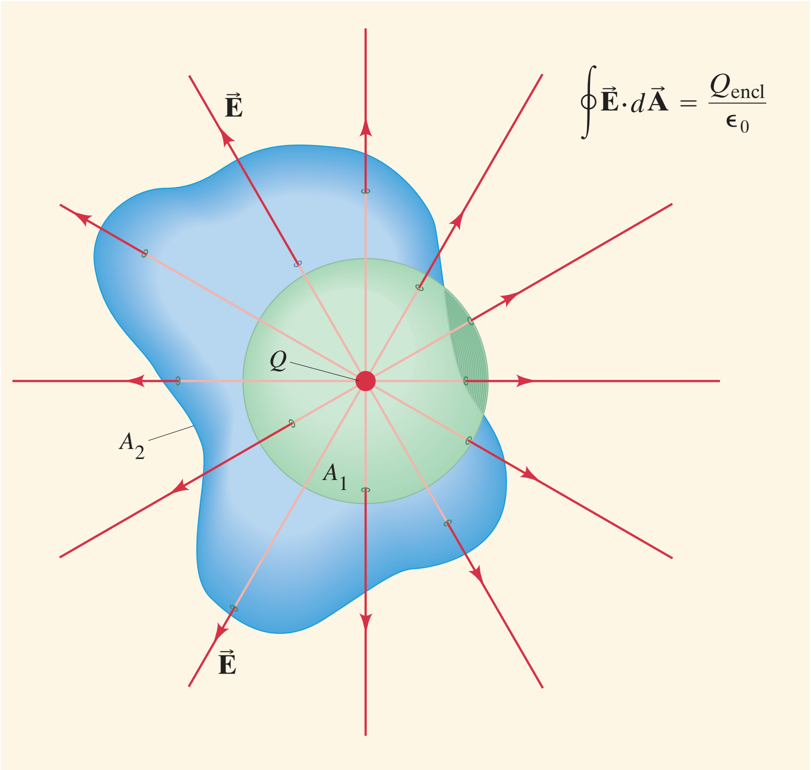



Gauss's law is an elegant relation between electric charge and electric field. It is more general than Coulomb's law. Gauss's law involves an integral of the electric field $\vec{E}$ at each point on a closed surface. The surface is only imaginary, and we choose the shape and placement of the surface so that we can evaluate the integral. In this drawing, two different 3-D surfaces are shown (one green, one blue), both enclosing a point charge $Q$. Gauss's law states that the product $\vec{E} \cdot d\vec{A}$, where $d\vec{A}$ is an infinitesimal area of the surface, integrated over the entire surface, equals the charge enclosed by the surface $Q_{encl}$ divided by $\epsilon_0$. Both surfaces here enclose the same charge $Q$. Hence $\oint \vec{E} \cdot d\vec{A}$ will give the same result for both surfaces.

# Gauss's Law

**CHAPTER-OPENING QUESTION — Guess now!**

[*Don't worry about getting the right answer now—the idea is to get your preconceived notions out on the table.*]

A nonconducting sphere has a uniform charge density throughout. How does the magnitude of the electric field vary inside with distance from the center?

**(a)** The electric field is zero throughout.
**(b)** The electric field is constant but nonzero throughout.
**(c)** The electric field is linearly increasing from the center to the outer edge.
**(d)** The electric field is exponentially increasing from the center to the outer edge.
**(e)** The electric field increases quadratically from the center to the outer edge.

CONTENTS



The great mathematician Karl Friedrich Gauss (1777–1855) developed an important relation, now known as Gauss's law, which we develop and discuss in this Chapter. It is a statement of the relation between electric charge and electric field and is a more general and elegant form of Coulomb's law.

We can, in principle, determine the electric field due to any given distribution of electric charge using Coulomb's law. The total electric field at any point will be the vector sum (or integral) of contributions from all charges present. Except for some simple cases, the sum or integral can be quite complicated to evaluate. For situations in which an analytic solution is not possible, a computer can be used.

In some cases, however, the electric field due to a given charge distribution can be calculated more easily or more elegantly using Gauss's law, as we shall see in this Chapter. But the major importance of Gauss's law is that it gives us additional insight into the nature of electrostatic fields, and a more general relationship between charge and field.

Before discussing Gauss's law itself, we first discuss the c oncept of *flux*.

Note: Sections marked with an asterisk (*) may be considered optional by the instructor.


From Chapter 22 of *Physics for Scientists & Engineers with Modern Physics*, Fourth Edition, Douglas C. Giancoli.

# 1 Electric Flux

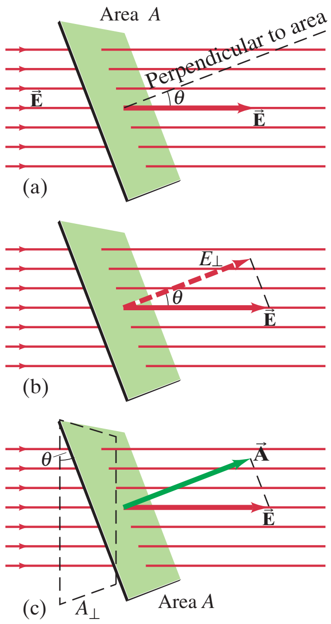



**FIGURE 1** (a) A uniform electric field $\vec{\mathbf{E}}$ passing through a flat area $A$. (b) $E_\perp = E\cos\theta$ is the component of $\vec{\mathbf{E}}$ perpendicular to the plane of area $A$. (c) $A_\perp = A\cos\theta$ is the projection (dashed) of the area $A$ perpendicular to the field $\vec{\mathbf{E}}$.

Gauss's law involves the concept of **electric flux**, which refers to the electric field passing through a given area. For a uniform electric field $\vec{\mathbf{E}}$ passing through an area $A$, as shown in Fig. 1a, the electric flux $\Phi_E$ is defined as

$$\Phi_E = EA\cos\theta,$$

where $\theta$ is the angle between the electric field direction and a line drawn perpendicular to the area. The flux can be written equivalently as

$$\Phi_E = E_\perp A = EA_\perp = EA\cos\theta, \qquad [\vec{\mathbf{E}} \text{ uniform}] \quad \textbf{(1a)}$$

where $E_\perp = E\cos\theta$ is the component of $\vec{\mathbf{E}}$ along the perpendicular to the area (Fig. 1b) and, similarly, $A_\perp = A\cos\theta$ is the projection of the area $A$ perpendicular to the field $\vec{\mathbf{E}}$ (Fig. 1c).

