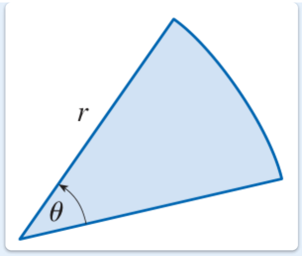
(A) Region $\alpha \le \theta \le \beta$ (B) Region divided into narrow sectors



To obtain a formula for the whole area, we need a formula for the area of each sector slice.

Area of sector slice

Let us verify that the area of a sector slice is $\frac{1}{2}r^2\theta$.



Take the angle θ in radians and divide by 2π to get the *fraction of the whole disk*.

Then multiply this fraction by πr^2 (whole disk area) to get the *area of the sector slice*.

$$\frac{\theta}{2\pi} \cdot \pi r^2 \quad \gg\gg \quad \frac{1}{2} r^2 \theta$$

Now use $d\theta$ and $r(\theta)$ for an infinitesimal sector slice, and integrate these to get the total area formula:

$$A \quad = \quad \int_\alpha^\beta \frac{1}{2} r(\theta)^2 \, d\theta$$

One easily verifies this formula for a circle.

Let $r(\theta) = R$ be a constant. Then:

$$\text{Area of circle} \quad = \quad \int_0^{2\pi} \frac{1}{2} R^2 \, d\theta \quad \gg\gg \quad \frac{1}{2} R^2 \theta \Big|_0^{2\pi} \quad \gg\gg \quad R^2 \pi$$

The sectorial area *between curves*:

Sectorial area between polar curves

$$A \quad = \quad \int_\alpha^\beta \frac{1}{2} \left(r_1(\theta)^2 - r_0(\theta)^2 \right) d\theta$$

⚠ Subtract *after* squaring, not before!

This aspect is *not* similar to the Cartesian version: $\displaystyle\int f - g \, dx$

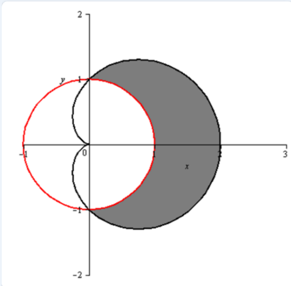
08 Illustration

Area between circle and limaçon

Find the area of the region enclosed between the circle $r_0(\theta) = 1$ and the limaçon $r_1(\theta) = 1 + \cos\theta$.

Solution

First draw the region:



The two curves intersect at $\theta = \pm\frac{\pi}{2}$. Therefore the area enclosed is given by integrating over $-\frac{\pi}{2} \leq \theta \leq +\frac{\pi}{2}$:

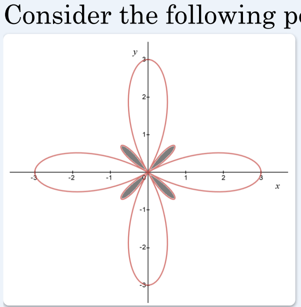
$$A = \int_a^b \frac{1}{2}(r_1^2 - r_0^2)\,d\theta \quad \gg\gg \quad \int_{-\pi/2}^{\pi/2} \frac{1}{2}\Big((1+\cos\theta)^2 - 1^2\Big)\,d\theta$$

$$\gg\gg \quad \frac{1}{2}\int_{-\pi/2}^{\pi/2} 2\cos\theta + \cos^2\theta\,d\theta \quad \gg\gg \quad \int_{-\pi/2}^{\pi/2} \cos\theta + \frac{1}{4}\big(1+\cos(2\theta)\big)\,d\theta$$

$$\gg\gg \quad \sin\theta + \frac{\theta}{4} + \frac{1}{8}\sin(2\theta)\Big|_{-\pi/2}^{\pi/2} \quad \gg\gg \quad 2 + \frac{\pi}{4}$$

Area of small loops

Consider the following polar graph of $r(\theta) = 1 + 2\cos(4\theta)$:



Find the area of the shaded region.

Solution

1. Bounds for one small loop.
   - Lower left loop occurs first.
   - This loop when $1 + 2\cos(4\theta) \leq 0$.
   - Solve this:

$$1 + 2\cos(4\theta) \leq 0 \quad \gg\gg \quad \cos(4\theta) \leq -\frac{1}{2}$$

$$\gg\gg \quad \frac{2\pi}{3} \leq 4\theta \leq \frac{4\pi}{3} \quad \gg\gg \quad \frac{\pi}{6} \leq \theta \leq \frac{\pi}{3}$$

2. Area integral.

- Arrange and expand area integral:

$$A = 4\int_\alpha^\beta \frac{1}{2} r(\theta)^2\,d\theta \quad \gg\gg \quad 4\int_{\pi/6}^{\pi/3} \frac{1}{2}\big(1+2\cos(4\theta)\big)^2\,d\theta$$

$$\gg\gg \quad 2\int_{\pi/6}^{\pi/3} 1+4\cos(4\theta)+4\cos^2(4\theta)\,d\theta$$

- Simplify integral using power-to-frequency: $\cos^2 A \rightsquigarrow \frac{1}{2}(1+\cos(2A))$ with $A=4\theta$:

$$\gg\gg \quad 2\int_{\pi/6}^{\pi/3} 1+4\cos(4\theta)+4\cdot\frac{1}{2}\big(1+\cos(8\theta)\big)\,d\theta$$

- Compute integral:

$$\gg\gg \quad 6\theta+2\sin(4\theta)+\frac{1}{4}\sin(8\theta)\Big|_{\pi/6}^{\pi/3}$$

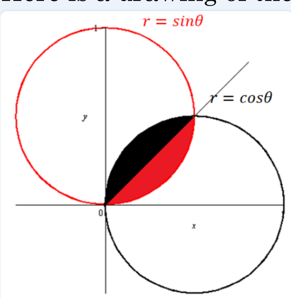
$$\gg\gg \quad \pi-\frac{3\sqrt{3}}{2}$$

Overlap area of circles

Compute the area of the overlap between crossing circles. For concreteness, suppose one of the circles is given by $r(\theta)=\sin\theta$ and the other is given by $r(\theta)=\cos\theta$.

Solution

Here is a drawing of the overlap:



1. Notice: total overlap area = 2× area of red region.
2. Bounds: $0\le\theta\le\frac{\pi}{4}$.
3. Apply area formula for the red region.
    - Area formula applied to $r(\theta)=\sin\theta$:

$$A=\int_\alpha^\beta \frac{1}{2}r(\theta)^2\,d\theta \qquad \gg\gg \qquad \int_0^{\pi/4}\frac{1}{2}\sin^2\theta\,d\theta$$

- Power-to-frequency: $\sin^2\theta \rightsquigarrow \frac{1}{2}\big(1-\cos(2\theta)\big)$:

$$\gg\gg \quad \int_0^{\pi/4} \frac{1}{4}\big(1-\cos(2\theta)\big)\,d\theta$$

$$\gg\gg \quad \frac{1}{4}\theta - \frac{1}{8}\sin(2\theta)\Big|_0^{\pi/4} \quad \gg\gg \quad \frac{\pi}{16} - \frac{1}{8}$$

- Double the result to include the black region:

$$\gg\gg \quad \frac{\pi}{8} - \frac{1}{4}$$