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FOUNDATION ac CONCEPTS

In previous chapters, we concentrated mostly on dc. We now turn our attention to ac (alternating current).

AC is important to us for a number of reasons. First, it is the basis of the electrical power system that supplies our homes and businesses with electrical energy. AC is used instead of dc because it has several important advantages, the chief one being that ac power can be transmitted easily and efficiently over long distances. However, the importance of ac extends far beyond its use in the electrical power industry. You will find, for example, that an understanding of ac concepts is necessary in virtually every branch of electronics that you will encounter, be it in the field of audio systems, communications systems, control systems, or any number of other areas. In fact, nearly every electrical and electronic device that we use in our daily lives operates from or involves the use of ac in some way.

We begin Part IV of this book with a look at fundamental ac concepts. We examine ways to generate ac voltages, methods used to represent ac voltages and currents, relationships between ac quantities in resistive, inductive, and capacitive circuits, and, finally, the meaning and representation of power in ac systems. This sets the stage for succeeding chapters that deal with ac circuit analysis techniques, including ac versions of the various methods that you have used for dc circuits throughout previous chapters of this book. ■

KEY TERMS

ac Current
ac Voltage
Alternating Current (ac)
Amplitude
Average Value
Cycle
Effective Value
Frequency
Hertz (Hz)
Instantaneous Value
Lag
Lead
Oscilloscope
Peak-to-Peak
Peak Value
Period (*T*)
Periodic Waveform
Phase Difference
Phase Shifts
Phasor
Radian Frequency
RMS (Root Mean Square) Values
Sine Wave
Waveform

OUTLINE

Introduction
Generating ac Voltages
Voltage and Current Conventions for ac
Frequency, Period, Amplitude, and Peak Value
Angular and Graphic Relationships for Sine Waves
Voltages and Currents as Functions of Time
Introduction to Phasors
ac Waveforms and Average Value
Effective (RMS) Values
Rate of Change of a Sine Wave (Derivative)
ac Voltage and Current Measurement
Circuit Analysis Using Computers

OBJECTIVES

After studying this chapter, you will be able to

- explain how ac voltages and currents differ from dc,
- draw waveforms for ac voltage and currents and explain what they mean,
- explain the voltage polarity and current direction conventions used for ac,
- describe the basic ac generator and explain how ac voltage is generated,
- define and compute frequency, period, amplitude, and peak-to-peak values,
- compute instantaneous sinusoidal voltage or current at any instant in time,
- define the relationships between ω, T, and f for a sine wave,
- define and compute phase differences between waveforms,
- use phasors to represent sinusoidal voltages and currents,
- determine phase relationships between waveforms using phasors,
- define and compute average values for time-varying waveforms,
- define and compute effective (rms) values for time-varying waveforms,
- use Multisim and PSpice to study ac waveforms.

1

ac FUNDAMENTALS

CHAPTER PREVIEW

Alternating currents (ac) are currents that alternate in direction (usually many times per second), passing first in one direction, then in the other through a circuit. Such currents are produced by voltage sources whose polarities alternate between positive and negative (rather than being fixed as with dc sources). By convention, alternating currents are called **ac currents** and alternating voltages are called **ac voltages.**

The variation of an ac voltage or current versus time is called its **waveform.** Since waveforms vary with time, they are designated by lowercase letters $v(t)$, $i(t)$, $e(t)$, and so on, rather than by uppercase letters V, I, and E as for dc. Often we drop the functional notation [e.g., $v(t)$] and just use the simpler notations v, i, and e instead.

While many waveforms are important to us, the most fundamental is the sine wave (also called sinusoidal ac). In fact, the sine wave is of such importance that many people associate the term ac with sinusoidal, even though ac refers to any quantity that alternates with time.

In this chapter, we look at basic ac principles, including the generation of ac voltages and ways to represent and manipulate ac quantities. These ideas are then used throughout the remainder of the book to develop methods of analysis for ac circuits. ■

Putting It in Perspective

Thomas Alva Edison

NOWADAYS WE TAKE IT FOR GRANTED that our electrical power systems are ac. (This is reinforced every time you see a piece of equipment rated "60 hertz ac.") However, this was not always the case. In the late 1800s, a fierce battle—the so-called "war of the currents"—raged in the emerging electrical power industry. The forces favoring the use of dc were led by Thomas Alva Edison, and those favoring the use of ac were led by George Westinghouse (Chapter 23) and Nikola Tesla (Chapter 24).

Edison, a prolific inventor who gave us the electric light, the phonograph, and many other great inventions as well, fought vigorously for dc. He had spent a considerable amount of time and money on the development of dc power and had a lot at stake, in terms of both money and prestige. So determined was Edison in this battle that he first persuaded the state of New York to adopt ac for its newly devised execution device, the electric chair, and then pointed at it with horror as an example of how deadly ac was. Ultimately, however, the combination of ac's advantages over dc and the stout opposition of Tesla and Westinghouse won the day for ac.

Edison was born in 1847 in Milan, Ohio. Most of his work was done at two sites in New Jersey—first at a laboratory in Menlo Park, and later at a much larger laboratory in West Orange, where his staff at one time numbered around 5000. He received patents as inventor or co-inventor on nearly 1300 inventions—an astonishing feat that made him probably the greatest inventor of all time.

Thomas Edison died at the age of 84 on October 18, 1931. ■

1.1 Introduction

Previously you learned that dc sources have fixed polarities and constant magnitudes and thus produce currents with constant value and unchanging direction, as illustrated in Figure 1–1. In contrast, the voltages of ac sources alternate in polarity and vary in magnitude and thus produce currents that vary in magnitude and alternate in direction.

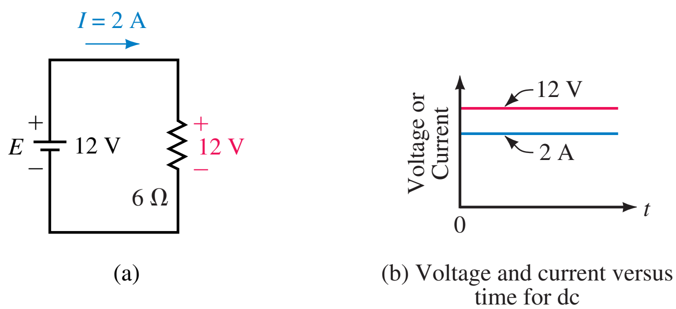


FIGURE 1–1 In a dc circuit, voltage polarities and current directions do not change.

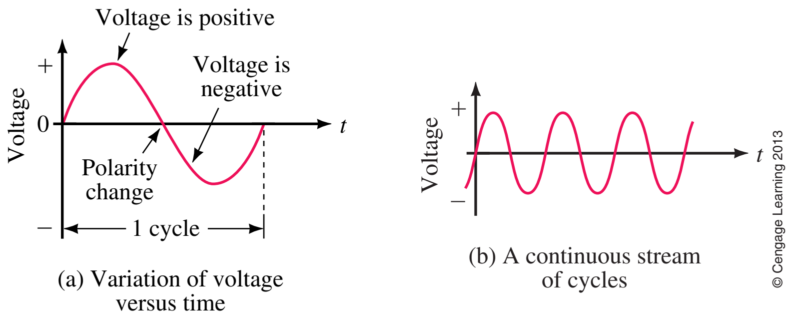


FIGURE 1–2 Sinusoidal ac waveforms. Values above the axis are positive while values below are negative.

CircuitSim 1-1

Sinusoidal ac Voltage

To illustrate, consider the voltage at the wall outlet in your home. Called a **sine wave** or **sinusoidal ac waveform** (for reasons discussed in Section 1.5), this voltage has the shape shown in Figure 1–2. Starting at zero, the voltage increases to a positive peak amplitude, decreases to zero, changes polarity, increases to a negative peak amplitude, then returns again to zero. One complete variation is referred to as a **cycle.** Since the waveform repeats itself at regular intervals as in (b), it is called a **periodic waveform.**

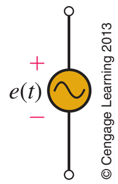


FIGURE 1–3 Symbol for a sinusoidal voltage source. Lowercase letter e is used to indicate that the voltage varies with time.

Symbol for an ac Voltage Source

The symbol for a sinusoidal voltage source is shown in Figure 1–3. Note that a lowercase e is used to represent voltage rather than E, since it is a function of time. Polarity marks are also shown although, since the polarity of the source varies, their meaning has yet to be established.

Sinusoidal ac Current

Figure 1–4 shows a resistor connected to an ac source. During the first half-cycle, the source voltage is positive; therefore, the current is in the clockwise direction. During the second half-cycle, the voltage polarity reverses; therefore, the current is in the counterclockwise direction. Since current is proportional to voltage, its shape is also sinusoidal (Figure 1–5).

CircuitSim 1-2

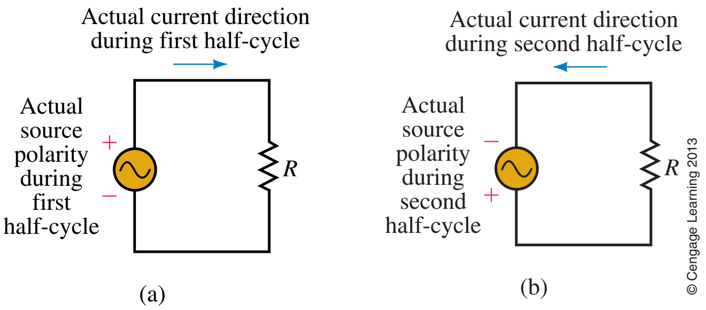


FIGURE 1–4 Current direction reverses when the source polarity reverses.

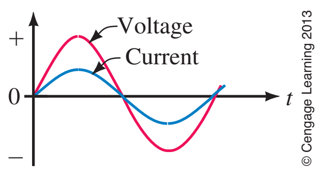


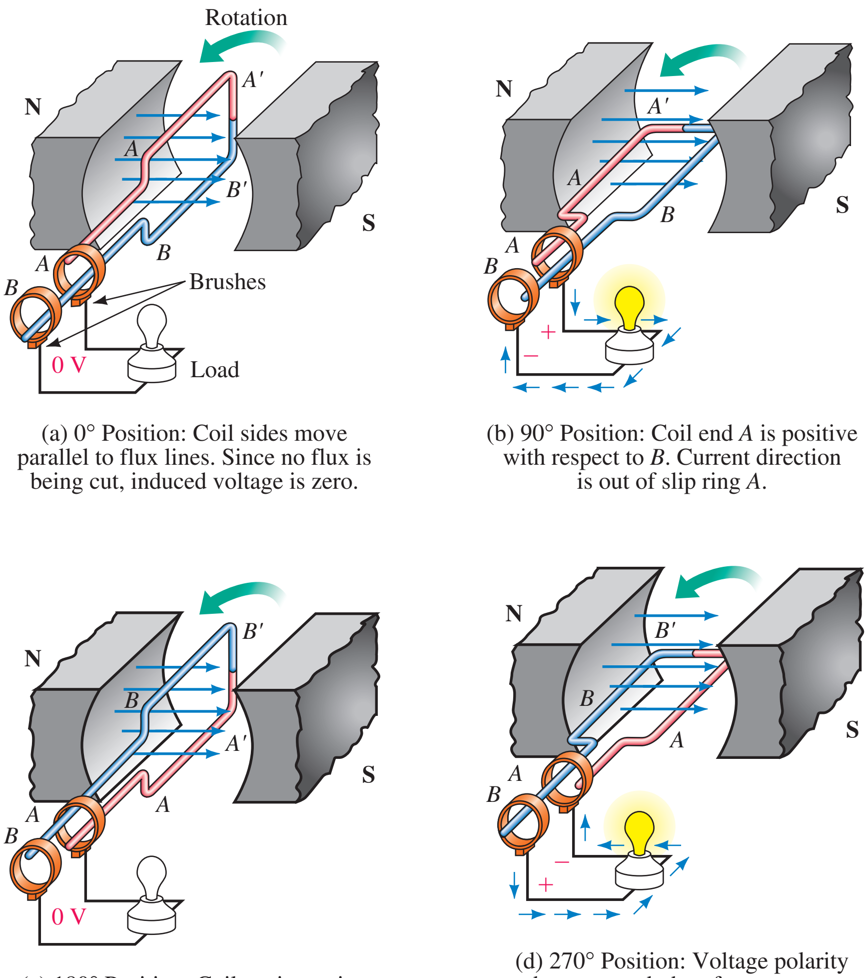
FIGURE 1–5 Current has the same wave-shape as voltage.

1.2 Generating ac Voltages

One way to generate an ac voltage is to rotate a coil of wire at constant angular velocity in a fixed magnetic field—see Note. As indicated in Figure 1–6, slip rings and brushes connect the coil to the load. The magnitude of

NOTES...

There are some good animations on the Internet that show the process of ac generation. Search for *Generator Animation.* (Some animations are better than others—look until you find a good one.)



(a) 0° Position: Coil sides move parallel to flux lines. Since no flux is being cut, induced voltage is zero.

(b) 90° Position: Coil end *A* is positive with respect to *B*. Current direction is out of slip ring *A*.

(c) 180° Position: Coil again cutting no flux. Induced voltage is zero.

(d) 270° Position: Voltage polarity has reversed, therefore, current direction has also reversed.

FIGURE 1–6 Generating an ac voltage. The 0° position of the coil is defined as in (a) where the coil sides move parallel to the flux lines. In practice, the rotations are so fast that the light does not have time to go out and so appears to be continuously on.

PRACTICAL NOTES...

In practice, the coil of Figure 1–6 consists of many turns wound on an iron core. The coil, core, and slip rings rotate as a unit.

In Figure 1–6, the magneticfield is fixed and the coil rotates.While small generators are built this way, large ac generators usually have the opposite construction, that is,their coils are fixed and the magnetic field is rotated instead. In addition,large ac generators are usually made as three-phase machines with threesets of coils instead of one. This is covered in Chapter 24. However, although its details are oversimplified, the generator of Figure 1–6 gives a true picture of the voltage produced by a real ac generator.

the resulting voltage is proportional to the rate at which flux lines are cut (Faraday's law, Chapter 13), and its polarity is dependent on the direction the coil sides move through the field. Since the rate of cutting flux varies with time, the resulting voltage will also vary with time. For example in Figure 1–6(a), since the coil sides are moving parallel to the field, no flux lines are being cut and the induced voltage at this instant (and hence the current) is zero. (This is defined as the 0° position of the coil.) As the coil rotates from the 0° position, coil sides *AA*′ and *BB*′ cut across flux lines; hence, voltage builds, reaching a peak when flux is cut at the maximum rate in the 90° position as in (b). Note the polarity of the voltage and the direction of current. As the coil rotates further, voltage decreases, reaching zero at the 180° position when the coil sides again move parallel to the field as in (c). At this point, the coil has gone through a half-revolution.

During the second half-revolution, coil sides cut flux in directions opposite to that which they did in the first half-revolution; hence, the polarity of the induced voltage reverses. As indicated in Figure 1–6(d), voltage reaches a peak at the 270° point, and, since the polarity of the voltage has changed,so has the direction of current. When the coil reaches the 360° position, volt-age is again zero and the cycle starts over. Figure 1–7 shows one cycle of

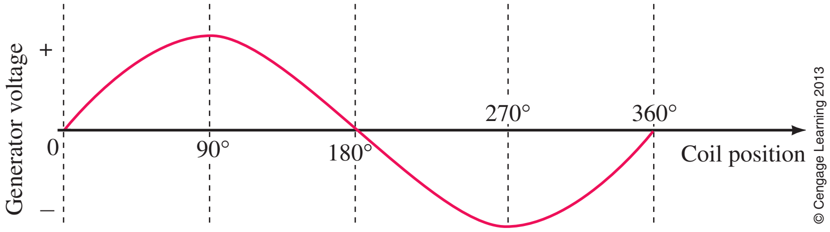


FIGURE 1–7 Coil voltage versus angular position.

the resulting waveform. Since the coil rotates continuously, the voltage produced will be a repetitive, periodic waveform as you saw in Figure 1–2(b). Current will be periodic also.

Time Scales

The horizontal axis of Figure 1–7 is scaled in degrees. Often we need it scaled in time. The length of time required to generate one cycle depends on the velocity of rotation. To illustrate, assume that the coil rotates at 600 rpm (revolutions per minute). Six hundred revolutions in 1 minute equals 600 rev/60 s = 10 revolutions in 1 second. At 10 revolutions per second, the time for 1 revolution is one-tenth of a second, that is, 100 ms. Since one cycle is 100 ms, a half-cycle is 50 ms, a quarter-cycle is 25 ms, and so on. Figure 1–8 shows the waveform rescaled in time.

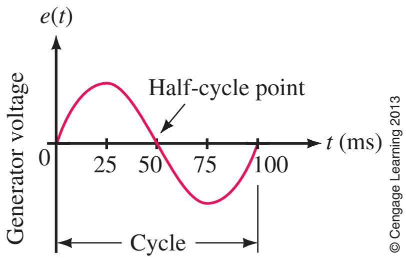
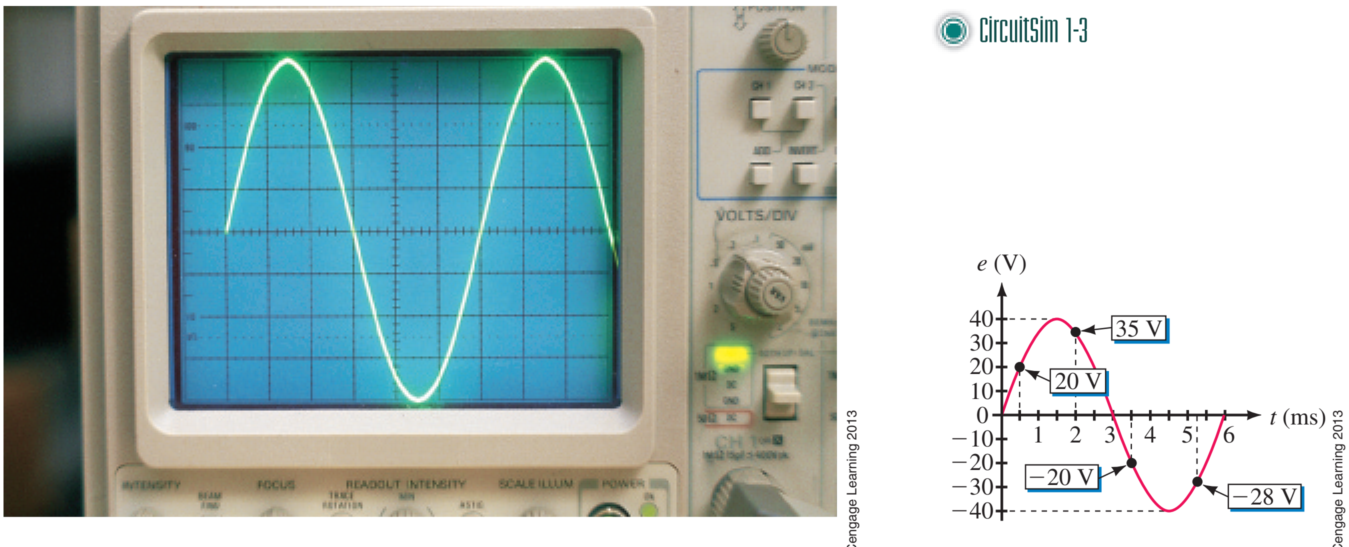


FIGURE 1–8 Cycle scaled in time. At 600 rpm, the cycle length is 100 ms.

Instantaneous Value

As Figure 1–8 shows, the coil voltage changes from instant to instant. The value of voltage at any point on the waveform is referred to as its **instantaneous value.** This is illustrated in Figure 1–9. Figure 1–9(a) shows a photograph of an actual waveform, and (b) shows it redrawn, with values scaled from the photo. For this example, the voltage has a peak value of 40 volts and a cycle time of 6 ms. From the graph, we see that at $t = 0$ ms, the voltage is zero. At $t = 0.5$ ms, it is 20 V. At $t = 2$ ms, it is 35 V. At $t = 3.5$ ms, it is -20 V, and so on.

CircuitSim 1-3



(a) Sinusoidal voltage

(b) Values scaled from the photograph

FIGURE 1–9 Instantaneous values.

Electronic Function Generators

AC waveforms may also be created electronically using function (also called signal) generators. In fact, with function generators, you are not limited to sinusoidal ac. The unit of Figure 1–10, for example, can produce a variety of variable-frequency waveforms, including sinusoidal, square wave, triangular, and so on. Waveforms such as these are commonly used to test electronic gear.

(a) Model 4045 Function Generator (b) Sample waveforms

FIGURE 1–10 Electronic function generators provide a variety of variable-frequency, variable-amplitude waveforms.

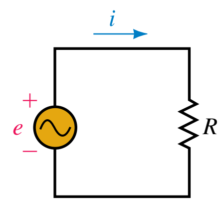
1.3 Voltage and Current Conventions for ac

In Section 1.1, we looked briefly at voltage polarities and current directions. At that time, we used separate diagrams for each half-cycle (Figure 1–4). However, this is unnecessary; one diagram and one set of references is all that is required. This is illustrated in Figure 1–11. First, we assign reference polarities for the source and a reference direction for the current. We then use the convention that, *when* e *has a positive value, its actual polarity is the same as the reference polarity, and when* e *has a negative value, its actual polarity is opposite to that of the reference.* For current, we use the convention that *when* i *has a positive value, its actual direction is the same as the reference arrow, and when* i *has a negative value, its actual direction is opposite to that of the reference.*

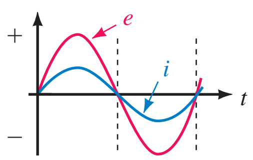
To illustrate, consider Figure 1–12. [Parts (b) and (c) show snapshots at two instants of time.] At time t_1, e has a value of 10 volts. This means that at this instant, the voltage of the source is 10 V and its top end is positive with respect to its bottom end as indicated in (b). With a voltage of 10 V and a resistance of 5 Ω, the instantaneous value of current is $i = e/R = 10 \text{ V}/5\ \Omega = 2$ A. Since i is positive, the current is in the direction of the reference arrow.

Now consider time t_2. Here, $e = -10$ V. This means that source voltage is again 10 V, but now its top end is negative with respect to its bottom end. Again applying Ohm's law, you get $i = e/R = -10 \text{ V}/5\ \Omega = -2$ A. Since i is negative, current is actually opposite in direction to the reference arrow. This is indicated in (c). (Recall our discussion of Section 4.2, Chapter 4 about the meaning of negative currents.)

The preceding concept is valid for any ac signal, regardless of waveshape. Let us illustrate.

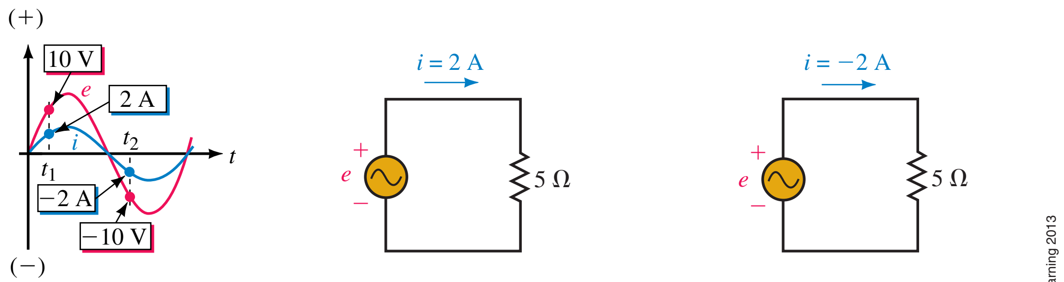


(a) References for voltage and current



(b) During the first half-cycle, voltage polarity and current direction are as shown in (a). Therefore, e and i are positive. During the second half-cycle, voltage polarity and current direction are opposite to that shown in (a). Therefore, e and i are negative.

FIGURE 1–11 ac Voltage and current reference conventions.



(b) Time t_1: $e = 10$ V and $i = 2$ A. Thus, voltage and current have the polarity and direction indicated.

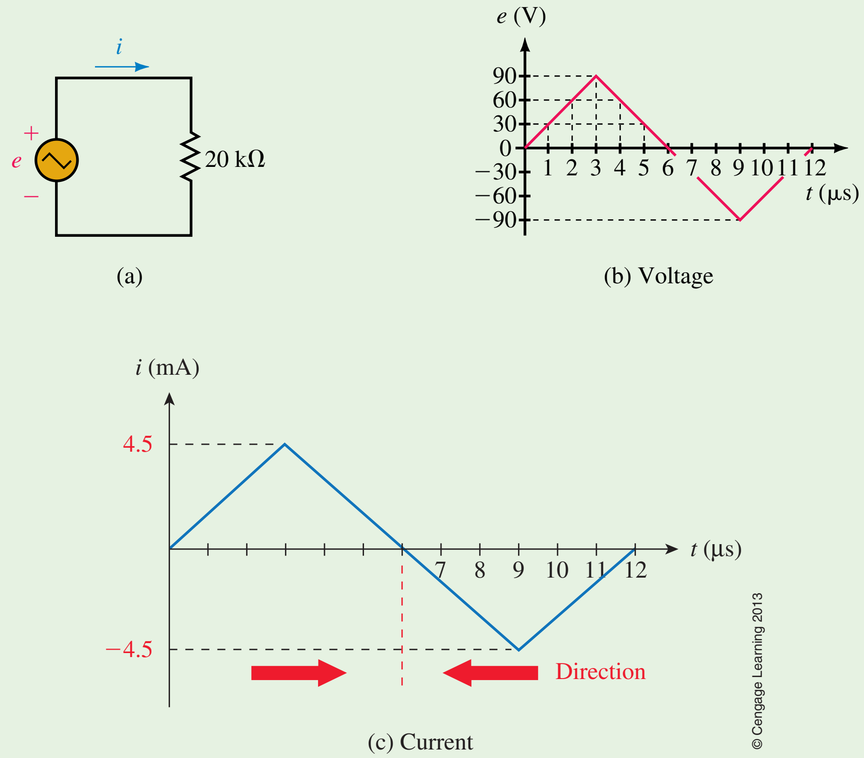
(c) Time t_2: $e = -10$ V and $i = -2$ A. Thus, voltage polarity is opposite to that indicated and current direction is opposite to the arrow direction.

