# CSE-170 Computer Graphics

## Lecture 16

## Lagrange and Hermite Interpolation

Dr. Renato Farias
rfarias2@ucmerced.edu

# Polynomials

- We can specify a polynomial curve in many ways:
  - Approximation
  - Control Points
  - Control Polygons
  - Tangents
  - etc.

$$f(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$$

$$f(t) = \sum_{i=0}^{n} a_i t^i$$

- How to solve interpolation with polynomials?

- Which degree do I need to use?
  - Quadratic?
  - Cubic?
  - Higher degree?

# Lagrange Polynomial

# Lagrange Interpolating Polynomial

## Lagrange Polynomial:

– Polynomial of degree *n*-1 for interpolating *n* points ( $x_i$, $y_i$ ), i=1,…,n

$$f(x) = \sum_{i=1}^{n} y_i b_i(x)$$

$$b_i(x) = \prod_{j=1,\, j \neq i}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

– Note: the above formula for *f* is expressed as a combination of blending functions

# Lagrange Interpolating Polynomial

## Lagrange:

– General formula:

$$f(x) = \sum_{i=1}^{n} y_i b_i(x) \qquad b_i(x) = \prod_{j=1,\, j\neq i}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

– Equivalent to:

$$f(x) =$$

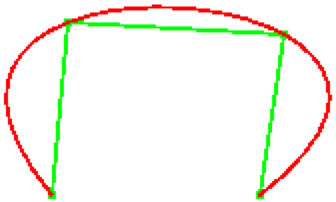
$$y_1 \frac{(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n)} + y_2 \frac{(x-x_1)(x-x_3)\cdots(x-x_n)}{(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)} + \ldots + y_n \frac{(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1})}$$

# Lagrange Interpolating Polynomial

- Parametric Form:

$$\mathbf{f}(t) = \sum_{i=1}^{n} \mathbf{p}_i b_i(t)$$

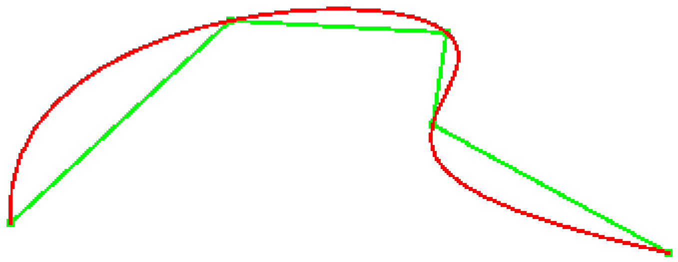
$$b_i(t) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{t - t_j}{t_i - t_j}$$

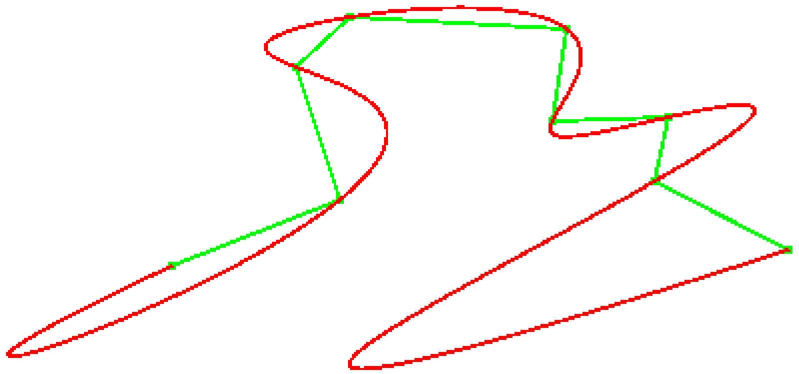


- $b_i(t)$: blending functions
  $\mathbf{p}_i$ : control points
- The parameterization can be chosen, usually: $f(t_i) = p_i$

# Lagrange Interpolating Polynomial

- Examples:





# Lagrange Interpolating Polynomial

- Lagrange polynomials are not useful in high degrees
  - Resulting curve may not be a reasonable/desired interpolation solution

- Solution
  - Piecewise simpler parts
  - A “simpler part” will be a polynomial of low degree:
    - Splines in their several forms

# Hermite Form

# (Charles Hermite 1822-1901)

# Cubics

- Cubics are the most practical polynomials to deal with
  - Simple and with enough flexibility

