University of Tehran

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Statistical Inference

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Assignment 5(Extra)

Bayesian Statistics

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Spring 2022

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Question 1 – Bayesian Elements

We know that the following expression is true according to bayes rule:

$$Posterior \propto Prior * \ Likelihood$$

which is equal to its mathematical presentation:

$$f(\theta|x) = \frac{f(\theta)f(x|\theta)}{f(x)}$$

In this question we want to investigate the effect of prior information on posterior probability density function in two different steps.

Suppose that $prior = Beta(\alpha, \beta)$ and $likelihood = Binomial(n, k)$ and the unknown parameter in both is θ.

A) Does posterior have Analytical Answer? If yes, please write it down.

B) How do you explain the results? Does the posterior depend on prior?

C) Repeat **(B)** for when the prior has a uniform distribution.

D) **(simulation)**Draw prior, likelihood and posterior in one figure using appropriate parameters.(For simplicity you can ignore $f(x)$ in drawing pdfs and only use $f(\theta|x) \propto f(\theta)f(x|\theta)$.).

Question 2 – Dice Game

Suppose we want to play a game using Bayes Rule! In this game we hold 2 dice In our hands. One with 12 sides and the other one with 6 sides . we call the 12-sided die "The Good Die". Now Note that:

H_1 : Good die on right hand(bad die on left)

H_2 : Good die on left hand(bad die on right)

Suppose that at the beginning $P(H_1) = P(H_2) = 0.5$.

Now the participant chooses one hand(No difference!). we roll the die on the selected hand and just announce whether or not the number is greater than or equal to 4. According to these explanations please answer the following questions:

A) Suppose that we chose right hand and the outcome is >=4. Update the prior belief about the probability of H_1 and H_2 according to the observation.

B) **(simulation)** now we want to repeat this 10 times and this time with the help of R (or any other way you want!) in this part first you must create a vector that defines the participant choice in each round and the second vector must define the outcome. As an example, your vectors must be in this manner for a 5-round game!
choice = c(0,0,0,1,1) so that 0 for "Right Hand" and 1s for "Left Hand"
outcome = c(1,0,0,1,0) so that 0 for ">=4" and 1s for "<4"

now create 2 binary vectors randomly and enjoy the game! You are required to plot the diagram for H_1(or H_2) probability according to each round.

C) Now that you've completed this game you have a rough understanding of how Bayesian statistics works. Compare it to the traditional frequentist statistics in simple words. (In only one or two lines)

Question 3 – MAP Estimation

In this question we want to be familiar with MAP[1] Estimation!

MAP estimation is one of the ways to obtain a point estimate by maximizing the posterior pdf. Curious? you can visit **this pdf file** or any other resources you may know for detailed information. Now answer the following question carefully:

A) Suppose that $X \sim N(0, \sigma_x^2)$ is transmitted over a communication channel. Assume that the received signal is given by: $Y = X + W$ where W is a gaussian noise with mean equal to zero and variance equal to σ_W^2 . find the MAP estimate of X, given Y = y observed.

B) What happens if $\sigma_W^2 \to \infty$ in the last part?

Hint: always identify prior and likelihood function and then start solving these types of questions. Also, you must maximize (likelihood * prior) at last!

[1] Maximum A Posteriori

Question 4 – Bayesian Hypothesis Testing

Suppose that the random variable X is transmitted over a communication channel. Assume that the received signal is given by $Y = 2X + W$.

Where $W \sim N(0, \sigma^2)$ is Independent of X.

Suppose that X=1 with probability p, and X=−1 with probability 1−p. The goal is to decide between X=1 and X=−1 by observing the random variable Y.

In a very simplified situation, in which the cost of making a wrong decision is not that important, we can use the concept "MAP Hypothesis Testing" in order to solve this question. With this detailed explanation, Now, find y (observed values) in which we can reject H_0 and accept H_1. (Note that H0: X = -1 ; H1: X = 1)

Question 5 – Metropolis Hastings Algorithm

You calculated posterior pdfs in previous questions analytically. In many real-life Applications you cannot calculate the posterior that easily. So, as a result there are some Methods developed to calculate the posterior numerically. One of the most famous Algorithms are those based on Monte Carlo simulation.

In this question you are asked to implement Metropolis Hastings Algorithm.

A) First explain about the algorithm briefly and then write a pseudocode that makes everything clear!

B) **(simulation)**Implement the algorithm with any code or library you may want to use. Set the parameters and functions by your own if needed.

Note! You can use any code/library from anywhere to implement this algorithm but your explanation is Important.

Question 6 – Medical Test Paradox

First watch this interesting video about the famous medical test problem. Then answer the following questions!

A) Why is the traditional way to solve this problem so counterintuitive?

B) What is the main idea and method of the video ?

C) Clearly Explain the concept of bayes factor and its relation to Test Sensitivity and Specificity and compare it to the original way used to solve this problem

D) Now we want to solve this problem again by using only odds and bayes factor! Suppose 8% of the population are infected with new variant of Covid-19. The rapid tests identify the infected person with 80% accuracy and the odds of not being infected but given positive results is 1:9. What are the chances that a person with positive test result truly has Covid-19?
(You must use the method introduced in the video. traditional way is unacceptable!)

Good Luck!