W13 Notes

Parametric curves

01 Theory

Parametric curves are curves traced by the path of a 'moving' point. An independent parameter, such as t for 'time', controls *both x and y* values through **Cartesian coordinate functions** $x(t)$ and $y(t)$. The coordinates of the moving point are $\big(x(t),\, y(t)\big)$.

Parametric curve

A **parametric curve** is a function from parameter space $\mathbb{R}$ to the plane $\mathbb{R}^2$ given in terms of coordinate functions:

$$t \longmapsto \big(\, x(t),\, y(t)\,\big)$$

Other notations

Be aware that sometimes the coordinate functions are written with f and g (or yet other letters) like this: $(x, y) = \big(\, f(t),\, g(t)\,\big)$

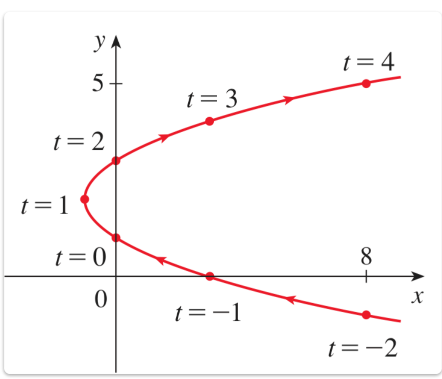
Or simply equating coordinate letters with functions: $x = f(t), \quad y = g(t)$

Sometimes a different parameter is used, like s or u.

For example, suppose:

$$x = t^2 - 2t, \qquad y = t + 1$$

The curve traced out is a parabola that opens horizontally:



Given a parametric curve, we can create an equation satisfied by x and y variables by solving for t in either coordinate function (inverting either f or g) and plugging the result into the other function.

In the example:

$$y = t + 1 \quad \gg\gg \quad t = y - 1$$

$$\gg\gg \quad x = t^2 - 2t \quad \gg\gg \quad x = (y-1)^2 - 2(y-1)$$

$$\gg\gg \quad x = y^2 - 4y + 3 \quad \gg\gg \quad x = (y-2)^2 - 1$$

This is the equation of a parabola centered at $(-1, 2)$ that opens to the right.

Image of a parametric curve

The **image** of a parametric curve is the *set* of output points $\big(x(t),\, y(t)\big)$ that are traversed by the moving point.

A parametric curve has *hidden information* that isn't contained in the image:

- The *time values* t when the moving point is found in various locations.
- The *speed* at which the curve is traversed.
- The *direction* in which the curve is traversed.

We can **reparametrize** a parametric curve to use a different parameter or different coordinate functions while leaving the *image unchanged*.

In the previous example, shift t by 1:

$$x = (t+1)^2 - 2(t+1), \qquad y = (t+1) + 1$$

$$\gg\gg \qquad x = t^2 - 1, \qquad y = t + 2$$

Since the parameter t and the parameter $t+1$ both cover the same values for $t \in (-\infty, \infty)$, the same curve is traversed. But the moving point in the second, shifted version reaches any given location *one unit earlier* in time. (When $t = -1$ in the second version, the input to $x(t)$ and $y(t)$ is the same as when $t = 0$ in the first one.)

02 Illustration

Example - Parametric circles

The standard equation of a circle of radius R centered at the point (h, k):

$$(x-h)^2 + (y-k)^2 = R^2$$

This equation says that the *distance* from a point (x, y) on the circle to the center point (h, k) equals R. This fact defines the circle.

Parametric coordinates for the circle:

$$x = h + R\cos t, \qquad y = k + R\sin t, \qquad t \in [0, 2\pi)$$

For example, the unit circle $x^2 + y^2 = 1$ is parametrized by $x = \cos t$ and $y = \sin t$.

☰ Example - Parametric lines

Parametric coordinate functions for a line:

$$x = a + rt, \qquad y = b + st, \qquad t \in (-\infty, +\infty)$$

Compare this to the graph of linear function:

$$y = mx + b \qquad \text{some } m, b$$

Vertical lines cannot be described as the graph of a function. We must use $x = a$.

Parametric lines can describe all lines equally well, including horizontal and vertical lines.

A vertical line $x = a$ is achieved by setting $s = 0$ and $r \neq 0$.

A horizontal line $y = b$ is achieved by setting $r = 0$ and $s \neq 0$.

