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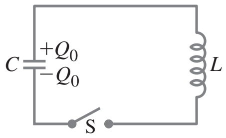
A spark plug in a car receives a high voltage, which produces a high enough electric field in the air across its gap to pull electrons off the atoms in the air–gasoline mixture and form a spark. The high voltage is produced, from the basic 12 V of the car battery, by an induction coil which is basically a transformer or mutual inductance. Any coil of wire has a self-inductance, and a changing current in it causes an emf to be induced. Such inductors are useful in many circuits.

# Inductance, Electromagnetic Oscillations, and AC Circuits

**CHAPTER-OPENING QUESTION—Guess now!**

[*Don't worry about getting the right answer now—the idea is to get your preconceived notions out on the table.*]

Consider a circuit with only a capacitor $C$ and a coil of many loops of wire (called an inductor, $L$) as shown. If the capacitor is initially charged $(Q = Q_0)$, what will happen when the switch S is closed?




**(a)** Nothing will happen—the capacitor will remain charged with charge $Q = Q_0$.
**(b)** The capacitor will quickly discharge and remain discharged $(Q = 0)$.
**(c)** Current will flow until the positive charge is on the opposite plate of the capacitor, and then will reverse—back and forth.
**(d)** The energy initially in the capacitor $\left(U_E = \frac{1}{2}Q_0^2/C\right)$ will all transfer to the coil and then remain that way.
**(e)** The system will quickly transfer half of the capacitor energy to the coil and then remain that way.

A changing magnetic flux through a circuit induces an emf in that circuit. An electric current produces a magnetic field. Combining these two ideas, we could predict that a changing current in one circuit ought to induce an emf and a current in a second nearby circuit, and even induce an emf in itself. One example is a transformer, but here we will treat this effect in a more general way in terms of what we will call mutual inductance and self-inductance. The concept of inductance also gives us a springboard to treat energy storage in a magnetic field. This Chapter concludes with an analysis of circuits that contain inductance as well as resistance and/or capacitance.

CONTENTS



Note: Sections marked with an asterisk (*) may be considered optional by the instructor.


From Chapter 30 of *Physics for Scientists & Engineers with Modern Physics*, Fourth Edition, Douglas C. Giancoli.

# 1 Mutual Inductance

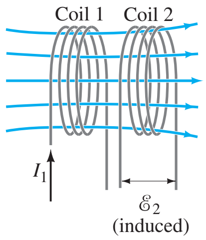



**FIGURE 1** A changing current in one coil will induce a current in the second coil.

If two coils of wire are placed near each other, as in Fig. 1, a changing current in one will induce an emf in the other. According to Faraday's law, the emf $\mathscr{E}_2$ induced in coil 2 is proportional to the rate of change of magnetic flux passing through it. This flux is due to the current $I_1$ in coil 1, and it is often convenient to express the emf in coil 2 in terms of the current in coil 1.

We let $\Phi_{21}$ be the magnetic flux in each loop of coil 2 created by the current in coil 1. If coil 2 contains $N_2$ closely wrapped loops, then $N_2\Phi_{21}$ is the total flux passing through coil 2. If the two coils are fixed in space, $N_2\Phi_{21}$ is proportional to the current $I_1$ in coil 1; the proportionality constant is called the **mutual inductance**, $M_{21}$, defined by

$$M_{21} = \frac{N_2\Phi_{21}}{I_1}. \tag{1}$$

The emf $\mathscr{E}_2$ induced in coil 2 due to a changing current in coil 1 is, by Faraday's law,

$$\mathscr{E}_2 = -N_2\frac{d\Phi_{21}}{dt}.$$

We combine this with Eq. 1 rewritten as $\Phi_{21} = M_{21}I_1/N_2$ (and take its derivative) and obtain

$$\mathscr{E}_2 = -M_{21}\frac{dI_1}{dt}. \tag{2}$$

This relates the change in current in coil 1 to the emf it induces in coil 2. The mutual inductance of coil 2 with respect to coil 1, $M_{21}$, is a "constant" in that it does not depend on $I_1$; $M_{21}$ depends on "geometric" factors such as the size, shape, number of turns, and relative positions of the two coils, and also on whether iron (or some other ferromagnetic material) is present. For example, if the two coils in Fig. 1 are farther apart, fewer lines of flux can pass through coil 2, so $M_{21}$ will be less. For some arrangements, the mutual inductance can be calculated (see Example 1). More often it is determined experimentally.

Suppose, now, we consider the reverse situation: when a changing current in coil 2 induces an emf in coil 1. In this case,

$$\mathscr{E}_1 = -M_{12}\frac{dI_2}{dt}$$

where $M_{12}$ is the mutual inductance of coil 1 with respect to coil 2. It is possible to show, although we will not prove it here, that $M_{12} = M_{21}$. Hence, for a given arrangement we do not need the subscripts and we can let

$$M = M_{12} = M_{21},$$

so that

$$\mathscr{E}_1 = -M\frac{dI_2}{dt} \tag{3a}$$

and

$$\mathscr{E}_2 = -M\frac{dI_1}{dt}. \tag{3b}$$

The SI unit for mutual inductance is the henry (H), where $1\,\text{H} = 1\,\text{V}\cdot\text{s/A} = 1\,\Omega\cdot\text{s}$.

**EXERCISE A** Two coils which are close together have a mutual inductance of 330 mH. (*a*) If the emf in coil 1 is 120 V, what is the rate of change of the current in coil 2? (*b*) If the rate of change of current in coil 1 is 36 A/s, what is the emf in coil 2?

**EXAMPLE 1** **Solenoid and coil.** A long thin solenoid of length $\ell$ and cross-sectional area $A$ contains $N_1$ closely packed turns of wire. Wrapped around it is an insulated coil of $N_2$ turns, Fig. 2. Assume all the flux from coil 1 (the solenoid) passes through coil 2, and calculate the mutual inductance.

**APPROACH** We first determine the flux produced by the solenoid, all of which passes uniformly through coil $N_2$, using the Equation for the magnetic field inside the solenoid:

$$B = \mu_0 \frac{N_1}{\ell} I_1,$$

where $n = N_1/\ell$ is the number of loops in the solenoid per unit length, and $I_1$ is the current in the solenoid.

**SOLUTION** The solenoid is closely packed, so we assume that all the flux in the solenoid stays inside the secondary coil. Then the flux $\Phi_{21}$ through coil 2 is

$$\Phi_{21} = BA = \mu_0 \frac{N_1}{\ell} I_1 A.$$

Then the mutual inductance is

$$M = \frac{N_2 \Phi_{21}}{I_1} = \frac{\mu_0 N_1 N_2 A}{\ell}.$$

**NOTE** We calculated $M_{21}$; if we had tried to calculate $M_{12}$, it would have been difficult. Given $M_{12} = M_{21} = M$, we did the simpler calculation to obtain $M$. Note again that $M$ depends only on geometric factors, and not on the currents.

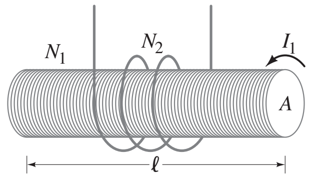


**FIGURE 2** Example 1.

**CONCEPTUAL EXAMPLE 2** **Reversing the coils.** How would Example 1 change if the coil with $N_2$ turns was inside the solenoid rather than outside the solenoid?

**RESPONSE** The magnetic field inside the solenoid would be unchanged. The flux through the coil would be $BA$ where $A$ is the area of the coil, not of the solenoid as in Example 1. Solving for $M$ would give the same formula except that $A$ would refer to the coil, and would be smaller.

**EXERCISE B** Which solenoid and coil combination shown in Fig. 3 has the largest mutual inductance? Assume each solenoid is the same.

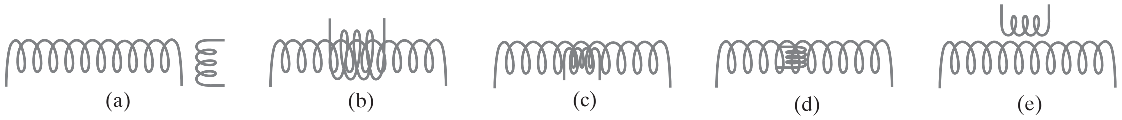


**FIGURE 3** Exercise B.

**PHYSICS APPLIED**
*Pacemaker*

A transformer is an example of mutual inductance in which the coupling is maximized so that nearly all flux lines pass through both coils. Mutual inductance has other uses as well, including some types of *pacemakers* used to maintain blood flow in heart patients. Power in an external coil is transmitted via mutual inductance to a second coil in the pacemaker at the heart. This type has the advantage over battery-powered pacemakers in that surgery is not needed to replace a battery when it wears out.

Mutual inductance can sometimes be a problem, however. Any changing current in a circuit can induce an emf in another part of the same circuit or in a different circuit even though the conductors are not in the shape of a coil. The mutual inductance $M$ is usually small unless coils with many turns and/or iron cores are involved. However, in situations where small voltages are being used, problems due to mutual inductance often arise. Shielded cable, in which an inner conductor is surrounded by a cylindrical grounded conductor, is often used to reduce the problem.

## 2 Self-Inductance

The concept of inductance applies also to a single isolated coil of $N$ turns. When a changing current passes through a coil (or solenoid), a changing magnetic flux is produced inside the coil, and this in turn induces an emf in that same coil. This induced emf opposes the change in flux (Lenz's law). For example, if the current through the coil is increasing, the increasing magnetic flux induces an emf that opposes the original current and tends to retard its increase. If the current is decreasing in the coil, the decreasing flux induces an emf in the same direction as the current, thus tending to maintain the original current.

