Lecture 02. Probability

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These slides are based on slides from Mahdi Roozbahani

Outline

- Probability Distributions
- Joint and Conditional Probability Distributions
- Bayes' Rule
- Mean and Variance
- Properties of Gaussian Distribution
- Maximum Likelihood Estimation

Probability

- A sample space S is the set of all possible outcomes of a conceptual or physical, repeatable experiment. (S can be finite or infinite.)
    - E.g., S may be the set of all possible outcomes of a dice roll: S $(1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6)$
    - E.g., S may be the set of all possible nucleotides of a DNA site: S $(A \quad C \quad G \quad T)$

    - E.g., S may be the set of all possible time-space positions of an aircraft on a radar screen.
- An Event A is any subset of S
    - Seeing "1" or "6" in a dice roll; observing a "G" at a site; UA007 in space-time interval

Three Key Ingredients in Probability Theory

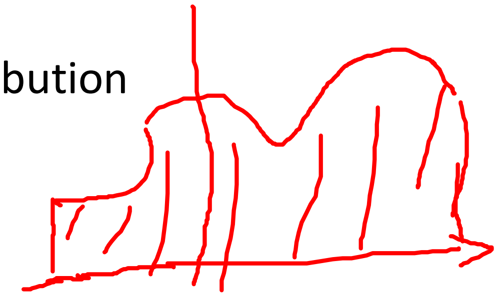
A sample space is a collection of all possible outcomes

Random variables X represents **outcomes** in sample space

Probability of a random variable to happen $p(x) = p(X = x)$

$$p(x) \geq 0$$

Continuous variable
Continuous probability distribution
Probability density function
Density or likelihood value
Temperature (real number)
Gaussian Distribution



$$\int_x p(x)dx = 1$$

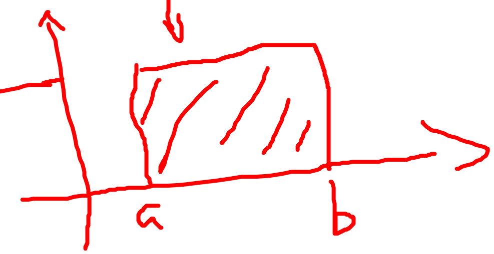
Discrete variable
Discrete probability distribution
Probability mass function
Probability value
Coin flip (integer)
Bernoulli distribution

$$\sum_{x \epsilon A} p(x) = 1$$

Continuous Probability Functions

- Examples:
  - Uniform Density Function:

$$f_x(x) = \begin{cases} \dfrac{1}{b-a} & for\; a \le x \le b \\ 0 & otherwise \end{cases}$$



  - Exponential Density Function:

$$f_x(x) = \frac{1}{\mu} e^{-\frac{x}{\mu}} \qquad for\; x \ge 0$$

$$F_x(x) = 1 - e^{\frac{-x}{\mu}} \qquad for\; x \ge 0$$

  - Gaussian(Normal) Density Function

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$N(\mu, \sigma)$

Discrete Probability Functions

- Examples:
  - Bernoulli Distribution:
    - $\begin{cases} 1-p & for\ x = 0 \\ p & for\ x = 1 \end{cases}$

1-Head

Tail

In Bernoulli, just a single trial is conducted

  - Binomial Distribution:
    - $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$

k is number of successes

n-k is number of failures

$\binom{n}{k}$ The total number of ways of selection **k** distinct combinations of **n** trials, irrespective of order.

Continuous Probability Functions

- A continuous random variable X is defined on a continuous sample space: an interval on the real line, a region in a high dimensional space, etc.
  - It is meaningless to talk about the probability of the random variable assuming a particular value --- P(x) = 0
  - Instead, we talk about the probability of the random variable assuming a value within a given interval, or half interval, or arbitrary Boolean combination of basic propositions.
  - Cumulative Distribution Function (CDF): $F_x(x) = \mathrm{P}[X \leq x]$
  - Probability Density Function (PDF): $F_x(x) = \int_{-\infty}^{x} f_x(x)\, dx$ or $f_x(x) = \frac{d\, F_x(x)}{dx}$
  - Properties: $f_x(x) \geq 0$ and $\int_{-\infty}^{\infty} f_x(x)\, dx = 1$
  - Interpretation: $f_x(x) = \mathrm{P}[X \in \frac{x, x+\Delta}{\Delta}]$