CircuitSim 1-4

FIGURE 1–12 Illustrating the ac voltage and current convention.

EXAMPLE 1–1

Figure 1–13(b) shows one cycle of a triangular voltage wave. Determine the current and its direction at t = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12 μs and sketch.



CircuitSim 1

FIGURE 1–13

TABLE 1 –1 Values for Example 15–1

t (μs)	e (V)	i (mA)
0	0	0
1	30	1.5
2	60	3.0
3	90	4.5
4	60	3.0
5	30	1.5
6	0	0
7	−30	−1.5
8	−60	−3.0
9	−90	−4.5
10	−60	−3.0
11	−30	−1.5
12	0	0

Solution Apply Ohm's law at each point in time. At $t = 0$ μs, $e = 0$ V, so $i = e/R = 0\text{ V}/20\text{ k}\Omega = 0$ mA. At $t = 1$ μs, $e = 30$ V. Thus, $i = e/R = 30\text{ V}/20\text{ k}\Omega = 1.5$ mA. At $t = 2$ μs, $e = 60$ V. Thus, $i = e/R = 60\text{ V}/20\text{ k}\Omega = 3$ mA. Continuing in this manner, you get the values shown in Table 1–1. The waveform is plotted as Figure 1–13(c).

PRACTICE PROBLEMS 1

1. Let the source voltage of Figure 1–11 be the waveform of Figure 1–9. If $R = 2.5\ \text{k}\Omega$, determine the current at $t = 0$, 0.5, 1, 1.5, 3, 4.5, and 5.25 ms.
2. For Figure 1–13, if $R = 180\ \Omega$, determine the current at $t = 1.5$, 3, 7.5, and 9 μs.

Answers
1. 0, 8, 14, 16, 0, −16, −11.2 (all mA)
2. 0.25, 0.5, −0.25, −0.5 (all A)

1.4 Frequency, Period, Amplitude, and Peak Value

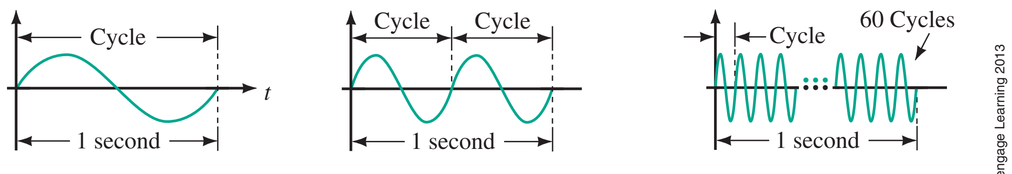
Periodic waveforms (i.e., waveforms that repeat at regular intervals), regardless of their waveshape, may be described by a group of attributes such as frequency, period, amplitude, peak value, and so on.

Frequency

The number of cycles per second of a waveform is defined as its **frequency.** In Figure 1–14(a), one cycle occurs in one second; thus its frequency is one cycle per second. Similarly, the frequency of (b) is two cycles per second and that of (c) is 60 cycles per second. Frequency is denoted by the lowercase letter *f*. In the SI system, its unit is the **hertz** (**Hz**, named in honor of pioneer researcher Heinrich Hertz, 1857–1894). By definition,

$$1\ \text{Hz} = 1 \text{ cycle per second} \tag{1–1}$$

Thus, the examples depicted in Figure 1–14 represent 1 Hz, 2 Hz, and 60 Hz, respectively.



(a) 1 cycle per second = 1 Hz (b) 2 cycles per second = 2 Hz (c) 60 cycles per second = 60 Hz

FIGURE 1–14 Frequency is measured in hertz (Hz).

The range of frequencies is immense. Power line frequencies, for example, are 60 Hz in many parts of the world (the USA and Canada for example) and 50 Hz in others. Audible sound frequencies range from about 20 Hz to about 20 kHz. The standard AM radio band occupies from 550 kHz to 1.6 MHz, while the FM band extends from 88 MHz to 108 MHz. TV transmissions occupy several bands in the 54-MHz to 890-MHz range. Above 300 GHz are optical and X-ray frequencies.

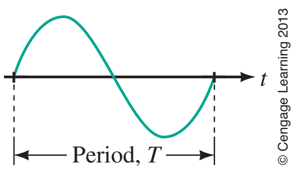


FIGURE 1–1 Period, *T* is the duration of one cycle, measured in seconds.

Period

The **period,** ***T,*** of a waveform (Figure 1–1) is the duration of one cycle. It is the inverse of frequency. To illustrate, consider again Figure 1–14. In (a), the frequency is 1 cycle per second; thus, the duration of each cycle is $T = 1$ s. In (b), the frequency is two cycles per second; thus, the duration of each cycle is $T = 1/2$ s, and so on. In general,

$$T = \frac{1}{f} \quad \text{(s)} \tag{1–2}$$

and

$$f = \frac{1}{T} \quad \text{(Hz)} \tag{1–3}$$

Note that these definitions are independent of waveshape.

EXAMPLE 1–2

a. What is the period of a 50-Hz voltage?
b. What is the period of a 1-MHz current?

Solution

a. $T = \dfrac{1}{f} = \dfrac{1}{50 \text{ Hz}} = 20 \text{ ms}$

b. $T = \dfrac{1}{f} = \dfrac{1}{1 \times 10^6 \text{ Hz}} = 1 \ \mu\text{s}$

EXAMPLE 1–3

Figure 1–16 shows an **oscilloscope** trace of a square wave. Controls have been set such that each horizontal division represents 50 μs. Determine the waveform's frequency.

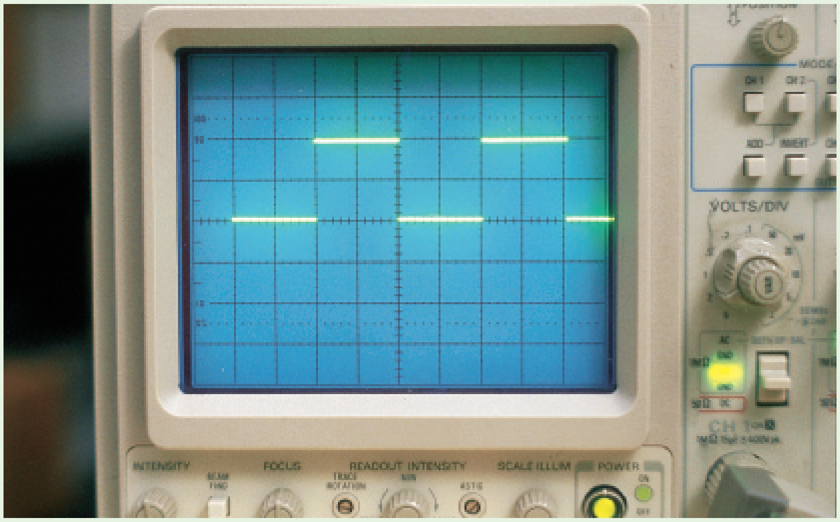


FIGURE 1–16 The concepts of frequency and period apply also to nonsinusoidal waveforms. Here, $T = 4 \text{ div} \times 50 \ \mu\text{s/div} = 200 \ \mu\text{s}$.

Solution Since the wave repeats itself every 200 μs, its period is 200 μs and

$$f = \frac{1}{200 \times 10^{-6} \text{ s}} = 5 \text{ kHz}$$

The period of a waveform can be measured between any two corresponding points (Figure 1–17). Often it is measured between zero points or peak points because they are easy to establish on an oscilloscope trace.

CircuitSim 1-6

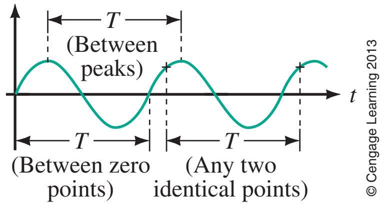


FIGURE 1–17 Period may be measured between any two corresponding points.

EXAMPLE 1–4

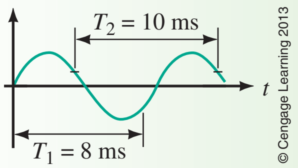


FIGURE 1 18

Determine the period and frequency of the waveform of Figure 1–18.

Solution Time interval T_1 does not represent a period as it is not measured between corresponding points. Interval T_2, however, is. Thus, T = 10 ms and

$$f = \frac{1}{T} = \frac{1}{10 \times 10^{-3}\ \text{s}} = 100\ \text{Hz}$$

Amplitude and Peak-to-Peak Value

The **amplitude** of a sine wave is the distance from its average to its peak. Thus, the amplitude of the voltage in Figures 1–19(a) and (b) is E_m.

Peak-to-peak voltage is also indicated in Figure 1–19(a). It is measured between peak and trough values. Peak-to-peak voltages are denoted $E_{p\text{-}p}$ or $V_{p\text{-}p}$ in this book. (Some authors use $V_{pk\text{-}pk}$ or the like.) Similarly, peak-to-peak currents are denoted as $I_{p\text{-}p}$. To illustrate, consider again Figure 1–9. The amplitude of this voltage is $E_m = 40$ V, and its peak-to-peak voltage is $E_{p\text{-}p} = 80$ V.

Peak Value

The **peak value** of a voltage or current is its maximum value with respect to zero. Consider Figure 1–19(b). Here, a sine wave rides on top of a dc value, yielding a peak that is the sum of the dc voltage and the ac waveform amplitude. For the case indicated, the peak voltage is $E + E_m$.

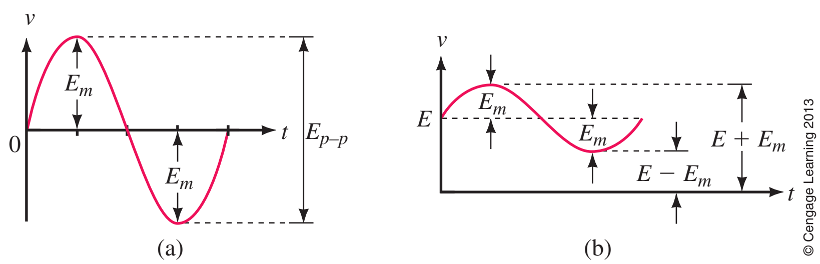


FIGURE 1–19 Definitions.

IN-PROCESS LEARNING CHECK 1

(Answers are at the end of the chapter.)

1. What is the period of an ac power system whose frequency is 60 Hz?
2. If you double the rotational speed of an ac generator, what happens to the frequency and period of the waveform?
3. If the generator of Figure 15–6 rotates at 3000 rpm, what is the period and frequency of the resulting voltage? Sketch four cycles and scale the horizontal axis in units of time.
4. For the waveform of Figure 15–9, list all values of time at which $e = 20$ V and $e = -35$ V. Hint: Sine waves are symmetrical.
5. Which of the waveform pairs of Figure 15–20 are valid combinations? Why?

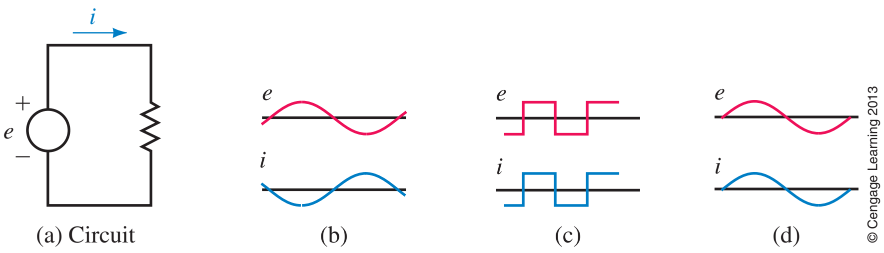


FIGURE 1–20 Which waveform pairs are valid?

6. For the waveform in Figure 1–21, determine the frequency.

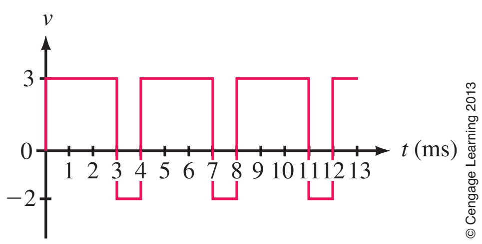


FIGURE 1–21

7. Two waveforms have periods of $T_1 = 10$ ms and $T_2 = 30$ ms. Which has the higher frequency? Compute the frequencies of both waveforms.
8. Two sources have frequencies f_1 and f_2, respectively. If $f_2 = 20f_1$, and T_2 is 1 μs, what is f_1? What is f_2?
9. Consider Figure 1–22. What is the frequency of the waveform?
10. For Figure 1–11, if $f = 20$ Hz, what is the current direction at $t = 12$ ms, 37 ms, and 60 ms? Hint: Sketch the waveform and scale the horizontal axis in ms. The answers should be apparent.
11. A 10-Hz sinusoidal current has a value of 5 amps at $t = 25$ ms. What is its value at $t = 75$ ms? See Hint in Problem 10.

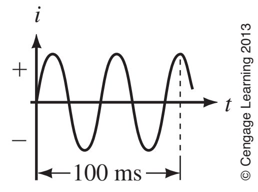


FIGURE 1–22

1.5 Angular and Graphic Relationships for Sine Waves

The Basic Sine Wave Equation

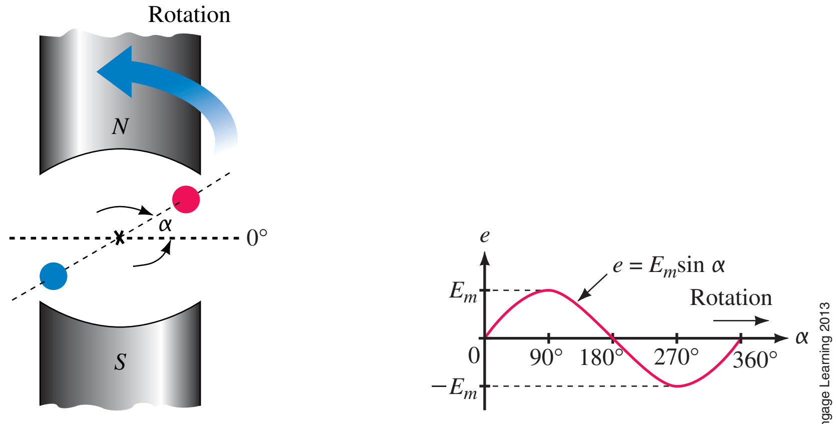
Consider again the generator of Figure 1–6, reoriented and redrawn in end view as Figure 1–23. As the coil rotates, the voltage produced is

$$e = E_m \sin \alpha \quad \text{(V)} \tag{1–4}$$

NOTES...

The derivation of Equation 1–4 may be found in many advanced physics books.

where E_m is the maximum coil voltage and α is the instantaneous angular position of the coil—see Note. (For a given generator and rotational velocity, E_m is constant.) Note that $\alpha = 0°$ represents the horizontal position of the coil and that one complete cycle corresponds to 360°. Equation 1–4 states that the voltage at any point on the sine wave may be found by multiplying E_m times the sine of the angle at that point.



(a) End view showing coil position (b) Voltage waveform

FIGURE 1–23 Coil voltage versus angular position.

EXAMPLE 1–5

If the amplitude of the waveform of Figure 1–23(b) is $E_m = 100$ V, determine the coil voltage at 30° and 330°.

Solution At $\alpha = 30°$, $e = E_m \sin \alpha = 100 \sin 30° = 50$ V. At 330°, $e = 100 \sin 330° = -50$ V. These are shown on the graph of Figure 1–24.

CircuitSim 1- 7

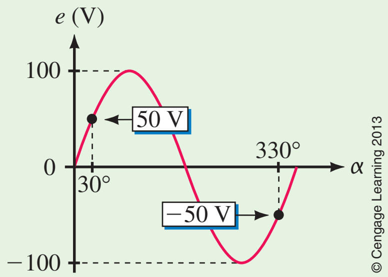


FIGURE 1–24

PRACTICE PROBLEMS 2

Table 1–2 is a tabulation of voltage versus angle computed from $e = 100 \sin \alpha$. Use your calculator to verify each value, then plot the result on graph paper. The resulting waveshape should look like Figure 1–24.

TABLE 1–2 Data for Plotting $e = 100 \sin \alpha$

Angle α	Voltage e
0	0
30	50
60	86.6
90	100
120	86.6
150	50
180	0
210	−50
240	−86.6
270	−100
300	−86.6
330	−50
360	0

Angular Velocity, ω

The rate at which the generator coil rotates is called its angular velocity. If the coil rotates through an angle of 30° in 1 second, for example, its angular velocity is 30° per second. Angular velocity is denoted by the Greek letter ω (omega). For the case cited, $\omega = 30°/\text{s}$. (Normally angular velocity is expressed in radians per second instead of degrees per second. We will make this change shortly—see Note.) When you know the angular velocity of a coil and the length of time that it has rotated, you can compute the angle through which it has turned. For example, a coil rotating at 30°/s rotates through an angle of 30° in 1 second, 60° in 2 seconds, 90° in 3 seconds, and so on. In general,

$$\alpha = \omega t \tag{1–5}$$

Expressions for t and ω can now be found. They are

$$t = \frac{\alpha}{\omega} \quad \text{(s)} \tag{1–6}$$

$$\omega = \frac{\alpha}{t} \tag{1–7}$$

NOTES...

Note as well that Equation 1–5to Equation 1–7 also hold when $\alpha$is expressed in radians and ω in radians/second. In fact, it is mostly with radian measure that we use these equations as you will soon see.

EXAMPLE 1–6

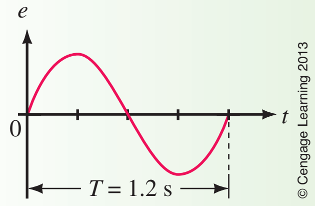


FIGURE 15–25

If the coil of Figure 1–23 rotates at $\omega = 300°/\text{s}$, how long does it take to complete one revolution?

Solution One revolution is 360°. Thus,

$$t = \frac{\alpha}{\omega} = \frac{360 \text{ degrees}}{300 \dfrac{\text{degrees}}{\text{s}}} = 1.2 \text{ s}$$

Since this is one period, we should use the symbol *T*. Thus, $T = 1.2$ s, as in Figure 1–25.

PRACTICE PROBLEMS 3

If the coil of Figure 1–23 rotates at 3600 rpm, determine its angular velocity, ω, in degrees per second.

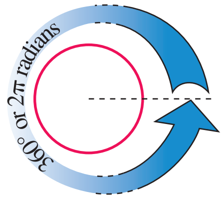
Answer
21 600 deg/s

Radian Measure

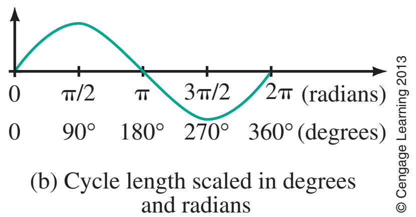
In practice, ω is usually expressed in radians per second, where radians and degrees are related by the identity

$$2\pi \text{ radians} = 360° \qquad \textbf{(1–8)}$$

One radian therefore equals $360°/2\pi = 57.296°$. A full circle, as shown in Figure 1–26(a), can be designated as either 360° or 2π radians. Likewise, the cycle length of a sinusoid, shown in Figure 1–26(b), can be stated as either 360° or 2π radians; a half-cycle as 180° or π radians, and so on.



(a) $360° = 2\pi$ radians



(b) Cycle length scaled in degrees and radians

FIGURE 1–26 Radian measure.

TABLE 1–3 Selected Angles in Degrees and Radians

Degrees	Radians
30	$\pi/6$
45	$\pi/4$
60	$\pi/3$
90	$\pi/2$
180	π
270	$3\pi/2$
360	2π

To convert from degrees to radians, multiply by $\pi/180$, while to convert from radians to degrees, multiply by $180/\pi$.

$$\alpha_{\text{radians}} = \frac{\pi}{180°} \times \alpha_{\text{degrees}} \qquad \textbf{(1–9)}$$

$$\alpha_{\text{degrees}} = \frac{180°}{\pi} \times \alpha_{\text{radians}} \qquad \textbf{(1–10)}$$

Table 1–3 shows selected angles in both measures.

Scientific calculators can perform these conversions directly; use yours to confirm the answers of Example 1–7.

EXAMPLE 1–7

a. Convert 315° to radians.

b. Convert $5\pi/4$ radians to degrees.

Solution

a. $\alpha_{radians} = (\pi/180°)(315°) = 5.5$ rad

b. $\alpha_{degrees} = (180°/\pi)(5\pi/4) = 225°$

Graphing Sine Waves

A sinusoidal waveform can be graphed with its horizontal axis scaled in degrees, radians, or time. When scaled in degrees or radians, one cycle is always 360° or 2π radians (Figure 1–27); when scaled in time, it is frequency dependent, since the length of a cycle depends on the coil's velocity of rotation as we saw in Figure 1–8. However, if scaled in terms of period T instead of in seconds, the waveform is also frequency independent, since one cycle is always T, as shown in Figure 1–27(c).

When graphing a sine wave, you don't need many points to get a good sketch: Values every 45° (one-eighth of a cycle) are generally adequate, Table 1–4. Often, you can simply "eyeball" the curve as illustrated next in Example 1–8.

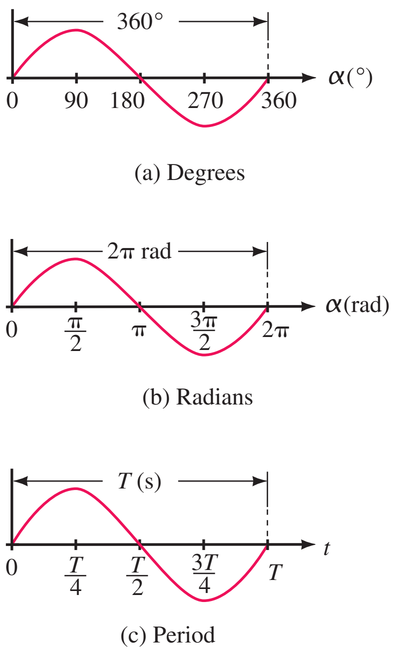


FIGURE 1–27 Comparison of various horizontal scales. Cycle length may be scaled in degrees, radians, or period. Each of these is independent of frequency.

TABLE 1–4 Values for Rapid Sketching

α (deg)	α (rad)	t (T)	Value of sin α
0	0	0	0.0
45	$\pi/4$	$T/8$	0.707
90	$\pi/2$	$T/4$	1.0
135	$3\pi/4$	$3T/8$	0.707
180	π	$T/2$	0.0
225	$5\pi/4$	$5T/8$	−0.707
270	$3\pi/2$	$3T/4$	−1.0
315	$7\pi/4$	$7T/8$	−0.707
360	2π	T	0.0

EXAMPLE 1–8

Sketch the waveform for a 25-kHz sinusoidal current that has an amplitude of 4 mA. Scale the axis in seconds.

Solution For this waveform, $T = 1/25$ kHz $= 40$ μs. Thus,

1. Lay out the time axis with the end of the cycle marked as 40 μs, the half-cycle point as 20 μs, the quarter-cycle point as 10 μs, and so on (Figure 1–28).
2. The peak value (i.e., 4 mA) occurs at the quarter-cycle point, which is 10 μs on the waveform. Likewise, −4 mA occurs at 30 μs. Now sketch.
3. Values at other time points can be determined easily if needed. For example, the value at 5 μs can be calculated by noting that 5 μs is one-eighth of a cycle, or 45°. Thus, $i = 4 \sin 45°$ mA $= 2.83$ mA. Alternately, from Table 1–4, at $T/8$, $i = (4\text{ mA})(0.707) = 2.83$ mA. As many points as you need can be computed and plotted in this manner.

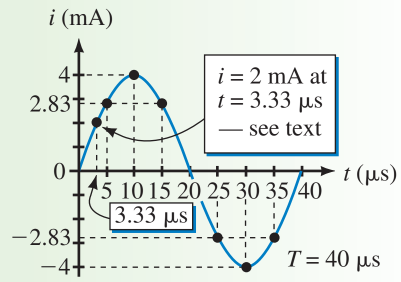


FIGURE 15–28

4. Values at particular angles can also be located easily. For instance, if you want a value at 30°, the required value is $i = 4 \sin 30°$ mA $= 2.0$ mA. To locate this point on the graph, note that 30° is one-twelfth of a cycle or $T/12 = (40\ \mu s)/12 = 3.33\ \mu s$. The point is shown on Figure 1–28.Note

that what we have done here is probably overkill, as you seldom need this much detail—it is often adequate to "eyeball" the curve as in Steps 1 and 2.

1.6 Voltages and Currents as Functions of Time

Relationship between ω, T, and f

Earlier you learned that one cycle of a sine wave may be represented as either $\alpha = 2\pi$ rads or $t = T$ s, Figure 1–27. Substituting these into $\alpha = \omega t$ (Equation 1–5), you get $2\pi = \omega T$. Transposing yields

$$\omega T = 2\pi \text{ (rad)} \qquad \textbf{(1–11)}$$

Thus,

$$\omega = \frac{2\pi}{T} \text{ (rad/s)} \qquad \textbf{(1–12)}$$

Recall that, $f = 1/T$ Hz. Substituting this into Equation 1–12 you get

$$\omega = 2\pi f \text{ (rad/s)} \qquad \textbf{(1–13)}$$

Note that ω is referred to as **radian frequency.**

EXAMPLE 1–9

In some parts of the world, the power system frequency is 60 Hz; in other parts, it is 50 Hz. Determine ω for each.

Solution For 60 Hz, $\omega = 2\pi f = 2\pi(60) = 377$ rad/s. For 50 Hz, $\omega = 2\pi f = 2\pi(50) = 314.2$ rad/s.

PRACTICE PROBLEMS 4

1. If $\omega = 240$ rad/s, what are T and f? How many cycles occur in 27 s?
2. If 56 000 cycles occur in 3.5 s, what is ω?

Answers

1. 26.18 ms, 38.2 Hz, 1031 cycles
2. 100.5×10^3 rad/s

Sinusoidal Voltages and Currents as Functions of Time

Recall from Equation 1–4, $e = E_m \sin \alpha$, and from Equation 1–5, $\alpha = \omega t$. Combining these equations yields

$$e = E_m \sin \omega t \qquad \textbf{(1–14a)}$$

Similarly,

$$v = V_m \sin \omega t \qquad \textbf{(1–14b)}$$

$$i = I_m \sin \omega t \qquad \textbf{(1–14c)}$$

EXAMPLE 1–10

A 100-Hz sinusoidal voltage source has an amplitude of 150 volts. Write the equation for e as a function of time.

