CSE-170 Computer Graphics

Lecture 20

Spline Interpolators

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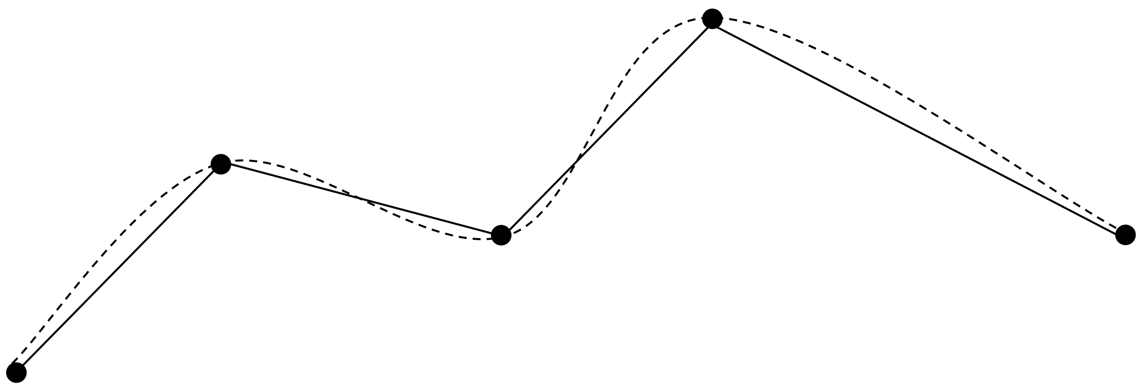
Catmull-Rom Splines

Catmull-Rom Splines

- A piecewise interpolating curve, given:
  - Points to interpolate: $\mathbf{p}_0, \cdots, \mathbf{p}_m$
  - And knot values: $u_0, \ldots, u_m$

  - Find spline $\mathbf{q}$ such that: $\mathbf{q}(u_i) = \mathbf{p}_i$

    How…?



Catmull-Rom Splines

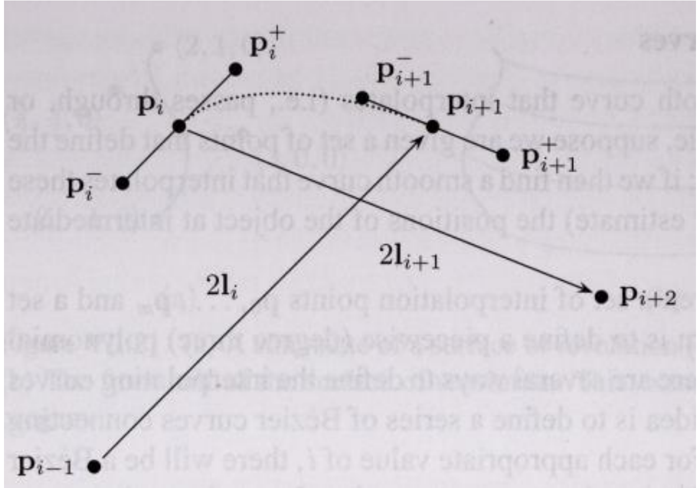
- Catmull-Rom solution:

$$\mathbf{I}_i = \frac{(\mathbf{p}_{i+1} - \mathbf{p}_{i-1})}{2}$$

$\leftarrow$ *half the vector from the previous point to the next point*

$$\mathbf{p}_i^- = \mathbf{p}_i - \frac{\mathbf{I}_i}{3} \qquad \mathbf{p}_i^+ = \mathbf{p}_i + \frac{\mathbf{I}_i}{3}$$

$\leftarrow$ *now add and subtract one third of the vector to each point in order to define all needed control points*



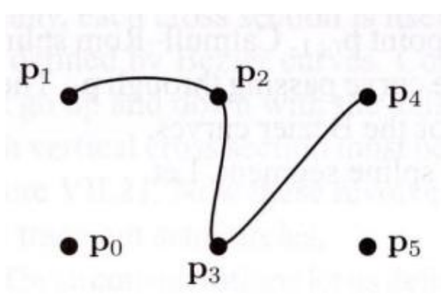
(Note: the "l" in the image is the "I" in the equations above)