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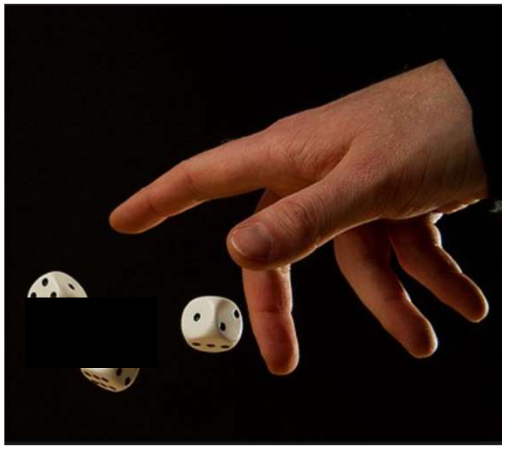
Example



X = Throw a dice

Y = Flip a coin

X and **Y** are random variables

N = total number of trials

$\boldsymbol{n_{ij}}$ = Number of occurrence

X

Y		$x_{i=1} = 1$	$x_{i=2} = 2$	$x_{i=3} = 3$	$x_{i=4} = 4$	$x_{i=5} = 5$	$x_{i=6} = 6$	C_j
	$y_{j=2} = tail$	$n_{ij} = 3$	$n_{ij} = 4$	$n_{ij} = 2$	$n_{ij} = 5$	$n_{ij} = 1$	$n_{ij} = 5$	20
	$y_{j=1} = head$	$n_{ij} = 2$	$n_{ij} = 2$	$n_{ij} = 4$	$n_{ij} = 2$	$n_{ij} = 4$	$n_{ij} = 1$	15
	C_i	5	6	6	7	5	6	N=35

X

Y		$x_{i=1} = 1$	$x_{i=2} = 2$	$x_{i=3} = 3$	$x_{i=4} = 4$	$x_{i=5} = 5$	$x_{i=6} = 6$	C_j
	$y_{j=2} = tail$	$n_{ij} = 3$	$n_{ij} = 4$	$n_{ij} = 2$	$n_{ij} = 5$	$n_{ij} = 1$	$n_{ij} = 5$	20
	$y_{j=1} = head$	$n_{ij} = 2$	$n_{ij} = 2$	$n_{ij} = 4$	$n_{ij} = 2$	$n_{ij} = 4$	$n_{ij} = 1$	15
	C_i	5	6	6	7	5	6	N=35

$$p(x = 1, y = tail) = \frac{3}{35}$$

$$p(y = tail|x = 1) = \frac{3}{5}$$

$$p(y = head) = \frac{15}{35}$$

Probability:

$$p(X = x_i) = \frac{c_i}{N}$$

Joint probability:

$$p(X = x_i, Y = y_j) = \frac{n_{ij}}{N}$$

Conditional probability:

$$p(Y = y_j | X = x_i) = \frac{n_{ij}}{c_i}$$

Sum rule

$$p(X = x_i) = \sum_{j=1}^{L} p\left(X = x_i, Y = y_j\right) \Rightarrow \qquad p(X) = \boxed{\sum_{Y} P(X, Y)}$$

Product rule

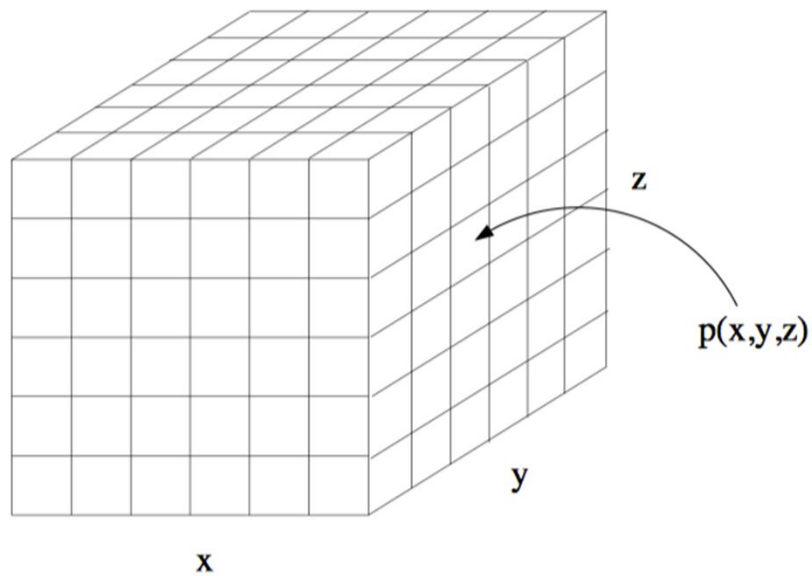
$$p\left(X = x_i, Y = y_j\right) = \frac{n_{ij}}{N} = \frac{n_{ij}}{c_i} \frac{c_i}{N} = p\left(Y = y_j \middle| X = x_i\right) p(X = x_i)$$