The area $A$ of a surface can be represented by a vector $\vec{\mathbf{A}}$ whose magnitude is $A$ and whose direction is perpendicular to the surface, as shown in Fig. 1c. The angle $\theta$ is the angle between $\vec{\mathbf{E}}$ and $\vec{\mathbf{A}}$, so the electric flux can also be written

$$\Phi_E = \vec{\mathbf{E}} \cdot \vec{\mathbf{A}}. \qquad [\vec{\mathbf{E}} \text{ uniform}] \quad \textbf{(1b)}$$

Electric flux has a simple intuitive interpretation in terms of field lines. Field lines can always be drawn so that the number ($N$) passing through unit area perpendicular to the field $(A_\perp)$ is proportional to the magnitude of the field ($E$): that is, $E \propto N/A_\perp$. Hence,

$$N \propto EA_\perp = \Phi_E,$$

so the flux through an area is proportional to the number of lines passing through that area.

**EXAMPLE 1** **Electric flux.** Calculate the electric flux through the rectangle shown in Fig. 1a. The rectangle is 10 cm by 20 cm, the electric field is uniform at 200 N/C, and the angle $\theta$ is 30°.

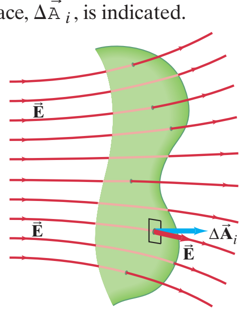
**APPROACH** We use the definition of flux, $\Phi_E = \vec{\mathbf{E}} \cdot \vec{\mathbf{A}} = EA\cos\theta$.

**SOLUTION** The electric flux is

$$\Phi_E = (200\,\mathrm{N/C})(0.10\,\mathrm{m} \times 0.20\,\mathrm{m})\cos 30° = 3.5\,\mathrm{N \cdot m^2/C}.$$

**EXERCISE A** Which of the following would cause a change in the electric flux through a circle lying in the $xz$ plane where the electric field is $(10\,\mathrm{N})\hat{\mathbf{j}}$? (*a*) Changing the magnitude of the electric field. (*b*) Changing the size of the circle. (*c*) Tipping the circle so that it is lying in the $xy$ plane. (*d*) All of the above. (*e*) None of the above.

**FIGURE 2** Electric flux through a curved surface. One small area of the surface, $\Delta\vec{\mathbf{A}}_i$, is indicated.



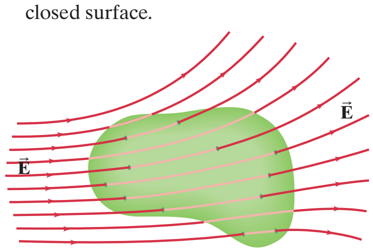

In the more general case, when the electric field $\vec{\mathbf{E}}$ is not uniform and the surface is not flat, Fig. 2, we divide up the chosen surface into $n$ small elements of surface whose areas are $\Delta A_1, \Delta A_2, \cdots, \Delta A_n$. We choose the division so that each $\Delta A_i$ is small enough that (1) it can be considered flat, and (2) the electric field varies so little over this small area that it can be considered uniform. Then the electric flux through the entire surface is approximately

$$\Phi_E \approx \sum_{i=1}^{n} \vec{\mathbf{E}}_i \cdot \Delta\vec{\mathbf{A}}_i,$$

where $\vec{\mathbf{E}}_i$ is the field passing through $\Delta\vec{\mathbf{A}}_i$. In the limit as we let $\Delta\vec{\mathbf{A}}_i \to 0$, the sum becomes an integral over the entire surface and the relation becomes mathematically exact:

$$\Phi_E = \int \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}. \quad \textbf{(2)}$$

**FIGURE 3** Electric flux through a closed surface.




Gauss's law involves the *total* flux through a *closed* surface — a surface of any shape that completely encloses a volume of space, such as that shown in Fig. 3. In this case, the net flux through the enclosing surface is given by

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}, \quad \textbf{(3)}$$

where the integral sign is written $\oint$ to indicate that the integral is over the value of $\vec{\mathbf{E}}$ at every point on an enclosing surface.

Up to now we have not been concerned with an ambiguity in the direction of the vector $\vec{\mathbf{A}}$ or $d\vec{\mathbf{A}}$ that represents a surface. For example, in Fig. 1c, the vector $\vec{\mathbf{A}}$ could point upward and to the right (as shown) or downward to the left and still be perpendicular to the surface. For a closed surface, we define (arbitrarily) the direction of $\vec{\mathbf{A}}$, or of $d\vec{\mathbf{A}}$, to point *outward* from the enclosed volume, Fig. 4. For an electric field line leaving the enclosed volume (on the right in Fig. 4), the angle $\theta$ between $\vec{\mathbf{E}}$ and $d\vec{\mathbf{A}}$ must be less than $\pi/2\,(=90°)$; hence $\cos\theta > 0$. For a line entering the volume (on the left in Fig. 4) $\theta > \pi/2$; hence $\cos\theta < 0$. Hence, *flux entering the enclosed volume is negative* $\left(\int E\cos\theta\, dA < 0\right)$, whereas *flux leaving the volume is positive*. Consequently, Eq. 3 gives the net flux *out* of the volume. If $\Phi_E$ is negative, there is a net flux *into* the volume.

In Figs. 3 and 4, each field line that enters the volume also leaves the volume. Hence $\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = 0.$ There is no net flux into or out of this enclosed surface. The flux, $\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$, will be nonzero only if one or more lines start or end within the surface. Since electric field lines start and stop only on electric charges, the flux will be nonzero only if the surface encloses a net charge. For example, the surface labeled $A_1$ in Fig. 5 encloses a positive charge and there is a net outward flux through this surface $\left(\Phi_E > 0\right)$. The surface $A_2$ encloses an equal magnitude negative charge and there is a net inward flux $\left(\Phi_E < 0\right)$. For the configuration shown in Fig. 6, the flux through the surface shown is negative (count the lines). The value of $\Phi_E$ depends on the charge enclosed by the surface, and this is what Gauss's law is all about.

