곱의 극한은 극한의 곱이다. (The limit of a product is the product of the limits.)





▶ Start ▶ End

$$\lim_{x \to a} f(x) = L$$

▶ Start ▶ End

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

▶ Start ▶ End

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \}$$

▶ Start ▶ End

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

$$\epsilon > 0$$

$$|f(x)g(x) - LM|$$

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

$$\epsilon > 0$$

$$|f(x)g(x) - LM| = |f(x)g(x) - f(x)M + f(x)M - LM|$$

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

$$\epsilon > 0$$

$$\begin{array}{lcl} |f(x)g(x) - LM| & = & |f(x)g(x) - f(x)M + f(x)M - LM| \\ & \leq & |f(x)g(x) - f(x)M| + |f(x)M - LM| \end{array}$$

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

$$\epsilon > 0$$

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

$$\begin{array}{lcl} |f(x)g(x)-LM| & = & |f(x)g(x)-f(x)M+f(x)M-LM| \\ \leq & |f(x)g(x)-f(x)M|+|f(x)M-LM| \text{ (\cdot` The Triangle Inequality)} \\ = & |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \end{array}$$

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

 $\epsilon > 0$

$$\begin{array}{lcl} |f(x)g(x)-LM| & = & |f(x)g(x)-f(x)M+f(x)M-LM| \\ \leq & |f(x)g(x)-f(x)M|+|f(x)M-LM| \; (\because \text{ The Triangle Inequality}) \\ = & |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \end{array}$$

 $\exists \delta_1 > 0$

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

 $\epsilon > 0$

$$\begin{array}{lcl} |f(x)g(x)-LM| & = & |f(x)g(x)-f(x)M+f(x)M-LM| \\ \leq & |f(x)g(x)-f(x)M|+|f(x)M-LM| \; (\because \text{ The Triangle Inequality}) \\ = & |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \end{array}$$

 $\exists \delta_1 > 0 \text{ s.t.}$

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

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Proof.

$$\begin{array}{lcl} |f(x)g(x)-LM| & = & |f(x)g(x)-f(x)M+f(x)M-LM| \\ \leq & |f(x)g(x)-f(x)M|+|f(x)M-LM| \; (\because \text{ The Triangle Inequality}) \\ = & |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \end{array}$$

$$\exists \delta_1 > 0 ext{ s.t. } 0 < |x-a| < \delta_1$$

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

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Proof.

$$\begin{array}{lcl} |f(x)g(x)-LM| & = & |f(x)g(x)-f(x)M+f(x)M-LM| \\ \leq & |f(x)g(x)-f(x)M|+|f(x)M-LM| \ (\because \ \text{The Triangle Inequality}) \\ = & |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \end{array}$$

$$\exists \delta_1 > 0 \text{ s.t. } 0 < |x-a| < \delta_1 \Rightarrow$$

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

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Proof.

$$\begin{array}{lcl} |f(x)g(x)-LM| & = & |f(x)g(x)-f(x)M+f(x)M-LM| \\ & \leq & |f(x)g(x)-f(x)M|+|f(x)M-LM| \text{ (`.` The Triangle Inequality)} \\ & = & |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \\ \exists \delta_1>0 \text{ s.t. } 0<|x-a|<\delta_1\Rightarrow|f(x)-L|<1 \end{array}$$

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

$$\begin{array}{lcl} |f(x)g(x)-LM| &=& |f(x)g(x)-f(x)M+f(x)M-LM| \\ &\leq& |f(x)g(x)-f(x)M|+|f(x)M-LM| \ (\because \text{ The Triangle Inequality}) \\ &=& |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \\ \\ \exists \delta_1>0 \text{ s.t. } 0<|x-a|<\delta_1\Rightarrow|f(x)-L|<1 \ (\because \lim_{x\to a}f(x)=L) \end{array}$$

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

$$\begin{array}{lll} |f(x)g(x)-LM| &=& |f(x)g(x)-f(x)M+f(x)M-LM|\\ &\leq& |f(x)g(x)-f(x)M|+|f(x)M-LM| \ (\because \ \mbox{The Triangle Inequality})\\ &=& |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \end{array}$$

$$\exists \delta_1>0 \ \mbox{s.t.} \ 0<|x-a|<\delta_1\Rightarrow|f(x)-L|<1 \ (\because \ \lim_{x\to a}f(x)=L) \ (\because \ \mbox{The Triangle Inequality})$$

