연쇄법칙 (The Chain Rule)









Theorem

g is differentiable at x



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g is differentiable at x f is differentiable at g(x)
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g is differentiable at x

f is differentiable at g(x)

F = f \circ g
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Theorem

g is differentiable at x f is differentiable at g(x) $F = f \circ g$, F(x) = f(g(x))



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g is differentiable at x
f is differentiable at g(x)
F = f \circ g, F(x) = f(g(x))
y = f(u)
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 \begin{cases} g \text{ is differentiable at } x \\ f \text{ is differentiable at } g(x) \\ F = f \circ g , F(x) = f(g(x)) \\ y = f(u) \\ u = g(x) \end{cases}
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$$\left[\begin{array}{l} g \text{ is differentiable at } x \\ f \text{ is differentiable at } g(x) \\ F = f \circ g \ , \ F(x) = f(g(x)) \\ y = f(u) \\ u = g(x) \end{array} \right] \Rightarrow$$



$$\begin{bmatrix} g \text{ is differentiable at } x \\ f \text{ is differentiable at } g(x) \\ F = f \circ g , F(x) = f(g(x)) \\ y = f(u) \\ u = g(x) \end{bmatrix} \Rightarrow \begin{bmatrix} f(x) & f(x) & f(x) \\ f(x) & f($$



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F is differentiable at x



$$\begin{bmatrix} g \text{ is differentiable at } x \\ f \text{ is differentiable at } g(x) \\ F = f \circ g , F(x) = f(g(x)) \\ y = f(u) \\ u = g(x) \end{bmatrix} \Rightarrow \begin{bmatrix} F \text{ is differentiable at } x \\ F'(x) = f'(g(x))g'(x) \end{bmatrix}$$

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$$F ext{ is differentiable at } x$$

$$F'(x) = f'(g(x))g'(x)$$

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Proof.

 $\varepsilon_1(h) =$



$$\varepsilon_1(h) = \left\{ \right.$$



$$\varepsilon_1(h) = \begin{cases} \frac{g(x+h) - g(x)}{h} - g'(x) & , \quad h \neq 0 \end{cases}$$



$$\varepsilon_1(h) = \begin{cases} \frac{g(x+h) - g(x)}{h} - g'(x) & , & h \neq 0 \\ 0 & , & h = 0 \end{cases}$$



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$$\varepsilon_1(h) = \left\{ \begin{array}{ccc} \frac{g(x+h)-g(x)}{h} - g'(x) & , & h \neq 0 \\ & & & \varepsilon_1 \text{ is continuous at } h = 0 \\ & 0 & , & h = 0 \end{array} \right.$$



$$\varepsilon_1(h) = \left\{ \begin{array}{ccc} \frac{g(x+h) - g(x)}{h} - g'(x) & , & h \neq 0 \\ & & & \varepsilon_1 \text{ is continuous at } h = 0 \text{ } (\because g \text{ is differentiable at } x) \\ & & & & & \end{array} \right.$$



$$\varepsilon_1(h) = \left\{ \begin{array}{ll} \frac{g(x+h) - g(x)}{h} - g'(x) & , \quad h \neq 0 \\ & \varepsilon_1 \text{ is continuous at } h = 0 \text{ ($\cdot \cdot \cdot$ g is differentiable at x)} \\ & h \cdot \varepsilon_1(h) \end{array} \right.$$



$$\varepsilon_1(h) = \begin{cases} \frac{g(x+h) - g(x)}{h} - g'(x) & , & h \neq 0 \\ 0 & & \varepsilon_1 \text{ is continuous at } h = 0 \text{ (\because g is differentiable at x)} \end{cases}$$

$$h \cdot \varepsilon_1(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h$$



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$$\varepsilon_2(k) =$$



$$\varepsilon_1(h) = \left\{ \begin{array}{l} \displaystyle \frac{g(x+h) - g(x)}{h} - g'(x) & , \quad h \neq 0 \\ \\ 0 & , \quad h = 0 \end{array} \right.$$

$$\varepsilon_1 \text{ is continuous at } h = 0 \text{ (\because g$ is differentiable at x)}$$

$$h \cdot \varepsilon_1(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h)$$