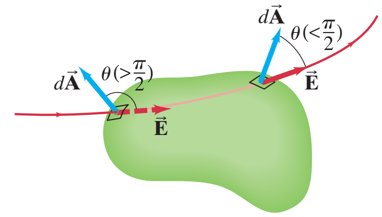
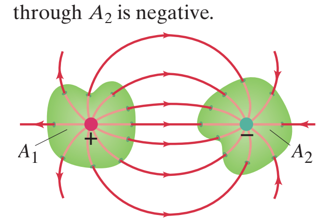



**FIGURE 4** The direction of an element of area $d\vec{\mathrm{A}}$ is taken to point outward from an enclosed surface.

**FIGURE 5** An electric dipole. Flux through surface $A_1$ is positive. Flux through $A_2$ is negative.



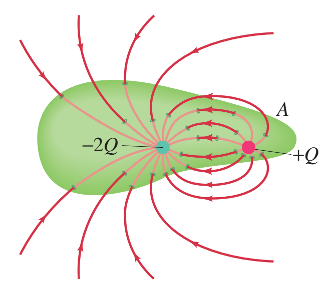

## 2 Gauss's Law

The precise relation between the electric flux through a closed surface and the net charge $Q_{\text{encl}}$ enclosed within that surface is given by **Gauss's law**:

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{\text{encl}}}{\epsilon_0}, \tag{4}$$

where $\epsilon_0$ is the same constant (permittivity of free space) that appears in Coulomb's law. The integral on the left is over the value of $\vec{\mathbf{E}}$ on any closed surface, and we choose that surface for our convenience in any given situation. The charge $Q_{\text{encl}}$ is the net charge *enclosed* by that surface. It doesn't matter where or how the charge is distributed within the surface. Any charge outside this surface must not be included. A charge outside the chosen surface may affect the position of the electric field lines, but will not affect the net number of lines entering or leaving the surface. For example, $Q_{\text{encl}}$ for the gaussian surface $A_1$ in Fig. 5 would be the positive charge enclosed by $A_1$; the negative charge does contribute to the electric field at $A_1$ but it is *not* enclosed by surface $A_1$ and so is not included in $Q_{\text{encl}}$.

**FIGURE 6** Net flux through surface $A$ is negative.




Now let us see how Gauss's law is related to Coulomb's law. First, we show that Coulomb's law follows from Gauss's law. In Fig. 7 we have a single isolated charge $Q$. For our "gaussian surface," we choose an imaginary sphere of radius $r$ centered on the charge. Because Gauss's law is supposed to be valid for any surface, we have chosen one that will make our calculation easy. Because of the *symmetry* of this (imaginary) sphere about the charge at its center, we know that $\vec{\mathbf{E}}$ must have the same magnitude at any point on the surface, and that $\vec{\mathbf{E}}$ points radially outward (inward for a negative charge) parallel to $d\vec{\mathbf{A}}$, an element of the surface area. Hence, we write the integral in Gauss's law as

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint E\, dA = E\oint dA = E\left(4\pi r^2\right)$$

since the surface area of a sphere of radius $r$ is $4\pi r^2$, and the magnitude of $\vec{\mathbf{E}}$ is the same at all points on this spherical surface. Then Gauss's law becomes, with $Q_{\text{encl}} = Q$,

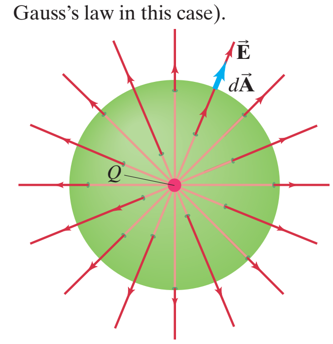
$$\frac{Q}{\epsilon_0} = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E\left(4\pi r^2\right)$$

because $\vec{\mathbf{E}}$ and $d\vec{\mathbf{A}}$ are both perpendicular to the surface at each point, and $\cos\theta = 1.$ Solving for $E$ we obtain

$$E = \frac{Q}{4\pi\epsilon_0 r^2},$$

which is the electric field form of Coulomb's law.

**FIGURE 7** A single point charge $Q$ at the center of an imaginary sphere of radius $r$ (our "gaussian surface"—that is, the closed surface we choose to use for applying Gauss's law in this case).



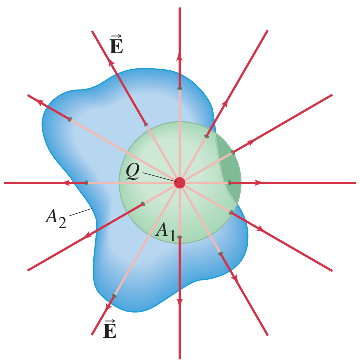



**FIGURE 8** A single point charge surrounded by a spherical surface, $A_1$, and an irregular surface, $A_2$.

Now let us do the reverse, and derive Gauss's law from Coulomb's law for static electric charges[†]. First we consider a single point charge $Q$ surrounded by an imaginary spherical surface as in Fig. 7(and shown again, green, in Fig. 8). Coulomb's law tells us that the electric field at the spherical surface is $E = (1/4\pi\epsilon_0)(Q/r^2)$. Reversing the argument we just used, we have

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} dA = \frac{Q}{4\pi\epsilon_0 r^2}(4\pi r^2) = \frac{Q}{\epsilon_0}.$$

This is Gauss's law, with $Q_{\text{encl}} = Q$, and we derived it for the special case of a spherical surface enclosing a point charge at its center. But what about some other surface, such as the irregular surface labeled $A_2$ in Fig. 8? The same number of field lines (due to our charge $Q$) pass through surface $A_2$, as pass through the spherical surface, $A_1$. Therefore, because the flux through a surface is proportional to the number of lines through it as we saw in Section 1, the flux through $A_2$ is the same as through $A_1$:

$$\oint_{A_2} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint_{A_1} \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q}{\epsilon_0}.$$

Hence, we can expect that

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q}{\epsilon_0}$$

would be valid for *any* surface surrounding a single point charge $Q$.