Solution $\omega = 2\pi f = 2\pi(100) = 628$ rad/s and $E_m = 150$ V. Thus, $e = E_m \sin \omega t = 150 \sin 628t$ V.

Equations 1–14 may be used to compute voltages or currents at any instant in time. Usually, ω is in radians per second, and thus ωt is in radians. You can work directly in radians or you can convert to degrees. For example, suppose you want to know the voltage at $t = 1.25$ ms for $e = 150 \sin 628t$ V.

Working in Rads. With your calculator in the RAD mode, $e = 150 \sin(628)(1.25 \times 10^{-3}) = 150 \sin 0.785 \text{ rad} = 106$ V.

CircuitSim 1-8

Working in Degrees. $0.785 \text{ rad} = 45°$. Thus, $e = 150 \sin 45° = 106$ V as before.

EXAMPLE 1–11

For $v = 170 \sin 2450t$, determine v at $t = 3.65$ ms and show the point on the v waveform.

Solution $\omega = 2450$ rad/s. Therefore, $\omega t = (2450)(3.65 \times 10^{-3}) = 8.943 \text{ rad} = 512.4°$. Thus, $v = 170 \sin 512.4° = 78.8$ V. Alternatively, $v = 170 \sin 8.943 \text{ rad} = 78.8$ V. The point is plotted on the waveform in Figure 1–29.

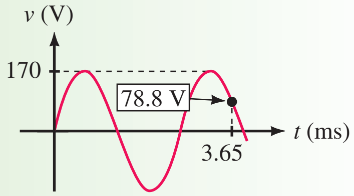


FIGURE 1–29

PRACTICE PROBLEMS 5

A sinusoidal current has a peak amplitude of 10 amps and a period of 120 ms.

a. Determine its equation as a function of time using Equation 1–14c.

b. Using this equation, compute a table of values at 10-ms intervals and plot one cycle of the waveform scaled in seconds.

c. Sketch one cycle of the waveform using the procedure of Example 1–8. (Note how much less work this is.)

Answers

a. $i = 10 \sin 52.36t$ A

c. Mark the end of the cycle as 120 ms, ½ cycle as 60 ms, ¼ cycle as 30 ms, and so on. Draw the sine wave so that it is zero at $t = 0$, 10 A at 30 ms, 0 A at 60 ms, -10 A at 90 ms, and ends at $t = 120$ ms. (See Figure 1–30.)

Determining When a Particular Value Occurs

Sometimes you need to know when a particular value of voltage or current occurs. Given $v = V_m \sin \alpha$, rewrite this as $\sin \alpha = v/V_m$. Then,

$$\alpha = \sin^{-1}\frac{v}{V_m} \qquad \textbf{(1–1)}$$

Compute the angle α at which the desired value occurs using the inverse sine function of your calculator, then determine the time from

$$t = \alpha/\omega$$

EXAMPLE 1–12

A sinusoidal current has an amplitude of 10 A and a period of 0.120 s. Determine the times at which

a. $i = 5.0$ A,
b. $i = -5$ A.

Solution

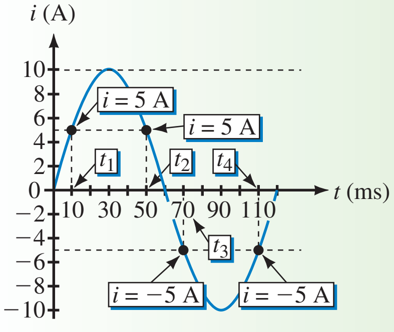


FIGURE 1–30

a. Consider Figure 1–30. As you can see, there are two points on the waveform where $i = 5$ A. Let these be denoted t_1 and t_2, respectively. First, determine ω:

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.120\ \text{s}} = 52.36\ \text{rad/s}$$

Let $i = 10 \sin \alpha$ A. Now, find the angle α_1 at which $i = 5$ A:

$$\alpha_1 = \sin^{-1}\frac{i}{I_m} = \sin^{-1}\frac{5\ \text{A}}{10\ \text{A}} = \sin^{-1}0.5 = 30° = 0.5236\ \text{rad}$$

Thus, $t_1 = \alpha_1/\omega = (0.5236\ \text{rad})/(52.36\ \text{rad/s}) = 0.01\ \text{s} = 10$ ms. This is indicated in Figure 1–30. Now consider t_2. Because of symmetry, t_2 is the same distance back from the half-cycle point as t_1 is in from the beginning of the cycle. Thus, $t_2 = 60\ \text{ms} - 10\ \text{ms} = 50$ ms.

b. Similarly, t_3 (the first point at which $i = -5$ A occurs) is 10 ms past mid-point, while t_4 is 10 ms back from the end of the cycle. Thus, $t_3 = 70$ ms and $t_4 = 110$ ms.

PRACTICE PROBLEMS 6

Given $v = 10 \sin 52.36t$, determine both occurrences of $v = -8.66$ V.

Answer
80 ms, 100 ms

Voltages and Currents with Phase Shifts

CircuitSim 1-9

If a sine wave does not pass through zero at $t = 0$ s as in Figure 1–30, it has a **phase shift.** Waveforms may be shifted to the left or to the right (see Figure 1–31). For a waveform shifted left as in (a),

$$v = V_m \sin(\omega t + \theta) \tag{1–16a}$$

while, for a waveform shifted right as in (b),

$$v = V_m \sin(\omega t - \theta) \tag{1–16b}$$

NOTES...

With equations such as 1–16(a) and (b), it is customary to express ωt in radians and θ in degrees, yielding mixed angular units (as indicated in the following examples). Although this is acceptable when the equations are written in symbolic form, you must convert both angles to the same unit (either degrees or radians) before you make numerical computations. This is illustrated in Example 1–14.

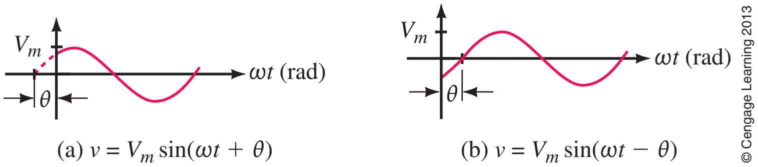


FIGURE 1–31 Waveforms with phase shifts. Angle θ is normally measured in degrees, yielding mixed angular units. (See Notes.)

EXAMPLE 1–13

Demonstrate that $v = 20 \sin(\omega t - 60°)$, where $\omega = \pi/6$ rad/s (i.e, $= 30°$/s), yields the shifted waveform shown in Figure 1–32.

Solution

1. Since ωt and 60° are both angles, $(\omega t - 60°)$ is also an angle. Let us define it as x. Then $v = 20 \sin x$, which means that the shifted wave is also sinusoidal.
2. Consider $v = \sin(\omega t - 60°)$. At $t = 0$ s, $v = 20 \sin(0 - 60°) = 20 \sin(-60°) = -17.3$ V as indicated in Figure 1–32.
3. Since $\omega = 30°$/s, it takes 2 s for ωt to reach 60°. Thus, at $t = 2$ s, $v = 20 \sin(60° - 60°) = 20 \sin 0° = 0$ V, and the waveform passes through zero at $t = 2$ s as indicated.

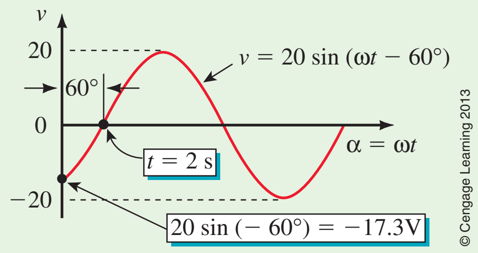


FIGURE 1–32

CircuitSim 1-10

As you can see, this example confirms that $\sin(\omega t - \theta)$ describes the wave-shape of Figure 1–31(b).

EXAMPLE 1–14

a. Determine the equation for the waveform of Figure 15–33(a), given $f = 60$ Hz. Compute current at $t = 4$ ms.

b. Repeat (a) for Figure 15–33(b).

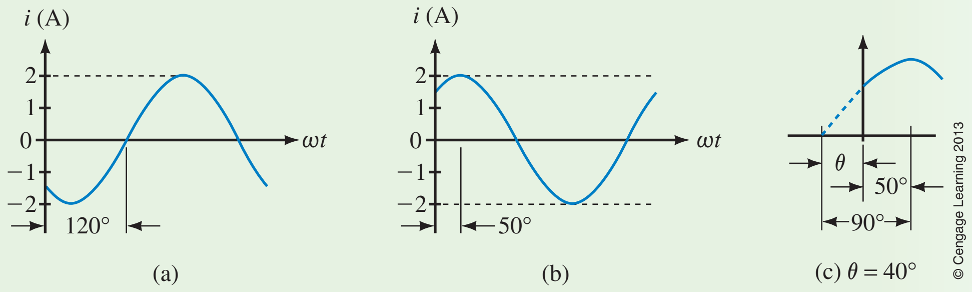


FIGURE 1–33

Solution

a. $I_m = 2$ A and $\omega = 2\pi(60) = 377$ rad/s. This waveform corresponds to Figure 1–31(b) with $\theta = 120°$. Therefore,

$$i = I_m \sin(\omega t - \theta) = 2 \sin(377t - 120°) \text{ A}$$

At $t = 4$ ms, current is

$$\begin{aligned} i &= 2 \sin(377 \times 4 \text{ ms} - 120°) = 2 \sin(1.508 \text{ rad} - 120°) \\ &= 2 \sin(86.4° - 120°) = 2 \sin(-33.64°) = -1.11 \text{ A}. \end{aligned}$$

b. This waveform matches Figure 15–31(a) as you can see if you extend the waveform back 90° from its peak as in (c). Note that $\theta = 40°$. Thus,

$$i = 2\sin(377t + 40°)\text{ A}$$

At $t = 4$ ms, current is

$$i = 2\sin(377 \times 4\text{ ms} + 40°) = 2\sin(126.4°)$$
$$= 1.61\text{ A.}$$

PRACTICE PROBLEMS 7

CircuitSim 1- 11

1. Given $i = 2\sin(377t + 60°)$, compute the current at $t = 3$ ms.
2. Sketch each of the following:
   a. $v = 10\sin(\omega t + 20°)$ V.
   b. $i = 80\sin(\omega t - 50°)$ A.
   c. $i = 50\sin(\omega t + 90°)$ A.
   d. $v = 5\sin(\omega t + 180°)$ V.
3. Given $i = 2\sin(377t + 60°)$, determine at what time $i = 1.8$ A.

Answers

1. 1.64 A
2. a. Same as Figure 15–31(a) with $V_m = 10$ V, $\theta = 20°$.
   b. Same as Figure 15–31(b) with $I_m = 80$ A, $\theta = 50°$.
   c. Same as Figure 15–39(b) except use $I_m = 50$ A instead of V_m.
   d. A negative sine wave with magnitude of 5 V.
3. 0.193 ms

Probably the easiest way to deal with shifted waveforms is to use phasors. We introduce the idea next.

1.7 Introduction to Phasors

A **phasor** is a rotating vector whose projection on a vertical axis can be used to represent sinusoidally varying quantities. To get at the idea, consider the red line of length V_m shown in Figure 1–34(a). (It is the phasor.) The vertical projection of this line (indicated in dotted red) is $V_m \sin \alpha$. Now, assume that the phasor rotates at angular velocity of ω rad/s in the counterclockwise direction. Then, $\alpha = \omega t$, and its vertical projection is $V_m \sin \omega t$. If we designate this projection (height) as v, we get $v = V_m \sin \omega t$, which is the familiar sinusoidal voltage equation.

If you plot a graph of this projection versus α, you get the sine wave of Figure 1–34(b). Figure 1–35 illustrates the graphing process—see Note.

NOTES...

There are some good animations on the Internet that show the process described here. Search for *Phasor Animation*. (Some animations are better than others. In addition, look for an animation that shows the generation of a sine wave, not a cosine wave.)

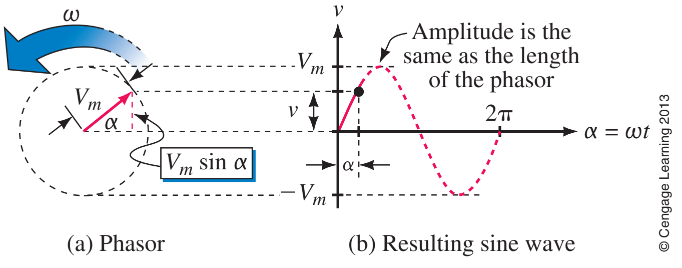


FIGURE 1–34 As the phasor rotates about the origin, its vertical projection creates a sine wave. (Figure 1–35 illustrates the process.)

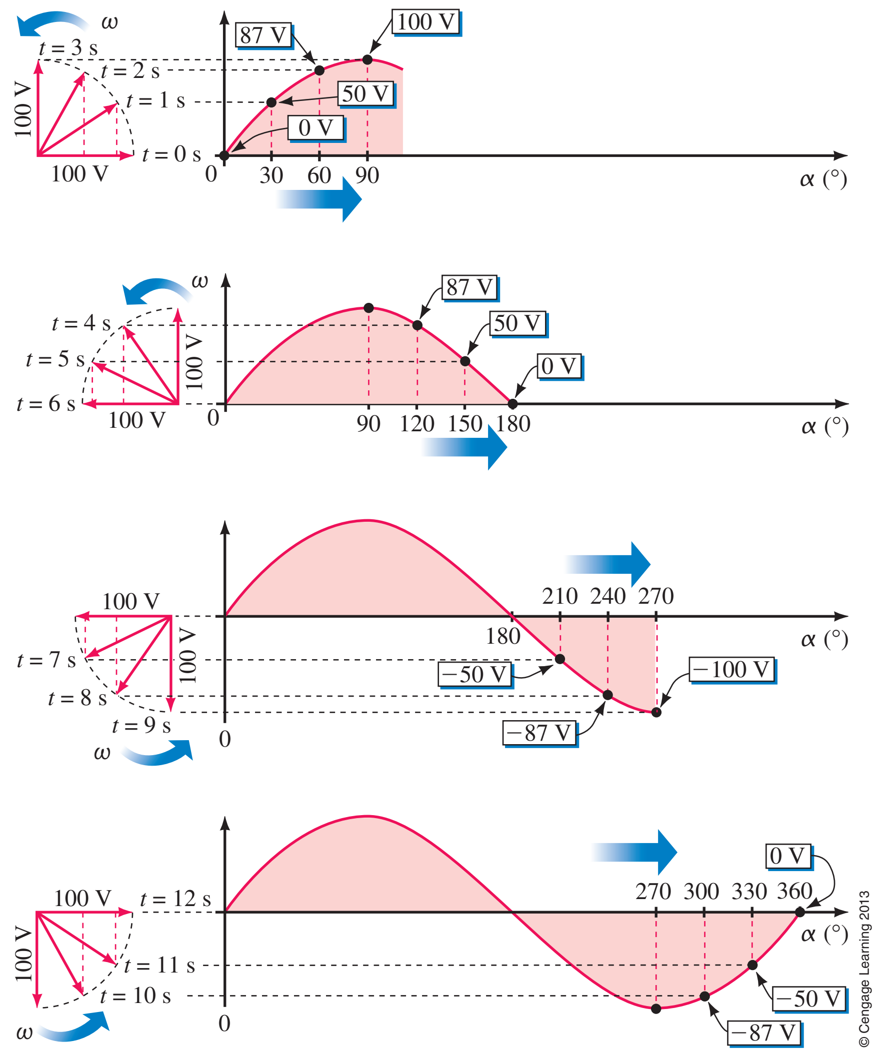


FIGURE 1–35 Evolution of the sine wave of Figure 1–34.

NOTES...

1. Although we have indicated phasor rotation in Figure 1–35 by a series of "snapshots," this is too cumbersome; in practice, we show only the phasor at its $t = 0$ s (reference) position and imply rotation rather than show it explicitly.
2. Although we are using maximum values (E_m and I_m) here, phasors are normally drawn in terms of effective (rms) values (considered in Section 1.9). For the moment, we will continue to use maximum values. We make the change to rms in Chapter 16.
3. Phasors apply only to sinusoidal waveforms.
4. In this discussion (for the sake of familiarity), we specify ω in degrees per second. Generally, however, you will use radians per second. We will make this change shortly.

It shows snapshots of the phasor and the evolving waveform at various instants of time for a phasor of magnitude $V_m = 100$ V rotating at $\omega = 30°/\text{s}$. For example, compute and plot voltage values at times $t = 0$, 1, 2, and 3 s:

1. At $t = 0$ s, $\alpha = 0$, the phasor is at its 0° position, and its vertical projection is $v = V_m \sin \omega t = 100 \sin 0° = 0$ V. The point is thus, at the origin.
2. At $t = 1$ s, the phasor has rotated 30° and its vertical projection is $v = 100 \sin 30° = 50$ V. This point is plotted at $\alpha = 30°$ on the horizontal axis.
3. At $t = 2$ s, $\alpha = 60°$, and $v = 100 \sin 60° = 87$ V, which is plotted at $\alpha = 60°$ on the horizontal axis. Similarly, at $t = 3$ s, $\alpha = 90°$, and $v = 100$ V. Continuing in this manner, the complete waveform is evolved.

From the foregoing, we conclude that *a sinusoidal waveform can be created by plotting the vertical projection of a phasor that rotates in the counterclockwise direction at constant angular velocity* ω. *If the phasor has a length of* V_m, *the waveform represents voltage; if the phasor has a length of* I_m, it represents current. Note carefully: **Phasors apply only to sinusoidal waveforms.**

EXAMPLE 1–1

Draw the phasor and waveform for current $i = 25 \sin \omega t$ mA for $f = 100$ Hz.

Solution The phasor has a length of 25 mA and is drawn at its $t = 0$ position, which is zero degrees as indicated in Figure 1–36. Since $f = 100$ Hz, the period is $T = 1/f = 10$ ms.

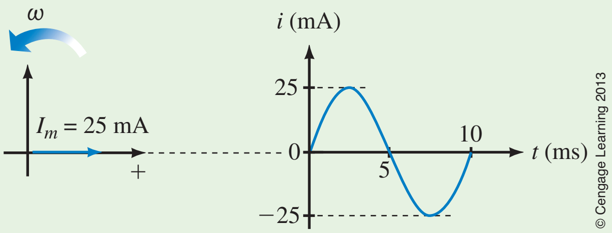
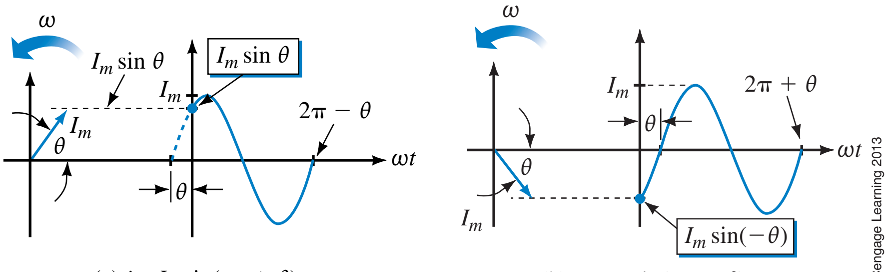


FIGURE 1–36 The reference position of the phasor is its $t = 0$ position.

Shifted Sine Waves

Phasors may be used to represent shifted waveforms, $v = V_m \sin(\omega t \pm \theta)$ or $i = I_m \sin(\omega t \pm \theta)$ as indicated in Figure 1–37. Angle θ is the position of the phasor at $t = 0$ s.

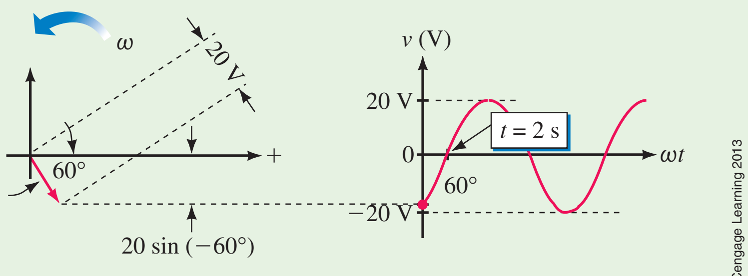


(a) $i = I_m \sin(\omega t + \theta)$ (b) $i = I_m \sin(\omega t - \theta)$

FIGURE 1–37 Phasors for shifted waveforms. Angle θ is the position of the phasor at $t = 0$ s.

EXAMPLE 1–16

Consider $v = 20 \sin(\omega t - 60°)$, where $\omega = \pi/6$ rad/s (i.e., 30°/s). Show that the phasor of Figure 1–38(a) represents this waveform.



(a) Phasor (b) $v = 20 \sin(\omega t - 60°)$, with $\omega = 30°$/s

FIGURE 1–38

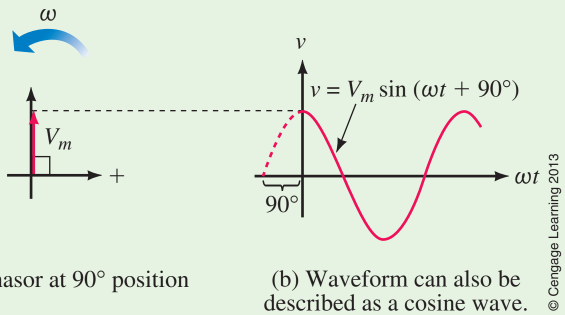
Solution The phasor has length 20 V and at time $t = 0$ is at –60° as indicated in (a). Now, as the phasor rotates, it generates a sinusoidal waveform, oscillating between ±20 V as indicated in (b). Note that the zero crossover point occurs at $t = 2$ s, since it takes 2 seconds for the phasor to rotate from –60° to 0° at 30 degrees per second. Now compare the waveform of (b) to the waveform of Figure 1–32, Example 1–13. They are identical. Thus, the phasor of (a) represents the shifted waveform $v = 20 \sin(\omega t - 60°)$.

EXAMPLE 1–17

With the aid of a phasor, sketch the waveform for $v = V_m \sin(\omega t + 90°)$.

Solution Place the phasor at 90° as in Figure 1–39(a). Note that the resultant waveform (b) is a cosine waveform, that is, $v = V_m \cos \omega t$. From this, we conclude that

$$\sin(\omega t + 90°) = \cos \omega t$$



(a) Phasor at 90° position

(b) Waveform can also be described as a cosine wave.

FIGURE 1–39 Demonstrating that $\sin(\omega t + 90°) = \cos \omega t$.

PRACTICE PROBLEMS 8

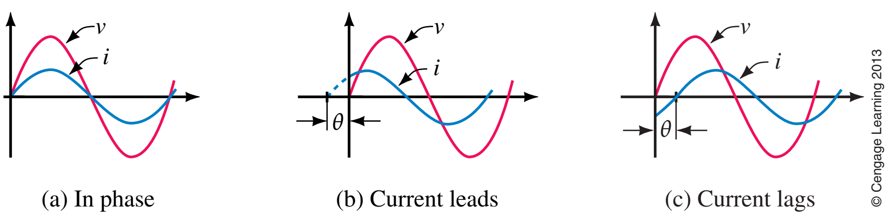
With the aid of phasors, show that

a. $\sin(\omega t - 90°) = -\cos \omega t$,
b. $\sin(\omega t \pm 180°) = -\sin \omega t$

Phase Difference

Phase difference refers to the angular displacement between different waveforms of the same frequency. Consider Figure 1–40. If the angular displacement is 0° as in (a), the waveforms are said to be in phase; otherwise, they are out of phase. When describing a phase difference, select one waveform as reference. Other waveforms then lead, lag, or are in phase with this reference. For example, in (b), for reasons to be discussed in the next paragraph, the current waveform is said to lead the voltage waveform, while in (c) the current waveform is said to lag.

CircuitSim 1-12



(a) In phase

(b) Current leads

(c) Current lags

FIGURE 1–40 Illustrating phase difference. In these examples, voltage is taken as reference.

NOTES...

1. To determine which waveform leads and which lags, make a quick sketch of their phasors, and the answer will be apparent. Note also that the terms *lead* and *lag* are relative. In Figure 1–41,we said that current leads volt-age; you can just as correctly say that voltage lags current.

2. When you plot two phasors as in Figure 1–43(a), the angle between them is their phase difference.

The terms **lead** and **lag** can be understood in terms of phasors. If you observe phasors rotating as in Figure 1–41(a), the one that you see passing first is leading and the other is lagging. By definition, *the waveform generated by the leading phasor leads the waveform generated by the lagging phasor and vice versa.* In Figure 1–41, phasor I_m leads phasor V_m; thus current $i(t)$ leads voltage $v(t)$.

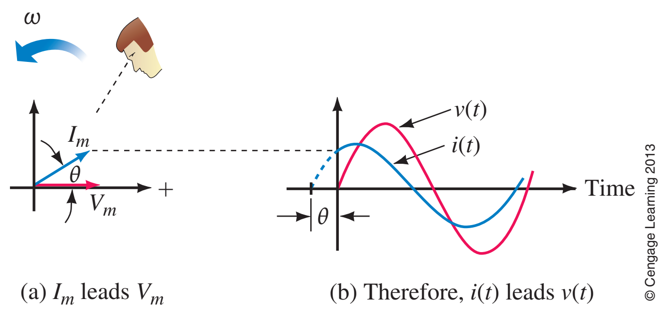


FIGURE 1–41 Defining lead and lag.

EXAMPLE 1–18

Voltage and current are out of phase by 40°, and voltage lags. Using current as the reference, sketch the phasor diagram and the corresponding waveforms.

Solution Since current is the reference, place its phasor in the 0° position and the voltage phasor at –40°. Figure 1–42 shows the phasors and corresponding waveforms.

