ĐẠI HỌC BÁCH KHOA – ĐẠI HỌC ĐÀ NẴNG
KHOA CÔNG NGHỆ THÔNG TIN

Môn học

Toán Ứng Dụng
Applied mathematics

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Bài 1:
Basic Number Theory

Basic Number Theory-1

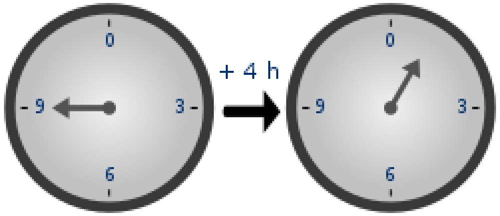
Introduction

- This article discusses topics that are frequently used to solve programming problems based on math. It includes the following topics:
  - Modular arithmetic (Số học mô đun)
  - Modular exponentiation (Phép luỹ thừa mô đun)
  - Greatest Common Divisor (GCD)
  - Extended Euclidean algorithm (Giải thuật Euclid mở rộng)
  - Modular multiplicative inverse (Nghịch đảo mô đun)

Modular arithmetic (Số học mô đun)

1. Modular arithmetic

- In mathematics, modular arithmetic is a system of arithmetic for integers, where numbers "wrap around" upon reaching a certain value—the modulus (plural moduli)



Time-keeping on this clock uses arithmetic modulo 12.

- A familiar use of modular arithmetic is in the 12-hour clock, in which the day is divided into two 12-hour periods.

1. Modular arithmetic

- When one number is divided by another, the modulo operation finds the remainder. It is denoted by the % symbol.
- Example
  - Assume that you have two numbers 5 and 2. 5%2 is 1 because when 5 is divided by 2, the remainder is 1.

1. Modular arithmetic

- Properties
  - $(a+b)\%c = ((a\%c) + (b\%c))\%c$
  - $(a*b)\%c = ((a\%c) * (b\%c))\%c$
  - $(a-b)\%c = ((a\%c) - (b\%c)+c)\%c$
  - $(a/b)\%c = ((a\%c) * (b^{-1}\%c))\%c$
  - Note: In the last property, b^{-1} is the multiplicative modulo inverse of b and c.
- Examples
  - If a=5, b=3, and c=2, then:
    - $(5+3)\%2 = 8\%2 = 0$; Similarly, $(5\%2+3\%2)\%2 = (1+1)\%2 = 0$.
    - $(5*3)\%2 = 15\%2 = 1$; Similarly, $(5\%2*3\%2)\%2 = (1*1)\%2 = 1$.

1. Modular arithmetic

- If a=12, b=15, and c=4, then the answer in some languages is
  - (12-15)%4 = (12%4 - 15%4)%4 = (0 - 3)%4 = -3.
- However, the answer of the % operator cannot be negative.
- Therefore, to make the answer positive, add c to the formula and compute it as follows:
  - (12-15)%4 = (12%4 - 15%4 **+ 4**)%4 = (0 – 3 + 4)%4 = 1.

When are these properties used?

- Assume that a = 10^{18}, b = 10^{18}, and c = 10^9+7. You have to find $(a*b)\%c$.

- When you multiply a with b, the answer is **10^{36}**, which does not conform with the standard integer data types. Therefore, to avoid this we used the properties.
  - $(a*b)\%c = ((a\%c)*(b\%c))\%c = (49*49)\%(10^9+7) = 2401$

Modular exponentiation (Luỹ thừa mô đun)

Modular exponentiation

- Exponentiation is a mathematical operation that is expressed as x^n and computed as
  - $x^n = x \cdot x \cdot \ldots \cdot x$ (n times).

- Basic method
  - While calculating x^n, the most basic solution is broken down into $x \cdot x^{n-1}$. The new problem is x^{n-1}, which is similar to the original problem. Therefore, like in original problem, it is further broken down to $x \cdot x \cdot x^{n-2}$.