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

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Proof.

$$\begin{array}{lcl} |f(x)g(x)-LM| & = & |f(x)g(x)-f(x)M+f(x)M-LM| \\ & \leq & |f(x)g(x)-f(x)M|+|f(x)M-LM| \ (\because \ \text{The Triangle Inequality}) \\ & = & |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \\ \\ \exists \delta_1>0 \text{ s.t. } 0<|x-a|<\delta_1\Rightarrow|f(x)-L|<1 \ (\because \lim_{x\to a}f(x)=L) \ (\because \ \text{The Triangle Inequality}) \\ |f(x)| \end{array}$$

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

$$|f(x)g(x)-LM| = |f(x)g(x)-f(x)M+f(x)M-LM| \le |f(x)g(x)-f(x)M|+|f(x)M-LM| \text{ (\cdot. The Triangle Inequality)}$$

$$= |f(x)|\cdot |g(x)-M|+|Mf(x)-ML| \text{ (\cdot. The Triangle Inequality)}$$

$$\exists \delta_1>0 \text{ s.t. } 0<|x-a|<\delta_1\Rightarrow |f(x)-L|<1 \text{ (\cdot. $\limbda_{x\to a}f(x)=L$) (\cdot. The Triangle Inequality)}$$

$$|f(x)|=|f(x)-L+L| \text{ (\cdot. \cdot. $\limbda_{x\to a}f(x)=L$) (\cdot. The Triangle Inequality)}$$

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

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Proof.

$$|f(x)g(x) - LM| = |f(x)g(x) - f(x)M + f(x)M - LM|$$

$$\leq |f(x)g(x) - f(x)M| + |f(x)M - LM| \text{ (`.' The Triangle Inequality)}$$

$$= |f(x)| \cdot |g(x) - M| + |Mf(x) - ML|$$

$$\exists \delta_1 > 0 \text{ s.t. } 0 < |x - a| < \delta_1 \Rightarrow |f(x) - L| < 1 \text{ (`.' } \lim_{x \to a} f(x) = L) \text{ (`.' The Triangle Inequality)}$$

$$|f(x)| = |f(x) - L + L| < |f(x) - L| + |L|$$

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

$$|f(x)g(x)-LM| = |f(x)g(x)-f(x)M+f(x)M-LM| \\ \leq |f(x)g(x)-f(x)M|+|f(x)M-LM| \text{ (\cdot,\cdot The Triangle Inequality)} \\ = |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \\ \exists \delta_1>0 \text{ s.t. } 0<|x-a|<\delta_1\Rightarrow|f(x)-L|<1 \text{ (\cdot,\cdot $\lim_{x\to a}f(x)=L$) (\cdot,\cdot The Triangle Inequality)} \\ |f(x)|=|f(x)-L+L|<|f(x)-L|+|L|<1+|L|$$

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

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Proof.

$$\begin{array}{lcl} |f(x)g(x)-LM| & = & |f(x)g(x)-f(x)M+f(x)M-LM| \\ \leq & |f(x)g(x)-f(x)M|+|f(x)M-LM| \text{ (\cdot} \cdot \text{ The Triangle Inequality)} \\ & = & |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \\ \exists \delta_1>0 \text{ s.t. } 0<|x-a|<\delta_1\Rightarrow|f(x)-L|<1 \text{ (\cdot} \cdot \lim_{x\to a}f(x)=L) \text{ (\cdot} \cdot \text{ The Triangle Inequality)} \end{array}$$

$$|f(x)| = |f(x) - L + L| \le |f(x) - L| + |L| < 1 + |L|$$

$$\exists \delta_2\,>\,0$$
 s.t.

