W08 Notes

Simple divergence test

01 Theory

Simple Divergence Test (SDT)

Applicability: *Any* series.

Test Statement:

$$\lim_{n\to\infty} a_n \neq 0 \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} a_n \quad \text{diverges}$$

- ⚠ The *converse is not valid*. For example, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges even though $\lim_{n\to\infty} \frac{1}{n} = 0$.

02 Illustration

Simple divergence test: examples

Consider: $\displaystyle\sum_{n=1}^{\infty} \frac{n}{4n+1}$

- This diverges by the SDT because $a_n \to \frac{1}{4}$ and not 0.

Consider: $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n+1}$

- This diverges by the SDT because $\lim_{n\to\infty} a_n = \text{DNE}$.
- We can say the terms "converge to the pattern $+1,\ -1,\ +1,\ -1,\ \ldots$," but that is not a limit value.

Positive series

03 Theory

Positive series

A series is called **positive** when its individual terms are positive, i.e. $a_n > 0$ for all n.

The partial sum sequence S_N is *monotone increasing* for a positive series.

By the monotonicity test for convergence of sequences, S_N therefore converges whenever it is *bounded above*. If S_N is not bounded above, then $\sum_{n=1}^{\infty} a_n$ diverges to $+\infty$.

Another test, called the **integral test**, studies the terms of a series as if they represent rectangles with upper corner pinned to the graph of a continuous function.

To apply the test, we must convert the integer index variable n in a_n into a continuous variable x. This is easy when we have a formula for a_n (provided it doesn't contain factorials or other elements dependent on integrality).

Integral Test (IT)

Applicability:

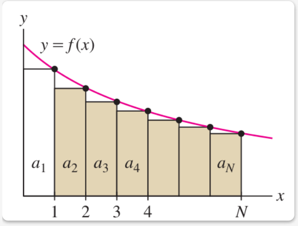
- (i) $f(x) > 0$
- (ii) $f(x)$ is continuous
- (iii) $f(x)$ is *monotone decreasing*

Test Statement:

$$\sum_{n=1}^{\infty} a_n \quad \frac{\text{converges}}{\text{diverges}} \qquad \Longleftrightarrow \qquad \int_1^{\infty} f(x)\,dx \quad \frac{\text{converges}}{\text{diverges}}$$

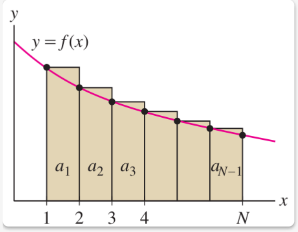
Extra - Integral test: explanation

To show that *integral convergence implies series convergence*, consider the diagram:



This shows that $\sum_{n=2}^{N} a_n \leq \int_1^N f(x)\,dx$ for any N. Therefore, if $\int_1^\infty f(x)\,dx$ converges, then $\int_1^N f(x)\,dx$ is bounded (independent of N) and so $\sum_{n=2}^{N} a_n$ is bounded by that inequality. But $\sum_{n=2}^{N} a_n = S_N - a_1$; so by adding a_1 to the bound, we see that S_N itself is bounded, which implies that $\sum_{n=1}^{\infty} a_n$ converges.

To show that *integral divergence implies series divergence*, consider a similar diagram:



This shows that $\sum_{n=1}^{N-1} a_n \geq \int_1^N f(x)\,dx$ for any N. Therefore, if $\int_1^\infty f(x)\,dx$ diverges, then $\int_1^N f(x)\,dx$ goes to $+\infty$ as $N \to \infty$, and so $\sum_{n=1}^{N-1} a_n$ goes to $+\infty$ as well. So $\sum_{n=1}^{\infty} a_n$ diverges.

- Notice: the picture shows $f(x)$ entirely above (or below) the rectangles.
  - This depends upon $f(x)$ being *monotone decreasing*, as well as $f(x) > 0$.
  - This explains the applicability conditions.