A non-vertical line $y = mx + b$ may be achieved by setting $s = m$ and $r = 1$, and $a = 0$.

Assuming that $r \neq 0$, the parametric coordinate functions describe a line satisfying:

$$y = b + s\left(\frac{x-a}{r}\right)$$

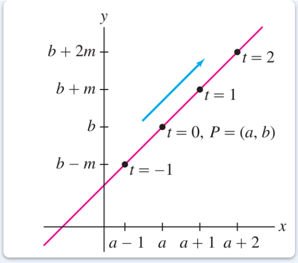
$$\gg\gg \quad y = \frac{s}{r} \cdot x + \left(b - \frac{s}{r} \cdot a\right)$$

and therefore the slope is $m = \frac{s}{r}$ and the y-intercept is $b - \frac{s}{r} \cdot a$.

The point-slope construction of a line has a parametric analogue:

point-slope line:

$$y - a = m(x - b) \qquad\qquad (x, y) = (a + t,\ b + mt)$$



Example - Parametric ellipses

The general equation of an ellipse centered at (h, k) with half-axes a and b is:

$$\left(\frac{x-h}{a}\right)^2 + \left(\frac{y-k}{b}\right)^2 = 1$$

This equation represents a *stretched unit circle*:

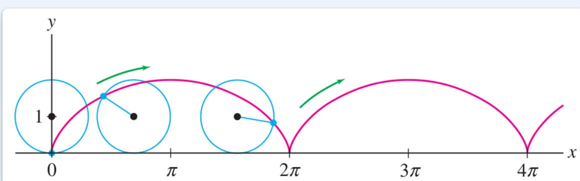
- by a in the x-axis
- by b in the y-axis

Parametric coordinate functions for the general ellipse:

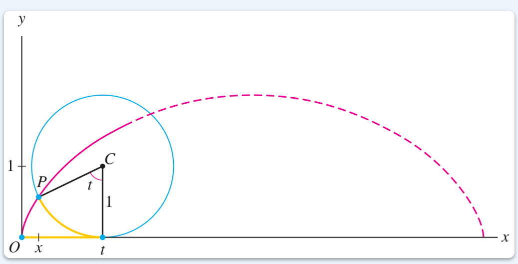
$$x = h + a\cos t, \qquad y = k + b\sin t, \qquad t \in [0, 2\pi)$$

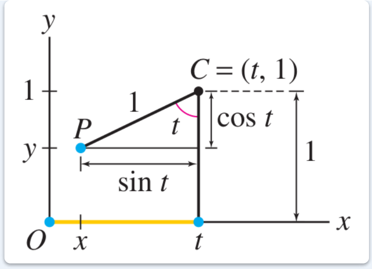
Example - Parametric cycloids

The cycloid is the curve traced by a pen attached to the rim of a wheel as it rolls.



It is easy to describe the cycloid parametrically. Consider the geometry of the situation:





The center C of the wheel is moving rightwards at a constant speed of 1, so its position is $(t, 1)$. The angle is revolving at the same constant rate of 1 (in *radians*) because the *radius* is 1.

The triangle shown has base $\sin t$, so the x coordinate is $t - \sin t$. The y coordinate is $1 - \cos t$.

So the coordinates of the point $P = (x, y)$ are given parametrically by:

$$x = t - \sin t, \qquad y = 1 - \cos t, \qquad t > 0$$

If the circle has another radius, say R, then the parametric formulas change to:

$$x = Rt - R\sin t, \qquad y = R - R\cos t, \qquad t > 0$$

Calculus with parametric curves

03 Theory - Slope, concavity

We can use $x(t)$ and $y(t)$ data to compute the slope of a parametric curve in terms of t.

Slope formula

Given a parametric curve $\big(x(t), y(t)\big)$, its slope satisfies:

$$\frac{dy}{dx} \quad = \quad \frac{y'(t)}{x'(t)} \qquad (\text{where } x'(t) \neq 0)$$

Concavity formula

Given a parametric curve $\big(x(t), y(t)\big)$, its concavity satisfies the formula:

$$\frac{d^2y}{dx^2} \quad = \quad \frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right) \cdot \frac{1}{x'(t)} \qquad (\text{where } x'(t) \neq 0)$$

Extra - Derivation of slope and concavity formulas

For both derivations, it is necessary to view t as a function of x through the inverse parameter function. For example if $x = f(t)$ is the parametrization, then $t = f^{-1}(x)$ is the inverse parameter function.