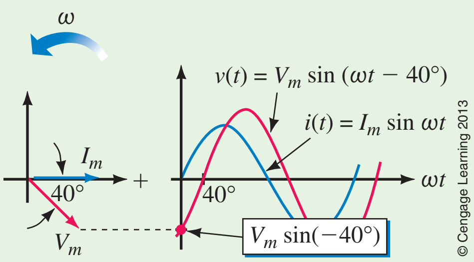


FIGURE 1–42

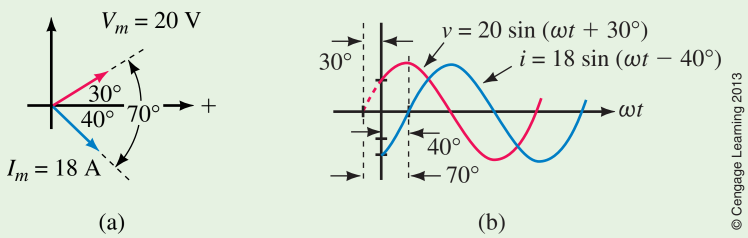
EXAMPLE 1–19

NOTES...

In all phasor diagrams to this point, we have used a stylized arrow to imply phasor rotation. In practice, however, this is not usually done—thus, from this point onward, we will omit it.

Given $v = 20 \sin(\omega t + 30°)$ and $i = 18 \sin(\omega t - 40°)$, draw the phasor diagram, determine phase relationships, and sketch the waveforms—see Note.

Solution The phasors are shown in Figure 1–43(a). From these, you can see that v leads i by 70°. The waveforms are shown in (b).



CircuitSim 15-13

FIGURE 1–43

EXAMPLE 1–20

Figure 1–44 shows a pair of waveforms v_1 and v_2 on an oscilloscope. Each major vertical division represents 20 V and each major division on the horizontal (time) scale represents 20 μs. Voltage v_1 leads. Prepare a phasor diagram using v_1 as reference. Determine equations for both voltages.

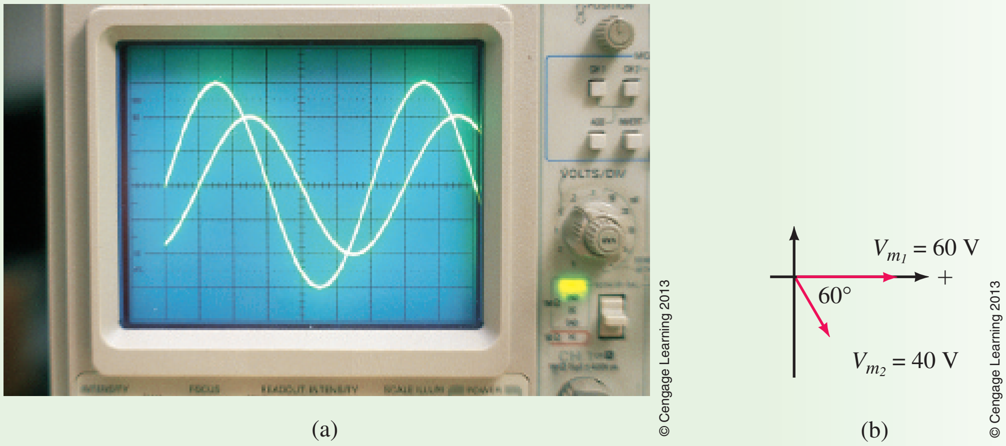


FIGURE 1–44

Solution From the photograph, the magnitude of v_1 is V_{m_1} = 3 div × 20 V/div = 60 V. Similarly, V_{m_2} = 40 V. Cycle length is $T = 6 \times 20$ μs = 120 μs, and the displacement between waveforms is 20 μs, which is ⅙ of a cycle (i.e., 60°). Selecting v_1 as reference and noting that v_2 lags yields the phasors shown in (b). Angular frequency $\omega = 2\pi/T = 2\pi/(120 \times 10^{-6}\text{ s}) = 52.36 \times 10^3$ rad/s. Thus, $v_1 = V_{m_1} \sin \omega t = 60 \sin(52.36 \times 10^3 t)$ V and $v_2 = 40 \sin(52.36 \times 10^3 t - 60°)$ V.

Sometimes voltages and currents are expressed in terms of cos ωt rather than sin ωt. As Example 1–17 shows, a cosine wave is a sine wave shifted by +90°, or alternatively, a sine wave is a cosine wave shifted by −90°. For sines or cosines with an angle, the following formulas apply.

$$\cos(\omega t + \theta) = \sin(\omega t + \theta + 90°) \tag{1–17a}$$

$$\sin(\omega t + \theta) = \cos(\omega t + \theta - 90°) \tag{1–17b}$$

To illustrate, consider $\cos(\omega t + 30°)$. From Equation 1–17a, $\cos(\omega t + 30°) = \sin(\omega t + 30° + 90°) = \sin(\omega t + 120°)$. Figure 1–45 illustrates this relationship graphically. The red phasor in (a) generates cos ωt as was shown in Example 1–17. Therefore, the green phasor generates a waveform that leads it by 30°, namely $\cos(\omega t + 30°)$. For (b), the red phasor generates sin ωt, and the green phasor generates a waveform that leads it by 120°, that is, $\sin(\omega t + 120°)$. Since the green phasor is the same in both cases, you can see that $\cos(\omega t + 30°) = \sin(\omega t + 120°)$. You may find this process easier to apply than trying to remember Equations 1–17(a) and (b).

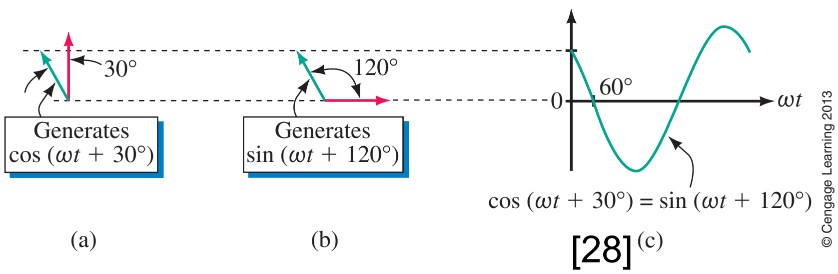


FIGURE 1–45 Using phasors to show that $\cos(\omega t + 30°) = \sin(\omega t + 120°)$.

EXAMPLE 1–21

Determine the phase angle between $v = 30\cos(\omega t + 20°)$ and $i = 25\sin(\omega t + 70°)$.

Solution $i = 25\sin(\omega t + 70°)$ may be represented by a phasor at 70°, and $v = 30\cos(\omega t + 20°)$ by a phasor at $(90° + 20°) = 110°$, Figure 1–46(a). Thus, v leads i by 40°. Waveforms are shown in (b).

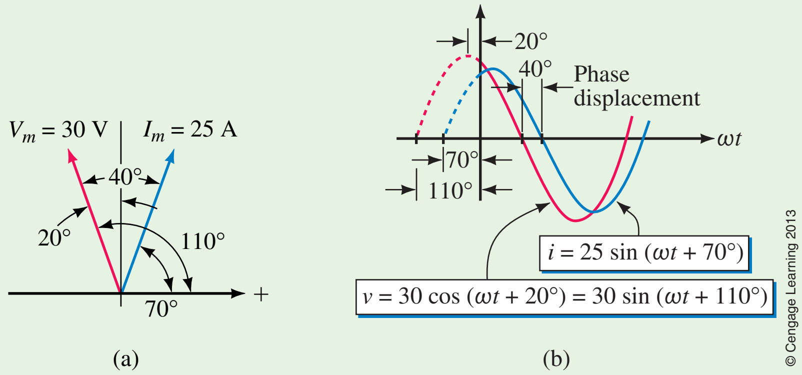


FIGURE 1–46

Sometimes you encounter negative waveforms such as $i = -I_m \sin \omega t$. To see how to handle these, refer back to Figure 1–36, which shows the waveform and phasor for $i = I_m \sin \omega t$. If you multiply this waveform by -1, you get the inverted waveform $-I_m \sin \omega t$ of Figure 1–47(a) with corresponding phasor (b). Note that the phasor is the same as the original phasor except that it is rotated by 180°. This is always true—thus, if you multiply a waveform by -1, the phasor for the new waveform is 180° rotated from the original phasor, regardless of the angle of the original phasor.

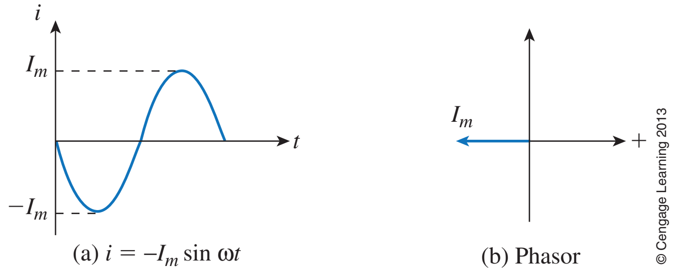


FIGURE 1–47 For a negative sine wave, the phasor is at 180°.

EXAMPLE 1–22

Find the phase relationship between $i = -4\sin(\omega t + 50°)$ and $v = 120\sin(\omega t - 60°)$.

Solution $i = -4\sin(\omega t + 50°)$ is represented by a phasor at $(50° - 180°) = -130°$ and $v = 120\sin(\omega t - 60°)$ by a phasor at $-60°$, Figure 1–48. The phase difference is 70° and voltage leads. From this, you can see that i can also be written as $i = 4\sin(\omega t - 130°)$.

Alternatively, you can add the 180° instead of subtracting it. This yields $50° + 180° = 230°$, which is the same as the $-130°$ that we obtained above.

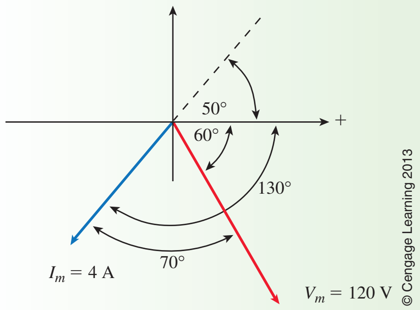


FIGURE 1 –48

The importance of phasors to ac circuit analysis cannot be overstated—you will find that they are one of your main tools for representing ideas and for solving problems in later chapters and in practice. (By the time you reach Section 18.2 of Chapter 18, analysis is done exclusively with phasors.) We will leave them for the moment, but pick them up again in Chapter 16.

IN-PROCESS LEARNING CHECK 2

(Answers are at the end of the chapter.)

1. If $i = 1 \sin \alpha$ mA, compute the current at $\alpha = 0°, 45°, 90°, 135°, 180°, 225°, 270°, 315°$, and $360°$.
2. Convert the following angles to radians:
   a. 20° c. 120°
   b. 50° d. 250°
3. If a coil rotates at $\omega = \pi/60$ radians per millisecond, how many degrees does it rotate through in 10 ms? In 40 ms? In 150 ms?
4. A current has an amplitude of 50 mA and $\omega = 0.2\pi$ rad/s. Sketch the waveform with the horizontal axis scaled in
   a. degrees b. radians c. seconds
5. If 2400 cycles of a waveform occur in 10 ms, what is ω in radians per second?
6. A sinusoidal current has a period of 40 ms and an amplitude of 8 A. Write its equation in the form of $i = I_m \sin \omega t$, with numerical values for I_m and ω.
7. A current $i = I_m \sin \omega t$ has a period of 90 ms. If $i = 3$ A at $t = 7.5$ ms, what is its equation?
8. Write equations for each of the waveforms in Figure 1–49 with the phase angle θ expressed in degrees and ω in rad/s.
9. Given $i = 10 \sin \omega t$, where $f = 50$ Hz, find all occurrences of
   a. $i = 8$ A between $t = 0$ and $t = 40$ ms
   b. $i = -5$ A between $t = 0$ and $t = 40$ ms
10. Sketch the following waveforms with the horizontal axis scaled in degrees:
   a. $v_1 = 80 \sin(\omega t + 45°)$ V c. $i_1 = 10 \cos \omega t$ mA
   b. $v_2 = 40 \sin(\omega t - 80°)$ V d. $i_2 = 5 \cos(\omega t - 20°)$ mA
11. Given $\omega = \pi/3$ rad/s, determine when voltage first crosses through 0 for
   a. $v_1 = 80 \sin(\omega t + 45°)$ V b. $v_2 = 40 \sin(\omega t - 80°)$ V
12. Consider the voltages of Question 10:
   a. Sketch phasors for v_1 and v_2.
   b. What is the phase difference between v_1 and v_2?
   c. Determine which voltage leads and which lags.
13. Repeat Question 12 for the currents of Question 10.

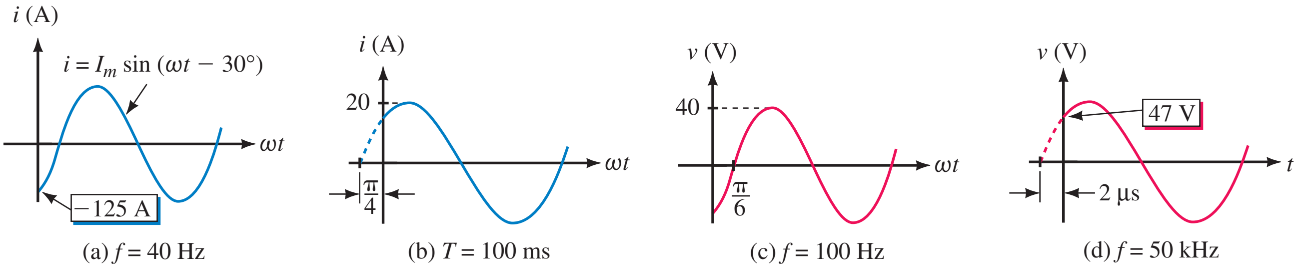


FIGURE 1–49

1.8 ac Waveforms and Average Value

While we can describe ac quantities in terms of frequency, period, instantaneous value, and other attributes, we do not yet have any way to give a meaningful value to an ac current or voltage in the same sense that we can say of a car battery that it has a voltage of 12 volts. This is because ac quantities constantly change, and thus there is no single numerical value that truly represents a waveform over its complete cycle. For this reason, ac quantities are generally described by a group of characteristics, including instantaneous, peak, average, and effective values. The first two of these we have already seen. In this section, we look at **average values;** in Section 1.9, we consider effective values.

Average Values

Many quantities are measured by their average, for instance, test and examination scores. To find the average of a set of marks for example, you add them, then divide by the number of items summed. For waveforms, the process is conceptually the same. For example, to find the average of a waveform, you can sum the instantaneous values over a full cycle, then divide by the number of points used. The trouble with this approach is that waveforms do not consist of discrete values.

Average in Terms of the Area under a Curve

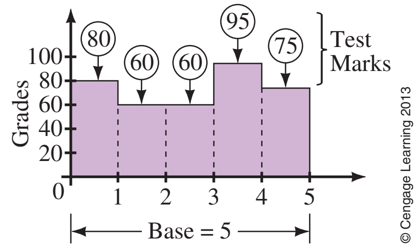


FIGURE 1–50 Determining average by area.

An approach more suitable for use with waveforms is to find the area under the curve, then divide by the baseline of the curve. To get at the idea, we can use an analogy. Consider again the technique of computing the average for a set of numbers. Assume that you earn marks of 80, 60, 60, 95, and 75 on a group of tests. Your average mark is therefore

$$\text{average} = (80 + 60 + 60 + 95 + 75)/5 = 74$$

An alternate way to view these marks is graphically as in Figure 1–50. The area under this curve can be computed as

$$\text{area} = (80 \times 1) + (60 \times 2) + (95 \times 1) + (75 \times 1)$$

Now divide this by the length of the base, namely 5. Thus,

$$\frac{(80 \times 1) + (60 \times 2) + (95 \times 1) + (75 \times 1)}{5} = 74$$

which is exactly the answer obtained above. That is,

$$\text{average} = \frac{\text{area under curve}}{\text{length of base}} \qquad \textbf{(1–18)}$$

This result is true in general. Thus, *to find the average value of a waveform, divide the area under the waveform by the length of its base. Areas above the axis are counted as positive, while areas below the axis are counted as negative.* This approach is valid regardless of waveshape.

Average values are also called dc values, because dc meters indicate average values rather than instantaneous values. Thus, if you measure a non-dc quantity with a dc meter, the meter will read the average of the waveform, that is, the value calculated according to Equation 1–18.

EXAMPLE 1–23

a. Compute the average for the current waveform of Figure 15–51.

b. If the negative portion of Figure 15–51 is −3 A instead of −1.5 A, what is the average?

c. If the current is measured by a dc ammeter, what will the ammeter indicate for each case?

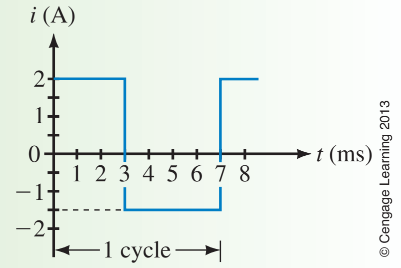


FIGURE 1 –51

Solution

a. The waveform repeats itself after 7 ms. Thus, $T = 7$ ms and the average is

$$I_{avg} = \frac{(2\text{ A} \times 3\text{ ms}) - (1.5\text{ A} \times 4\text{ ms})}{7\text{ ms}} = \frac{6-6}{7} = 0\text{ A}$$

b. $$I_{avg} = \frac{(2\text{ A} \times 3\text{ ms}) - (3\text{ A} \times 4\text{ ms})}{7\text{ ms}} = \frac{-6\text{ A}}{7} = -0.857\text{ A}$$

c. A dc ammeter measuring (a) will indicate zero, while for (b) it will indicate −0.857 A.

EXAMPLE 1–24

Compute the average value for the waveforms of Figures 1–52(a) and (c). Sketch the averages for each.

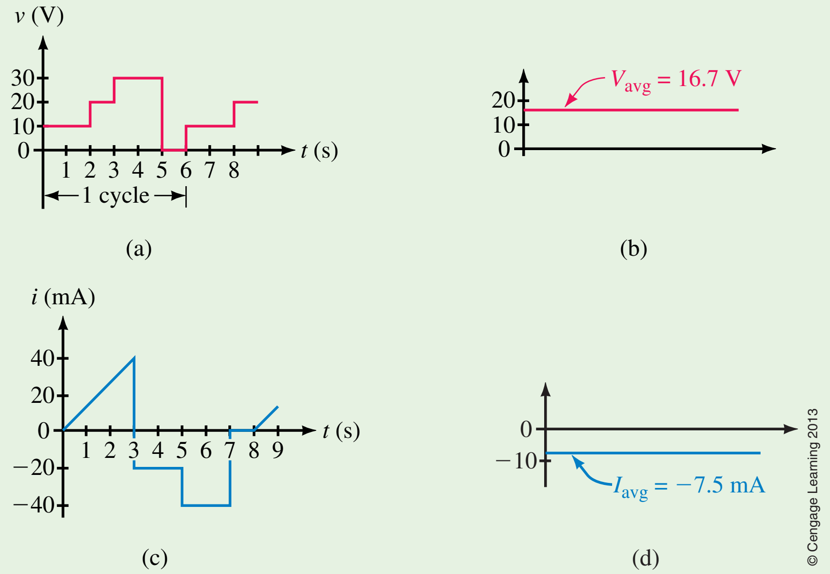


FIGURE 1–52

Solution For the waveform of (a), $T = 6$ s. Thus,

$$V_{avg} = \frac{(10\text{V} \times 2\text{ s}) + (20\text{V} \times 1\text{ s}) + (30\text{V} \times 2\text{ s}) + (0\text{V} \times 1\text{ s})}{6\text{ s}} = \frac{100\text{ V-s}}{6\text{ s}} = 16.7\text{V}$$

CircuitSim 15-14

The average is shown as (b). For the waveform of (c), $T = 8$ s and

$$I_{avg} = \frac{\frac{1}{2}(40\text{ mA} \times 3\text{ s}) - (20\text{ mA} \times 2\text{ s}) - (40\text{ mA} \times 2\text{ s})}{8\text{ s}} = \frac{-60}{8}\text{ mA} = -7.5\text{ mA}$$

PRACTICE PROBLEMS 9

Determine the averages for Figures 1–53(a) and (b).

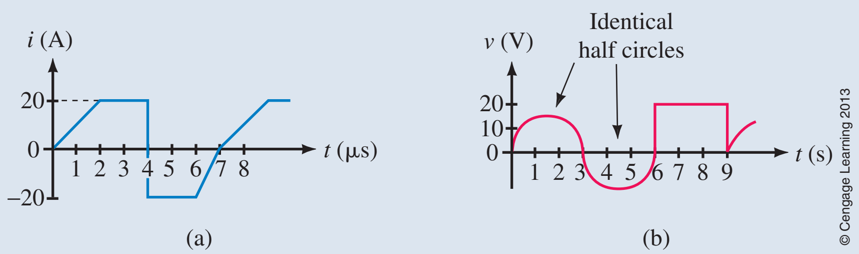


FIGURE 1–53

Answers
a. 1.43 A; b. 6.67 V

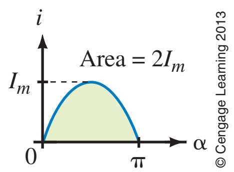


FIGURE 1–54 Area under a half-cycle.

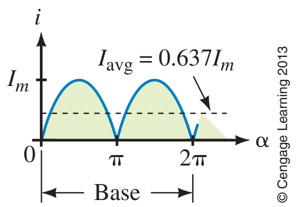


FIGURE 1–55 Full-wave average.

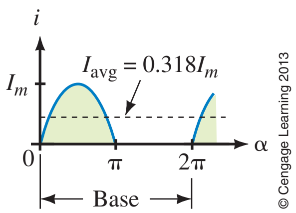


FIGURE 1–56 Half-wave average.

Sine Wave Averages

Because a sine wave is symmetrical, its area below the horizontal axis is the same as its area above the axis; thus, over a full cycle its net area is zero, independent of frequency and phase angle. Thus, the average of $\sin \omega t$, $\sin(\omega t \pm \theta)$, $\sin 2\omega t$, $\cos \omega t$, $\cos(\omega t \pm \theta)$, $\cos 2\omega t$, and so on are each zero. The average of half a sine wave, however, is not zero. Consider Figure 1–54. The area under the half-cycle may be found using calculus as

$$\text{area} = \int_0^{\pi} I_m \sin\alpha \, d\alpha = \Big[-I_m \cos \alpha\Big]_0^{\pi} = 2I_m \qquad \textbf{(1–19)}$$

Similarly, the area under a half-cycle of voltage is $2V_m$. (If you haven't studied calculus, you can approximate this area using numerical methods as described later in this section.)

Two cases are important in electronics; full-wave average and half-wave average. The full-wave case is illustrated in Figure 1–55. The area from 0 to 2π is $2(2I_m)$ and the base is 2π. Thus, the average is

$$I_{avg} = \frac{2(2I_m)}{2\,\pi} = \frac{2I_m}{\pi} = 0.637I_m$$

For the half-wave case (Figure 1–56),

$$I_{avg} = \frac{2I_m}{2\,\pi} = \frac{I_m}{\pi} = 0.318I_m$$

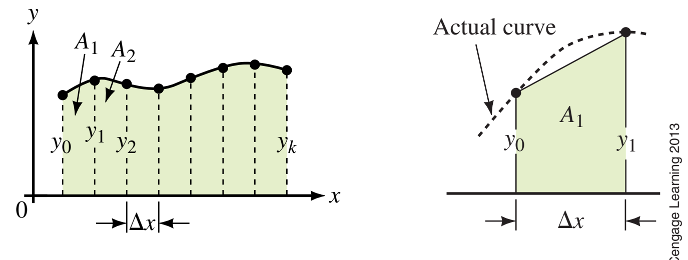
The corresponding expressions for voltage are

$$V_{avg} = 0.637V_m \text{ (full-wave)}$$

$$V_{avg} = 0.318V_m \text{ (half-wave)}$$

Numerical Methods

If the area under a curve cannot be computed exactly, it can be approximated. One method is to approximate the curve by straight line segments as in Figure 1–57. (If the straight lines closely fit the curve, the accuracy is very good.) Each element of area is a trapezoid (b) whose area is its average height



(a) Approximating the curve (b) Element of area

FIGURE 1–57 Determining area using the trapezoidal rule.

times its base. Thus, $A_1 = ½(y_0 + y_1)\Delta x$, $A_2 = ½(y_1 + y_2)\Delta x$, and so on. Total area is $A_1 + A_2 + \cdots + A_k = [½(y_0 + y_1) + ½(y_1 + y_2) + \cdots]\Delta x$ and so on. Combining terms yields

$$\text{area} = \left(\frac{y_0}{2} + y_1 + y_2 + \cdots + y_{k-1} + \frac{y_k}{2}\right)\Delta x \qquad \textbf{(1–20)}$$

This result is known as the trapezoidal rule. Example 1–25 illustrates its use.

EXAMPLE 1–25

Approximate the area under $y = \sin(\omega t - 30°)$, Figure 1–58. Use an increment size of $\pi/6$ rad, that is, 30°.

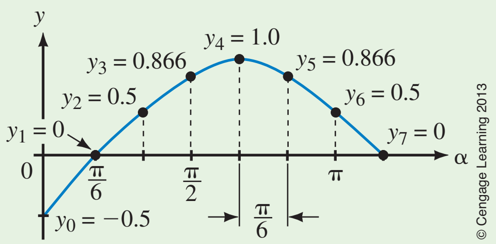


FIGURE 1–58

Solution Points on the curve $\sin(\omega t - 30°)$ have been computed by calculator and plotted as Figure 1–58. Substituting these values into Equation 1–20 yields

$$\text{area} = \left(\frac{1}{2}(-0.5) + 0 + 0.5 + 0.866 + 1.0 + 0.866 + 0.5 + \frac{1}{2}(0)\right)\left(\frac{\pi}{6}\right) = 1.823$$

The exact area (found using calculus) is 1.866; thus, the approximation of Example 1–25 is in error by 2.3%.

PRACTICE PROBLEMS 10

1. Repeat Example 15–25 using an increment size of $\pi/12$ rad. What is the percent error?
2. Approximate the area under $v = 50 \sin(\omega t + 30°)$ from $\omega t = 0°$ to $\omega t = 210°$. Use an increment size of $\pi/12$ rad.