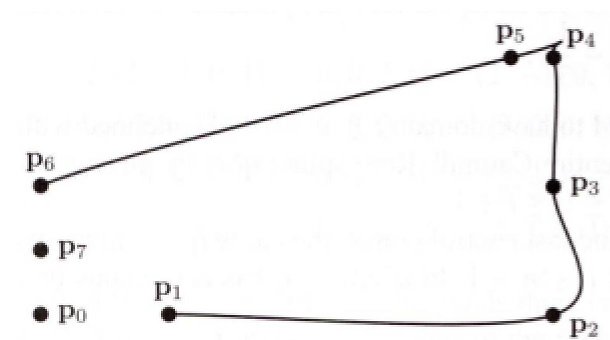
  - Once new points are created, just compute a cubic Bezier for the control polygon formed by each: P_i, P_i^+, P^-_{i+1}, P_{i+1}

Catmull-Rom Splines

- Note that first and last points are not interpolated
  - But this is not really a problem
  - Allows to model initial/final curve shape direction

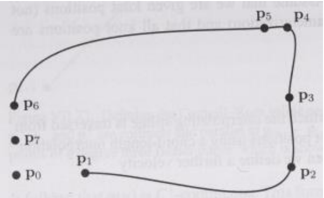
- Possible undesired effect: overshooting
  - How to avoid overshooting…?





Bessel-Overhauser Splines

Bessel-Overhauser Splines



- Overshooting can be solved with a parameterization close to chord-length:
  1. Choose knot vectors ***u*** close to chord-length
     (common to use distance between control points)
  2. Weight intermediate points according to distances:

$$\mathbf{v}_{i^+} = \frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{u_{i+1} - u_i} \qquad \mathbf{v}_{i^-} = \frac{\mathbf{p}_i - \mathbf{p}_{i-1}}{u_i - u_{i-1}}$$

$\longleftarrow$ *vectors scaled to have similar lengths*

$$\mathbf{v}_i = \frac{(u_{i+1} - u_i)\mathbf{v}_{i^-} + (u_i - u_{i-1})\mathbf{v}_{i^+}}{u_{i+1} - u_{i-1}}$$

$\longleftarrow$ *weighted average of vectors give the direction of the adjacent control points*

$$\mathbf{p}_i^- = \mathbf{p}_i - \frac{1}{3}(u_i - u_{i-1})\mathbf{v}_i$$

$\longleftarrow$ *previous control point is scaled by the "distance" to the original previous point*

$$\mathbf{p}_i^+ = \mathbf{p}_i + \frac{1}{3}(u_{i+1} - u_i)\mathbf{v}_i$$

$\longleftarrow$ *next control point is scaled by the "distance" to the original next point*

Bessel-Overhauser Splines

- Ex.: using distances between the points:

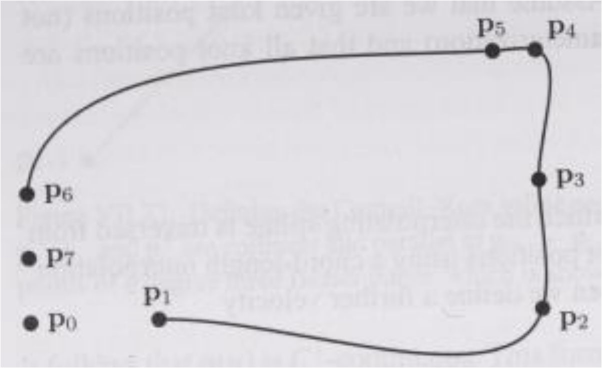
$$d_{i^-} = \|\mathbf{p}_i - \mathbf{p}_{i-1}\|, \quad d_{i^+} = \|\mathbf{p}_{i+1} - \mathbf{p}_i\|$$

$$\hat{\mathbf{v}}_{i^+} = \frac{\mathbf{p}_{i+1} - \mathbf{p}_i}{d_{i^+}} \qquad \hat{\mathbf{v}}_{i^-} = \frac{\mathbf{p}_i - \mathbf{p}_{i-1}}{d_{i^-}}$$

$$\mathbf{v}_i = \frac{d_{i^+}\hat{\mathbf{v}}_{i^-} + d_{i^-}\hat{\mathbf{v}}_{i^+}}{d_{i^+} + d_{i^-}}$$