Modular exponentiation

- Basic method (con't)
  - This is a recursive way of determining the answer to x^n. However, sometimes an equation cannot be broken down any further as in the case of n=0. A C++ code for this solution, considering n≥0 is as follows:

```
int recursivePower(int x,int n)
{
    if(n==0)
        return 1;
    return x*recursivePower(x,n-1);
}
```

Modular exponentiation

- Basic method (con’t)
  - The recursive method aligns with the explanation, however, the solution can also be written in an iterative format, which is quite ad hoc. A variable 'result', to which x is multiplied for n number of times, is maintained. The iterative code is as follows:

```
int iterativePower(int x,int n) {
    int result=1;
    while(n>0) {
        result=result*x;
        n--;
    }
    return result;
}
```

Modular exponentiation

- Time complexity
  - With respect to time complexity, it is a fairly efficient *O(n)* solution. However, when it comes to finding x^n, where n can be as large as **10^{18}**, this solution will not be suitable.
- Optimized method
  - While calculating x^n, the basis of Binary Exponentiation relies on whether n is odd or even.
  - If n is even, then x^n can be broken down to $(x^2)^{n/2}$. Programmatically, finding x^2 is a one-step process. However, the problem is to find $(x^2)^{n/2}$.

Modular exponentiation

- Optimized method (con't)
  - Notice how the computation steps were reduced from n to n/2 in just **one** step? You can continue to divide the power by 2 as long as it is even.
  - When n is odd, try and convert it into an even value. x^n can be written as $x \cdot x^{n-1}$. This ensures that $n-1$ is even.
    - If n is even, replace x^n by $(x^2)^{n/2}$.
    - If n is odd, replace x^n by $x \cdot x^{n-1}$. $n-1$ becomes even and you can apply the relevant formula.

Modular exponentiation

- Example
  - You are required to compute 3^{10}. The steps are as follows:
    - **Step 1**: The power of 3 is 10, which is even. Break it down as follows: $3^{10} \rightarrow (3^2)^5 \rightarrow 9^5$
    - **Step 2**: Find 9^5. The power of 9 is 5, which is odd. Convert it into an even power and then apply the following formula:
      $$9^5 \rightarrow 9 \cdot 9^4 \rightarrow 9 \cdot (9^2)^2 \rightarrow 9 \cdot (81^2)$$
    - **Step 3**: 81^2 is a one-step computation process
  - The result is $9 \cdot 81 \cdot 81 = 59049$.
  - This is an efficient method and the *ten-step process* of determining 3^{10} is reduced to a *three-step process*.

Modular exponentiation

- At every step, n is divided by 2. Therefore, the time complexity is *O(log N)*.
- The code for the process is as follows:

```
int binaryExponentiation(int x,int n){
    if(n==0)
        return 1;
    else if(n%2 == 0)        //n is even
        return binaryExponentiation(x*x,n/2);
    else                     //n is odd
        return x*binaryExponentiation(x*x,(n-1)/2);
}
```

Modular exponentiation

- An iterative version of this method is as follows:

```
int binaryExponentiation(int x,int n){
    int result=1;
    while(n>0)
    {
        if(n % 2 ==1)
            result=result * x;
        x=x*x;
        n=n/2;
    }
    return result;
}
```

Modular exponentiation

- However, storing answers that are too large for their respective datatypes is an issue with this method. In some languages the answer will exceed the range of the datatype while in other languages it will timeout due to large number multiplications. In such instances, you must use modulus (%). Instead of finding x^n, you must find (x^n) % m.

Modular exponentiation

- For example, run the implementation of the method to find 2^{1e+9}. The O(N) solution will timeout, while the O(logN) solution will run in time but it will produce garbage values.
- To fix this you must use the modulo operation i.e. % M in those lines where a temporary answer is computed.