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

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Proof.

$$\begin{array}{lcl} |f(x)g(x)-LM| & = & |f(x)g(x)-f(x)M+f(x)M-LM| \\ \leq & |f(x)g(x)-f(x)M|+|f(x)M-LM| \text{ (\cdot} \cdot \text{ The Triangle Inequality)} \\ & = & |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \\ \exists \delta_1>0 \text{ s.t. } 0<|x-a|<\delta_1\Rightarrow|f(x)-L|<1 \text{ (\cdot} \cdot \lim_{x\to a}f(x)=L) \text{ (\cdot} \cdot \text{ The Triangle Inequality)} \end{array}$$

$$|f(x)| = |f(x) - L + L| \le |f(x) - L| + |L| < 1 + |L|$$

$$\exists \delta_2 > 0 ext{ s.t. } 0 < |x-a| < \delta_2$$

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

$$|f(x)g(x)-LM| = |f(x)g(x)-f(x)M+f(x)M-LM| \\ \leq |f(x)g(x)-f(x)M|+|f(x)M-LM| \text{ (\cdot,\cdot The Triangle Inequality)} \\ = |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \\ \exists \delta_1>0 \text{ s.t. } 0<|x-a|<\delta_1\Rightarrow|f(x)-L|<1 \text{ (\cdot,\cdot $\lim_{x\to a}f(x)=L$) (\cdot,\cdot The Triangle Inequality)} \\ |f(x)|=|f(x)-L+L|\leq|f(x)-L|+|L|<1+|L| \\ \exists \delta_2>0 \text{ s.t. } 0<|x-a|<\delta_2\Rightarrow$$

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

$$\begin{array}{lll} |f(x)g(x)-LM| &=& |f(x)g(x)-f(x)M+f(x)M-LM|\\ &\leq& |f(x)g(x)-f(x)M|+|f(x)M-LM| \ (\because \ \mbox{The Triangle Inequality})\\ &=& |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \ \\ \exists \delta_1>0 \ \mbox{s.t.} \ 0<|x-a|<\delta_1\Rightarrow|f(x)-L|<1 \ (\because \ \lim_{x\to a}f(x)=L) \ (\because \ \mbox{The Triangle Inequality})\\ |f(x)|&=|f(x)-L+L|\leq|f(x)-L|+|L|<1+|L|\\ \exists \delta_2>0 \ \mbox{s.t.} \ 0<|x-a|<\delta_2\Rightarrow|g(x)-M|<\frac{\epsilon}{2(1+|L|)} \end{array}$$

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

$$\begin{array}{lll} |f(x)g(x)-LM| &=& |f(x)g(x)-f(x)M+f(x)M-LM|\\ &\leq& |f(x)g(x)-f(x)M|+|f(x)M-LM| \ (\because \ \mbox{The Triangle Inequality})\\ &=& |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \ \\ \exists \delta_1>0 \ \mbox{s.t.} \ 0<|x-a|<\delta_1\Rightarrow|f(x)-L|<1 \ (\because \lim_{x\to a}f(x)=L) \ (\because \ \mbox{The Triangle Inequality})\\ |f(x)|=|f(x)-L+L|\leq|f(x)-L|+|L|<1+|L|\\ \exists \delta_2>0 \ \mbox{s.t.} \ 0<|x-a|<\delta_2\Rightarrow|g(x)-M|<\frac{\epsilon}{2(1+|L|)} \ (\because \lim_{x\to a}g(x)=M) \end{array}$$

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

$$\begin{split} |f(x)g(x)-LM| &= |f(x)g(x)-f(x)M+f(x)M-LM| \\ &\leq |f(x)g(x)-f(x)M|+|f(x)M-LM| \ (\because \text{ The Triangle Inequality}) \\ &= |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \\ \exists \delta_1>0 \text{ s.t. } 0<|x-a|<\delta_1\Rightarrow|f(x)-L|<1 \ (\because \lim_{x\to a}f(x)=L) \ (\because \text{ The Triangle Inequality}) \\ |f(x)|&=|f(x)-L+L|\leq|f(x)-L|+|L|<1+|L| \\ \exists \delta_2>0 \text{ s.t. } 0<|x-a|<\delta_2\Rightarrow|g(x)-M|<\frac{\epsilon}{2(1+|L|)} \ (\because \lim_{x\to a}g(x)=M) \end{split}$$

$$\exists \delta_3 > 0$$

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

$$|f(x)g(x) - LM| = |f(x)g(x) - f(x)M + f(x)M - LM|$$

$$\leq |f(x)g(x) - f(x)M| + |f(x)M - LM| \text{ (\cdot} \cdot \text{ The Triangle Inequality})$$

