# W02 - Examples

## Bayes' Theorem

**Example - Bayes' Theorem - COVID tests**

Assume that 0.5% of people have COVID. Suppose a COVID test gives a (true) positive on 96% of patients who have COVID, but gives a (false) positive on 2% of patients who do not have COVID. Bob tests positive. What is the probability that Bob has COVID?

**Solution**

1. Label events.
    - Event $A_P$: Bob is actually positive for COVID
    - Event $A_N$: Bob is actually negative; note $A_N = A_P^c$
    - Event $T_P$: Bob tests positive
    - Event $T_N$: Bob tests negative; note $T_N = T_P^c$
2. Identify knowns.
    - Know: $P[T_P \mid A_P] = 96\%$
    - Know: $P[T_P \mid A_N] = 2\%$
    - Know: $P[A_P] = 0.5\%$ and therefore $P[A_N] = 99.5\%$
    - We seek: $P[A_P \mid T_P]$
3. Translate Bayes' Theorem.
    - Using $A = T_P$ and $B = A_P$ in the formula:

    $$P[A_P \mid T_P] = P[T_P \mid A_P] \cdot \frac{P[A_P]}{P[T_P]}$$

    - We know all values on the right except $P[T_P]$
4. Use Division into Cases.
    - Observe:

    $$T_P = T_P \cap A_P \bigcup T_P \cap A_N$$

    - Division into Cases yields:

    $$P[T_P] = P[A_P] \cdot P[T_P \mid A_P] + P[A_N] \cdot P[T_P \mid A_N]$$

    - Important to notice this technique!
        - It is a common element of Bayes' Theorem application problems.
        - It is frequently needed *for the denominator*.
    - Plug in data and compute:

    $$\gg\gg \quad P[T_P] = \frac{5}{1000} \cdot \frac{96}{100} + \frac{995}{1000} \cdot \frac{2}{100} \quad \gg\gg \quad \approx 0.0247$$

5. Compute answer.

- Plug in and compute:

$$P[A_P \mid T_P] = P[T_P \mid A_P] \cdot \frac{P[A_P]}{P[T_P]}$$

$$\gg\gg \quad 0.96 \cdot \frac{0.005}{0.0247} \quad \gg\gg \quad \approx 19\%$$

**Intuition - COVID testing**

Some people find the low number surprising. In order to repair your intuition, think about it like this: roughly 2.5% of tests are positive, with roughly 2% coming from *false* positives, and roughly 0.5% from *true* positives. The true ones make up only 1/5 of the positive results!

(This rough approximation is by assuming $96\% = 100\%$.)

If *two* tests both come back positive, the odds of COVID are now 98%.

If *only people with symptoms* are tested, so that, say, 20% of those tested have COVID, that is, $P[A_P \mid T_P] = 20\%$, then one positive test implies a COVID probability of 92%.

**Exercise - Bayes' Theorem and Multiplication: Inferring bin from marble**

There are marbles in bins in a room:

- Bin 1 holds 7 red and 5 green marbles.
- Bin 2 holds 4 red and 3 green marbles.

Your friend goes in the room, shuts the door, and selects a random bin, then draws a random marble. (Equal odds for each bin, then equal odds for each marble in that bin.) He comes out and shows you a red marble.

What is the probability that this red marble was taken from Bin 1?

Solution

## Independence

**Exercise - Independence and complements**

Prove that these are logically equivalent statements:

- $A$ and $B$ are independent
- $A$ and $B^c$ are independent
- $A^c$ and $B^c$ are independent

Make sure you demonstrate both directions of each equivalency.

Solution

### Example - Checking independence by hand

A bin contains 4 red and 7 green marbles. Two marbles are drawn.

Let $R_1$ be the event that the first marble is red, and let $G_2$ be the event that the second marble is green.

- (a) Show that $R_1$ and $G_2$ are independent if the marbles are drawn *with replacement*.
- (b) Show that $R_1$ and $G_2$ are not independent if the marbles are drawn *without replacement*.

**Solution**

(a) With replacement.

1. Identify knowns.
   - Know: $P[R_1] = \frac{4}{11}$
   - Know: $P[G_2] = \frac{7}{11}$
2. Compute both sides of independence relation.
   - Relation is $P[R_1G_2] = P[R_1] \cdot P[G_2]$
   - Right side is $\frac{4}{11} \cdot \frac{7}{11}$
   - For $P[R_1G_2]$, have $4 \cdot 7$ ways to get $R_1G_2$, and $11^2$ total outcomes.
   - So left side is $\frac{4 \cdot 7}{11^2}$, which equals the right side.

(b) Without replacement.

1. Identify knowns.
   - Know: $P[R_1] = \frac{4}{11}$ and therefore $P[R_1^c] = \frac{7}{11}$
   - We seek: $P[G_2]$ and $P[R_1G_2]$
2. Find $P[G_2]$ using Division into Cases.
   - Division into cases:

   $$G_2 = G_2 \cap R_1 \bigcup G_2 \cap R_1^c$$

   - Therefore:

   $$P[G_2] = P[R_1] \cdot P[G_2 \mid R_1] + P[R_1^c] \cdot P[G_2 \mid R_1^c]$$

   - Find these by counting and compute:

   $$\gg\gg \quad P[G_2] = \frac{4}{11} \cdot \frac{7}{10} + \frac{7}{11} \cdot \frac{6}{10} \quad \gg\gg \quad \frac{70}{110}$$

3. Find $P[R_1G_2]$ using Multiplication rule.
   - Multiplication rule (implicitly used above already):

   $$P[R_1G_2] = P[R_1] \cdot P[G_2 \mid R_1] \quad \gg\gg \quad \frac{4}{11} \cdot \frac{7}{10} \quad \gg\gg \quad \frac{28}{110}$$

4. Compare both sides.
   - Left side: $P[R_1G_2] = \frac{28}{110}$

- Whereas, right side:

$$P[R_1]\cdot P[G_2]=\frac{4}{11}\cdot\frac{70}{110}=\frac{28}{121}$$

- But $\frac{28}{110}\neq\frac{28}{121}$ so $P[R_1G_2]\neq P[R_1]\cdot P[G_2]$ and they are *not independent.*

## Tree diagrams

**Example - Tree diagrams: Marble transferred, marble drawn**

Setup:

- Bin 1 holds five red and four green marbles.
- Bin 2 holds four red and five green marbles.

Experiment:

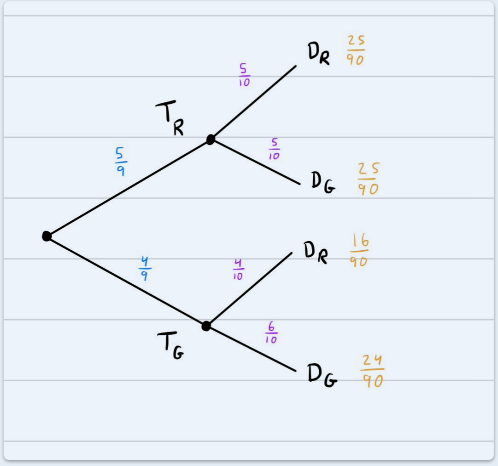
- You take a random marble from Bin 1 and put it in Bin 2 and shake Bin 2.
- Then you draw a random marble from Bin 2 and look at it.

Questions:

- (a) What is the probability you *draw* a red marble?
- (b) Supposing that you drew a red marble, what is the probability that a red marble was *transferred*?

**Solution**

1. Construct the tree diagram.
    - Identify sub-experiments, label events, compute probabilities:




2. For (a), compute $P[D_R]$.
    - Add up leaf numbers for $D_R$ at leaf:

$$P[D_R]=\frac{25}{90}+\frac{16}{90}=\frac{41}{90}$$

3. For (b), compute $P[T_R\mid D_R]$.

- Conditional probability:

$$P[T_R \mid D_R] = \frac{P[T_R D_R]}{P[D_R]}$$

- Plug in data and compute:

$$\gg\gg \quad \frac{25/90}{41/90} \quad \gg\gg \quad \frac{25}{41}$$

- Interpretation: mass of desired pathway over mass of possible pathways.

## Counting

### Combinations: Counting teams with Cooper

A team of 3 student volunteers is formed at random from a class of 40. What is the probability that Cooper is on the team?

**Solution**

There are $\binom{39}{2}$ teams that include Cooper, and $\binom{40}{3}$ teams in total. So we have:

$$P = \frac{39!}{2!37!} \Big/ \frac{40!}{3!37!} = \frac{3}{40}$$

### Combinations: Groups with Haley and Hugo

The class has 40 students. Suppose the professor chooses 3 students Wednesday at random, and again 3 on Friday. What is the probability that Haley is chosen today and Hugo on Friday?

**Solution**

1. Count total outcomes.
    - Have $\binom{40}{3}$ possible groups chosen Wednesday.
    - Have $\binom{40}{3}$ possible groups chosen Friday.
    - Therefore $\binom{40}{3} \times \binom{40}{3}$ possible groups in total.
2. Count desired outcomes.
    - Groups of 3 with Haley are same as groups of 2 taken from others.
    - Therefore have $\binom{39}{2}$ groups that contain Haley.
    - Have $\binom{39}{2}$ groups that contain Hugo.
    - Therefore $\binom{39}{2} \times \binom{39}{2}$ total desired outcomes.
3. Compute probability.
    - Let $E$ label the desired event.

- Use formula:

$$P[E] = \frac{|E|}{|S|}$$

- Therefore:

$$P[E] \quad \gg\gg \quad \frac{\binom{39}{2} \times \binom{39}{2}}{\binom{40}{3} \times \binom{40}{3}}$$

$$\gg\gg \quad \left(\frac{\frac{39 \cdot 38}{2!}}{\frac{40 \cdot 39 \cdot 38}{3!}}\right)^2 \quad \gg\gg \quad \left(\frac{3}{40}\right)^2$$