Modular exponentiation

- The recursive method:

```
int modularExponentiation(int x,int n,int M){
    if(n==0)
        return 1;
    else if(n%2 == 0)        //n is even
        return modularExponentiation((x*x)%M,n/2,M);
    else                                   //n is odd
        return (x*modularExponentiation((x*x)%M,(n-1)/2,M))%M;

}
```

Modular exponentiation

- Similarly, the iterative binary exponentiation method can be modified as follows:

```
int modularExponentiation(int x,int n,int M){
    int result=1;
    while(n>0){
        if(n % 2 ==1)
            result=(result * x)%M;
        x=(x*x)%M;
        n=n/2;
    }
    return result;
}
```

Solution analysis

- Recursive solution analysis
  - Time complexity: O(log N)
  - Memory complexity: O(log N) because a function call consumes memory and log N recursive function calls are made
- Iterative solution analysis
  - Time complexity: O(log N)
  - Memory complexity: O(1)

Greatest Common Divisor (GCD)

Greatest Common Divisor (GCD)

- The GCD of two or more numbers is the largest positive number that divides all the numbers that are considered.

- For example, the GCD of 6 and 10 is 2 because it is the largest positive number that can divide both 6 and 10.

Naive approach

- Traverse all the numbers from min(A, B) to 1 and check whether the current number divides both A and B. If yes, it is the GCD of A and B.

```
int GCD(int A, int B) {
    int m = min(A, B), gcd;
    for(int i = m; i > 0; --i)
        if(A % i == 0 && B % i == 0) {
            gcd = i;
            return gcd;
        }
}
```

- The time complexity of this function is
  - O(min(A, B)).

Euclid's algorithm

- The idea behind this algorithm is GCD(A,B) = GCD(B,A%B). It will recurse until A%B=0.

```
int GCD(int A, int B) {
    return (B?0) A : GCD(B, A % B);
}
```

- Example
  - If a = 16 and B = 10, then the GCD is computed:
    - GCD(16, 10) = GCD(10, 16 % 10) = GCD(10, 6)
    - GCD(10, 6) = GCD(6, 10 % 6) = GCD(6, 4)
    - GCD(6, 4) = GCD(4, 6 % 4) = GCD(4, 2)
    - GCD(4, 2) = GCD(2, 4 % 2) = GCD(2, 0)
  - Since B = 0, GCD(2,0) will return 2.
- The time complexity is O(log(max(A, B))).

Extended Euclidean algorithm (Giải thuật Euclid mở rộng)

Extended Euclidean algorithm

- This algorithm is an extended form of Euclid's algorithm. GCD(A,B) has a special property so that it can always be represented in the form of an equation i.e. Ax+By=GCD(A,B).
- The coefficients (x and y) of this equation will be used to find the modular multiplicative inverse. The coefficients can be zero, positive or negative in value.
- This algorithm takes two inputs as A and B and returns GCD(A,B) and coefficients of the above equation as output.

Extended Euclidean algorithm

- Example
  - If A=30 and B=20, then $30*(1)+20*(-1)=10$ where 10 is the GCD of 20 and 30.
- Key idea
  - $A.x+B.y=GCD(A,B)$ --- (1)
  - GCD(A,B)=GCD(B,A%B). Therefore, we can write the equation as follows: $B.x_1+ (A \% B).y_1=GCD(A,B)$ ---(2)
  - We can write $A\%B=A-B*\lfloor A/B \rfloor$ where $\lfloor\rfloor$ means floor value. B and substitute it in equation 2. Your equation will be as follows: $B.x_1+ (A - \lfloor A/B \rfloor.B).y_1=GCD(A,B)$
  - Thus, $B.(x_1 - \lfloor A/B \rfloor.y_1)+A.y_1=GCD(A,B)$. ---(3)

Extended Euclidean algorithm

- From:
  - A.x+B.y=GCD(A,B) --- (1)
  - $B.(x_1 - \lfloor A/B \rfloor.y_1)+A.y_1=GCD(A,B)$. ---(3)
- We get:
  - $x=y_1$
  - $y=x_1 - \lfloor A/B \rfloor.y_1$
- These equations are key in understanding the extended Euclidean algorithm.
- Recursive calls are made to GCD(B, A%B). The values that are returned from recursive calls are x_1 and y_1, which are used to get x and y.