Finally, let us look at the case of more than one charge. For each charge, $Q_i$, enclosed by the chosen surface,

$$\oint \vec{\mathbf{E}}_i \cdot d\vec{\mathbf{A}} = \frac{Q_i}{\epsilon_0},$$

where $\vec{\mathbf{E}}_i$ refers to the electric field produced by $Q_i$ alone. By the superposition principle for electric fields, the total field $\vec{\mathbf{E}}$ is equal to the sum of the fields due to each separate charge, $\vec{\mathbf{E}} = \Sigma\vec{\mathbf{E}}_i$. Hence

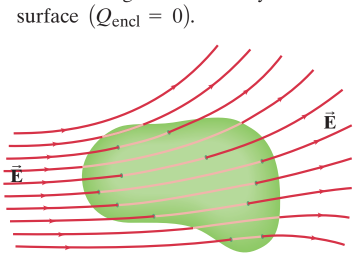
$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint (\Sigma\vec{\mathbf{E}}_i) \cdot d\vec{\mathbf{A}} = \Sigma \frac{Q_i}{\epsilon_0} = \frac{Q_{\text{encl}}}{\epsilon_0},$$

where $Q_{\text{encl}} = \Sigma Q_i$ is the total net charge enclosed within the surface. Thus we see, based on this simple argument, that Gauss's law follows from Coulomb's law for any distribution of static electric charge enclosed within a closed surface of any shape.

The derivation of Gauss's law from Coulomb's law is valid for electric fields produced by static electric charges. We will see later that electric fields can also be produced by changing magnetic fields. Coulomb's law cannot be used to describe such electric fields. But Gauss's law *is* found to hold also for electric fields produced in any of these ways. Hence *Gauss's law is a more general law than Coulomb's law*. It holds for any electric field whatsoever.

Even for the case of static electric fields that we are considering in this Chapter, it is important to recognize that $\vec{\mathbf{E}}$ on the left side of Gauss's law is not necessarily due only to the charge $Q_{\text{encl}}$ that appears on the right. For example, in Fig. 9 there is an electric field $\vec{\mathbf{E}}$ at all points on the imaginary gaussian surface, but it is not due to the charge enclosed by the surface (which is $Q_{\text{encl}} = 0$ in this case). The electric field $\vec{\mathbf{E}}$ which appears on the left side of Gauss's law is the *total* electric field at each point, on the gaussian surface chosen, not just that due to the charge $Q_{\text{encl}}$, which appears on the right side. Gauss's law has been found to be valid for the total field at any surface. It tells us that any *difference* between the input and output flux of the electric field over any surface is due to charge within that surface.

**FIGURE 9** Electric flux through a closed surface. (Same as Fig. 3.) No electric charge is enclosed by this surface $(Q_{\text{encl}} = 0)$.




[†]Note that Gauss's law would look more complicated in terms of the constant $k = 1/4\pi\epsilon_0$ than in Coulomb's law:

| *Coulomb's law* | *Gauss's law* |
|---|---|
| $E = k\dfrac{Q}{r^2}$ | $\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = 4\pi kQ$ |
| $E = \dfrac{1}{4\pi\epsilon_0}\dfrac{Q}{r^2}$ | $\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \dfrac{Q}{\epsilon_0}.$ |

Gauss's law has a simpler form using $\epsilon_0$; Coulomb's law is simpler using $k$. The normal convention is to use $\epsilon_0$ rather than $k$ because Gauss's law is considered more general and therefore it is preferable to have it in simpler form.

**CONCEPTUAL EXAMPLE 2** **Flux from Gauss's law.** Consider the two gaussian surfaces, $A_1$ and $A_2$, shown in Fig. 10. The only charge present is the charge $Q$ at the center of surface $A_1$. What is the net flux through each surface, $A_1$ and $A_2$?

**RESPONSE** The surface $A_1$ encloses the charge $+Q$. By Gauss's law, the net flux through $A_1$ is then $Q/\epsilon_0$. For surface $A_2$, the charge $+Q$ is outside the surface. Surface $A_2$ encloses zero net charge, so the net electric flux through $A_2$ is zero, by Gauss's law. Note that all field lines that enter the volume enclosed by surface $A_2$ also leave it.

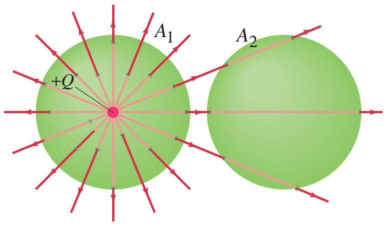



**FIGURE 10** Example 2. Two gaussian surfaces.

**EXERCISE B** A point charge $Q$ is at the center of a spherical gaussian surface $A$. When a second charge $Q$ is placed just outside $A$, the total flux through this spherical surface $A$ is (*a*) unchanged, (*b*) doubled, (*c*) halved, (*d*) none of these.

**EXERCISE C** Three $2.95\ \mu\text{C}$ charges are in a small box. What is the net flux leaving the box? (*a*) $3.3 \times 10^{12}\,\text{N}\cdot\text{m}^2/\text{C}$, (*b*) $3.3 \times 10^{5}\,\text{N}\cdot\text{m}^2/\text{C}$, (*c*) $1.0 \times 10^{12}\,\text{N}\cdot\text{m}^2/\text{C}$, (*d*) $1.0 \times 10^{6}\,\text{N}\cdot\text{m}^2/\text{C}$, (*e*) $6.7 \times 10^{6}\,\text{N}\cdot\text{m}^2/\text{C}$.

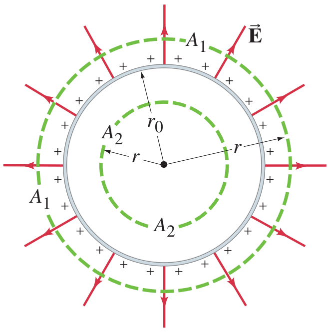
We note that the integral in Gauss's law is often rather difficult to carry out in practice. We rarely need to do it except for some fairly simple situations that we now discuss.