The magnetic flux $\Phi_B$ passing through the $N$ turns of a coil is proportional to the current $I$ in the coil, so we define the **self-inductance** $L$ (in analogy to mutual inductance, Eq. 1) as

$$L = \frac{N\Phi_B}{I}. \qquad \textbf{(4)}$$

Then the emf $\mathscr{E}$ induced in a coil of self-inductance $L$ is, from Faraday's law,

$$\mathscr{E} = -N\frac{d\Phi_B}{dt} = -L\frac{dI}{dt}. \qquad \textbf{(5)}$$

Like mutual inductance, self-inductance is measured in henrys. The magnitude of $L$ depends on the geometry and on the presence of a ferromagnetic material. Self-inductance (inductance, for short) can be defined, as above, for any circuit or part of a circuit.

Circuits always contain some inductance, but often it is quite small unless the circuit contains a coil of many turns. A coil that has significant self-inductance $L$ is called an **inductor**. Inductance is shown on circuit diagrams by the symbol

; [inductor symbol]

any resistance an inductor has should also be shown separately. Inductance can serve a useful purpose in certain circuits. Often, however, inductance is to be avoided in a circuit. Precision resistors are normally wire wound and thus would have inductance as well as resistance. The inductance can be minimized by winding the insulated wire back on itself in the opposite sense so that the current going in opposite directions produces little net magnetic flux; this is called a **noninductive winding**.

If an inductor has negligible resistance, it is the inductance (or induced emf) that controls a changing current. If a source of changing or alternating voltage is applied to the coil, this applied voltage will just be balanced by the induced emf of the coil (Eq. 5). Thus we can see from Eq. 5 that, for a given $\mathscr{E}$, if the inductance $L$ is large, the change in the current will be small, and therefore the current itself if it is ac will be small. The greater the inductance, the less the ac current. An inductance thus acts something like a resistance to impede the flow of alternating current. We use the term *reactance* or *impedance* for this quality of an inductor. We will discuss reactance and impedance more fully in Sections 7 and 8. We shall see that reactance depends not only on the inductance $L$, but also on the frequency. Here we mention one example of its importance. The resistance of the primary in a transformer is usually quite small, perhaps less than $1\,\Omega$. If resistance alone limited the current in a transformer, tremendous currents would flow when a high voltage was applied. Indeed, a dc voltage applied to a transformer can burn it out. It is the induced emf (or reactance) of the coil that limits the current to a reasonable value.

Common inductors have inductances in the range from about $1\,\mu\text{H}$ to about $1\,\text{H}$ (where $1\,\text{H} = 1\,\text{henry} = 1\,\Omega\cdot\text{s}$).

**EXAMPLE 3** **Solenoid inductance.** (*a*) Determine a formula for the self-inductance $L$ of a tightly wrapped and long solenoid containing $N$ turns of wire in its length $\ell$ and whose cross-sectional area is $A$. (*b*) Calculate the value of $L$ if $N = 100$, $\ell = 5.0\,\text{cm}$, $A = 0.30\,\text{cm}^2$ and the solenoid is air filled.

**APPROACH** To determine the inductance $L$, it is usually simplest to start with Eq. 4, so we need to first determine the flux.

**SOLUTION** ($a$) The magnetic field inside a solenoid (ignoring end effects) is constant: $B = \mu_0 nI$ where $n = N/\ell$. The flux is $\Phi_B = BA = \mu_0 NIA/\ell$, so

$$L = \frac{N\Phi_B}{I} = \frac{\mu_0 N^2 A}{\ell}.$$

($b$) Since $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{T\cdot m/A}$, then

$$L = \frac{(4\pi \times 10^{-7}\,\mathrm{T\cdot m/A})(100)^2(3.0 \times 10^{-5}\,\mathrm{m^2})}{(5.0 \times 10^{-2}\,\mathrm{m})} = 7.5\,\mu\mathrm{H}.$$

**NOTE** Magnetic field lines "stray" out of the solenoid, especially near the ends, so our formula is only an approximation.

**CONCEPTUAL EXAMPLE 4** **Direction of emf in inductor.** Current passes through the coil in Fig. 4 from left to right as shown. ($a$) If the current is increasing with time, in which direction is the induced emf? ($b$) If the current is decreasing in time, what then is the direction of the induced emf?

**RESPONSE** ($a$) From Lenz's law we know that the induced emf must oppose the change in magnetic flux. If the current is increasing, so is the magnetic flux. The induced emf acts to oppose the increasing flux, which means it acts like a source of emf that opposes the outside source of emf driving the current. So the induced emf in the coil acts to oppose $I$ in Fig. 4a. In other words, the inductor might be thought of as a battery with a positive terminal at point A (tending to block the current entering at A), and negative at point B. ($b$) If the current is decreasing, then by Lenz's law the induced emf acts to bolster the flux—like a source of emf reinforcing the external emf. The induced emf acts to increase $I$ in Fig. 4b, so in this situation you can think of the induced emf as a battery with its negative terminal at point A to attract more (+) current to move to the right.

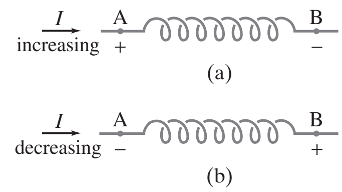



**FIGURE 4** Example 4. The + and − signs refer to the induced emf due to the changing current, as if points A and B were the terminals of a battery (and the coiled loops were the inside of the battery).

**EXAMPLE 5** **Coaxial cable inductance.** Determine the inductance per unit length of a coaxial cable whose inner conductor has a radius $r_1$ and the outer conductor has a radius $r_2$, Fig. 5. Assume the conductors are thin hollow tubes so there is no magnetic field within the inner conductor, and the magnetic field inside both thin conductors can be ignored. The conductors carry equal currents $I$ in opposite directions.

**APPROACH** We need to find the magnetic flux, $\Phi_B = \int \vec{\mathbf{B}} \cdot d\vec{\mathbf{A}}$, between the conductors. The lines of $\vec{\mathbf{B}}$ are circles surrounding the inner conductor (only one is shown in Fig. 5a). From Ampère's law, $\oint \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = \mu_0 I$, the magnitude of the field along the circle at a distance $r$ from the center, when the inner conductor carries a current $I$, is:

$$B = \frac{\mu_0 I}{2\pi r}.$$

The magnetic flux through a rectangle of width $dr$ and length $\ell$ (along the cable, Fig. 5b), a distance $r$ from the center, is

$$d\Phi_B = B(\ell\, dr) = \frac{\mu_0 I}{2\pi r}\ell\, dr.$$

**SOLUTION** The total flux in a length $\ell$ of cable is

$$\Phi_B = \int d\Phi_B = \frac{\mu_0 I\ell}{2\pi}\int_{r_1}^{r_2}\frac{dr}{r} = \frac{\mu_0 I\ell}{2\pi}\ln\frac{r_2}{r_1}.$$

Since the current $I$ all flows in one direction in the inner conductor, and the same current $I$ all flows in the opposite direction in the outer conductor, we have only one turn, so $N = 1$ in Eq. 4. Hence the self-inductance for a length $\ell$ is

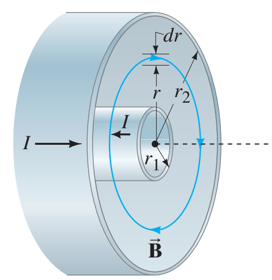
$$L = \frac{\Phi_B}{I} = \frac{\mu_0 \ell}{2\pi}\ln\frac{r_2}{r_1}.$$

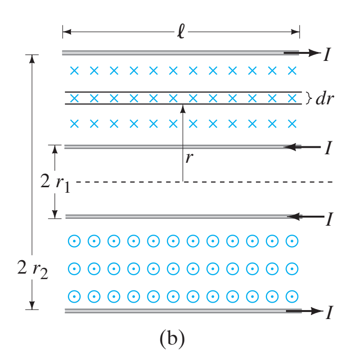
The inductance per unit length is

$$\frac{L}{\ell} = \frac{\mu_0}{2\pi}\ln\frac{r_2}{r_1}.$$

Note that $L$ depends only on geometric factors and not on the current $I$.

**FIGURE 5** Example 5. Coaxial cable: (a) end view, (b) side view (cross section).



# 3 Energy Stored in a Magnetic Field

When an inductor of inductance $L$ is carrying a current $I$ which is changing at a rate $dI/dt$, energy is being supplied to the inductor at a rate

$$P = I\mathscr{E} = LI\frac{dI}{dt}$$

where $P$ stands for power and we used† Eq. 5. Let us calculate the work needed to increase the current in an inductor from zero to some value $I$. Using this last equation, the work $dW$ done in a time $dt$ is

$$dW = P\,dt = LI\,dI.$$

Then the total work done to increase the current from zero to $I$ is

$$W = \int dW = \int_0^I LI\,dI = \tfrac{1}{2}LI^2.$$

This work done is equal to the energy $U$ stored in the inductor when it is carrying a current $I$ (and we take $U = 0$ when $I = 0$):

$$U = \tfrac{1}{2}LI^2. \qquad \textbf{(6)}$$

This can be compared to the energy stored in a capacitor, $C$, when the potential difference across it is $V$:

$$U = \tfrac{1}{2}CV^2.$$

**EXERCISE C** What is the inductance of an inductor if it has a stored energy of 1.5 J when there is a current of 2.5 A in it? (*a*) 0.48 H, (*b*) 1.2 H, (*c*) 2.1 H, (*d*) 4.7 H, (*e*) 19 H.

Just as the energy stored in a capacitor can be considered to reside in the electric field between its plates, so the energy in an inductor can be considered to be stored in its magnetic field. To write the energy in terms of the magnetic field, let us use the result of Example 3, that the inductance of an ideal solenoid (end effects ignored) is $L = \mu_0 N^2 A/\ell$. Because the magnetic field $B$ in a solenoid is related to the current $I$ by $B = \mu_0 NI/\ell$, we have

$$\begin{aligned} U &= \tfrac{1}{2}LI^2 = \frac{1}{2}\left(\frac{\mu_0 N^2 A}{\ell}\right)\left(\frac{B\ell}{\mu_0 N}\right)^2 \\ &= \frac{1}{2}\frac{B^2}{\mu_0}A\ell. \end{aligned}$$

We can think of this energy as residing in the volume enclosed by the windings, which is $A\ell$. Then the energy per unit volume or **energy density** is

$$u = \text{energy density} = \frac{1}{2}\frac{B^2}{\mu_0}. \qquad \textbf{(7)}$$

This formula, which was derived for the special case of a solenoid, can be shown to be valid for any region of space where a magnetic field exists. If a ferromagnetic material is present, $\mu_0$ is replaced by $\mu$. This equation is analogous to that for an electric field, $\frac{1}{2}\epsilon_0 E^2$.

