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VCM - INTRO ROBOTICS
2023/02/27 — Homework 1

1 Composite Rotations

Two frames A and B are initially coincident. Frame B then undergoes the following sequence of transformations:

1. a rotation of $\pi/4$ about the $\mathbf{y}$ axis (fixed);
2. a rotation of $\pi/2$ about the $\mathbf{x}$ axis (fixed);
3. a rotation of $\pi/6$ about the $\mathbf{z}$ axis (moving);
4. a rotation of $\pi/3$ about the $\mathbf{x}$ axis (fixed);
5. a rotation of $\pi/3$ about the $\mathbf{y}$ axis (moving).

Write the final rotation matrix ${}^{A}_{B}\mathbf{R}$ describing the orientation of B with respect to A.
Note: you do not need to compute the final matrix by performing all intermediate multiplications. All that matters here is the order, so you can leave matrices in their symbolic form (as long as it is correct).

- Counterclockwise rotation around x-axis

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

- Counterclockwise rotation around y-axis

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

- Counterclockwise rotation around z-axis

$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

...owing three figures show what positive rotations look like

moving = append

fixed = prepend

$\boxed{1.}$ ${}^{A}_{B}B' = R_y(\pi/4)$

$\boxed{2}$ ${}^{A}_{B}B'' = R_x\left(\frac{\pi}{2}\right) R_y\left(\frac{\pi}{4}\right)$

$\boxed{3}$ ${}^{A}_{B}B''' = R_x\left(\frac{\pi}{2}\right) R_y\left(\frac{\pi}{4}\right) R_z\left(\frac{\pi}{6}\right)$

$\boxed{4}$ ${}^{A}_{B}B^{IV} = R_x\left(\frac{\pi}{3}\right) R_x\left(\frac{\pi}{2}\right) R_y\left(\frac{\pi}{4}\right) R_z\left(\frac{\pi}{6}\right)$

$\boxed{5}$ ${}^{A}_{B}B^{V} = R_x\left(\frac{\pi}{3}\right) R_x\left(\frac{\pi}{2}\right) R_y\left(\frac{\pi}{4}\right) R_z\left(\frac{\pi}{6}\right) R_y\left(\frac{\pi}{3}\right)$

$\boxed{\text{END P1}}$

2 Transformation Matrices

Two frames A and B are initially coincident. Frame B then undergoes the following transformations:

1. a rotation of $\pi/2$ about the $\mathbf{x}$ axis;
2. a translation of 3 units about the $\mathbf{y}$;
3. a rotation of $\pi/2$ about the $\mathbf{z}$ axis (fixed frame).

Write the transformation matrices ${}^A_B\mathbf{T}$ and ${}^B_A\mathbf{T}$.

1

$S_1 \Rightarrow R_x$

$S_2 \Rightarrow T_{y3} R_x$

$S_3 \Rightarrow R_z [T_{y3} R_x]$

$${}^A_B B' = R_x\left(\frac{\pi}{2}\right) \rightarrow \boxed{1}$$

$${}^A_B B'' = T_y(3)\, R_x\left(\frac{\pi}{2}\right) \rightarrow \boxed{2}$$

$${}^A_B B''' = R_z\left(\frac{\pi}{2}\right) T_y(3)\, R_x\left(\frac{\pi}{2}\right) \rightarrow \boxed{3}$$

$\boxed{1}$

$$\boxed{R_x\left(\frac{\pi}{2}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}}$$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\boxed{2}$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\alpha = 0$

$\beta = 3$

$\gamma = 0$

$$T = \begin{bmatrix} R_N & \begin{matrix} P_x \\ P_y \\ P_z \end{matrix} \\ 0 \; 0 \; 0 & 1 \end{bmatrix} = B$$

$$\boxed{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}$$

$\boxed{3}$

$$R_z\left(\frac{\pi}{2}\right) T_y(3)\, R_x\left(\frac{\pi}{2}\right) \Rightarrow \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left[\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \right]$$

$$R_z\left(\frac{\pi}{2}\right) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R^4} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\boxed{\text{P2}}$ continued

$${}^B_A R = {}^A_B R^T$$

$$R_z\left(\tfrac{\pi}{2}\right) R_x\left(\tfrac{\pi}{2}\right) T_y(3) \Longrightarrow \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \right)$$

$$\boxed{{}^A_B T \Rightarrow \begin{bmatrix} 0 & 0 & 1 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