$$p(X, Y) = p(Y|X)p(X)$$

Joint Distribution

- Key concept: two or more random variables may interact. Thus, the probability of one taking on a certain value depends on which value(s) the others are taking.
- We call this a joint ensemble and write

$$p(x, y) = \text{prob}(X = x \text{ and } Y = y)$$

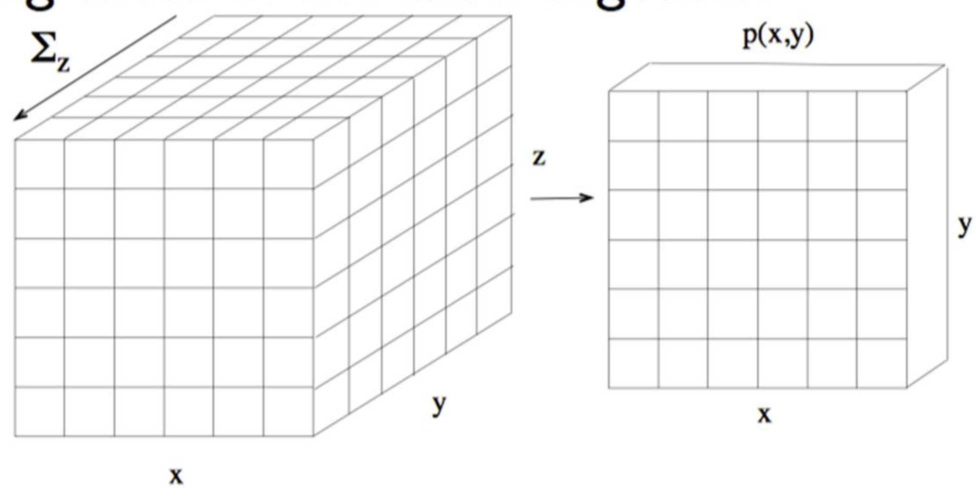


Marginal Distribution

- We can "sum out" part of a joint distribution to get the *marginal distribution* of a subset of variables:

$$p(x) = \sum_y p(x,y)$$

- This is like adding slices of the table together.

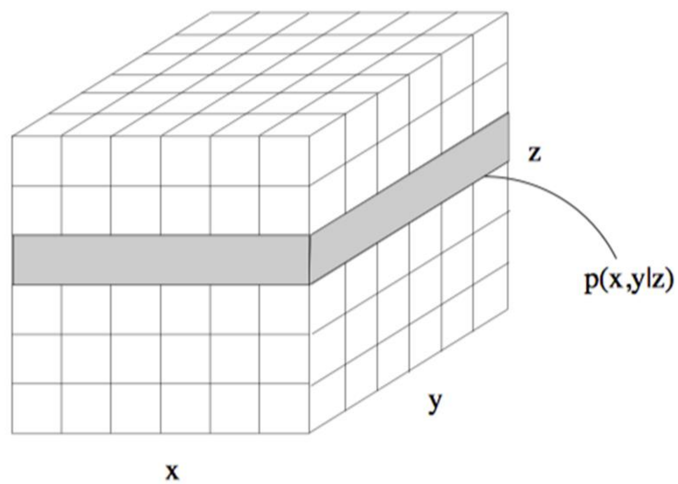


- Another equivalent definition: $p(x) = \sum_y p(x|y)p(y)$.

Conditional Distribution

- If we know that some event has occurred, it changes our belief about the probability of other events.
- This is like taking a "slice" through the joint table.