Next we use the integral test to evaluate the family of ***p*-series**, and later we can use p-series in comparison tests without repeating the work of the integral test.

p-series

A ***p*-series** is a series of this form: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$

Convergence properties:

$$p > 1 : \text{series converges} \qquad\qquad p \leq 1 : \text{series diverges}$$

Extra - Proof of p-series convergence

To verify the convergence properties of p-series, apply the integral test:

1. Applicability: verify it's continuous, positive, decreasing.
   - Convert n to x to obtain the function $f(x) = \frac{1}{x^p}$.
   - Indeed $\frac{1}{x^p}$ is continuous and positive and decreasing as x increases.
2. Apply the integral test.
   - Integrate, assuming $p \neq 1$:

$$\int_1^{\infty} \frac{1}{x^p}\,dx \quad \gg\gg \quad \lim_{R\to\infty} \left.\frac{x^{p-1}}{p-1}\right|_1^R$$

$$\gg\gg \quad \lim_{R\to\infty} \left(\frac{R^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1}\right)$$

   - When $p > 1$ we have $\lim_{R\to\infty} \frac{R^{-p+1}}{-p+1} = 0$
   - When $p < 1$ we have $\lim_{R\to\infty} \frac{R^{-p+1}}{-p+1} = \infty$
   - When $p = 1$, integrate a second time:

$$\int_1^{\infty} \frac{1}{x}\,dx \quad \gg\gg \quad \lim_{R\to\infty} \ln x\Big|_1^R$$

$$\gg\gg \quad \lim_{R\to\infty} \ln R - \ln 1 \quad \gg\gg \quad \infty$$

3. Conclude: the integral converges when $p > 1$ and diverges when $p \leq 1$.

- Supplement: we could instead immediately refer to the convergence results for *p-integrals* instead of reproving them here.

04 Illustration

p-series examples

By finding p and applying the p-series convergence properties:

We see that $\sum_{n=1}^{\infty} \frac{1}{n^{1.1}}$ converges: $p = 1.1$ so $p > 1$

But $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges: $p = 1/2$ so $p < 1$

Integral test - pushing the envelope of convergence

Does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ converge?

Does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converge?

Notice that $\ln n$ grows *very slowly* with n, so $\frac{1}{n \ln n}$ is just a *little* smaller than $\frac{1}{n}$ for large n, and similarly $\frac{1}{n(\ln n)^2}$ is just a little smaller still.

Solution

1. The two series lead to the two functions $f(x) = \frac{1}{x \ln x}$ and $g(x) = \frac{1}{x(\ln x)^2}$.
2. Check applicability.
   - Clearly $f(x)$ and $g(x)$ are both continuous, positive, decreasing functions on $x \in [2, \infty]$.
3. Apply the integral test to $f(x)$.
   - Integrate $f(x)$:

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx \quad \gg\gg \quad \int_{u=\ln 2}^{\infty} \frac{1}{u}\, du$$

$$\gg\gg \quad \lim_{R \to \infty} \ln u \Big|_{\ln 1}^{R} \quad \gg\gg \quad \infty$$

4. Conclude: $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ *diverges*.
5. Apply the integral test to $g(x)$.
   - Integrate $g(x)$:

$$\int_M^{\infty} \frac{1}{x(\ln x)^2}\, dx \quad \gg\gg \quad \int_{u=\ln M}^{\infty} \frac{1}{u^2}\, du$$

$$\gg\gg \quad \lim_{R \to \infty} -u^{-1} \Big|_{\ln M}^{R} \quad \gg\gg \quad \frac{1}{\ln M}$$

6. Conclude: $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ *converges*.

05 Theory

Direct Comparison Test (DCT)

Applicability: Both series are positive: $a_n > 0$ and $b_n > 0$.

Test Statement: Suppose $a_n \leq b_n$ for large enough n.
(Meaning: for $n \geq N$ with some given N.) Then:

- Smaller pushes up bigger:

$$\sum_{n=1}^{\infty} a_n \quad \text{diverges} \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} b_n \quad \text{diverges}$$

- Bigger controls smaller:

$$\sum_{n=1}^{\infty} b_n \quad \text{converges} \qquad \Longrightarrow \qquad \sum_{n=1}^{\infty} a_n \quad \text{converges}$$

06 Illustration

Direct comparison test: rational functions

The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\,3^n}$ *converges* by the DCT.

- Choose: $a_n = \frac{1}{\sqrt{n}\,3^n}$ and $b_n = \frac{1}{3^n}$
- Check: $0 \leq \frac{1}{\sqrt{n}\,3^n} \leq \frac{1}{3^n}$
- Observe: $\sum \frac{1}{3^n}$ is a convergent geometric series

The series $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^3}$ *converges* by the DCT.