# 3 Applications of Gauss's Law

Gauss's law is a very compact and elegant way to write the relation between electric charge and electric field. It also offers a simple way to determine the electric field when the charge distribution is simple and/or possesses a high degree of *symmetry*. In order to apply Gauss's law, however, we must choose the "gaussian" surface very carefully (for the integral on the left side of Gauss's law) so we can determine $\vec{\mathbf{E}}$. We normally try to think of a surface that has just the symmetry needed so that $E$ will be constant on all or on parts of its surface. Sometimes we choose a surface so the flux through part of the surface is zero.

**EXAMPLE 3** **Spherical conductor.** A thin spherical shell of radius $r_0$ possesses a total net charge $Q$ that is uniformly distributed on it (Fig. 11). Determine the electric field at points (*a*) outside the shell, and (*b*) inside the shell. (*c*) What if the conductor were a solid sphere?

**FIGURE 11** Cross-sectional drawing of a thin spherical shell of radius $r_0$, carrying a net charge $Q$ uniformly distributed. $A_1$ and $A_2$ represent two gaussian surfaces we use to determine $\vec{\mathbf{E}}$. Example 3.




**APPROACH** Because the charge is distributed symmetrically, the electric field must also be *symmetric*. Thus the field outside the sphere must be directed radially outward (inward if $Q < 0$) and must depend only on $r$, not on angle (spherical coordinates).

**SOLUTION** (*a*) The electric field will have the same magnitude at all points on an imaginary gaussian surface, if we choose that surface as a sphere of radius $r$ $(r > r_0)$ concentric with the shell, and shown in Fig. 11 as the dashed circle $A_1$. Because $\vec{\mathbf{E}}$ is perpendicular to this surface, the cosine of the angle between $\vec{\mathbf{E}}$ and $d\vec{\mathbf{A}}$ is always 1. Gauss's law then gives (with $Q_{\text{encl}} = Q$ in Eq. 4)

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E(4\pi r^2) = \frac{Q}{\epsilon_0},$$

where $4\pi r^2$ is the surface area of our sphere (gaussian surface) of radius $r$. Thus

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}. \qquad [r > r_0]$$

Thus the field outside a uniformly charged spherical shell is the same as if all the charge were concentrated at the center as a point charge.

(*b*) Inside the shell, the electric field must also be symmetric. So $E$ must again have the same value at all points on a spherical gaussian surface ($A_2$ in Fig. 11) concentric with the shell. Thus $E$ can be factored out of the integral and, with $Q_{\text{encl}} = 0$ because the charge enclosed within the sphere $A_2$ is zero, we have

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E(4\pi r^2) = 0.$$

Hence

$$E = 0 \qquad [r < r_0]$$

inside a uniform spherical shell of charge.

(*c*) These same results also apply to a uniformly charged solid spherical conductor, since all the charge would lie in a thin layer at the surface.

**EXERCISE D** A charge $Q$ is placed on a hollow metal ball. The charge is all on the surface of the ball because metal is a conductor. How does the charge distribute itself on the ball? (*a*) Half on the inside surface and half on the outside surface. (*b*) Part on each surface in inverse proportion to the two radii. (*c*) Part on each surface but with a more complicated dependence on the radii than in answer (*b*). (*d*) All on the inside surface. (*e*) All on the outside surface.

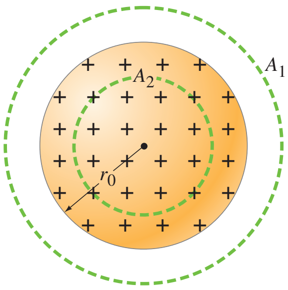



**FIGURE 12** A solid sphere of uniform charge density. Example 4.

**EXAMPLE 4** **Solid sphere of charge.** An electric charge $Q$ is distributed uniformly throughout a nonconducting sphere of radius $r_0$, Fig. 12. Determine the electric field (*a*) outside the sphere $(r > r_0)$ and (*b*) inside the sphere $(r < r_0)$.

**APPROACH** Since the charge is distributed symmetrically in the sphere, the electric field at all points must again be *symmetric*. $\vec{\mathbf{E}}$ depends only on $r$ and is directed radially outward (or inward if $Q < 0$).

**SOLUTION** (*a*) For our gaussian surface we choose a sphere of radius $r$ $(r > r_0)$, labeled $A_1$ in Fig. 12. Since $E$ depends only on $r$, Gauss's law gives, with $Q_{\text{encl}} = Q$,

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E(4\pi r^2) = \frac{Q}{\epsilon_0}$$

or

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}.$$

Again, the field outside a spherically symmetric distribution of charge is the same as that for a point charge of the same magnitude located at the center of the sphere.

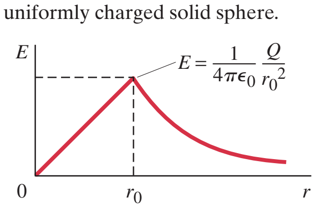
(*b*) Inside the sphere, we choose for our gaussian surface a concentric sphere of radius $r$ $(r < r_0)$, labeled $A_2$ in Fig. 12. From symmetry, the magnitude of $\vec{\mathbf{E}}$ is the same at all points on $A_2$, and $\vec{\mathbf{E}}$ is perpendicular to the surface, so

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E(4\pi r^2).$$

We must equate this to $Q_{\text{encl}}/\epsilon_0$ where $Q_{\text{encl}}$ is the charge enclosed by $A_2$. $Q_{\text{encl}}$ is not the total charge $Q$ but only a portion of it. We define the **charge density**, $\rho_E$, as the charge per unit volume $(\rho_E = dQ/dV)$, and here we are given that $\rho_E =$ constant. So the charge enclosed by the gaussian surface $A_2$, a sphere of radius $r$, is

$$Q_{\text{encl}} = \left(\frac{\frac{4}{3}\pi r^3 \rho_E}{\frac{4}{3}\pi r_0^3 \rho_E}\right) Q = \frac{r^3}{r_0^3} Q.$$

**FIGURE 13** Magnitude of the electric field as a function of the distance $r$ from the center of a uniformly charged solid sphere.