We will need the derivative $\frac{dt}{dx}$ in terms of t. For this we use the formula for derivative of inverse functions:

$$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$$

Given all this, both formulas are simple applications of the chain rule.

For the slope:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} \qquad \gg\gg \qquad y'(t) \cdot \frac{1}{dx/dt}$$

$$\gg\gg \qquad \frac{y'(t)}{x'(t)}$$

For the concavity:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \qquad \gg\gg \qquad \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$$

$$\gg\gg \qquad \frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right) \cdot \frac{1}{x'(t)}$$

(In the second step we inserted the formula for $\frac{dy}{dx}$ from the slope.)

Pure vertical, Pure horizontal movement

In view of the formula $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$, we see:

- Pure vertical: when $x'(t) = 0$ and yet $y'(t) \neq 0$
- Pure horizontal: when $y'(t) = 0$ and yet $x'(t) \neq 0$

When $x'(t_0) = y'(t_0) = 0$ for the same $t = t_0$, we have a **stationary point**, which might subsequently progress into pure vertical, pure horizontal, or neither.

04 Illustration

Example - Tangent to a cycloid

Find the tangent line (described parametrically) to the cycloid $(4t - 4\sin t,\, 4 - 4\cos t)$ when $t = \pi/4$.

Solution

Compute x' and y'.

Find $x'(t)$:

$$x(t) = 4t - 4\sin t \quad \gg\gg \quad x'(t) = 4 - 4\cos t$$

Find $y'(t)$:

$$y(t) = 4 - 4\cos t \quad \gg\gg \quad y'(t) = 4\sin t$$

Plug in $t = \pi/4$:

$$x'(\pi/4) \quad \gg\gg \quad 4 - 4\cos(\pi/4) \quad \gg\gg \quad 4 - 2\sqrt{2}$$

Plug in $t = \pi/4$:

$$y'(\pi/4) \quad \gg\gg \quad 4\sin(\pi/4) \quad \gg\gg \quad 2\sqrt{2}$$

Apply formula: $\dfrac{dy}{dx} = \dfrac{y'}{x'}$:

Calculate $\frac{dy}{dx}$ at $t = \pi/4$:

$$\frac{dy}{dx}(\pi/4) = \frac{y'(\pi/4)}{x'(\pi/4)} \qquad \gg\gg \qquad \frac{2\sqrt{2}}{4 - 2\sqrt{2}}$$

Simplify:

$$\gg\gg \qquad \frac{2\sqrt{2}}{4 - 2\sqrt{2}} \cdot \frac{4 + 2\sqrt{2}}{4 + 2\sqrt{2}}$$

$$\gg\gg \qquad \frac{8\sqrt{2} + 8}{16 - 8} \quad \gg\gg \quad \sqrt{2} + 1$$

So:

$$\left.\frac{dy}{dx}\right|_{t=\pi/4} = \sqrt{2} + 1$$

This is the slope m for our line.

Need the point P for our line. Find (x, y) at $t = \pi/4$.

Plug $t = \pi/4$ into parametric formulas:

$$\big(x(t),\, y(t)\big)\Big|_{t=\pi/4} \quad \gg\gg \quad \left(4\frac{\pi}{4} - 4\sin(\pi/4),\ 4 - 4\cos(\pi/4)\right)$$

$$\gg\gg \quad \left(\pi - 2\sqrt{2}, 4 - 2\sqrt{2}\right)$$

Point-slope formulation of tangent line:

$$x = a + t, \quad y = b + mt$$

Inserting our data:

$$x \ = \ (\pi - 2\sqrt{2}) + t, \qquad y \ = \ (4 - 2\sqrt{2}) + (\sqrt{2} + 1)t$$

☷ Example - Vertical and horizontal tangents of the circle

Consider the circle parametrized by $x = \cos t$ and $y = \sin t$. Find the points where the tangent lines are vertical or horizontal.

Solution

For the points with vertical tangent line, we find where the moving point has $x'(t) = 0$ (purely vertical motion):

$$x'(t) = -\sin t,$$

$$x'(t) = 0 \qquad \gg\gg \qquad -\sin t = 0 \qquad \gg\gg \qquad t = 0, \pi$$

The moving point is at $(1, 0)$ when $t = 0$, and at $(-1, 0)$ when $t = \pi$.

