ĐẠI HỌC BÁCH KHOA – ĐẠI HỌC ĐÀ NẴNG
KHOA CÔNG NGHỆ THÔNG TIN

IT FACULTY

Môn học

Toán Ứng Dụng
Applied mathematics

Giảng Viên: Phạm Minh Tuấn

Giới thiệu

- Phạm Minh Tuấn
- E-mail: pmtuan@dut.udn.vn
- Tel: 0913230910
- Khoa Công nghệ thông tin –Trường ĐHBK – ĐHĐN

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Bài 5:
Inclusion-exclusion
(Bao hàm-Loại trừ)

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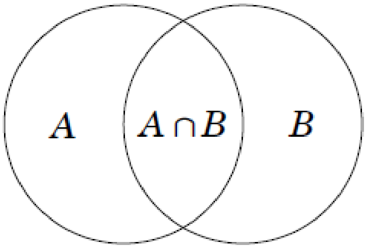
Introduction

- **Inclusion-exclusion** is a technique that can be used for counting the size of a union of sets when the sizes of the intersections are known, and vice versa. A simple example of the technique is the formula

$$|A \cup B| = |A| + |B| - |A \cap B|$$

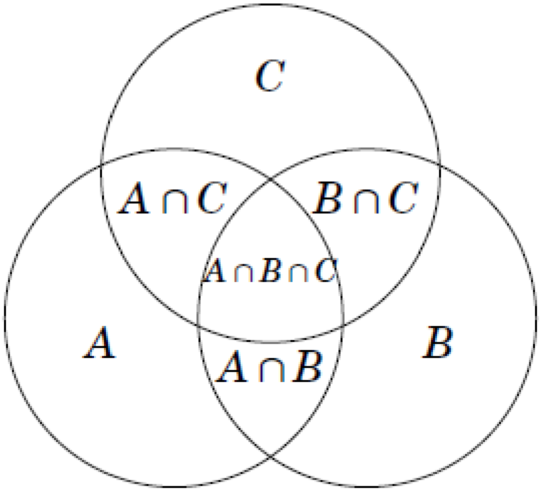
where A and B are sets and $|X|$ denotes the size of X. The formula can be illustrated as follows:



- Our goal is to calculate the size of the union $A \cup B$ that corresponds to the area of the region that belongs to at least one circle. The picture shows that we can calculate the area of $A \cup B$ by first summing the areas of A and B and then subtracting the area of $A \cap B$.

Introduction

- The same idea can be applied when the number of sets is larger. When there are three sets, the inclusion-exclusion formula is
  $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
- And the corresponding picture is



General case

- In the general case, the size of the union $X_1 \cup X_2 \cup \cdots \cup X_n$ can be calculated by going through all possible intersections that contain some of the sets $X_1, X_2, \cdots, X_n$. If the intersection contains an odd number of sets, its size is added to the answer, and otherwise its size is subtracted from the answer.
- Note that there are similar formulas for calculating the size of an intersection from the sizes of unions. For example,
  $|A \cup B| = |A| + |B| - |A \cap B|$
  and
  $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

Example

- Let A be a set of 4 sets:
  - $A_1 = 1{,}2{,}3$
  - $A_2 = 2{,}3{,}4$
  - $A_3 = 1{,}3{,}5$
  - $A_4 = 2{,}3$
- The goal is to compute the cardinality of sum of all the above subsets. Then the formula for the above example looks like this:
- $|A_1| + |A_2| + |A_3| + |A_4| - |A_{12}| - |A_{13}| - |A_{14}| - |A_{23}| - |A_{24}| - |A_{34}| + |A_{123}| + |A_{124}| + |A_{134}| + |A_{234}| - |A_{1234}| = 3 + 3 + 3 + 2 - 2 - 2 - 2 - 1 - 2 - 1 + 1 + 2 + 1 + 1 - 1 = 5$

TEST YOUR UNDERSTANDING

- For a given number n $(1 \le n \le 10^{18})$. the goal is to compute the probability that randomly chosen integer from a range [1, n] with uniform distribution is divisible by at least one of integers 2,3,11 or 13.
- Input
  - In the first and only line of the input there is one integer n.
- Output
  - In a single line, output two space separated integers *p* and *q*, such that the probability o choosing an integer from a range [1, n] which is divisible by either 2,3,11 or 13 is equal to p/q and *p* and *q* don't have common divisors greater than 1.

