SGN – Assignment #1

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1 Periodic orbit

Exercise 1

Consider the 3D Earth–Moon Circular Restricted Three-Body Problem with $\mu = 0.012150$. Note that the CRTBP has an integral of motion, that is, the Jacobi constant

$$J(x, y, z, v_x, x_y, v_z) := 2\Omega(x, y, z) - v^2 = C$$

where $\Omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1}{2}\mu(1-\mu)$ and $v^2 = v_x^2 + v_y^2 + v_z^2$.

1) Find the coordinates of the five Lagrange points L_i in the rotating, adimensional reference frame with at least 10-digit accuracy and report their Jacobi constant C_i.

Solutions to the 3D CRTBP satisfy the symmetry

$$\mathcal{S} : (x, y, z, \dot{x}, \dot{y}, \dot{z}, t) \to (x, -y, z, -\dot{x}, \dot{y}, -\dot{z}, -t).$$

Thus, a trajectory that crosses perpendicularly the $y = 0$ plane twice is a periodic orbit.

2) Given the initial guess $\mathbf{x}_0 = (x_0, y_0, z_0, v_{x0}, v_{y0}, v_{z0})$, with

$$\begin{aligned} x_0 &= 1.068792441776 \\ y_0 &= 0 \\ z_0 &= 0.071093328515 \\ v_{x0} &= 0 \\ v_{y0} &= 0.319422926485 \\ v_{z0} &= 0 \end{aligned}$$

Find the periodic halo orbit having a Jacobi Constant $C = 3.09$; that is, develop the theoretical framework and implement a differential correction scheme that uses the STM, either approximated through finite differences **or** achieved by integrating the variational equation.
Hint: Consider working on $\varphi(\mathbf{x} + \Delta\mathbf{x}, t + \Delta t)$ and $J(\mathbf{x} + \Delta\mathbf{x})$ and then enforce perpendicular cross of $y = 0$ and Jacobi energy.

The periodic orbits in the CRTBP exist in families. These can be computed by 'continuing' the orbits along one coordinate or one parameter, e.g., the Jacobi energy C. The *numerical continuation* is an iterative process in which the desired variable is *gradually* varied, while the rest of the initial guess is taken from the solution of the previous iteration, thus aiding the convergence process.

3) By gradually decreasing C and using numerical continuation, compute the families of halo orbits until $C = 3.04$.

(8 points)

To prepare the ZIP file for the submission of the Assignment:

- Complete your answers on the Overleaf project you created.
- Red text indicates where answers are needed; be sure there is no red stuff in your report.

- In your answers, be concise: to the point.
- Download the PDF from the Main menu on Overleaf.
- Create a single .zip file containing both the report in PDF and the MATLAB files. The name shall be `lastname123456_Assign1.zip`.
- **Deadline for the submission: Dec 20 2024, 23:59**.
- **Load the compressed file to the Assignments folder on Webeep**.

Write your answer here

- Develop the exercise in the file `lastname123456_Assign1_Ex1.m`
- Organize the script in sections, one for each point; use local functions if needed.

Fill the table with the required results. Use 10-digits

	L_1	L_2	L_3	L_4	L_5
x	± 0.0000000000	± 0.0000000000	± 0.0000000000	± 0.0000000000	± 0.0000000000
y	± 0.0000000000	± 0.0000000000	± 0.0000000000	± 0.0000000000	± 0.0000000000
C	± 0.0000000000	± 0.0000000000	± 0.0000000000	± 0.0000000000	± 0.0000000000

Table 1: Lagrangian points coordinates and Jacobi constants

x	y	z
± 0.0000000000	± 0.0000000000	± 0.0000000000

v_x	v_x	v_x
± 0.0000000000	± 0.0000000000	± 0.0000000000

Table 2: Corrected initial state of the halo orbit with $C = 3.09$

x	y	z
± 0.0000000000	± 0.0000000000	± 0.0000000000

v_x	v_y	v_x
± 0.0000000000	± 0.0000000000	± 0.0000000000

Table 3: Corrected initial state of the halo orbit with $C = 3.04$

2 Impulsive guidance

Exercise 2

Consider the two-impulse transfer problem stated in Section 3.1 (Topputo, 2013)*.