- Choose: $a_n = \frac{\cos^2 n}{n^3}$ and $b_n = \frac{1}{n^3}$.
- Check: $0 \leq \frac{\cos^2 n}{n^3} \leq \frac{1}{n^3}$
- Observe: $\sum \frac{1}{n^3}$ is a convergent p-series

The series $\sum_{n=1}^{\infty} \frac{n}{n^3+1}$ *converges* by the DCT.

- Choose: $a_n = \frac{n}{n^3+1}$ and $b_n = \frac{1}{n^2}$
- Check: $0 \leq \frac{n}{n^3+1} \leq \frac{1}{n^2}$ (notice that $\frac{n}{n^3+1} \leq \frac{n}{n^3}$)
- Observe: $\sum \frac{1}{n^2}$ is a convergent p-series

The series $\sum_{n=2}^{\infty} \frac{1}{n-1}$ *diverges* by the DCT.

- Choose: $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n-1}$
- Check: $0 \leq \frac{1}{n} \leq \frac{1}{n-1}$
- Observe: $\sum \frac{1}{n}$ is a divergent p-series

07 Theory

Some series can be compared using the DCT after applying certain manipulations and tricks.

For example, consider the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$. We suspect convergence because $a_n \approx \frac{1}{n^2}$ for *large* n. But unfortunately, $a_n > \frac{1}{n^2}$ always, so we cannot apply the DCT.

We could make some *ad hoc* arguments that do use the DCT, eventually:

- Method 1:
  - Observe that for $n > 1$ we have $\frac{1}{n^2-1} \leq \frac{10}{n^2}$. (Check it!)

- But $\sum \frac{10}{n^2}$ converges, indeed its value is $10 \cdot \sum \frac{1}{n^2}$, which is $\frac{10\pi^2}{6}$.
- So the series $\sum \frac{1}{n^2-1}$ converges.

- Method 2:
  - Observe that we can change the letter n to $n+1$ by starting the new n at $n=1$.
  - Then we have:

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} \quad = \quad \sum_{n=1}^{\infty} \frac{1}{(n+1)^2-1} \quad = \quad \sum_{n=1}^{\infty} \frac{1}{n^2+2n}$$

  - This last series has terms smaller than $\frac{1}{n^2}$ so the DCT with $b_n = \frac{1}{n^2}$ (a convergent p-series) shows that the original series converges too.

These convoluted arguments suggest that a more general version of Comparison is possible.

Indeed, it is sufficient to compare the *limiting behavior* of two series. The limit of *ratios* (limit of 'comparison') links up the convergence / divergence of $\sum a_n$ and $\sum b_n$.

Limit Comparison Test (LCT)

Applicability: Both series are positive: $a_n > 0$ and $b_n > 0$.

Test Statement: Suppose that $\lim_{n\to\infty} \frac{a_n}{b_n} = L$. Then:

- If $0 < L < \infty$:

$$\sum a_n \quad \text{converges} \qquad \iff \qquad \sum b_n \quad \text{converges}$$

If $L = 0$ or $L = \infty$, we can still draw an inference, but in only one direction:

- If $L = 0$:

$$\sum b_n \quad \text{converges} \qquad \Longrightarrow \qquad \sum a_n \quad \text{converges}$$

- If $L = \infty$:

$$\sum b_n \quad \text{diverges} \qquad \Longrightarrow \qquad \sum a_n \quad \text{diverges}$$

Extra - Limit Comparison Test: Theory

Suppose $a_n/b_n \to L$ and $0 < L < \infty$. Then for n sufficiently large, we know $a_n/b_n < L+1$.

Doing some algebra, we get $a_n < (L+1)b_n$ for n large.

If $\sum b_n$ converges, then $\sum (L+1)b_n$ also converges (constant multiple), and then the DCT implies that $\sum a_n$ converges.

Conversely: we also know that $b_n/a_n \to 1/L$, so $b_n < (1/L+1)a_n$ for all n sufficiently large. Thus if $\sum a_n$ converges, $\sum (1/L+1)a_n$ also converges, and by the DCT again $\sum b_n$ converges too.

The cases with $L = 0$ or $L = \infty$ are handled similarly.

08 Illustration

☷ Limit Comparison Test examples

The series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ *converges* by the LCT.