Derangements

- As an example, let us count the number of **derangements (hoán vị hoàn toàn)** of elements {1,2,...,n}, i.e., permutations where no element remains in its ***original place***. For example, when $n = 3$, there are two derangements: (2,3,1) and (3,1,2).
- One approach for solving the problem is to use inclusion-exclusion. Let X_k be the set of permutations that contain the element k at position k. For example, when $n = 3$, the sets are as follows:
  - $X_1 = \{(1,2,3), (1,3,2)\}$
  - $X_2 = \{(1,2,3), (3,2,1)\}$
  - $X_1 = \{(1,2,3), (2,1,3)\}$

Derangements

- Using these sets, the number of derangements equals
$$n! - |X_1 \cup X_2 \cup \cdots \cup X_n|$$
- so it suffices to calculate the size of the union. Using inclusion-exclusion, this reduces to calculating sizes of intersections which can be done efficiently. For example, when $n = 3$, the size of $|X_1 \cup X_2 \cup X_3|$ is $|X_1 \cup X_2 \cup X_3| = |X_1| + |X_2| + |X_3| - |X_1 \cap X_2| - |X_1 \cap X_3| - |X_2 \cap X_3| + |X_1 \cap X_2 \cap X_3| = 2 + 2 + 2 - 1 - 1 - 1 + 1 = 4$
- so the number of solutions is 3! - 4 = 2.

Other solution

- It turns out that the problem can also be solved without using inclusion-exclusion. Let $f(n)$ denote the number of derangements for $\{1,2,\ldots, n\}$. We can use the following recursive formula:

$$f(n) = \begin{cases} 0 & n = 1 \\ 1 & n = 2 \\ (n-1)\big(f(n-2) + (n-1)\big) & n > 2 \end{cases}$$

- The formula can be derived by considering the possibilities how the element 1 changes in the derangement. There are $(n-1)$ ways to choose an element x that replaces the element 1. In each such choice, there are two options
  - We also replace the element x with the element 1. After this, the remaining task is to construct a derangement of $(n-2)$ elements
  - We replace the element x with some other element than 1. Now we have to construct a derangement of $(n-1)$ element, because we cannot replace the element x with the element 1, and all other elements must be changed.

Burnside's lemma

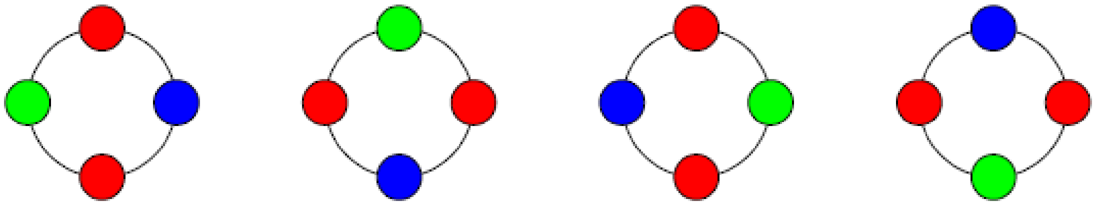
- **Burnside's lemma** can be used to count the number of combinations so that only one representative is counted for each group of symmetric combinations. Burnside's lemma states that the number of combinations is

$$\sum_{k=1}^{n} \frac{c(k)}{n}$$

where there are n ways to change the position of a combination, and there are $c(k)$ combinations that remain unchanged when the kth way is applied.