# 4 *LR* Circuits

Any inductor will have some resistance. We represent this situation by drawing its inductance $L$ and its resistance $R$ separately, as in Fig. 6a. The resistance $R$ could also include any other resistance present in the circuit. Now we ask, what happens when a battery or other source of dc voltage $V_0$ is connected in series to such an $LR$ circuit?

†No minus sign here because we are supplying power to oppose the emf of the inductor.

At the instant the switch connecting the battery is closed, the current starts to flow. It is opposed by the induced emf in the inductor which means point B in Fig. 6a is positive relative to point C. However, as soon as current starts to flow, there is also a voltage drop of magnitude $IR$ across the resistance. Hence the voltage applied across the inductance is reduced and the current increases less rapidly. The current thus rises gradually as shown in Fig. 6b, and approaches the steady value $I_{max} = V_0/R_0$, for which all the voltage drop is across the resistance.

We can show this analytically by applying Kirchhoff's loop rule to the circuit of Fig. 6a. The emfs in the circuit are the battery voltage $V_0$ and the emf $\mathscr{E} = -L(dI/dt)$ in the inductor opposing the increasing current. Hence the sum of the potential changes around the loop is

$$V_0 - IR - L\frac{dI}{dt} = 0,$$

where $I$ is the current in the circuit at any instant. We rearrange this to obtain

$$L\frac{dI}{dt} + RI = V_0. \tag{8}$$

This is a linear differential equation and can be integrated in the same way for an $RC$ circuit. We rewrite Eq. 8 and then integrate:

$$\int_{I=0}^{I} \frac{dI}{V_0 - IR} = \int_0^t \frac{dt}{L}.$$

Then

$$-\frac{1}{R}\ln\left(\frac{V_0 - IR}{V_0}\right) = \frac{t}{L}$$

or

$$I = \frac{V_0}{R}\left(1 - e^{-t/\tau}\right) \tag{9}$$

where

$$\tau = \frac{L}{R} \tag{10}$$

is the **time constant** of the $LR$ circuit. The symbol $\tau$ represents the time required for the current $I$ to reach $(1 - 1/e) = 0.63$ or 63% of its maximum value $(V_0/R)$. Equation 9 is plotted in Fig. 6b. (Compare to the $RC$ circuit.)

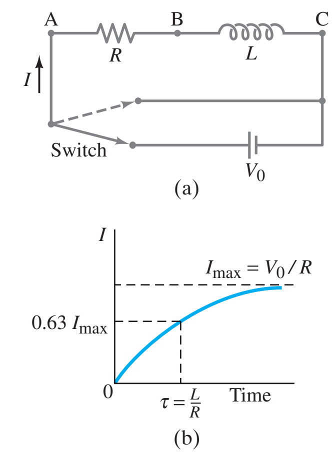



**FIGURE 6** (a) $LR$ circuit; (b) growth of current when connected to battery.

**EXERCISE D** Show that $L/R$ does have dimensions of time.

Now let us flip the switch in Fig. 6a so that the battery is taken out of the circuit, and points A and C are connected together as shown in Fig. 7 at the moment when the switching occurs (call it $t = 0$) and the current is $I_0$. Then the differential equation (Eq. 8) becomes (since $V_0 = 0$):

$$L\frac{dI}{dt} + RI = 0.$$

We rearrange this equation and integrate:

$$\int_{I_0}^{I} \frac{dI}{I} = -\int_0^t \frac{R}{L}\,dt$$

where $I = I_0$ at $t = 0$, and $I = I$ at time $t$.

We integrate this last equation to obtain

$$\ln\frac{I}{I_0} = -\frac{R}{L}t$$

or

$$I = I_0 e^{-t/\tau} \tag{11}$$

where again the time constant is $\tau = L/R$. The current thus decays exponentially to zero as shown in Fig. 8.

This analysis shows that there is always some "reaction time" when an electromagnet, for example, is turned on or off. We also see that an $LR$ circuit has properties similar to an $RC$ circuit. Unlike the capacitor case, however, the time constant here is *inversely* proportional to $R$.

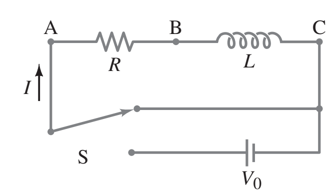
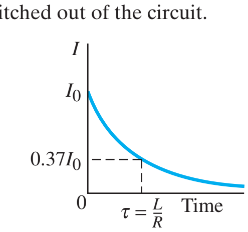



**FIGURE 7** The switch is flipped quickly so the battery is removed but we still have a circuit. The current at this moment (call it $t = 0$) is $I_0$.

**FIGURE 8** Decay of the current in Fig. 7 in time after the battery is switched out of the circuit.



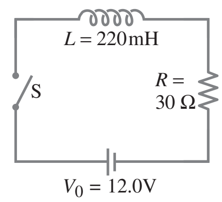



**FIGURE 9** Example 6.

**EXAMPLE 6** **An *LR* circuit.** At $t = 0$, a 12.0-V battery is connected in series with a 220-mH inductor and a total of 30-Ω resistance, as shown in Fig. 9. (*a*) What is the current at $t = 0$? (*b*) What is the time constant? (*c*) What is the maximum current? (*d*) How long will it take the current to reach half its maximum possible value? (*e*) At this instant, at what rate is energy being delivered by the battery, and (*f*) at what rate is energy being stored in the inductor's magnetic field?

**APPROACH** We have the situation shown in Figs. 6a and b, and we can apply the equations we just developed.

**SOLUTION** (*a*) The current cannot instantaneously jump from zero to some other value when the switch is closed because the inductor opposes the change $\left(\mathscr{E}_L = -L(dI/dt)\right)$. Hence just after the switch is closed, $I$ is still zero at $t = 0$ and then begins to increase.

(*b*) The time constant is, from Eq. 10, $\tau = L/R = (0.22\,\mathrm{H})/(30\,\Omega) = 7.3\,\mathrm{ms}$.

(*c*) The current reaches its maximum steady value after a long time, when $dI/dt = 0$ so $I_{max} = V_0/R = 12.0\,\mathrm{V}/30\,\Omega = 0.40\,\mathrm{A}$.

(*d*) We set $I = \frac{1}{2}I_{max} = V_0/2R$ in Eq. 9, which gives us

$$1 - e^{-t/\tau} = \tfrac{1}{2}$$

or

$$e^{-t/\tau} = \tfrac{1}{2}.$$

We solve for $t$:

$$t = \tau \ln 2 = \left(7.3 \times 10^{-3}\,\mathrm{s}\right)(0.69) = 5.0\,\mathrm{ms}.$$

(*e*) At this instant, $I = I_{max}/2 = 200\,\mathrm{mA}$, so the power being delivered by the battery is

$$P = IV = (0.20\,\mathrm{A})(12\,\mathrm{V}) = 2.4\,\mathrm{W}.$$

(*f*) From Eq. 6, the energy stored in an inductor $L$ at any instant is

$$U = \tfrac{1}{2}LI^2$$

where $I$ is the current in the inductor at that instant. The *rate* at which the energy changes is

$$\frac{dU}{dt} = LI\frac{dI}{dt}.$$

We can differentiate Eq. 9 to obtain $dI/dt$, or use the differential equation, Eq. 8, directly:

$$\begin{aligned}\frac{dU}{dt} &= I\left(L\frac{dI}{dt}\right) = I\left(V_0 - RI\right)\\ &= (0.20\,\mathrm{A})[12\,\mathrm{V} - (30\,\Omega)(0.20\,\mathrm{A})] = 1.2\,\mathrm{W}.\end{aligned}$$

Since only part of the battery's power is feeding the inductor at this instant, where is the rest going?

**EXERCISE E** A resistor in series with an inductor has a time constant of 10 ms. When the same resistor is placed in series with a 5-$\mu$F capacitor, the time constant is $5 \times 10^{-6}$ s. What is the value of the inductor? (*a*) 5 $\mu$H; (*b*) 10 $\mu$H; (*c*) 5 mH; (*d*) 10 mH; (*e*) not enough information to determine it.

**PHYSICS APPLIED**
*Surge protection*

An inductor can act as a "surge protector" for sensitive electronic equipment that can be damaged by high currents. If equipment is plugged into a standard wall plug, a sudden "surge," or increase, in voltage will normally cause a corresponding large change in current and damage the electronics. However, if there is an inductor in series with the voltage to the device, the sudden change in current produces an opposing emf preventing the current from reaching dangerous levels.