THRU FORMULA

$${}^B_A T = \begin{bmatrix} {}^B_A R & -{}^B_A R \, {}^A O' \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$$

$$\boxed{T T^{-1} = I}$$

$${}^B_A R = {}^A_B R^T$$

$$\Longrightarrow \begin{bmatrix} 0 & 1 & 0 & -(0) \\ 0 & 0 & 1 & -(0) \\ 1 & 0 & 0 & -(3) \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^B_A T}$$

Verify

$$T^B_A T^A_B = I$$

$$\begin{bmatrix} 0 & 0 & 1 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \checkmark$$

T^A_B & T^B_A are valid

$\boxed{\text{END P2}}$

3 Quaternions to Rotations

Let $\mathbf{q} = a + bi + cj + dk$ be a unit quaternion. In the lecture notes it is stated that its associated rotation matrix is

$$\mathbf{R} = \begin{bmatrix} 2(a^2+b^2)-1 & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & 2(a^2+c^2)-1 & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & 2(a^2+d^2)-1 \end{bmatrix}.$$

Show that $\mathbf{R}$ is a rotation matrix.

- PROVE R is a rotation matrix

$\boxed{1}$ each of the columns has length 1

$\boxed{2}$ its columns are mutually orthogonal

$\boxed{3}$ its determinant is 1

$\boxed{3}$ $\det(M) = 1$

$$\Rightarrow 2(a^2+b^2)-1 \begin{vmatrix} 2(a^2+c^2)-1 & 2(cd-ab) \\ 2(cd+ab) & 2(a^2+d^2)-1 \end{vmatrix} - 2(bc-ad) \begin{vmatrix} 2(bc+ad) & 2(cd-ab) \\ 2(bd-ac) & 2(a^2+d^2)-1 \end{vmatrix} + 2(bd+ac) \begin{vmatrix} 2(bc-ad) & 2(a^2+c^2)-1 \\ 2(bd-ac) & 2(cd+ab) \end{vmatrix}$$

$$\Rightarrow 2(a^2+b^2)-1 \left[\left[(2a^2+c^2)-1\right]\left[2(a^2+d^2)-1\right] - (2cd-2ab)(2cd+2ab) \right]$$

$\boxed{1}$

$\boxed{2}$

$$-2(bc-ad)\left[(2bc+2ad)\left[(2a^2+2d^2)-1\right] - (2bd-2ac)(2cd-2ab)\right]$$

$\boxed{3}$

$$+2(bd+ac)\left[(2bc-2ad)(2cd+2ab) - (2bd-2ac)\left[2a^2+2c^2-1\right]\right]$$

$\boxed{1}$

$$2a^2+2b^2-1 \left[\begin{array}{c|ccc} & 2a^2 & +2c^2 & -1 \\ \hline 2a^2 & 4a^4 & +4a^2c^2 & -2a^2 \\ 2d^2 & 4a^2d^2 & 4c^2d^2 & -2d^2 \\ -1 & -2a^2 & -2c^2 & 1 \end{array} - \begin{array}{c|cc} & 2cd & -2ab \\ \hline 2cd & 4c^2d^2 & -\cancel{4abcd} \\ 2ab & \cancel{4abcd} & -4a^2b^2 \end{array} \right]$$

$\boxed{1}$

$$2a^2+2b^2-1 \left[\cancel{4a^4} + \cancel{4a^2c^2} - \cancel{2a^2} + 4a^2d^2 + \cancel{4c^2d^2} - 2d^2 - \cancel{2a^2} - 2c^2 + 1 - \left[\cancel{4c^2d^2} - \cancel{4a^2b^2}\right] \right]$$

$$\left[2a^2+2b^2-1\right]\left[4a^4 + 4a^2b^2 + 4a^2c^2 + 4a^2d^2 - 4a^2 - 2c^2 - 2d^2 + 1\right]$$

