CSE-170 Computer Graphics

Lecture 15

Parametric Curves

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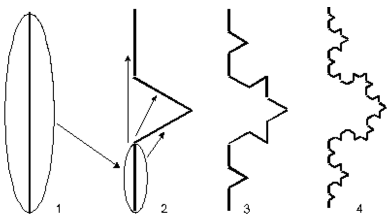
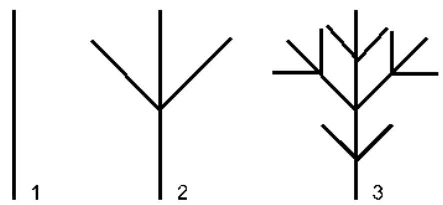
Curves

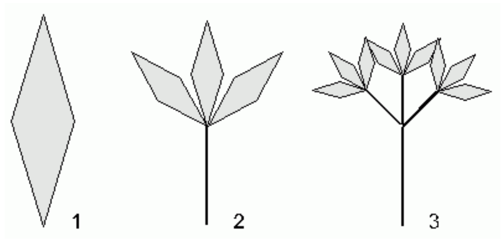
- Curves:
  - can be defined in any dimensions
    - 2D curves define points in the 2D plane
    - 3D curves define points in the 3D space
  - are "infinitesimally thin"
  - will be used to outline the border of objects
  - can be open or closed
  - examples
    - Segments
    - Circles
    - Free-form shapes

Curves

- Main approaches to define curves:
  - Procedural curves: fractals, subdivision rules







  - Parametric curves: are mappings
    - Ex.: Continuous map from 1D space to n-D space

      f(t)=(x,y), ex: f(t)=(cos t, sin t)

  - Implicit curves: defined by an equation
    - Described by all points satisfying an equation

      f(x,y)=0 ex: $x^2+y^2-1=0$

Parametric Curves

- Parameterized curves are very useful
  - Points along the curve can be easily generated
- How do we specify generic parametric curves?
  - Control Points or Control Polygons
    - By interpolation or approximation
  - Tangents
  - etc.
- What type of functions should we use?
  - In most cases polynomials
    - Simple and fast to evaluate and can be flexible enough
  - Piecewise polynomial functions very common

Parameterization

Parametric Curves

- A curve f is often defined in an interval
- For example, if it is defined in [0,1], then:

  f(t = 0) is the beginning of the curve, and

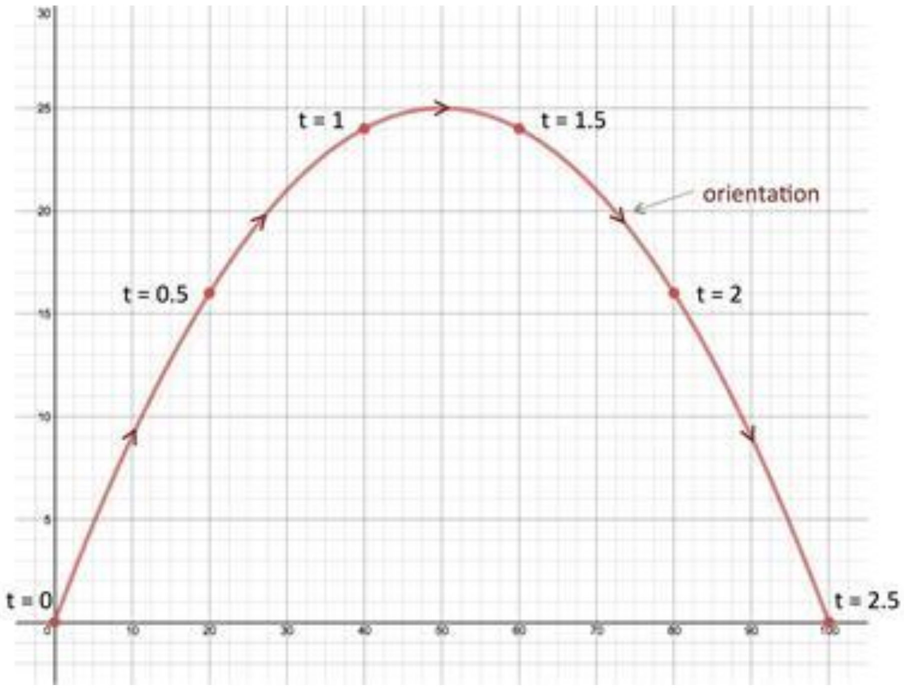
  f(t = 1) is its end.

  t can be seen as "time", as in a trajectory

Parametric Curves

- Ex: $f(t) = (\ 40t, -16t^2 + 40t\ )$, t in [0,2.5]
- Equivalent to: $x(t) = 40t$

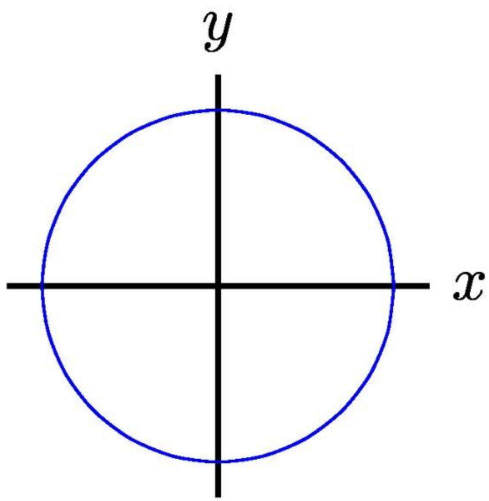
$$y(t) = -16t^2 + 40t$$



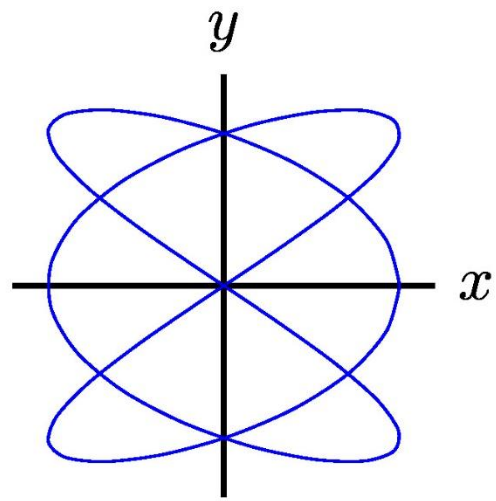
https://www.softschools.com/math/pre_calculus/parametric_equations_introduction/