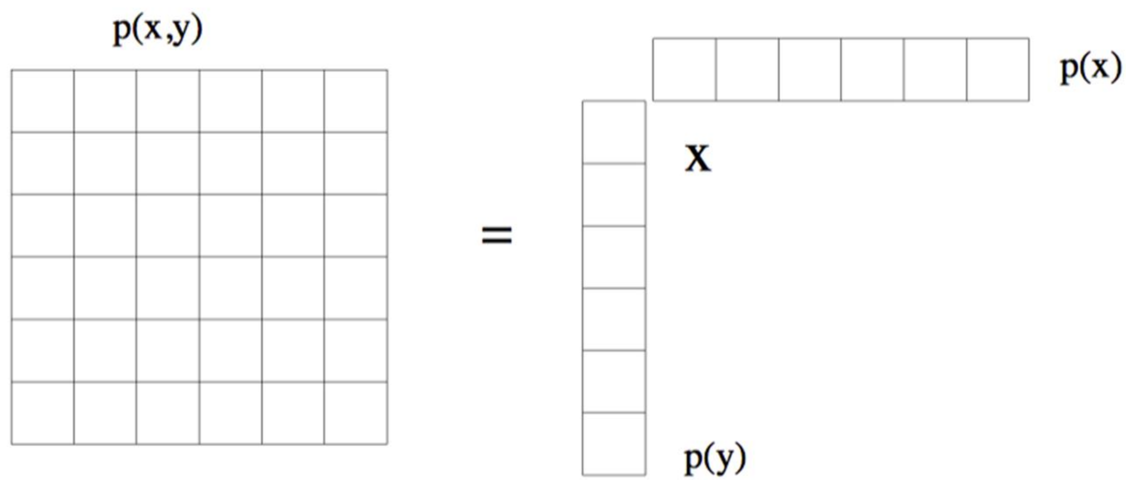
$$\boxed{p(x|y) = p(x, y)/p(y)}$$



Independence & Conditional Independence

- Two variables are independent iff their joint factors:

$$p(x, y) = p(x)p(y)$$



- Two variables are conditionally independent given a third one if for all values of the conditioning variable, the resulting slice factors:

$$p(x, y|z) = p(x|z)p(y|z) \qquad \forall z$$

Conditional Independence

- Examples:

P(Virus | ~~Drink Beer~~) = P(Virus)
iff Virus is independent of Drink Beer

P(Flu | Virus;~~DrinkBeer~~) = P(Flu|Virus)
iff Flu is independent of Drink Beer, given Virus

P(Headache | Flu;~~Virus~~;DrinkBeer) =
P(Headache|Flu;DrinkBeer)
iff Headache is independent of Virus, given Flu and Drink Beer

Assume the above independence, we obtain:

P(Headache;Flu;Virus;DrinkBeer)
=P(Headache | Flu;Virus;DrinkBeer) P(Flu | Virus;DrinkBeer)
P(Virus | Drink Beer) P(DrinkBeer)
=P(Headache|Flu;DrinkBeer) P(Flu|Virus) P(Virus) P(DrinkBeer)

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Bayes' Rule

- P(X|Y)= Fraction of the worlds in which X is true given that Y is also true.

- For example:
  - H="Having a headache"
  - F="Coming down with flu"
  - $P(Headche|Flu)$ = fraction of flu-inflicted worlds in which you have a headache. How to calculate?

- Definition:

$$P(X|Y) = \frac{P(X,Y)}{P(Y)} = \frac{\mathrm{P(Y|X)P(X)}}{\mathrm{P(Y)}}$$

Corollary:

$$P(X,Y) = P(Y|X)P(X)$$

This is called Bayes Rule

Bayes' Rule

- $P(Headache|Flu) = \frac{P(Headache,Flu)}{P(Flu)}$

$$= \frac{P(Flu|Headache)P(Headache)}{P(Flu)}$$

Other cases:

- $P(Y|X) = \frac{P(X|Y)P(Y)}{P(X|Y)P(Y)+P(X|\neg Y)P(\neg Y)}$
- $P(Y=y_i|X) = \frac{P(X|Y)P(Y)}{\sum_{i\in S} P(X|Y=y_i)P(Y=y_i)}$ p(x)
- $P(Y|X,Z) = \frac{P(X|Y,Z)P(Y,Z)}{P(X,Z)} =$

$$\frac{P(X|Y,Z)P(Y,Z)}{P(X|Y,Z)P(Y,Z)+P(X|\neg Y,Z)P(\neg Y,Z)}$$

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Mean and Variance

- Expectation: The mean value, center of mass, first moment:

$E(X) =$

$$E_X[g(X)] = \int_{-\infty}^{\infty} g(x) p_X(x) dx = \mu$$

- N-th moment: $g(x) = x^n$
- N-th central moment: $g(x) = (x - \mu)^n$

$N(\mu, \sigma)$

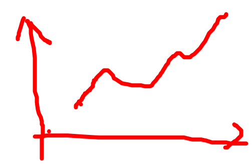
- Mean: $E_X[X] = \int_{-\infty}^{\infty} x p_X(x) dx$
  - $E[\alpha X] = \alpha E[X]$
  - $E[\alpha + X] = \alpha + E[X]$

$E(\alpha) = \alpha$

$E(X^2) = \mu^2 + \sigma$

- Variance(Second central moment): $Var(x) = E_X[(X - E_X[X])^2] = E_X[X^2] - E_X[X]^2$ $= Var(X)$
  - $Var(\alpha X) = \alpha^2 Var(X)$
  - $Var(\alpha + X) = Var(X)$

$Var(a) = 0$



For Joint Distributions

- Expectation and Covariance:
  - $E[X + Y] = E[X] + E[Y]$
  - $cov(X, Y) = E[(X - E_X[X])(Y - E_Y(Y)] = E[XY] - E[X]E[Y]$
  - $Var(X + Y) = Var(X) + 2cov(X, Y) + Var(Y)$

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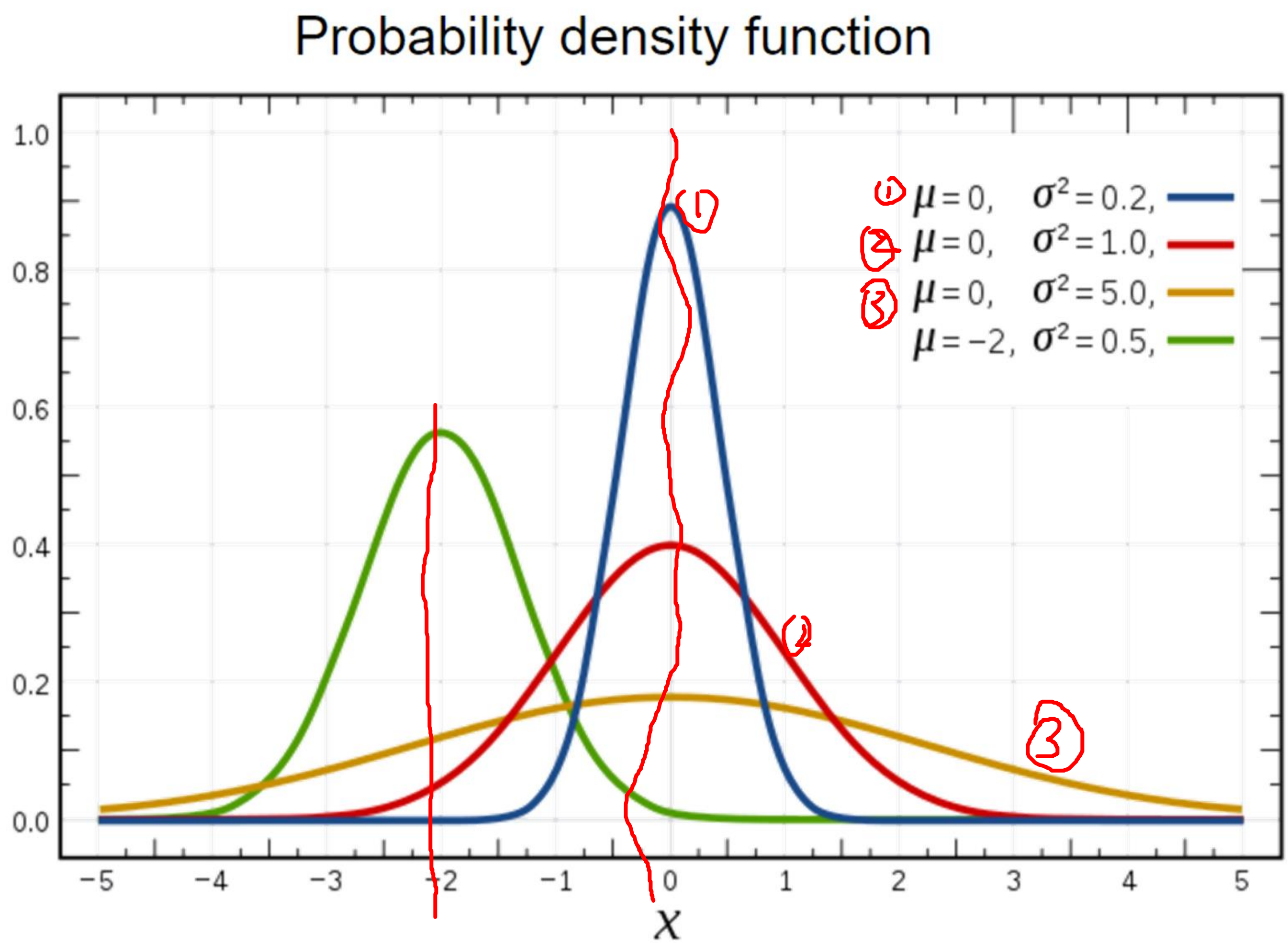
Gaussian Distribution