$$[2a^2+2b^2-1]\left[4a^4+4a^2b^2+4a^2c^2+4a^2d^2-4a^2-2c^2-2d^2+1\right]$$

$\boxed{2}$

$$-2(bc-ad)\left[(2bc+2ad)\left((2a^2+2d^2)-1\right)-(2bd-2ac)(2cd-2ab)\right]$$

$\boxed{3}$

$$+2(bd+ac)\left[(2bc-2ad)(2cd+2ab)-(2bd-2ac)\left[(2a^2+2c^2)-1\right]\right]$$

$$-2bc+2ad\left[\begin{array}{c|cc|c} & 2bc & 2ad & -1 \\ \hline 2a^2 & 4a^2bc & 4a^3d & -2a^2 \\ 2d^2 & 4bcd^2 & 4ad^3 & -2d^2 \end{array}\qquad \begin{array}{c|cc} & 2bd & -2ac \\ \hline 2cd & 4bcd^2 & -4ac^2d \\ -2ab & -4ab^2d & 4a^2bc \end{array}\right.$$

$$4a^2bc+4a^3d+4bcd^2+4ad^3-2a^2-2d^2 - \left[4bcd^2-4ac^2d-4ab^2d+4a^2bc\right]$$

$\boxed{2}$

$$\boxed{(-2bc+2ad)\left(4a^2bc+4a^3d+4bcd^2+4ad^3-2a^2-2d^2-4bcd^2+4ac^2d+4ab^2d-4a^2bc\right)}$$

$\boxed{3}$

$$2bd+2ac\left[\begin{array}{c|cc} & 2bc & -2ad \\ \hline 2cd & 4bc^2d & -4acd^2 \\ +2ab & 4ab^2c & -4a^2bd \end{array}\qquad \begin{array}{c|ccc} & 2a^2 & 2c^2 & -1 \\ \hline 2bd & 4a^2bd & 4bc^2d & -2bd \\ -2ac & 4a^3c & -4ac^3 & 2ac \end{array}\right]$$

$$\boxed{2bd+2ac\left[4bc^2d-4acd^2+4ab^2c-4a^2bd-4a^2bd-4bc^2d+2bd+4a^3c+4ac^3-2ac\right]}$$

REDUCE

$$[2a^2+2b^2-1]\left[4a^4+4a^2b^2+4a^2c^2+4a^2d^2-4a^2-2c^2-2d^2+1\right]$$

$$\boxed{(-2bc+2ad)\left(4a^2bc+4a^3d+4bcd^2+4ad^3-2a^2-2d^2-4bcd^2+4ac^2d+4ab^2d-4a^2bc\right)}$$

$$+\ \boxed{2bd+2ac\left[4bc^2d-4acd^2+4ab^2c-4a^2bd-4a^2bd-4bc^2d+2bd+4a^3c+4ac^3-2ac\right]}$$

$$[2a^2+2b^2-1]\left[4a^4+4a^2b^2+4a^2c^2+4a^2d^2-4a^2-2c^2-2d^2+1\right]$$

$$\boxed{(-2bc+2ad)\left[4a^2bc+4a^3d+4bcd^2+4ad^3-2a^2-2d^2-4bcd^2+4ac^2d+4ab^2d-4a^2bc\right]}$$

$$+\ \boxed{2bd+2ac\left[4bc^2d-4acd^2+4ab^2c-4a^2bd-4a^2bd-4bc^2d+2bd+4a^3c+4ac^3-2ac\right]}$$

unfinished

[1] columns of M are all = 1

3 Quaternions to Rotations

Let $\mathbf{q} = a + bi + cj + dk$ be a unit quaternion. In the lecture notes it is stated that its associated rotation matrix is

$$\mathbf{R} = \begin{bmatrix} 2(a^2+b^2)-1 & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & 2(a^2+c^2)-1 & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & 2(a^2+d^2)-1 \end{bmatrix}.$$

Show that $\mathbf{R}$ is a rotation matrix.

$\boxed{\text{Column}=1}$

$$|q| = a^2+b^2+c^2+d^2 = 1$$

$$a^2-1+c^2+d^2 = -b^2$$

$$1 = \left[2(a^2+b^2)-1\right]^2 + \left[2(bc+ad)\right]^2 + \left[2(bd-ac)\right]^2$$

using wolfram Alpha to expand

$$1 = 4a^4 + 8a^2b^2 + 4a^2c^2 + 4a^2d^2 - 4a^2 + 4b^4 - 4b^2 + 4b^2c^2 + 4b^2d^2 + 1$$