*F. Topputo, "On optimal two-impulse Earth–Moon transfers in a four-body model", Celestial Mechanics and Dynamical Astronomy, Vol. 117, pp. 279–313, 2013, DOI: 10.1007/s10569-013-9513-8.

1) Using the procedure in Section 3.2, produce a first guess solution using $\alpha = 0.2\pi$, $\beta = 1.41$, $\delta = 4$, and $t_i = 2$. Plot the solution in both the rotating frame and Earth-centered inertial frame (see Appendix 1 in (Topputo, 2013)). Consider the parameters listed in Table 4 and extract the radius and gravitational parameters of the Earth and Moon from the provided kernels and use the latter to compute the parameter μ.

Symbol	Value	Units	Meaning
m_s	3.28900541×10^5	-	Scaled mass of the Sun
ρ	3.88811143×10^2	-	Scaled Sun-(Earth+Moon) distance
ω_s	$-9.25195985 \times 10^{-1}$	-	Scaled angular velocity of the Sun
ω_{em}	$2.66186135 \times 10^{-1}$	s^{-1}	Earth-Moon angular velocity
l_{em}	3.84405×10^8	m	Earth-Moon distance
h_i	167	km	Altitude of departure orbit
h_f	100	km	Altitude of arrival orbit
DU	3.84405000×10^5	km	Distance Unit
TU	4.34256461	days	Time Unit
VU	1.02454018	km/s	Velocity Unit

Table 4: Constants to be considered to solve the PBRFBP. The units of distance, time, and velocity are used to map scaled quantities into physical units.

2) Considering the first guess in 1) and using $\{\mathbf{x}_i, t_i, t_f\}$ as variables, solve the problem in Section 3.1 with simple shooting in the following cases

   a) without providing any derivative to the solver, and

   b) by providing the derivatives and by estimating the state transition matrix with variational equations.

3) Considering the first guess solution in 1) and the procedure in Section 3.3, solve the problem with multiple shooting taking $N = 4$ and using the variational equation to compute the Jacobian of the nonlinear equality constraints.

4) Perform an n-body propagation using the solution $\{\mathbf{x}_i, t_i, t_f\}$ obtained in point 2), transformed in Earth-centered inertial frame and into physical units. To move from 2-D to 3-D, assume that the position and velocity components in inertial frame are $r_z(t_i) = 0$ and $v_z(t_i) = 0$. To perform the propagation it is necessary to identify the epoch t_i. This can be done by mapping the relative position of the Earth, Moon and Sun in the PCRTBP to a similar condition in the real world:

   a) Consider the definition of $\theta(t)$ provided in Section 2.2 to compute the angle $\theta_i = \theta(t_i)$. Note that this angle corresponds to the angle between the rotating frame x-axis, aligned to the position vector from the Earth-Moon System Barycenter (EMB) to the Moon, and the Sun direction.

   b) The angle θ ranges between $[0, 2\pi]$ and it covers this domain in approximately the revolution period of the Moon around the Earth.

   c) Solve a zero-finding problem to determine the epoch at which the angle Moon-EMB-Sun is equal to θ_i, considering as starting epoch `2024 Sep 28 00:00:00.000 TDB`. **Hints**: Exploit the SPK kernels to define the orientation of the rotating frame axes in the inertial frame for an epoch t. Consider only the projection of the EMB-Sun position vector onto the so-defined x-y plane to compute the angle (planar motion).

   Plot the propagated orbit and compare it to the previously found solutions.

(11 points)

Write your answer here

- Develop the exercise in the file `lastname123456_Assign1_Ex2.m`
- Organize the script in sections, one for each point; use local functions if needed.