- **Gaussian Distribution:** $f(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$



[Probability versus likelihood]

Multivariate Gaussian Distribution

$$p(x|\mu,\Sigma) = \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}}\exp\{-\frac{1}{2}(x-\mu)^{\top}\Sigma^{-1}(x-\mu)\}$$

- Moment Parameterization $\mu = E(X)$

$$\Sigma = Cov(X) = E[(X-\mu)(X-\mu)^{\top}]$$

- Mahalanobis Distance $\Delta^2 = (x-\mu)^{\top}\Sigma^{-1}(x-\mu)$

- Tons of applications (MoG, FA, PPCA, Kalman filter,...)

Multivariate Gaussian Distribution

- Joint Gaussian $P(X_1, X_2)$

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$$

- Marginal Gaussian

$$\mu_2^m = \mu_2 \qquad \Sigma_2^m = \Sigma_2$$

- Conditional Gaussian $P(X_1 | X_2 = x_2)$

$$\mu_{1|2} = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2)$$

$$\Sigma_{1|2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

Properties of Gaussian Distribution

- The linear transform of a Gaussian r.v. is a Gaussian. Remember that no matter how x is distributed

$$E(AX + b) = AE(X) + b$$

$$Cov(AX + b) = ACov(X)A^{\top}$$

this means that for Gaussian distributed quantities:

$$X \sim N(\mu, \Sigma) \;\rightarrow\; AX + b \sim N(A\mu + b, A\Sigma A^{\top})$$

- The sum of two independent Gaussian r.v. is a Gaussian

$$\mathrm{Y} = \mathrm{X}_1 + \mathrm{X}_2,\; \mathrm{X}_1 \perp X_2 \;\rightarrow \mu_y = \mu_1 + \mu_2, \Sigma_y = \Sigma_1 + \Sigma_2$$

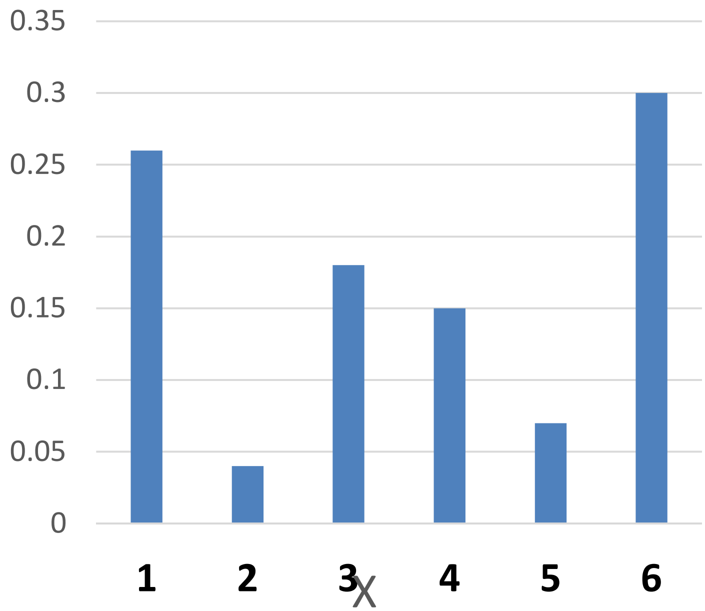
- The multiplication of two Gaussian functions is another Gaussian function (although no longer normalized)

$$N(a, A)N(b, B) \propto N(c, C),$$

$$where\; C = (A^{-1} + B^{-1})^{-1}, c = \mathrm{C}\mathrm{A}^{-1}a + CB^{-1}b$$