# 5 *LC* Circuits and Electromagnetic Oscillations

In any electric circuit, there can be three basic components: resistance, capacitance, and inductance, in addition to a source of emf. (There can also be more complex components, such as diodes or transistors.) We have previously discussed both $RC$ and $LR$ circuits. Now we look at an $LC$ circuit, one that contains only a capacitance $C$ and an inductance, $L$, Fig. 10. This is an idealized circuit in which we assume there is no resistance; in the next Section we will include resistance. Let us suppose the capacitor in Fig. 10 is initially charged so that one plate has charge $Q_0$ and the other plate has charge $-Q_0$, and the potential difference across it is $V = Q/C$. Suppose that at $t = 0$, the switch is closed. The capacitor immediately begins to discharge. As it does so, the current $I$ through the inductor increases. We now apply Kirchhoff's loop rule (sum of potential changes around a loop is zero):

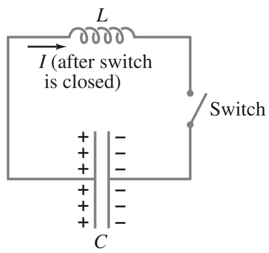



**FIGURE 10** An $LC$ circuit.

$$-L\frac{dI}{dt} + \frac{Q}{C} = 0.$$

Because charge leaves the positive plate on the capacitor to produce the current $I$ as shown in Fig. 10, the charge $Q$ on the (positive) plate of the capacitor is decreasing, so $I = -dQ/dt$. We can then rewrite the above equation as

$$\frac{d^2Q}{dt^2} + \frac{Q}{LC} = 0. \tag{12}$$

This is a familiar differential equation. It has the same form as the equation for simple harmonic motion. The solution of Eq. 12 can be written as

$$Q = Q_0\cos(\omega t + \phi) \tag{13}$$

where $Q_0$ and $\phi$ are constants that depend on the initial conditions. We insert Eq. 13 into Eq. 12, noting that $d^2Q/dt^2 = -\omega^2 Q_0\cos(\omega t + \phi)$; thus

$$-\omega^2 Q_0\cos(\omega t + \phi) + \frac{1}{LC}Q_0\cos(\omega t + \phi) = 0$$

or

$$\left(-\omega^2 + \frac{1}{LC}\right)\cos(\omega t + \phi) = 0.$$

This relation can be true for all times $t$ only if $\left(-\omega^2 + 1/LC\right) = 0$, which tells us that

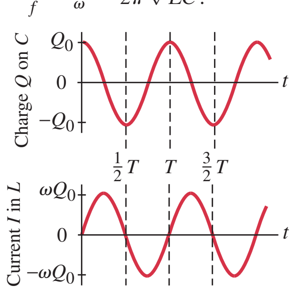
$$\omega = 2\pi f = \sqrt{\frac{1}{LC}}. \tag{14}$$

Equation 13 shows that the charge on the capacitor in an $LC$ circuit oscillates sinusoidally. The current in the inductor is

$$I = -\frac{dQ}{dt} = \omega Q_0\sin(\omega t + \phi)$$
$$= I_0\sin(\omega t + \phi); \tag{15}$$

so the current too is sinusoidal. The maximum value of $I$ is $I_0 = \omega Q_0 = Q_0/\sqrt{LC}$. Equations 13 and 15 for $Q$ and $I$ when $\phi = 0$ are plotted in Fig. 11.

**FIGURE 11** Charge $Q$ and current $I$ in an $LC$ circuit. The period $T = \frac{1}{f} = \frac{2\pi}{\omega} = 2\pi\sqrt{LC}$.



Now let us look at $LC$ oscillations from the point of view of energy. The energy stored in the electric field of the capacitor at any time $t$ is:

$$U_E = \frac{1}{2}\frac{Q^2}{C} = \frac{Q_0^2}{2C}\cos^2(\omega t + \phi).$$

The energy stored in the magnetic field of the inductor at the same instant is (Eq. 6)

$$U_B = \frac{1}{2}LI^2 = \frac{L\omega^2 Q_0^2}{2}\sin^2(\omega t + \phi) = \frac{Q_0^2}{2C}\sin^2(\omega t + \phi)$$

where we used Eq. 14. If we let $\phi = 0$, then at times $t = 0$, $t = \frac{1}{2}T$, $t = T$, and so on (where $T$ is the period $= 1/f = 2\pi/\omega$), we have $U_E = Q_0^2/2C$ and $U_B = 0$. That is, all the energy is stored in the electric field of the capacitor. But at $t = \frac{1}{4}T, \frac{3}{4}T$, and so on, $U_E = 0$ and $U_B = Q_0^2/2C$, and so all the energy is stored in the magnetic field of the inductor. At any time $t$, the total energy is

$$U = U_E + U_B = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}LI^2$$
$$= \frac{Q_0^2}{2C}\left[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)\right] = \frac{Q_0^2}{2C}. \qquad \textbf{(16)}$$

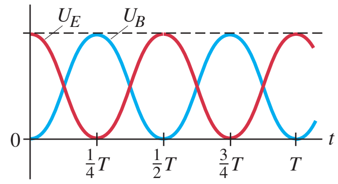



**FIGURE 12** Energy $U_E$ (red line) and $U_B$ (blue line) stored in the capacitor and the inductor as a function of time. Note how the energy oscillates between electric and magnetic. The dashed line at the top is the (constant) total energy $U = U_E + U_B$.

Hence the total energy is constant, and energy is conserved.

What we have in this $LC$ circuit is an ***LC* oscillator** or **electromagnetic oscillation**. The charge $Q$ oscillates back and forth, from one plate of the capacitor to the other, and repeats this continuously. Likewise, the current oscillates back and forth as well. They are also energy oscillations: when $Q$ is a maximum, the energy is all stored in the electric field of the capacitor; but when $Q$ reaches zero, the current $I$ is a maximum and all the energy is stored in the magnetic field of the inductor. Thus the energy oscillates between being stored in the electric field of the capacitor and in the magnetic field of the inductor. See Fig. 12.

**EXERCISE F** Return to the Chapter-Opening Question and answer it again now. Try to explain why you may have answered differently the first time.

**EXAMPLE 7** ***LC* circuit.** A 1200-pF capacitor is fully charged by a 500-V dc power supply. It is disconnected from the power supply and is connected, at $t = 0$, to a 75-mH inductor. Determine: ($a$) the initial charge on the capacitor; ($b$) the maximum current; ($c$) the frequency $f$ and period $T$ of oscillation; and ($d$) the total energy oscillating in the system.

**APPROACH** We use the analysis above, and the definition of capacitance $Q = CV$.

**SOLUTION** ($a$) The 500-V power supply, before being disconnected, charged the capacitor to a charge of

$$Q_0 = CV = (1.2 \times 10^{-9}\,\mathrm{F})(500\,\mathrm{V}) = 6.0 \times 10^{-7}\,\mathrm{C}.$$

($b$) The maximum current, $I_{\max}$, is (see Eqs. 14 and 15)

$$I_{\max} = \omega Q_0 = \frac{Q_0}{\sqrt{LC}} = \frac{(6.0 \times 10^{-7}\,\mathrm{C})}{\sqrt{(0.075\,\mathrm{H})(1.2 \times 10^{-9}\,\mathrm{F})}} = 63\,\mathrm{mA}.$$

($c$) Equation 14 gives us the frequency:

$$f = \frac{\omega}{2\pi} = \frac{1}{(2\pi\sqrt{LC})} = 17\,\mathrm{kHz},$$

and the period $T$ is

$$T = \frac{1}{f} = 6.0 \times 10^{-5}\,\mathrm{s}.$$

($d$) Finally the total energy (Eq. 16) is

$$U = \frac{Q_0^2}{2C} = \frac{(6.0 \times 10^{-7}\,\mathrm{C})^2}{2(1.2 \times 10^{-9}\,\mathrm{F})} = 1.5 \times 10^{-4}\,\mathrm{J}.$$

# 6 *LC* Oscillations with Resistance (*LRC* Circuit)

The $LC$ circuit discussed in the previous Section is an idealization. There is always some resistance $R$ in any circuit, and so we now discuss such a simple $LRC$ circuit, Fig. 13.

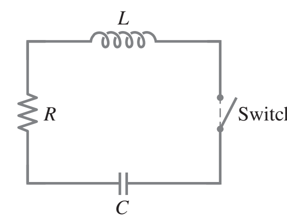



**FIGURE 13** An $LRC$ circuit.

Suppose again that the capacitor is initially given a charge $Q_0$ and the battery or other source is then removed from the circuit. The switch is closed at $t = 0$. Since there is now a resistance in the circuit, we expect some of the energy to be converted to thermal energy, and so we don't expect undamped oscillations as in a pure $LC$ circuit. Indeed, if we use Kirchhoff's loop rule around this circuit, we obtain

$$-L\frac{dI}{dt} - IR + \frac{Q}{C} = 0,$$

which is the same equation we had in Section 5 with the addition of the voltage drop $IR$ across the resistor. Since $I = -dQ/dt,$ as we saw in Section 5, this equation becomes

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0. \tag{17}$$

This second-order differential equation in the variable $Q$ has precisely the same form as that for the damped harmonic oscillator:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0.$$

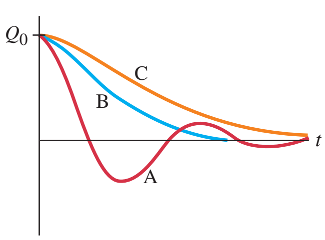
Hence we can analyze our $LRC$ circuit in the same way as for damped harmonic motion. Our system may undergo damped oscillations, curve A in Fig. 14 (underdamped system), or it may be critically damped (curve B), or overdamped (curve C), depending on the relative values of $R$, $L$, and $C$. With $m$ replaced by $L$, $b$ by $R$, and $k$ by $C^{-1}$, we find that the system will be underdamped when

$$R^2 < \frac{4L}{C},$$

and overdamped for $R^2 > 4L/C$. Critical damping (curve B in Fig. 14) occurs when $R^2 = 4L/C$. If $R$ is smaller than $\sqrt{4L/C}$, the angular frequency, $\omega'$, will be

$$\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}. \tag{18}$$

**FIGURE 14** Charge $Q$ on the capacitor in an $LRC$ circuit as a function of time: curve A is for underdamped oscillation $(R^2 < 4L/C)$, curve B is for critically damped $(R^2 = 4L/C)$, and curve C is for overdamped $(R^2 > 4L/C)$.




And the charge $Q$ as a function of time will be

$$Q = Q_0 e^{-\frac{R}{2L}t}\cos(\omega' t + \phi) \tag{19}$$

where $\phi$ is a phase constant.

Oscillators are an important element in many electronic devices: radios and television sets use them for tuning, tape recorders use them (the "bias frequency") when recording, and so on. Because some resistance is always present, electrical oscillators generally need a periodic input of power to compensate for the energy converted to thermal energy in the resistance.