$$= |f(x)| \cdot |g(x) - M| + |Mf(x) - ML| \text{ (\cdot} \cdot \text{ The Triangle Inequality})$$

$$|\delta_1 > 0 \text{ s.t. } 0 < |x - a| < \delta_1 \Rightarrow |f(x) - L| < 1 \text{ (\cdot} \cdot \text{ Iim } f(x) = L) \text{ (\cdot} \cdot \text{ The Triangle Inequality})$$

$$\exists \delta_1 > 0 \text{ s.t. } 0 < |x - a| < \delta_1 \Rightarrow |f(x) - L| < 1 \ (\because \lim_{x \to a} f(x) = L) \ (\because \text{ The Triangle Inequality})$$

$$|f(x)| = |f(x) - L + L| \le |f(x) - L| + |L| < 1 + |L|$$

$$\exists \delta_2 > 0 \text{ s.t. } 0 < |x - a| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2(1 + |L|)} \left(\because \lim_{x \to a} g(x) = M \right)$$

$$\exists \delta_3 \, > \, 0 \; \text{s.t.}$$

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

$$\begin{array}{lcl} |f(x)g(x)-LM| & = & |f(x)g(x)-f(x)M+f(x)M-LM| \\ \leq & |f(x)g(x)-f(x)M|+|f(x)M-LM| \; (\because \text{ The Triangle Inequality}) \\ = & |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \end{array}$$

$$\exists \delta_1>0 \text{ s.t. } 0<|x-a|<\delta_1\Rightarrow |f(x)-L|<1 \ (\because \lim_{x\to a}f(x)=L) \ (\because \text{ The Triangle Inequality})$$

$$|f(x)| = |f(x) - L + L| \le |f(x) - L| + |L| < 1 + |L|$$

$$\exists \delta_2 > 0 \text{ s.t. } 0 < |x - a| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2(1 + |L|)} \ (\because \lim_{x \to a} g(x) = M)$$

$$\exists \delta_3 > 0 \text{ s.t. } 0 < |x-a| < \delta_3$$

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

$$\exists \delta_1>0$$
 s.t. $0<|x-a|<\delta_1\Rightarrow |f(x)-L|<1$ ($\because\lim_{x\to a}f(x)=L$) (\because The Triangle Inequality) $|f(x)|=|f(x)-L+L|\leq |f(x)-L|+|L|<1+|L|$

$$\exists \delta_2 > 0 \text{ s.t. } 0 < |x - a| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2(1 + |L|)} \left(\because \lim_{x \to a} g(x) = M \right)$$

$$\exists \delta_3 > 0 \text{ s.t. } 0 < |x-a| < \delta_3 \Rightarrow$$

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

▶ Start ▶ End

Theorem

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$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

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$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

$$\begin{split} |f(x)g(x)-LM| &= |f(x)g(x)-f(x)M+f(x)M-LM| \\ &\leq |f(x)g(x)-f(x)M|+|f(x)M-LM| \ (\because \text{ The Triangle Inequality}) \\ &= |f(x)|\cdot |g(x)-M|+|Mf(x)-ML| \\ \exists \delta_1>0 \text{ s.t. } 0<|x-a|<\delta_1\Rightarrow |f(x)-L|<1 \ (\because \lim_{x\to a}f(x)=L) \ (\because \text{ The Triangle Inequality}) \\ |f(x)|&=|f(x)-L+L|\leq |f(x)-L|+|L|<1+|L| \\ \exists \delta_2>0 \text{ s.t. } 0<|x-a|<\delta_2\Rightarrow |g(x)-M|<\frac{\epsilon}{2(1+|L|)} \ (\because \lim_{x\to a}g(x)=M) \\ \exists \delta_3>0 \text{ s.t. } 0<|x-a|<\delta_3\Rightarrow |Mf(x)-ML|<\frac{\epsilon}{2} \ (\because \lim_{x\to a}Mf(x)=ML) \\ \delta \end{split}$$