$4a^2 \quad 4b^2$

$$4a^2(a^2 + 2b^2 + c^2 + d^2 - 1) + 4b^2(b^2 - 1 + c^2 + d^2) + 1$$

$$4a^2(+b^2) + 4b^2(-a^2) + 1$$

$$4a^2(+b^2) + 4b^2(-a^2)$$

$$1 = 4a^2b^2 - 4b^2a^2 + 1$$

$$1 = 1 \checkmark$$

$$\begin{array}{c|ccc} & 2a^2 & 2b^2 & -1 \\ \hline 2a^2 & 4a^4 & 4a^2b^2 & -2a^2 \\ 2b^2 & 4a^2b^2 & 4b^4 & -2b^2 \\ -1 & -2a^2 & -2b^2 & 1 \end{array}$$

$a^2+b^2+c^2+d^2=1$

b^2

$a^2+b^2+c^2+d^2-1 =$

1

$a^2+2b^2+c^2+d^2-1 =$

$0 - b$

$\boxed{\text{COLUMN}=2}$

$$1 = (2(bc-ad))^2 + (2(a^2+c^2)-1)^2 + [2(cd+ab)]^2$$

a^2

w/ wolfram alpha

$$1 = 4c^4 + 4c^2a^2 + 8c^2a^2 + 4c^2b^2 - 4c^2 + 4a^4 - 4a^2 + 4d^2a^2 + 4a^2b^2 + 1$$

$$1 = 4c^2\left[c^2 + d^2 + 2a^2 + b^2 - 1\right] + 4a^2\left[a^2 - 1 + d^2 + b^2\right]$$

$-c^2$

$$1 = 4c^2(a^2) - 4a^2c^2 + 1$$

$$\boxed{1=1} \checkmark$$

$\boxed{\text{COLUMN}=3}$

$$1 = (2(bd+ac))^2 + 2(cd-ab)^2 + \left[2(a^2+d^2)-1\right]^2$$

w/ wolfram $\Rightarrow$

$$1 = 4d^4 + 4b^2d^2 + 8d^2a^2 + 4d^2c^2 - 4d^2 + 4a^4 - 4a^2 + 4b^2a^2 + 4a^2c^2 + 1$$

$$4d^2(d^2 + b^2 + 2a^2 + c^2 - 1) + 4a^2\left[a^2 - 1 + b^2 + c^2\right] + 1 \longrightarrow$$

$-d^2$

$$1 = 4d^2(a^2) - 4a^2d^2 + 1 \Rightarrow \boxed{1=1} \checkmark$$

$\therefore$ we have proved that all columns of M are equal to 1

$$\mathbf{R} = \begin{bmatrix} 2(a^2+b^2)-1 & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & 2(a^2+c^2)-1 & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & 2(a^2+d^2)-1 \end{bmatrix}.$$

	$2a^2$	$2b^2$	-1
$2bc$	$4a^2bc$	$4b^3c$	$-2bc$
$-2ad$	$-4a^2d$	$-4ab^2d$	$2ad$

2 THE COLUMNS ARE MUTUALLY ORTHOGONAL

$$\boxed{Col_1 \cdot Col_2} = (2a^2+2b^2-1)(2bc-2ad) + (2bc+2ad)(2a^2+2c^2-1) + (2bd+2ac)(2cd+2ab)$$

thru symbolab check

$$= 4bc^3 - 4bc + 4b^3c + 4bca^2 + 4bcd^2$$

$$= 4bc(c^2 - 1 + b^2 + a^2 + d^2)$$

reduction 1

~~where~~

$$\boxed{a^2+b^2+c^2+d^2=1}$$

$$= 4bc\,(1-1)$$

$$= 4bc(0) = \boxed{0} \quad \therefore \text{orthogonal}$$

$$\boxed{Col_1 \cdot Col_3} = 2(a^2+b^2)-1\,(2bd+2ac) + 2(bc+ad)(2cd-2ab) + (2bd-2ac)(2(a^2+d^2)-1)$$

check thru symbolab

$$= 4b^3d - 4bd + 4bd^3 + 4bda^2 + 4bdc^2$$

$$= 4bd(b^2 - 1 + d^2 + a^2 + c^2)$$

reduction 1

$$= 4bd\,(1-1)$$

$$= 4bd(0) = \boxed{0} \quad \therefore \text{orthogonal}$$

$$\boxed{Col_2 \cdot Col_3} = 2(bc-ad)\,2(bd+ac) + (2(a^2+c^2)-1)(2cd-ab) + 2(cd+ab)(2(a^2+d^2)-1)$$