**EXAMPLE 8** **Damped oscillations.** At $t = 0$, a 40-mH inductor is placed in series with a resistance $R = 3.0\,\Omega$ and a charged capacitor $C = 4.8\,\mu\text{F}$. (*a*) Show that this circuit will oscillate. (*b*) Determine the frequency. (*c*) What is the time required for the charge amplitude to drop to half its starting value? (*d*) What value of $R$ will make the circuit nonoscillating?

**APPROACH** We first check $R^2$ vs. $4L/C$; then use Eqs. 18 and 19.

**SOLUTION** (*a*) In order to oscillate, the circuit must be underdamped, so we must have $R^2 < 4L/C$. Since $R^2 = 9.0\,\Omega^2$ and $4L/C = 4(0.040\,\text{H})/(4.8 \times 10^{-6}\,\text{F}) = 3.3 \times 10^4\,\Omega^2$, this relation is satisfied, so the circuit will oscillate.

(*b*) We use Eq. 18:

$$f' = \frac{\omega'}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = 3.6 \times 10^2\,\text{Hz}.$$

(*c*) From Eq. 19, the amplitude will be half when

$$e^{-\frac{R}{2L}t} = \tfrac{1}{2}$$

or

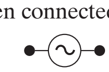
$$t = \frac{2L}{R}\ln 2 = 18\,\text{ms}.$$

(*d*) To make the circuit critically damped or overdamped, we must use the criterion $R^2 \geq 4L/C = 3.3 \times 10^4\,\Omega^2$. Hence we must have $R \geq 180\,\Omega$.

## 7 AC Circuits with AC Source

We have previously discussed circuits that contain combinations of resistor, capacitor, and inductor, but only when they are connected to a dc source of emf or to no source. Now we discuss these circuit elements when they are connected to a source of alternating voltage that produces an alternating current (ac).

First we examine, one at a time, how a resistor, a capacitor, and an inductor behave when connected to a source of alternating voltage, represented by the symbol

 [alternating voltage]

which produces a sinusoidal voltage of frequency $f$. We assume in each case that the emf gives rise to a current

$$I = I_0 \cos 2\pi f t = I_0 \cos \omega t \tag{20}$$

where $t$ is time and $I_0$ is the peak current. Remember that $V_{\text{rms}} = V_0/\sqrt{2}$ and $I_{\text{rms}} = I_0/\sqrt{2}$.

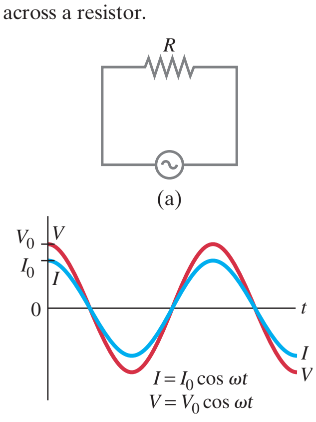



**FIGURE 15** (a) Resistor connected to an ac source. (b) Current (blue curve) is in phase with the voltage (red) across a resistor.

### Resistor

When an ac source is connected to a resistor as in Fig. 15a, the current increases and decreases with the alternating voltage according to Ohm's law

$$V = IR = I_0 R \cos \omega t = V_0 \cos \omega t$$

where $V_0 = I_0 R$ is the peak voltage as a function of time. Figure 15b shows the voltage (red curve) and the current (blue curve). Because the current is zero when the voltage is zero and the current reaches a peak when the voltage does, we say that the current and voltage are **in phase**. Energy is transformed into heat at an average rate

$$\overline{P} = \overline{IV} = I_{\text{rms}}^2 R = V_{\text{rms}}^2/R.$$

### Inductor

In Fig. 16a an inductor of inductance $L$ (symbol ) is connected to the ac source. We ignore any resistance it might have (it is usually small). The voltage applied to the inductor will be equal to the "back" emf generated in the inductor by the changing current as given by Eq. 5. This is because the sum of the electric potential changes around any closed circuit must add up to zero, according to Kirchhoff's rule.

Thus

$$V - L\frac{dI}{dt} = 0$$

or (inserting Eq. 20)

$$V = L\frac{dI}{dt} = -\omega L I_0 \sin \omega t. \tag{21}$$

Using the identity $\sin\theta = -\cos(\theta + 90°)$ we can write

$$V = \omega L I_0 \cos(\omega t + 90°) = V_0 \cos(\omega t + 90°) \tag{22a}$$

where

$$V_0 = I_0 \omega L \tag{22b}$$

is the peak voltage. The current $I$ and voltage $V$ as a function of time are graphed for the inductor in Fig. 16b. It is clear from this graph, as well as from Eqs. 22, that the current and voltage are out of phase by a quarter cycle, which is equivalent to $\pi/2$ radians or 90°. We see from the graph that

**the current lags the voltage by 90° in an inductor.**

That is, the current in an inductor reaches its peaks a quarter cycle later than the voltage does. Alternatively, we can say that the voltage leads the current by 90°.

Because the current and voltage in an inductor are out of phase by 90°, the product $IV$ (= power) is as often positive as it is negative (Fig. 16b). So no energy is transformed in an inductor on the average; and no energy is dissipated as thermal energy.

Just as a resistor impedes the flow of charge, so too an inductor impedes the flow of charge in an alternating current due to the back emf produced. For a resistor $R$, the peak current and peak voltage are related by $V_0 = I_0 R$. We can write a similar relation for an inductor:

$$V_0 = I_0 X_L \qquad \left[\begin{array}{c}\text{maximum or rms values,}\\ \text{not at any instant}\end{array}\right] \tag{23a}$$

where, from Eq. 22b (and using $\omega = 2\pi f$ where $f$ is the frequency of the ac),

$$X_L = \omega L = 2\pi f L. \tag{23b}$$

The term $X_L$ is called the **inductive reactance** of the inductor, and has units of ohms. The greater $X_L$ is, the more it impedes the flow of charge and the smaller the current. $X_L$ is larger for higher frequencies $f$ and larger inductance $L$.

Equation 23a is valid for peak values $I_0$ and $V_0$; it is also valid for rms values, $V_{rms} = I_{rms} X_L$. Because the peak values of current and voltage are not reached at the same time, Eq. 23a is *not valid at a particular instant*, as is the case for a resistor $(V = IR)$.

Note from Eq. 23b that if $\omega = 2\pi f = 0$ (so the current is dc), there is no back emf and no impedance to the flow of charge.

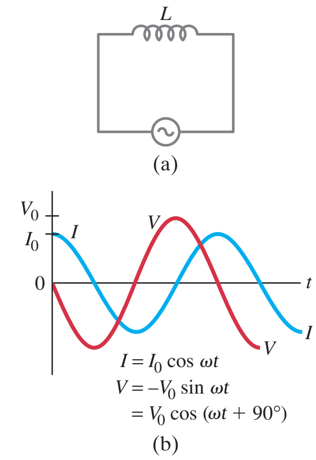



**FIGURE 16** (a) Inductor connected to an ac source. (b) Current (blue curve) lags voltage (red curve) by a quarter cycle or 90°.

**EXAMPLE 9** **Reactance of a coil.** A coil has a resistance $R = 1.00\,\Omega$ and an inductance of 0.300 H. Determine the current in the coil if (*a*) 120-V dc is applied to it, (*b*) 120-V ac (rms) at 60.0 Hz is applied.

**APPROACH** When the voltage is dc, there is no inductive reactance ($X_L = 2\pi f L = 0$ since $f = 0$), so we apply Ohm's law for the resistance. When the voltage is ac, we calculate the reactance $X_L$ and then use Eq. 23a.

**SOLUTION** (*a*) With dc, we have no $X_L$ so we simply apply Ohm's law:

$$I = \frac{V}{R} = \frac{120\text{ V}}{1.00\,\Omega} = 120\text{ A}.$$

(*b*) The inductive reactance is

$$X_L = 2\pi f L = (6.283)(60.0\text{ s}^{-1})(0.300\text{ H}) = 113\,\Omega.$$

In comparison to this, the resistance can be ignored. Thus,

$$I_{rms} = \frac{V_{rms}}{X_L} = \frac{120\text{ V}}{113\,\Omega} = 1.06\text{ A}.$$

**NOTE** It might be tempting to say that the total impedance is $113\,\Omega + 1\,\Omega = 114\,\Omega$. This might imply that about 1% of the voltage drop is across the resistor, or about 1 V; and that across the inductance is 119 V. Although the 1 V across the resistor is correct, the other statements are not true because of the alteration in phase in an inductor. This will be discussed in the next Section.

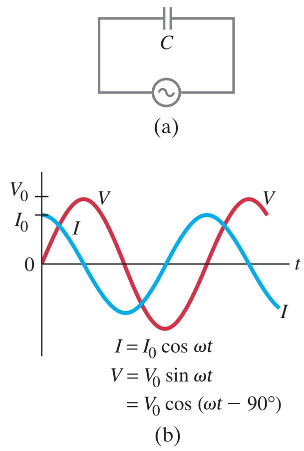



**FIGURE 17** (a) Capacitor connected to an ac source. (b) Current leads voltage by a quarter cycle, or 90°.

## Capacitor

When a capacitor is connected to a battery, the capacitor plates quickly acquire equal and opposite charges; but no steady current flows in the circuit. A capacitor prevents the flow of a dc current. But if a capacitor is connected to an alternating source of voltage, as in Fig. 17a, an alternating current will flow continuously. This can happen because when the ac voltage is first turned on, charge begins to flow and one plate acquires a negative charge and the other a positive charge. But when the voltage reverses itself, the charges flow in the opposite direction. Thus, for an alternating applied voltage, an ac current is present in the circuit continuously.

