연쇄법칙 (The Chain Rule)









Theorem

g is differentiable at x



```
g is differentiable at x f is differentiable at g(x)
```



```
g is differentiable at x

f is differentiable at g(x)

F = f \circ g
```



Theorem

g is differentiable at x f is differentiable at g(x) $F = f \circ g$, F(x) = f(g(x))



```
g is differentiable at x
f is differentiable at g(x)
F = f \circ g, F(x) = f(g(x))
y = f(u)
```



```
 \begin{cases} g \text{ is differentiable at } x \\ f \text{ is differentiable at } g(x) \\ F = f \circ g , F(x) = f(g(x)) \\ y = f(u) \\ u = g(x) \end{cases}
```





$$\left[\begin{array}{l} g \text{ is differentiable at } x \\ f \text{ is differentiable at } g(x) \\ F = f \circ g \ , \ F(x) = f(g(x)) \\ y = f(u) \\ u = g(x) \end{array} \right] \Rightarrow$$



$$\begin{bmatrix} g \text{ is differentiable at } x \\ f \text{ is differentiable at } g(x) \\ F = f \circ g , F(x) = f(g(x)) \\ y = f(u) \\ u = g(x) \end{bmatrix} \Rightarrow \begin{bmatrix} f(x) & f(x) & f(x) \\ f(x) & f($$



$$\begin{bmatrix} g \text{ is differentiable at } x \\ f \text{ is differentiable at } g(x) \\ F = f \circ g , F(x) = f(g(x)) \\ y = f(u) \\ u = g(x) \end{bmatrix} \Rightarrow \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}$$

F is differentiable at x



$$\begin{bmatrix} g \text{ is differentiable at } x \\ f \text{ is differentiable at } g(x) \\ F = f \circ g , F(x) = f(g(x)) \\ y = f(u) \\ u = g(x) \end{bmatrix} \Rightarrow \begin{bmatrix} F \text{ is differentiable at } x \\ F'(x) = f'(g(x))g'(x) \end{bmatrix}$$

$$\begin{cases} F \text{ is differentiable at } x \\ F'(x) = f'(g(x))g'(x) \end{cases}$$



$$\begin{bmatrix} g \text{ is differentiable at } x \\ f \text{ is differentiable at } g(x) \\ F = f \circ g \\ y = f(u) \\ u = g(x) \end{bmatrix} \Rightarrow \begin{bmatrix} F \text{ is differentiable at } x \\ F'(x) = f'(g(x))g'(x) \\ \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \end{bmatrix}$$

$$F ext{ is differentiable at } x$$

$$F'(x) = f'(g(x))g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$



$$\begin{bmatrix} g \text{ is differentiable at } x \\ f \text{ is differentiable at } g(x) \\ F = f \circ g \text{ , } F(x) = f(g(x)) \\ y = f(u) \\ u = g(x) \end{bmatrix} \Rightarrow \begin{bmatrix} F \text{ is differentiable at } x \\ F'(x) = f'(g(x))g'(x) \\ \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \end{bmatrix}$$

$$F ext{ is differentiable at } x$$

$$F'(x) = f'(g(x))g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$



$$\begin{bmatrix} g \text{ is differentiable at } x \\ f \text{ is differentiable at } g(x) \\ F = f \circ g \text{ , } F(x) = f(g(x)) \\ y = f(u) \\ u = g(x) \end{bmatrix} \Rightarrow \begin{bmatrix} F \text{ is differentiable at } x \\ F'(x) = f'(g(x))g'(x) \\ \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \end{bmatrix}$$

$$F ext{ is differentiable at } x$$

$$F'(x) = f'(g(x))g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$



Proof.