For the points with horizontal tangent line, we find where the moving point has $y'(t) = 0$ (purely horizontal motion):

$$y'(t) = \cos t,$$

$$y'(t) = 0 \qquad \gg\gg \qquad \cos t = 0$$

$$\gg\gg \qquad t = \frac{\pi}{2}, \frac{3\pi}{2}$$

The moving point is at $(0, 1)$ when $t = \pi/2$, and at $(0, -1)$ when $t = 3\pi/2$.

Example - Finding the point with specified slope

Consider the parametric curve given by $(x, y) = (t^2, t^3)$. Find the point where the slope of the tangent line to this curve equals 5.

Solution

Compute the derivatives:

$$x'(t) \;=\; 2t, \qquad y'(t) \;=\; 3t^2$$

Therefore the slope of the tangent line, in terms of t:

$$m = \frac{dy}{dx} \;=\; \frac{y'(t)}{x'(t)}$$

$$\gg\gg \quad \frac{3t^2}{2t} \quad \gg\gg \quad \frac{3}{2}t$$

Set up equation:

$$m \quad = \quad 5$$

$$\frac{3}{2}t \quad = \quad 5$$

Solve. Obtain $t = \frac{10}{3}$.

Find the point:

$$(x,y)\Big|_{t=10/3} \quad \gg\gg \quad \left(\frac{100}{9}, \frac{1000}{27}\right)$$

05 Theory - Arclength

Arclength formula

The **arclength** of a parametric curve with coordinate functions $x(t)$ and $y(t)$ is:

$$L = \int_a^b \sqrt{(x')^2 + (y')^2}\, dt$$

This formula assumes the curve is traversed one time as t increases from a to b.

⚠ Counts total traversal

This formula applies when the curve image is traversed *one time* by the moving point.

Sometimes a parametric curve traverses its image with repetitions. The arclength formula would add length from each repetition!

Extra - Derivation of arclength formula

The arclength of a parametric curve is calculated by integrating the infinitesimal arc element:

$$ds = \sqrt{dx^2 + dy^2}$$

$$L = \int_a^b ds$$

In order to integrate ds in the t variable, as we must for parametric curves, we convert ds to a function of t:

$$ds = \sqrt{dx^2 + dy^2} \qquad \gg\gg \qquad \sqrt{\frac{1}{dt^2}\cdot(dx^2+dy^2)\cdot dt^2}$$

$$\gg\gg \qquad \sqrt{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}}\cdot\sqrt{dt^2} \qquad \gg\gg \qquad \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\gg\gg \qquad ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

So we obtain $ds = \sqrt{(x')^2 + (y')^2}\, dt$ and the arclength formula follows from this:

$$L = \int_a^b \sqrt{(x')^2 + (y')^2}\, dt$$

06 Illustration

Example - Perimeter of a circle

The perimeter of the circle $(R\cos t, R\sin t)$ is easily found. We have $(x', y') = (-R\sin t, R\cos t)$, and therefore:

$$(x')^2 + (y')^2 = (-R\sin t)^2 + (R\cos t)^2$$

$$\gg\gg \qquad R^2\sin^2 t + R^2\cos^2 t \qquad \gg\gg \qquad R^2$$

$$ds = \sqrt{(x')^2 + (y')^2}\, dt = R\, dt$$

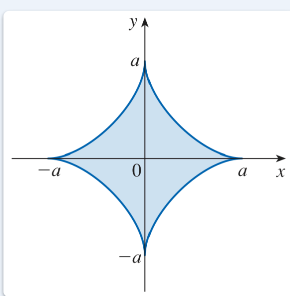
Integrate around the circle:

$$\text{Perimeter} \quad = \quad \int_0^{2\pi} ds \qquad \gg\gg \qquad \int_0^{2\pi} R\, dt$$

$$\gg\gg \qquad Rt\Big|_0^{2\pi} = 2\pi R$$

Example - Perimeter of an asteroid

Find the perimeter length of the 'asteroid' given parametrically by $(x, y) = \left(a\cos^3\theta,\ a\sin^3\theta\right)$ for $a = 2$.