Hence, from Gauss's law,

$$E(4\pi r^2) = \frac{Q_{\text{encl}}}{\epsilon_0} = \frac{r^3}{r_0^3}\frac{Q}{\epsilon_0}$$

or

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r_0^3} r. \qquad [r < r_0]$$

Thus the field increases linearly with $r$, until $r = r_0$. It then decreases as $1/r^2$, as plotted in Fig. 13.

**EXERCISE E** Return to the Chapter-Opening Question and answer it again now. Try to explain why you may have answered differently the first time.

The results in Example 4 would have been difficult to obtain from Coulomb's law by integrating over the sphere. Using Gauss's law and the *symmetry* of the situation, this result is obtained rather easily, and shows the great power of Gauss's law. However, its use in this way is limited mainly to cases where the charge distribution has a high degree of symmetry. In such cases, we *choose* a simple surface on which $E =$ constant, so the integration is simple. Gauss's law holds, of course, for any surface.

**EXAMPLE 5** **Nonuniformly charged solid sphere.** Suppose the charge density of the solid sphere in Fig. 12, Example 4, is given by $\rho_E = \alpha r^2$, where $\alpha$ is a constant. (*a*) Find $\alpha$ in terms of the total charge $Q$ on the sphere and its radius $r_0$. (*b*) Find the electric field as a function of $r$ inside the sphere.

**APPROACH** We divide the sphere up into concentric thin shells of thickness $dr$ as shown in Fig. 14, and integrate (*a*) setting $Q = \int \rho_E \, dV$ and (*b*) using Gauss's law.

**SOLUTION** (*a*) A thin shell of radius $r$ and thickness $dr$ (Fig. 14) has volume $dV = 4\pi r^2 \, dr$. The total charge is given by

$$Q = \int \rho_E \, dV = \int_0^{r_0} (\alpha r^2)(4\pi r^2 \, dr) = 4\pi\alpha \int_0^{r_0} r^4 \, dr = \frac{4\pi\alpha}{5} r_0^5.$$

Thus $\alpha = 5Q/4\pi r_0^5$.

(*b*) To find $E$ inside the sphere at distance $r$ from its center, we apply Gauss's law to an imaginary sphere of radius $r$ which will enclose a charge

$$Q_{\text{encl}} = \int_0^r \rho_E \, dV = \int_0^r (\alpha r^2)\, 4\pi r^2 \, dr = \int_0^r \left(\frac{5Q}{4\pi r_0^5} r^2\right) 4\pi r^2 \, dr = Q\frac{r^5}{r_0^5}.$$

By *symmetry*, $E$ will be the same at all points on the surface of a sphere of radius $r$, so Gauss's law gives

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{\text{encl}}}{\epsilon_0}$$

$$(E)(4\pi r^2) = Q\frac{r^5}{\epsilon_0 r_0^5},$$

so

$$E = \frac{Qr^3}{4\pi\epsilon_0 r_0^5}.$$

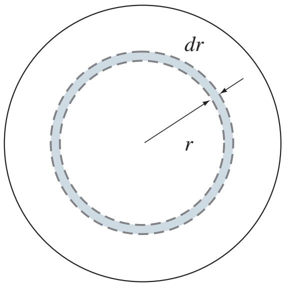



**FIGURE 14** Example 5.

**EXAMPLE 6** **Long uniform line of charge.** A very long straight wire possesses a uniform positive charge per unit length, $\lambda$. Calculate the electric field at points near (but outside) the wire, far from the ends.

**APPROACH** Because of the *symmetry*, we expect the field to be directed radially outward and to depend only on the perpendicular distance, $R$, from the wire. Because of the cylindrical symmetry, the field will be the same at all points on a gaussian surface that is a cylinder with the wire along its axis, Fig. 15. $\vec{\mathbf{E}}$ is perpendicular to this surface at all points. For Gauss's law, we need a closed surface, so we include the flat ends of the cylinder. Since $\vec{\mathbf{E}}$ is parallel to the ends, there is no flux through the ends (the cosine of the angle between $\vec{\mathbf{E}}$ and $d\vec{\mathbf{A}}$ on the ends is $\cos 90° = 0$).

**SOLUTION** For our chosen gaussian surface Gauss's law gives

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E(2\pi R\ell) = \frac{Q_{\text{encl}}}{\epsilon_0} = \frac{\lambda\ell}{\epsilon_0},$$

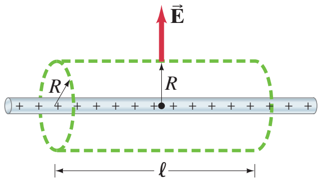
where $\ell$ is the length of our chosen gaussian surface ($\ell \ll$ length of wire), and $2\pi R$ is its circumference. Hence

$$E = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{R}.$$

**NOTE** This is the same result as when using Coulomb's law ($x$ there instead of $R$), but here it takes much less effort. Again we see the great power of Gauss's law.[†]

**NOTE** Recall that we use $R$ for the distance of a particle from an axis (cylindrical symmetry), but lower case $r$ for the distance from a point (usually the origin 0).

**FIGURE 15** Calculation of $\vec{\mathbf{E}}$ due to a very long line of charge. Example 6.




[†]But note that the method using Coulomb's law allows calculation of $E$ also for a short line of charge by using the appropriate limits for the integral, whereas Gauss's law is not readily adapted due to lack of symmetry.

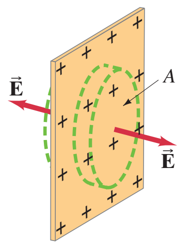



**FIGURE 16** Calculation of the electric field outside a large uniformly charged nonconducting plane surface. Example 7.

**EXAMPLE 7** **Infinite plane of charge.** Charge is distributed uniformly, with a surface charge density $\sigma$ ($\sigma$ = charge per unit area $= dQ/dA$), over a very large but very thin nonconducting flat plane surface. Determine the electric field at points near the plane.

**APPROACH** We choose as our gaussian surface a small closed cylinder whose axis is perpendicular to the plane and which extends through the plane as shown in Fig. 16. Because of the symmetry, we expect $\vec{\mathbf{E}}$ to be directed perpendicular to the plane on both sides as shown, and to be uniform over the end caps of the cylinder, each of whose area is $A$.