Let us look at this in more detail. By Kirchhoff's loop rule, the applied source voltage must equal the voltage $V$ across the capacitor at any moment:

$$V = \frac{Q}{C}$$

where $C$ is the capacitance and $Q$ is the charge on the capacitor plates. The current $I$ at any instant (given as $I = I_0 \cos \omega t$, Eq. 20) is

$$I = \frac{dQ}{dt} = I_0 \cos \omega t.$$

Hence the charge $Q$ on the plates at any instant is given by

$$Q = \int_0^t dQ = \int_0^t I_0 \cos \omega t \, dt = \frac{I_0}{\omega} \sin \omega t.$$

Then the voltage across the capacitor is

$$V = \frac{Q}{C} = I_0 \left( \frac{1}{\omega C} \right) \sin \omega t.$$

Using the trigonometric identity $\sin \theta = \cos(90° - \theta) = \cos(\theta - 90°)$, we can rewrite this as

$$V = I_0 \left( \frac{1}{\omega C} \right) \cos(\omega t - 90°) = V_0 \cos(\omega t - 90°) \tag{24a}$$

where

$$V_0 = I_0 \left( \frac{1}{\omega C} \right) \tag{24b}$$

is the peak voltage. The current $I \left(= I_0 \cos \omega t\right)$ and voltage $V$ (Eq. 24a) across the capacitor are graphed in Fig. 17b. It is clear from this graph, as well as a comparison of Eq. 24a with Eq. 20, that the current and voltage are out of phase by a quarter cycle or 90° ($\pi/2$ radians):

**The current leads the voltage across a capacitor by 90°.**

Alternatively we can say that the voltage lags the current by 90°. This is the opposite of what happens for an inductor.

Because the current and voltage are out of phase by 90°, the average power dissipated is zero, just as for an inductor. Energy from the source is fed to the capacitor, where it is stored in the electric field between its plates. As the field decreases, the energy returns to the source. Thus *only a resistance will dissipate energy* as thermal energy in an ac circuit.

A relationship between the applied voltage and the current in a capacitor can be written just as for an inductance:

$$V_0 = I_0 X_C \qquad \left[\begin{matrix}\text{maximum or rms values,} \\ \text{not at any instant}\end{matrix}\right] \tag{25a}$$

where $X_C$ is the **capacitive reactance** of the capacitor, and has units of ohms; $X_C$ is given by (see Eq. 24b):

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}. \tag{25b}$$

When frequency $f$ and/or capacitance $C$ are smaller, $X_C$ is larger and thus impedes the flow of charge more. That is, when $X_C$ is larger, the current is smaller (Eq. 25a). In the next Section we use the term **impedance** to represent reactances and resistance.

Equation 25a relates the peak values of $V$ and $I$, or the rms values $\left(V_{rms} = I_{rms} X_C\right)$. But it is not valid at a particular instant because $I$ and $V$ are not in phase.

Note from Eq. 25b that for dc conditions, $\omega = 2\pi f = 0$ and $X_C$ becomes infinite. This is as it should be, since a pure capacitor does not pass dc current. Also, note that the reactance of an inductor increases with frequency, but that of a capacitor decreases with frequency.

**EXAMPLE 10** **Capacitor reactance.** What is the rms current in the circuit of Fig. 17a if $C = 1.0\,\mu\text{F}$ and $V_{\text{rms}} = 120\,\text{V}$? Calculate (*a*) for $f = 60\,\text{Hz}$, and then (*b*) for $f = 6.0 \times 10^5\,\text{Hz}$.

**APPROACH** We find the reactance using Eq. 25b, and solve for current in the equivalent form of Ohm's law, Eq. 25a.

**SOLUTION** (*a*) $X_C = 1/2\pi fC = 1/(6.28)(60\,\text{s}^{-1})(1.0 \times 10^{-6}\,\text{F}) = 2.7\,\text{k}\Omega.$ The rms current is (Eq. 25a):

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{X_C} = \frac{120\,\text{V}}{2.7 \times 10^3\,\Omega} = 44\,\text{mA}.$$

(*b*) For $f = 6.0 \times 10^5\,\text{Hz}$, $X_C$ will be $0.27\,\Omega$ and $I_{\text{rms}} = 440\,\text{A}$, vastly larger!

**NOTE** The dependence on $f$ is dramatic. For high frequencies, the capacitive reactance is very small, and the current can be large.

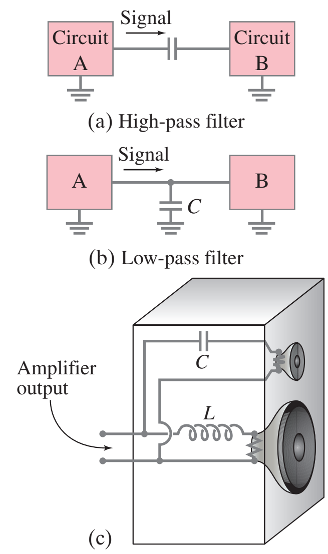



**FIGURE 18** (a) and (b) Two common uses for a capacitor as a filter. (c) Simple loudspeaker cross-over.

Two common applications of capacitors are illustrated in Fig. 18a and b. In Fig. 18a, circuit A is said to be capacitively coupled to circuit B. The purpose of the capacitor is to prevent a dc voltage from passing from A to B but allowing an ac signal to pass relatively unimpeded (if $C$ is sufficiently large, Eq. 25b). The capacitor in Fig. 18a is called a **high-pass filter** because it allows high-frequency ac to pass easily, but not dc.

PHYSICS APPLIED
*Capacitors as filters*

In Fig. 18b, the capacitor passes ac to ground. In this case, a dc voltage can be maintained between circuits A and B, but an ac signal leaving A passes to ground instead of into B. Thus the capacitor in Fig. 18b acts like a **low-pass filter** when a constant dc voltage is required; any high-frequency variation in voltage will pass to ground instead of into circuit B. (Very low-frequency ac will also be able to reach circuit B, at least in part.)

PHYSICS APPLIED
*Loudspeaker cross-over*

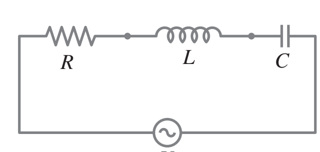
Loudspeakers having separate "woofer" (low-frequency speaker) and "tweeter" (high-frequency speaker) may use a simple "cross-over" that consists of a capacitor in the tweeter circuit to impede low-frequency signals, and an inductor in the woofer circuit to impede high-frequency signals $(X_L = 2\pi fL)$. Hence mainly low-frequency sounds reach and are emitted by the woofer. See Fig. 18c.

**EXERCISE G** At what frequency is the reactance of a 1.0-$\mu$F capacitor equal to $500\,\Omega$? (*a*) 320 Hz, (*b*) 500 Hz, (*c*) 640 Hz, (*d*) 2000 Hz, (*e*) 4000 Hz.

**EXERCISE H** At what frequency is the reactance of a 1.0-$\mu$H inductor equal to $500\,\Omega$? (*a*) 80 Hz, (*b*) 500 Hz, (*c*) 80 MHz, (*d*) 160 MHz, (*e*) 500 MHz.

# 8 *LRC* Series AC Circuit

**FIGURE 19** An *LRC* circuit.




Let us examine a circuit containing all three elements in series: a resistor $R$, an inductor $L$, and a capacitor $C$, Fig. 19. If a given circuit contains only two of these elements, we can still use the results of this Section by setting $R = 0$, $X_L = 0$, or $X_C = 0$, as needed. We let $V_R$, $V_L$, and $V_C$ represent the voltage across each element at a *given instant* in time; and $V_{R0}$, $V_{L0}$, and $V_{C0}$ represent the *maximum* (peak) values of these voltages. The voltage across each of the elements will follow the phase relations we discussed in the previous Section. At any instant the voltage $V$ supplied by the source will be, by Kirchhoff's loop rule,

$$V = V_R + V_L + V_C. \tag{26}$$

CAUTION
*Peak voltages do not add to yield source voltage*

Because the various voltages are not in phase, they do not reach their peak values at the same time, so the peak voltage of the source $V_0$ will *not* equal $V_{R0} + V_{L0} + V_{C0}$.

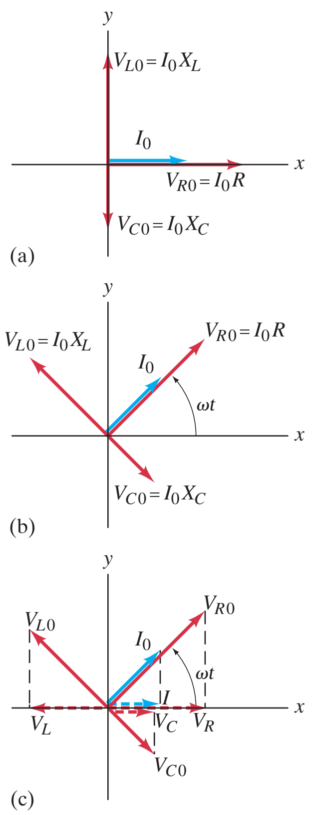
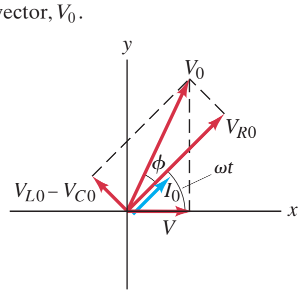



**FIGURE 20** Phasor diagram for a series $LRC$ circuit at (a) $t = 0$, (b) a time $t$ later. (c) Projections on $x$ axis reflect Eqs. 20, 22a, and 24a.