Solution

Notice: Throughout this problem we use the parameter θ instead of t. This does *not* mean we are using polar coordinates!

Compute the derivatives in θ:

$$\left(x', y'\right) = \left(3a\cos^2\theta\sin\theta,\ 3a\sin^2\theta\cos\theta\right)$$

Compute the infinitesimal arc element.

$$(x')^2 + (y')^2 = 9a^2 \cos^4\theta \sin^2\theta + 9a^2 \sin^4\theta \cos^2\theta$$

$$\gg\gg \qquad 9a^2 \sin^2\theta \cos^2\theta \left(\cos^2\theta + \sin^2\theta\right)$$

$$\gg\gg \qquad 9a^2 \sin^2\theta \cos^2\theta$$

Plug into the arc element, simplify:

$$ds = \sqrt{(x')^2 + y')^2}\, d\theta = \sqrt{9a^2 \sin^2\theta \cos^2\theta}\, d\theta$$

$$\gg\gg \qquad ds = 3a\big|\sin\theta\cos\theta\big|\, d\theta$$

Bounds of integration?

Easiest to use $\theta \in [0, \pi/2]$. This covers one edge of the asteroid. Then multiply by 4 for the final answer.

On the interval $\theta \in [0, \pi/2]$, the factor $3a \sin\theta \cos\theta$ is *positive*. So we can drop the absolute value and integrate directly.

> ⚠ **Absolute values matter!**
>
> If we tried to integrate on the whole range $\theta \in [0, 2\pi]$, then $3a \sin\theta \cos\theta$ really does change sign.
>
> To perform integration properly with these absolute values, we'd need to convert to a piecewise function by adding appropriate minus signs.

Integrate the arc element:

$$\int_0^{\pi/2} ds \quad \gg\gg \quad \int_0^{\pi/2} 3a \sin\theta \cos\theta \, d\theta$$

$$\gg\gg \quad 3a \int_{u=0}^{1} u\, du \qquad\qquad (u = \sin\theta)$$

$$\gg\gg \quad 3a \frac{u^2}{2}\bigg|_0^1 \quad \gg\gg \quad \frac{3a}{2}$$

Finally, multiply by 4 to get the total perimeter: L=6a

07 Theory - Distance, speed

Distance function

The **distance function** $s(t)$ returns the total distance traveled by the particle from a chosen starting time t_0 up to the (input) time t:

$$s(t) \;=\; \int_{t_0}^{t} ds \quad = \quad \int_{t_0}^{t} \sqrt{x'(u)^2 + y'(u)^2}\, du$$

We need the dummy variable u so that the integration process does not conflict with t in the upper bound.

Speed function

The **speed** of a moving particle is the *rate of change of distance*:

$$v(t) \;=\; s'(t) \quad = \quad \sqrt{x'(t)^2 + y'(t)^2}$$

This formula can be explained in either of two ways:

1. Apply the Fundamental Theorem of Calculus to the integral formula for $s(t)$.
2. Consider $ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt$ for a small change dt: so the *rate of change* of arclength is $\frac{ds}{dt}$, in other words $s'(t)$.

08 Illustration

Example - Speed, distance, displacement

The parametric curve $\left(t,\, \frac{2}{3}t^{3/2}\right)$ describes the position of a moving particle (t measuring seconds).
(a) What is the speed function?

Suppose the particle travels for 8 seconds starting at $t = 0$.
(b) What is the total distance traveled?
(c) What is the total displacement?

Solution

(a)
Compute *derivatives*:

$$(x',\, y') = \left(1,\, t^{1/2}\right)$$

Compute the *speed*.

Find sum of squares:

$$(x')^2 + (y')^2 = 1 + (t^{1/2})^2 = 1 + t$$

Get the speed function:

$$v(t) = \sqrt{(x')^2 + (y')^2} = \sqrt{1+t}$$

(b)

Distance traveled by using *speed*.

Compute total distance traveled function:

$$s(t) = \int_{u=0}^{t} \sqrt{1+u}\, du$$

Integrate.

Substitute $w = 1 + u$ and $dw = du$.

New bounds are 1 and $1 + t$.

Calculate:

$$\gg\gg \quad \int_1^{1+t} \sqrt{w}\, dw$$

$$\gg\gg \quad \frac{2}{3} w^{3/2} \Big|_1^{1+t} \quad \gg\gg \quad \frac{2}{3}\left((1+t)^{3/2} - 1\right)$$

Insert $t = 8$ for the answer.