**SOLUTION** Since no flux passes through the curved sides of our chosen cylindrical surface, all the flux is through the two end caps. So Gauss's law gives

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = 2EA = \frac{Q_{\text{encl}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0},$$

where $Q_{\text{encl}} = \sigma A$ is the charge enclosed by our gaussian cylinder. The electric field is then

$$E = \frac{\sigma}{2\epsilon_0}.$$

**NOTE** The field is uniform for points far from the ends of the plane, and close to its surface.

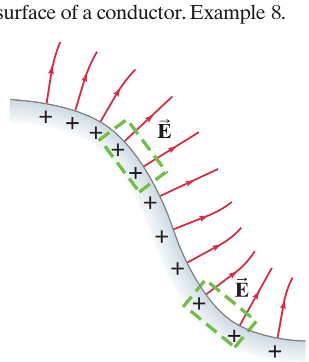
**EXAMPLE 8** **Electric field near any conducting surface.** Show that the electric field just outside the surface of any good conductor of arbitrary shape is given by

$$E = \frac{\sigma}{\epsilon_0},$$

where $\sigma$ is the surface charge density on the conductor's surface at that point.

**APPROACH** We choose as our gaussian surface a small cylindrical box, as we did in the previous Example. We choose the cylinder to be very small in height, so that one of its circular ends is just above the conductor (Fig. 17). The other end is just below the conductor's surface, and the sides are perpendicular to it.

**FIGURE 17** Electric field near surface of a conductor. Example 8.




**SOLUTION** The electric field is zero inside a conductor and is perpendicular to the surface just outside it, so electric flux passes only through the outside end of our cylindrical box; no flux passes through the short sides or inside end. We choose the area $A$ (of the flat cylinder end) small enough so that $E$ is essentially uniform over it. Then Gauss's law gives

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = \frac{Q_{\text{encl}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0},$$

so that

$$E = \frac{\sigma}{\epsilon_0}. \qquad \text{[at surface of conductor]} \quad \textbf{(5)}$$

**NOTE** This useful result applies for a conductor of any shape.

**CAUTION**
*When is $E = \sigma/\epsilon_0$ and when is $E = \sigma/2\epsilon_0$*

Why is it that the field outside a large plane nonconductor is $E = \sigma/2\epsilon_0$ (Example 7) whereas outside a conductor it is $E = \sigma/\epsilon_0$ (Example 8)? The reason for the factor of 2 comes not from conductor verses nonconductor. It comes instead from how we define charge per unit area $\sigma$. For a thin flat nonconductor, Fig. 16, the charge may be distributed throughout the volume (not only on the surface, as for a conductor). The charge per unit area $\sigma$ represents all the charge throughout the thickness of the thin nonconductor. Also our gaussian surface has its ends outside the nonconductor on each side, so as to include all this charge.

For a conductor, on the other hand, the charge accumulates on the outer surfaces only. For a thin flat conductor, as shown in Fig. 18, the charge accumulates on both surfaces, and using the same small gaussian surface we did in Fig. 17, with one end inside and the other end outside the conductor, we came up with the result, $E = \sigma/\epsilon_0$. If we defined $\sigma$ for a conductor, as we did for a nonconductor, $\sigma$ would represent the charge per area for the entire conductor. Then Fig. 18 would show $\sigma/2$ as the surface charge on each surface, and Gauss's law would give $\int \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = (\sigma/2)A/\epsilon_0 = \sigma A/2\epsilon_0$ so $E = \sigma/2\epsilon_0$, just as for a nonconductor. We need to be careful about how we define charge per unit area $\sigma$.

In the static situation, the electric field inside any conductor must be zero even if it has a net charge. (Otherwise, the free charges in the conductor would move—until the net force on each, and hence $\vec{\mathbf{E}}$, were zero.) We also mentioned there that any net electric charge on a conductor must all reside on its outer surface. This is readily shown using Gauss's law. Consider any charged conductor of any shape, such as that shown in Fig. 19, which carries a net charge $Q$. We choose the gaussian surface, shown dashed in the diagram, so that it all lies just below the surface of the conductor and encloses essentially the whole volume of the conductor. Our gaussian surface can be arbitrarily close to the surface, but still *inside* the conductor. The electric field is zero at all points on this gaussian surface since it is inside the conductor. Hence, from Gauss's law, Eq. 4, the net charge within the surface must be zero. Thus, there can be no net charge within the conductor. Any net charge must lie on the surface of the conductor.

If there is an empty cavity inside a conductor, can charge accumulate on that (inner) surface too? As shown in Fig. 20, if we imagine a gaussian surface (shown dashed) just inside the conductor above the cavity, we know that $\vec{\mathbf{E}}$ must be zero everywhere on this surface since it is inside the conductor. Hence, by Gauss's law, *there can be no net charge at the surface of the cavity*.

But what if the cavity is not empty and there is a charge inside it?

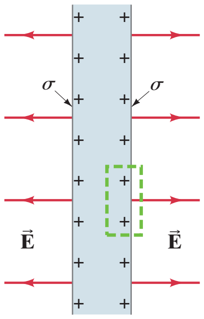
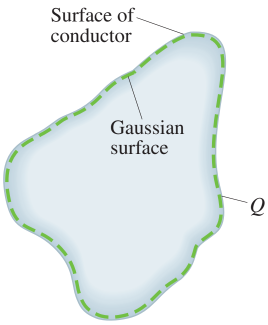



**FIGURE 18** Thin flat charged conductor with surface charge density $\sigma$ at each surface. For the conductor as a whole, the charge density is $\sigma' = 2\sigma$.

**FIGURE 19** An insulated charged conductor of arbitrary shape, showing a gaussian surface (dashed) just below the surface of the conductor.