$r_{x,0}$ $[DU]$	$r_{y,0}$ $[DU]$	$v_{x,0}$ $[VU]$	$v_{y,0}$ $[VU]$
$\pm$0.000000	$\pm$0.000000	$\pm$0.000000	$\pm$0.000000

Table 5: Initial guess in Earth-Moon rotating frame.

Gradients	$r_{x,0}$ $[DU]$	$r_{y,0}$ $[DU]$	$v_{x,0}$ $[VU]$	$v_{y,0}$ $[VU]$	t_i $[TU]$	t_f $[TU]$
False	$\pm$ 0.000000	$\pm$0.000000	$\pm$0.000000	$\pm$0.000000	0.000	0.000
True	$\pm$ 0.000000	$\pm$0.000000	$\pm$0.000000	$\pm$0.000000	0.000	0.000

Table 6: Simple shooting solutions in the Earth-Moon rotating frame.

$r_{x,0}$ $[DU]$	$r_{y,0}$ $[DU]$	$v_{x,0}$ $[VU]$	$v_{y,0}$ $[VU]$	t_i $[TU]$	t_f $[TU]$
$\pm$ 0.000000	$\pm$0.000000	$\pm$0.000000	$\pm$0.000000	0.000	0.000

Table 7: Multiple shooting solution in the Earth-Moon rotating frame.

Symbol	**Calendar epoch** (UTC)
t_i	YYYY-MM-DDTHH:MM:SS.sss
t_f	YYYY-MM-DDTHH:MM:SS.sss

$r_{x,0}$ $[km]$	$r_{y,0}$ $[km]$	$r_{z,0}$ $[km]$
±0.00000000	±0.00000000	0.0

$v_{x,0}$ $[km/s]$	$v_{y,0}$ $[km/s]$	$v_{z,0}$ $[km/s]$
±0.00000000	±0.00000000	0.0

Table 8: Initial epoch, final epoch, and initial state in Earth-centered inertial frame.

3 Continuous guidance

Exercise 3

A low-thrust option is being considered to perform an orbit raising maneuver using a low-thrust propulsion system in Earth orbit. The spacecraft is released on a circular orbit on the equatorial plane at an altitude of 800 km and has to reach an orbit inclined by 0.75 deg on the equatorial plane at 1000 km. This orbital regime is characterized by a large population of resident space objects and debris, whose spatial density q can be expressed as:

$$q(\rho) = \frac{k_1}{k_2 + \left(\dfrac{\rho - \rho_0}{DU}\right)^2}$$

where ρ is the distance from the Earth center. The objective is to design an optimal orbit raising that minimizes the risk of impact, that is to minimize the following objective function

$$F(t) = \int_{t_0}^{t_f} q(\rho(t))\, \mathrm{d}t.$$

The parameters and reference Distance Unit to be considered are provided in Table 9.

Symbol	Value	Units	Meaning
h_i	800	km	Altitude of departure orbit
h_f	1000	km	Altitude of arrival orbit
Δi	0.75	deg	Inclination change
R_e	6378.1366	km	Earth radius
μ	398600.435	km^3/s^2	Earth gravitational parameter
ρ_0	$750 + R_e$	km	Reference radius for debris flux
k_1	1×10^{-5}	DU^{-1}	Debris spatial density constant 1
k_2	1×10^{-4}	DU^2	Debris spatial density constant 2
m_0	1000	kg	Initial mass
$T_{\max}$	3.000	N	Maximum thrust
I_{sp}	3120	s	Specific impulse
DU	7178.1366	km	Distance Unit
MU	m_0	kg	Mass Unit

Table 9: Problem parameters and constants. The units of time TU and velocity VU can be computed imposing that the scaled gravitational parameter $\overline{\mu} = 1$.

1) Plot the debris spatial density $q(\rho) \in [h_i - 100;\, h_f + 100]$ km and compute the initial state and target orbital state, knowing that: i) the initial and final state are located on the x-axis of the equatorial J2000 reference frame; ii) the rotation of the angle Δi is performed around the x-axis of the equatorial J2000 reference frame (RAAN = 0).