Example

- Let us calculate the number of necklaces of n pearls, where each pearl has m possible colors. Two necklaces are symmetric if they are similar after rotating them. For example, the necklaces



- There are n ways to change the position of a necklace, because we can rotate it 0,1,..., $n-1$ steps clockwise. If the number of steps is 0, all m^n necklaces remain the same, and if the number of steps is 1, only the m necklaces where each pearl has the same color remain the same.

More generally

- when the number of steps is k, a total of $m^{\gcd(k,n)}$ necklaces remain the same. The reason for this is that blocks of pearls of size $\gcd(k,n)$ will replace each other. Thus, according to Burnside's lemma, the number of necklaces is

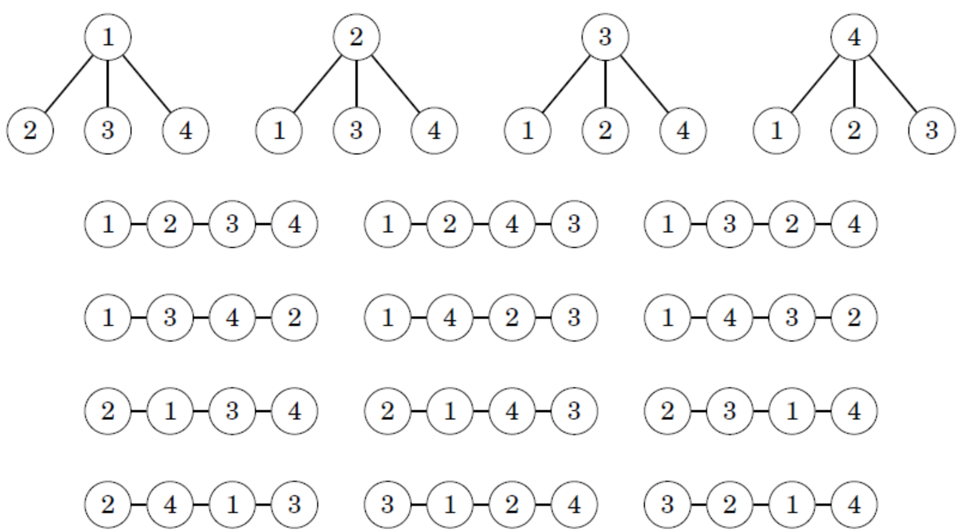
$$\sum_{k=1}^{n} \frac{m^{\gcd(k,n)}}{n}$$

- For example, the number of necklaces of length 4 with 3 colors is

$$\frac{3^4 + 3 + 3^2 + 3}{4} = 24$$

Cayley's formula

- **Cayley's formula** states that there are n^{n-2} labeled trees that contain n nodes. The nodes are labeled 1,2,..., n, and two trees are different if either their structure or labeling is different.
- For example, when $n = 4$, the number of labeled trees is 4^{4-2} :

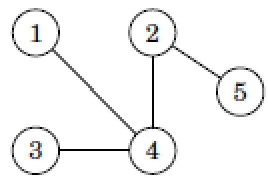


Prüfer code

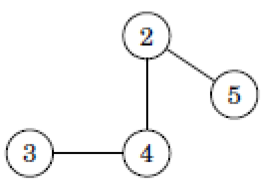
- Cayley's formula can be derived using Prüfer codes.
- A **Prüfer code** is a sequence of $n-2$ numbers that describes a labeled tree. The code is constructed by following a process that removes $n-2$ leaves from the tree. At each step, the leaf with the smallest label is removed, and the label of its only neighbor is added to the code.

Prüfer code

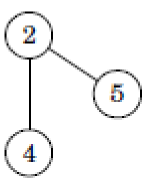
- For example, let us calculate the Prüfer code of the following graph



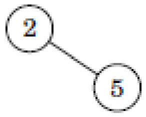
First we remove node 1 and add node 4 to the code:



Then we remove node 3 and add node 4 to the code:



Finally we remove node 4 and add node 2 to the code:



Thus, the Prüfer code of the graph is [4,4,2].

Problems