Answers

1. 1.855; 0.59%
2. 67.9 (exact 68.3; error = 0.6%)

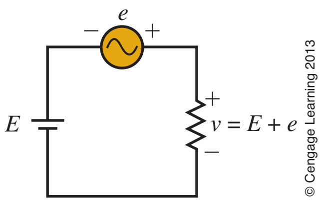
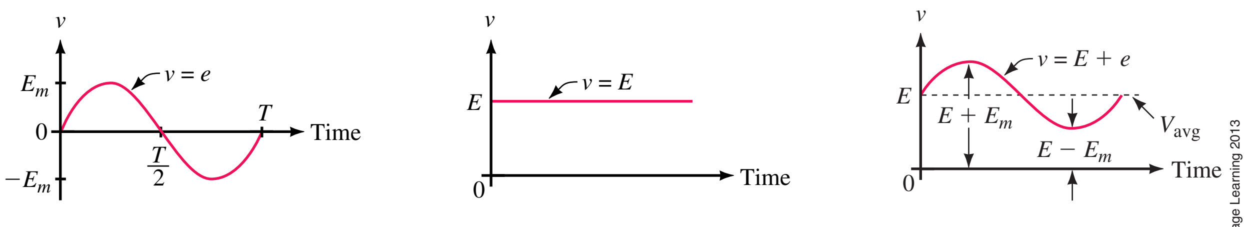


FIGURE 1–59

Superimposed ac and dc

Sometimes ac and dc are used in the same circuit. For example, amplifiers are powered by dc, but the signals they amplify are ac. Figure 1–59 shows a simple circuit with combined ac and dc.

Figure 1–60(c) shows superimposed ac and dc. Since we know that the average of a sine wave is zero, the average value of the combined waveform will be its dc component, E. However, peak voltages depend on both components as illustrated in (c). Note for the case illustrated that although the waveform varies sinusoidally, it does not alternate in polarity since it never changes polarity to become negative.



(a) AC alone. $E = 0$ V. $V_{avg} = 0$ V

(b) DC alone. $e = 0$ V. $V_{avg} = E$

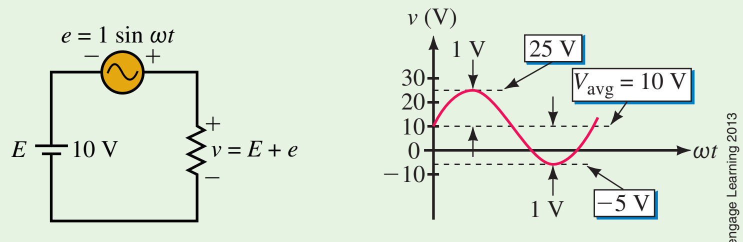
(c) Superimposed ac and dc. $V_{avg} = E$

FIGURE 1–60 Superimposed dc and ac.

EXAMPLE 1–26

Draw the waveform for voltage v for the circuit of Figure 1–61(a). Determine its average, peak, and trough voltages.

CircuitSim 15-15



(a) (b)

FIGURE 1–61 $v = 10 + 1 \sin \omega t$.

Solution The waveform consists of a 10-V dc value with 1-V ac riding on top of it. The average is the dc value, $V_{avg} = 10$ V. The peak voltage is $10 + 1 = 25$ V, while the trough voltage is $10 - 1 = -5$ V. This waveform alternates in polarity, although not symmetrically (as is the case when there is no dc component).

PRACTICE PROBLEMS 11

Repeat Example 1–26 if the dc source of Figure 1–61 is $E = -5$ V.

Answers

$V_{avg} = -5$ V; positive peak $= 10$ V; negative trough $= -20$ V

1.9 Effective (RMS) Values

While instantaneous, peak, and average values provide useful information about a waveform, none of them truly represents the ability of the waveform to do useful work. In this section, we look at a representation that does. It is called the waveform's **effective value.** The concept of effective value is an important one; in practice, most ac voltages and currents are expressed as effective values. Effective values are also called **rms values** for reasons discussed shortly.

What Is an Effective Value?

An effective value is an equivalent dc value: it tells you how many volts or amps of dc that a time-varying waveform is equal to in terms of its ability to produce average power. Effective values depend on the waveform. A familiar example of such a value is the value of the voltage at the wall outlet in your home. In North America its value is 120 Vac. This means that the sinusoidal voltage at the wall outlets of your home is capable of producing the same average power as 120 volts of steady dc.

Effective Values for Sine Waves

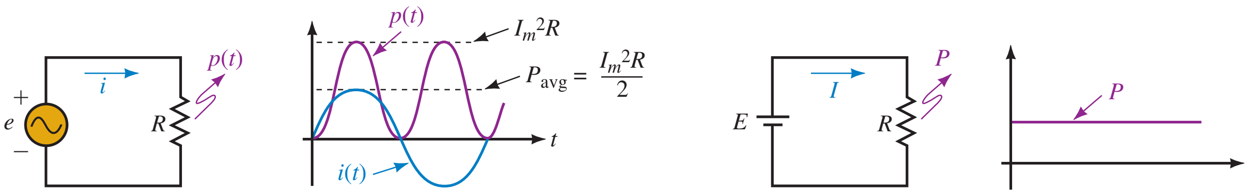
The effective value of a waveform can be determined using the circuits of Figure 1–62. Consider a sinusoidally varying current, $i(t)$. By definition, the effective value of i is that value of dc current that produces the same average power. Consider (b). Let the dc source be adjusted until its average power is the same as the average power in (a). The resulting dc current is then the effective value of the current of (a). To determine this value, determine the average power for both cases, then equate them.

First, consider the dc case. Since current is constant, power is constant, and average power is

$$P_{\text{avg}} = P = I^2R \qquad \textbf{(1–21)}$$

Now consider the ac case. Power to the resistor at any value of time is $p(t) = i^2R$, where i is the instantaneous value of current. A sketch of $p(t)$ is shown in Figure 1–62(a), obtained by squaring values of current at various points along the axis, then multiplying by R. Average power is the average of $p(t)$. Since $i = I_m \sin \omega t$,

$$\begin{aligned} p(t) &= i^2R \\ &= (I_m \sin \omega t)^2R = I_m{}^2R \sin^2\omega t \\ &= I_m{}^2R\left[\frac{1}{2}(1 - \cos 2\omega t)\right] \end{aligned} \qquad \textbf{(1–22)}$$



$p(t) = i^2R$. Therefore, $p(t)$ varies cyclically.

(a) ac Circuit

$P = I^2R$. Therefore, P is constant.

(b) dc Circuit

FIGURE 1–62 Determining the effective value of sinusoidal ac.

NOTES...

Because ac currents alternate in direction, you might expect average power to be zero, with power during the negative half-cycle being equal and opposite to power during the positive half-cycle and hence canceling. However this is not true since, as Equation 1–22 shows, current is squared, and hencepower is never negative. This is con-sistent with the idea that insofar as power dissipation is concerned, the direction of current through a resistor does not matter (Figure 1–63).

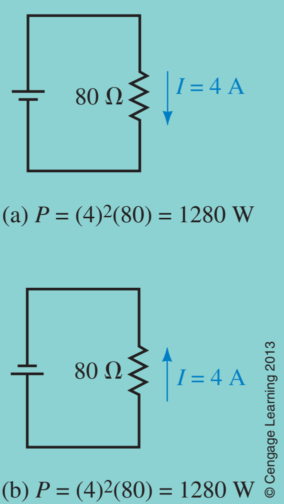


FIGURE 1–63 Since power depends only on current magnitude, it is the same for both current directions.

CircuitSim 1-16

where we have used the trigonometric identity $\sin^2 \omega t = ½(1 - \cos 2\omega t)$, from the mathematics tables inside the front cover of this book. Thus,

$$p(t) = \frac{I_m^2 R}{2} - \frac{I_m^2 R}{2}\cos 2\omega t \qquad \textbf{(1–23)}$$

To get the average of $p(t)$, note that the average of $\cos 2\omega t$ is zero and thus the last term of Equation 1–23 drops off, leaving

$$P_{avg} = \text{average of } p(t) = \frac{I_m^2 R}{2} \qquad \textbf{(1–24)}$$

Now equate Equations 1–21 and 1–24, then cancel R.

$$I^2 = \frac{I_m^2}{2}$$

Now take the square root of both sides. Thus,

$$I = \sqrt{\frac{I_m^2}{2}} = \frac{I_m}{\sqrt{2}} = 0.707 I_m$$

Current I is the value that we are looking for; it is the effective value of current i. To emphasize that it is an effective value, we will initially use subscripted notation I_{eff}. Thus,

$$I_{eff} = \frac{I_m}{\sqrt{2}} = 0.707 I_m \qquad \textbf{(1–25)}$$

Effective values for voltage are found in the same way:

$$E_{eff} = \frac{E_m}{\sqrt{2}} = 0.707 E_m \qquad \textbf{(1–26a)}$$

$$V_{eff} = \frac{V_m}{\sqrt{2}} = 0.707 V_m \qquad \textbf{(1–26b)}$$

As you can see, *effective values for sinusoidal waveforms depend only on magnitude.*

EXAMPLE 1–27

Determine the effective values of

a. $i = 10 \sin \omega t$ A
b. $i = 50 \sin(\omega t + 20°)$ mA
c. $v = 100 \cos 2\omega t$ V

Solution Since effective values depend only on magnitude,

a. $I_{eff} = (0.707)(10\text{ A}) = 7.07$ A
b. $I_{eff} = (0.707)(50\text{ mA}) = 35.35$ mA
c. $V_{eff} = (0.707)(100\text{ V}) = 70.7$ V

To obtain peak values from effective values, rewrite Equations 1–25 and15–26. Thus,

$$I_m = \sqrt{2}I_{\text{eff}} = 1.414I_{\text{eff}} \tag{1–27}$$

$$E_m = \sqrt{2}E_{\text{eff}} = 1.414V_{\text{eff}} \tag{1–28a}$$

$$V_m = \sqrt{2}V_{\text{eff}} = 1.414V_{\text{eff}} \tag{1–28b}$$

It is important to note that these relationships hold only for sinusoidal waveforms. However, the concept of effective value applies to all waveforms, as we shall soon see.

Consider again the ac voltage at the wall outlet in your home. Since $E_{\text{eff}} = 120$ V, $E_m = (\sqrt{2})(120\text{ V}) = 170$ V. This means that a sinusoidal voltage alternating between ± 170 V produces the same average power in a resistive circuit as 120 V of steady dc (Figure 1–64).

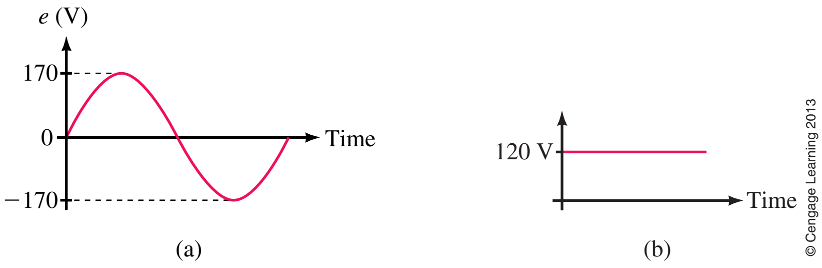


FIGURE 1–64 120 V of steady dc is capable of producing the same average power as sinusoidal ac with $E_m = 170$ V.

General Equation for Effective Values

The $\sqrt{2}$ relationship holds only for sinusoidal waveforms. For other waveforms, you need a more general formula. Using calculus, it can be shown that for any waveform

$$I_{\text{eff}} = \sqrt{\frac{1}{T}\int_0^T i^2 dt} \tag{1–29}$$

with a similar equation for voltage. This equation can be used to compute effective values for any waveform, including sinusoidal. In addition, it leads to a graphic approach to finding effective values. In Equation 1–29, the integral of i^2 represents the area under the i^2 waveform. Thus,

$$I_{\text{eff}} = \sqrt{\frac{\text{area under the } i^2 \text{ curve}}{\text{base}}} \tag{1–30}$$

To compute effective values using this equation, do the following:

Step 1: Square the current (or voltage) curve.

Step 2: Find the area under the squared curve.

Step 3: Divide the area by the length of the curve.

Step 4: Find the square root of the value from Step 3.

This process is easily carried out for rectangular-shaped waveforms since the area under their squared curves is easy to compute. For other waveforms, you have to use calculus or approximate the area using numerical methods as in end-of-chapter Problem 62. For the special case of superimposed ac and dc (Figure 1–60), Equation 1–29 leads to the following formula:

$$I_{\text{eff}} = \sqrt{I_{\text{dc}}^2 + I_{\text{ac}}^2} \tag{1–31}$$

where I_{dc} is the dc current value, I_{ac} is the effective value of the ac component, and I_{eff} is the effective value of the combined ac and dc currents. Equations 1–30 and 1–31 also hold for voltage when V is substituted for I.

RMS Values

Consider again Equation 1–30. To use this equation, we compute the root of the mean square to obtain the effective value. For this reason, effective values are called **root mean square** or **rms values,** and **the terms *effective* and *rms* are synonymous.** Since, in practice, sinusoidal ac quantities are almost always expressed and measured as rms values, we shall assume from here on that, unless otherwise noted, *all sinusoidal ac voltages and currents are rms values.*

EXAMPLE 1–28

One cycle of a voltage waveform is shown in Figure 1–65(a). Determine its effective (rms) value.

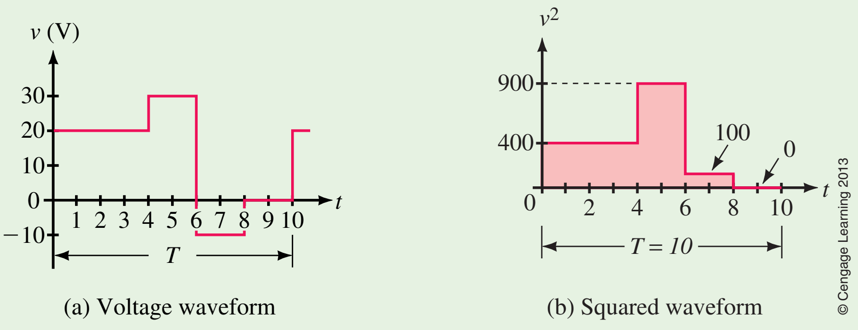


FIGURE 1–65

Solution Square the voltage waveform point by point, and plot it as in (b). Apply Equation 1–30:

$$V_{\text{eff}} = \sqrt{\frac{(400 \times 4) + (900 \times 2) + (100 \times 2) + (0 \times 2)}{10}}$$

$$= \sqrt{\frac{3600}{10}} = 19.0 \text{ V}$$

Thus, the waveform of Figure 1–65(a) has the same effective value as 19.0 V of steady dc.

EXAMPLE 1–29

Determine the effective (rms) value of the waveform of Figure 1–66(a).

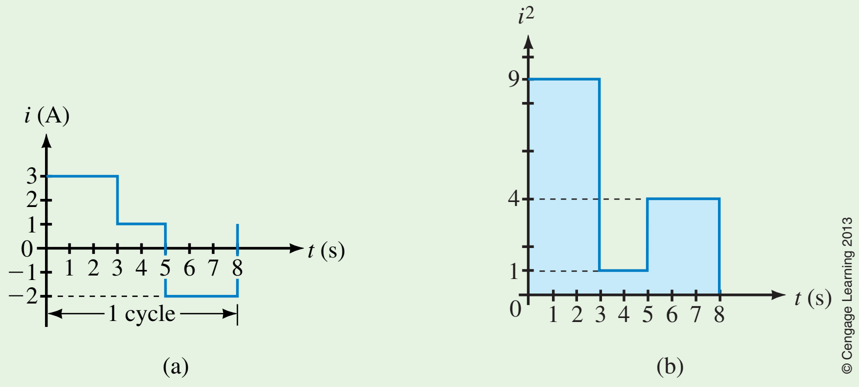


FIGURE 1–66

Solution Square the curve, then apply Equation 1–30. Thus,

$$I_{\text{eff}} = \sqrt{\frac{(9 \times 3) + (1 \times 2) + (4 \times 3)}{8}}$$

$$= \sqrt{\frac{41}{8}} = 2.26 \text{ A}$$

EXAMPLE 1–30

Compute the rms value of the waveform of Figure 1–61(b).

Solution Use Equation 1–31 (with I replaced by V). First, compute the rms value of the ac component. $V_{\text{ac}} = 0.707 \times 1 = 10.61$ V. Now substitute this into Equation 1–31. Thus,

$$V_{\text{rms}} = \sqrt{V_{\text{dc}}^2 + V_{\text{ac}}^2} = \sqrt{(10)^2 + (10.61)^2} = 14.6 \text{ V}$$

PRACTICE PROBLEMS 12

1. Determine the rms value of the current of Figure 15–51.
2. Repeat for the voltage graphed in Figure 15–52(a).

Answers
1. 1.73 A; 2. 20 V

Dropping the *eff* and *rms* Notation

The subscripts *eff* and *rms* are not used in practice. Once the concept is familiar, we drop them. From now on, they are implied rather than stated; thus, we use I instead of I_{eff} and V instead of V_{eff}.

1.10 Rate of Change of a Sine Wave (Derivative)

As you will see in Chapter 16, several important circuit effects depend on the rate of change of sinusoidal quantities. The rate of change of a quantity is the slope (i.e., derivative) of its waveform versus time. Consider the waveform of Figure 1–67. As indicated, the slope is maximum positive at the beginning of the cycle, zero at both its peaks, maximum negative at the half-cycle crossover point, and maximum positive at the end of the cycle. This slope is plotted in Figure 1–68. Note that it is also sinusoidal, but it leads the original waveform by 90°. Thus, if A is a sine wave, its slope B is a cosine wave—see Note.

NOTES...

The Derivative of a Sine Wave
The result developed intuitively here can be proven easily using calculus. To illustrate, consider the waveform $\sin \omega t$ shown in Figure 1–67. The slope of this function is its derivative. Thus,

$$\text{Slope} = \frac{d}{dt}\sin \omega t = \omega \cos \omega t$$

Therefore, the slope of a sine wave is a cosine wave as depicted in Figure 1–68.

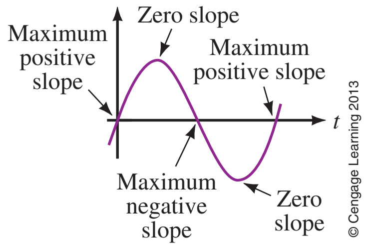


FIGURE 1–67 Slope at various places for a sine wave.

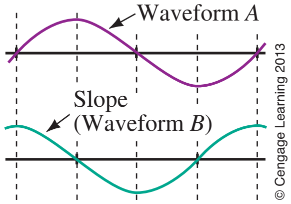


FIGURE 1–68 Showing the 90° phase shift.

CircuitSim 1-17

1.11 ac Voltage and Current Measurement

Two of the most important instruments for measuring ac quantities are the multimeter and the oscilloscope. Multimeters read the magnitude of ac voltage and current, and sometimes frequency. Oscilloscopes show waveshape and period and permit determination of frequency, phase difference, and so on.

Meters for Voltage and Current Measurement

There are two basic classes of ac meters: one measures rms correctly for sinusoidal waveforms only (called "average responding" instruments); the other measures rms correctly regardless of waveform (called "true rms" meters). Most common meters are average responding meters.

Average Responding Meters

Average responding meters use a rectifier circuit to convert incoming ac to dc. They then respond to the average value of the rectified input. Internal circuitry then rescales the measured value and displays it as an rms value. However, since these meters are calibrated for sinusoidal ac only, their readings are meaningless for all other waveforms. Most low-cost multimeters are average responding meters.

True RMS Measurement

If you need to measure the rms value of a nonsinusoidal waveform, you need a true rms meter. A true rms meter indicates true rms voltages and currents regardless of waveform. For example, for the waveform of Figure 1–64(a), any ac meter will correctly read 120 V (since it is a sine wave). For the waveform considered in Example 1–30, a true rms meter (such as that of Figure 1–69) will correctly read 14.6 V, but an average responding meter will yield only a meaningless value.

Courtesy of Fluke Corporation

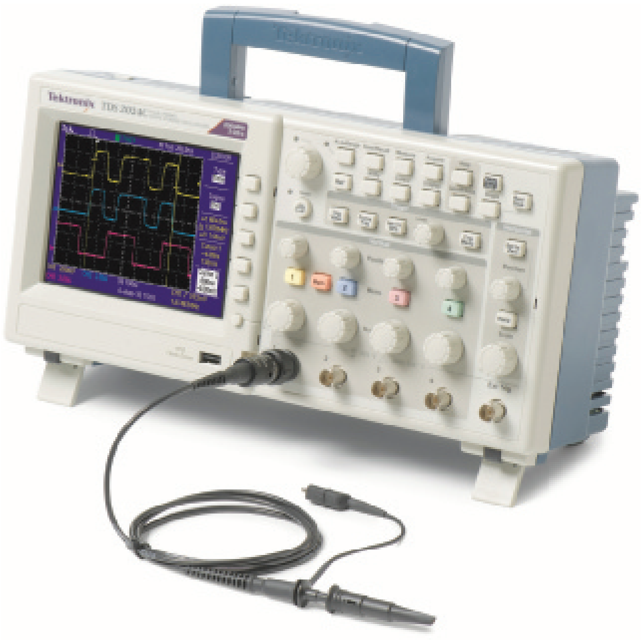
FIGURE 1–69 A true rms digital multimeter (DMM).

Connecting ac Meters to Measure Voltage and Current

Unlike dc meters, ac meters are not polarity sensitive—that is, you do not need to be concerned about polarity when connecting an ac meter. For example, suppose you connect an ac voltmeter to read voltage, then disconnect it, reverse its leads, reconnect, then read voltage again. Both connections will yield a positive reading.

Oscilloscopes

Oscilloscopes (frequently referred to as scopes) are used for time domain measurement, that is, waveshape, frequency, period, phase difference, and so on, as illustrated earlier in Figures 1–16 and 1–44. As indicated, scope screens are scaled in volts vertically and in time horizontally to permit visual study of waveforms and (with the aid of cursors), measurement of waveform values—see Note. Most modern oscilloscopes use digital technology (rather than the bulkyCRT designs of a few years ago) and have many more features. The scope ofFigure 1–70 for example, is a multichannel digital instrument with multicolored traces that permits the observation and analysis of many signals simultaneously.



CircuitSim 1-18

FIGURE 1–70 An oscilloscope displaying multiple waveforms simultaneously.

Oscilloscopes measure voltage. To measure current, you need to convert it to an equivalent voltage. The most inexpensive way to do this is to place a resistor in the circuit (small enough so as to not appreciably disturb the circuit), measure the voltage across this resistor with the oscilloscope, then use Ohm's law to convert the waveform to current. Alternatively, you can use a current probe that clips over the wire and monitors its magnetic field. One type of current probe uses a Hall effect sensor and can measure a wide range of currents over a considerable frequency range, including dc. However, these are quite expensive.

NOTES...

A series of hands-on, computer-based oscilloscope tutorials using circuit simulation techniques may be found on our Web site. Go to the *Student Premium Website* at cengagebrain.com and follow the links to *Lab Pre-Study Simulations—The Oscilloscope*. The simulations use Multisim.

Meter Frequency Considerations

AC meters measure voltage and current only over a limited frequency range, typically up to a few hundred kilohertz. Note, however, that accuracy may become poorer as frequency gets higher—check your manual. Oscilloscopes, on the other hand, measure very high frequencies; even moderately priced oscilloscopes work at frequencies up to hundreds of MHz.

1.12 Circuit Analysis Using Computers

MultiSim

PSPICE

Multisim and PSpice both provide a convenient way to study phase relationships and other aspects of ac waveforms as they both incorporate easy-to-use graphing facilities. To illustrate, let us graph $e_1 = 100 \sin \omega t$ V and $e_2 = 80 \sin(\omega t + 60°)$ V and confirm the phase difference by measurement. (Since the measurement of phase difference is independent of frequency, we can choose any convenient frequency we want for this simulation. Let us use 500 Hz.)

Multisim

Select an ac signal voltage source and place it on your screen as in Figure 1–71 (see Note 1). Click Place, select *Junction*, and add a junction dot above the source. Repeat for the second source, add grounds, then wire the circuits as shown. Read Note 2, double-click source 1, then set voltage to 100 and frequency to 500. For source 2, set voltage to 80, frequency to 500, and phase to 60. Click *Options, Sheet Properties* then *Show All* to display your circuit's node numbers. Click *Simulate* and select *Transient Analysis;* in the dialog box, set *TSTOP* to 0.002 (to run the solution to 2 ms so that you see a full cycle) and *Minimum number of time points* to 200 (see Note 3). Click the *Output* tab and select voltage variables V(1) and V(2) (or nodes $1 and $2 for legacy users), and click *Add*—see Notes 4 and 5. When you click *Simulate,* you should get the waveforms of Figure 1–71(b) on your screen. (They may appear on a black background, but you can change that if you like.)

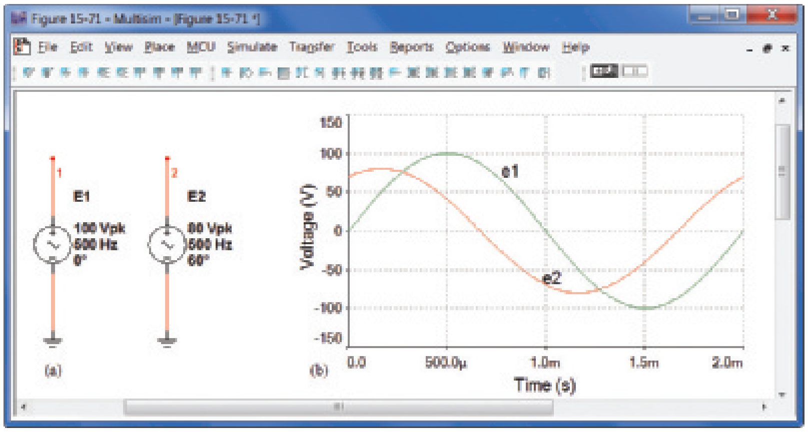


FIGURE 1–71 Studying phase relationships using Multisim.

You can verify the angle between the waveforms using cursors. First, note that the period $T = 2$ ms $= 2000$ µs. (This corresponds to 360°.) Click the *Show Grid* icon, then the *Show Cursors* icon. Using the cursors, measure the time between crossover points as indicated in Figure 1–72. You should get 333 µs. This yields an angular displacement of

$$\theta = \frac{333\ \mu\text{s}}{2000\ \mu\text{s}} \times 360° = 60°$$

as expected.