 $\varepsilon_1(h) =$



$$\varepsilon_1(h) = \left\{ \right.$$



$$\varepsilon_1(h) = \begin{cases} \frac{g(x+h) - g(x)}{h} - g'(x) & , \quad h \neq 0 \end{cases}$$



$$\varepsilon_1(h) = \begin{cases} \frac{g(x+h) - g(x)}{h} - g'(x) & , & h \neq 0 \\ 0 & , & h = 0 \end{cases}$$



$$\varepsilon_1(h) = \begin{cases} \frac{g(x+h) - g(x)}{h} - g'(x) & , & h \neq 0 \\ 0 & , & h = 0 \end{cases}$$



$$\varepsilon_1(h) = \left\{ \begin{array}{ccc} \frac{g(x+h)-g(x)}{h} - g'(x) & , & h \neq 0 \\ & & & \varepsilon_1 \text{ is continuous at } h = 0 \\ & 0 & , & h = 0 \end{array} \right.$$



$$\varepsilon_1(h) = \left\{ \begin{array}{ccc} \frac{g(x+h) - g(x)}{h} - g'(x) & , & h \neq 0 \\ & & & \varepsilon_1 \text{ is continuous at } h = 0 \text{ } (\because g \text{ is differentiable at } x) \\ & & & & & \end{array} \right.$$



$$\varepsilon_1(h) = \left\{ \begin{array}{ll} \frac{g(x+h) - g(x)}{h} - g'(x) & , \quad h \neq 0 \\ \\ 0 & , \quad h = 0 \end{array} \right.$$

$$\varepsilon_1 \text{ is continuous at } h = 0 \text{ ($\cdot \cdot \cdot$ g is differentiable at x)}$$

$$h \cdot \varepsilon_1(h)$$



$$\varepsilon_1(h) = \begin{cases} \frac{g(x+h) - g(x)}{h} - g'(x) & , & h \neq 0 \\ 0 & & \varepsilon_1 \text{ is continuous at } h = 0 \text{ (\because g is differentiable at x)} \end{cases}$$

$$h \cdot \varepsilon_1(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h$$



$$\varepsilon_1(h) = \left\{ \begin{array}{l} \frac{g(x+h) - g(x)}{h} - g'(x) & , \quad h \neq 0 \\ 0 & \varepsilon_1 \text{ is continuous at } h = 0 \text{ } (\because g \text{ is differentiable at } x) \\ \\ h \cdot \varepsilon_1(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h \text{ } , \text{ } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h) \end{array} \right.$$



$$\varepsilon_1(h) = \left\{ \begin{array}{l} \frac{g(x+h) - g(x)}{h} - g'(x) & , \quad h \neq 0 \\ 0 & \varepsilon_1 \text{ is continuous at } h = 0 \text{ } (\because g \text{ is differentiable at } x) \\ \\ h \cdot \varepsilon_1(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h \text{ } , \text{ } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h) \end{array} \right.$$

$$\varepsilon_2(h) =$$



$$\varepsilon_1(h) = \left\{ \begin{array}{l} \displaystyle \frac{g(x+h) - g(x)}{h} - g'(x) & , \quad h \neq 0 \\ \\ 0 & , \quad h = 0 \end{array} \right.$$

$$\varepsilon_1 \text{ is continuous at } h = 0 \text{ ($\cdot \cdot$ g is differentiable at x)}$$

$$h \cdot \varepsilon_1(h) = \left\{ g(x+h) - g(x) \right\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h)$$

$$\varepsilon_2(h) = \left\{ \begin{array}{l} \\ \\ \end{array} \right.$$



$$\varepsilon_1(h) = \left\{ \begin{array}{l} \frac{g(x+h) - g(x)}{h} - g'(x) & , \quad h \neq 0 \\ 0 & \varepsilon_1 \text{ is continuous at } h = 0 \text{ } (\because g \text{ is differentiable at } x) \\ \\ h \cdot \varepsilon_1(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h \text{ } , \text{ } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h) \\ \\ f(u+k) - f(u) & g'(x) \text{ } (h+k) = 0 \end{array} \right.$$



$$\varepsilon_1(h) = \left\{ \begin{array}{l} \displaystyle \frac{g(x+h) - g(x)}{h} - g'(x) & , \quad h \neq 0 \\ \\ 0 & , \quad h = 0 \end{array} \right.$$