2) Adimensionalize the problem using as reference length $DU = \rho_i = h_i + R_e$ and reference mass $MU = m_0$, imposing that $\mu = 1$. Report all the adimensionalized parameters.

3) Using the PMP, write down the spacecraft equations of motion, the costate dynamics, and the zero-finding problem for the unknowns $\{\boldsymbol{\lambda}_0, t_f\}$ with the appropriate transversality condition. **Hint**: the spacecraft has to reach the target state computed in point 1).

4) Solve the problem considering the data provided in Table 9. To obtain an initial guess for the costate, generate random numbers such that $\lambda_{0,i} \in [-250; +250]$ while $t_f \approx 10\pi$.

Report the obtained solution in terms of $\{\boldsymbol{\lambda}_0, t_f\}$ and the error with respect to the target. Assess your results exploiting the properties of the Hamiltonian in problems that are not time-dependent and time-optimal solution. Plot the evolution of the components of the primer vector $\boldsymbol{\alpha}$ in a NTW reference frame†.

5) Solve the problem for a lower thrust level $T_{\max} = 2.860$ N. Compare the new solution with the one obtained in the previous point. **Hint**: exploit numerical continuation.

(11 points)

Write your answer here

- Develop the exercise in the file `lastname123456_Assign1_Ex3.m`
- Organize the script in sections, one for each point; use local functions if needed.

$r_{x,i}$ $[km]$	$r_{y,i}$ $[km]$	$r_{z,i}$ $[km]$	$v_{x,i}$ $[km/s]$	$v_{y,i}$ $[km/s]$	$v_{z,i}$ $[km/s]$
±0000.000000	±0000.000000	±0000.000000	±0.00000000	±0.00000000	±0.00000000

$r_{x,f}$ $[km]$	$r_{y,f}$ $[km]$	$r_{z,f}$ $[km]$	$v_{x,f}$ $[km/s]$	$v_{y,f}$ $[km/s]$	$v_{z,f}$ $[km/s]$
±0000.000000	±0000.000000	±0000.000000	±0.00000000	±0.00000000	±0.00000000

Table 10: Initial and target state in Earth-centered equatorial J2000 intertial frame.

t_f [mins]	m_f $[kg]$
0000.0000	000.0000

λ_{0,r_x}	λ_{0,r_y}	λ_{0,r_z}	λ_{0,v_x}	λ_{0,v_y}	λ_{0,v_z}	$\lambda_{0,m}$
±000.0000	±000.0000	±000.0000	±000.0000	±000.0000	±000.0000	±000.0000

Table 11: Optimal orbit raising transfer solution ($T_{\max} = 3.000$ N).

Error	**Value**	**Units**
$\|\|\mathbf{r}(t_f) - \mathbf{r}_f\|\|$	0.0000	km
$\|\|\mathbf{v}(t_f) - \mathbf{v}_f\|\|$	0.0000	m/s

Table 12: Final state error with respect to target position and velocity ($T_{\max} = 3.000$ N).

†The T-axis is aligned with the velocity, the N-axis is normal to the angular momentum, while the W-axis is pointing inwards, i.e., towards the Earth.

t_f [mins]	m_f $[kg]$
0000.0000	000.0000

λ_{0,r_x}	λ_{0,r_y}	λ_{0,r_z}	λ_{0,v_x}	λ_{0,v_y}	λ_{0,v_z}	$\lambda_{0,m}$
±000.0000	±000.0000	±000.0000	±000.0000	±000.0000	±000.0000	±000.0000

Table 13: Optimal orbit raising transfer solution ($T_{\text{max}} = 2.860$ N).

Error	**Value**	**Units**
$\lvert\lvert\mathbf{r}(t_f) - \mathbf{r}_f\rvert\rvert$	0.0000	km
$\lvert\lvert\mathbf{v}(t_f) - \mathbf{v}_f\rvert\rvert$	0.0000	m/s

Table 14: Final state error with respect to target position and velocity ($T_{\text{max}} = 2.860$ N).