NOTES...

1. For this example, use the ac signal source from the Signal Sources component bin. To access it, right-click *View* and ensure that *Signal Source Components* is selected. Use your mouse to locate its icon, then click and position it on your screen.
2. To ensure that your source phase angles are set up correctly, click *Options/Global Preferences* then select the *Simulation* tab (or the *Parts* tab if you are using legacy software). Look for the small sinusoidal waveform. Ensure that the *Shift Right* button is selected.
3. Multisim generates time steps automatically when it plots waveforms. Sometimes, however, it does not generate enough points, and you get a jagged curve. If this happens, click *Minimum number of time points* and type in a larger number (say 200).

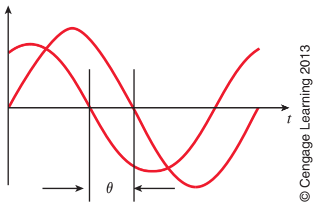


FIGURE 1–72

4. Node numbers in Figure 1–71 depend on the order in which you connect wire—thus, your node numbers may be different from those shown. However, for purposes of discussion, we will use the node numbers as illustrated.

5. Node numbers help you identify variables—for example, V(1) is the voltage (with respect to ground) at Node 1, V(2) is the voltage at Node 2, and so on. [For legacy software users, you need to use $1 instead of V(1), etc.]

PSpice

Click the *Place Part* icon, ensure that the SOURCE library is selected, type **VSIN** in the Place Part box, then click the *Place Part* button. Build the circuit as shown in Figure 1–73 on your screen. Note the empty parameter boxes beside each source. Double-click each in turn and enter 0V for the offset, 100V for the amplitude, and 500Hz for the frequency for Source 1. Similarly enter values for Source 2. Now double-click the V2 source symbol and in its Property editor window, ensure that the *Parts* tab is selected, scroll until you find a cell labelled *PHASE*, type in **60,** click *Apply,* then close. (You don't need to do this for Source 1 because PSpice will automatically use the default value of zero degrees.) Click the *New Simulations* icon and enter the file name. When the dialog box opens, select *Time Domain*, set *TSTOP* to 2 ms (to display a full cycle), then *OK*. Place voltage markers as shown to automatically plot the traces, then click *Run*. When the simulation is complete, the waveforms shown appear. (They may be on a black background, but you can change that if you like.)

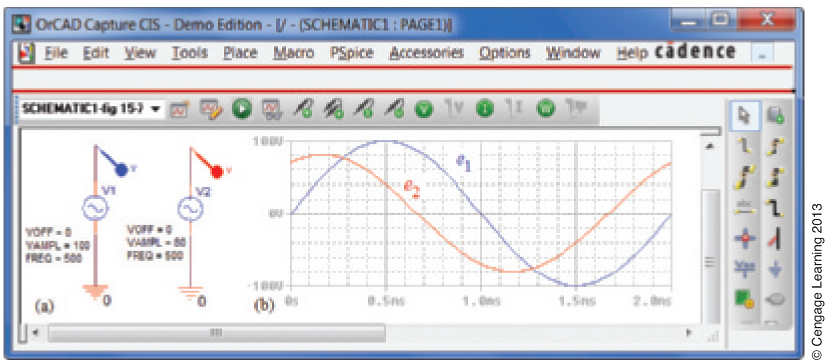


FIGURE 1–73 Studying phase relationships using PSpice.

You can verify the angle between the waveforms using cursors. First, note that the period $T = 2$ ms $= 2000$ μs. (This corresponds to 360°.) Now using the cursors, measure the time between crossover points as indicated in Figure 1–72. You should get 333 μs. This yields an angular displacement of

$$\theta = \frac{333\ \mu\text{s}}{2000\ \mu\text{s}} \times 360° = 60°$$

which agrees with the given sources.

Problems

1.1 Introduction

1. What do we mean by "ac voltage?" By "ac current?"

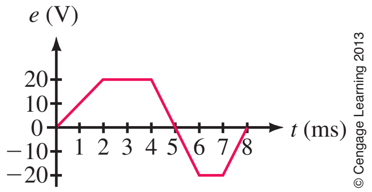


FIGURE 1–74

1.2 Generating ac Voltages

2. The waveform of Figure 1–8 is created by a 600-rpm generator. If the speed of the generator changes so that its cycle time is 50 ms, what is its new speed?

3. a. What do we mean by instantaneous value? b. For Figure 1–74, determine instantaneous voltages at $t = 0$, 1, 2, 3, 4, 5, 6, 7, and 8 ms.

1.3 Voltage and Current Conventions for ac

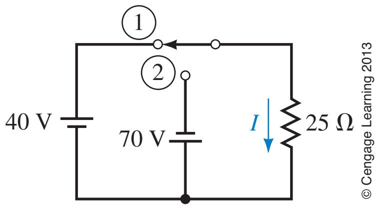


FIGURE 1–75

4. Consider the circuit of Figure 1–75. By moving the switch back and forth between positions 1 and 2, you can create a rectangular ac waveform. Suppose you let the switch remain in each position for 1 second,
   a. Sketch the voltage waveform across the load, complete with numeric values. Don't forget to scale the time axis.
   b. Sketch the load current, complete with numerical values.

5. The source of Figure 1–76 has the waveform of Figure 1–74. Determine the current at $t = 0$, 1, 2, 3, 4, 5, 6, 7, and 8 ms. Include signs.

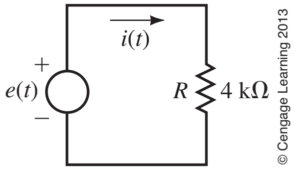


FIGURE 1–76

1.4 Frequency, Period, Amplitude, and Peak Value

6. For each of the following, determine the period:
   a. $f = 100$ Hz
   b. $f = 40$ kHz
   c. $f = 200$ MHz
7. For each of the following, determine the frequency:
   a. $T = 0.5$ s
   b. $T = 100$ ms
   c. $5T = 80$ μs
8. For a triangular wave, $f = 1.25$ MHz. What is its period? How long does it take to go through 8×10^7 cycles? 9. Determine the period and frequency

for the waveform of Figure 1–77.10. Determine the period and frequency for the waveform of Figure 1–78. How
many cycles are shown?

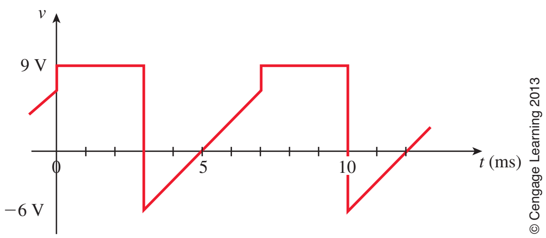


FIGURE 1–77

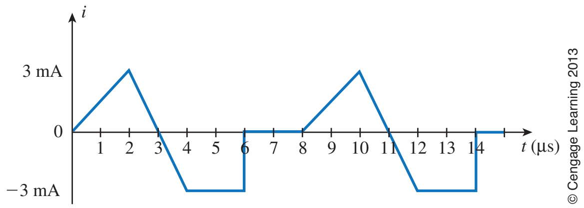


FIGURE 1–78

11. What is the peak-to-peak voltage for Figure 1–77? What is the peak-to-peak current of Figure 1–78?
12. For a certain waveform, $625T = 12.5$ ms. What is the waveform's period and frequency?
13. A square wave with a frequency of 847 Hz goes through how many cycles in 2 minutes and 57 seconds?
14. For the waveform of Figure 1–79, determine
    a. period b. frequency c. peak-to-peak value
15. Two waveforms have periods of T_1 and T_2. If $T_1 = 0.25\ T_2$ and $f_1 = 10$ kHz, what are T_1, T_2, and f_2?
16. Two waveforms have frequencies f_1 and f_2. If $T_1 = 4\ T_2$ and waveform 1 is as shown in Figure 1–77, what is f_2?

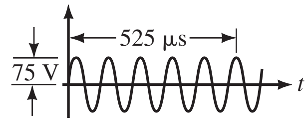


FIGURE 1–79

1.5 Angular and Graphic Relationships for Sine Waves

17. Given voltage $v = V_m \sin \alpha$. If $V_m = 240$ V, what is v at $\alpha = 37°$?
18. For the sinusoidal waveform of Figure 1–80,
    a. Determine the equation for i.
    b.D etermine current at all points marked.
19. A sinusoidal voltage has a value of 50 V at $\alpha = 150°$. What is V_m?
20. Convert the following angles from radians to degrees:
    a. $\pi/12$ d. 1.43
    b. $\pi/1.5$ e. 17
    c. $3\pi/2$ f. 32π
21. Convert the following angles from degrees to radians:
    a. 10° d. 150°
    b. 25° e. 350°
    c. 80° f. 620°
22. A 50-kHz sine wave has an amplitude of 150 V. Sketch the waveform with its axis scaled in microseconds.
23. If the period of the waveform in Figure 1–80 is 180 ms, compute current at $t = 30$, 75, 140, and 315 ms.
24. A sinusoidal waveform has a period of 60 μs and $V_m = 80$ V. Sketch the waveform. What is its voltage at 4 μs?
25. A 20-kHz sine wave has a value of 50 volts at $t = 5$ μs. Determine V_m and sketch the waveform.
26. For the waveform of Figure 1–81, determine v_2.

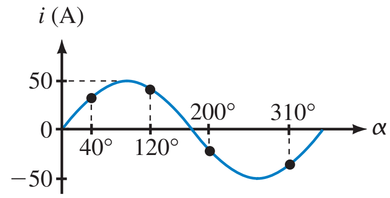


FIGURE 1–80

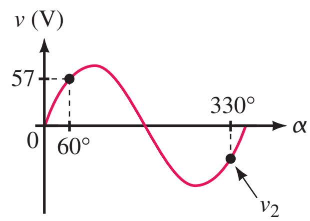


FIGURE 1–81

1.6 Voltages and Currents as Functions of Time

27. Calculate ω in radians per second for each of the following:
    a. $T = 100$ ns d. period = 20 ms
    b. $f = 30$ Hz e. 5 periods in 20 ms
    c. 100 cycles in 4 s
28. For each of the following values of ω, compute f and T:
    a. 100 rad/s b. 40 rad in 20 ms c. 34×10^3 rad/s
29. Determine equations for sine waves with the following:
    a. $V_m = 170$ V, $f = 60$ Hz c. $T = 120$ μs, $v = 10$ V at $t = 12$ μs
    b. $I_m = 40$ μA, $T = 10$ ms

30. Determine f, T, and amplitude for each of the following:
    a. $v = 75 \sin 200\pi t$ b. $i = 8 \sin 300t$
31. A sine wave has a peak-to-peak voltage of 40 V and $T = 50$ ms. Determine its equation.
32. Sketch the following waveforms with the horizontal axis scaled in degrees, radians, and seconds:
    a. $v = 100 \sin 200\pi t$ V
    b. $i = 90 \sin \omega t$ mA, $T = 80\ \mu s$
33. Given $i = 47 \sin 8260t$ mA, determine current at $t = 0$ s, 80 μs, 410 μs, and 1200 μs.
34. Given $v = 100 \sin \alpha$. Sketch one cycle.
    a. Determine at which two angles $v = 86.6$ V.
    b. If $\omega = 100\pi/60$ rad/s, at which times do these occur?
35. Write equations for the waveforms of Figure 15–82. Express the phase angle in degrees.

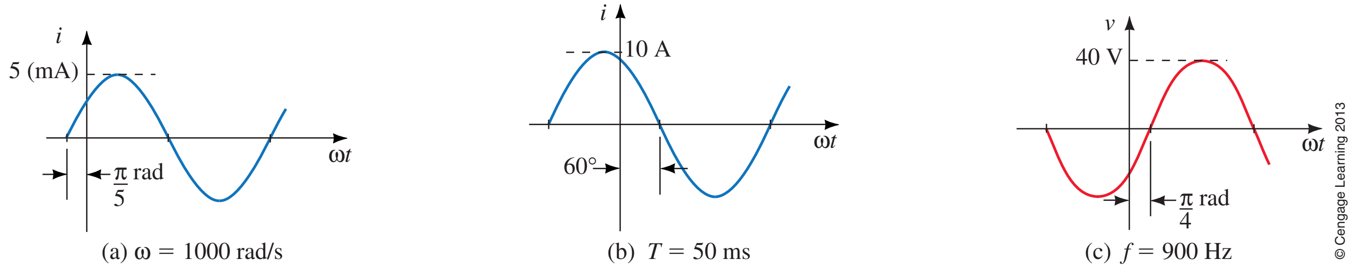


FIGURE 1–82

36. Sketch the following waveforms with the horizontal axis scaled in degrees and seconds:
    a. $v = 100 \sin(232.7t + 40°)$ V
    b. $i = 20 \sin(\omega t - 60°)$ mA, $f = 200$ Hz
37. Given $v = 5 \sin(\omega t + 45°)$. If $\omega = 20\pi$ rad/s, what is v at $t = 20$, 75, and 90 ms?
38. Repeat Problem 35 for the waveforms of Figure 1–83.

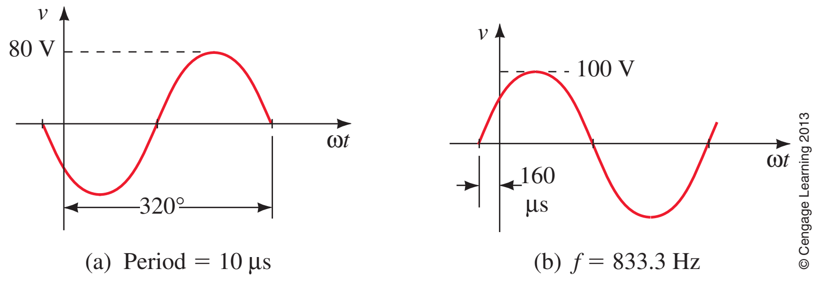


FIGURE 1–83

39. Determine the equation for the waveform shown in Figure 1–84.
40. For the waveform of Figure 1–85, determine i_2.
41. Given $v = 30 \sin(\omega t - 45°)$ where $\omega = 40\pi$ rad/s. Sketch the waveform. At what time does v reach 0 V? At what time does it reach 23 V and −23 V?

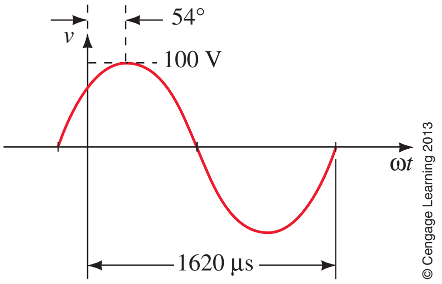


FIGURE 1–84

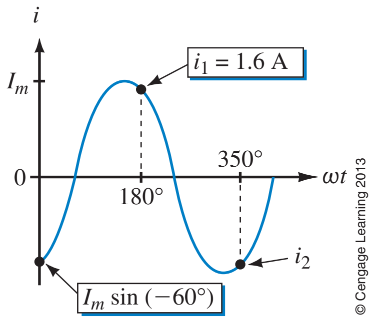


FIGURE 1–85

1.7 Introduction to Phasors

42. For each of the phasors of Figure 15–86, determine the equation for $v(t)$ or $i(t)$ as applicable, and sketch the waveform.

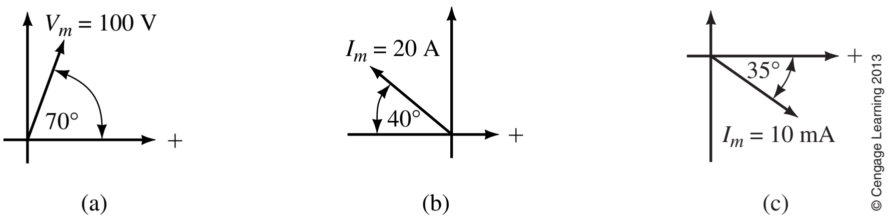


FIGURE 1–86

43. With the aid of phasors, sketch the waveforms for each of the following pairs and determine the phase difference and which waveform leads:
    a. $v = 100 \sin \omega t$
       $i = 80 \sin(\omega t + 20°)$
    b. $v_1 = 200 \sin(\omega t - 30°)$
       $v_2 = 150 \sin(\omega t - 30°)$
    c. $i_1 = 40 \sin(\omega t + 30°)$
       $i_2 = 50 \sin(\omega t - 20°)$
    d. $v = 100 \sin(\omega t + 140°)$
       $i = 80 \sin(\omega t - 160°)$
44. Repeat Problem 43 for the following:
    a. $i = 40 \sin(\omega t + 80°)$
       $v = -30 \sin(\omega t - 70°)$
    b. $v = 20 \cos(\omega t + 10°)i$
       $= 1 \sin(\omega t - 10°)$
    c. $v = 20 \cos(\omega t + 10°)$
       $i = 1 \sin(\omega t + 120°)$
    d.$v = 80 \cos(\omega t + 30°)$
       $i = 10 \cos(\omega t - 1°)$
45. For the waveforms in Figure 1–87, determine the phase differences. Which waveform leads?
46. Draw phasors for the waveforms of Figure 1–87.

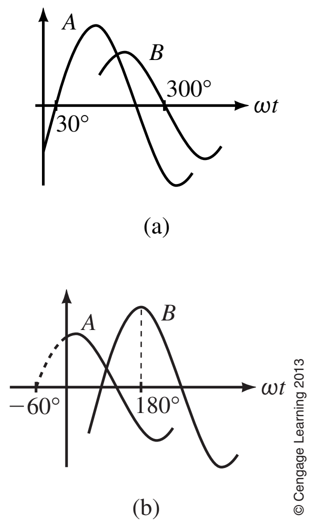


FIGURE 1–87

1.8 ac Waveforms and Average Value

47. What is the average value of each of the following over an integral number of cycles?
    a. $i = 5 \sin \omega t$
    b. $i = 40 \cos \omega t$
    c. $v = 400 \sin(\omega t + 30°)$
    d. $v = 20 \cos 2\omega t$
48. Using Equation 1–20, compute the area under the half-cycle of Figure 1–54using increments of $\pi/12$ rad.
49. Compute I_{avg} or V_{avg} for the waveforms of Figure 1–88.

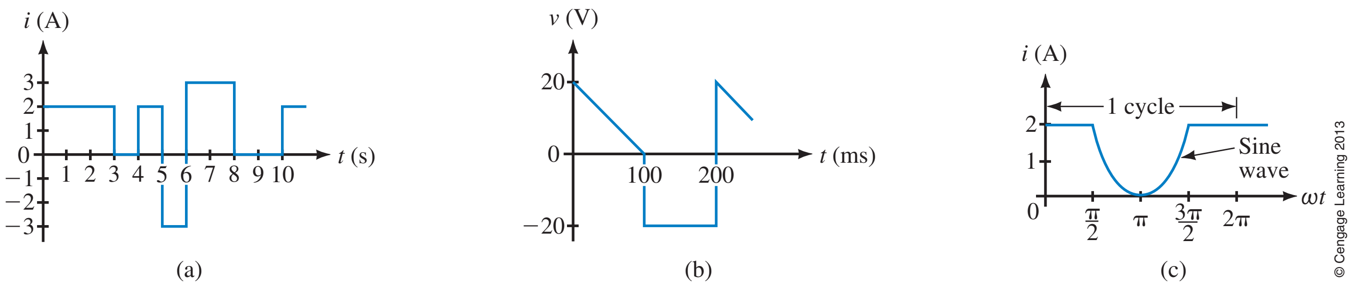


FIGURE 1–88

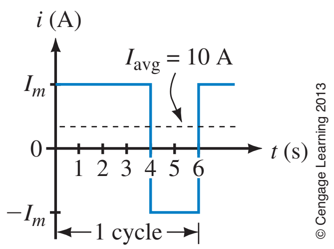


FIGURE 1–89

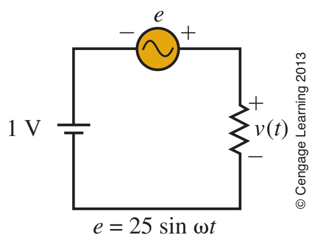


FIGURE 1–90

50. For the waveform of Figure 1–89, compute I_m.
51. For the circuit of Figure 1–90, $e = 25 \sin \omega t$ V and period $T = 120$ ms.
    a. Sketch voltage $v(t)$ with the axis scaled in milliseconds.
    b. Determine the peak and minimum voltages.
    c. Compute v at $t = 10$, 20, 70, and 100 ms.
    d. Determine V_{avg}.
52. Using numerical methods for the curved part of the waveform (with increment size $\Delta t = 0.25$ s), determine the area and the average value for the waveform of Figure 1–91.
53. Using calculus, find the average value for Figure 1–91.

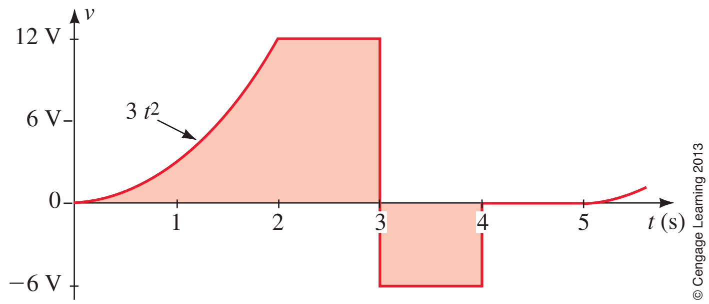


FIGURE 1–91

1.9 Effective (RMS) Values

54. Determine the effective values of each of the following:
    a. $v = 100 \sin \omega t$ V
    b. $i = 8 \sin 377t$ A
    c. $v = 40 \sin(\omega t + 40°)$ V
    d. $i = 120 \cos \omega t$ mA
55. Determine the rms values of each for the following:
    a. A 12-V battery
    b. $-24 \sin(\omega t + 73°)$ mA
    c. $10 + 24 \sin \omega t$ V
    d. $45 - 27 \cos 2\omega t$ V
56. For a sine wave, $V_{eff} = 9$ V. What is its amplitude?
57. Determine the root mean square values for
    a. $i = 3 + \sqrt{2}(4) \sin(\omega t + 44°)$ mA
    b. Voltage v of Figure 1–92 with $e = 25 \sin \omega t$ V
58. Compute the rms values for Figure 1–88(a) and Figure 1–89. For Figure 1–89, use $I_m = 30$ A.
59. Compute the rms values for the waveforms of Figure 1–92.

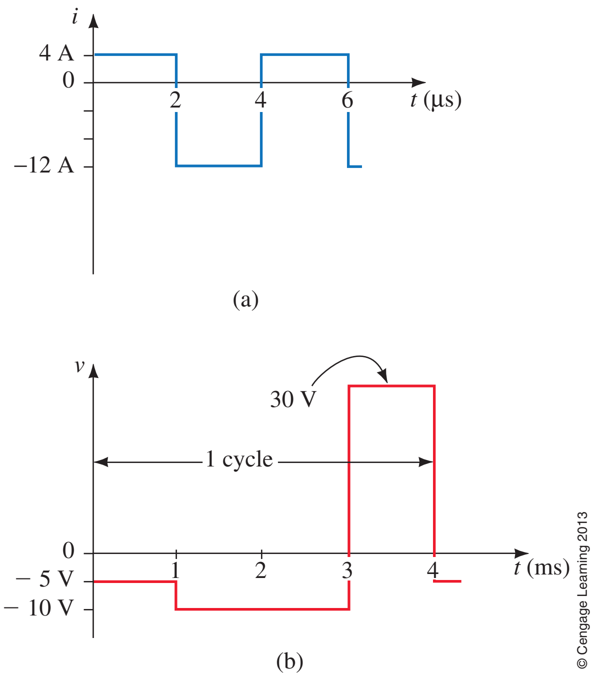


FIGURE 1–92

60. Compute the effective value for Figure 1–93.
61. Determine the rms value of the waveform of Figure 1–94. Why is it the same as that of a 24-V battery?

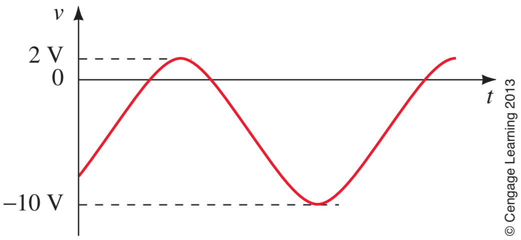


FIGURE 1–93

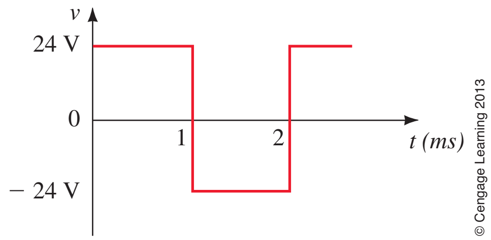


FIGURE 1–94

62. Compute the rms value of the waveform of Figure 1–52(c). To handle the triangular portion, use Equation 1–20. Use a time interval $\Delta t = 1$ s.
63. Repeat Problem 62, using calculus to handle the triangular portion.

1.11 ac Voltage and Current Measurement

64. Determine the reading of an average responding ac meter for each of the following cases. (Note: Meaningless is a valid answer if applicable.) Assume the frequency is within the range of the instrument.
    a. $v = 153 \sin \omega t$ V
    b. $v = \sqrt{2}(120) \sin(\omega t + 30°)$ V
    c. The waveform of Figure 15–61
    d. $v = 597 \cos \omega t$ V
65. Repeat Problem 64 using a true rms meter.

1.12 Circuit Analysis Using Computers

Use Multisim or PSpice for the following:

66. Plot the waveform of Problem 37 and, using the cursor, determine voltage at the times indicated. Compare to answers in Appendix D. Don't forget to convert the frequency to Hz.
67. Plot the waveform of Problem 41. Using the cursor, determine the times at which v reaches 0 V, 23 V, and −23 V. Compare to answers in Appendix D.
68. Assume the equations of Problem 43 all represent voltages. For each case, plot the waveforms, then use the cursor to determine the phase difference between waveforms. Use $f = 100$ Hz. Compare to answers in Appendix D.