$$\varepsilon_1 \text{ is continuous at } h = 0 \text{ ($\cdot \cdot$ g is differentiable at x)}$$

$$h \cdot \varepsilon_1(h) = \left\{ g(x+h) - g(x) \right\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h)$$

$$\varepsilon_2(h) = \left\{ \begin{array}{l} \displaystyle \frac{f(u+k) - f(u)}{h} - f'(u) & , \quad k \neq 0 \\ \\ 0 & , \quad k = 0 \end{array} \right.$$



$$\varepsilon_1(h) = \left\{ \begin{array}{l} \displaystyle \frac{g(x+h) - g(x)}{h} - g'(x) & , \quad h \neq 0 \\ \\ 0 & , \quad h = 0 \end{array} \right.$$

$$\varepsilon_1 \text{ is continuous at } h = 0 \text{ ($\cdot \cdot$ g is differentiable at x)}$$

$$h \cdot \varepsilon_1(h) = \left\{ g(x+h) - g(x) \right\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h)$$

$$\varepsilon_2(h) = \left\{ \begin{array}{l} \displaystyle \frac{f(u+k) - f(u)}{h} - f'(u) & , \quad k \neq 0 \\ \\ 0 & , \quad k = 0 \end{array} \right.$$



$$\varepsilon_1(h) = \begin{cases} \frac{g(x+h) - g(x)}{h} - g'(x) &, & h \neq 0 \\ 0 & \varepsilon_1 \text{ is continuous at } h = 0 \text{ (\because g is differentiable at x)} \end{cases}$$

$$h \cdot \varepsilon_1(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h) \}$$

$$\varepsilon_2(h) = \left\{ \begin{array}{ll} \displaystyle \frac{f(u+k)-f(u)}{h} - f'(u) & , \quad k \neq 0 \\ \\ 0 & , \quad k = 0 \end{array} \right.$$
 ε_2 is continuous at $k=0$



$$\varepsilon_1(h) = \left\{ \begin{array}{l} \displaystyle \frac{g(x+h) - g(x)}{h} - g'(x) & , \quad h \neq 0 \\ \\ 0 & , \quad h = 0 \end{array} \right.$$

$$\varepsilon_1 \text{ is continuous at } h = 0 \text{ (\because g$ is differentiable at x)}$$

$$h \cdot \varepsilon_1(h) = \left\{ g(x+h) - g(x) \right\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h)$$

$$\varepsilon_2(h) = \left\{ \begin{array}{l} \displaystyle \frac{f(u+k) - f(u)}{h} - f'(u) & , \quad k \neq 0 \\ \\ 0 & , \quad k = 0 \end{array} \right.$$

$$\varepsilon_2 \text{ is continuous at } k = 0 \text{ (\because f$ is differentiable at u)}$$



$$\varepsilon_1(h) = \left\{ \begin{array}{l} \frac{g(x+h) - g(x)}{h} - g'(x) & , \quad h \neq 0 \\ 0 & , \quad h = 0 \end{array} \right.$$

$$\varepsilon_1 \text{ is continuous at } h = 0 \text{ (\cdot: g is differentiable at x)}$$

$$h \cdot \varepsilon_1(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h)$$

$$\varepsilon_2(h) = \left\{ \begin{array}{l} \frac{f(u+k) - f(u)}{h} - f'(u) & , \quad k \neq 0 \\ 0 & , \quad k = 0 \end{array} \right.$$

$$\varepsilon_2 \text{ is continuous at $k = 0$ (\cdot: f is differentiable at u)}$$

$$k \cdot \varepsilon_2(k)$$



$$\varepsilon_1(h) = \left\{ \begin{array}{l} \frac{g(x+h) - g(x)}{h} - g'(x) & , \quad h \neq 0 \\ 0 & , \quad h = 0 \end{array} \right.$$

$$\varepsilon_1 \text{ is continuous at } h = 0 \text{ (\cdot: g is differentiable at x)}$$