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

$$\begin{split} |f(x)g(x)-LM| &= |f(x)g(x)-f(x)M+f(x)M-LM| \\ &\leq |f(x)g(x)-f(x)M|+|f(x)M-LM| \text{ (\cdot. The Triangle Inequality)} \\ &= |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \text{ (\cdot. The Triangle Inequality)} \\ \exists \delta_1>0 \text{ s.t. } 0<|x-a|<\delta_1\Rightarrow|f(x)-L|<1 \text{ (\cdot. } \lim_{x\to a}f(x)=L) \text{ (\cdot. The Triangle Inequality)} \\ |f(x)|&=|f(x)-L+L|\leq|f(x)-L|+|L|<1+|L| \\ \exists \delta_2>0 \text{ s.t. } 0<|x-a|<\delta_2\Rightarrow|g(x)-M|<\frac{\epsilon}{2(1+|L|)} \text{ (\cdot. } \lim_{x\to a}g(x)=M) \\ \exists \delta_3>0 \text{ s.t. } 0<|x-a|<\delta_3\Rightarrow|Mf(x)-ML|<\frac{\epsilon}{2} \text{ (\cdot. } \lim_{x\to a}Mf(x)=ML) \\ \delta=\min(\delta_1,\delta_2,\delta_3) \end{split}$$

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

$$|f(x)g(x)-LM| = |f(x)g(x)-f(x)M+f(x)M-LM| \leq |f(x)g(x)-f(x)M|+|f(x)M-LM| (\because \text{ The Triangle Inequality}) \\ = |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| (\because \text{ The Triangle Inequality}) \\ = |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| (\because \lim_{x\to a}f(x)=L) (\because \text{ The Triangle Inequality}) \\ |f(x)|=|f(x)-L+L|\leq |f(x)-L|+|L|<1+|L| \\ \exists \delta_2>0 \text{ s.t. } 0<|x-a|<\delta_2\Rightarrow|g(x)-M|<\frac{\epsilon}{2(1+|L|)} (\because \lim_{x\to a}Mf(x)=ML) \\ \exists \delta_3>0 \text{ s.t. } 0<|x-a|<\delta_3\Rightarrow|Mf(x)-ML|<\frac{\epsilon}{2} (\because \lim_{x\to a}Mf(x)=ML) \\ \delta=\min(\delta_1,\delta_2,\delta_3) \\ 0<|x-a|<\delta$$

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

$$\begin{split} |f(x)g(x)-LM| &= |f(x)g(x)-f(x)M+f(x)M-LM| \\ &\leq |f(x)g(x)-f(x)M|+|f(x)M-LM| \ (\because \text{ The Triangle Inequality}) \\ &= |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \ (\because \text{ The Triangle Inequality}) \\ \exists \delta_1>0 \text{ s.t. } 0<|x-a|<\delta_1\Rightarrow|f(x)-L|<1 \ (\because \lim_{x\to a}f(x)=L) \ (\because \text{ The Triangle Inequality}) \\ |f(x)|&=|f(x)-L+L|\leq|f(x)-L|+|L|<1+|L| \\ \exists \delta_2>0 \text{ s.t. } 0<|x-a|<\delta_2\Rightarrow|g(x)-M|<\frac{\epsilon}{2(1+|L|)} \ (\because \lim_{x\to a}g(x)=M) \\ \exists \delta_3>0 \text{ s.t. } 0<|x-a|<\delta_3\Rightarrow|Mf(x)-ML|<\frac{\epsilon}{2} \ (\because \lim_{x\to a}Mf(x)=ML) \\ \delta=\min(\delta_1,\delta_2,\delta_3) \\ 0<|x-a|<\delta\Rightarrow \end{split}$$

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

$$|f(x)g(x)-LM| = |f(x)g(x)-f(x)M+f(x)M-LM| \\ \leq |f(x)g(x)-f(x)M|+|f(x)M-LM| \\ \leq |f(x)g(x)-f(x)M|+|f(x)M-LM| \\ (\because \text{ The Triangle Inequality}) \\ = |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \\ \exists \delta_1>0 \text{ s.t. } 0<|x-a|<\delta_1\Rightarrow|f(x)-L|<1 \\ (\because \lim_{x\to a}f(x)=L) \\ (\because \text{ The Triangle Inequality}) \\ |f(x)|=|f(x)-L+L|\leq|f(x)-L|+|L|<1+|L| \\ \exists \delta_2>0 \text{ s.t. } 0<|x-a|<\delta_2\Rightarrow|g(x)-M|<\frac{\epsilon}{2(1+|L|)} \\ (\because \lim_{x\to a}g(x)=M) \\ \exists \delta_3>0 \text{ s.t. } 0<|x-a|<\delta_3\Rightarrow|Mf(x)-ML|<\frac{\epsilon}{2} \\ (\because \lim_{x\to a}Mf(x)=ML) \\ \delta=\min(\delta_1,\delta_2,\delta_3) \\ 0<|x-a|<\delta\Rightarrow|f(x)g(x)-LM|<\epsilon \\ \end{cases}$$