- Canonical form:

$$f(t) = a + bt + ct^2 + dt^3$$

- But several other forms can be used to represent cubics

# Hermite

- A cubic has four coefficients:
  *a*, *b*, *c*, and *d*

$$f(x) = a + bx + cx^2 + dx^3$$
$$f'(x) = b + 2cx + 3dx^2$$

- Four constraints can be given:
  - 1st point $(x_0, y_0) \rightarrow$ $\quad y_0 = f(x_0) = a + bx_0 + cx_0^2 + dx_0^3$
  - Derivative at 1st point, $m_0 \rightarrow$ $\quad m_0 = f'(x_0) = b + 2cx_0 + 3dx_0^2$
  - 2nd point $(x_1, y_1) \rightarrow$ $\quad y_1 = f(x_1) = a + bx_1 + cx_1^2 + dx_1^3$
  - Derivative at 2nd point, $m_1 \rightarrow$ $\quad m_1 = f'(x_1) = b + 2cx_1 + 3dx_1^2$

- In the Hermite form we consider fixed:
  - $x_0 = 0$ and $x_1 = 1$

# Hermite

- Case where $x_0 = 0$ and $x_1 = 1$:
  - From:

$$y_0 = f(x_0) = a + bx_0 + cx_0^2 + dx_0^3$$

$$m_0 = f'(x_0) = b + 2cx_0 + 3dx_0^2$$

$$y_1 = f(x_1) = a + bx_1 + cx_1^2 + dx_1^3$$

$$m_1 = f'(x_1) = b + 2cx_1 + 3dx_1^2$$

  - By replacing $x_0 = 0$ and $x_1 = 1$ we have:

$$y_0 = f(0) = a$$

$$m_0 = f'(0) = b$$

$$y_1 = f(1) = a + b + c + d$$

$$m_1 = f'(1) = b + 2c + 3d$$

How to write $f$ in terms of the constraints instead of $a$, $b$, $c$, and $d$?

# Hermite

- Case where $x_0 = 0$ and $x_1 = 1$:

$$y_0 = f(0) = a$$
$$m_0 = f'(0) = b$$
$$y_1 = f(1) = a + b + c + d$$
$$m_1 = f'(1) = b + 2c + 3d$$

$$\downarrow$$

$$\boxed{a = y_0}$$
$$\boxed{b = m_0}$$
$$y_1 = y_0 + m_0 + c + d$$
$$m_1 = m_0 + 2c + 3d$$

$$y_1 - y_0 - m_0 - d = c$$
$$m_1 = m_0 + 2(y_1 - y_0 - m_0 - d) + 3d$$
$$m_1 = 2y_1 - 2y_0 - m_0 + d$$
$$\boxed{d = m_1 + m_0 - 2y_1 + 2y_0}$$

$$\downarrow$$

$$y_1 - y_0 - m_0 - (m_1 + m_0 - 2y_1 + 2y_0) = c$$
$$\boxed{c = 3y_1 - 3y_0 - m_1 - 2m_0}$$

$$\downarrow$$

$$f(x) = a + bx + cx^2 + dx^3$$

$$\downarrow$$

$$f(x) = y_0 + m_0 x + (3y_1 - 3y_0 - m_1 - 2m_0)x^2 + (m_1 + m_0 - 2y_1 + 2y_0)x^3$$

$$\downarrow$$

$$f(x) = (1 - 3x^2 + 2x^3)y_0 + (x - 2x^2 + x^3)m_0 + (3x^2 - 2x^3)y_1 + (-x^2 + x^3)m_1$$

# Hermite

- Case where $x_0 = 0$ and $x_1 = 1$:

$$y_0 = f(0) = a$$
$$m_0 = f'(0) = b$$
$$y_1 = f(1) = a + b + c + d$$
$$m_1 = f'(1) = b + 2c + 3d$$

$$\downarrow$$

$$\boxed{a = y_0}$$
$$\boxed{b = m_0}$$
$$y_1 = y_0 + m_0 + c + d$$
$$m_1 = m_0 + 2c + 3d$$

$$y_1 - y_0 - m_0 - d = c$$
$$m_1 = m_0 + 2(y_1 - y_0 - m_0 - d) + 3d$$
$$m_1 = 2y_1 - 2y_0 - m_0 + d$$
$$\boxed{d = m_1 + m_0 - 2y_1 + 2y_0}$$

$$\downarrow$$

$$y_1 - y_0 - m_0 - (m_1 + m_0 - 2y_1 + 2y_0) = c$$
$$\boxed{c = 3y_1 - 3y_0 - m_1 - 2m_0}$$

$$\downarrow$$

$$f(x) = a + bx + cx^2 + dx^3$$

$$\downarrow$$

$$f(x) = y_0 + m_0 x + (3y_1 - 3y_0 - m_1 - 2m_0)x^2 + (m_1 + m_0 - 2y_1 + 2y_0)x^3$$

$$\downarrow$$

$$f(x) = \boxed{(1 - 3x^2 + 2x^3)}\,y_0 + \boxed{(x - 2x^2 + x^3)}\,m_0 + \boxed{(3x^2 - 2x^3)}\,y_1 + \boxed{(-x^2 + x^3)}\,m_1$$

*Hermite basis functions*

# Hermite

- Hermite parametric form
  - t in [0,1]

$$\mathbf{p}_0 = \mathbf{f}(0)$$
$$\mathbf{p}_1 = \mathbf{f}'(0)$$
$$\mathbf{p}_2 = \mathbf{f}(1)$$
$$\mathbf{p}_3 = \mathbf{f}'(1)$$

$$f(t) = (1-3t^2+2t^3)\mathbf{p}_0 + (t-2t^2+t^3)\mathbf{p}_1 + (3t^2-2t^3)\mathbf{p}_2 + (-t^2+t^3)\mathbf{p}_3$$

$$\boxed{\mathbf{f}(t) = \sum_{i=0}^{3} \mathbf{p}_i H_i(t)}$$

$$H_0(t) = 1-3t^2+2t^3$$
$$H_1(t) = t-2t^2+t^3$$
$$H_2(t) = 3t^2-2t^3$$
$$H_3(t) = -t^2+t^3$$

$\longrightarrow$ *Hermite basis functions*

# Hermite

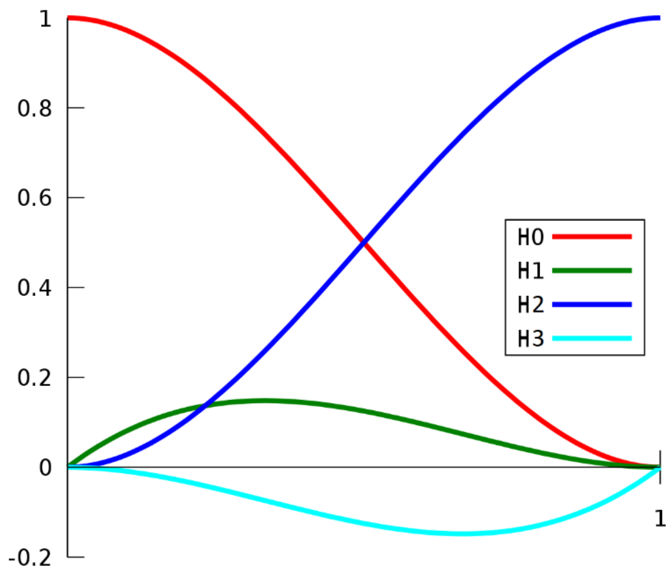
- Hermite parametric form
  - t in [0,1]

$$\mathbf{p}_0 = \mathbf{f}(0)$$
$$\mathbf{p}_1 = \mathbf{f}'(0)$$
$$\mathbf{p}_2 = \mathbf{f}(1)$$
$$\mathbf{p}_3 = \mathbf{f}'(1)$$

$$\boxed{\mathbf{f}(t) = \sum_{i=0}^{3} \mathbf{p}_i H_i(t)}$$

$$H_0(t) = 1 - 3t^2 + 2t^3$$
$$H_1(t) = t - 2t^2 + t^3$$
$$H_2(t) = 3t^2 - 2t^3$$
$$H_3(t) = -t^2 + t^3$$

Hermite basis functions:



# Hermite

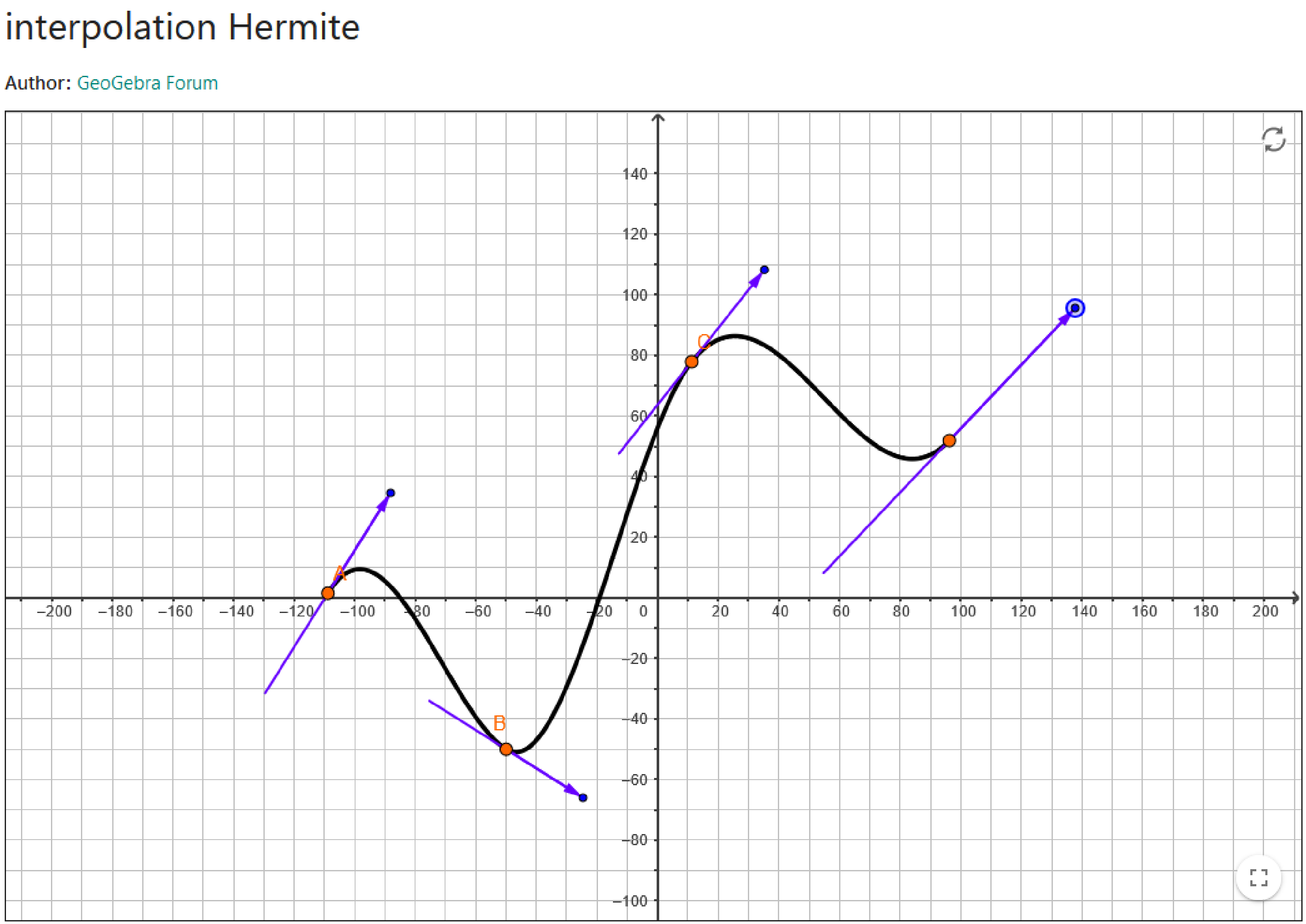
- Parametric form can also be expressed in matrix notation:
  - t in [0,1]

$$\mathbf{p}_0 = \mathbf{f}(0) = \mathbf{a}$$
$$\mathbf{p}_1 = \mathbf{f}'(0) = \mathbf{b}$$
$$\mathbf{p}_2 = \mathbf{f}(1) = \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}$$
$$\mathbf{p}_3 = \mathbf{f}'(1) = \mathbf{a} + \mathbf{c} + \mathbf{d}$$

$$\begin{pmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & -2 & 3 & -1 \\ 2 & 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{p}_0 \\ \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{pmatrix} = \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \\ \mathbf{c} \\ \mathbf{d} \end{pmatrix}$$

$$\boxed{\mathbf{f}(t) = \mathbf{a} + \mathbf{b}t + \mathbf{c}t^2 + \mathbf{d}t^3}$$

# Hermite




https://www.geogebra.org/m/wTneDg9v