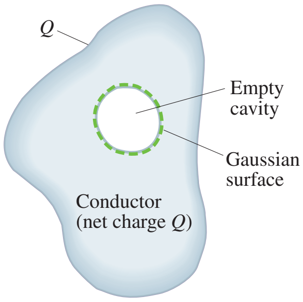
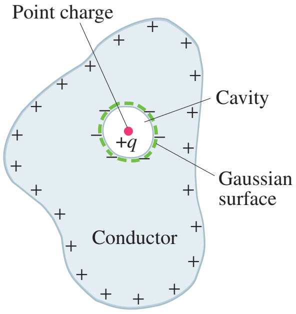




**FIGURE 20** An empty cavity inside a charged conductor carries zero net charge.

**FIGURE 21** Example 9.




**CONCEPTUAL EXAMPLE 9** **Conductor with charge inside a cavity.** Suppose a conductor carries a net charge $+Q$ and contains a cavity, inside of which resides a point charge $+q$. What can you say about the charges on the inner and outer surfaces of the conductor?

**RESPONSE** As shown in Fig. 21, a gaussian surface just inside the conductor surrounding the cavity must contain zero net charge ($E = 0$ in a conductor). Thus a net charge of $-q$ must exist on the cavity surface. The conductor itself carries a net charge $+Q$, so its outer surface must carry a charge equal to $Q + q$. These results apply to a cavity of any shape.

**EXERCISE F** Which of the following statements about Gauss's law is correct? (*a*) If we know the charge enclosed by a surface, we always know the electric field everywhere at the surface. (*b*) When finding the electric field with Gauss's law, we always use a sphere for the gaussian surface. (*c*) If we know the total flux through a surface, we also know the total charge inside the surface. (*d*) We can only use Gauss's law if the electric field is constant in space.

PROBLEM SOLVING

### Gauss's Law for Symmetric Charge Distributions

1. First identify the **symmetry** of the charge distribution: spherical, cylindrical, planar. This identification should suggest a gaussian surface for which $\vec{\mathbf{E}}$ will be constant and/or zero on all or on parts of the surface: a sphere for spherical symmetry, a cylinder for cylindrical symmetry and a small cylinder or "pillbox" for planar symmetry.
2. Draw the appropriate gaussian surface making sure it passes through the point where you want to know the electric field.
3. Use the symmetry of the charge distribution to determine the direction of $\vec{\mathbf{E}}$ at points on the gaussian surface.
4. Evaluate the flux, $\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$. With an appropriate gaussian surface, the dot product $\vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$ should be zero or equal to $\pm E\,dA$, with the magnitude of $E$ being constant over all or parts of the surface.
5. Calculate the charge *enclosed* by the gaussian surface. Remember it's the enclosed charge that matters. Ignore all the charge outside the gaussian surface.
6. Equate the flux to the enclosed charge and solve for $E$.

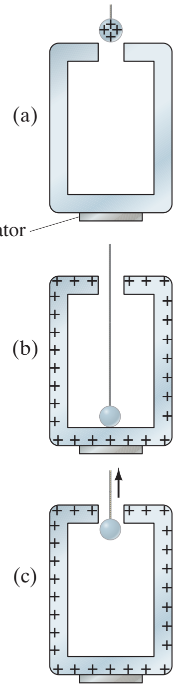



**FIGURE 22** (a) A charged conductor (metal ball) is lowered into an insulated metal can (a good conductor) carrying zero net charge. (b) The charged ball is touched to the can and all of its charge quickly flows to the outer surface of the can. (c) When the ball is then removed, it is found to carry zero net charge.

## *4 Experimental Basis of Gauss's and Coulomb's Laws

Gauss's law predicts that any net charge on a conductor must lie only on its surface. But is this true in real life? Let us see how it can be verified experimentally. And in confirming this prediction of Gauss's law, Coulomb's law is also confirmed since the latter follows from Gauss's law, as we saw in Section 2. Indeed, the earliest observation that charge resides only on the outside of a conductor was recorded by Benjamin Franklin some 30 years before Coulomb stated his law.

A simple experiment is illustrated in Fig. 22. A metal can with a small opening at the top rests on an insulator. The can, a conductor, is initially uncharged (Fig. 22a). A charged metal ball (also a conductor) is lowered by an insulating thread into the can, and is allowed to touch the can (Fig. 22b). The ball and can now form a single conductor. Gauss's law, as discussed above, predicts that all the charge will flow to the outer surface of the can. (The flow of charge in such situations does not occur instantaneously, but the time involved is usually negligible). These predictions are confirmed in experiments by (1) connecting an electroscope to the can, which will show that the can is charged, and (2) connecting an electroscope to the ball after it has been withdrawn from the can (Fig. 22c), which will show that the ball carries zero charge.

The precision with which Coulomb's and Gauss's laws hold can be stated quantitatively by writing Coulomb's law as

$$F = k\frac{Q_1 Q_2}{r^{2+\delta}}.$$

For a perfect inverse-square law, $\delta = 0$. The most recent and precise experiments (1971) give $\delta = (2.7 \pm 3.1) \times 10^{-16}$. Thus Coulomb's and Gauss's laws are found to be valid to an extremely high precision!

# Summary


The **electric flux** passing through a flat area $A$ for a uniform electric field $\vec{\mathbf{E}}$ is

$$\Phi_E = \vec{\mathbf{E}} \cdot \vec{\mathbf{A}}. \tag{1b}$$

If the field is not uniform, the flux is determined from the integral

$$\Phi_E = \int \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}. \tag{2}$$

The direction of the vector $\vec{\mathbf{A}}$ or $d\vec{\mathbf{A}}$ is chosen to be perpendicular to the surface whose area is $A$ or $dA$, and points outward from an enclosed surface. The flux through a surface is proportional to the number of field lines passing through it.

**Gauss's law** states that the net flux passing through any closed surface is equal to the net charge $Q_{\text{encl}}$ enclosed by the surface divided by $\epsilon_0$:

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q_{\text{encl}}}{\epsilon_0}. \tag{4}$$

Gauss's law can in principle be used to determine the electric field due to a given charge distribution, but its usefulness is mainly limited to a small number of cases, usually where the charge distribution displays much symmetry. The real importance of Gauss's law is that it is a more general and elegant statement (than Coulomb's law) for the relation between electric charge and electric field. It is one of the basic equations of electromagnetism.


# Answers to Exercises

**A:** (*d*).

**B:** (*a*).

**C:** (*d*).

**D:** (*e*).

**E:** (*c*).

**F:** (*c*).