ANSWERS TO IN-PROCESS LEARNING CHECKS

IN-PROCESS LEARNING CHECK 1

1. 16.7 ms
2. Frequency doubles, period halves
3. 20 ms; 50 Hz
4. 20 V; 0.5 ms and 2.5 ms; −35 V: 4 ms and 5 ms
5. (c) and (d); Since current is directly proportional to the voltage, it will have the same waveshape.
6. 250 Hz
7. $f_1 = 100$ Hz; $f_2 = 33.3$ Hz
8. 50 kHz and 1 MHz
9. 22.5 Hz
10. At 12 ms, direction →; at 37 ms, direction ←; at 60 ms, →
11. At 75 ms, $i = -5$ A

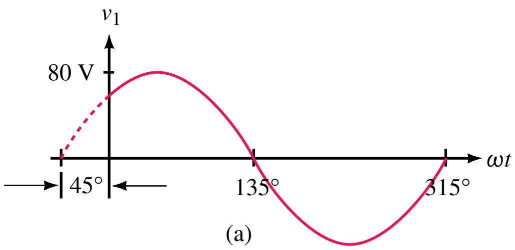
IN-PROCESS LEARNING CHECK 2

1.

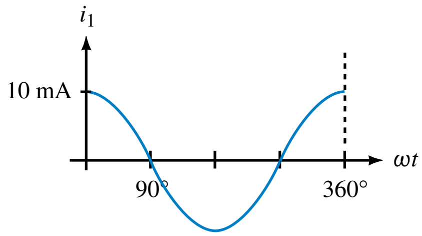
α (deg)	0	45	90	135	180	225	270	315	360
i (mA)	0	10.6	1	10.6	0	−10.6	−1	−10.6	0

2. a. 0.349 c. 2.09
   b. 0.873 d. 4.36
3. 30°; 120°; 450°
4. Same as Figure 15–27 with $T = 10$ s and amplitude = 50 mA.
5. 1.508×10^6 rad/s
6. $i = 8 \sin 157t$ A
7. $i = 6 \sin 69.81t$ A
8. a. $i = 250 \sin(251t - 30°)$ A
   b. $i = 20 \sin(62.8t + 45°)$ A
   c. $v = 40 \sin(628t - 30°)$ V
   d. $v = 80 \sin(314 \times 10^3 t + 36°)$ V
9. a. 2.95 ms; 7.05 ms; 22.95 ms; 27.05 ms
   b. 11.67 ms; 18.33 ms; 31.67 ms; 38.33 ms

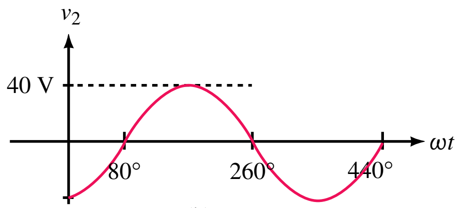
10.



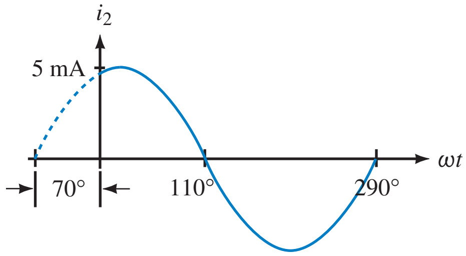
(a)



(c)



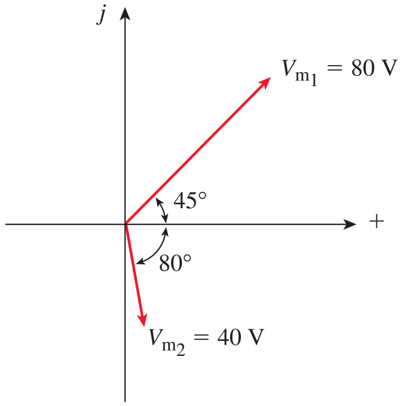
(b)



(d)

11. a. 2.25 s   b. 1.33 s

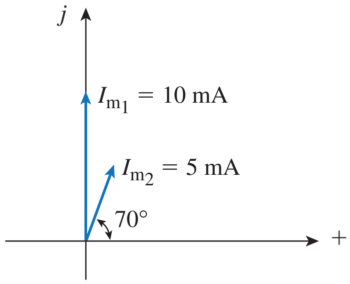
12. a.



b. 125°

c. v_1 leads

13. a.



b. 20°

c. i_1 leads

KEY TERMS

Capacitive Reactance

Complex Number

Impedance

Inductive Reactance

Phasor Domain

Polar Form

Rectangular Form

Time Domain

OUTLINE

Complex Number Review

Complex Numbers in ac Analysis

R, L, and *C* Circuits with Sinusoidal Excitation

Resistance and Sinusoidal ac

Inductance and Sinusoidal ac

Capacitance and Sinusoidal ac

The Impedance Concept

Computer Analysis of ac Circuits

OBJECTIVES

After studying this chapter, you will be able to

- express complex numbers in rectangular and polar forms,
- represent ac voltage and current phasors as complex numbers,
- represent ac sources in transformed form,
- add and subtract currents and voltages using phasors,
- compute inductive and capacitive reactance,
- determine voltages and currents in simple ac circuits,
- explain the impedance concept,
- determine impedance for *R, L,* and *C* circuit elements,
- determine voltages and currents in simple ac circuits using the impedance concept,
- use Multisim and PSpice to solve simple ac circuit problems.

2

R, L, AND *C* ELEMENTS AND THE IMPEDANCE CONCEPT

CHAPTER PREVIEW

In Chapter 1, you learned how to analyze a few simple ac circuits in the time domain using voltages and currents expressed as functions of time. However, this is not a very practical approach. A more practical approach is to represent ac voltages and currents as phasors and circuit elements as impedances, and to analyze circuits in the phasor domain using complex algebra. With this approach, ac circuit analysis is handled much like dc circuit analysis, and all basic relationships and theorems—Ohm's law, Kirchhoff's laws, mesh and nodal analysis, superposition, and so on—apply. The major difference is that ac quantities are complex numbers, rather than real numbers, as with dc. While this complicates computational details, it does not alter basic circuit principles. This is the approach used in practice. The basic ideas are developed in this chapter.

Since phasor analysis and the impedance concept require a familiarity with complex numbers, we begin with a short review. ■

Putting It in Perspective

Charles Proteus Steinmetz

CHARLES STEINMETZ WAS BORN IN BRESLAU, Germany, in 1865 and emigrated to the United States in 1889. In 1892, he began working for the General Electric Company in Schenectady, New York, where he stayed until his death in 1923, and it was there that his work revolutionized ac circuit analysis. Prior to his time, this analysis had to be carried out using calculus, a difficult and time-consuming process. By 1893, however, Steinmetz had reduced the very complex alternating-current theory to, in his words, "a simple problem in algebra." The key concept in this simplification was the phasor—a representation based on complex numbers. By representing voltages and currents as phasors, Steinmetz was able to define a quantity called impedance and then use it to determine voltage and current magnitude and phase relationships in one algebraic operation.

Steinmetz wrote the seminal textbook on ac analysis based on his method, but at the time he introduced it, it was said that he was practically the only person who understood it. Now, however, it is common knowledge, and the methods devised by Steinmetz form the basis for nearly all ac circuit analysis techniques in use today. In this chapter, we learn the method and illustrate its application to the solution of basic ac circuit problems.

In addition to his work for GE, Charles Steinmetz was a professor of electrical engineering (1902–1913) and electrophysics (1913–1923) at Union College in Schenectady. ■

2.1 Complex Number Review

A **complex number** is a number of the form $\mathbf{C} = a + jb$, where a and b are real numbers and $j = \sqrt{-1}$. The number a is called the **real** part of $\mathbf{C}$ and b is called its **imaginary** part. (In circuit theory, j is used to denote the imaginary component rather than i to avoid confusion with current i.)

Geometrical Representation

Complex numbers may be represented geometrically, either in rectangular form or in polar form as points on a two-dimensional plane called the complex plane (Figure 2–1). The complex number $\mathbf{C} = 6 + j8$, for example, represents a point whose coordinate on the real axis is 6 and whose coordinate on the imaginary axis is 8. This form of representation is called the **rectangular form.**

Complex numbers may also be represented in **polar form** by magnitude and angle. Thus, $\mathbf{C} = 10 \angle 53.13°$ (Figure 2–2) is a complex number with magnitude 10 and angle 53.13°. This magnitude and angle representation is just an alternate way of specifying the location of the point represented by $\mathbf{C} = a + jb$.

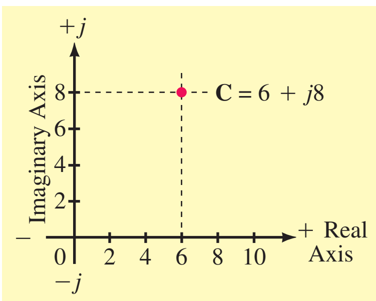


FIGURE 2–1 A complex number in rectangular form.

Conversion between Rectangular and Polar Forms

To convert between forms, note from Figure 2–3 that

$$\mathbf{C} = a + jb \quad \text{(rectangular form)} \tag{2–1}$$

$$\mathbf{C} = C\angle\theta \quad \text{(polar form)} \tag{2–2}$$

where C is the magnitude of **C**. From the geometry of the triangle,

$$a = C\cos\theta \tag{2–3a}$$

$$b = C\sin\theta \tag{2–3b}$$

where

$$C = \sqrt{a^2 + b^2} \tag{2–4a}$$

and

$$\theta = \tan^{-1}\frac{b}{a} \tag{2–4b}$$

While Equations 2–3 and 2–4 permit conversion between forms, they are not used much in computations anymore because modern calculators do the job automatically. However, we will do one example using them to illustrate the principles involved.

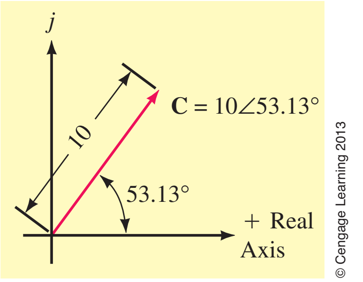


FIGURE 2–2 A complex number in polar form.

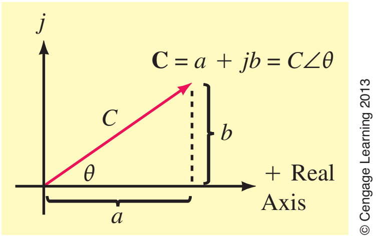
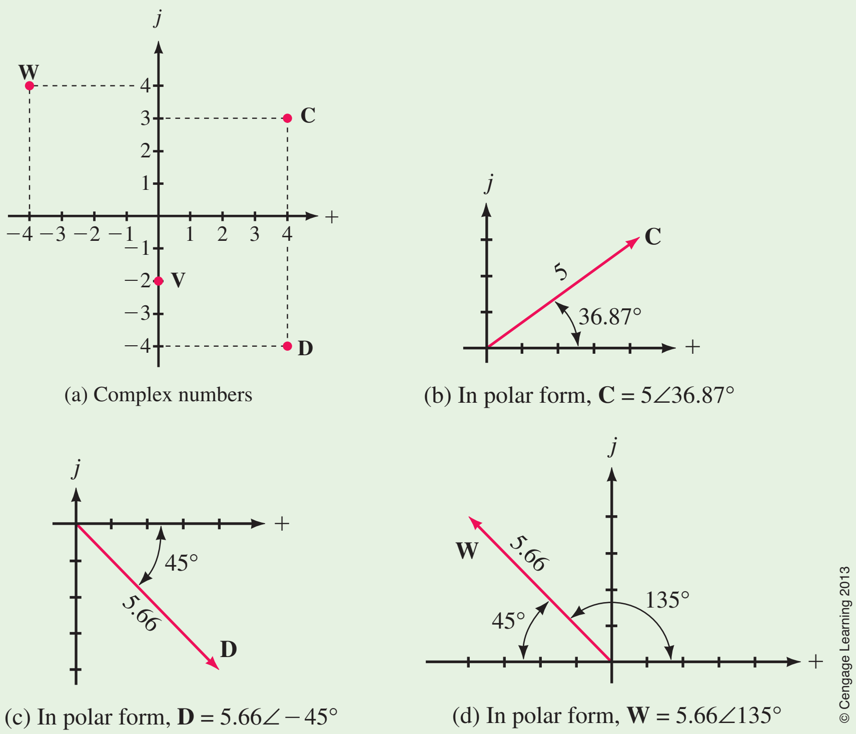


FIGURE 2–3 Polar and rectangular equivalence.

EXAMPLE 2–1

Determine rectangular and polar forms for the complex numbers **C**, **D**, **V**, and **W** of Figure 2–4(a).



(a) Complex numbers

(b) In polar form, $\mathbf{C} = 5\angle 36.87°$

(c) In polar form, $\mathbf{D} = 5.66\angle -45°$

(d) In polar form, $\mathbf{W} = 5.66\angle 135°$

FIGURE 2–4

Solution *Point C:* Real part = 4; imaginary part = 3. Thus, C = 4 + $j3$. In polar form, $C = \sqrt{4^2 + 3^2} = 5$ and $\theta_C = \tan^{-1}(3/4) = 36.87°$. Thus, C = $5\angle 36.87°$ as indicated in (b).

Point D: In rectangular form, $\mathbf{D} = 4 - j4$. Thus, $D = \sqrt{4^2 + 4^2} = 5.66$ and $\theta_D = \tan^{-1}(-4/4) = -45°$. Therefore, $\mathbf{D} = 5.66\angle -45°$, as shown in (c).

Point V: In rectangular form, $\mathbf{V} = -j2$. In polar form, $\mathbf{V} = 2\angle -90°$.

Point W: In rectangular form, $\mathbf{W} = -4 + j4$. Thus, $W = \sqrt{4^2 + 4^2} = 5.66$ and $\tan^{-1}(-4/4) = -45°$. Inspection of Figure 2–4(d) shows, however, that this 45° angle is the supplementary angle. The actual angle (measured from the positive horizontal axis) is 135°. Thus, $\mathbf{W} = 5.66\angle 135°$.

As discussed later in this section, inexpensive calculators are available that perform such conversions directly—you simply enter the complex number components and press the conversion key. With these, the problem of determining angles for numbers such as **W** in Example 2–1 does not occur; you just enter $-4 + j4$ and the calculator returns $5.66\angle 135°$.

Powers of j

Powers of j are frequently required in calculations. Here are some useful powers:

$$
\begin{aligned}
j^2 &= (\sqrt{-1})(\sqrt{-1}) = -1 \\
j^3 &= j^2 j = -j \\
j^4 &= j^2 j^2 = (-1)(-1) = 1 \\
(-j)j &= -j^2 = 1 \\
\frac{1}{j} &= \frac{1}{j} \times \frac{j}{j} = \frac{j}{j^2} = -j
\end{aligned}
\qquad \textbf{(2–5)}
$$

Addition and Subtraction of Complex Numbers

Addition and subtraction of complex numbers can be performed analytically or graphically. Analytic addition and subtraction are most easily illustrated in rectangular form, while graphical addition and subtraction are best illustrated in polar form. For analytic addition, add real and imaginary parts separately. Similarly for subtraction. For graphical addition, add vectorially as in Figure 2–5(a); for subtraction, change the sign of the subtrahend, then add, as in Figure 2–5(b).

EXAMPLE 2–2

Given $\mathbf{A} = 2 + j1$ and $\mathbf{B} = 1 + j3$, determine their sum and difference analytically and graphically.

Solution

$$\mathbf{A} + \mathbf{B} = (2 + j1) + (1 + j3) = (2 + 1) + j(1 + 3) = 3 + j4.$$

$$\mathbf{A} - \mathbf{B} = (2 + j1) - (1 + j3) = (2 - 1) + j(1 - 3) = 1 - j2.$$

Graphical addition and subtraction are shown in Figure 2–5.

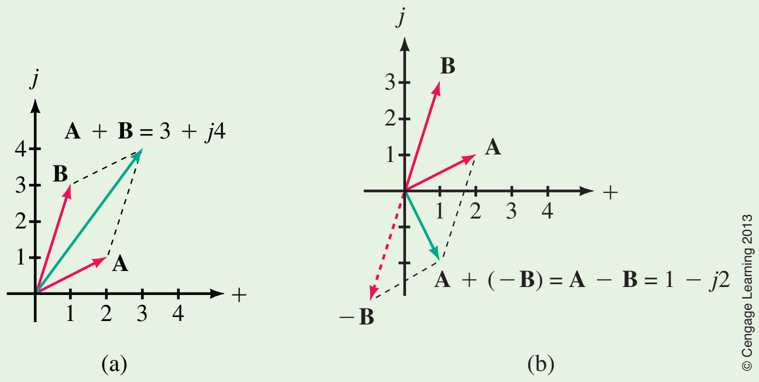


FIGURE 2–5

Multiplication and Division of Complex Numbers

These operations are best illustrated using polar representation. For multiplication, multiply magnitudes and add angles algebraically. For division, divide the magnitude of the denominator into the magnitude of the numerator, then subtract algebraically the angle of the denominator from that of the numerator. Thus, given $\mathbf{A} = A\angle\theta_A$ and $\mathbf{B} = B\angle\theta_B$,

$$\mathbf{A} \cdot \mathbf{B} = AB\angle\theta_A + \theta_B \tag{2–6}$$

$$\mathbf{A}/\mathbf{B} = A/B\angle\theta_A - \theta_B \tag{2–7}$$

EXAMPLE 2–3

Given $\mathbf{A} = 3\angle 35°$ and $\mathbf{B} = 2\angle -20°$, determine the product $\mathbf{A} \cdot \mathbf{B}$ and the quotient $\mathbf{A}/\mathbf{B}$.

Solution

$$\mathbf{A} \cdot \mathbf{B} = (3\angle 35°)(2\angle -20°) = (3)(2)\angle 35° - 20° = 6\angle 15°$$

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{(3\angle 35°)}{(2\angle -20°)} = \frac{3}{2}\angle 35° - (-20°) = 1.5\angle 55°$$

EXAMPLE 2–4

For computations involving purely real, purely imaginary, or small integer numbers, it is sometimes more convenient to multiply directly in rectangular form than it is to convert to polar. Compute the following directly:

a. $(-j3)(2 + j4)$
b. $(2 + j3)(1 + j5)$

Solution

a. $(-j3)(2 + j4) = (-j3)(2) + (-j3)(j4) = -j6 - j^2 12 = 12 - j6$
b. $(2 + j3)(1 + j5) = (2)(1) + (2)(j5) + (j3)(1) + (j3)(j5)$
$\quad = 2 + j10 + j3 + j^2 15 = 2 + j13 - 15 = -13 + j13$

PRACTICE PROBLEMS 1

1. Polar numbers with the same angle can be added or subtracted directly without conversion to rectangular form. For example, the sum of $6\angle 36.87°$ and $4\angle 36.87°$ is $10\angle 36.87°$, while the difference is $6\angle 36.87° - 4\angle 36.87° = 2\angle 36.87°$. By means of sketches, indicate why this procedure is valid.
2. To compare methods of multiplication with small integer values, convert the numbers of Example 2–4 to polar form, multiply them, then convert the answers back to rectangular form.

Answers

1. Since the numbers have the same angle, their sum also has the same angle and thus, their magnitudes simply add (or subtract).

NOTES...

A common mistake when determining reciprocals in rectangular form is to write the reciprocal of $a + jb$ as $\frac{1}{a + jb} = \frac{1}{a} + \frac{1}{jb}$. This is not correct. To illustrate, consider $\mathbf{C} = 3 + j4$. To find its reciprocal, enter $3 + j4$ into your calculator (see section entitled *Calculators for ac Analysis*), then press the inverse key. The result is $\frac{1}{\mathbf{C}} = \frac{1}{3 + j4} = 0.12 - j0.2$. Clearly, this is not equal to $\frac{1}{3} + \frac{1}{j4} = 0.333 - j0.25$.

Reciprocals

In polar form, the reciprocal of a complex number $\mathbf{C} = C\angle\theta$ is

$$\frac{1}{\mathbf{C}\angle\theta} = \frac{1}{C}\angle -\theta \tag{2–8}$$

Thus,

$$\frac{1}{20\angle 30°} = 0.05\angle -30°$$

When you work in rectangular form, you must be somewhat careful—see the Note.

Complex Conjugates

The conjugate of a complex number (denoted by an asterisk *) is a complex number with the same real part but the opposite imaginary part. Thus, the conjugate of $\mathbf{C} = C\angle\theta = a + jb$ is $\mathbf{C}^* = C\angle -\theta = a - jb$. For example, if $\mathbf{C} = 3 + j4 = 5\angle 53.13°$, then $\mathbf{C}^* = 3 - j4 = 5\angle -53.13°$.

Calculators for ac Analysis

The analysis of ac circuits involves a considerable amount of complex number arithmetic, and you will need a calculator that works easily in both rectangular and polar forms, and (preferably) displays results in standard (or near-standard) notation as in Figure 2–6. (There are a number of calculators on the market, so be sure to check with your professors or instructors before buying to determine what make and model is recommended for your school. For purposes of illustration, we have selected the Sharp EL-506W—see Note. If you have a different calculator, consult its instruction manual, then follow through the examples. Although details differ, the concepts will generally be the same.) Computations that you will need to do in circuit analysis range from simple conversions between forms (as illustrated in Example 2–5) to rather complex computations (as illustrated in Example 2–6).

NOTES...

While preparing this edition, we surveyed current users, but found no consensus about the choice of calculator. From among the various choices, we arbitrarily selected the Sharp EL-506W for illustrative purposes.

EXAMPLE 2–5

Using your calculator,

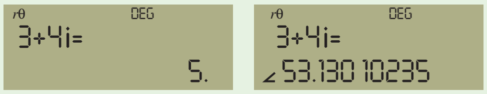
a. determine the polar form of $\mathbf{C} = 3 + j\,4$,

b.d etermine the inverse of $3 - j\,4$ with the answer expressed in rectangular form.

Solution Consult your manual, then set your calculator for complex number operation with angles expressed in degrees. Using the following as general guidelines, proceed as shown:

a. Key in 3, press the + key, key in 4, press the imaginary operator key "*i*." Typically, your display will show $3 + 4i$. Now convert to polar form using the $\rightarrow r\theta$ procedure detailed in your manual. Depending on your calculator, the complete answer $5\ \angle 53.13°$ may appear on your screen, or you may have to extract the angle part as a second step—check your manual for details.

Sharp EL-506W Procedure: To select the complex mode, press the following keys: $\boxed{\text{Mode}}\ \boxed{3}$. Key in 3, press the +, key in 4, press the "imaginary" key, press key 2nd F, press $\rightarrow r\theta$ (i.e., key 8), then the = key. The magnitude 5 appears, Figure 2–6(a). To display the angle, press 2nd F, then $\leftarrow, \rightarrow$(i.e., the Exp key). The angle part of the answer appears, Figure 2–6(b). Thus, $3 + j4 = 5\ \angle 53.13°$ (rounded to a sensible number of digits).



(a) (b)

FIGURE 2–6 Converting rectangular to polar form on the Sharp EL-506W. The magnitude and angle parts of the answer must be retrieved separately. (a) Showing the magnitude part of answer (b) Showing the angle part of answer.

b. Key in a left bracket, key in 3, press the minus key (not the ± key), key in 4, press the "imaginary" key, then add a right bracket. You should see $(3 - 4i)$ on your display. Now invoke the inverse operation. Your calculator should show $(3 - 4i)^{-1}$ (or something similar). Press the = key to get the answer $0.12 + 0.16i$.

Sharp EL-506W Procedure: After you enter $(3 - 4i)$, press 2nd F, then x^{-1} (i.e., key 2). This yields $(3 - 4i)^{-1}$. Press the = key. If your calculator is set to rectangular, the real part of the answer, 0.12, will appear on your display. Press 2nd F, then the $\leftarrow, \rightarrow$ key to view the imaginary part, $0.16i$. If your calculator is set to polar, you will get $0.2\ \angle 53.13°$. To convert this to rectangular, press 2nd F, then $\rightarrow xy$.

While all calculators with complex number capabilities can easily perform basic calculations such as those of Example 2–5, there is a great variation in how easily they handle complex calculations. To illustrate, consider the expression $(9 + j2) + 10\angle 30°$. On some calculators, you need intermediate steps, for example, you need to convert $10\angle 30°$ to rectangular in order to perform the addition. On other calculators, you can freely mix rectangular and polar numbers on input. This is illustrated in Figure 2–7. The answer will be displayed in either rectangular or polar, depending on what mode you have set.

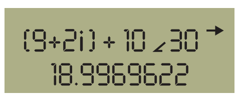


FIGURE 2–7 On this calculator, rectangular and polar forms can be mixed freely on input. (Only the magnitude part of the answer is shown, as the angle part must be retrieved separately—recall Figure 2–6.)

(In polar form, the answer is $18.997 \angle 21.62°$. Note that only the magnitude of the answer is shown in Figure 2–7. A second step is required to extract the angle.) Now consider Example 2–6.

EXAMPLE 2–6

Reduce the following to polar form.

$$(6 + j5) + \frac{(3 - j4)(10\angle 40°)}{6 + 30\angle 53.13°}$$

Solution

a. Using a calculator with basic capabilities requires a number of intermediate steps, some of which are shown below.

$$\text{answer} = (6 + j5) + \frac{(5\angle -53.13)(10\angle 40)}{6 + (18 + j24)}$$

$$= (6 + j5) + \frac{(5\angle -53.13)(10\angle 40)}{24 + j24}$$

$$= (6 + j5) + \frac{(5\angle -53.13)(10\angle 40)}{33.94\angle 45}$$

$$= (6 + j5) + 1.473\angle -58.13 = (6 + j5) + (0.778 - j1.251)$$

$$= 6.778 + j3.749 = 7.75\angle 28.95°$$

b. While all calculators permit you to solve the problem in small steps as in (a), others (such as the Sharp EL-506W and the TI-89) permit you to solve the problem in one step. Using brackets to avoid ambiguity, you can, for example, key the expression in as follows:

$$(6 + 5i) + ((3 - 4i)(10\angle 40)) \div (6 + 30\angle 53.13)$$

When you press the = key (or whatever key your calculator manual specifies), the expression is evaluated and the answer displayed. If your calculator is set to polar mode, you will get an answer of $7.75\angle 28.95°$; if your calculator is set to rectangular mode, you will get $6.778 + 3.749i$.