**FIGURE 21** Phasor diagram for a series $LRC$ circuit showing the sum vector, $V_0$.




Let us now find the impedance of an $LRC$ circuit as a whole (the effect of $R$, $X_C$, and $X_L$), as well as the peak current $I_0$, and the phase relation between $V$ and $I$. The current at any instant must be the same at all points in the circuit. Thus the *currents in each element are in phase with each other, even though the voltages are not.* We choose our origin in time $(t = 0)$ so that the current $I$ at any time $t$ is (as in Eq. 20)

$$I = I_0 \cos \omega t.$$

We analyze an $LRC$ circuit using† a **phasor diagram**. Arrows (treated like vectors) are drawn in an $xy$ coordinate system to represent each voltage. The *length of each arrow represents the magnitude of the peak voltage across each element*:

$$V_{R0} = I_0 R, \qquad V_{L0} = I_0 X_L, \qquad \text{and} \qquad V_{C0} = I_0 X_C.$$

$V_{R0}$ is in phase with the current and is initially $(t = 0)$ drawn along the positive $x$ axis, as is the current (Fig. 20a). $V_{L0}$ leads the current by 90°, so it leads $V_{R0}$ by 90° and is initially drawn along the positive $y$ axis. $V_{C0}$ lags the current by 90°, so $V_{C0}$ is drawn initially along the negative $y$ axis, Fig. 20a.

If we let the vector diagram rotate counterclockwise at frequency $f$, we get the diagram shown in Fig. 20b; after a time, $t$, each arrow has rotated through an angle $\omega t$. Then the *projections of each arrow on the x axis represent the voltages across each element* at the instant $t$ (Fig. 20c). For example $I = I_0 \cos \omega t$. Compare Eqs. 22a and 24a with Fig. 20c to confirm the validity of the phasor diagram.

The sum of the projections of the three voltage vectors represents the instantaneous voltage across the whole circuit, $V$. Therefore, the vector sum of these vectors will be the vector that represents the peak source voltage, $V_0$, as shown in Fig. 21 where it is seen that $V_0$ makes an angle $\phi$ with $I_0$ and $V_{R0}$. As time passes, $V_0$ rotates with the other vectors, so the instantaneous voltage $V$ (projection of $V_0$ on the $x$ axis) is (see Fig. 21)

$$V = V_0 \cos(\omega t + \phi).$$

The voltage $V$ across the whole circuit must equal the source voltage (Fig. 19). Thus the voltage from the source is out of phase‡ with the current by an angle $\phi$.

From this analysis we can now determine the total **impedance** $Z$ of the circuit, which is defined in analogy to resistance and reactance as

$$V_{\text{rms}} = I_{\text{rms}} Z, \qquad \text{or} \qquad V_0 = I_0 Z. \tag{27}$$

From Fig. 21 we see, using the Pythagorean theorem ($V_0$ is the hypotenuse of a right triangle), that

$$\begin{aligned} V_0 &= \sqrt{V_{R0}^2 + (V_{L0} - V_{C0})^2} \\ &= I_0 \sqrt{R^2 + (X_L - X_C)^2}. \end{aligned}$$

Thus, from Eq. 27, the total impedance $Z$ is

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \tag{28a}$$

$$= \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}. \tag{28b}$$

Also from Fig. 21, we can find the phase angle $\phi$ between voltage and current:

$$\tan \phi = \frac{V_{L0} - V_{C0}}{V_{R0}} = \frac{I_0 (X_L - X_C)}{I_0 R} = \frac{X_L - X_C}{R}. \tag{29a}$$

†We could instead do our analysis by rewriting Eq. 26 as a differential equation (setting $V_C = Q/C$, $V_R = IR = (dQ/dt)R$, and $V_L = L\, dI/dt$) and trying to solve the differential equation. The differential equation we would get would look like $m\, d^2x/dt^2 + b\, dx/dt + kx = F_0 \cos \omega t$ (on forced vibrations), and would be solved in the same way. Phasor diagrams are easier, and at the same time give us some physical insight.

‡As a check, note that if $R = X_C = 0$, then $\phi = 90°$, and $V_0$ would lead the current by 90°, as it must for an inductor alone. Similarly, if $R = L = 0$, $\phi = -90°$ and $V_0$ would lag the current by 90°, as it must for a capacitor alone.

We can also write

$$\cos\phi = \frac{V_{R0}}{V_0} = \frac{I_0 R}{I_0 Z} = \frac{R}{Z}. \tag{29b}$$

Figure 21 was drawn for the case $X_L > X_C$, and the current lags the source voltage by $\phi$. When the reverse is true, $X_L < X_C$, then $\phi$ in Eqs. 29 is less than zero, and the current leads the source voltage.

We saw earlier that power is dissipated only by a resistance; none is dissipated by inductance or capacitance. Therefore, the average power $\overline{P} = I_{\text{rms}}^2 R$. But from Eq. 29b, $R = Z\cos\phi$. Therefore

$$\overline{P} = I_{\text{rms}}^2 Z\cos\phi = I_{\text{rms}} V_{\text{rms}} \cos\phi. \tag{30}$$

The factor $\cos\phi$ is referred to as the **power factor** of the circuit. For a pure resistor, $\cos\phi = 1$ and $\overline{P} = I_{\text{rms}} V_{\text{rms}}$. For a capacitor or inductor alone, $\phi = -90°$ or $+90°$, respectively, so $\cos\phi = 0$ and no power is dissipated.

**EXAMPLE 11** ***LRC* circuit.** Suppose $R = 25.0\,\Omega$, $L = 30.0\,\text{mH}$, and $C = 12.0\,\mu\text{F}$ in Fig. 19, and they are connected in series to a 90.0-V ac (rms) 500-Hz source. Calculate (*a*) the current in the circuit, (*b*) the voltmeter readings (rms) across each element, (*c*) the phase angle $\phi$, and (*d*) the power dissipated in the circuit.

**APPROACH** To obtain the current we need to determine the impedance (Eq. 28 plus Eqs. 23b and 25b), and then use $I_{\text{rms}} = V_{\text{rms}}/Z$. Voltage drops across each element are found using Ohm's law or equivalent for each element: $V_R = IR$, $V_L = IX_L$, and $V_C = IX_C$.

**SOLUTION** (*a*) First, we find the reactance of the inductor and capacitor at $f = 500\,\text{Hz} = 500\,\text{s}^{-1}$:

$$X_L = 2\pi f L = 94.2\,\Omega, \qquad X_C = \frac{1}{2\pi f C} = 26.5\,\Omega.$$

Then the total impedance is

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{(25.0\,\Omega)^2 + (94.2\,\Omega - 26.5\,\Omega)^2} = 72.2\,\Omega.$$

From the impedance version of Ohm's law, Eq. 27,

$$I_{\text{rms}} = \frac{V_{\text{rms}}}{Z} = \frac{90.0\,\text{V}}{72.2\,\Omega} = 1.25\,\text{A}.$$

(*b*) The rms voltage across each element is

$$(V_R)_{\text{rms}} = I_{\text{rms}} R = (1.25\,\text{A})(25.0\,\Omega) = 31.2\,\text{V}$$
$$(V_L)_{\text{rms}} = I_{\text{rms}} X_L = (1.25\,\text{A})(94.2\,\Omega) = 118\,\text{V}$$
$$(V_C)_{\text{rms}} = I_{\text{rms}} X_C = (1.25\,\text{A})(26.5\,\Omega) = 33.1\,\text{V}.$$

**NOTE** These voltages do *not* add up to the source voltage, 90.0 V (rms). Indeed, the rms voltage across the inductance *exceeds* the source voltage. This can happen because the different voltages are out of phase with each other, and at any instant one voltage can be negative, to compensate for a large positive voltage of another. The rms voltages, however, are always positive by definition. Although the rms voltages need not add up to the source voltage, the instantaneous voltages at any time must add up to the source voltage at that instant.

**CAUTION**

*Individual peak or rms voltages do NOT add up to source voltage (due to phase differences)*

(*c*) The phase angle $\phi$ is given by Eq. 29b,

$$\cos\phi = \frac{R}{Z} = \frac{25.0\,\Omega}{72.2\,\Omega} = 0.346,$$

so $\phi = 69.7°$. Note that $\phi$ is positive because $X_L > X_C$ in this case, so $V_{L0} > V_{C0}$ in Fig. 21.

(*d*) $\overline{P} = I_{\text{rms}} V_{\text{rms}} \cos\phi = (1.25\,\text{A})(90.0\,\text{V})(25.0\,\Omega/72.2\,\Omega) = 39.0\,\text{W}.$

# 9 Resonance in AC Circuits

The rms current in an $LRC$ series circuit is given by (see Eqs. 27 and 28b):

$$I_{\mathrm{rms}} = \frac{V_{\mathrm{rms}}}{Z} = \frac{V_{\mathrm{rms}}}{\sqrt{R^2 + \left(\omega L - \dfrac{1}{\omega C}\right)^2}}. \tag{31}$$

Because the reactance of inductors and capacitors depends on the frequency $f$ $(= \omega/2\pi)$ of the source, the current in an $LRC$ circuit will depend on frequency. From Eq. 31 we can see that the current will be maximum at a frequency that satisfies

$$\left(\omega L - \frac{1}{\omega C}\right) = 0.$$

We solve this for $\omega$ and call the solution $\omega_0$:

$$\omega_0 = \sqrt{\frac{1}{LC}}. \qquad \text{[resonance]} \quad \textbf{(32)}$$

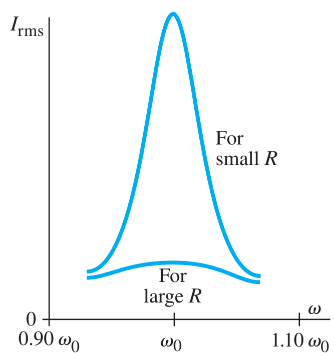



**FIGURE 22** Current in $LRC$ circuit as a function of angular frequency, $\omega$, showing resonance peak at $\omega = \omega_0 = \sqrt{1/LC}$.

When $\omega = \omega_0$, the circuit is in **resonance**, and $f_0 = \omega_0/2\pi$ is the **resonant frequency** of the circuit. At this frequency, $X_C = X_L$, so the impedance is purely resistive and $\cos\phi = 1$. A graph of $I_{\mathrm{rms}}$ versus $\omega$ is shown in Fig. 22 for particular values of $R$, $L$, and $C$. For small $R$ compared to $X_L$ and $X_C$, the resonance peak will be higher and sharper. When $R$ is very small, the circuit approaches the pure $LC$ circuit we discussed in Section 5. When $R$ is large compared to $X_L$ and $X_C$, the resonance curve is relatively flat—there is little frequency dependence.