- Choose: $a_n = \frac{1}{2^n-1}$ and $b_n = \frac{1}{2^n}$.
- Compare in the limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} \quad \gg\gg \quad \lim_{n\to\infty} \frac{2^n}{2^n-1} \quad \gg\gg \quad 1$$

- Observe: $\sum \frac{1}{2^n}$ is a convergent geometric series

The series $\sum_{n=1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$ *diverges* by the LCT.

- Choose: $a_n = \frac{2n^2+3n}{\sqrt{5+n^5}}$, $b_n = n^{-1/2}$
- Compare in the limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} \quad \gg\gg \quad \lim_{n\to\infty} \frac{(2n^2+3n)\sqrt{n}}{\sqrt{5+n^5}}$$

$$\frac{(2n^2+3n)\sqrt{n}}{\sqrt{5+n^5}} \quad \xrightarrow{n\to\infty} \quad \frac{2n^{5/2}}{n^{5/2}} \to 2$$

- Observe: $\sum n^{-1/2}$ is a divergent p-series

The series $\sum_{n=2}^{\infty} \frac{n^2}{n^4-n-1}$ *converges* by the LCT.

- Choose: $a_n = \frac{n^2}{n^4-n-1}$ and $b_n = n^{-2}$
- Compare in the limit:

$$\lim_{n\to\infty} \frac{a_n}{b_n} \quad \gg\gg \quad \lim_{n\to\infty} \frac{n^4}{n^4-n-1} \quad \gg\gg \quad 1$$

- Observe: $\sum_{n=2}^{\infty} n^{-2}$ is a converging p-series

Alternating series

09 Theory

Consider these series:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\cdots \quad = \quad \infty$$

$$-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\frac{1}{5}-\frac{1}{6}-\frac{1}{7}-\cdots \quad = \quad -\infty$$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\cdots \quad = \quad \ln 2$$

$$1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}-\frac{1}{6}+\frac{1}{7}+\cdots \quad = \quad ?$$

The absolute values of terms are the same between these series, only the signs of terms change.

The first is a **positive series** because there are no negative terms.

The second series is the negation of a positive series – the study of such series is equivalent to that of positive series, just add a negative sign everywhere. These signs can be factored out of the series. (For example $\sum -\frac{1}{n} = -\sum \frac{1}{n}$.)

The third series is an **alternating series** because the signs alternate in a strict pattern, every other sign being negative.

The fourth series is *not* alternating, nor is it positive, nor negative: it has a mysterious or unknown pattern of signs.

A series with any negative signs present, call it $\sum_{n=1}^{\infty} a_n$, **converges absolutely** when the positive series of absolute values of terms, namely $\sum_{n=1}^{\infty} |a_n|$, converges.

THEOREM: Absolute implies ordinary

If a series $\sum_{n=1}^{\infty} a_n$ converges absolutely, then it also converges as it stands.

A series might converge due to the presence of negative terms and yet *not* converge absolutely:

A series $\sum_{n=1}^{\infty} a_n$ is said to be **converge conditionally** when the series converges as it stands, but the series produced by inserting absolute values, namely $\sum_{n=1}^{\infty} |a_n|$, diverges.

The alternating harmonic series above, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2$, is therefore conditionally convergent. Let us see why it converges. We can group the terms to create new sequences of *pairs*, each pair being a positive term. This can be done in two ways. The first creates an increasing sequence, the second a decreasing sequence:

$$\text{increasing from 0:} \qquad \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + \cdots$$

$$\text{decreasing from 1:} \qquad 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \left(\frac{1}{6} - \frac{1}{7}\right) - \cdots$$

Suppose S_N gives the sequence of partial sums of the original series. Then S_{2N} gives the first sequence of pairs, namely $S_2, S_4, S_6, \ldots$. And S_{2N-1} gives the second sequence of pairs, namely $S_1, S_3, S_5, \ldots$.

The second sequence shows that S_N is bounded above by 1, so S_{2N} is monotone increasing and bounded above, so it converges. Similarly S_{2N-1} is monotone decreasing and bounded below, so it converges too, and of course they must converge to the same thing.

The fact that the terms were *decreasing in magnitude* was an essential ingredient of the argument for convergence. This fact ensured that the parenthetical pairs were positive numbers.