The distance traveled up to $t = 8$ is:

$$s(8) = \frac{2}{3}\left(9^{3/2} - 1\right) \quad \gg\gg \quad \frac{2}{3}(27 - 1) \quad \gg\gg \quad \frac{52}{3}$$

This is our final answer.

(c)

Displacement formula: $d = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$

Pythagorean formula for distance between given points.

Compute starting and ending points.

For starting point, insert $t = 0$:

$$\big(x(t), y(t)\big)\Big|_{t=0} \qquad \gg\gg \qquad \left(t, \frac{2}{3} t^{3/2}\right)\Bigg|_{t=0} \qquad \gg\gg \qquad (0, 0)$$

For ending point, insert $t = 8$:

$$\left(x(t), y(t)\right)\Big|_{t=8} \quad \gg\gg \quad \left(t, \frac{2}{3}t^{3/2}\right)\Big|_{t=8}$$

$$\gg\gg \quad \left(8, \frac{2}{3}8^{3/2}\right) \quad \gg\gg \quad \left(8, \frac{32\sqrt{2}}{3}\right)$$

Plug points into distance formula.

Insert $(0,0)$ and $\left(8, 32\sqrt{2}/3\right)$:

$$\sqrt{8^2 + \left(\frac{32\sqrt{2}}{3}\right)^2} \quad \gg\gg \quad \sqrt{64 + \frac{2048}{9}}$$

$$\gg\gg \quad \frac{\sqrt{2624}}{3}$$

This is our final answer.

09 Theory - Surface area of revolutions

Surface area of a surface of revolution: thin bands

Suppose a parametric curve $\big(x(t),\, y(t)\big)$ is revolved around the x-axis or the y-axis.

The surface area is:

$$A \quad = \quad \int_a^b 2\pi R(t)\, \sqrt{(x')^2 + (y')^2}\, dt$$

The radius $R(t)$ should be the distance to the axis:

$$\begin{aligned} R(t) &= y(t) \qquad \text{revolution about } x\text{-axis} \\ R(t) &= x(t) \qquad \text{revolution about } y\text{-axis} \end{aligned}$$

This formulas adds the areas of thin bands, but the bands are demarcated using parametric functions instead of input values of a graphed function.

The formula assumes that the curve is traversed one time as t increases from a to b.

10 Illustration

Example - Surface of revolution - parametric circle

By revolving the unit upper semicircle about the x-axis, we can compute the surface area of the unit sphere.

The parametrization of the unit upper semicircle is: $(x, y) = (\cos t,\, \sin t)$.

The derivative is: $(x', y') = (-\sin t,\, \cos t)$.

Therefore, the arc element:

$$ds = \sqrt{(x')^2 + (y')^2}\, dt$$

$$\gg\gg \quad \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt \quad \gg\gg \quad dt$$

Now for R we choose $R = y(t) = \sin t$ because we are revolving about the x-axis.

Plugging all this into the integral formula and evaluating gives:

$$A = \int_0^\pi 2\pi \sin t\, dt \quad \gg\gg \quad -2\pi \cos t\Big|_0^\pi \quad \gg\gg \quad 4\pi$$

Notice: This method is a little easier than the method using the graph $y = \sqrt{1 - x^2}$.

Example - Surface of revolution - parametric curve

Set up the integral which computes the surface area of the surface generated by revolving about the x-axis the curve $(t^3,\, t^2 - 1)$ for $0 \le t \le 1$.

Solution

For revolution about the x-axis, we set $R = y(t) = t^2 - 1$.

Then compute ds:

$$ds = \sqrt{(x')^2 + (y')^2} \quad \gg\gg \quad \sqrt{(3t^2)^2 + (2t)^2} \quad \gg\gg \quad \sqrt{9t^4 + 4t^2}$$

$$\gg\gg \quad \sqrt{t^2(9t^2 + 4)} \quad \gg\gg \quad t\sqrt{9t^2 + 4}$$

Therefore the desired integral is:

$$A = \int_0^1 2\pi R\, ds \quad \gg\gg \quad \int_0^1 2\pi(t^2 - 1)t\sqrt{9t^2 + 4}\, dt$$