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$$\varepsilon_2(k) = \left\{ \begin{array}{l} \displaystyle \frac{f(u+k) - f(u)}{k} - f'(u) & , \quad k \neq 0 \end{array} \right.$$



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$$h \cdot \varepsilon_1(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h) \}$$

$$\varepsilon_2(k) = \left\{ \begin{array}{ccc} \frac{f(u+k)-f(u)}{k} & -f'(u) & , & k \neq 0 \\ & & & \varepsilon_2 \text{ is continuous at } k = 0 \\ \\ 0 & & , & k = 0 \end{array} \right.$$



$$\begin{split} \varepsilon_1(h) &= \left\{ \begin{array}{l} \frac{g(x+h)-g(x)}{h} - g'(x) & , \quad h \neq 0 \\ 0 & , \quad h = 0 \end{array} \right. \\ \\ \left. b \cdot \varepsilon_1(h) = \left\{ g(x+h)-g(x) \right\} - g'(x) \cdot h \ , \ g(x+h)-g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h) \right. \\ \\ \left. \varepsilon_2(k) = \left\{ \begin{array}{l} \frac{f(u+k)-f(u)}{k} - f'(u) & , \quad k \neq 0 \\ 0 & , \quad k = 0 \end{array} \right. \\ \\ \left. \varepsilon_2 \text{ is continuous at } k = 0 \text{ (\because f is differentiable at u)} \right. \end{split}$$



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$$f(u+k) - f(u)$$



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$$f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k)$$

$$\text{(Let $k = g(x+h) - g(x) \text{ , } g(x+h)$)}$$



$$\varepsilon_1(h) = \begin{cases} \frac{g(x+h) - g(x)}{h} - g'(x) &, & h \neq 0 \\ 0 &, & h = 0 \end{cases}$$

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$$f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k)$$

$$\text{(Let $k = g(x+h) - g(x)$, $g(x+h) = g(x) + k$}$$



$$\varepsilon_1(h) = \left\{ \begin{array}{l} \frac{g(x+h) - g(x)}{h} - g'(x) & , \quad h \neq 0 \\ 0 & , \quad h = 0 \end{array} \right.$$

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$$\varepsilon_2(k) = \left\{ \begin{array}{l} \frac{f(u+k) - f(u)}{k} - f'(u) & , \quad k \neq 0 \\ 0 & , \quad k = 0 \end{array} \right.$$

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$$\text{(Let $k = g(x+h) - g(x)$, } g(x+h) = g(x) + k = u + k)$$



$$\varepsilon_1(h) = \left\{ \begin{array}{l} \displaystyle \frac{g(x+h) - g(x)}{h} - g'(x) & , \quad h \neq 0 \\ 0 & , \quad h = 0 \end{array} \right.$$

$$\varepsilon_1 \text{ is continuous at } h = 0 \text{ ($\cdot \cdot \cdot$ g is differentiable at x)}$$

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$$\varepsilon_2(k) = \left\{ \begin{array}{l} \displaystyle \frac{f(u+k) - f(u)}{k} - f'(u) & , \quad k \neq 0 \\ 0 & , \quad k = 0 \end{array} \right.$$

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$$f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k) \text{ (Let $k = g(x+h) - g(x)$) , } g(x+h) = g(x) + k = u + k)$$



$$\varepsilon_1(h) = \left\{ \begin{array}{l} \frac{g(x+h) - g(x)}{h} - g'(x) & , \quad h \neq 0 \\ 0 & , \quad h = 0 \end{array} \right.$$

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$$\varepsilon_2 \text{ is continuous at } k = 0 \text{ (\because f$ is differentiable at u)}$$