Parametric Curves

- t in [0,2π]



$$x(t) = \cos(t)$$
$$y(t) = \sin(t)$$



$$x(t) = 3\cos(t)$$
$$y(t) = 2\sin(t)$$

https://mathbooks.unl.edu/Calculus/sec-3-7-parametric.html

Parametric Curves

- Reparameterization

  Consider $g(u)$, where u is defined in $[a,b]$

  A new parameterization can be defined in [0,1]:

  $f(t) = g(\ a+(b-a)t\ ),\ t$ in $[0,1]$

  (we can also write: $f(t) = g(\ r(t)\ ),\ r(t) = a+(b-a)t$)

- Different parameterizations possible

  For ex: $r(t) = (t^2)$

Parametric Curves

- Arc-length parameterization
  - parameter tells you the length traveled so far
  - an increment in parameter space generates the same increment along the curve

- Transforming to arc-length parameterization is sometimes very difficult

$$s(t) = \int_{t_0}^{t} |a'(t)| dt$$

Piecewise Curves

Piecewise Parametric Curves

- Piecewise Curves:
  - are composed of pieces which have different equations
- How to break a curve into simple pieces?
  - Define pieces in parametric space, for ex.:

$$f(u) = f_1(u), \text{ if } u < 0.5$$
$$\quad\quad\; f_2(u), \text{ if } u \geq 0.5$$

Piecewise Parametric Curves

- Why break a curve into simple pieces?
  - Simpler pieces are easier to define/model
  - Easier to control
  - Easier to guarantee "local control behavior"
  - Knot control (control at connection points)
  - etc.

- Splines:
  - also known as a flexible physical device
  - are modeled as piecewise polynomial functions
  - will be our main tool

Continuity

Continuity

- Let f be a piecewise curve composed of continuous pieces
- For f to be continuous, continuity at every connection point is needed:

Ex.: $f(u) = f_1(u)$ if $u < 0.5$

$f_2(u)$ if $u \geq 0.5$

Continuity test: Is $f_1(0.5) = f_2(0.5)$?

Note: f_1 and f_2 are known to be continuous here, but if this is not the case, we need to verify continuity for all points along the curve.

Continuity

Continuity test at 0.5:   Is $f_1(0.5) = f_2(0.5)$?

Let *u* denote a connection point:

C^0 continuity: is $f_1(u) = f_2(u)$, for all *u*?

C^1 continuity: is it C^0 and $f_1'(u) = f_2'(u)$, for all *u*?

C^2 continuity: is it C^1 and $f_1''(u) = f_2''(u)$, for all *u*?

C^n continuity: …

etc.

Continuity

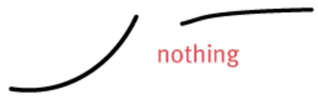
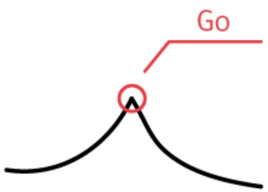
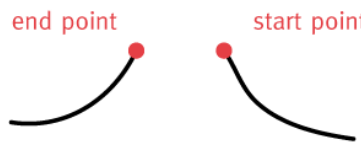
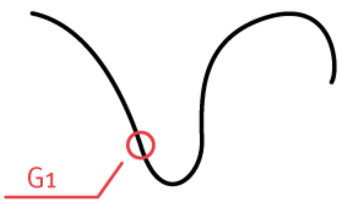
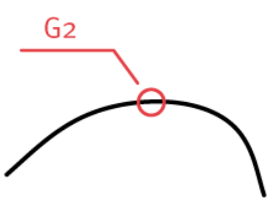
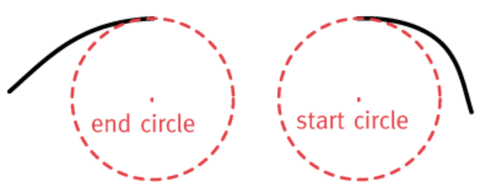
- Geometric Continuity
  - A "visual" definition of continuity
    - The parametrization may have a "sudden change in velocity", but the resulting curve is a continuous curve
  - Example:

    C^1 continuity requires $f_1'(u) = f_2'(u)$. Instead,

    G^1 continuity requires $\exists k : f_1'(u) = k\, f_2'(u)$.

  - Geometric continuity G^1, G^2, G^3, … are defined equivalently to the definition of C^n continuity

Continuity

	continuity type	adjacency analysis	math. analysis	aestetical impression
disjoint (no continuity)			$f(x) \neq g(x)$ also: domains are not adjacent	you need to move the end points to join.
position			$f(x) = g(x)$	curves will be continuous, but will probably have a visible crease.
tangency (position variation)			$f'(x) = g'(x)$	curves will appear smooth. their reflection, though, could have sudden changes.
curvature (tangency variation)			$f''(x) = g''(x)$	reflections will change smoothly as well. ambient shadows will be gradual.

https://www.giuliopiacentino.com/curves-continuity/