Implementation

```
int d, x, y;
void extendedEuclid(int A, int B) {
    if(B == 0) {
        d = A;
        x = 1;
        y = 0;
    }
    else {
        extendedEuclid(B, A%B);
        int temp = x;
        x = y;
        y = temp - (A/B)*y;
    }
}
```

- The time complexity of the extended Euclidean algorithm is O(log(max(A,B))).

Implementation

```
int main( ) {
	extendedEuclid(16, 10);
	cout << ”The GCD of 16 and 10 is ” << d << endl;
	cout << ”Coefficients x and y are ”<< x <<  “and  “ << y << endl;
	return 0;
}
```

Output

```
The GCD of 16 and 10 is 2.
Coefficients x and y are 2 and -3.
```

Extended Euclidean algorithm

- Initially, the extended Euclidean algorithm will run as Euclid's algorithm until you determine GCD(A,B) or until B = 0. It will then assign x = 1 and y = 0.
- In the current scenario, since B = 0 and GCD(A,B) is A, the equation Ax+By=GCD(A,B) will be changed to $A*1+0*0=A$.
- The values of d, x, and y in the process of the extendedEuclid( ) function are as follows:
  - $d=2, x=1, y=0$
  - $d=2, x=0, y=1-(4/2)*0=1$
  - $d=2, x=1, y=0-(6/4)*1=-1$
  - $d=2, x=-1, y=1-(10/6)*-1=2$
  - $d=2, x=2, y=-1-(16/10)*2=-3$

When is this algorithm used?

- This algorithm is used when A and B are co-prime.
- In such cases, x becomes the multiplicative modulo inverse of A under modulo B, and y becomes the multiplicative modulo inverse of B under modulo A.
- This has been explained in detail in the ***Modular multiplicative inverse*** section.

Modular multiplicative inverse (Nghịch đảo mô đun)

Modular multiplicative inverse

- What is a multiplicative inverse?
  - If A.B=1, you are required to find B such that it satisfies the equation. The solution is simple. The value of B is 1/A or A^{-1}. Here, B is the multiplicative inverse of A.
- What is modular multiplicative inverse?
  - If you have two numbers A and M, you are required to find B such it that satisfies the following equation: (A.B)%M=1
  - Here B is the modular multiplicative inverse of A under modulo M.

Modular multiplicative inverse

- Formally, if you have two integers A and M, B is said to be modular multiplicative inverse of A under modulo M if it satisfies the following equation:
  - $A.B \equiv 1 \pmod{M}$. where B is in the range [1,M-1]
- ***Why is B in the range [1,M-1]?***
  - $(A*B)\%M=(A\%M*B\%M)\%M$
  - Since we have B%M, the inverse must be in the range [0,M-1]. However, since 0 is invalid, the inverse must be in the range [1,M-1].

Existence of modular multiplicative inverse

- An inverse exists only when A and M are coprime i.e. GCD(A,M)=1.
- For example, if A=5 and M=12, then $(A*5)\%M=(5*5)\%12=1$. Here, 5 is the modular multiplicative inverse of 5 under modulo 12.
- Though $(5*17)\%12=1$, but since $17 > 12$, it isn't considered.
- Therefore, the answer is 5.

Approach 1 (naive approach)

```
int modInverse(int A,int M){
    A=A%M;
    for(int B=1;B<M;B++)
        if((A*B)%M)==1)
            return B;
}
```

- The algorithm mentioned above runs in *O(M)*.