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$$f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k)$$

$$\text{(Let } k = g(x+h) - g(x), \quad g(x+h) = g(x) + k = u + k)$$

$$f(g(x+h)) - f(g(x)) = \{f'(u) + \varepsilon_2(g(x+h) - g(x))\} \{g(x+h) - g(x)\}$$



$$\begin{split} \varepsilon_1(h) &= \left\{ \begin{array}{l} \frac{g(x+h) - g(x)}{h} - g'(x) &, \quad h \neq 0 \\ 0 &, \quad h = 0 \end{array} \right. \\ & \left. \varepsilon_1 \text{ is continuous at } h = 0 \text{ } (\because g \text{ is differentiable at } x) \right. \\ & \left. h \cdot \varepsilon_1(h) = \left\{ g(x+h) - g(x) \right\} - g'(x) \cdot h \text{ }, \text{ } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h) \right. \\ & \left. \varepsilon_2(k) = \left\{ \begin{array}{l} \frac{f(u+k) - f(u)}{k} - f'(u) &, \quad k \neq 0 \\ 0 &, \quad k = 0 \end{array} \right. \\ & \left. k \cdot \varepsilon_2(k) = \left\{ f(u+k) - f(u) \right\} - f'(u) \cdot k \text{ }, \text{ } f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k) \right. \\ & \left. f(u+k) - f(u) \right. & \left. = f'(u) \cdot k + k \cdot \varepsilon_2(k) \right. \\ & \left. \left(\text{Let } k = g(x+h) - g(x) \right) \cdot g(x+h) = g(x) + k = u + k \right) \\ & \left. f(g(x+h)) - f(g(x)) \right. & \left. \left\{ f'(u) + \varepsilon_2(g(x+h) - g(x)) \right\} \left\{ g(x+h) - g(x) \right\} \\ & \left. \left\{ f'(u) + \varepsilon_2(g(x+h) - g(x)) \right\} \left\{ g'(x) + \varepsilon_1(h) \right\} \cdot h \end{split} \right. \end{split}$$



Proof.

$$\varepsilon_1(h) = \begin{cases} \frac{g(x+h) - g(x)}{h} - g'(x) &, & h \neq 0 \\ 0 &, & h = 0 \end{cases}$$

$$\varepsilon_1 \text{ is continuous at } h = 0 \text{ (}\because g \text{ is differentiable at } x \text{)}$$

$$h \cdot \varepsilon_1(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h) \}$$

$$\varepsilon_2(k) = \begin{cases} \frac{f(u+k) - f(u)}{k} - f'(u) &, & k \neq 0 \\ 0 &, & k = 0 \end{cases}$$

$$\varepsilon_2 \text{ is continuous at } k = 0 \text{ (}\because f \text{ is differentiable at } u \text{)}$$

$$k \cdot \varepsilon_2(k) = \{f(u+k) - f(u)\} - f'(u) \cdot k \text{ , } f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k) \}$$

$$f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k) \text{ (Let } k = g(x+h) - g(x) \text{ , } g(x+h) = g(x) + k = u + k) \}$$

$$f(g(x+h)) - f(g(x)) = \{f'(u) + \varepsilon_2(g(x+h) - g(x))\} \{g(x+h) - g(x)\}$$

$$= \{f'(u) + \varepsilon_2(g(x+h) - g(x))\} \{g'(x) + \varepsilon_1(h)\} \cdot h \}$$

F'(x)



$$\varepsilon_{1}(h) = \begin{cases} \frac{g(x+h) - g(x)}{h} - g'(x) &, & h \neq 0 \\ 0 &, & h = 0 \end{cases}$$

$$\varepsilon_{1} \text{ is continuous at } h = 0 \text{ (}\because\text{ } g \text{ is differentiable at } x \text{)}$$

$$h \cdot \varepsilon_{1}(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_{1}(h)$$

$$\varepsilon_{2}(k) = \begin{cases} \frac{f(u+k) - f(u)}{k} - f'(u) &, & k \neq 0 \\ 0 & \varepsilon_{2} \text{ is continuous at } k = 0 \text{ (}\because\text{ } f \text{ is differentiable at } u \text{)} \end{cases}$$

$$k \cdot \varepsilon_{2}(k) = \{f(u+k) - f(u)\} - f'(u) \cdot k \text{ , } f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_{2}(k)$$

$$f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_{2}(k)$$

$$(\text{Let } k = g(x+h) - g(x) \text{ , } g(x+h) = g(x) + k = u + k)$$

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END