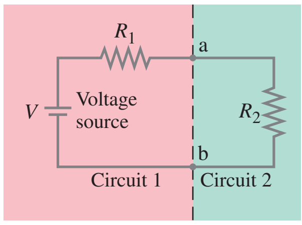
This electrical resonance is analogous to mechanical resonance. The energy transferred to the system by the source is a maximum at resonance whether it is electrical resonance, the oscillation of a spring, or pushing a child on a swing. That this is true in the electrical case can be seen from Eq. 30; at resonance, $\cos\phi = 1$, and power $\overline{P}$ is a maximum. A graph of power versus frequency peaks like that for the current, Fig. 22.

Electric resonance is used in many circuits. Radio and TV sets, for example, use resonant circuits for tuning in a station. Many frequencies reach the circuit, but a significant current flows only for those at or near the resonant frequency. Either $L$ or $C$ is variable so that different stations can be tuned in.

# *10 Impedance Matching

It is common to connect one electric circuit to a second circuit. For example, a TV antenna is connected to a TV receiver, an amplifier is connected to a loudspeaker; electrodes for an electrocardiogram are connected to a recorder. Maximum power is transferred from one to the other, with a minimum of loss, when the output impedance of the one device matches the input impedance of the second.

**FIGURE 23** Output of the circuit on the left is input to the circuit on the right.




To show why, we consider simple circuits that contain only resistance. In Fig. 23 the source in circuit 1 could represent the signal from an antenna or a laboratory probe, and $R_1$ represents its resistance including internal resistance of the source. $R_1$ is called the output impedance (or resistance) of circuit 1. The output of circuit 1 is across the terminals a and b which are connected to the input of circuit 2 which may be very complicated. We let $R_2$ be the equivalent "input resistance" of circuit 2.

The power delivered to circuit 2 is $P = I^2R_2$ where $I = V/(R_1 + R_2)$. Thus

$$P = I^2R_2 = \frac{V^2R_2}{(R_1 + R_2)^2}.$$

If the resistance of the source is $R_1$, what value should $R_2$ have so that the maximum power is transferred to circuit 2? To determine this, we take the derivative of $P$ with respect to $R_2$ and set it equal to zero, which gives

$$V^2\left[\frac{1}{(R_1 + R_2)^2} - \frac{2R_2}{(R_1 + R_2)^3}\right] = 0$$

or

$$R_2 = R_1.$$

Thus, the maximum power is transmitted when the *output impedance* of one device *equals the input impedance* of the second. This is called **impedance matching**.

In an ac circuit that contains capacitors and inductors, the different phases are important and the analysis is more complicated. However, the same result holds: to maximize power transfer it is important to match impedances $(Z_2 = Z_1)$.

In addition, it is possible to seriously distort a signal if impedances do not match, and this can lead to meaningless or erroneous experimental results.

CAUTION

*Erroneous results can occur if impedances don't match*

## *11 Three-Phase AC

Transmission lines typically consist of four wires, rather than two. One of these wires is the ground; the remaining three are used to transmit three-phase ac power which is a superposition of three ac voltages 120° out of phase with each other:

$$V_1 = V_0 \sin \omega t$$
$$V_2 = V_0 \sin(\omega t + 2\pi/3)$$
$$V_3 = V_0 \sin(\omega t + 4\pi/3).$$

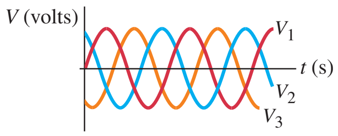



**FIGURE 24** The three voltages, out of phase by 120° $(= \frac{2}{3}\pi$ radians), in a three-phase power line.

(See Fig. 24.) Why is three-phase power used? Single-phase ac (i.e., the voltage $V_1$ by itself) delivers power to the load in pulses. A much smoother flow of power can be delivered using three-phase power. Suppose that each of the three voltages making up the three-phase source is hooked up to a resistor $R$. Then the power delivered is:

$$P = \frac{1}{R}(V_1^2 + V_2^2 + V_3^2).$$

You can show that this power is a constant equal to $3V_0^2/2R$, which is three times the rms power delivered by a single-phase source. This smooth flow of power makes electrical equipment run smoothly. Although houses use single-phase ac power, most industrial-grade machinery is wired for three-phase power.

**EXAMPLE 12** **Three-phase circuit.** In a three-phase circuit, 266 V rms exists between line 1 and ground. What is the rms voltage between lines 2 and 3?

**SOLUTION** We are given $V_{\text{rms}} = V_0/\sqrt{2} = 266\text{ V}$. Hence $V_0 = 376\text{ V}$. Now $V_3 - V_2 = V_0[\sin(\omega t + 4\pi/3) - \sin(\omega t + 2\pi/3)] = 2V_0 \sin\frac{1}{2}(\frac{2\pi}{3})\cos\frac{1}{2}(2\omega t)$ where we used the identity: $\sin A - \sin B = 2\sin\frac{1}{2}(A - B)\cos\frac{1}{2}(A + B)$. The rms voltage is

$$(V_3 - V_2)_{\text{rms}} = \frac{1}{\sqrt{2}}2V_0 \sin\frac{\pi}{3} = \sqrt{2}(376\text{ V})(0.866) = 460\text{ V (rms)}.$$

# Summary


A changing current in a coil of wire will induce an emf in a second coil placed nearby. The **mutual inductance**, $M$, is defined as the proportionality constant between the induced emf $\mathscr{E}_2$ in the second coil and the time rate of change of current in the first:

$$\mathscr{E}_2 = -M\, dI_1/dt. \tag{3b}$$

We can also write $M$ as

$$M = \frac{N_2 \Phi_{21}}{I_1} \tag{1}$$

where $\Phi_{21}$ is the magnetic flux through coil 2 with $N_2$ loops, produced by the current $I_1$ in another coil (coil 1).

Within a single coil, a changing current induces an opposing emf, $\mathscr{E}$, so a coil has a **self-inductance** $L$ defined by

$$\mathscr{E} = -L\, dI/dt. \tag{5}$$

This induced emf acts as an *impedance* to the flow of an alternating current. We can also write $L$ as

$$L = N\frac{\Phi_B}{I} \tag{4}$$

where $\Phi_B$ is the flux through the inductance when a current $I$ flows in its $N$ loops.

When the current in an inductance $L$ is $I$, the energy stored in the inductance is given by

$$U = \tfrac{1}{2}LI^2. \tag{6}$$

This energy can be thought of as being stored in the magnetic field of the inductor. The energy density $u$ in any magnetic field $B$ is given by

$$u = \frac{1}{2}\frac{B^2}{\mu_0}, \tag{7}$$

where $\mu_0$ is replaced by $\mu$ if a ferromagnetic material is present.

When an inductance $L$ and resistor $R$ are connected in series to a constant source of emf, $V_0$, the current rises according to an exponential of the form

$$I = \frac{V_0}{R}\left(1 - e^{-t/\tau}\right), \tag{9}$$

where

$$\tau = L/R \tag{10}$$

is the **time constant**. The current eventually levels out at $I = V_0/R$. If the battery is suddenly switched out of the ***LR* circuit**, and the circuit remains complete, the current drops exponentially, $I = I_0 e^{-t/\tau}$, with the same time constant $\tau$.

The current in a pure ***LC* circuit** (or charge on the capacitor) would oscillate sinusoidally. The energy too would oscillate back and forth between electric and magnetic, from the capacitor to the inductor, and back again. If such a circuit has resistance ($LRC$), and the capacitor at some instant is charged, it can undergo damped oscillations or exhibit critically damped or overdamped behavior.

Capacitance and inductance offer *impedance* to the flow of alternating current just as resistance does. This impedance is referred to as **reactance**, $X$, and is defined (as for resistors) as the proportionality constant between voltage and current (either the rms or peak values). Across an inductor,

$$V_0 = I_0 X_L, \tag{23a}$$

and across a capacitor,

$$V_0 = I_0 X_C. \tag{25a}$$

The reactance of an inductor increases with frequency:

$$X_L = \omega L. \tag{23b}$$

where $\omega = 2\pi f$ and $f$ is the frequency of the ac. The reactance of a capacitor decreases with frequency:

$$X_C = \frac{1}{\omega C}. \tag{25b}$$

Whereas the current through a resistor is always in phase with the voltage across it, this is not true for inductors and capacitors: in an inductor, the current lags the voltage by 90°, and in a capacitor the current leads the voltage by 90°.

In an ac ***LRC* series circuit**, the total **impedance** $Z$ is defined by the equivalent of $V = IR$ for resistance: namely $V_0 = I_0 Z$ or $V_{\rm rms} = I_{\rm rms} Z$. The impedance $Z$ is related to $R$, $C$, and $L$ by

$$Z = \sqrt{R^2 + (X_L - X_C)^2}. \tag{28a}$$

The current in the circuit lags (or leads) the source voltage by an angle $\phi$ given by $\cos\phi = R/Z$. Only the resistor in an ac $LRC$ circuit dissipates energy, and at a rate

$$\overline{P} = I_{\rm rms}^2 Z\cos\phi \tag{30}$$

where the factor $\cos\phi$ is referred to as the **power factor**.

An $LRC$ series circuit **resonates** at a frequency given by

$$\omega_0 = \sqrt{\frac{1}{LC}}. \tag{32}$$

The rms current in the circuit is largest when the applied voltage has a frequency equal to $f_0\ (= \omega_0/2\pi)$. The lower the resistance $R$, the higher and sharper the resonance peak.


# Answers to Exercises

**A:** (*a*) 360 A/s; (*b*) 12 V.

**B:** (*b*).

**C:** (*a*).

**D:** From Eq. 5, $L$ has dimensions $VT/A$ so $(L/R)$ has dimensions $(VT/A)/(V/A) = T$.

**E:** (*d*).

**F:** (*c*)

**G:** (*a*).

**H:** (*c*).