1. Representation of Curves. Give one curve example for each of the following ways to describe curves: implicit, parametric and procedural. Identify suitable algorithms for drawing the curves on each representation.

   Solution:

   Implicit: $(x-1)^2 + y^2 - 10 + cos(x) = 0$; marching squares.

   Parametric: $f(t) = (\sin(t), \cos(t))$; approximation with segments by evaluating t.

   Procedural: circle; split and tweak seed triangle to converge to a circle as much as desired.

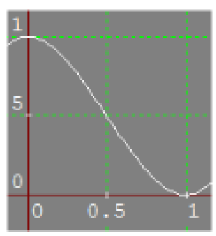
   □

2. Hermite Polynomial. Let $y = f(x)$ be a cubic polynomial curve in the canonical form.
   a) Find coefficients a, b, c, d of the cubic such that: $f(0) = 1, f'(0) = 0, f(1) = 0, f'(1) = 0$.
   b) Sketch how the curve will look like in the XY plane.

   Solution:

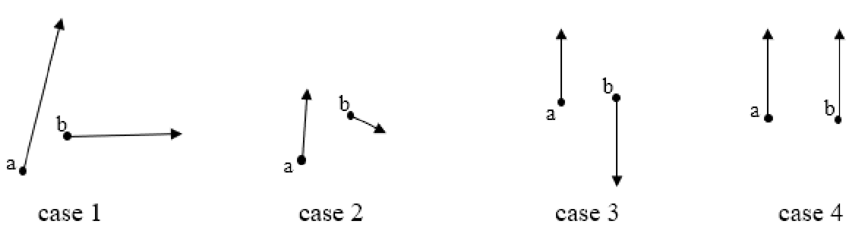
   a) The canonical form is $f(x) = ax^3 + bx^2 + cx + d$. The four given conditions lead to four equations with four unknowns, allowing us to find the coefficients:
   $f(0) = 1 \Rightarrow 0 + 0 + 0 + d = 1 \Rightarrow d = 1$;
   $f'(0) = 0 \Rightarrow 0 + 0 + c = 0 \Rightarrow c = 0$;
   $f(1) = 0 \Rightarrow a + b + c + d = 0 \Rightarrow a + b + 0 + 1 = 0 \Rightarrow a = -b - 1$;
   $f'(1) = 0 \Rightarrow 3a + 2b + c = 0 \Rightarrow 3(-b - 1) + 2b + 0 = 0 \Rightarrow b = -3$.
   The answer is thus: a=2, b=-3, c=0, d=1.

   b) From the constraints it is possible to see that the curve starts at $y = 1$, ends at $y = 0$ and has horizontal tangents at both $x = 0$ and $x = 1$. It is thus straightforward to sketch its shape:



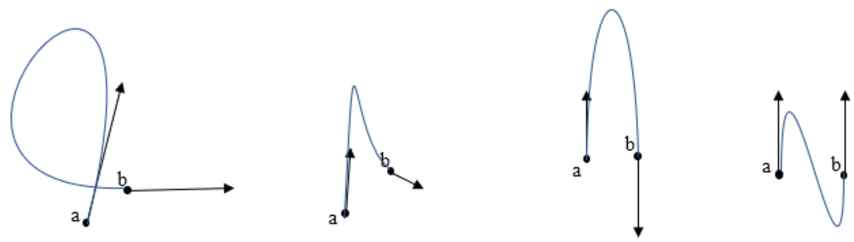
   □

3. Parametric Hermite. Given the point constraints $f(0) = \mathbf{a}$ and $f(1) = \mathbf{b}$, and the tangent constraints as shown below in four different cases, draw a sketch of how the parametric Hermite curve should look like in each case.



   Solution:

   □



4. Cubic Bézier Curves.

a) Find the equation of the 2D parametric cubic Bézier curve defined by the following four control points: $\mathbf{p}_0 = (0,0)$, $\mathbf{p}_1 = (2,2)$, $\mathbf{p}_2 = (4,0)$, and $\mathbf{p}_3 = (6,2)$. Write the equation in its simplest possible form. As a reminder, the Bernstein polynomials for the cubic case are: $B_{0,3}(t) = (1-t)^3$, $B_{1,3}(t) = 3t(1-t)^2$, $B_{2,3}(t) = 3t^2(1-t)$, $B_{3,3}(t) = t^3$.

b) Verify that your equation interpolates the first ant last points.

c) Verify that your curve is tangent to the first and last segments of the control polygon.

Solution:

a) $f(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} B_{0,3}(t) + \begin{pmatrix} 2 \\ 2 \end{pmatrix} B_{1,3}(t) + \begin{pmatrix} 4 \\ 0 \end{pmatrix} B_{2,3}(t) + \begin{pmatrix} 6 \\ 2 \end{pmatrix} B_{3,3}(t) =$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 6t(1-t)^2 \\ 6t(1-t)^2 \end{pmatrix} + \begin{pmatrix} 12t^2(1-t) \\ 0 \end{pmatrix} + \begin{pmatrix} 6t^3 \\ 2t^3 \end{pmatrix} =$$

$$\begin{pmatrix} 6t(1-t)^2 + 12t^2(1-t) + 6t^3 \\ 6t(1-t)^2 + 2t^3 \end{pmatrix} = \begin{pmatrix} (6t - 6t^2 + 12t^2)(1-t) + 6t^3 \\ 6t(1 - 2t + t^2) + 2t^3 \end{pmatrix} =$$

$$\begin{pmatrix} (6t + 6t^2)(1-t) + 6t^3 \\ 6t - 12t^2 + 6t^3 + 2t^3 \end{pmatrix} = \begin{pmatrix} 6t + 6t^2 - 6t^2 - 6t^3 + 6t^3 \\ 6t - 12t^2 + 8t^3 \end{pmatrix} = \begin{pmatrix} 6t \\ 8t^3 - 12t^2 + 6t \end{pmatrix}.$$

b) $f(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{p}_0$, $f(1) = \begin{pmatrix} 6 \\ 8 - 12 + 6 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \mathbf{p}_3$;

c) Tangent vectors are given by the curve derivative: $f'(t) = \begin{pmatrix} 6 \\ 24t^2 - 24t + 6 \end{pmatrix}$.