Central Limit Theorem

Probability mass function of a biased dice



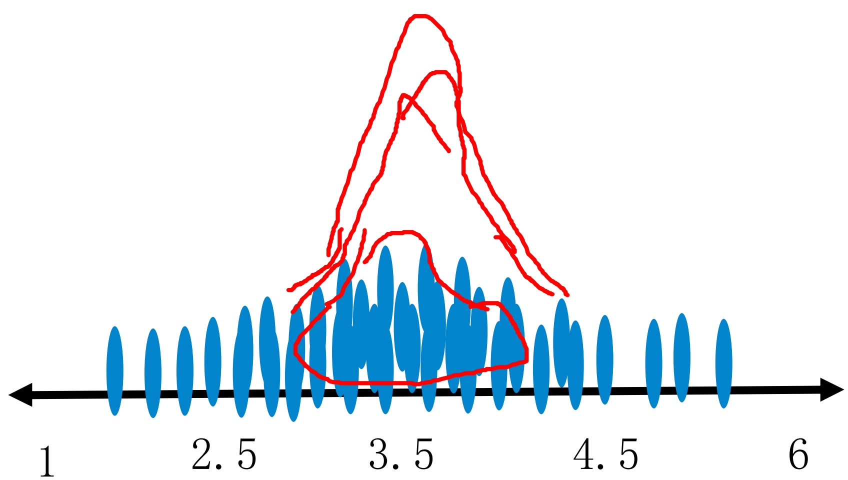
Let's say, I am going to get a sample from this pmf having a size of $n = 4$

$$S_1 = \{1,1,1,6\} \Rightarrow E(S_1) = 2.25$$

$$S_2 = \{1,1,3,6\} \Rightarrow E(S_2) = 2.75$$

$$\vdots$$

$$S_m = \{1,4,6,6\} \Rightarrow E(S_m) = 4.25$$



According to CLT, it will follow a bell curve distribution (normal distribution)

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Maximum Likelihood Estimation

Data — Model

M ← M =

$N(\mu, \sigma)$

- Probability: inferring probabilistic quantities for data given fixed models (e.g. prob. of events, marginals, conditionals, etc).
- Statistics: inferring a model given fixed data observations (e.g. clustering, classification, regression).

Main assumption:

Independent and identically distributed random variables

i.i.d

Maximum Likelihood Estimation

For Bernoulli (i.e. flip a coin):

Objective function: $f(x_i;p) = p^{x_i}(1-p)^{1-x_i}$ $\quad x_i \in \{0,1\}$ or $\{head, tail\}$

$[x_1 \quad x_2 \; \cdots \; x_n]$

$[0 \quad 1 \quad 0$

$[1 \quad 1 \quad 0 \quad 1]$

$$L(p) = p(X = x_1, X = x_2, X = x_3, \ldots, X = x_n)$$

i.i.d assumption

$$= p(X = x_1)p(X = x_2)\ldots p(X = x_n) = f(x_1;p)f(x_2;p)\ldots f(x_n;p)$$

Max

$$L(p) = \prod_{i=1}^{n} f(x_i;p) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$$

$$L(p) = p^{x_1}(1-p)^{1-x_1} \times p^{x_2}(1-p)^{1-x_2} \ldots \times p^{x_n}(1-p)^{1-x_n} =$$

$$= p^{\sum x_i}(1-p)^{\sum(1-x_i)}$$

We don't like multiplication, let's convert it into summation

What's the trick?

Take the log

$$L(p) = p^{\sum x_i}(1-p)^{\sum(1-x_i)}$$

max $$logL(p) = l(p) = \log(p)\sum_{i=1}^{n} x_i + \log(1-p)\sum_{i=1}^{n}(1-x_i)$$

How to optimize p?

$$\frac{\partial l(p)}{\partial p} = 0$$

$$\frac{\sum_{i=1}^{n} x_i}{p} - \frac{\sum_{i=1}^{n}(1-x_i)}{1-p} = 0$$

a b

$$\sum^{n} 1$$

$$\frac{a(1-p) - bp}{p(1-p)} = 0$$

$$a - ap - bp = 0$$

$$p = \frac{a}{a+b} = \frac{\sum x_i}{n}$$

$$p = \frac{1}{n}\sum_{i=1}^{n} x_i$$