$$= 4c^3d - 4cd + 4cd^3 + 4b^2cd + 4ca^2d$$

$$= 4cd\left[c^2 - 1 + d^2 + b^2 + a^2\right]$$

use reduction 1

$$= 4cd\,(1-1)$$

$$= 4cd(0) = \boxed{0} \quad \therefore \text{orthogonal}$$

WHERE, IN ORDER TO PROVE THAT R IS A valid rotation matrix the following have to be satisfied:

$\boxed{1}$ each of the columns has length 1
$\boxed{2}$ its columns are mutually orthogonal
$\boxed{3}$ its determinant is 1

✓
✓

thru symbolab only → not able to get final reduction

$\boxed{\text{END P3}}$

4 Change of Coordinates

Three vectors are operating in a shared space. Let A, B, and C the three frames attached to the robots, and let W be a world frame. Assume that the following transformation matrices are known: ${}^{B}_{A}\mathbf{T}$, ${}^{C}_{W}\mathbf{T}$, ${}^{B}_{C}\mathbf{T}$, ${}^{W}_{B}\mathbf{T}$ Assume robot A perceives a point of interest whose coordinates are ${}^{A}\mathbf{p}$. Can you determine any of the following: ${}^{B}\mathbf{p}$, ${}^{C}\mathbf{p}$. ${}^{W}\mathbf{p}$? For each of the required points, if the answer is positive, show how it can be computed, and if the answer is negative explain why it cannot be computed.

FIND Bp, Cp, Wp

$T(A \to B)$
$T(W \to C)$
$T(C \to B)$
$T(B \to W)$

find Bp

$$Bp = \left[T(A \to B)\right] Ap = {}^{B}_{A}T \cdot Ap$$

Find Cp

$$Cp = \left[T(W \to \boxed{C}) \cdot T(B \to W) \cdot T(A \to B)\right] Ap =$$

$$\left[{}^{C}_{W}T \; {}^{W}_{B}T \; {}^{B}_{A}T\right] \cdot Ap$$

Find Wp

$$Wp = \left[T(B \to W) \cdot T(A \to B)\right] Ap = \left[{}^{W}_{B}T \; {}^{B}_{A}T\right] \cdot Ap$$

END P4

5 Quaternions

Quaternions can be multiplied following rules similar to those we follow for complex numbers. A fundamental thing to remember is that **quaternions product is not commutative**. When multiplying two quaternions, keep in mind the following definitions about products between their imaginary coefficients i, j, k:

- $i^2 = j^2 = k^2 = ijk = -1$
- $ij = k$, $ji = -k$
- $jk = i$, $kj = -i$
- $ki = j$, $ik = -j$

Consider the following two quaternions:

$$\mathbf{p} = 1 + 2i - 3k$$

$$\mathbf{q} = 5 + 4j + 2k.$$

Compute:

1. the product $\mathbf{pq}$.
2. the norm of the product $\mathbf{pq}$.

Note: show the intermediate steps; if you just write the result, you will not get any point.

$\boxed{1.}$ $pq = (1 + 2i - 3k)(5 + 4j + 2k)$

$= 5 + 4j + 2k + 10i + 8\overset{k}{ij} + 4\overset{-j}{ik} - 15k - 12\overset{-i}{kj} - 6\overset{-1}{k^2}$

$= \cancel{5} + \cancel{4j} + \cancel{2k} + \cancel{10}i + 8k - \cancel{4j} - 15k + \cancel{12i} + \cancel{6}$

$= 11 + 22i + 0j - 5k$

$\boxed{= 11 + 22i - 5k}$

	q: 5	4j	2k
p: 1	5	4j	2k
2i	10i	8ij	4ik
-3k	-15k	-12kj	-6k²

$\sqrt{121 + 484\cancel{i^2} - 25\cancel{k^2}}$

$\boxed{2}$ norm of $pq = |pq|$

$= \sqrt{pq \circ pq^*} \Rightarrow$

$= \sqrt{11^2 + 22^2 + -5)^2} = \sqrt{121 + 484 + 25} = \sqrt{630}$

$= 25.0798$

END PS