The first segment vector is $\mathbf{p}_1 - \mathbf{p}_0 = (2,2)$, and the tangent at the first point is collinear: $f'(0) = (6,6)$. The last segment vector is $\mathbf{p}_3 - \mathbf{p}_2 = (2,2)$, which is also collinear to the curve derivative at the last curve point:$f'(1) = (6,6)$.

□

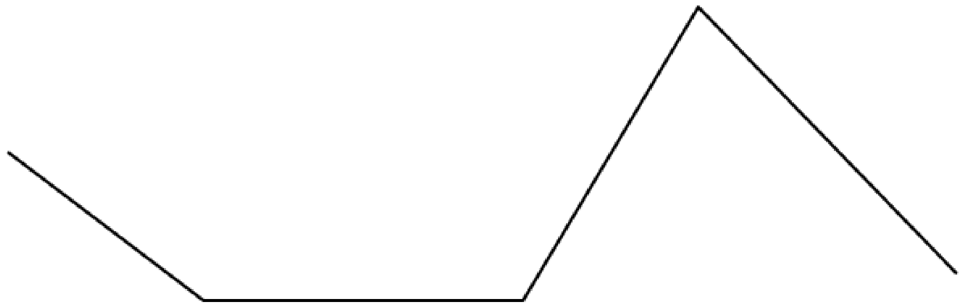
5. De Casteujau Algorithm.

a) Given the control polygon below of a quintic Bézier curve q, draw approximately the points $q(0.25)$, $q(0.5)$, and $q(0.75)$ using the De Casteujau algorithm. Draw as well the lines used for obtaining the points. After determining the position of the three points, sketch how you think the final curve should look like.

b) Now suppose that we want to obtain a similar curve but ensuring that it is tangent to the third segment of the control polygon. Draw two new control polygons that obtain the desired curve with two adjacent Bézier curves. What are the conditions to ensure that the curve remains C^1 at the junction point?

Solution:

a) Just review the de Casteujau algorithm and remember that a Bézier curve will always start at the first control point, end at the last control point, and be tangent to the first and last segments of the



control polygon.

b) The new control polygons P_A and P_B can be obtained by separating the original control polygon in two with a junction point placed at the middle of the 3rd segment. In this way, the last point of P_A will be the junction point, which will also be the first point of P_B. In addition, the last segment of P_A will be co-linear and have the same length as the first segment of P_B. In this way the conditions to ensure a C^1 connection at the junction point are met.

□

6. B-Splines.
   a) List the main advantages of B-Splines in comparison to Bézier curves?
   b) How can B-Splines have corners with non-continuous derivatives?
   c) How should the control polygon of a quadratic B-Spline be in order to achieve a smooth C^2 closed curve?
   d) Draw a control polygon with 8 points and sketch the respective quadratic and cubic B-Splines. What is the main difference between the 2 curves in your sketch?
   e) Sketch now a Bézier curve in the same control polygon used in the previous item. What are the main differences between this Bézier curve and the B-Splines of the previous item? What is the degree of the obtained Bézier curve?

   Solution:

   a) The degree of the polynomials produced by B-Splines can be chosen independently of the number of control points, B-Splines have local control support, and B-Splines allow additional shape control with non-uniform knot vectors.

   b) With non-uniform knot values with consecutive duplicated values.

   c) The first and final control segments must perfectly overlap, with the last point becoming coincident to the second point.

   d) The quadratic curve is tangent at the midpoints of the control polygon, with endpoints starting and ending at the midpoint of the first and last control segments. The cubic curve is farther away from the control polygon and does not necessarily touches the control polygon, not even at the endpoints.

   e) The Bézier curve will start at the first point and being tangent to the first segment of the control polygon. It will also end at the last point and being tangent to the last segment of the control polygon. The 2nd and 3rd degree B-Splines do not touch the end points of the control polygons and are not tangent at the same points. The degree will be 7.

□

7. Catmull-Rom Splines. Sketch points **a**, **b**, **c**, and **d** to be interpolated by a Catmull-Rom Spline. Draw in your sketch the two extra points $\mathbf{p}_1$ and $\mathbf{p}_2$ needed to form the local control polygon (**b**, $\mathbf{p}_1$, $\mathbf{p}_2$,

c) of the Bézier piece connecting points **b** and **c**. Draw the control polygon in the figure and give the expressions defining points $\mathbf{p}_1$ and $\mathbf{p}_2$.

Solution:

Your sketch should correctly depict the control polygon with two segments parallel to the lines $\{\mathbf{a}, \mathbf{c}\}$ and $\{\mathbf{b}, \mathbf{d}\}$. The points are defined as: $\mathbf{p}_1 = \mathbf{b} + (\mathbf{c} - \mathbf{a})/6$, $\mathbf{p}_2 = \mathbf{c} - (\mathbf{d} - \mathbf{b})/6$.

□

8. Answer the questions:

a) What is the main advantage of the Bessel-Overhauser formulation over the Catmull-Rom formulation?

b) What are the two main steps in a refinement of subdivision surfaces?

c) Name three different methods for implementing subdivision surfaces?

Solution:

a) It prevents overshooting.

b) Split and tweak.

c) Doo-Sabin, Catmull-Clark, Loop.

□