2.2 Complex Numbers in ac Analysis

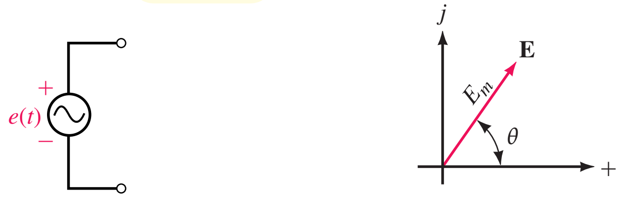
Representing ac Voltages and Currents by Complex Numbers

As you learned in Chapter 15, ac voltages and currents can be represented as phasors (see Note 1). Since phasors have magnitude and angle, they can be viewed as complex numbers. To get at the idea, consider the voltage source of Figure 2–8(a). Its phasor equivalent (b) has magnitude E_m and angle θ. It therefore can be viewed as the complex number

$$\mathbf{E} = E_m\angle\theta \qquad \textbf{(2–9)}$$

NOTES...

1. As you saw in Chapter 15, the basic definition of a phasor is a rotating radius vector with a length equal to the amplitude (E_m or I_m) of the voltage or current that it represents. In practice, however, phasor voltages and phasor currents are always



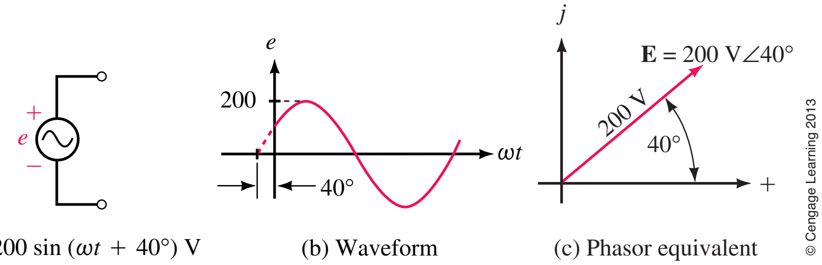
(a) $e(t) = E_m \sin(\omega t + \theta)$ (b) $\mathbf{E} = E_m\angle\theta$

FIGURE 2–8 Representation of a sinusoidal source voltage as a complex number.

expressed as rms values. We will make this change shortly; in the meantime, there are a few ideas that we need to explore using the phasor's fundamental definition.

2. Quantities expressed as time functions are said to be in the *time domain,* while quantities expressed as phasors are said to be in the *phasor domain.* Thus, $e = 200 \sin(\omega t + 40°)$ is in the time domain, while its phasor equivalent $\mathbf{E} = 200\text{ V}\angle 40°$ is in the phasor domain.
3. The phasor domain is sometimes referred to as the *frequency domain.*

From this point of view, the sinusoidal voltage $e(t) = 200 \sin(\omega t + 40°)$ of Figure 2–9(a) and (b) can be represented by its phasor equivalent, $\mathbf{E} = 200\text{ V}\angle 40°$, as in (c).

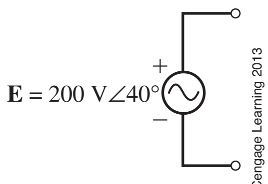


(a) $e = 200 \sin(\omega t + 40°)$ V (b) Waveform (c) Phasor equivalent

FIGURE 2–9 Transforming $e = 200 \sin(\omega t + 40°)$ V to $\mathbf{E} = 200\text{ V}\angle 40°$.

We can take advantage of this equivalence. *Rather than show a source as a time-varying voltage* e(t) *that we subsequently convert to a phasor, we can represent the source by its phasor equivalent right from the start.* This viewpoint is illustrated in Figure 2–10. Since $\mathbf{E} = 200\text{ V}\angle 40°$, this representation retains all the original information of Figure 2–9 since the sinusoidal time variation and its associated angle as illustrated in Figure 2–9(b) are implicit in the definition of the phasor.

The idea illustrated in Figure 2–10 is of fundamental importance to circuit theory. *By replacing the time function* e(t) *with its phasor equivalent* **E**, *we have transformed the source from the time domain to the phasor domain*—see Notes 2 and 3. (This is one of Steinmetz's great contributions to electrical science.)



Transformed source

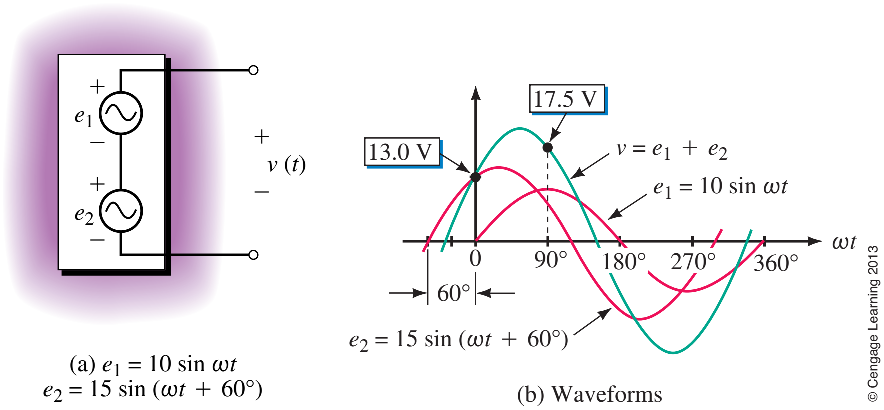
FIGURE 2–10 Direct transformation of the source.

Circuit Relationships in the Phasor Domain

Before we move on, we should note that both Kirchhoff's voltage law (KVL) and Kirchhoff's current law (KCL) apply in the **time domain** and in the **phasor domain**—see Notes 2 and 3. For example, $e = v_1 + v_2$ in the time domain can be transformed to $\mathbf{E} = \mathbf{V}_1 + \mathbf{V}_2$ in the phasor domain and vice versa. Similarly for currents. (Although we have stated the preceding result without proof, it can be proven rigorously—as is done in some advanced circuit theory textbooks.) The value of this approach is illustrated next.

Summing ac Voltages and Currents

Sinusoidal quantities must sometimes be added or subtracted as in Figure 2–11. Here, we want the sum of e_1 and e_2, where $e_1 = 10 \sin \omega t$ and $e_2 = 15 \sin(\omega t + 60°)$. First, let us illustrate the process in the time domain. The sum of



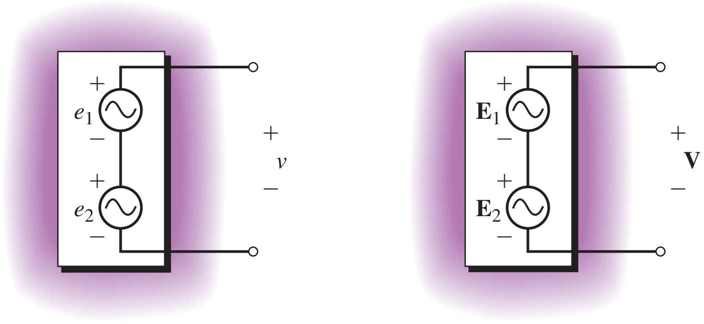
(a) $e_1 = 10 \sin \omega t$
$e_2 = 15 \sin(\omega t + 60°)$

(b) Waveforms

FIGURE 2–11 Summing waveforms point by point.

e_1 and e_2 can be found by adding waveforms point by point as in Figure 2–11(b). For example, at $\omega t = 0°$, $e_1 = 10 \sin 0° = 0$, and $e_2 = 15 \sin(0° + 60°) = 13$ V, and their sum is 13 V. Similarly, at $\omega t = 90°$, $e_1 = 10 \sin 90° = 10$ V, and $e_2 = 15 \sin(90° + 60°) = 15 \sin 150° = 7.5$ V, and their sum is 17.5 V. Continuing in this manner, the sum of $e_1 + e_2$ (the green waveform) is obtained.

As you can see, the process is tedious and provides no analytic expression for the resulting voltage; thus, it is seldom used. A better way is to transform the sources and use complex numbers to perform the addition. This is shown in Figure 2–12. Here, we have replaced voltages e_1 and e_2 with their phasor equivalents, $\mathbf{E}_1$ and $\mathbf{E}_2$, and v with its phasor equivalent, $\mathbf{V}$. Since $v = e_1 + e_2$, replacing v, e_1, and e_2 with their phasor equivalents yields $\mathbf{V} = \mathbf{E}_1 + \mathbf{E}_2$. Now $\mathbf{V}$ can be found by adding $\mathbf{E}_1$ and $\mathbf{E}_2$ as complex numbers. Once $\mathbf{V}$ is known, its corresponding time equation and companion waveform can be determined. This is the procedure used in practice.



(a) Original network.
$v(t) = e_1(t) + e_2(t)$

(b) Transformed network.
$\mathbf{V} = \mathbf{E}_1 + \mathbf{E}_2$

FIGURE 2–12 Transformed circuit. This is one of the key ideas of sinusoidal circuit analysis.

EXAMPLE 2–7

Given $e_1 = 10 \sin \omega t$ V and $e_2 = 15 \sin(\omega t + 60°)$ V as before, determine v and sketch it.

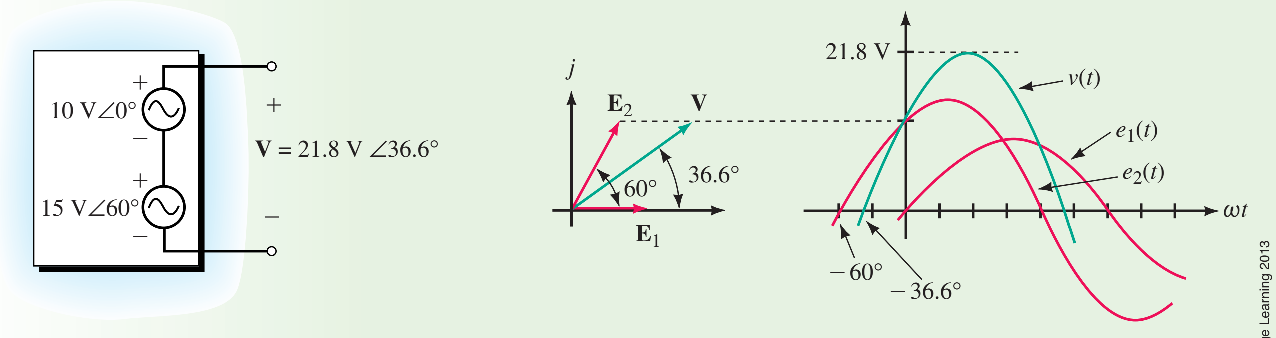
CircuitSim 2 -1

Solution

$e_1 = 10 \sin \omega t$ V. Thus, $\mathbf{E}_1 = 10\text{ V}\angle 0°$.

$e_2 = 15 \sin(\omega t + 60°)$ V. Thus, $\mathbf{E}_2 = 15\text{ V}\angle 60°$.Transformed

sources are shown in Figure 2–13(a) and phasors in (b).



(a) Phasor summation

(b) Phasors

(c) Waveforms: Compare $v(t)$ here to $v(t)$ in Figure 2–11(b)

FIGURE 16–13 Note that $v(t)$, determined from phasor $\mathbf{V}$ gives the same result as adding e_1 and e_2 point by point.

$$\mathbf{V} = \mathbf{E}_1 + \mathbf{E}_2 = 10\angle 0° + 15\angle 60° = 21.8\ \text{V}\angle 36.6°$$

Thus, $v = 21.8 \sin(\omega t + 36.6°)$ V.

Waveforms are shown in (c). (To verify that this produces the same result as adding the waveforms point by point, see Practice Problems 2 or the Multisim simulation CircuitSim 2–1.)

PRACTICE PROBLEMS 2

Verify by direct substitution that $v = 21.8 \sin(\omega t + 36.6°)$ V, as in Figure 2–13, is the sum of e_1 and e_2. To do this, compute e_1 and e_2 at a point, add them, then compare the sum to $21.8 \sin(\omega t + 36.6°)$ V computed at the same point. Perform this computation at $\omega t = 30°$ intervals over the complete cycle to satisfy yourself that the result is true everywhere. [For example, at $\omega t = 0°$, $v = 21.8 \sin(\omega t + 36.6°) = 21.8 \sin(36.6°) = 13$ V, as we saw in Figure 2–11.]

Answers

Here are the points on the graph at 30° intervals:

ωt	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
v	13	20	21.7	17.5	8.66	−2.5	−13	−20	−21.7	−17.5	−8.66	2.5	13

Representing Phasors as RMS Values

To this point, we have used peak values such as E_m, V_m, and I_m to represent the magnitudes of phasor voltages and currents, as this has been most convenient for our purposes. However, in practice, phasors are always expressed as rms values. Accordingly, we will now make the change. Thus, from now on, a phasor such as $\mathbf{V} = 120\ \text{V}\angle 0°$ will be taken to mean 120 V rms at an angle of zero degrees. If you need to convert this to a time function, first multiply the rms value by $\sqrt{2}$, then follow the usual procedure. Thus, the phasor $\mathbf{V} = 120\ \text{V}\angle 0°$, when converted to the time domain, yields $v(t) = \sqrt{2}(120\ \text{V}) \sin \omega t = 170 \sin \omega t$ V. See Note 1.

NOTES...

1. Although we use phasors to represent sinusoidal waveforms, it should be noted that sine waves and phasors are not the same thing. Sinusoidal voltages and currents are real—they are the actual quantities that you measure with meters and whose waveforms you see on oscilloscopes. Phasors, on the other hand, are mathematical abstractions that we use to help visualize relationships and solve problems.
2. To add or subtract sinusoidal voltages or currents, follow the three steps outlined in Example 2–7. That is,
   - convert sine waves to phasors and express them in complex number form,
   - add or subtract the complex numbers,
   - convert back to time functions if desired.

EXAMPLE 2–8

Express the voltage and current of Figure 2–14 in both the time and the phasor domains. (Remember to use rms values in the phasor domain.)

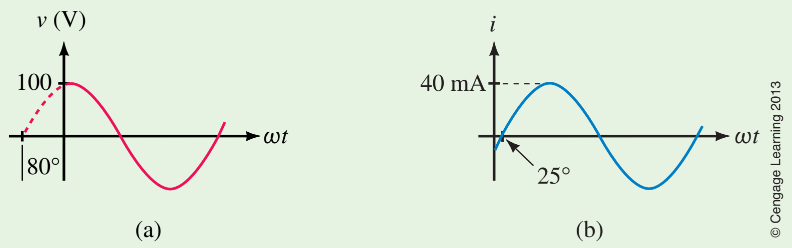


FIGURE 2–14

Solution

a. Time domain: $v = 100 \sin(\omega t + 80°)$ volts.
Phasor domain: $\mathbf{V} = (0.707)(100\text{ V }\angle 80°) = 70.7\text{ V}\angle 80°$ (rms).

b. Time domain: $i = 40 \sin(\omega t - 25°)$ mA.
Phasor domain: $\mathbf{I} = (0.707)(40\text{ mA}\angle -25°) = 28.3\text{ mA}\angle -25°$ (rms).

EXAMPLE 2–9

Consider two currents $i1$ and $i2$. If $i_1 = 14.14 \sin(\omega t - 55°)$ A and $i_2 = 4 \sin(\omega t + 15°)$ A:

CircuitSim 2- 2

a. Determine their sum, i. Work with rms values.

b. If the currents are measured with ammeters, what will be their readings?

Solution

a.
$$\mathbf{I}_1 = (0.707)(14.14\text{ A})\angle -55° = 10\text{ A}\angle -55°$$
$$\mathbf{I}_2 = (0.707)(4\text{ A})\angle 15° = 2.828\text{ A}\angle 15°$$
$$= \mathbf{I}_1 + \mathbf{I}_2 = 10\text{ A}\angle - 55° + 2.828\text{ A}\angle 15°$$
$$= 11.3\text{ A}\angle -41.4$$
$i(t) = \sqrt{2}(11.3) \sin(\omega t - 41.4°) = 2 \sin(\omega t - 41.4°)$ Ab. 10 A, 2.828 A, and 11.3 A.

Example 2–9 illustrates a primary reason why we use rms—it ties in with standard practice in the sense that all voltmeters and ammeters display their measurements in rms. Additionally, in real life (after you have gotten through the learning process), you seldom need to deal explicitly with time functions; all you really need is magnitude and angle. Thus, very soon, we will stop working in the time domain entirely and work only with phasors. At that point, the solution will be complete when we have the answer in the form $\mathbf{I} = 11.3\text{A}\angle -41.4°$. (To help focus on rms, voltages and currents in the next two examples are expressed as an rms value times $\sqrt{2}$.)

EXAMPLE 2–10

For Figure 2–15, $v_1 = \sqrt{2}(2)\ \sin \omega t$ V, $v_2 = \sqrt{2}(24)\ \sin(\omega t + 90°)$ and $v_3 = \sqrt{2}(15)\ \sin(\omega t - 90°)$ V. Determine source voltage e. If measurements are made with voltmeters, what will the meters indicate?

Solution The answer can be obtained by KVL. First, convert to phasors. Thus, $\mathbf{V}_1 = 2\text{ V} \angle 0°$, $\mathbf{V}_2 = 24\text{ V} \angle 90°$, and $\mathbf{V}_3 = 15\text{ V} \angle{-90°}$. KVL yields $\mathbf{E} = \mathbf{V}_1 + \mathbf{V}_2 + \mathbf{V}_3 = 2\text{ V} \angle 0° + 24\text{ V} \angle 90° + 15\text{ V} \angle{-90°} = 18.4\text{ V} \angle 29.4°$. Converting back to a function of time yields $e = \sqrt{2}(18.4)\ \sin(\omega t + 29.4°)$ V. Voltmeter readings will be 2 V, 24 V, and 15 V for the voltage drops and 18.4 V for the source.

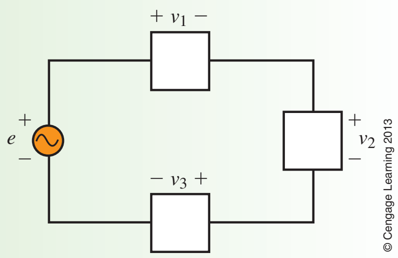


FIGURE 16–15

EXAMPLE 2–11

For Figure 2–2, $i_1 = \sqrt{2}(23)\ \sin \omega t$ mA, $i_2 = \sqrt{2}(0.29)\ \sin(\omega t + 63°)$ A, and $i_3 = \sqrt{2}(127) \times 10^{-3}\ \sin(\omega t - 72°)$ A. Determine current i_T. If currents are measured with ammeters, what will they indicate?

Solution Convert to phasors. Thus, $\mathbf{I}_1 = 23\text{ mA}\angle 0°$, $\mathbf{I}_2 = 0.29\text{ A}\angle 63°$, and $\mathbf{I}_3 = 127 \times 10^{-3}\text{ A}\angle{-72°}$. KCL yields $\mathbf{I}_T = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 = 23\text{ mA}\angle 0° + 290\text{ mA}\angle 63° + 127\text{ mA}\angle{-72°} = 238\text{ mA}\angle 35.4°$. Converting back to a function of time yields $i_T = \sqrt{2}(238)\ \sin(\omega t + 35.4°)$ mA. Ammeters would read 23 mA, 290 mA, and 127 mA for the branch currents and 238 mA for the total current.

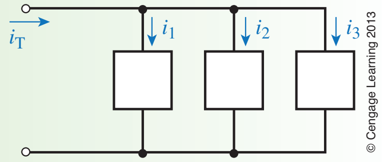


FIGURE 2 –16

PRACTICE PROBLEMS 3

1. Convert the following to time functions. Values are rms.

   a. $\mathbf{E} = 500\text{ mV}\angle{-20°}$ b. $\mathbf{I} = 80\text{ A}\angle 40°$

2. For the circuit of Figure 16–17, using phasors, determine voltage e_1.
   If voltages are measured with voltmeters, what will the meters indicate?

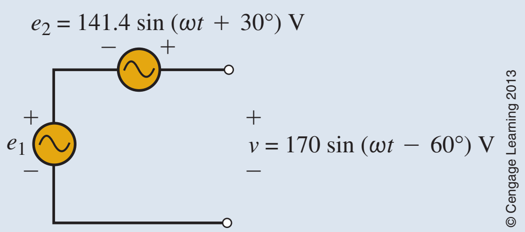


FIGURE 2–17

Answers

1. a. $e = 707 \sin(\omega t - 20°)$ mV b. $i = 113 \sin(\omega t + 40°)$ A
2. $e_1 = 221 \sin(\omega t - 99.8°)$ V, 156.3 V, 100 V for the sources and 120.2 for their sum.

IN-PROCESS LEARNING CHECK 1

(Answers are at the end of the chapter.)

1. Convert the following to polar form:

   a. $j6$ b. $-j4$ c. $3 + j3$ d. $4 - j6$

   e. $-5 + j8$ f. $1 - j2$ g. $-2 - j3$

2. Convert the following to rectangular form:

   a. $4\angle 90°$ b. $3\angle 0°$ c. $2\angle -90°$ d. $5\angle 40°$

   e. $6\angle 120°$ f. $2.5\angle -20°$ g. $1.75\angle -160°$

3. If $-\mathbf{C} = 12\angle -140°$, what is $\mathbf{C}$?
4. Given: $\mathbf{C}_1 = 36 + j4$ and $\mathbf{C}_2 = 52 - j11$. Determine $\mathbf{C}_1 + \mathbf{C}_2$, $\mathbf{C}_1 - \mathbf{C}_2$, $1/(\mathbf{C}_1 + \mathbf{C}_2)$, and $1/(\mathbf{C}_1 - \mathbf{C}_2)$. Express in rectangular form.
5. Given: $\mathbf{C}_1 = 24\angle 25°$ and $\mathbf{C}_2 = 12\angle -125°$. Determine $\mathbf{C}_1 \cdot \mathbf{C}_2$ and $\mathbf{C}_1/\mathbf{C}_2$.
6. Compute the following and express answers in rectangular form:

   a. $\dfrac{6 + j4}{10\angle 20°} + (14 + j2)$ b. $(1 + j6) + \left[2 + \dfrac{(12\angle 0°)(14 + j2)}{6 - (10\angle 20°)(2\angle -10°)}\right]$

7. For Figure 16–18, determine i_T where $i_1 = 10 \sin \omega t$, $i_2 = 20 \sin(\omega t - 90°)$, and $i_3 = 5 \sin(\omega t + 90°)$.
8. If meters are used to measure the currents of Figure 16–18, what will be their readings?

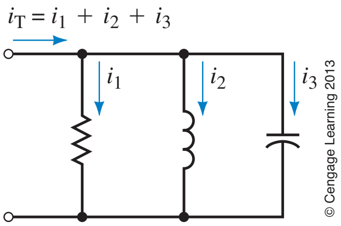


FIGURE 2–18

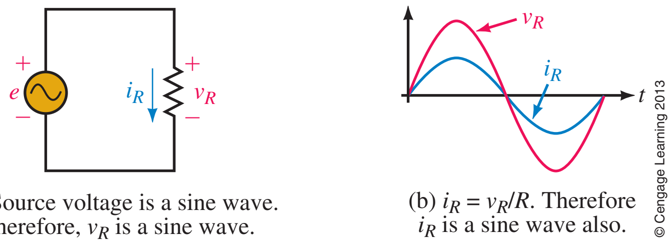
2.3 *R, L,* and *C* Circuits with Sinusoidal Excitation

R, *L*, and *C* circuit elements each have quite different electrical properties. Resistance, for example, opposes current, while inductance opposes changes in current, and capacitance opposes changes in voltage. These differences result in quite different voltage-current relationships as you saw earlier. We now investigate these relationships for the case of sinusoidal ac. Sine waves have several important characteristics that you will discover from this investigation:

1. When a circuit consisting of linear circuit elements *R*, *L*, and *C* is connected to a sinusoidal source, all currents and voltages in the circuit will be sinusoidal and of the same frequency as the source.
2. Since all voltages and currents are sinusoidal functions of the same frequency as the source, it is necessary only to determine their magnitudes and phase angles to complete the solution.

2.4 Resistance and Sinusoidal ac

We begin with a purely resistive circuit. Here, Ohm's law applies, and thus current is directly proportional to voltage. Current variations therefore follow voltage variations, reaching their peak when voltage reaches its peak, changing direction when voltage changes polarity, and so on (Figure 2–19). From this, we conclude that *for a purely resistive circuit, current and voltage are in phase.* Since voltage and current waveforms coincide, their phasors also coincide (Figure 2–20).



(a) Source voltage is a sine wave. Therefore, v_R is a sine wave.

(b) $i_R = v_R/R$. Therefore i_R is a sine wave also.

FIGURE 2–19 Ohm's law applies. Note that current and voltage are in phase.

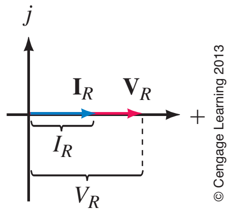


FIGURE 2–20 For a resistor, voltage and current phasors are in phase.

The relationship illustrated in Figure 2–19 may be stated mathematically as

$$i_R = \frac{v_R}{R} = \frac{V_m \sin \omega t}{R} = \frac{V_m}{R} \sin \omega t = I_m \sin \omega t \qquad \textbf{(2–10)}$$

where

$$I_m = V_m/R \qquad \textbf{(2–11)}$$

Transposing,

$$V_m = I_m R \qquad \textbf{(2–12)}$$

The in-phase relationship is true regardless of reference. Thus, if $v_R = V_m \sin(\omega t + \theta)$, then $i_R = I_m \sin(\omega t + \theta)$.

EXAMPLE 2–12

For the circuit of Figure 2–19(a), if $R = 5\ \Omega$ and $i_R = 12 \sin(\omega t - 18°)$ A, determine v_R.

Solution $v_R = Ri_R = 5 \times 12 \sin(\omega t - 18°) = 60 \sin(\omega t - 18°)$ V. The waveforms are shown in Figure 2–21.

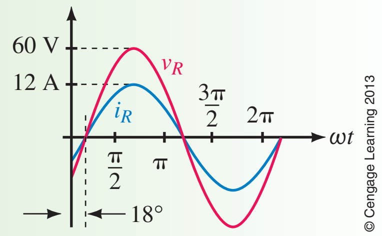


FIGURE 2 –21

PRACTICE PROBLEMS 4

1. If $v_R = 150 \cos \omega t$ V and $R = 25$ kