Specimen Geometriae Luciferae

G.W. Leibniz, 1695
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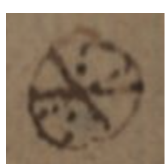
[Manuscript]
[Typescript]

It has often been observed by men endowed with keen judgement that Geometers, though they deliver the truest and most certain things, and confirm them so that one cannot withhold assent, yet they neither sufficiently enlighten the mind, nor open the fountains of discovery, while the reader feels themselves captured and bound, but not sufficiently able to grasp how they have fallen into this trap. This issue makes people admire more than understand the demonstrations of Geometers, and not perceive enough fruit for the improvement of the intellect, also profitable for other disciplines, and which seems to me in fact to be the most powerful use of Mathematical demonstrations. Then, as I often pondered these matters, very many things occurred to me that seemed to help restore the causes and reopen the fountains, so I decided to write down a sample of them with an informal style and freer structure, just as it comes now to mind, saving a more rigorous method of explaining them for another time.

Geometers use, or can use, various concepts taken from elsewhere, namely about what is same and what distinct, or i.e. coincident and non-coincident, about what is-in[1] or not is-in, about determined and undetermined, about congruent and incongruent, about similar and dissimilar, about whole and part, about equal, greater, and lesser, about continuous and interrupted, about change, and finally, what is properly their own, about situs and extension.

The doctrine about coincident and non-coincident is itself the doctrine of logic about the forms of syllogisms. Hence, we take it that things which coincide with the same third thing coincide with each other; if one of two coincidents did not coincide with the third, neither would the other coincide with it.

A Geometer shows thus that the point where two diameters of a circle (that is, straight lines dividing the circle into two congruent parts) intersect coincides with the point where another two diameters of the same circle intersect. See Fig. 1].



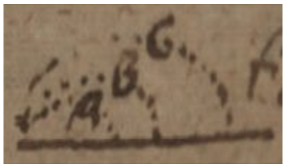
[Fig. 1]

Some part of the doctrine about what is-in something else was even involved in demonstrations by Aristotle in his Prior Analytics, for he observed that the predicate is-in the subject, that is, the notion of the predicate [is-in] the notion of the subject, even though on the other hand individuals of the subject are-in individuals of the predicate. And at

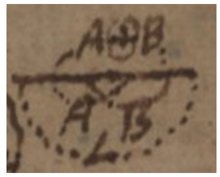
[1]Here and throughout, we use the hyphenated expression to represent Leibniz's term of art *inesse*.

this point more universal things could be demonstrated about that containing and that contained, or being-in, which would be as useful in matters of Logic as in Geometry.

I gave a sample of these when I demonstrated in Fig. 2] that if A is in B and B is in C, then A also is in C; in Fig. 3] that if A is in L and B is in L, then the composite of A and B also is in L; in Fig. 4] that if A is in B and B is in A, then A and B coincide. I also solved



[Fig. 2]

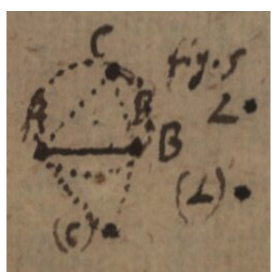


[Fig. 3]

[Fig. 4]

the problem of finding arbitrarily many things such that nothing new can be composed from them, which happens if they are-in each other mutually, successively [*continue*]; as when A is in B and B in C and C in D etc., then nothing new can be composed from these. This can also be exhibited in another way, as when there are five things A, B, C, D, E, and $A \oplus B$ coincides with C, and A is in D, and finally $B \oplus D$ coincides with E, then nothing new can be composed from them however they may be combined. From this I also show how more things of a given number should relate as to coincidence and existence-in, so that useful combinations could be arranged from this for composing something new. And part of the general Combinatorial Science of universally accepted formulas is involved in these things, to which not only Geometry, but also Logistics or the universal Mathematics treating of Magnitudes and Ratios in general, is elsewhere shown to be subordinate.

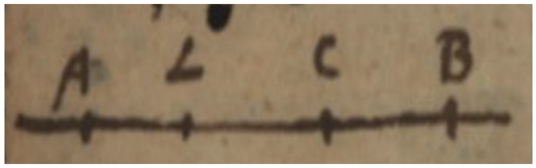
Next is the doctrine of the determined and the undetermined, when of course, from certain givens a requirement is so circumscribed that only a unique thing can be found which satisfies these conditions. There is also semidetermined, when indeed not a unique thing but multiple, of fixed number, or i.e. finite in number, can be exhibited that satisfy them. Thus, given two points A, B, the line AB or i.e. the minimal path from one to the other is determined (Fig. 5]); but if a point C is sought in the plane whose distances from



[Fig. 5]

the given points A and B are of a given magnitude, the problem is semidetermined, since two points in the same plane can be found, say C and (C), that satisfy the requirement. But only a unique circle can be found whose circumference passes through three given points A, B, C. And hence if two circles are proposed, and it is found in the course of argument that each of them passes through three proposed points, it is certain that those circles, which are two in name, are really one and the same or coincide. Whether I hold the given conditions to be determining can be recognized from them themselves, when they are such that they contain the generation or production of the thing sought, or at least that they demonstrate its possibility, and in the course of generating or demonstrating one always proceeds in a determinate manner, so that nothing is left up to decision or choice.[A] If, indeed, one does arrive at the generation of the thing or the demonstration of its possibility by proceeding in this way, then certainly the problem is thoroughly determined.

From here I deduced many remarkable and very useful Axioms, which nevertheless I don't see observed often enough. The most powerful of these is that a determiner can be substituted for a determined in a new determination in which the determined determines something in turn, while preserving this determination. Thus, if we say that the indefinite line passing through A and B (Fig. 6]) is the locus of all points relating in a determinate

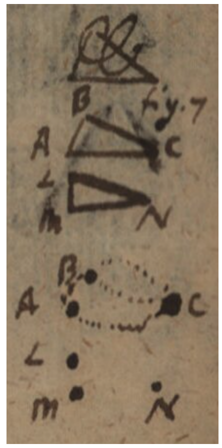


[Fig. 6]

way to A and B, or i.e. unique with their situses to A and B, I will demonstrate from there that, for two other points taken on the same line, say C and A (taking now one of the earlier points for the sake of ease and brevity), the same line is determined also by these two points, or i.e., that any point in the same line is unique with its situs to A and C. The demonstration is like this: Let there be a line through A and B, each point of which, say L, is unique with its situs to A and B, so that no other point can be found relating in the same way to A and B (which is a property of the line), or i.e. $A.B.L$ un. (this is how I will write determination), and take another point C on the same line; I claim that any point of the line, such as L, is also unique with its situs to A and C, or i.e. $A.C.L$ un. Indeed, $A.B.L$ un. (by hypothesis) and $A.B.C.$ un. (since C is on the line through A, B); now remove B in the latter determination by means of the prior determination, substituting $A.L$ for B (by the present *axiom*, because B is determined by $A.L^{\mathrm{B}}$); and so in the latter determination, instead of $A.B.C$ we will have $A.A.L.C$ un. But the repetition of A here is useless, that is, if $A.A.L.C$ is un., then $A.L.C$ also is un., or i.e. L is unique with its situs to A and C, which is what we set out to demonstrate.

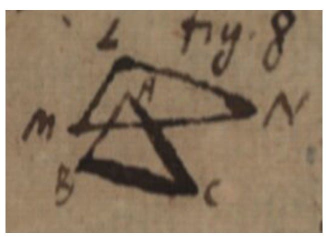
Whence from this example we see a new kind of calculus is born, never until now exploited by a mortal, which magnitudes do not enter, but rather points, and where calculation is not done by equations, but through determinations, or congruences and coincidences. Determination can indeed be resolved into coincidence by means of congruence in this way: $A.B.L.$ un. means: if the situs $A.B.L$ is congruent to the situs $A.B.Y$, [then] L and Y coincide. Now I usually denote coincidence by such a sign: ∞, and congruence alone by such a sign: $\propto$. And hence $A.B.L.$ un. means the same thing as the following conditional proposition: If $A.B.L \propto A.B.Y$, then $L \infty Y$, where I use the letter Y for an indefinite point, in imitation of the Algebraists, for whom the last letters, such as x, y, usually signify indefinite magnitudes. For, whatever point you take, say Y, which relates in the same way to the points A and B as L relates to the points A and B, it necessarily coincides with L, supposing of course that the situs of L to A and B is unique, or i.e. that L is on the line passing through A and B.

Let us pass on, therefore, to explain congruences. Things are *congruent* that cannot be distinguished in any way if they are observed by themselves, like the two triangles ABC and $AB(C)$ in Fig. 5], for which nothing prevents us placing the one onto the other so that they coincide. So now they are only distinguished by position, or the relation to something else already given in position, as for instance, given another point L, it can happen that ABC relates in a different way to L than $AB(C)$ relates to L, for example if L is closer to C than to (C). It is necessary, though, that another (L) could be found that relates in the same way to $AB(C)$ as L relates to ABC, so that $ABCL$ and $AB(C)(L)$ are congruent; otherwise, if something like this could not be done for $AB(C)$ which can be done for ABC (so that (L) could not be found for the former as L for the latter), eo ipso ABC and $AB(C)$ could be distinguished, or i.e. would not be congruent. And this itself is an axiom of the greatest moment, that if two things ABC and $AB(C)$ are congruent and some L is found relating in a certain way to the one ABC, then also another (L) exists[2], or is possible, that relates in the same way to the other $AB(C)$. Now I use this notation (Fig. 7]), $A.B.C \infty L.M.N$,



[Fig. 7]

which signifies that the three points A, B, C are situated among themselves in the same way as the three points L, M, N. But this is to be understood respectively according to the prescribed order, so of course when $A.B.C$ and $L.M.N$ are understood to be congruent, or to coincide, or to be able to be placed onto each other, A coincides with L, and B with M, and C with N. Hence if $A.B.C \infty L.M.N$, it follows that also $A.B \infty L.M$, and likewise for the others. But in order to obtain $A.B.C \infty L.M.N$, we must first check that $A.B \infty L.M$ and $A.C \infty L.N$ and $B.C \infty M.N$, and then finally indeed by composing we may safely say that $A.B.C \infty L.M.N$. So we see (Fig. 8]) that triangles ABC and LMN may have two equal



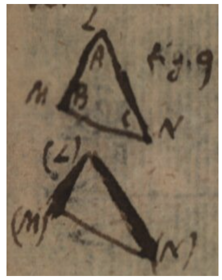
[Fig. 8]

sides, AB [equal] to LM and AC [equal] to LN, but nevertheless not be congruent because they do not have equal third sides, BC and MN. Now the way that in general a congruence of combinations of higher degree can be obtained from congruences of combinations of lower degree, and that one does not need all triples to find the congruence of a quadruple, but only three, and for obtaining a congruence of quintuples, five triples, and of sextuples, seven triples, and so forth to infinity, will appear below when we talk about similarities.

Now it is also clear that in general from all combinations of one degree being respectively congruent, one can always conclude that all combinations of another degree are congruent, for example all triples from all pairs, since from all combinations of one degree, for example from all the pairs from four things being congruent, one can conclude that the whole combination of the four things itself, or the quadruple $A.B.C.D$, is congruent with $L.M.N.P$. Now from the congruence of the whole combination it follows that any lower combination, or any triple, is congruent to the corresponding one; therefore from all pairs, all triples.

[2]Sometimes using "exists" or "there is" for "detur" in modern mathematical idiom.

From these we learn the remarkable difference of congruences from coincidences and existences-in or i.e. containments. For (Fig. 9]) if the line AB coincides with the line LM,



[Fig. 9]

and at the same time the line AC coincides with LN, then the line BC also coincides with the line MN. When AB and LM coincide, eo ipso the point A also coincides with L and B with M; and when AC and LN coincide, eo ipso the point C also coincides with the point N; since, therefore, the points A, B, C coincide with L, M, N respectively, and hence B, C with M, N,[3] the lines BC and MN also coincide. From the nature of the line as to existences-in, I showed elsewhere that, if A is-in L and B [is-in] M, then $A \oplus B$ will also be-in $L \oplus M$, and if $A \oplus B$ is-in $L \oplus M$, and $A \oplus C$ is-in $L \oplus N$, then $A \oplus B \oplus C$ will also be-in $L \oplus M \oplus N$, which mode of arguing cannot be imitated with congruences and similarities.

Now from these things that we just said about the difference between coincidences and congruences flows in turn the reason why triangles ABC and $(L)(M)(N)$ (Fig. 9]) are congruent if the sides AB and $(L)(M)$ as well as AC and $(L)(N)$ are congruent, possibly not mentioning the third ones AC and $(M)(N)$, provided that the angles at A and (L) are congruent. For if the line $(L)(M)$ is congruent to the line AB and the line $(L)(N)$ to the line AC, and also the angle at (L) to the angle at A, then the lines $(L)(M)$ and $(L)(N)$ can be transferred onto AB and AC, with their situs preserved, and so $(L)(M)(N)$ can be placed onto ABC, such that AB and LM as well as AC and LN coincide; therefore, by the nature of coincidence, BC and MN also coincide; and so if the enclosing lines as well as their angles are congruent, then the bases will also be congruent, and so the whole triangle [congruent] to the triangle.

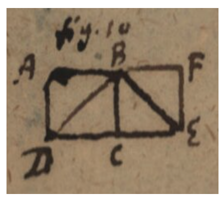
And from this very example we can illustrate this remarkable and very useful Axiom: Things determined in the same way from congruent things are congruent. Thus, since, in general, from two lines given in magnitude and their angle given in magnitude and position, a triangle is determined or i.e. given in position, hence if two triangles ABC and $(L)(M)(N)$ are given, having legs AB congruent with (LM) and AC with $(L)(N)$, as well as a congruent angle that they enclose, angle A with angle (L), the triangles themselves will be congruent. Similarly, since from three lines given in magnitude the angles of a triangle are also given in magnitude, and so everything is determined which would prevent congruence by being different, hence if two triangles have three lines respectively equal and hence congruent (since equal lines are congruent), the triangles themselves will be congruent. And this, on closer consideration, will be found to coincide with Euclid's method of superposition.

Other axioms are also relevant here, such as, things congruent to the same thing are congruent to each other, and of things congruent to each other, if one is incongruent to a third, then the other will be also incongruent to it, which are nevertheless just corollaries of the axioms about same and different. For in things that are congruent, everything is the same, except position, so that they differ only in number. And in general, whatever can be done or said of one of the congruents can also be done and said of the other, with this one exception, that the things which apply in the one differ in number or position from those which apply in the other. Thus we will understand not only two cubits or two feet

[3] MS has "L, M".

to be congruent, but also two pounds, taken abstractly, two hours, two equal degrees of speed. It is also noteworthy that if the peripheries of two bodies are congruent, then also the bodies themselves are congruent, because if the boundaries are congruent in actuality or i.e. coincide, the bodies also coincide. But it is not necessary for surfaces and curves to coincide or be congruent whose extremes coincide or are congruent. It can nonetheless be said in general, that two extensions coincide or are congruent if the things in it [sic] that can be touched from the outside, or i.e. that can be common to itself and the outside,[C] coincide or are congruent. Hence, because surfaces and curves (but not solids) can be touched everywhere from the outside, it is not sufficient for their boundaries to be congruent or coincident. But in general it is the nature of[4] space, of extension (and so also of body, insofar as we conceive nothing other than space to be present in it), that in the inside it is everywhere congruent and indistinguishable (such as if I move in the middle of water, or feel in the middle of darkness, and do not hit anything) and it can only be distinguished through those things that can be touched from the outside, or i.e. are common to it and another thing [alio] (with which it may not have any common part). Hence also if two surfaces or curves are found to be uniform, with their extremes congruent or even congruous in actuality, then they themselves will be congruent or coincide[D] in actuality.

Equals arise from congruents. Namely, what things are congruent, or can be rendered congruent by transformation if necessary, are called equal. Thus in Fig. 10] triangles BAD,



[Fig. 10]

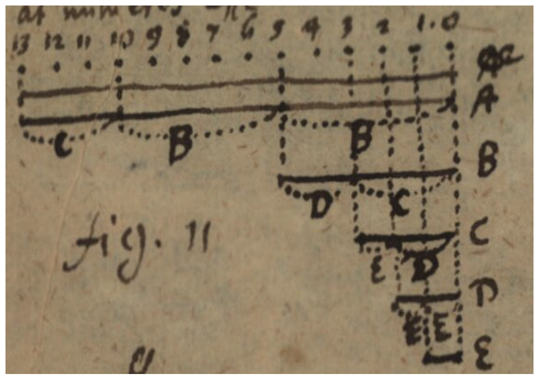
BCD, BCE, BFE are congruent and therefore equal; since triangle EBD is also equal to the square $ABCD$, though indeed the triangle and square are not congruent, nevertheless in this case a square congruent to the former can be made from the triangle by a transposition of its parts, for if you transfer the one part BCD of triangle EBD onto the congruent BFE, with the other part ECB remaining, then from BFE and ECB the square $BCEF$ is formed congruent to the square $ABCD$. Now we usually denote equality with the sign =, that is, $A = B$ signifies that A and B are equal.

Things can also be called equal whose magnitude is the same. And magnitude is a certain attribute of things, a given species [certa species] of which cannot be determined by any definition or by any given concepts [certis notionibus], but rather some fixed measure is needed which one may consult, and consequently if God rendered the entire world with all its parts larger, preserving the same proportion, there would be no basis for noting it. But with one fixed thing taken, as it were a measure, the magnitude of other things can also be ascertained [cognosco] by applying it to the others and using the numbers of repetitions. And so magnitude is determined by the number of parts that are equal to each other, or unequal by some given rule. And although some thing may be incommensurable with respect to a measure or with respect to things to which the measure repeated is exactly congruent, yet by infinitely continued subtraction of the thing from the measure or the measure from the thing as many times as possible, and of the remainder from what was subtracted, then the quantity of the thing with respect to the measure is ascertained from the progression of the numbers expressing repetitions. And consequently those things are equal that relate in the same way to the measure with respect to repetition, and eo ipso it is clear that they can be

[4]Leibniz began to write here "extensi ut ubique simul congr–s" (it is the nature of extent that... everywhere at the same time congruent...) and then crossed this out in favor of the longer statement about space, extent, and bodies.

made congruent, since they are resolved in the same way into parts respectively congruent to each other.

From this one also understands what Mathematicians call ratio or proportion. For if A and B are two things, and the one A is accepted as the measure, then the *magnitude* of the other B is expressed by some number (or series of numbers proceeding according to a given law), setting A to be expressed by unity[5]. But if neither is the measure, then the number expressing B by A, as if A were the measure or unit, expresses the ratio or proportion of A to B. And in general the expression of one thing by another homogeneous one (or i.e. one resolvable into congruent things) expresses the ratio of one to the other, and hence ratio is the simplest relation of the two as to magnitude, in which no third thing homogeneous to them is assumed for expressing the magnitude of the one from the magnitude of the other by its value. For example, let A and B be two magnitudes (see Fig. 11]), and let us aim to



[Fig. 11]

determine their ratio to each other; let us suppose A is greater and B lesser, and therefore subtract B from A as many times as possible, for example 2 times, and suppose C remains; this C is necessarily smaller than B, and so let C be subtracted again from B as many times as possible; now suppose it can be subtracted 1 time and the remainder is D, and D can be subtracted from C again 1 time and the remainder is E, and finally D can be subtracted from E 2 times and the remainder is Nothing. Clearly, $A \overset{(1)}{=} 2B + C$ and $B = 1C + D$ (2); therefore by substituting for B in eqn. 1 the value expressed in eqn. 2, $A = 2C + 2D + 1C$ (3), or $A = 3C + 2D$ (4). Again $C = 1D + E$ (5); therefore (from eqns. 4 and 5), $A = 5D + 3E$ (6), and (from eqns. 2 and 5), $B = 2D + E$ (7). Finally $D = 2E$ (8). Therefore (from eqns. 6 and 8) comes $A = 13E$ (9) and (from eqns. 7 and 8), $B = 5E$ (10). From this we see that E is the greatest measure common to all, and setting E as the unit, we have $A = 13$ and $B = 5$. But whatever unit is assumed, A and B will still be to each other as the numbers 13 and 5, and A will be thirteen fifths of B or $A = \frac{13}{5}B$ (that is $A = \frac{13}{5}$ if B were the unit), namely A is $13E$, while E is a fifth of B; on the other hand, B will be five thirteenths of A or $B = \frac{5}{13}A$, for $B = 5E$ whereas E is one thirteenth of A. Now it is clear that the quantities homogeneous to A and B arising here are, in order,

$$\begin{array}{ccccc} A & B & C & D & E \\ 13E & 5E & 3E & 2E & 1E, \end{array}$$

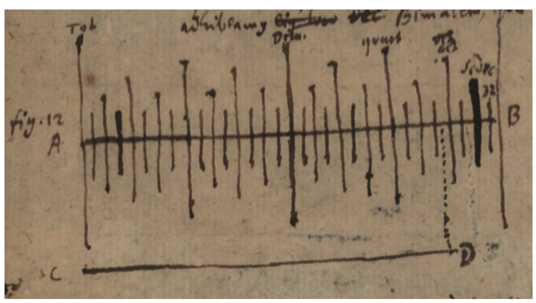
and the numbers of subtractions or *quotients* are 2, 1, 1, 2. And if we cannot arrive at some final thing (like E here) that measures all the others by exact repetitions of itself, so A and B cannot be resolved into parts congruent to this measure itself, and thus also to each other, then we will not arrive at values expressed by numbers of this kind, which a mere repetition of units produces; however, from the progression of quotients itself we can ascertain and determine a species of ratio; just as here given the series of quotients 2, 1, 1, 2 the ratio of

[5]The Latin "unitas" is translated sometimes by "unit" and sometimes by "unity".

A and B is given, where such a series of quotients results from the subtractions performed, so also if the series proceeds to infinity, which happens for those magnitudes which are said to be incommensurable to each other, still if the mere progression of the series is given, *eo ipso* the ratio of the magnitudes will be given, and the longer we continue the series, the closer we will approach [to it].

There are, however, infinitely many other ways of expressing magnitudes, whether through a series or through certain operations or certain motions. In such a manner I discovered that with the square of the diameter being $\frac{1}{1}$, the circle is $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11}$, etc. That is, if the square of the diameter is set to be a square foot (the diameter being a foot), the Circle is the square of the diameter one time, minus (because we have taken too much) a third part of it, plus (because we have removed too much) a fifth part of it, minus (because we have readded too much) a seventh part. And so on, being understood to continue according to a series of odd numbers,[E] the series differs from the magnitude of the circle less than any given quantity, and hence coincides with it. For if we say $1 - \frac{1}{3}$, the error is less than $\frac{1}{5}$, since otherwise we would not have added too much by adding $\frac{1}{5}$; and again if we say $1 - \frac{1}{3} + \frac{1}{5}$, the error is less than $\frac{1}{7}$, since otherwise we would not have subtracted too much by subtracting $\frac{1}{7}$, and so on. Therefore, by continuing[6] for some way, the error is always less than the fraction following next; and if any quantity is given no matter how small, some fraction can be found expressing something even smaller.

But the common use of calculating in numbers, and practice, is primarily served by the expression of magnitudes by the number of parts of a geometric progression, for instance decimal. But since that cannot be expressed well in a small figure, we will use the Binary [Bimal], which is both naturally the first and the simplest. Namely let us divide the line AB in Fig. 12] into two equal parts or two halves, and each half again into two equal parts,



[Fig. 12]

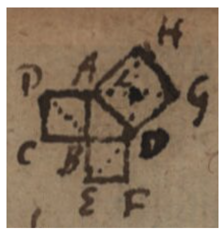
so we will have four quarters, and bisecting the quarters again, eight eighths, and so on, sixteen sixteenths, etc. In the same way we could divide the line into 10, 100, 1000, 10000, etc. parts. Now let CD be a quantity to be estimated by the scale of equal parts that we made descending according to the geometric progression. Let us place CD onto the scale AB and C of course onto A, and let us see where in our scale the other extremity D falls. And let us compare D first with the points of the larger division, proceeding from there step by step to the smaller ones. And since CD is less than the scale AB (for if it were greater, then we would have first subtracted the scale as many times as possible), D will fall between A and B; now we see that $CD = \frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{1}{32}$ and something still further, yet smaller than $\frac{1}{32}$; and so if the scale is not further subdivided, that expression is sufficient at least for this, for the error to be less than $\frac{1}{32}$. And if we subdivide once more, we can have such an expression of CD by the scale AB that the error is less than $\frac{1}{64}$. And so on. Thus similarly, if the scale is divided into 10, 100, 1000, 10000 parts and so on, we can arrange that the error is less than $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, $\frac{1}{10000}$ etc.

By this method arises the remarkable advantage that all quantities which would be

[6] MS is not clear on this word.

expressed by fractions can be expressed in integers as precisely as desired. Indeed, let there be a seventh part of a foot, or whatever other portion or fraction. Let us take 100000 etc. and divide it by 7 continuing as far as desired, the result will be 1428571428571428 etc. or $\frac{1}{7} = \frac{1}{10} + \frac{4}{100} + \frac{2}{1000} + \frac{8}{10000}$ etc. or $1x + 4x^2 + 2x^3 + 8x^4$ etc. setting $x = \frac{1}{10}$ and x^2 to be $\frac{1}{100}$ or the square of $\frac{1}{10}$, and x^3 to be the cube of $\frac{1}{10}$, and so on. And the error is always smaller than one of the last portions, where we stopped, in this case less than $\frac{1}{10000}$. It should also be noted here, above all, that a period always results when the quantity is commensurable to the proposed unit, as in this case 142847 repeats to infinity. Hence the nature of the progression is perfectly known [cognoscitur]. It is clear, moreover, that these things have their place whether we estimate a magnitude by calculation or by actual application to the proposed scale. But the Binary progression has this remarkable [property], that the coefficients, or the numbers by which the powers x, x^2, x^3 etc. are multiplied, are just 1 or 0.

There are still other ways of expressing magnitudes, for although they may be incommensurable with the unit, it can happen nonetheless that certain powers of them or something generated from them can be co-measured with the unit or scale. That this may appear by an example, consider Fig. 13], where AB is a line, for instance a foot, and its square, or i.e.



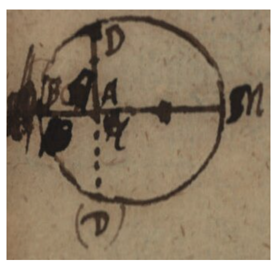
[Fig. 13]

a square foot, is $ABCD$.[F] Let another line BD be equal to AB, so that the angle ABD at B is right, and let the line AD be drawn. And upon the line BD (= AB) let there be a square $BEFD$, equal to the square $ABCD$ (or i.e. AC[G]), and finally upon the line AD let there be a square $ADGH$. Now it is well established, not only from Euclid's Elements, but even by mere inspection of the figure, that the square $ADGH$ is twice the square $ABCD$, or is equal to the squares AC and BF taken together. Indeed, drawing the diagonals AG, DH, intersecting at L, the square $ADGH$ will be resolved into four triangles ALD, DLG, GLH, and HLA, equal and congruent to each other, and the square AC is resolved into two such triangles by drawing the diagonal DB; therefore, the square $ADGH$ is twice the square AC, and hence the square or i.e. power of the line AB (namely the square AC) can be co-measured with the square or i.e. power of the line AD (namely with the square $ADGH$). But let us see now whether the lines AB and AD themselves can be co-measured, or both expressed by numbers, meaning of course rational numbers, which can be expressed by repetition of the unit or of some fixed some-numbereth [aliquotae] portion of that unit (which exhausts the unit by its own repetition). Let us set AB to be 1 (namely one foot), and ask what is AD; it should be a number which multiplied by itself (or squared) produces 2, namely twice what AB squared produces. In fact such a number cannot be an integer. For it must be less than 2 (because 2, 3, and other larger numbers squared, or multiplied by themselves[7], produce more than 2, for 2 by 2 gives 4, and 3 by 3 gives 9 etc.), but still it must be larger than 1 (because 1 by 1 gives 1, not 2); therefore it falls between 1 and 2, so it cannot be an integer, but rather a fraction. In fact neither does any fractional number perform better. For the square of every fractional number is a fractional number, whereas 2 is indeed an integer that should be the square of AD, and so AD is neither an integer nor a fractional number, and thus not rational, but surd.[H] And thus it is expressed either geometrically by drawing lines, as in the figure, or by calculation, and indeed either mechan-

[7] Lit. "drawn on themselves"

ically by approximation, or exactly, as if I said that it was $\frac{1414}{1000}$ or 1414 thousandths of a foot, or more accurately $\frac{14142136}{10000000}$ (or 14142136 ten-thousand-thousandths), for this fraction multiplied by itself will yield $\frac{20000001}{10000000}$ and a little more, so that its difference from 2 is less than one ten-thousand-thousandth. AD is expressed exactly either in common numbers by an infinite series, or by surds. How AD is expressed from AB by an infinite series would be too lengthy to explain here. Algebraically or with surds AD is expressed by the notation for doing an extraction of the square root from 2, or i.e. putting $AB = 1$, then AD will be $\sqrt[2]{2}$, that is the square root of 2, or the number whose square is 2. This surd notation is useful in calculation, since it vanishes by multiplication [of the number] by itself, which cannot equally be said of the Notation of Trisection of an Angle or anything else that has nothing in common with calculation.

Now it will be worthwhile to uncover here the true source of incommensurable quantities, namely, where they come from in the nature of things. Their cause, then, is *ambiguity*, or when the thing sought is semidetermined by the givens (on this [see] above) so that several (but finitely many) things satisfy them, and no method [ratio] of distinguishing one from the other[I] can be applied to the givens. Let us show this in the very example of the preceding paragraph, where we were seeking a number that multiplied by itself makes 2. But it should be known that such numbers always come in pairs. Indeed, 4 can be produced from +2 times +2 as well as from −2 times −2. And so $\sqrt[2]{4}$ is an ambiguous number, and it signifies +2 as well as −2; similarly, $\sqrt[2]{9}$ is an ambiguous number and signifies +3 as well as −3. Therefore also $\sqrt[2]{2}$ is an ambiguous number, and $\frac{1414}{1000}$ satisfies it as well as $-\frac{1414}{1000}$. Thus, by its nature, or i.e. in general, $\sqrt[2]{a}$ cannot be reduced to something rational since everything rational is determinate; nevertheless the extraction proceeds through accident [per accidens[J]], that is, in certain numbers that arose of course through such involution. Ambiguity can be shown also with curves. Let there be (Fig. 14]) a circle whose diameter BM is 3 and a portion



[Fig. 14]

AB of it be 1. Let AD be drawn at right angles from the point A meeting the circle at D; then $AD = \sqrt[2]{2}$, or the square of AD will be 2. For from the nature of a circle the square of AD is equal to the rectangle under BA or i.e. 1 and under AM or 2, which rectangle is 2. But this very construction shows that by the same law that we found the point D, we could have also found a point (D) by drawing out the line from A in the opposite direction, and so if AD is $\frac{1414}{1000}$, then $A(D)$ will be $-\frac{1414}{1000}$. This is also the reason why such problems cannot be solved by straight lines alone, since a line intersects a line in just one point, but a circle is intersected by a line in two points, and hence solves ambiguous problems of this kind.

Actually these surd expressions also provide us with a way of expressing impossible or imaginary quantities through calculation. For every line intersects another line of the same plane (unless they are parallel); but a circle does not intersect a line whose distance from the center is greater than the radius, and a problem that should be solved by such an intersection is imaginary or impossible. Indeed, in the value of the quantity we seek, $\sqrt[2]{-aa}$ (or something similar) occurs, whose square is $-aa$, which is then impossible because such a number $\sqrt[2]{-aa}$ is neither positive nor privative, or i.e. the line that is sought cannot be exhibited by motion either forward or backward. For if it were either positive or privative, the square would nonetheless be positive, as we pointed out already before; whereas its square actually comes out negative. Nonetheless, even these imaginary quantities are of service for expressing real quantities, to the extent that some real quantities could not be expressed by calculation except by the intervention of the imaginary ones, as is shown elsewhere, but then the imaginary ones are eliminated functionally [virtualiter].[K]

But having explained the nature of magnitude and measure well enough, let us return to the consideration of equality, where it should be noted that two things can be shown to be equals if it is shown that one is neither lesser nor greater than the other, and yet they are homogeneous, or one can be transformed into the other. Thus Archimedes exhibits a certain cylinder equal to a sphere, a triangle equal to a parabola; now it is clear that a sphere can be transformed into a cylinder if liquid filling the sphere is poured into a cylinder. That a parabola can be transformed into a triangle, or that a triangle and a parabola are homogeneous, can be shown, since their ratio can be found to be the same as that of a line to a line. I prove it as follows: Let there be (Fig. 15]) two prisms or cylindrical bodies AE

[Fig. 15]

and LQ, let the base, or section parallel to the horizontal, of the one AE be the parabola, say CDE (or others congruent to it), and let the base of the other LQ be the triangle NPQ. Suppose first that AE is full of liquid up to the altitude AB, which, if it is poured from there into LQ, we suppose fills it up to the altitude LM, and that the filled portion LMR of LQ is equal to the portion of AE filled at first with the same liquid, namely ABF. Now the quantities of such cylindrical portions come from the altitude multiplied by the base, or are in a composite ratio of the altitudes and bases, therefore when the portions are equal, the bases will be reciprocally as the altitudes, or the parabola CDE will be to the triangle NPQ as the line LM to the line AB; thus if another triangle is made which also is to NPQ as the line AB to the line LM,[8] which it is well-established can be done by common Geometry [communem Geometriam] (and it is also understood at first glance of the mind from the nature of similar triangles, about which more soon), it is clear that a triangle equal to this parabola is given, or that the parabola can be transformed into a triangle.

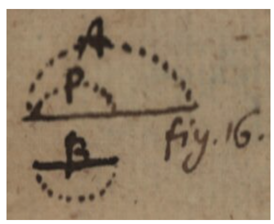
We also know [cognoscimus] magnitudes from generation or motion, as in this case a method is given of estimating a cylindrical body from the motion of the base along the altitude by which such a body is generated; thus the rectangle spanned by two lines is estimated from the product [ex ductu][L] of the line by the line. By this method a surface and also solids generated by rotation are estimated, and to this pertains that spectacular theorem[M] that the thing generated by the motion of some extension is equal to the thing generated by that extension multiplied over [ducto in] the path of the center of gravity, certain rather wonderful generalizations of which I gave elsewhere. However, these truths

[8]These segments are swapped in the MS, presumably by accident.

can be demonstrated by reductio ad absurdum, or by applying the preceding method, when it is shown that something cannot be greater or lesser than is asserted.

Also the method through indivisibles and infinites, or rather through the infinitely small or infinitely large, or through the infinitesimal and infinituple, is spectacularly useful. For it contains a certain resolution as it were into a common measure, though smaller than any given quantity; or a means by which it is shown, by neglecting some things which make an error smaller than anything given and thus nothing, that of two things which are to be compared, one is transformable into the other by transposing. But one should realize that a curve is not composed of points, nor a surface of curves, nor a body of surfaces, but a curve of little curves, a surface of little surfaces, and a body of little bodies indefinitely small; that is, it is shown that two extensions can be compared by resolving them into little parts equal or congruent to each other, of whatever smallness, just as into a common measure, and the error is always smaller than one of these little parts, or at least of a constant or decreasing finite ratio to one, whence it is clear that the error of such a comparison is smaller than anything given. The Method of Exhaustions, somewhat different than the previous one, is also pertinent here, though they come together at the root. There it is shown how there is a certain infinite sequence of magnitudes, of which a first and a final can be obtained,[N] [and] which continuously approach some proposed [magnitude], such that the difference eventually becomes less than [anything] given, and so in the end nothing, or i.e. it is exhausted. And thus the final magnitude of this sequence (which we had said was obtained) is equal to the proposed Magnitude; but here it seemed good only to touch on these things.

We have not yet defined what is greater and lesser, which by all means must be done. Therefore I say, a Lesser than something is what is equal to a part of it, or (Fig. 16]) if



[Fig. 16]

there are two things A and B, and p is a part of A equal to B, then we call *Greater* and B *Lesser*. From here that celebrated Axiom is immediately demonstrated, that the whole is greater than its part, assuming only the other axiom, true in itself [per se] or identical, that certainly each thing endowed with quantity is as great as it is, or is equal to itself, or that every three-foot thing is three feet, etc. The demonstration, comprising a single syllogism, is thus: *Whatever is equal to a part p of the whole A, that thing is less than the whole A* (from the definition of lesser); *now the part p of the whole A is equal to the part p of the whole A, that is, to itself,* (by the Axiom identical or true in itself [per se]), therefore *the part p of the whole A is less than the whole A*, or the whole is greater than the part.

But here we already need to explain something of what whole and part are. Of course it is clear that a part is-in the whole, or i.e. the whole being posited, *eo ipso* the part is immediately posited, or i.e. by positing the part along with certain other parts, *eo ipso* the whole is posited, so that the parts, taken together with their position,[9] differ from the whole only nominally [nomine tenus], and the name of the whole is only put in place of them in reckonings for abbreviation. There are, however, also some things that are-in it, even though they are not parts, such as points that can be taken on a line, a diameter that can be taken on a circle; and therefore the part ought to be Homogeneous with the whole; and hence if two things A and B are homogeneous and A is-in B, then A will be the whole and B the part, and so the demonstrations I gave elsewhere about containing and contained or

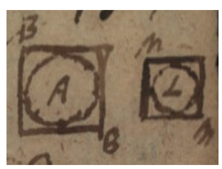
[9]Latin: "partes una cum sua positione sumtae".

i.e. existing-in can be transferred to whole and part. But what Homogeneous is, partly we have touched upon and partly we will explain more fully.

From these definitions of equal, greater, lesser, whole, and part, very many axioms can be demonstrated, which were assumed by Euclid. We have already shown that the whole is greater than its part. That a whole can be composed from its parts in some way, or i.e. that parts can be assigned which coincide with it when taken together, is clear from what was said in the previous paragraph, that is, from the nature of things existing-in. What is less than the lesser is less than the greater, or if A is less than B, and B less than C, then A will be less than C, or $A + L = B$ and $B + M = C$, therefore $A + L + M = C$. Now these axioms, that from adding or subtracting equals from equals, equals result, and others of this kind, are immediately demonstrated from this, that Equals are those which are the same in magnitude, or which can be substituted for each other with the magnitude preserved, and if things are treated in the same way with respect to magnitude (according to all determinate methods of treating by which only a unique thing is produced), then equals will result. From here it immediately appears that equals will become equals by the addition, subtraction, and multiplication of equals; but if roots of the same denomination are extracted from equals, whether pure or afflicted, then it is not necessary that equals immediately result because the problem of extracting roots is, by its nature and absolutely speaking, ambiguous. And so one may not say that those things are equal which produce equals when multiplied by themselves or with the same thing in the same way. Thus, two unequal numbers can be given (namely 1 and 2) whose remainder taken from 3 (2 or 1) being multiplied by the number itself (1 or 2) makes an equal, namely 2.

It is now time, after speaking of magnitude and equals, to speak also of shape or form and similars; the usefulness of similarity in Geometry is indeed very great, but its nature is not considered to have been explained adequately, hence many things are demonstrated in a roundabout way which are immediately clear on the first observation to one rightly considering them. It is well known from Euclid's book of Givens, what things are given in position, what things in magnitude, and finally what things in shape. If something is given in *position* from certain given things, then something else which is given from the same things in the same (determinate) way will be coincident with the first, or the same in number;[O] if something is given in *magnitude* from certain things, and something else is given from the same or equal things in the same (determinate) way, then it will be equal to the first; if something is given in *shape* from certain things, and something else is given from the same or similar things in the same determinate way, it will have the same shape as the first or be similar. Finally, those things which are similar and equal are congruent. And things that are given in both magnitude and shape, they can be said to be given in *pattern* [exemplo] or *type*, so that those things which are of the same type or example, that is, [the same] in both quality or form and quantity, are called congruent. Further, those which cannot be distinguished in any way, neither through themselves nor through other things, certainly are the same or coincident, and such things, in objects for which nothing other than extension is considered, are those that have the same position or that are actually congruent with the same locus. But there are other things which agree in all respects or i.e. are of the same type or pattern, yet still differ in number, such as right angles, two eggs similar in all respects, two seals of the same type expressed in uniform wax. It is clear that these things, if looked at in themselves, cannot be distinguished in any way, even if they are compared with each other. They are only distinguished with respect to situs toward external things. Thus if two eggs are perfectly similar and equal, and are located next to each other, it can only be noted that one is east or west of the other, or north or south, or is above or below, or is closer to some other body placed outside of them. And these things are called congruent, which are such that nothing at all can be affirmed about the one that is not able to be understood also regarding the other, with a distinction only of number or individual, or i.e. of the position which one has at some given time, since multiple things are not in the same place at the same time, nor one thing in multiple [places]. But those things are similar whose shape or definition is the same, or which are of the same lowest shape [speciei infimae], as any circles whatsoever are of the same shape, and the same definition fits each, neither can a circle be divided into distinct shapes that differ according to some definition. Indeed, although one circle could be one foot, another half a foot, etc., nevertheless no definition can be given of a foot, but we need some fixed and permanent type; hence, measures of things tend to be made from durable material, and thus someone proposed that the pyramids of Egypt be used, which have already endured so many centuries and likely will endure a long time yet. In this way, as long as we suppose that neither the globe of the earth, nor the motions of the stars will noticeably change, the same quantity of the tilt of the earth can be investigated by future generations as by us. And if some shapes keep the same magnitude in the whole world over many centuries, as the cells of bees seem to do to some people, a constant measure could be taken from this also. Finally, as long as we suppose that nothing will markedly change in the cause of gravity, nor in the motion of the stars, future generations can learn our measures with the aid of a pendulum. But if, as I already said elsewhere, God changed everything with the same proportion being preserved, every measure would be lost to us, and neither could we know how much things had changed, because a measure cannot be comprehended by any fixed definition and so cannot be retained in memory either, rather we need a real conservation of it. From all these things I judge that the difference between magnitude and shape, or between quantity and quality, is manifest.

And thus, if two things are similar, they cannot be distinguished in themselves separately [per se sigillatim]. For example, two unequal circles will never be distinguished as long as each of them is viewed separately. All theorems, all constructions, all properties, proportions, aspects that can be noted in one circle can also be noted in the other. As the diameter relates to the side of a certain regular polygon inscribed or circumscribed in the one, so also it will relate in the other; as the one circle relates to its circumscribed square, so also will the other to its; whence it is immediately clear, by permuting, that circles are as the square of the diameters, for because A is to B as L is to M (Fig. 17]), by permuting A will

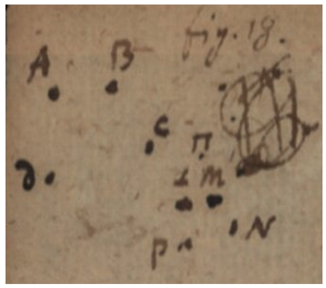


[Fig. 17]

be to L as B to M. And hence it is clear in general that similar surfaces are as the squares of homologous lines, and similar bodies as the cubes of homologous lines. From this also Archimedes took it that the centers of gravity of similar figures are similarly situated. And so, in order to distinguish two similars, for instance two circles, we need not only to view them separately and do it by memory, but we need to view them simultaneously and move them to each other in reality, or apply to them some common real measure, or something already measured or to be measured by application of a real measure, bringing it from one to the other. And so at last it will appear whether they are congruent or not. Indeed, if some homologous things from the two similars are congruent, e.g. the diameters of the two circles, or the parameters[P] of two parabolas, it is clearly necessary that the similars themselves are congruent as well, and so also equal. It is not true that if similars are added to or subtracted from similars, then similars will come out, unless they are added or subtracted in the same way in both cases. And in general whatever is determined from similars similarly, or in the same way, those are similar; whereas if they are semidetermined, when the problem is ambiguous, [then] at least to each of the semidetermined things on one side will correspond one of the semidetermined things on the other, which will be similar to it. This can also be said of equals, congruents, and coincidents. If two homologous things from two similars coincide, the two similars will only be congruent, since things which coincide are congruent; whereas when there exist congruent homologous things from similars, they are congruent.

Similarity, moreover, I customarily denote in this way ~, and $A \sim B$ signifies A sim. B.

From things separately similar, however, as I said, one may not infer that the composites are also similar, and it may be that $AB \sim LM$ and $AC \sim LN$ and $BC \sim MN$, but one may not conclude $ABC \sim LMN$; otherwise since any line is similar to any [other line], one could conclude that any figure at all is similar to any other, even though such a method of argumentation does proceed for congruences. But in combinations of three or higher such argumentation proceeds, which is remarkable. Namely, if all triples on the one side are similar to all triples on the other side, the quadruples, quintuples, etc. assembled from them will also be similar, or i.e. if (Fig. 18]) $ABC \sim LMN$ and $ABD \sim LMP$ and $ACD \sim LNP$



[Fig. 18]

and $BCD \sim MNP$, then $ABCD \sim LMNP$. But as to whether one of the triples can be omitted or can be concluded from the others, let us, for example, see whether $BCD \sim MNP$ can be omitted. Take LMN similar to triangle ABC and LMP similar to ABD, it is clear that given $ABCD$ and assuming LMN (which is given in shape) is given arbitrarily in magnitude and position and LMP in shape and magnitude, and since we also have LM in position (because LM is assumed in LMN) it is clear that P falls on the circle described by the motion of triangle LMP around LM as the axis. In this plane, however, we can take P, with L and M staying the same, only twice, let us say at P or at π (because the circumference of this circle punctures the plane in two points). The third similarity, namely $ACD \sim LNP$, shows that P ought to be chosen from these, excluding π, since $ACD \sim LN\pi$ is not [true]. And so, in the plane, in this way everything is determined, or i.e. from only three similarities of corresponding triples one infers the similarity also of the fourth triple and so too of the whole quadruple; and in the attached figure, since A, B, C, D are in the same plane, certainly L, M, N, P will also be in the same plane. But absolutely, in space, if A, B, C, D are understood to be placed anywhere, let us see what will become of the similarities of the triples for inferring the similarity of the whole quadruples. And so, as from the first two similarities we have two things, LMN (assumed in position and magnitude, given in shape) and the circle with axis LM, described by the point P attached rigidly to the axis [and] rotated about the axis, then, from $ACD \sim LNP$, because LP and NP are given, having already LN, the circle with axis LN, described by the point P attached to the axis [and] rotated about the same, is also given. These two circles are not in the same plane, nonetheless they are both in planes orthogonal to the plane LMN, or i.e. they are themselves both orthogonal to the plane LMN. They must necessarily meet each other, else the thing sought would be impossible which nonetheless is elsewhere established to be possible (from general postulates, since it is possible to have anywhere something similar to anything), and so these two circles meet each other. But two circles orthogonal to a plane in which they have their centers relate in the same way with respect to the plane, as much above the plane as below the plane, therefore when they meet each other, they meet each other as much above as below the plane, and so in two points. Now there remains $BCD \sim MNP$, where since MN is given in position and MNP in shape, certainly MNP will be given in type, or magnitude and shape, or again a circle will be given, described by the point P with axis MN. And because [the circle] intersects each of them [the other circles] in two points, and at least one intersection coincides with both, or i.e. [the circle] is incident on a point where the two previous circles intersect each other else the problem would turn out to be impossible, it is necessary that both intersections coincide with the previous intersections. Hence the third circle exhibits nothing new, and therefore the three triples suffice to conclude the fourth; however, the problem is semidetermined, and the matter reduces to the same as if it were proposed, given the distances of one point from three points, to find that fourth point, a problem which is semidetermined. But the way in which we demonstrated it here is extraordinary, as well as mental, and the very method by which we formulated arguments for similarity from it is also extraordinary, since in part we assumed three points initially, in part we obtained such things as were needed, whence the problem for the fourth is determined, so that a quadruple is similar to a quadruple. For finding that a quintuple is similar to another, let a similar quadruple be found first, which is done with three triangles or triples. One point remains for this, and clearly it is determined from the givens, namely its given distances from the four points; and so there is need of only two more triples or triangles that the new point enters into. More precisely, as we have shown,

$$\begin{array}{ccccc} \text{Let} & ABC,\, ABD,\, ACD & & ABCD & \text{And so also} \;\; BCD \\ \text{be similar to} & LMN,\, LMP,\, LNP; & \text{will be similar to} & LMNP, & \text{similar to} \;\; MNP. \end{array}$$

We ask from what additional things we may conclude that $ABCDE$ is similar to $LMNPQ$. We found a little earlier that $LMNP$ is similar to $ABCD$, hence because $LMNP$ is given in position, and thus in magnitude all the more, and $LMNPQ$ is given in shape (because it is given that it is similar to $ABCDE$), it is necessary that $LMNPQ$ is also given in magnitude, or i.e. the lines LQ, MQ, NQ, PQ are given in magnitude; therefore the point Q is given in position, for it has been shown elsewhere that a point with given situs to four points not placed in the same plane is determined or unique. But to return to our triples, it suffices to add these to the three previous similarities of triples:

$$\begin{array}{cccc} \text{That} & LMQ,\, NPQ & \text{So that} & ABCDE \\ \text{be similar to} & ABE,\, CDE, & \text{becomes similar to} & LMNPQ, \end{array}$$

thus indeed from $ABE \sim LMQ$, because ABE and LM are given, LQ and MQ will be given, and from $CDE \sim NPQ$, because CDE and NP are given, NQ and PQ will be given. For the two $ABE \sim LMQ$ and $CDE \sim NPQ$, we could have also used $ACE \sim LNQ$ and $BDE \sim LPQ$, or $ADE \sim LPQ$ and $BCE \sim MNQ$, maintaining that in the two similarities we have conjoined there be nothing in common except E and Q. Hence it is also clear that from the similarity of three quadruples, the similarity of a quintuple is given. Indeed from these five similarities of triples I infer three quadruples as follows:

$$\begin{array}{c|c|c} \text{From } \underline{ABC},\, \underline{ABD},\, \underline{ACD} & \text{From } \underline{ABE},\, ACE,\, BCE & \text{From } ACE,\, \underline{ADE},\, CDE \\ \text{sim. to } LMN,\, LMP,\, LNP, & \text{sim. to } LMQ,\, LNQ,\, MNQ, & \text{sim. to } LNQ,\, LPQ,\, NPQ, \\ \text{infer } ABCD \sim LMNP. & \text{infer } ABCE \sim LMNQ. & \text{infer } ACDE \sim LNPQ. \end{array}$$

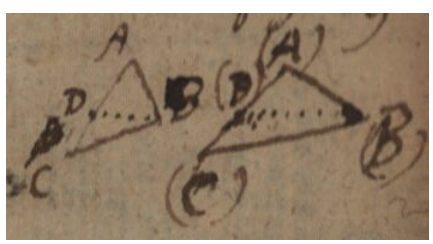
Certainly at least three quadruples are needed to obtain the five triples that suffice for the quintuple, which [triples] we have noted with small lines drawn underneath. For a similarity of a sextuple, if we want that $ABCDEF$ be $\sim LMNPQR$, let us make $ABCDE \sim LMNPQ$, for which there is need of the five triples specified above. Then because every point is sufficiently determined from its situs to four other points being given, we only need to find LR, MR, NR, PR, which will happen the same way as above by assuming only two similarities of triples having nothing besides F and R in common, namely [assuming] that LMR, NPR are similar to $\underline{ABF}$, $\underline{CDF}$, from which, together with the five similarities above, we infer the sextuple $ABCDEF \sim LMNPQR$. And thus from three similar triples or triangles we can infer the similarity of two quadruples or pyramids assembled from them; from five similar triples or triangles (or from three similar pyramids) we can infer the similarity of two quintuples or i.e. the pentagonal[10] solids assembled from them; from seven similar triples

[10]This refers to a five-cornered solid, not a solid with pentagonal faces. A similar comment applies to tetragonal, hexagonal, and other polygons below. Leibniz uses 'polygon' to refer to a many-cornered figure in space as well as in the plane.

or triangles we can infer the similarity of the hexagonal solids assembled from them, and so forth to infinity, supposing that more than three of the points are not in one plane. From one, three, five, seven, nine, etc. triples or similar triangles, we infer the similarity of two triples, quadruples, quintuples, sextuples, septuples, etc. assembled from them, or i.e. the tetragonal (or pyramidal), pentagonal, hexagonal, septagonal etc. solids. Note, here, that the number of faces of a solid is not immediately defined from the number of corners. But it will also be worthwhile to investigate the progression by which it is shown how the higher combinations are inferred sufficiently from quadruples or pyramids, and from quintuples or pentagonal solids, and so forth, which are so rapidly established by means of the sufficient triples already found.

But here it should be noted, chiefly, that the same things we said about similarities, regarding inferring similarities of higher combinations from triples, quadruples, quintuples, etc., can be directly applied to congruences. Indeed, $LMPN$ is found to be congruent to $ABCD$ (Fig. 18]) in the same way in which $LMPN$ is found to be similar to $ABCD$, the only difference being that, while for finding similarity one could assume the first line LM arbitrarily, for finding congruence one must assume that LM is equal to AB; having now LM, from this the triangle LMN is obtained now in type (being similar, of course, to the given ABC), which can then be assumed in position and placed wherever one likes. Now from this, since the distances of the point P from the points L, M, N are given, the point P can be obtained, and $LMNP$ (a pyramidal solid) becomes similar or also congruent to $ABCD$. And this method should be noted, that indeed whatever things suffice for constructing something according to a prescribed condition, in this case similarity or congruence, those things also suffice for inferring from them that very condition. Congruences have at least this privilege, that they can be inferred from congruences of pairs or i.e. lines, but for new similarities nothing can be inferred from similarities of pairs or lines, indeed all lines are similar to each other; but from the similarities of triangles or triples one can infer the similarities of other polygons, even solid ones. And because just as many similarities of triangles are needed for concluding the similarity of a tetragon in a plane as a tetragon in a solid, perhaps also the same number of similar triangles is needed for inferring similarity in higher polygons whether in a plane or in a solid, something which we do not have leisure to examine now.

In another regard, for two figures to be similar, it is necessary for their angles to be congruent, which I show like this, since otherwise if they did not have corresponding or homologous angles that were equal and thus congruent, then they could be distinguished in themselves separately. Indeed if (Fig. 19]) angle A is not congruent to angle (A), hence in



[Fig. 19]

AC taking $AD = AB$ and adjoining DB, and similarly in $(A)(C)$ taking $(A)(D) = (A)(B)$ and adjoining $(D)(B)$, the ratio of DB to AB will not be the same as that of $(D)(B)$ to $(A)(B)$, therefore or hence ABC and $(A)(B)(C)$ can be distinguished. On the other hand, if all the angles are the same, one shows that the triangles themselves are similar in this way, that a triangle is given from being given one side and all angles, and now side is similar to side (of course any line [is similar] to any line) and angle is congruent to angle, therefore the triangles are determined in the same way from similar and congruent things, and so they are similar. For making tetragons, pentagons, etc. similar (whether in a plane or in a solid), we do not merely need that all angles are equal, because a polygon higher than a triangle is not immediately given from being given one side and all angles, and so however many sides are needed for determining a tetragon, pentagon, etc. with all angles being given, the ratio of those sides can also be assumed to be the same as in the tetragon and other given polygon, and from that, with the angles being the same, the figure is similar, since from these sides and angles the figure can even be constructed; and in general, if all sides and all angles [are given][11] or only some sides and some angles, provided the givens are sufficient for constructing the figure, and the problem is completely determined from them (or else is semidetermined such that the several things satisfying it are congruent or similar to each other), then it is sufficient that no dissimilarity can be noted in these givens, and thus that the angles in both are equal while the corresponding given sides in both are proportional, in order to know that similar figures arise from both. But it was already noted above that if some (or one) homologous things in two similar figures are congruent, then all the rest are congruent. On the other hand, coincidence cannot be inferred completely from one coincidence of homologous things, but according to the nature of a figure more or fewer coincidences of homologous things are needed to infer complete coincidence.

By this technique, since the corresponding angles of similar figures are necessarily equal and thus congruent, Geometers have made it so that they have no need for special rules about similarity, and in fact so that everything in Geometry that can be asserted about similarity can be demonstrated through congruences. This is admittedly helpful for demonstrations that compel the intellect, but in that way there is often need for long detours, whereas, through the consideration of similarity itself, one can foresee the same things by a shorthand and a simple intuition of the mind, by a certain mental analysis depending less on the inspection of figures and on images.

Now then, Homogeneous things arise from similars in almost the same way as equals arise from congruents, which is worth noting, for just as equals are things which either are congruent or can be rendered congruent by transformation, so Homogeneous things are those which either are similar (whose homogeneity is self-evident, like that of two squares to each other or two circles to each other) or at least can be rendered similar by transformation; now such transformation occurs if nothing is taken away or added but nonetheless it becomes something else, when a certain transformation occurs with certain parts preserved, as when we cut the square $ABCD$ (in Fig. 10]) into two triangles ABD and BCD, and by rejoining them differently (for instance by transferring ABD into BCE) from there we form triangle DBE; but certain transformations do not preserve any parts, as when a straight line is to be transformed into a curve, a rounded surface into a plane, and something completely rectilinear into something curvilinear or vice versa; then therefore only the minima are preserved,[Q] and it is a transformation where from one thing another is made with at least the minima remaining the same; and it is thus preserved in a perfect real transformation in a flexible thing or a liquid. But in mental transformation quasi-minima can be used in place of minima, that is, things indefinitely small, to make a quasi-transformation; considering also that a quasi-curvilinear thing is used instead of a curvilinear thing, namely a rectilinear polygon with an arbitrarily large number of sides, if then the quasi-transformation we seek continues in this manner or the error or difference between the quasi-transformation and the true one becomes ever smaller and smaller, so that it eventually becomes smaller than any given, one can conclude the true transformation. And because things are equal of which one is made from the other by transformation, it is also clear that things Homogeneous to each other are those which are themselves similar, or equals of which, at least, are similar.

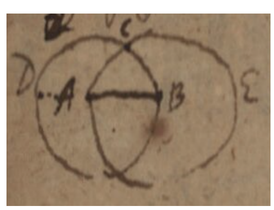
It also clear that Homogeneous things are those which are generated by continuous increment or decrement of the same thing, the minima and maxima at least, or extrema, being excepted. Thus if we suppose that a path or curve grows continuously by the motion of a point, [then] the curves described by a single point are homogeneous to each other, and doubtless even curves generated by distinct points, since although they are dissimilar, it is

[11] The sentence construction is somewhat loose, and a few words are ambiguous or possibly erroneous. The main verb of the protasis appears to be missing.

clear that this dissimilarity arises from certain particular obstructions that cannot change homogeneity. And the same holds for things that are described by the motion of a curve or surface. One should understand, however, a motion by which one describing point does not pass through the traces of another describing point. Doubtless we can also imagine that homogeneous things are made from each other in a continuous fashion, as a circle transformed continuously into various ellipses can pass through infinitely many ellipses of all possible shapes. And in general in Homogeneous things that axiom has its place, that whatever passes continuously from one extreme to another passes through all intermediates; which does not apply, however, to the angle of contact, which is really not a mediate but rather is of another and clearly heterogeneous nature.

Euclid defines Homogeneous things in another way, of course, as things for which, one being subtracted from the other and again the remainder from the one subtracted and continuing forever, there remains either nothing or a quantity smaller than a given one. It is true that this given quantity, than which a smaller [quantity] should remain, must also be of a previously ascertained homogeneity; it will, however, be of an ascertained homogeneity if it is similar to either one, or if it measures either of them by repetition. And so if for two given quantities a common quasi-measure can be found smaller than a true measure of either of them assumed however small, then it can be said that the two are homogeneous to each other. This definition is indeed correct and useful for putting together coercive demonstrations, but it does not equally illuminate the mind as the one that is taken from the consideration of similarity. And one does follow from the other, since from such a quasi-resolution into a quasi-common measure it is shown that one can be transformed into the other, or at least into something similar to it, such that the error is smaller than any given. Indeed, it is evident that all things having a common measure certainly can be transformed such that one becomes similar to the other.

As for the rest, something must be said about the Continuum and about Change before we come to explaining Extension and Motion (which are species of them). A Continuum is a whole, of which any two co-integrant parts (i.e. parts which taken together coincide with the whole) have something in common, and even if they are not redundant or have no common part, or if the aggregate of their magnitudes is equal to the aggregate of the whole, then at least they have in common some boundary. And thus, if one is to pass from one to the other continuously, not indeed by a jump, it is necessary to pass through that common boundary. Whence is demonstrated what Euclid tacitly assumed without demonstration in the first [proposition] of the first [book][12], that two circles in the same plane one of which is partly inside and partly outside the other, intersect each other somewhere, such as if one circle (Fig. 20]) is described by radius AC, the other by radius BC, and AC and BC are



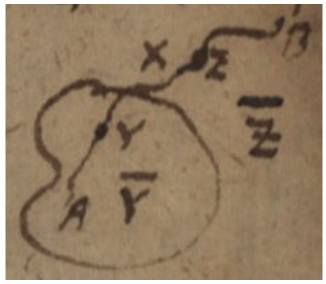
[Fig. 20]

equal to each other and to AB, then it is evident that some B which is in one circumference DCB falls inside the other circle ACE, because B is its center, but in turn it is clear that D, where the extended line BA meets the circumference DCB, falls outside the circle ACE, and so the circumference DCB, since it is continuous and is found partly inside the circle ACE and partly outside, intersects its circumference somewhere. And in general, if some continuous curve is on some surface, and it is partly inside and partly outside a part of that surface, it intersects the perimeter of this part somewhere. And if some continuous surface is

[12]MS: "prima primi", referring to Proposition I of Book I of Euclid's Elements.

partly inside some solid and partly outside, it necessarily intersects the boundary [ambitus] of the solid somewhere. If it is only outside or only inside, and yet meets the perimeter or boundary [terminus] of the one, then it is said to be tangent, that is, the intersections coincide with each other.[R]

We can even express this by some kind of calculus, as when a part of some extension is $\overline{Y}$ (Fig. 21]) and each point falling in this part $\overline{Y}$ is called by one generic name Y, while every



[Fig. 21]

point of that extension falling outside this part is called by one generic name Z, and so the whole extension taken outside that part $\overline{Y}$ is called $\overline{Z}$; it is clear that points falling on the boundary [ambitum] of the part $\overline{Y}$ are common to $\overline{Y}$ and $\overline{Z}$ or can be called partly Y and Z, that is, it can be said that some Y are Z and some Z are Y. Now the whole extension is composed of $\overline{Y}$ and $\overline{Z}$[13] simultaneously, or is $\overline{Y} \oplus \overline{Z}$, as every point of it is either Y or Z, allowing that some are both Y and Z. Let us now suppose some other new extension is given, say AXB, existing in the proposed extension $\overline{Y} \oplus \overline{Z}$, and this new extension we will call generically $\overline{X}$, so that any point of it will be X; it is clear first of all that every X is either Y or Z. But if it is established from the givens that some X is Y (for instance A which falls inside $\overline{Y}$) and again that some X is Z (for instance B which falls outside $\overline{Y}$ and thus in $\overline{Z}$), it follows that some X is both Y and Z at the same time. Hence, although in general, in other situations, nothing follows in this way from particulars, yet in a continuum such a thing can be inferred from them because of the special nature of continuity. So, to collect briefly the inference: If there are three continua $\overline{X}$, $\overline{Y}$, $\overline{Z}$, and every X is either Y or Z, and some X is Y and some X[14] is Z, then some X will be Y and Z at the same time. Hence one also infers that $\overline{X} \oplus \overline{Y}$ comprises some new continuum, because some Y is Z or some Z is Y.[S]

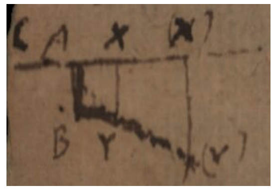
We can understand something of a continuum not only in things that exist at the same time, indeed not only in time and space, but also in a change and the aggregate of all states of some continuous change, e.g. if we suppose that a circle is continuously transformed and passes through all shapes [species] of Ellipses while preserving its magnitude, the aggregate of all these states[T] or all these Ellipses can be imagined in the form of [instar] a continuum, even though all these Ellipses are not placed beside each other, and indeed neither do they exist at the same time, rather one is made from another. We can nonetheless take ones congruent to them instead of them, or compose a solid consisting[15] of all those Ellipses, or whose sections parallel to the base are all those Ellipses taken in order. But if we imagine a sphere being transformed into equal Spheroids in order, then we cannot exhibit a real continuum assembled in this way out of all those spheroids, because we do not have more than three dimensions in extension alone. But if we are willing to use some new consideration, for example weight, we can exhibit a fourth dimension, and thus exhibit a real solid, albeit heterogeneous or with parts of distinct weight, which represents all the spheroids by its sections parallel to the same base. But there is not even need to ascend to

[13]MS lacks the overline on Y, but it was inserted as a marginal note and probably the overlines were forgotten.

[14]MS shows "some Y is Z".

[15]Latin *constans*, 'consisting of' or 'constituted by'.

a fourth dimension or use weights in addition to extension, for instead of the spheroids, let us take only right figures proportional to them, which certainly can be done, and a plane can be assembled from them, whose sections parallel to the base are proportional to the spheroids corresponding in order and so will represent[U] the continuous transmutation of the sphere into spheroids. Indeed, it suffices for us that some line AX can be assumed (Fig. 22]) which is traversed by a movable point X starting from A, and let us suppose

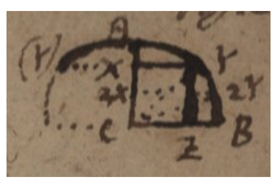


[Fig. 22]

that, corresponding to each portion of the line or abscissa AX, we can exhibit a state of the sphere continuously transmuted into spheroids with the magnitude preserved, represented by the line XY, or i.e. such that the ordinate lines XY correspond to the spheroids in order, or so that XY is to AB in order as the ratios of the conjugate axes (by which the spheroid is determined when the magnitude is given, which is always the same here) are to unity (for the ratio in the sphere is that of equality). In this way it is clear how the continuous change is represented by the line AX and the curve BY, or by the plane figure $BAXYB$; whereas if we had not changed the shape retaining the magnitude, but rather the magnitude retaining the shape, those XY would have been proportional to those magnitudes or states. But now when the shape is changed, they are at least proportional to something determining the shape. Weighing the matter, though, shows that the line AX alone is sufficient, so that we may imagine a portion of the line can be taken corresponding to each logarithm of the ratio of the conjugate axes, which vanishes at A or in the case of equality. But if we want to take abscissas corresponding not to the logarithms but to the ratios, then for the case of the sphere or the circle an abscissa CA should be taken, representing unity, which will continuously grow as the ratios of the axes grows. It continuously shrinks, on the other hand, when the ratios shrink, and it vanishes at C when the circle is transformed into an Ellipse or the sphere into a spheroid of infinitely small longitude. And this is if in transmuting the change occurs according to only one consideration, as here only the ratio of the axes changes, since Ellipses can vary in only one way with magnitude preserved; but if we were told to vary the circle in infinity times infinity ways, namely according to magnitude as well as according to shape, so that it must pass through all types of Ellipses, then that change would have to be represented not by a line or a curve, but by some surface; it is the same as if the magnitude of the circle needed to be preserved, but it had to be transformed into Ellipses of second degree, of which there are not only infinitely many shapes [species], but also infinitely many genera, and under each genus infinitely many shapes, and so infinity times infinity shapes. And if you made the circle not only to be transmuted through all shapes of Ellipses of second degree but also to vary in magnitude, and so to pass through all types of Ellipses of second degree, then the states of the circle would be infinity times infinity times infinity, and all the changes would have to be represented by some solid. And if the circle had to pass through all types of Ellipses or Ovals of third degree, the variations could not all be exhibited in one continuum except through a fourth dimension, using for instance weight or another heterogeneity of extension. And so on. In this way it is necessary that at one moment infinitely many changes occur, indeed sometimes infinity times infinitely many, otherwise one eternity would not suffice for traversing all the variations.

And so from these things the nature of continuous change is also understood, and truly it is not enough for it that between any states an intermediate one can be found; indeed, certain progressions can be contrived in which such interpolation proceeds perpetually, yet a continuum cannot be assembled from them; rather it is necessary that a continuous cause be understood which operates at each moment, or that for each point of some indefinite line, a corresponding state can be assigned as we said.[V] And such changes can be understood with respect to place, shape, magnitude, velocity, and even other qualities that are not of this regard, such as heat and light. Thus also the Angle of contact is in no way homogeneous to a common angle, indeed it is not even *syngeneous*[16] to it, as a point to a curve, but it relates to it in some measure as an angle to a line; indeed a continuous generation with a fixed law cannot be contrived which equally passes through angles of contact and angles of straight lines. It is the same regarding the angle of osculation that I devised, and other older things. Of course the angle of intersection of two curves that intersect each other is the same as that of the lines tangent to them; the angle of contact of two curves tangent to each other is the same as the angle of contact of the two osculating circles to the curves, as I showed elsewhere.

Before we move on from here, something else must be said about Relation or the disposition [habitudine] of things to each other, which differs greatly from ratio or proportion which, to be sure, is just one simpler species of it. Now there are perfect or determining relations, through which one thing can be found from others; there are indeterminate relations, when something relates to [se habere] another such that knowledge of its disposition still does not suffice for determining the one from the other being given, unless new things or new conditions are added. Sometimes only new conditions are added, but sometimes also new things. In relations one can also consider homeoptosis and hetereoptosis.[17] Namely, if there is a certain relation between homogeneous things A, B, C, and each one of these three things relates in the same way, so that by permuting their places in the formula none other than the previous relation would arise, then the relation will be a certain absolute Homeoptosis; it can also happen, however, that only some of the homogeneous things falling into the relation relate by homeoptosis, for instance A and B, allowing C to relate differently than A and B. And this Homeoptosis is of the greatest importance in reasoning. It can also happen that there is a certain relation between A and B (where however one still requires that other things homogeneous to them enter the relation) where A is determined from B being given, but B is only semidetermined from A, indeed it may even be undetermined. I would like to illustrate these things by example. Let $ABCYA$ be the quadrant of a circle (Fig. 23][18]), the magnitude of whose radius AC or CB or CY will be called a, and the magnitude of



[Fig. 23]

the right sine YX will be called y, and the magnitude of the sine of the complement CX will be called x. It is clear that the square of CY equals the squares of CX and of YX together, or the equation $xx + yy = aa$ holds, which expresses the relation among these three homogeneous things x, y, and a, by means of which y or the right sine can be obtained from the given a and x or from the given radius and sine of the complement. In this relation, it is clear that x and y relate by homeoptosis, and a relates in a different way from them. It is also clear that the relation is semidetermining with respect to position, even though it is absolutely determining with respect to size; indeed, $y = \sqrt{aa - xx}$, which is ambiguous and signifies $y = +\sqrt{aa - xx}$ as much as $y = -\sqrt{aa - xx}$, of which the former y signifies XY and the latter signifies $X(Y)$, but XY and $X(Y)$ are congruent, or equal in size. It is also clear

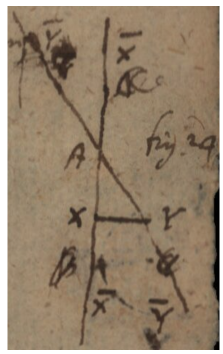
[16]MS has this word in Greek.

[17]From the Greek "homoeo-" and "heteroeo-"

[18]This figure is labelled '24' in the MS, but a prior '23' was crossed out and the subsequent also labelled '24'.

that a, or the magnitude of the radius, is constant or relates in the same way, and any x and y are indefinite, for as the radius is obtained from the given CX and XY (by extracting the root from the sum of their squares[19]), so the radius is obtained in the same way from C_2X and ${}_2X_2Y$. Such constant magnitudes relating in the same way to other indefinite ones are usually called parameters.

Even as we have explained here the relation of the points Y of the quadrant to the points X of the line, or the way by which, with the magnitude of the radius being given and the points A, B, C being given in position, a corresponding point Y of the circle can be found from the point X of the line (albeit in a dual manner, or semideterminately), so also we will be able to give another simpler relation through which, from the points of one line given in position, the corresponding points of another line, also given in position, in the same plane, can be determined in order, which relation we will find is much simpler. In Fig. 24, let



[Fig. 24]

there be lines $\overline{X}$ and $\overline{Y}$ in the same plane intersecting each other at the point A, so that some X is A and some Y is also A, and in that case $X \infty Y$. Given now the lines $\overline{X}$ and $\overline{Y}$ in position and a common point A, the angle they form will also be given, and so too the ratio of the lines AX and XY supposing XY is the normal ordinate to AX; let this ratio be expressed by some number n, and the equation of AX to XY (or x to y) will be as 1 to n or as unity to this number, and $y = nx$ will hold. Whence it is clear that this relation between x and y is so simple that there is no need to assume some third thing homogeneous to them, or i.e. some other line, much less a higher extension; indeed the n we assumed is a number only or a magnitude not needing any position, determined rather by species or concept alone, and not homogeneous to those lines either. And this simple relation of two Homogeneous magnitudes is nothing but a ratio; that is, the relation is given between these two lines existing in the same given plane, $\overline{X}$ and $\overline{Y}$, since if one of them is given in position, and their common point A is given, and finally the ratio between XY and AX or between the ordinate y and the abscissa x is the same as between the number n and the unity 1, then the other line will also be given in position.

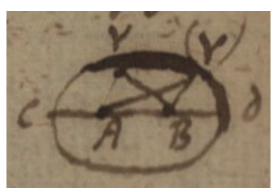
I will show now by an example that every relation between two homogeneous things alone or i.e. between only two homogeneous things endowed with magnitude, such that nothing else enters besides numbers, is a ratio or proportion, even if sometimes it is convoluted such that it appears to be of another nature. Let the equation $x^2 + 2xy \overset{(1)}{=} yy$ hold, in which no other real magnitude enters than these two, x and y, homogeneous to each other, and let us suppose they are lines, and therefore let us write $\frac{y}{x} \overset{(2)}{=} n$ so that n is the ratio of x to y, or at least the quotient or number expressing that relation. Now equation (1) divided by xx yields $1 + \frac{2y}{x} \overset{(3)}{=} \frac{yy}{xx}$ which is (by equation (2)) $1 + 2n \overset{(4)}{=} nn$; the matter is therefore reduced to a ratio alone, or a number to be found that expresses it; and so from equation (1) nothing else is given than the ratio between y and x, although that is given here surdly or ambiguously,

[19] The Latin, "extrahendo radicem ex quadratorum ab his summis", seems confused on a few counts.

for indeed it becomes $nn - 2n + 1 \overset{(5)}{=} 2$ or, extracting the root, $-n + 1 = \sqrt[2]{2}$ or $n \overset{(6)}{=} 1 \pm \sqrt{2}$. Whence we can deduce a method of this kind for finding y or the magnitude of CY or $C(Y)$ from the given x or magnitude of CX (Fig. 25]). Let there be a right isosceles triangle CXA

[Fig. 25]

whose base is $CX = x$, and let a circle $X(Y)Y$ be described with center A and radius AX, bisecting the extended line CY or $C(Y)$, namely in Y and in (Y); I say that the line CY or $C(Y)$ is what we seek, or its magnitude is expressed by y in the equation $xx + 2xy = yy$. If CX is x, then CY or $C(Y)$ will be y; indeed, CY is to CX as $\sqrt{2}+1$ to 1 and $C(Y)$ is to CX as $\sqrt{2}$–1 to 1, or setting CX as unity or 1, then $CY = CA(\sqrt{2}) + AY$ (or 1) $= \sqrt{2}+1$ and $C(Y) = CA\ (\sqrt{2})\ - A(Y)$(or -1) $= \sqrt{2} - 1$. And thus by setting x as unity, y will be the sum or difference of these two, $\sqrt{2}$ and 1, where it should nonetheless be noted that one root must be understood as privative or false, that is the size of $C(Y)$ will be $\sqrt{2} - 1$, but the − sign must be prefixed to it, so it becomes $-\sqrt{2}+1$. Hence, y is either $1 + \sqrt{2}$ or $1 - \sqrt{2}$. Furthermore, it is clear from this that the locus of all points Y is the line CY, if the locus of all points X is the line CX, provided that the angle of the lines is such that, any ${}_2X_2Y$ being drawn parallel to the first XY already found, C_2X is always to C_2Y as we said, or according to the ratio which equation (1) expresses or which the ratio found in equation (6) expresses. But the relations of distinct curves to each other can be expressed not only through parallel lines drawn from one to the other, but also through lines converging to one point, and often one relation is simpler than the other. Thus, if (Fig. 26]) there is an Ellipse



[Fig. 26]

whose two foci are A and B, and any point Y in the Ellipse is taken, then it is a property of the Ellipse that $AY + BY$ is always equal to a constant line, namely the major axis CD of the Ellipse, and hence that $AY + BY$ and $A(Y) + B(Y)$ are equal to each other.

Furthermore, just as the nature of the curve AYB (Fig. 23]) is conveniently expressed by two normal lines YX and YZ emanating from one point of it Y to a certain two lines given in position, normal to each other, CA and CB, so too the nature of the curve $Y(Y)$ (Fig. 27]), which does not remain in any fixed plane, can be expressed, if from any point of it, say Y, set above,[W] three normal lines are drawn to three planes CXA, CZB, CVD normal to each other, namely YX, YZ, and YV (which we will call x, z, v). And now if two equations are given, for instance one between x and z, the other between x and v, the nature of the curve $Y(Y)$ will be sufficiently determined. The former equation will express the nature of the curve $Z(Z)$ projected onto the plane CZB from the curve $Y(Y)$, the latter the nature of the curve $V(V)$ projected onto the plane CVD from the same curve $Y(Y)$. In fact the three planes could be not just normal to each other, but indeed at any

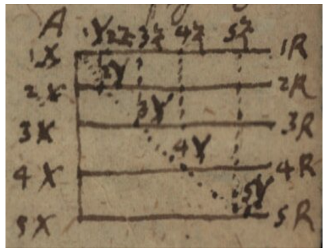
[Fig. 27]

given angle, hence if at least two planes are assumed normal, but a third such as CVD at an indefinite angle, we can find whether the whole curve $Y(Y)$ does not fall in some plane, which will happen if the arbitrary plane CVD can be taken such that the curve $V(V)$ and the curve $Y(Y)$ coincide, or that the lines v become infinitely small or vanish.

Hence the nature of loci is also clear if, namely, the point Y is placed in the plane (Fig. 23]) and its distances YX and YZ are given from two indefinite lines CX and CZ given in position in the same plane, the problem is determined, albeit ambiguous,[X] that is, certain points are given in the same plane, four in number, which can satisfy it. If in fact the distances themselves are not given, but only their relation to each other, by means of which one is determined from the other being given, then the problem is undetermined, or it becomes a locus, for instance the circle in Fig. 23]; and we say that all the points Y are on the circle if they are of such a nature that when normal conjugate ordinates YX and YZ are drawn from each of them to the two lines CX and CZ normal to each other, the squares of the conjugate ordinates taken together are always the same amount or are equal to the same constant square, for the locus of such points will be on a circle whose center is C and whose radius is the side of a constant power or square. Similarly in a solid (Fig. 27]), if the distances YX, YZ, YV of a point Y from three planes CXA, CZB, CVD are given, the problem is determined, albeit ambiguous, since certain points finite in number (namely eight[20]) satisfy it. But it must be understood that the magnitudes are given, some unit being assumed, if there are as many given equations as there are unknowns, and thus if, to find the three lines x, z, v, also three equations are given (independent of each other), [the lines] themselves will be understood to be given, and the problem will be determined; but if only two equations are given, the problem will be undetermined in the first degree, or i.e. we will not determinately obtain an unknown point Y, but rather $\overline{Y}$ or the locus of all Y or the curve $Y(Y)$ of which every point satisfies these conditions. But if for finding these three magnitudes or lines only one equation is given to us into which these three lines enter, then the problem is infinity times undetermined, or it is undetermined in the second degree, and the locus is on a surface, or some determinate surface is obtained (or semideterminate or ambiguous, namely twin or triplet or quadruplet etc.) of which every point satisfies this condition, or the relation expressed by this equation. Hence we now understand what the loci for a point, a curve, and a surface are, and how, when equations or relations expressed by equations are given, points, curves, and surfaces are determined.

These same things can be explained also through compositions of rectilinear motions. Indeed (Fig. 28]) if a ruler RX moves over a line $\overline{X}$, always in the same plane and always preserving the same angle, and meanwhile some point Y is moved on the ruler itself, such that if the motion of each of them begins at a point A, or X or Y, and then when the ruler arrives at ${}_2X$, ${}_3X$ etc. the point arrives at ${}_2Y$, ${}_3Y$, ${}_4Y$ (that is, in ${}_2Z$, ${}_3Z$, ${}_4Z$ if the ruler had stopped in the first place [situs] A_1R), some curve $\overline{Y}$ or ${}_1Y_2Y_3Y$ etc. will be described by this composite motion whose nature is given from the given relation between the corresponding AX and AZ; for example, if the AZ are proportional to the AX, or i.e. if A_2X is to A_2Z (or to ${}_2X_2Y$) as A_3X is to A_3Z, and so on, or if A_2X, A_3X, A_4X are as A_2Z, A_3Z, A_4Z, the curve AYY or $\overline{Y}$ will be a line; if the AZ are in duplicate ratio of the AX or as their squares, the curve $\overline{Y}$ will be a quadratic parabola; if in triplicate, it will be a cubic parabola

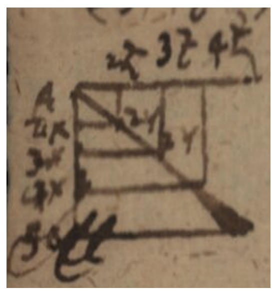
[20] MS has "quatuor".



[Fig. 28]

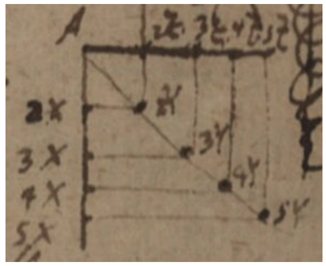
etc. If the AZ are as the reciprocal of the AX, or i.e. A_2X to A_3X is as A_3Z to A_2Z, and the same everywhere, then the curve $\overline{Y}$ will be a Hyperbola whose asymptotes are $\overline{X}$ and $\overline{Z}$. And so on in this way one or another curve can arise which is not to be pursued here.

It is advantageous to note in general how one understands from this motion what sides a curve turns its convexity or concavity towards, whether it has a contrary flexion,[Y] a vertex or point of reversal, maximal and minimal abscissae or ordinates of its period. First let us suppose in Fig. 29] that the velocities of the ruler or instantaneous increments ${}_2X_3X$, ${}_3X_4X$



[Fig. 29]

etc. (which are indefinitely small) of the abscissae AX are proportional to the corresponding velocities or instantaneous increments ${}_2Z_3Z$, ${}_3Z_4Z$ etc. of the point or i.e. of the conjugate abscissae AZ (or the ordinates XY); then AYY is a straight line; otherwise, it will be curved. But now (Fig. 28]) if we suppose that, while the velocity of the ruler remains uniform or the instantaneous increments ${}_2X_3X$, ${}_3X_4X$ etc. of the abscissae AX remain equal, the velocity of the point grows or the increments of the conjugate abscissae or instantaneous increments ${}_2Z_3Z$, ${}_3Z_4Z$ etc. of the ordinates AZ grow; or [if, while] the velocity of the ruler grows, the velocity of the point, which earlier was doing the same thing as the velocity of the ruler, grows more, or while the instantaneous increments of the abscissae grow, the instantaneous increments of the ordinates grow even more; then the curve AYY (Fig. 28]) turns its convexity toward the directrix AX, if both are simultaneously growing, namely the abscissae as well as the conjugate abscissae or i.e. the recessions from the fixed point A of the ruler as well as of the moving point on the ruler; this must be supposed from the beginning if indeed in the beginning the ruler as well as the moving point on it are understood to recede from A. And it is the same if on the other hand both the ruler as well as the point on the ruler are understood continuously to approach A, and the velocity of the ruler or the instantaneous approachings to A remain the same, or grow less [quickly] than the velocities or instantaneous increments of the point on the ruler. But since in this way the point is understood merely to retrace the previous path, this remark will not matter in the future. But if it happens that (Fig. 30]), with the velocities of the ruler or the instantaneous increments of the abscissae, namely ${}_2X_3X$ etc., decreasing, the velocities of the point on the ruler or the instantaneous increments of the ordinates ${}_2Z_3Z$ etc. remain uniform, or grow, or at least decrease less than ${}_2X_3X$ etc., then too the curve AYY turns its convexity toward

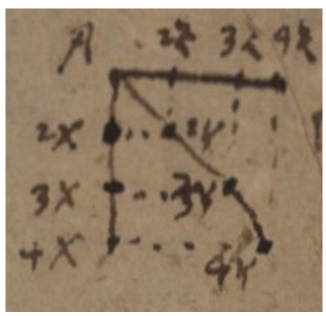


[Fig. 30]

the directrix AX.

On the other hand, from these things it is immediately clear that if the instantaneous increments of the abscissae increase more, or decrease less, than the instantaneous increments of the conjugate abscissae or ordinates, then the curve turns its concavity toward the directrix (or the lines in which the abscissae are taken) provided we suppose that the curve recedes both from the directrix AX and the conjugate directrix AZ, or approaches them, that is, in the one directrix as well as the other it recedes from their common point A or approaches it. This is clear, I say, from the foregoing, if we merely change the directrix and its abscissae into the conjugate directrix and the conjugate abscissae in Fig. 28] or Fig. 30], or conversely; indeed, it is evident that if a curve turns its concavity to one directrix, it turns its convexity to the conjugate one, and conversely; when, of course, it is receding from both at once.

From here it is also clear how a contrary flexion of a curve arises. Indeed, in Fig. 31], if, as the points X of the directrix recede from A, the corresponding points Z of the conjugate directrix also recede from A, and, whereas previously the increments ${}_2Z_3Z$ etc. of the conjugate abscissae increased more or decreased less than the increments ${}_2X_3X$ etc. of the principal abscissae from A up until ${}_3Y$, at[21] ${}_3Y$ the contrary begins to happen, there the curve has a contrary flexion and from concave becomes convex, toward the same parts. That is, if we suppose the rectangle ${}_4X_4Z$ is divided by the curve A_2Y_4Y into two parts $A_4X_4Y_3YA$ and $A_4Z_4Y_3YA$, then while the part A_3Y of the dividing curve turned its concavity to the posterior part of the space, the other part ${}_3Y_4Y$ will turn its concavity[22] to the prior part of the space, that is, while each straight line or chord in the part A_3Y of the curve, such as A_2Y, ${}_2Y_3Y$, fell in the posterior part of the space, now each chord in the part ${}_3Y_4Y$ of the curve falls in the prior part of the space.

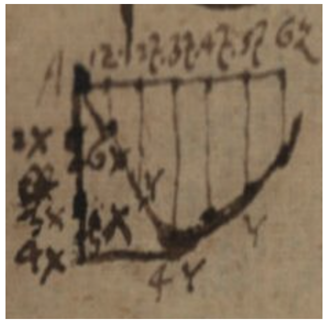


[Fig. 31]

But if we further suppose that the increments of both abscissae, namely the principal and

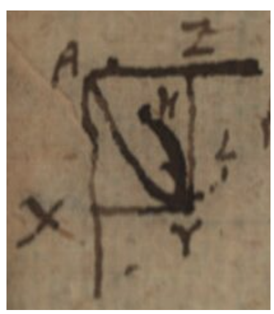
[21] The MS includes also "but", but that seems not to fit the initial sentence structure. The MS shows evidence of draft revision that probably led to this.

[22] MS has "convexity", but it should be "concavity", and Leibniz's error explained by the fact that the sentence was revised a few times.



[Fig. 32]

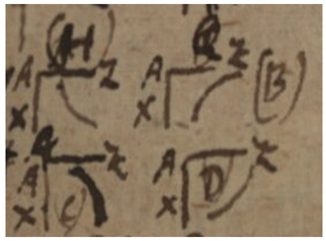
conjugate, or at least of one of them, decrease continuously, and we take the one which alone decreases, or at least decreases more, and we suppose that its velocity eventually vanishes, and thus furthermore changes to the contrary by continuous change, that is the curved line does not recede from A anymore with respect to its abscissa but rather approaches A, then we have there points of reversal. For example in Fig. 32] the velocity of X decreases until ${}_4X$ where it vanishes, namely ${}_1X_2X$, ${}_2X_3X$, ${}_3X_4X$, which represent velocities, continuously decrease until they vanish at ${}_4X$, where the velocity of progressing forward changes into regression, and X tends from ${}_4X$ to ${}_5X$, ${}_6X$ and approaches A again, as the velocity of regression grows in turn (at least for some time), and all the while Z progresses forward with a uniform velocity; moreover the ordinate ${}_4X_4Y$, drawn from the place of reversal of the point X, namely from ${}_4X$, to the curve, is tangent to it at ${}_4Y$. It can happen that the points X and Z reverse toward A simultaneously, but this is rather singular, and in that case at the point of reversal the curve has infinitely many tangents, as is clear in Fig. 33] where the two lines XY and ZY perpendicular to each other are simultaneously tangent to



[Fig. 33]

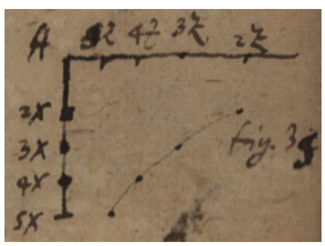
the curve AYH; whence it is clear, since the whole curve falls inside the rectangle XZ, that every line drawn through Y falling outside the triangle is therefore tangent to the curve, and it seems one can doubt whether there is one curve or rather two, AY and HY intersecting each other at H; but because such generating processes [generationes] could be contrived for a single curve, and we have an example in secondary cycloids, nothing prevents the whole AYH being taken as a single curve. But if the curve does not have infinitely many tangents, or X and Z do not reverse simultaneously, or in Fig. 33] if the curve AY does not tend to H, but to L, then it is clear that with the one ordinate from X, namely XY, being tangent to the curve at Y, the other ZY, which of course is perpendicular to XY and thus to the tangent, is also perpendicular to the curve AYL, and so [the ordinate ZY] is the maximum or minimum of the ordinates of this period, indeed the maximum when the curve turns its concavity at Y to the directrix AZ, and the minimum when it turns its convexity to it.

Next let us combine both variations of a curve, one which is according to convexity and concavity, and another which is according to approach and recession with respect to a directrix. A curve can indeed approach as well as recede with respect to a directrix to which it turns either its concavity or its convexity, as in Fig. 34] in (H) concave receding,

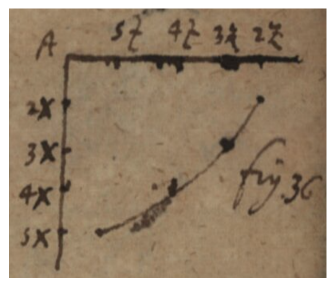


[Fig. 34]

in (B) concave approaching, in (C) convex receding, in (D) convex approaching; but if one relates it to both directrices simultaneously, then when it recedes from both, it turns its convexity to one and concavity to the other, as in (H) and in (C); but when it approaches one and recedes from the other, then it turns its concavity to both or convexity to both, as in (B) and (D). And so now we should come to the case in which the curve recedes from one directrix and approaches the other, or where X recedes from A but Z approaches A, where the curve Y turns its concavity or convexity to both directrices; its convexity, as in Fig. 35], if the ratio of ${}_2X_3X$ to ${}_3X_4X$ receding from A is smaller than that of ${}_2Z_3Z$ to



[Fig. 35]

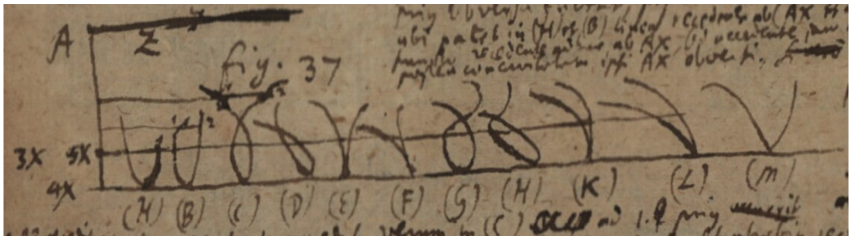


[Fig. 36]

${}_3Z_4Z$ approaching A, or if, while the velocities of recession in one directrix either increase or remain [constant] or decrease, the velocities of approaching in the other [directrix] increase less or decrease more. On the other hand, in Fig. 36] the curve turns its concavity to both directrices, if the ratio of ${}_2X_3X$ to ${}_3X_4X$ receding from A is greater than the ratio of ${}_2Z_3Z$ to ${}_3Z_4Z$ approaching A, or if, while the velocities of recession in one directrix either increase or remain [constant] or decrease, the velocities of approach in the other [directrix] increase less or decrease more.