$$h \cdot \varepsilon_1(h) = \left\{ g(x+h) - g(x) \right\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h) \right.$$

$$\varepsilon_2(h) = \left\{ \begin{array}{l} \frac{f(u+k) - f(u)}{h} - f'(u) & , \quad k \neq 0 \\ 0 & , \quad k = 0 \end{array} \right.$$

$$\varepsilon_2 \text{ is continuous at $k = 0$ (\cdot: f is differentiable at u)}$$

$$k \cdot \varepsilon_2(k) = \left\{ f(u+k) - f(u) \right\} - f'(u) \cdot k \right.$$



$$\varepsilon_1(h) = \left\{ \begin{array}{l} \displaystyle \frac{g(x+h) - g(x)}{h} - g'(x) & , \quad h \neq 0 \\ \\ 0 & , \quad h = 0 \end{array} \right.$$

$$\varepsilon_1 \text{ is continuous at } h = 0 \text{ (}\because g \text{ is differentiable at } x \text{)}$$

$$h \cdot \varepsilon_1(h) = \left\{ g(x+h) - g(x) \right\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h) \right.$$

$$\varepsilon_2(h) = \left\{ \begin{array}{l} \displaystyle \frac{f(u+k) - f(u)}{h} - f'(u) & , \quad k \neq 0 \\ \\ 0 & , \quad k = 0 \end{array} \right.$$

$$\varepsilon_2 \text{ is continuous at } k = 0 \text{ (}\because f \text{ is differentiable at } u \text{)}$$

$$k \cdot \varepsilon_2(k) = \left\{ f(u+k) - f(u) \right\} - f'(u) \cdot k \text{ , } f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k) \right.$$



$$\varepsilon_1(h) = \left\{ \begin{array}{l} \frac{g(x+h) - g(x)}{h} - g'(x) & , \quad h \neq 0 \\ 0 & , \quad h = 0 \end{array} \right.$$

$$\varepsilon_1 \text{ is continuous at } h = 0 \text{ (}\because\text{ } g \text{ is differentiable at } x \text{)}$$

$$h \cdot \varepsilon_1(h) = \left\{ g(x+h) - g(x) \right\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h) \right.$$

$$\varepsilon_2(h) = \left\{ \begin{array}{l} \frac{f(u+k) - f(u)}{h} - f'(u) & , \quad k \neq 0 \\ 0 & , \quad k = 0 \end{array} \right.$$

$$\varepsilon_2 \text{ is continuous at } k = 0 \text{ (}\because\text{ } f \text{ is differentiable at } u \text{)}$$

$$k \cdot \varepsilon_2(k) = \left\{ f(u+k) - f(u) \right\} - f'(u) \cdot k \text{ , } f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k) \right.$$

$$f(u+k) - f(u)$$



$$\varepsilon_1(h) = \begin{cases} \frac{g(x+h) - g(x)}{h} - g'(x) &, & h \neq 0 \\ 0 &, & h = 0 \end{cases}$$

$$\varepsilon_1 \text{ is continuous at } h = 0 \text{ (}\because g \text{ is differentiable at } x \text{)}$$

$$h \cdot \varepsilon_1(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h)$$

$$\varepsilon_2(h) = \begin{cases} \frac{f(u+k) - f(u)}{h} - f'(u) &, & k \neq 0 \\ 0 &, & k = 0 \end{cases}$$

$$\varepsilon_2 \text{ is continuous at } k = 0 \text{ (}\because f \text{ is differentiable at } u \text{)}$$

$$k \cdot \varepsilon_2(k) = \{f(u+k) - f(u)\} - f'(u) \cdot k \text{ , } f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k)$$

$$f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k)$$



$$\varepsilon_1(h) = \begin{cases} \frac{g(x+h) - g(x)}{h} - g'(x) &, & h \neq 0 \\ 0 &, & h = 0 \end{cases}$$

$$\varepsilon_1 \text{ is continuous at } h = 0 \text{ (\because g$ is differentiable at x)}$$

$$h \cdot \varepsilon_1(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h)$$

$$\varepsilon_2(h) = \begin{cases} \frac{f(u+k) - f(u)}{h} - f'(u) &, & k \neq 0 \\ 0 &, & k = 0 \end{cases}$$

$$\varepsilon_2 \text{ is continuous at } k = 0 \text{ (\because f$ is differentiable at u)}$$

$$k \cdot \varepsilon_2(k) = \{f(u+k) - f(u)\} - f'(u) \cdot k \text{ , } f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k)$$

$$f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k)$$

$$\text{(Let $k = g(x+h) - g(x) \text{ , } g(x+h)$)}$$



$$\varepsilon_1(h) = \begin{cases} \frac{g(x+h) - g(x)}{h} - g'(x) &, & h \neq 0 \\ 0 &, & h = 0 \end{cases}$$

$$h \cdot \varepsilon_1(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h , & g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h) \end{cases}$$

$$\varepsilon_2(h) = \begin{cases} \frac{f(u+k) - f(u)}{h} - f'(u) &, & k \neq 0 \\ 0 &, & k = 0 \end{cases}$$

$$k \cdot \varepsilon_2(k) = \{f(u+k) - f(u)\} - f'(u) \cdot k , & f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k)$$

$$f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k)$$

$$(\text{Let } k = g(x+h) - g(x) , & g(x+h) = g(x) + k \end{cases}$$



$$\varepsilon_1(h) = \left\{ \begin{array}{l} \frac{g(x+h) - g(x)}{h} - g'(x) & , \quad h \neq 0 \\ 0 & , \quad h = 0 \end{array} \right.$$

$$\varepsilon_1 \text{ is continuous at } h = 0 \text{ (}\because\text{ } g \text{ is differentiable at } x \text{)}$$

$$h \cdot \varepsilon_1(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h)$$

$$\varepsilon_2(h) = \left\{ \begin{array}{l} \frac{f(u+k) - f(u)}{h} - f'(u) & , \quad k \neq 0 \\ 0 & , \quad k = 0 \end{array} \right.$$

$$\varepsilon_2 \text{ is continuous at } k = 0 \text{ (}\because\text{ } f \text{ is differentiable at } u \text{)}$$

$$k \cdot \varepsilon_2(k) = \{f(u+k) - f(u)\} - f'(u) \cdot k \text{ , } f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k)$$

$$f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k)$$

$$\text{(Let } k = g(x+h) - g(x) \text{ , } g(x+h) = g(x) + k = u + k)$$



$$\varepsilon_1(h) = \left\{ \begin{array}{l} \displaystyle \frac{g(x+h) - g(x)}{h} - g'(x) & , \quad h \neq 0 \\ 0 & , \quad h = 0 \end{array} \right.$$

$$\varepsilon_1 \text{ is continuous at } h = 0 \text{ (\because g$ is differentiable at x)}$$

$$h \cdot \varepsilon_1(h) = \left\{ g(x+h) - g(x) \right\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h)$$

$$\varepsilon_2(h) = \left\{ \begin{array}{l} \displaystyle \frac{f(u+k) - f(u)}{h} - f'(u) & , \quad k \neq 0 \\ 0 & , \quad k = 0 \end{array} \right.$$

$$\varepsilon_2 \text{ is continuous at } k = 0 \text{ (\because f$ is differentiable at u)}$$

$$k \cdot \varepsilon_2(k) = \left\{ f(u+k) - f(u) \right\} - f'(u) \cdot k \text{ , } f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k)$$

$$f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k) \text{ (Let $k = g(x+h) - g(x)$) , } g(x+h) = g(x) + k = u + k)$$



$$\varepsilon_1(h) = \begin{cases} \frac{g(x+h) - g(x)}{h} - g'(x) &, & h \neq 0 \\ 0 &, & h = 0 \end{cases}$$

$$\varepsilon_1 \text{ is continuous at } h = 0 \text{ (\because g$ is differentiable at x)}$$

$$h \cdot \varepsilon_1(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h) \}$$

$$\varepsilon_2(h) = \begin{cases} \frac{f(u+k) - f(u)}{h} - f'(u) &, & k \neq 0 \\ 0 &, & k = 0 \end{cases}$$

$$\varepsilon_2 \text{ is continuous at } k = 0 \text{ (\because f$ is differentiable at u)}$$

$$k \cdot \varepsilon_2(k) = \{f(u+k) - f(u)\} - f'(u) \cdot k \text{ , } f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k) \}$$

$$f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k)$$

$$\text{(Let $k = g(x+h) - g(x)$) } \text{(Let $k = g(x+h) - g(x)$) } \{g(x+h) - g(x)\}$$



$$\begin{split} \varepsilon_1(h) &= \left\{ \begin{array}{l} \frac{g(x+h) - g(x)}{h} - g'(x) & , \quad h \neq 0 \\ 0 & , \quad h = 0 \end{array} \right. \\ & \left. \begin{array}{l} \varepsilon_1 \text{ is continuous at } h = 0 \text{ (\because g is differentiable at x)} \\ 0 & , \quad h = 0 \end{array} \right. \\ & \left. \begin{array}{l} h \cdot \varepsilon_1(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h) \end{array} \right. \\ & \left. \begin{array}{l} \varepsilon_2(h) = \left\{ \begin{array}{l} \frac{f(u+k) - f(u)}{h} - f'(u) & , \quad k \neq 0 \\ 0 & , \quad k = 0 \end{array} \right. \\ & \left. \begin{array}{l} \varepsilon_2 \text{ is continuous at } k = 0 \text{ (\because f is differentiable at u)} \\ & \left. \begin{array}{l} k \cdot \varepsilon_2(k) = \{f(u+k) - f(u)\} - f'(u) \cdot k \text{ , } f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k) \\ & \left. \begin{array}{l} \left(\text{Let } k = g(x+h) - g(x) \text{ , } g(x+h) = g(x) + k = u + k \right) \\ f(g(x+h)) - f(g(x)) & = \quad \{f'(u) + \varepsilon_2(g(x+h) - g(x))\} \{g(x+h) - g(x)\} \\ & = \quad \{f'(u) + \varepsilon_2(g(x+h) - g(x))\} \{g'(x) + \varepsilon_1(h)\} \cdot h \end{array} \right. \end{split}$$



Proof.

$$\varepsilon_1(h) = \begin{cases} \frac{g(x+h) - g(x)}{h} - g'(x) &, & h \neq 0 \\ 0 &, & h = 0 \end{cases}$$

$$\varepsilon_1 \text{ is continuous at } h = 0 \text{ ($\cdot \cdot : g is differentiable at x)}$$

$$h \cdot \varepsilon_1(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h)$$

$$\varepsilon_2(h) = \begin{cases} \frac{f(u+k) - f(u)}{h} - f'(u) &, & k \neq 0 \\ 0 & \varepsilon_2 \text{ is continuous at } k = 0 \text{ ($\cdot \cdot : f is differentiable at u)} \end{cases}$$

$$k \cdot \varepsilon_2(h) = \{f(u+k) - f(u)\} - f'(u) \cdot k \text{ , } f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k)$$

$$f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k)$$

$$(\text{Let } k = g(x+h) - g(x) \text{ , } g(x+h) = g(x) + k = u + k)$$

$$f(g(x+h)) - f(g(x)) = \{f'(u) + \varepsilon_2(g(x+h) - g(x))\} \{g(x+h) - g(x)\}$$

$$= \{f'(u) + \varepsilon_2(g(x+h) - g(x))\} \{g'(x) + \varepsilon_1(h)\} \cdot h$$

F'(x)



$$\varepsilon_1(h) = \begin{cases} \frac{g(x+h) - g(x)}{h} - g'(x) &, & h \neq 0 \\ 0 &, & h = 0 \end{cases}$$

$$h \cdot \varepsilon_1(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h , & g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_1(h) \end{cases}$$

$$\varepsilon_2(h) = \begin{cases} \frac{f(u+k) - f(u)}{h} - f'(u) &, & k \neq 0 \\ 0 &, & k = 0 \end{cases}$$

$$\varepsilon_2 \text{ is continuous at } k = 0 \text{ ($:$f$ is differentiable at } u\text{)}$$

$$k \cdot \varepsilon_2(k) = \{f(u+k) - f(u)\} - f'(u) \cdot k , & f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k) \end{cases}$$

$$f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_2(k)$$

$$(\text{Let } k = g(x+h) - g(x), & g(x+h) = g(x) + k = u + k)$$

$$f(g(x+h)) - f(g(x)) = \{f'(u) + \varepsilon_2(g(x+h) - g(x))\} \{g(x+h) - g(x)\}$$

$$= \{f'(u) + \varepsilon_2(g(x+h) - g(x))\} \{g'(x) + \varepsilon_1(h)\} \cdot h$$

$$F'(x) = \lim_{k \to 0} \frac{f(g(x+h)) - f(g(x))}{k}$$



$$\varepsilon_{1}(h) = \begin{cases} \frac{g(x+h) - g(x)}{h} - g'(x) &, h \neq 0 \\ 0 &, h = 0 \end{cases}$$

$$h \cdot \varepsilon_{1}(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h , g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_{1}(h) \end{cases}$$

$$\varepsilon_{2}(h) = \begin{cases} \frac{f(u+k) - f(u)}{h} - f'(u) &, k \neq 0 \\ 0 &, k = 0 \end{cases}$$

$$\varepsilon_{2} \text{ is continuous at } k = 0 \text{ (\because f$ is differentiable at u)}$$

$$k \cdot \varepsilon_{2}(k) = \{f(u+k) - f(u)\} - f'(u) \cdot k , f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_{2}(k) \}$$

$$f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_{2}(k) \text{ (Let } k = g(x+h) - g(x), g(x+h) = g(x) + k = u + k)}$$

$$f(g(x+h)) - f(g(x)) = \{f'(u) + \varepsilon_{2}(g(x+h) - g(x))\} \{g(x+h) - g(x)\} \cdot h \}$$

$$F'(x) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h \to 0} \{f'(u) + \varepsilon_{2}(g(x+h) - g(x))\} \{g'(x) + \varepsilon_{1}(h)\}$$



$$\varepsilon_{1}(h) = \begin{cases} \frac{g(x+h) - g(x)}{h} - g'(x) &, & h \neq 0 \\ 0 &, & h = 0 \end{cases}$$

$$\varepsilon_{1} \text{ is continuous at } h = 0 \text{ (}\because g \text{ is differentiable at } x \text{)}$$

$$h \cdot \varepsilon_{1}(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_{1}(h) \}$$

$$\varepsilon_{2}(h) = \begin{cases} \frac{f(u+k) - f(u)}{h} - f'(u) &, & k \neq 0 \\ 0 &, & k = 0 \end{cases}$$

$$\varepsilon_{2} \text{ is continuous at } k = 0 \text{ (}\because f \text{ is differentiable at } u \text{)}$$

$$k \cdot \varepsilon_{2}(k) = \{f(u+k) - f(u)\} - f'(u) \cdot k \text{ , } f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_{2}(k) \}$$

$$f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_{2}(k) \text{ (Let } k = g(x+h) - g(x) \text{ , } g(x+h) = g(x) + k = u + k) \}$$

$$f(g(x+h)) - f(g(x)) = \{f'(u) + \varepsilon_{2}(g(x+h) - g(x))\} \{g(x+h) - g(x)\} \cdot h \}$$

$$= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h \to 0} \{f'(u) + \varepsilon_{2}(g(x+h) - g(x))\} \{g'(x) + \varepsilon_{1}(h)\}$$



$$\varepsilon_{1}(h) = \begin{cases} \frac{g(x+h) - g(x)}{h} - g'(x) &, & h \neq 0 \\ 0 &, & h = 0 \end{cases}$$

$$\varepsilon_{1} \text{ is continuous at } h = 0 \text{ (}\because g \text{ is differentiable at } x \text{)}$$

$$h \cdot \varepsilon_{1}(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_{1}(h) \}$$

$$\varepsilon_{2}(h) = \begin{cases} \frac{f(u+k) - f(u)}{h} - f'(u) &, & k \neq 0 \\ 0 &, & k = 0 \end{cases}$$

$$\varepsilon_{2} \text{ is continuous at } k = 0 \text{ (}\because f \text{ is differentiable at } u \text{)}$$

$$k \cdot \varepsilon_{2}(k) = \{f(u+k) - f(u)\} - f'(u) \cdot k \text{ , } f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_{2}(k) \}$$

$$f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_{2}(k) \text{ (Let } k = g(x+h) - g(x) \text{ , } g(x+h) = g(x) + k = u + k) \}$$

$$f(g(x+h)) - f(g(x)) = \{f'(u) + \varepsilon_{2}(g(x+h) - g(x))\} \{g(x+h) - g(x)\} \text{ } f'(u) + \varepsilon_{2}(g(x+h) - g(x))\} \{g'(x) + \varepsilon_{1}(h)\} \cdot h \}$$

$$F'(x) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} \text{ } h$$

$$= \lim_{h \to 0} \{f'(u) + \varepsilon_{2}(g(x+h) - g(x))\} \{g'(x) + \varepsilon_{1}(h)\} \text{ } f'(u) + \varepsilon_{2}(g(x+h) - g(x))\} \{g'(x) + \varepsilon_{1}(h)\} \text{ } f'(u) + \varepsilon_{2}(g(x+h) - g(x))\} \{g'(x) + \varepsilon_{1}(h)\} \}$$



$$\varepsilon_{1}(h) = \begin{cases} \frac{g(x+h) - g(x)}{h} - g'(x) &, & h \neq 0 \\ 0 &, & h = 0 \end{cases}$$

$$\varepsilon_{1} \text{ is continuous at } h = 0 \text{ (}\because g \text{ is differentiable at } x \text{)}$$

$$h \cdot \varepsilon_{1}(h) = \{g(x+h) - g(x)\} - g'(x) \cdot h \text{ , } g(x+h) - g(x) = g'(x) \cdot h + h \cdot \varepsilon_{1}(h) \}$$

$$\varepsilon_{2}(h) = \begin{cases} \frac{f(u+k) - f(u)}{h} - f'(u) &, & k \neq 0 \\ 0 &, & k = 0 \end{cases}$$

$$\varepsilon_{2} \text{ is continuous at } k = 0 \text{ (}\because f \text{ is differentiable at } u \text{)}$$

$$k \cdot \varepsilon_{2}(k) = \{f(u+k) - f(u)\} - f'(u) \cdot k \text{ , } f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_{2}(k) \}$$

$$f(u+k) - f(u) = f'(u) \cdot k + k \cdot \varepsilon_{2}(k) \text{ (Let } k = g(x+h) - g(x) \text{ , } g(x+h) = g(x) + k = u + k) \}$$

$$f(g(x+h)) - f(g(x)) = \{f'(u) + \varepsilon_{2}(g(x+h) - g(x))\} \{g(x+h) - g(x)\} \text{ } f'(u) + \varepsilon_{2}(g(x+h) - g(x))\} \{g'(x) + \varepsilon_{1}(h)\} \cdot h \}$$

$$F'(x) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} \text{ } h$$

$$= \lim_{h \to 0} \{f'(u) + \varepsilon_{2}(g(x+h) - g(x))\} \{g'(x) + \varepsilon_{1}(h)\} \text{ } f'(u) + \varepsilon_{2}(g(x+h) - g(x))\} \{g'(x) + \varepsilon_{1}(h)\} \text{ } f'(u) + \varepsilon_{2}(g(x+h) - g(x))\} \{g'(x) + \varepsilon_{1}(h)\} \}$$



END