W02 - Examples

Trig power products

Example - Trig power product with an odd power

Compute the integral:

$$\int \cos^2 x \cdot \sin^5 x \, dx$$

Solution

1. Swap over the even bunch.
    - Max even bunch leaving power-one is $\sin^4 x$:
    $$\sin^5 x \quad \gg\gg \quad \sin x \left(\sin^2 x\right)^2 \quad \gg\gg \quad \sin x \left(1 - \cos^2 x\right)^2$$
    - Apply to $\sin^5 x$ in the integrand:
    $$\int \cos^2 x \cdot \sin^5 x \, dx \quad \gg\gg \quad \int \cos^2 x \cdot \sin x \left(1 - \cos^2 x\right)^2 dx$$
2. Perform u-substitution on the power-one integrand.
    - Set $u = \cos x$.
    - Hence $du = \sin x \, dx$. Recognize this in the integrand.
    - Convert the integrand:
    $$\int \cos^2 x \cdot \sin x \left(1 - \cos^2 x\right)^2 dx \quad \gg\gg \quad \int \cos^2 x \cdot \left(1 - \cos^2 x\right)^2 \left(\sin x \, dx\right)$$
    $$\gg\gg \quad \int u^2 \cdot (1 - u^2)^2 \, du$$
3. Perform the integral.
    - Expand integrand and use power rule to obtain:
    $$\int u^2 \cdot (1 - u^2)^2 \, du = \frac{1}{3} u^3 - \frac{2}{5} u^5 + \frac{1}{7} u^7 + C$$
    - Insert definition $u = \cos x$:
    $$\int \cos^2 x \cdot \sin^5 x \, dx \quad \gg\gg \quad \int u^2 \cdot (1 - u^2)^2 \, du$$
    $$\gg\gg \quad \frac{1}{3} \cos^3 x - \frac{2}{5} \cos^5 x + \frac{1}{7} \cos^7 x + C$$
4. This is our final answer.

Example - Trig power product with tan and sec

Compute the integral:

$$\int \tan^5 x \cdot \sec^3 x \, dx$$

Solution

1. Try $du = \sec^2 x \, dx$.
   - Factor du out of the integrand:

     $$\int \tan^5 x \cdot \sec^3 x \, dx \qquad \gg\gg \qquad \int \tan^5 x \cdot \sec x \left(\sec^2 x \, dx \right)$$

   - We then must swap over remaining $\sec x$ into the $\tan x$ type.
   - Cannot do this because $\sec x$ has odd power. Need even to swap.
2. Try $du = \sec x \, \tan x \, dx$.
   - Factor du out of the integrand:

     $$\int \tan^5 x \cdot \sec^3 x \, dx \qquad \gg\gg \qquad \int \tan^4 x \cdot \sec^2 x \left(\sec x \, \tan x \, dx \right)$$

   - Swap remaining $\tan x$ into $\sec x$ type:

     $$\int (\tan^2 x)^2 \cdot \sec^2 x \left(\sec x \, \tan x \, dx \right)$$

     $$\gg\gg \qquad \int \left(\sec^2 x - 1 \right)^2 \cdot \sec^2 x \left(\sec x \, \tan x \, dx \right)$$

   - Substitute $u = \sec x$ and $du = \sec x \, \tan x \, dx$:

     $$\gg\gg \qquad \int (u^2 - 1)^2 \cdot u^2 \, du$$
3. Compute the integral in u and convert back to x.
   - Expand the integrand:

     $$\gg\gg \qquad \int u^6 - 2u^4 + u^2 \, du$$

   - Apply power rule:

     $$\gg\gg \qquad \frac{u^7}{7} - 2\frac{u^5}{5} + \frac{u^3}{3} + C$$

   - Plug back in, $u = \sec x$:

     $$\gg\gg \qquad \frac{\sec^7 x}{7} - 2\frac{\sec^5 x}{5} + \frac{\sec^3 x}{3} + C$$
4. This is our final answer.

Trig substitution

Example - Trig sub in quadratic: completing the square

Compute the integral:

$$\int \frac{dx}{\sqrt{x^2 - 6x + 11}}$$

Solution

1. Notice square root of a quadratic.
2. Complete the square to obtain Pythagorean form.
   - Find constant term for a complete square:

     $$x^2 - 6x + \left(\frac{-6}{2}\right)^2 = x^2 - 6x + 9 = (x-3)^2$$

   - Add and subtract desired constant term:

     $$x^2 - 6x + 11 \qquad \gg\gg \qquad x^2 - 6x + 9 - 9 + 11$$

   - Simplify:

     $$x^2 - 6x + 9 - 9 + 11 \qquad \gg\gg \qquad (x-3)^2 + 2$$

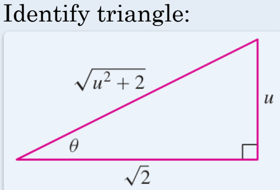
3. Perform shift substitution.
   - Set $u = x - 3$ as inside the square:

     $$(x-3)^2 + 2 = u^2 + 2$$

   - Infer $du = dx$.
   - Plug into integrand:

     $$\int \frac{dx}{\sqrt{x^2 - 6x + 11}} \qquad \gg\gg \qquad \int \frac{du}{\sqrt{u^2 + 2}}$$

4. Trig sub with $\tan\theta$.
   - Identify triangle:



   - Use substitution $u = \sqrt{2}\tan\theta$. (From triangle or memorized tip.)
   - Infer $du = \sqrt{2}\sec^2\theta\, d\theta$.
   - Plug in data:

     $$\int \frac{du}{\sqrt{u^2+2}} \qquad \gg\gg \qquad \int \frac{\sec^2\theta}{\sec\theta}\, d\theta = \int \sec\theta\, d\theta$$

5. Compute trig integral.
   - Use ad hoc formula:

     $$\int \sec\theta\, d\theta = \ln|\tan\theta + \sec\theta| + C$$

6. Convert trig back to x.

- First in terms of u, referring to the triangle:

$$\tan\theta = \frac{u}{\sqrt{2}}, \qquad \sec\theta = \frac{\sqrt{u^2+2}}{\sqrt{2}}$$

- Then in terms of x using $u = x - 3$.
- Plug everything in:

$$\ln|\tan\theta + \sec\theta| + C \qquad \gg\gg \qquad \ln\left|\frac{x-3}{\sqrt{2}} + \frac{\sqrt{(x-3)^2+2}}{\sqrt{2}}\right| + C$$

7. Simplify using log rules.

- Log rule for division gives us:

$$\ln\frac{f(x)}{a} = \ln f(x) - \ln a$$

- The common denominator $\frac{1}{\sqrt{2}}$ can be pulled outside as $-\ln\sqrt{2}$.
- The new term $-\ln\sqrt{2}$ can be “absorbed into the constant” (redefine C).
- So we write our final answer thus:

$$\ln\left|x - 3 + \sqrt{(x-3)^2+2}\right| + C$$