Alternating Series Test (AST) - "Leibniz Test"

Applicability: Alternating series only: $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ with $a_n > 0$

Test Statement:
If:

- (1) a_n are *decreasing*, so $a_1 > a_2 > a_3 > a_4 > \cdots > 0$
- (2) $a_n \to 0$ as $n \to \infty$ (i.e. it passes the SDT)

Then:

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n \quad \text{converges}$$

Furthermore, partial sum *errors* are bounded by "subsequent terms":

$$|S - S_N| \leq a_{N+1}$$

Extra - Alternating Series Test: Theory

Just as for the alternating harmonic series, we can form *positive* paired-up series because the terms are decreasing:

$$(a_1 - a_2) + (a_3 - a_4) + (a_5 - a_6) + \cdots$$

$$a_1 - (a_2 - a_3) - (a_4 - a_5) - (a_6 - a_7) - \cdots$$

The first sequence S_{2N} is monotone increasing from 0, and the second S_{2N-1} is decreasing from a_1. The first is therefore also bounded above by a_1. So it converges. Similarly, the second converges. Their difference at any point is $S_{2N} - S_{2N-1}$ which is equal to $-a_{2N}$, and this goes to zero. So the two sequences must converge to the same thing.

By considering these paired-up sequences and the effect of adding each new term one after the other, we obtain the following order relations:

$$0 < S_2 < S_4 < S_6 < \ \cdots \ < S < \ \cdots \ < S_5 < S_3 < S_1 = a_1$$

Therefore $S_{2N} < S < S_{2M-1}$ for any N and M.

By setting $M = N$ and then subtracting S_{2N-1} from each expression, we have $S_{2N} - S_{2N-1} < S - S_{2N-1} < 0$. Note that $-a_{2N} = S_{2N} - S_{2N-1}$.

By setting $M = N + 1$ and then subtracting S_{2N} from each expression, we have $0 < S - S_{2N} < S_{2N+1} - S_{2N}$. Note that $a_{2N+1} = S_{2N+1} - S_{2N}$.

Lastly, plugging in these expressions for a_{2N} and a_{2N+1} and taking absolute values, we have $|S - S_{2N-1}| < a_{2N}$ from the first, and $|S - S_{2N}| < a_{2N+1}$ from the second. These two formulas give the same fact, the first for the odd and the second for the even partial sums. Combining the two cases into one expression gives the desired result: $|S - S_N| \leq a_{N+1}$.

10 Illustration

Alternating Series Test: Basic illustration

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ converges by the AST.

- Notice that $\sum \frac{1}{\sqrt{n}}$ diverges as a p-series with $p = 1/2 < 1$.
- Therefore the first series converges *conditionally*.

(b) $\sum_{n=1}^{\infty} \dfrac{\cos n\pi}{n^2}$ converges by the AST.

- Notice the funny notation: $\cos n\pi = (-1)^n$.
- This series converges *absolutely* because $\left|\frac{\cos n\pi}{n^2}\right| = \frac{1}{n^2}$, which is a p-series with $p = 2 > 1$.

Approximating π

The Taylor series for $\arctan x$ is given by:

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

Use this series to approximate $\pi/4$ with an error less than 0.001.

Solution

The main idea is to use $\tan \frac{\pi}{4} = 1$ and thus $\arctan 1 = \frac{\pi}{4}$. Therefore:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

and thus:

$$\pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \cdots$$

Write E_n for the error of the approximation, meaning $E_n = S - S_n$.

By the AST error formula, we have $|E_n| < a_{n+1}$.

We desire n such that $|E_n| < 0.001$. So we find n such that $a_{n+1} < 0.001$ and this will imply:

$$|E_n| < a_{n+1} < 0.001$$

The formula is $a_n = \frac{4}{2n-1}$. Plug in $n+1$ in place of n to find $a_{n+1} = \frac{4}{2n+1}$. Then we solve:

$$\begin{aligned} a_{n+1} = \frac{4}{2n+1} &< 0.001 \\ \ggg \qquad \frac{4}{0.001} &< 2n+1 \\ \ggg \qquad 3999 &< 2n \\ \ggg \qquad 2000 &\leq n \end{aligned}$$

So we conclude that at least 2000 terms are necessary for the approximation of π to be accurate to within 0.001.