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

$$\begin{split} |f(x)g(x)-LM| &= |f(x)g(x)-f(x)M+f(x)M-LM| \\ &\leq |f(x)g(x)-f(x)M|+|f(x)M-LM| \ (\because \text{ The Triangle Inequality}) \\ &= |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \ (\because \text{ The Triangle Inequality}) \\ \exists \delta_1>0 \text{ s.t. } 0<|x-a|<\delta_1\Rightarrow|f(x)-L|<1 \ (\because \lim_{x\to a}f(x)=L) \ (\because \text{ The Triangle Inequality}) \\ |f(x)|&=|f(x)-L+L|\leq|f(x)-L|+|L|<1+|L| \ \exists \delta_2>0 \text{ s.t. } 0<|x-a|<\delta_2\Rightarrow|g(x)-M|<\frac{\epsilon}{2(1+|L|)} \ (\because \lim_{x\to a}g(x)=M) \ \exists \delta_3>0 \text{ s.t. } 0<|x-a|<\delta_3\Rightarrow|Mf(x)-ML|<\frac{\epsilon}{2} \ (\because \lim_{x\to a}Mf(x)=ML) \ \delta=\min(\delta_1,\delta_2,\delta_3) \ 0<|x-a|<\delta\Rightarrow|f(x)g(x)-LM|<\epsilon \end{split}$$

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

$$|f(x)g(x) - LM| = |f(x)g(x) - f(x)M + f(x)M - LM| \le |f(x)g(x) - f(x)M| + |f(x)M - LM| (\because \text{ The Triangle Inequality})$$

$$= |f(x)| \cdot |g(x) - M| + |Mf(x) - ML| (\because \text{ The Triangle Inequality})$$

$$\exists \delta_1 > 0 \text{ s.t. } 0 < |x - a| < \delta_1 \Rightarrow |f(x) - L| < 1 (\because \lim_{x \to a} f(x) = L) (\because \text{ The Triangle Inequality})$$

$$|f(x)| = |f(x) - L + L| \le |f(x) - L| + |L| < 1 + |L|$$

$$\exists \delta_2 > 0 \text{ s.t. } 0 < |x - a| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2(1 + |L|)} (\because \lim_{x \to a} g(x) = M)$$

$$\exists \delta_3 > 0 \text{ s.t. } 0 < |x - a| < \delta_3 \Rightarrow |Mf(x) - ML| < \frac{\epsilon}{2} (\because \lim_{x \to a} Mf(x) = ML)$$

$$\delta = \min(\delta_1, \delta_2, \delta_3)$$

$$0 < |x - a| < \delta \Rightarrow |f(x)g(x) - LM| < \epsilon$$

$$\therefore \forall \epsilon > 0$$

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

$$|f(x)g(x) - LM| = |f(x)g(x) - f(x)M + f(x)M - LM| \le |f(x)g(x) - f(x)M| + |f(x)M - LM| (\because \text{ The Triangle Inequality})$$

$$= |f(x)| \cdot |g(x) - M| + |Mf(x) - ML| (\because \text{ The Triangle Inequality})$$

$$\exists \delta_1 > 0 \text{ s.t. } 0 < |x - a| < \delta_1 \Rightarrow |f(x) - L| < 1 (\because \lim_{x \to a} f(x) = L) (\because \text{ The Triangle Inequality})$$

$$|f(x)| = |f(x) - L + L| \le |f(x) - L| + |L| < 1 + |L|$$

$$\exists \delta_2 > 0 \text{ s.t. } 0 < |x - a| < \delta_2 \Rightarrow |g(x) - M| < \frac{\epsilon}{2(1 + |L|)} (\because \lim_{x \to a} g(x) = M)$$

$$\exists \delta_3 > 0 \text{ s.t. } 0 < |x - a| < \delta_3 \Rightarrow |Mf(x) - ML| < \frac{\epsilon}{2} (\because \lim_{x \to a} Mf(x) = ML)$$

$$\delta = \min(\delta_1, \delta_2, \delta_3)$$

$$0 < |x - a| < \delta \Rightarrow |f(x)g(x) - LM| < \epsilon$$

$$\therefore \forall \epsilon > 0 . \exists \delta > 0$$

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

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Proof.

$$|f(x)g(x)-LM| = |f(x)g(x)-f(x)M+f(x)M-LM| \le |f(x)g(x)-f(x)M|+|f(x)M-LM| (\because \text{ The Triangle Inequality})$$

$$= |f(x)| \cdot |g(x)-M|+|Mf(x)-ML| (\because \text{ The Triangle Inequality})$$

$$\exists \delta_1>0 \text{ s.t. } 0<|x-a|<\delta_1\Rightarrow |f(x)-L|<1 (\because \lim_{x\to a}f(x)=L) (\because \text{ The Triangle Inequality})$$

$$|f(x)|=|f(x)-L+L|\leq |f(x)-L|+|L|<1+|L|$$

$$\exists \delta_2>0 \text{ s.t. } 0<|x-a|<\delta_2\Rightarrow |g(x)-M|<\frac{\epsilon}{2(1+|L|)} (\because \lim_{x\to a}g(x)=M)$$

$$\exists \delta_3>0 \text{ s.t. } 0<|x-a|<\delta_3\Rightarrow |Mf(x)-ML|<\frac{\epsilon}{2} (\because \lim_{x\to a}Mf(x)=ML)$$

$$\delta=\min(\delta_1,\delta_2,\delta_3)$$

$$0<|x-a|<\delta\Rightarrow |f(x)g(x)-LM|<\epsilon$$

$$\therefore \forall \epsilon>0. \ \exists \delta>0 \text{ s.t.}$$

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

 $\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

$$\begin{split} |f(x)g(x)-LM| &= |f(x)g(x)-f(x)M+f(x)M-LM| \\ &\leq |f(x)g(x)-f(x)M|+|f(x)M-LM| \text{ ($\cdot\cdot$ The Triangle Inequality)} \\ &= |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \text{ ($\cdot\cdot$ The Triangle Inequality)} \\ \exists \delta_1>0 \text{ s.t. } 0<|x-a|<\delta_1\Rightarrow|f(x)-L|<1 \text{ ($\cdot\cdot$ $$} \frac{1}{x\to a}f(x)=L \text{) ($\cdot\cdot$ The Triangle Inequality)} \\ |f(x)|=|f(x)-L+L|\leq|f(x)-L|+|L|<1+|L|\\ \exists \delta_2>0 \text{ s.t. } 0<|x-a|<\delta_2\Rightarrow|g(x)-M|<\frac{\epsilon}{2(1+|L|)} \text{ ($\cdot\cdot$ $$} \frac{1}{x\to a}Mf(x)=M \text{)} \\ \exists \delta_3>0 \text{ s.t. } 0<|x-a|<\delta_3\Rightarrow|Mf(x)-ML|<\frac{\epsilon}{2} \text{ ($\cdot\cdot$ $$} \frac{1}{x\to a}Mf(x)=M \text{)} \\ \delta=\min(\delta_1,\delta_2,\delta_3) \\ 0<|x-a|<\delta\Rightarrow|f(x)g(x)-LM|<\epsilon \end{split}$$

▶ Start ▶ End

Theorem

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$$|f(x)g(x)-LM| = |f(x)g(x)-f(x)M+f(x)M-LM| \leq |f(x)g(x)-f(x)M|+|f(x)M-LM| (\because \text{ The Triangle Inequality}) \\ = |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| (\because \text{ The Triangle Inequality}) \\ = |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| (\because \text{ Imp }f(x)=L) (\because \text{ The Triangle Inequality}) \\ |f(x)|=|f(x)-L+L|\leq|f(x)-L|+|L|<1+|L|\\ |\exists \delta_2>0 \text{ s.t. } 0<|x-a|<\delta_2\Rightarrow|g(x)-M|<\frac{\epsilon}{2(1+|L|)} (\because \lim_{x\to a}g(x)=M) \\ |\exists \delta_3>0 \text{ s.t. } 0<|x-a|<\delta_3\Rightarrow|Mf(x)-ML|<\frac{\epsilon}{2} (\because \lim_{x\to a}Mf(x)=ML) \\ \delta=\min(\delta_1,\delta_2,\delta_3) \\ 0<|x-a|<\delta\Rightarrow|f(x)g(x)-LM|<\epsilon \\ \because \forall \epsilon>0,\exists \delta>0 \text{ s.t. } 0<|x-a|<\delta\Rightarrow$$

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

 $\epsilon > 0$

$$\begin{split} |f(x)g(x)-LM| &= |f(x)g(x)-f(x)M+f(x)M-LM| \\ &\leq |f(x)g(x)-f(x)M|+|f(x)M-LM| \ (\because \text{ The Triangle Inequality}) \\ &= |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \ (\because \text{ The Triangle Inequality}) \\ \exists \delta_1>0 \text{ s.t. } 0<|x-a|<\delta_1\Rightarrow|f(x)-L|<1 \ (\because \lim_{x\to a}f(x)=L) \ (\because \text{ The Triangle Inequality}) \\ |f(x)|&=|f(x)-L+L|\leq|f(x)-L|+|L|<1+|L|\\ \exists \delta_2>0 \text{ s.t. } 0<|x-a|<\delta_2\Rightarrow|g(x)-M|<\frac{\epsilon}{2(1+|L|)} \ (\because \lim_{x\to a}g(x)=M) \\ \exists \delta_3>0 \text{ s.t. } 0<|x-a|<\delta_3\Rightarrow|Mf(x)-ML|<\frac{\epsilon}{2} \ (\because \lim_{x\to a}Mf(x)=ML) \\ \delta=\min(\delta_1,\delta_2,\delta_3) \\ 0<|x-a|<\delta\Rightarrow|f(x)g(x)-LM|<\epsilon \end{split}$$

 $\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } 0 < |x - a| < \delta \Rightarrow |f(x)g(x) - LM| < \epsilon$

▶ Start ▶ End

Theorem

$$\lim_{x \to a} f(x) = L, \lim_{x \to a} g(x) = M$$

$$\lim_{x \to a} \{ f(x)g(x) \} = LM$$

Proof.

$$\begin{split} |f(x)g(x)-LM| &= |f(x)g(x)-f(x)M+f(x)M-LM| \\ &\leq |f(x)g(x)-f(x)M|+|f(x)M-LM| \text{ (\cdot `The Triangle Inequality)} \\ &= |f(x)|\cdot|g(x)-M|+|Mf(x)-ML| \text{ (\cdot `The Triangle Inequality)} \\ \exists \delta_1>0 \text{ s.t. } 0<|x-a|<\delta_1\Rightarrow|f(x)-L|<1 \text{ (\cdot : $\lim_{x\to a}f(x)=L$) (\cdot ` The Triangle Inequality)} \\ |f(x)|&=|f(x)-L+L|\leq|f(x)-L|+|L|<1+|L|\\ \exists \delta_2>0 \text{ s.t. } 0<|x-a|<\delta_2\Rightarrow|g(x)-M|<\frac{2}{2(1+|L|)} \text{ (\cdot : $\lim_{x\to a}g(x)=M$)} \\ \exists \delta_3>0 \text{ s.t. } 0<|x-a|<\delta_3\Rightarrow|Mf(x)-ML|<\frac{\epsilon}{2} \text{ (\cdot : $\lim_{x\to a}Mf(x)=ML$)} \\ \delta=\min(\delta_1,\delta_2,\delta_3) \\ 0<|x-a|<\delta\Rightarrow|f(x)g(x)-LM|<\epsilon \\ \therefore \forall \epsilon>0, \exists \delta>0 \text{ s.t. } 0<|x-a|<\delta\Rightarrow|f(x)g(x)-LM|<\epsilon \end{split}$$

Github:

https://min7014.github.io/math20240104001.html

Click or paste URL into the URL search bar, and you can see a picture moving.