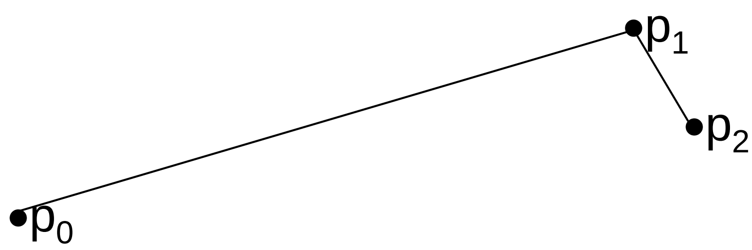
$$\mathbf{p}_i^- = \mathbf{p}_i - \frac{1}{3} d_{i^-}\mathbf{v}_i$$

$$\mathbf{p}_i^+ = \mathbf{p}_i + \frac{1}{3} d_{i^+}\mathbf{v}_i$$



Bessel-Overhauser Splines

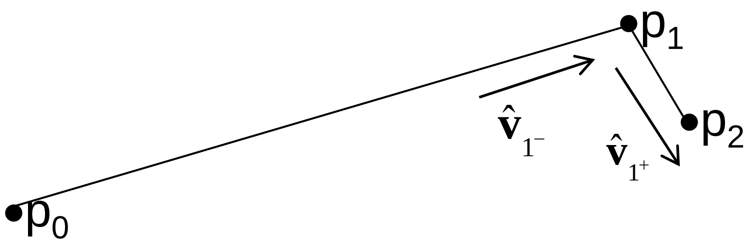
- Example:



- Compute control segments before and after p_1

Bessel-Overhauser Splines

- Example:

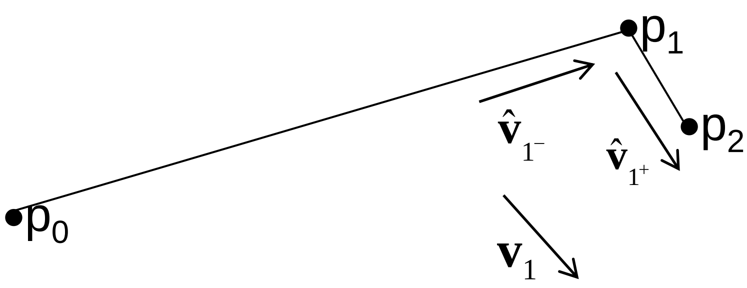


$$d_{1^-} = \|\mathbf{p}_1 - \mathbf{p}_0\| = 5, \quad d_{1^+} = \|\mathbf{p}_2 - \mathbf{p}_1\| = 1$$

$$\hat{\mathbf{v}}_{1^+} = \frac{\mathbf{p}_2 - \mathbf{p}_1}{1} \qquad \hat{\mathbf{v}}_{1^-} = \frac{\mathbf{p}_1 - \mathbf{p}_0}{5}$$

Bessel-Overhauser Splines

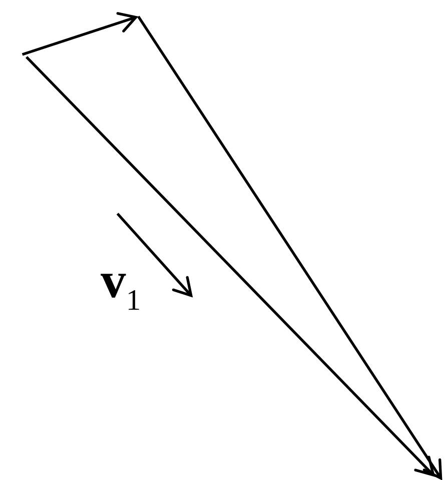
- Example:



$$d_{1^-} = \left\| \mathbf{p}_1 - \mathbf{p}_0 \right\| = 5, \quad d_{1^+} = \left\| \mathbf{p}_2 - \mathbf{p}_1 \right\| = 1$$