From this one understands how it can happen that a curve which previously turned its convexity to the directrix now turns its concavity to it or vice versa, even if it does not have a contrary flexion but remains concave to the same sides, namely when a reversal occurs in that directrix, as in Fig. 37] if the motion of X is receding from A in ${}_3X_4X$ and approaching A in ${}_4X_5X$, where it is clear from (H), (B), (C), (D), (E), (F), (G), (K) how a reversal can occur in various ways, so that to the same line AX to which concavity was turned at first, afterward convexity is turned, or vice versa, where it is clear in (H) and (B) that, with the curve receding from AX and from AZ and in the point of reversal still receding from AX but now approaching AZ, it turns its convexity at first and its concavity afterward to AX; and the same in (B) where the curve approaches AX at first then always recedes from it, it approaches at 1, recedes at (B) and at 2; whereas it recedes from AZ up until (B) then approaches[23] it. But in (C) at 1 the concavity is turned to AX at first, then at 2 the

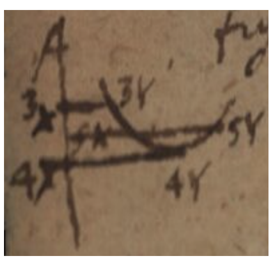
[23] MS has "recedit". Probably a slip.



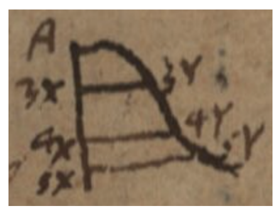
[Fig. 37]

convexity, and it recedes both times, which is obtained by means of a loop[24] that contains one regression with respect to AZ but two regressions with respect to AX. Such also in (D) which is placed at an incline. Moreover, (E) arises from (C), and (D) from (F), when the loop vanishes in a point, and so the reversals according to AZ as well as according to AX coincide there, whence in that point there can be infinitely many tangents, such as we have already touched upon above. But if the same loop contains a contrary flexion at the same time, as in (G) and (K), then when that loop vanishes so that (K) or (L) or (M) arises, and thus the contrary flexion coincides with the point of reversal, it happens that the curve turns its convexity or concavity to where it was at first without the reversal preventing it, since when a double cause for changing the facing[25] coalesces, they cancel each other and the facing remains such as it was previously with respect to the directrix AX, namely in (K), (L), (M) they turn their concavity to it after the regression as well as before; if they were inverted, they would turn their convexity to it after the regression as well as before.

One understands from this, besides, that there is a double cause for the curve changing its facing and what previously turned its concavity to the directrix AX now turning its convexity: one, a reversal of the point X moved in that directrix, (as in Fig. 38]) the curve



[Fig. 38]



[Fig. 39]

YY from ${}_3Y$ to ${}_4Y$ turns its convexity to AX, but after the regression at ${}_4Y$ it turns its concavity to it at ${}_5Y$, since the point X from ${}_3X$ to ${}_4X$ recedes from A, but it approaches A or regresses from ${}_4X$ to ${}_5X$; the other cause, of course, is a contrary flexion, when the curve itself actually becomes concave from convex[Z] or vice versa, as in Fig. 39] where the curve has a contrary flexion at ${}_4Y$, so that the tangent line, whereas previously it fell on one side of the curve, after ${}_4Y$ falls on the other side, but at the point ${}_4Y$ itself there is no tangent,[AA] or rather the tangent and a secant[AB] coincide, indeed (Fig. 40]) the tangent line intersects the curve endowed with a contrary flexion at L [and] the same [curve] somewhere else at M, and since L and M can be moved continuously closer and closer together, it comes about that they coincide at last at N, where there is no tangent, or rather in a certain respect the tangent and secant are the same thing at the same time, whence also in the point of contrary flexion three points of the curve, which elsewhere are distinct, coincide in one, two from the tangent (for every tangent is understood to intersect the curve in two coincident points), one from the secant. And it is apparent that LN and MN coincide at the point of flexion

[24]The Latin is "ventris", literally "belly" or "womb".
[25]MS: "obversio". The English "turns" is used in this vicinity to render the verb form, "obvertere".

[Fig. 40]

N of the two parts, just as if two distinct curves LNS, MNR with opposite convexities are tangent to each other at N, whence by crossing over from one to the other, the flexed LNM or the flexed RNS can be made.

Now from these two ways, distinct from each other, in which the facing of a curve to some directrix changes, we will be able to define a period inside which one understands there to be some maxima and minima, since when a curve has many contrary flexions and many points of reversal, it has distinct maxima and minima for each of its periods. Evidently the curve Y in Fig. 41] recedes from its directrix AX up until B, then approaches it again, and

[Fig. 41]

so the ordinate is maximal at B (if the curve turns its concavity to the directrix there); after that the curve approaches the directrix AX from B and simultaneously recedes from the directrix AZ up until C, where there is a point of reversal, or where it still approaches AX but no longer recedes from AZ; but from C (where the ordinate to AX is tangent to the curve) up until D it approaches the directrix AX and the directrix AZ simultaneously, where again it begins to recede from the directrix AX but still continues to approach AZ up until E, where it recedes from AZ again as well as from AX. Therefore, the points of reversal, which change the facing, make periods. Thus the first period is ABC, in which the curve turns its concavity to the directrix AX, of which period the maximal ordinate is at B; another period is CDE, where the curve turns its convexity to the directrix AX, and of which the minimal ordinate is at D. Furthermore the curve CDE, being drawn out, can intersect itself at F. And if the whole loop is understood to coincide in a point, then the double reversal with respect to the directrix AZ coincides with the simple one with respect to the directrix AX. And so, since the doubled reversals cancel each other, it can happen in this way that the curve $(Y)(B)(F)(G)$ (in the same Fig. 41]), which approached[26] the directrix AX from (B) up until (F), recedes from it again after (F), without any contrary flexion and likewise without any reversal with respect to the other, conjugate, directrix AZ; the alternation of these is, however, elsewhere necessary for a curve to recede again from a directrix which it had approached. But let us return to the first curve $AYBCDEFG$, and after two periods ABC and CDE, let us seek the third EGH from the most recent point of reversal E to the next point of contrary flexion H, of which period the maximal ordinate is at G. The fourth period is HJK from the point of contrary flexion H to the new point of reversal K, of which period the minimum is at the point J. Here it should be noted that, although two periods immediately [adjacent] to each other, of which each has its own maximum or minimum with respect to the same directrix AX, should be distinguished from each other either by some point of reversal with respect to the conjugate directrix AZ or by some point of contrary flexion on the curve itself, nevertheless neither a point of reversal of the conjugate directrix nor a point of contrary flexion immediately creates a period that has a maximum or a minimum, indeed not even multiple points of contrary flexion necessarily create a new period, as is clear from the serpentine KLM; however, multiple new points of reversal with respect to the conjugate directrix AZ do necessarily make a new period or new periods of maxima or minima for this directrix AX, if points of contrary flexion are absent from the curve. I demonstrate it like this: because the ordinates of points of reversal with respect to the conjugate directrix are maximal and minimal to the conjugate directrix, therefore if multiple points of reversal with respect to the conjugate directrix are given, then multiple such ordinates to the conjugate directrix are given, and therefore also [multiple] periods of maxima and minima for the conjugate directrix, since each maximum or minimum has its own period; now these periods with respect to the conjugate directrix AZ are necessarily delimited either by points of contrary flexion, or by points of reversal with respect to the first directrix AX, but points of contrary flexion are absent here by hypothesis, therefore points of reversal with respect to the directrix AX must be present, and so both maxima and minima and so also periods with respect to the directrix AX, which is what was claimed. Finally one should note that periods (with respect to the same directrix) are, as a rule, such that maxima and minima succeed each other alternately, but in certain cases there is an exception, such as on the curve $(Y)(B)(F)(G)$ in the same Fig. 41] two maxima immediately succeed each other, the ordinate from (B)[27] to AX and the ordinate from (G) to AX (unless we want to compute the ordinate from (F) at the same time, which, however, does not have its own period because it vanished), and the reason is that two points of reversal are implicit there or mutually suppress each other, which if they were understood to be expressed, and counted, then the rule of alternation would remain true. Similarly, it can happen that a point of reversal and a contrary flexion coincide, and in that way the alternation [happens]. As in the same figure if there were a new period $KLMNP$ from the point of reversal K to the point of contrary flexion P, and the maximal ordinate of this period is the one from N to the directrix AX, and again a new period PQR from the point of contrary flexion P to the point of reversal R, of which period the minimal[28] ordinate is the one from Q to the directrix AX,[29] from there again a new period RST from the point of reversal R to the point T (and what kind it is should be clear from the continuation of the curve), of which period the maximal ordinate is the one from S to the directrix AX. And up to now the alternation of maxima and minima is always preserved; but if we suppose that the whole loop $VPQRV$ vanishes in a point V,

[26] MS has "recessit ad", presumably a typo (based on Leibniz's figure).
[27] Leibniz omitted the parentheses here and in (G) and (F) next.
[28] Reading minimal for "maxima".
[29] MS breaks the sentence here, but the next sentence is fragmentary.

then the ordinate from V to AX could not be called a maximum or minimum of ordinates because the curve $NVST$ does not cross[30] but is tangent; therefore the maximal ordinate of the period MNV, namely from N in the direction of AX, is immediately succeeded by the maximal ordinate of the period VST, namely from S to the same directrix, because of course R and Q, the points of reversal and contrary flexion that coincide in one, mutually compensate each other and cancel.

And so we have scattered here some seeds from which certain general elements of curves spring forth, and the curves may be separated into certain fixed classes by their form. Many other things can be demonstrated from these principles, such as that the direction of the point describing a curve is the same as that of the tangent line; one could also expound the elements of curved lines that are described in a solid by the composition of three motions, when of course (Fig. 27]) one plane CD advances along another CB from CE toward BF, and in the plane CD[31] the ruler CG moves, approaches ED or recedes from there, and in the ruler CG the point C moves toward G or recedes from G. One can also draw from these things the manner of drawing the tangents of curves and finding maxima and minima; but we do not pursue this here, nor give a full treatment, just a certain foretaste and introduction.

So much for this time.

[30] Literally, "is not secant" [non secat].

[31] MS has CG for both the plane and the ruler, but they are inserted after the fact and likely the first was confused.

Notes

[A]Leibniz is approaching the concept of 'natural' or 'canonical' objects. These are mathematical objects which are uniquely determined in a structure or situation.

[B]Leibniz seems to assume that if $A.B.C$ is unique (meaning C is unique with its situs to $A.B$), then $A.B.C$ is unique (meaning B is unique with its situs to $A.C$), and similar permutations of the meaning. But this does not seem more obvious than his main claim (and it fails in other geometries), so it does not seem that this use of the axiom of determiners is successful.

[C]Leibniz expresses a concept close to the modern concept of topological boundary.

[D]Leibniz sometimes uses "congruent in actuality" [actu congrentia] as a synonym for "coincident".

[E]Latin: "seriem numerorum imparium continuatim intelligendo"

[F]Leibniz has labeled two different points both D in his picture.

[G]Here Leibniz uses the diagonal as a shorthand for the whole square.

[H]'Surd' has often been written 'irrational' in modern English. It was frequently used in particular for irrational roots of integers.

[I]One should consider how it is that semidetermined items can be distinguished, so that this construction does not violate the PII.

[J]"Per accidens" refers to the concept of accidental properties, those that do not follow from the definition ("by nature", "in general"). This may be the accidental properties of the number we want to take the square root of, namely positivity or being a rational square. Or it may be the accidental properties of the square root itself, that is, whether it is positive or negative, which does not follow from the property "squaring to a".

[K]Leibniz seems to refer here to a functional perspective on imaginary numbers. The "virtue" of an imaginary number may refer to its power to produce certain real numbers within the apparatus of the algebraic calculus.

[L]The Latin "ductu" is the ancestor of "product" and in Latin carries both the meanings of "drawing along" and "product" (meaning: multiplication by) in the modern sense. These are the same operation in geometry.

[M]This is known as Pappus's theorem.

[N]"A final magnitude can be obtained" may be referring to a limit in the mathematical sense rather than a last element, since he says the series goes on infinitely. The modern mathematical definition of limit would not be formalized until [YEAR?].

[O]"Different in number" meaning: differing only in which is which; they are not identical, not the same individual thing.

[P]The 'parameter' of a parabola is the semi-latus rectum, the distance from focus to directrix.

[Q]Refers to the smallest parts, which don't quite exist for the abstract continuum so "quasi-minimia", which probably are infinitesimals, are needed. The minima are minimal things, meaning that they should not strictly contain smaller things.

[R]The curve of intersections collapses to a point when it becomes tangent.

[S]The reason given here does not seem to make sense.

[T]Leibniz has the idea of a moduli space.

[U]Leibniz has some concept of decompositions of moduli spaces.

[V]Leibniz reduces temporal continuity to spatial continuity.

[W]This probably indicates that the point should be considered to be over the plane to which a perpendicular is dropped.

[X]Elsewhere Leibniz has used "semidetermined" for this situation.

[Y]The concept of "contrary flexion" is closely related to the modern inflection point. It seems that "contrary flexion" refers most consistently to points where the curvature switches signs. The modern inflection point refers to the point on the graph of a function where the second derivative switches signs. These concepts frequently agree.

[Z]This seems to mean the sign of the curvature (or side with the osculating circle) switches, which fact is independent of a choice of directrix.

[AA]The sense of "tangent" here in use implies touching without crossing. A secant, then, is a crossing line.

[AB]Leibniz seems to use *tangens* and *una secans* here metonymically referring to the points where the lines are tangent and secant, not the lines themselves. In the subsequent illustration, as the points L and M move together, the (single) line between them, intersecting the curve as tangent at L and as secant at M, converges to the tangent line at N.