Approach 2

- A and M are coprime i.e. $Ax+My=1$. In the extended Euclidean algorithm, x is the modular multiplicative inverse of A under modulo M. Therefore, the answer is x. You can use the extended Euclidean algorithm to find the multiplicative inverse.
- For example, if $A=5$ and $M=12$, then $GCD(A,B)=1$. Therefore, the inverse exists.
- $5*(5)+12*(-2)=1$ (which comes from the extended Euclidean algorithm). Therefore, 5 is the inverse of $A=5$ under modulo 12.

Algorithm

```
int d,x,y;
int modInverse(int A, int M)
{
    extendedEuclid(A,M);
    return (x%M+M)%M;    //x may be negative
}
```

- Time complexity
  - O(log(max(A,M)))

Approach 3 (used only when M is prime)

- This approach uses Fermat's Little Theorem.
- The theorem specifies the following:
  - $A^{M-1} \equiv 1 (\text{mod } M)$
- By multiplying with A^{-1} both sides, the equation can be rewritten as follows:
  - $A^{-1} \equiv A^{M-2} (\text{mod } M)$
- The formula for A^{-1} is in the form of exponents. Therefore, modular exponentiation can be used to determine the result.
- For example, if A=5 and M=11 then $A^{M-2}(\text{mod } M) = 5^9(\text{mod}11) = 9$ is the inverse of 5 under modulo 11.

Algorithm

```
int modInverse(int A,int M)
{
    return modularExponentiation(A,M-2,M);
}
```

- ***Time complexity** O(log M)*
- ***When is modular inverse used?***
  - Modular inverse is used to solve (A/B)%M as follows: $(A/B)\%M=(A*B^{-1})\%M$.
  - After you find the inverse, you can solve this equation easily.

TEST YOUR UNDERSTANDING

- Find the answer of $(A^B/C)\%M$.
- **Input**:
  - The only line of input consists of four integers *A*,*B*,*C* and *M*. The input always has C and M such that gcd(C,M)=1.
- **Output**:
  - Print the answer as asked in the problem statement.
- **Constraints**:
  - $1 \le A,B,C,M \le 10^9$
  - gcd(C,M)=1.

Solutions

- By dividing the problem, it can be re-written as $(A^B)\%M*(C^{-1}\%M)$, lets call it *p*. The value of *p* can be larger than *M*, so it should be again modulo by *M*.
- The first part, that is $ans_1=(A^B)\%M$, can be easily calculated using Modular Exponentiation.
- The second part of the problem, $ans_2=(C^{-1}\%M)$ can be calculated using Extended Euclidean Algorithm as GCD(C,M) =1.
- **Note:** We can't use Fermat's Little Theorem here, as It is not given that, M will be a prime number.
- The required answer of the problem will be $(ans_1*ans_2)\%M$.

Code

```
#include <iostream>
using namespace std;
long long mE(long long x,long long n,int M)
{
    long long result=1;
    while(n>0)
    {
        if(n % 2 ==1)
            result=(result * x)%M;
        x=(x*x)%M;
        n=n/2;
    }
    return result;
}
long long a,b,c,m;
int main() {
            cin >> a >> b >> c >> m;
            cout <<
(mE(a,b,m)*modInverse(c,m))%m;
}
```

```
long long d, x, y;
void extendedEuclid(long long A, long long B) {
    if(B == 0) {
        d = A;
        x = 1;
        y = 0;
    }
    else {
        extendedEuclid(B, A%B);
        long long temp = x;
        x = y;
        y = temp - (A/B)*y;
    }
}

long long modInverse(long long A, long long M)
{
    extendedEuclid(A,M);
    return (x%M+M)%M;    //x may be negative
}
```

Problems

- **Mảng hằng Modulo**
- **Trang trí**
- Xếp Quà Tặng
- Thi đấu vòng bảng
- Mũ là ước chung lớn nhất
- Số có số ước là nguyên tố