4.1

$$Y = \sqrt{|X|} \quad \text{for } 0 \le y \le 1$$
$$F_Y(y) = P(Y \le y) = P\left(-y^2 \le X \le y^2\right) = y^2,$$
$$f_Y(y) = 2y$$
$$Y = -\ln|X| \quad \text{y} \ge 0$$
$$F_Y(y) = P(Y \le y) = P\left(X \ge e^{-y}\right) + P\left(X \le -e^{-y}\right) = 1 - e^{-y},$$
$$f_Y(y) = e^{-y},$$

4.5

$$A = |X - Y|$$
$$F_A(a) = P(|X - Y| \le a) = 1 - (1-a)^2.$$
$$f_A(a) = \begin{cases} 2(1-z) & \text{if } 0 \le z \le 1 \\ 0 & \text{else} \end{cases}$$

4.9

$$Z = X - Y$$

for $z \ge 0$

$$\begin{aligned} F_Z(z) &= P(X - Y \le z) \\ &= 1 - P(X - Y > z) \\ &= 1 - \int_0^\infty \mu e^{-\mu y} \left(\int_{z+y}^\infty \lambda e^{-\lambda x} dx \right) dy \\ &= 1 - e^{-\lambda z} \int_0^\infty \mu e^{-(\lambda+\mu) y} dy \\ &= 1 - \frac{\mu}{\lambda + \mu} e^{-\lambda z} \end{aligned}$$

for $z < 0$

$$F_Z(z) = 1 - F_Z(-z) = 1 - \left(1 - \frac{\lambda}{\lambda+\mu} e^{-\mu(-z)}\right) = \frac{\lambda}{\lambda+\mu} e^{\mu z}$$

so

$$F_Z(z) = \begin{cases} 1 - \frac{\mu}{\lambda+\mu} e^{-\lambda z} & z \ge 0 \\ \frac{\lambda}{\lambda+\mu} e^{\mu z} & z < 0 \end{cases}$$

PDF is:

$$f_Z(z) = \begin{cases} \frac{\lambda\mu}{\lambda+\mu} e^{-\lambda z} & z \ge 0 \\ \frac{\lambda\mu}{\lambda+\mu} e^{\mu z} & z < 0 \end{cases}$$

4.13

$$X - Y = X + Z - (a+b)$$
$$Z = a + b - Y$$

the PDF of $X+Z$ equals PDF of $X+Y$
the PDF of $X-Y$: shift the PDF of $X+Y$ to the left by $a+b$.

4.17

when a constant is added to x or y covariance is constant so their mean is 0.

$\operatorname{cov}(X-Y, X+Y) = E[(X-Y)(X+Y)] = E\left[X^2\right] - E\left[Y^2\right] = \operatorname{var}(X) - \operatorname{var}(Y) = 0$

4.29

$$\begin{aligned}
M(s) = E\left[e^{sX}\right] &= \frac{1}{2}e^{s} + \frac{1}{4}e^{2s} + \frac{1}{4}e^{3s} \\
E[X] = \frac{d}{ds}M(s)\bigg|_{s=0} &= \frac{1}{2} + \frac{2}{4} + \frac{3}{4} = \frac{7}{4} \\
E\left[X^2\right] = \frac{d^2}{ds^2}M(s)\bigg|_{s=0} &= \frac{1}{2} + \frac{4}{4} + \frac{9}{4} = \frac{15}{4} \\
E\left[X^3\right] = \frac{d^3}{ds^3}M(s)\bigg|_{s=0} &= \frac{1}{2} + \frac{8}{4} + \frac{27}{4} = \frac{37}{4}
\end{aligned}$$

4.33

$f_X(x) = \begin{cases} \frac{1}{3} \cdot 2e^{-2x} + \frac{2}{3} \cdot 3e^{-3x} & x \geq 0 \\ 0 & \text{else} \end{cases}$

4.37

X : number of types of ordered pizza
X_i : $X_i = \begin{cases} 1 & \text{a type } i \text{ pizza ordered at least once} \\ 0 & \text{else.} \end{cases}$
$X = X_1 + \cdots + X_n$

$$E[X] = E[E[X \mid K]] = E\left[E\left[X_1 + \cdots + X_n \mid K\right]\right] = nE\left[E\left[X_1 \mid K\right]\right]$$

p(if a customer does'ntordertype 1) : $(n-1)/n$

$$E\left[X_1 \mid K = k\right] = 1 - \left(\frac{n-1}{n}\right)^k$$

$$E\left[X_1 \mid K\right] = 1 - \left(\frac{n-1}{n}\right)^K$$

$$p = \frac{n-1}{n}$$

$$E[X] = nE\left[1 - p^K\right] = n - nE\left[p^K\right] = n - nE\left[e^{K \log p}\right] = n - nM_K(\log p)$$

4.41

(a) E : number of people
$M_E(s) = e^{\lambda(e^s - 1)}$ X_i uniformly distributed so

$$M_X(s) = \frac{e^s - 1}{s}.$$

$$M_Y(s) = e^{\lambda(M_X(s) - 1)} = e^{\lambda\left(\frac{e^s - 1}{s} - 1\right)}$$

(b) with chain rule

$$E[Y] = \frac{d}{ds} M_Y(s)\bigg|_{s=0} = \frac{d}{ds} M_X(s)\bigg|_{s=0} \cdot \lambda e^{\lambda(M_X(s)-1)}\bigg|_{s=0} = \frac{1}{2} \cdot \lambda = \frac{\lambda}{2}$$

(c) with iterated expectations law

$$E[Y] = E[E[Y \mid N]] = E[NE[X]] = E[N]E[X] = \frac{\lambda}{2}$$