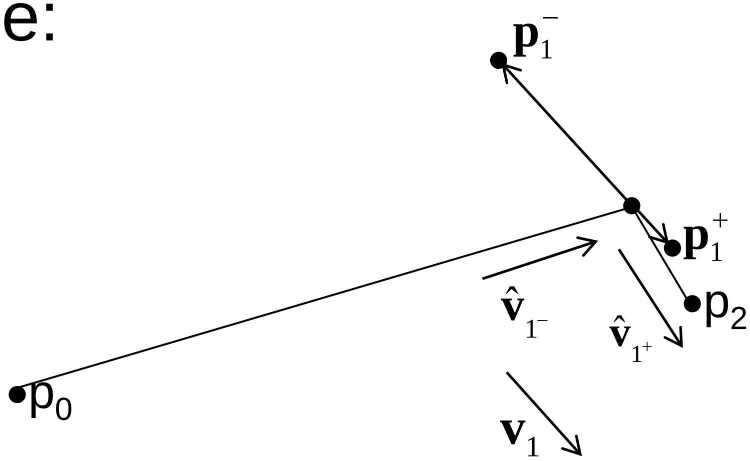
$$\hat{\mathbf{v}}_{1^+} = \frac{\mathbf{p}_2 - \mathbf{p}_1}{1} \qquad \hat{\mathbf{v}}_{1^-} = \frac{\mathbf{p}_1 - \mathbf{p}_0}{5}$$

$$\mathbf{v}_1 = \frac{1\hat{\mathbf{v}}_{i^-} + 5\hat{\mathbf{v}}_{i^+}}{5+1}$$



Bessel-Overhauser Splines

- Example:

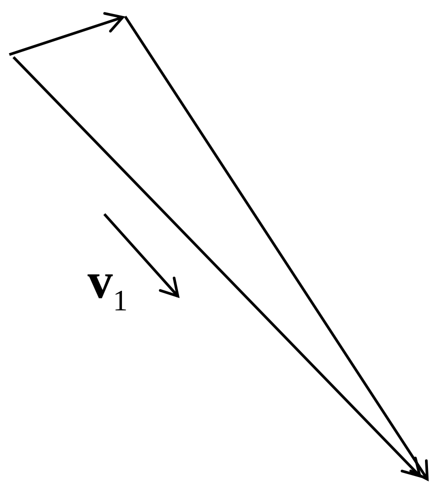


$$d_{1^-} = \left\| \mathbf{p}_1 - \mathbf{p}_0 \right\| = 5, \quad d_{1^+} = \left\| \mathbf{p}_2 - \mathbf{p}_1 \right\| = 1$$

$$\hat{\mathbf{v}}_{1^+} = \frac{\mathbf{p}_2 - \mathbf{p}_1}{1} \qquad \hat{\mathbf{v}}_{1^-} = \frac{\mathbf{p}_1 - \mathbf{p}_0}{5}$$

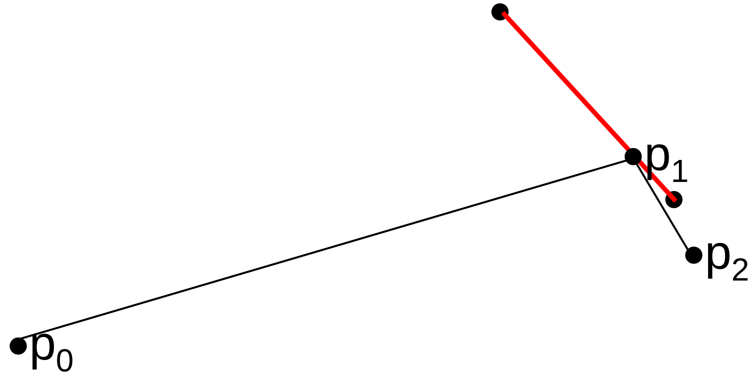
$$\mathbf{v}_1 = \frac{1\hat{\mathbf{v}}_{i^-} + 5\hat{\mathbf{v}}_{i^+}}{5+1}$$

$$\mathbf{p}_1^- = \mathbf{p}_1 - \frac{1}{3} 5\mathbf{v}_1 \qquad \mathbf{p}_1^+ = \mathbf{p}_1 + \frac{1}{3}\mathbf{v}_1$$



Bessel-Overhauser Splines

- Result:



Extension to Surfaces

Surface Interpolation

- The interpolators we have seen are defined by Bézier curves
  - The same strategy used to define Bézier surface patches can be used to interpolate given points in 3D
  - There are analogous formulations for the surface interpolation cases of Catmull-Rom splines and Bessel-Overhauser splines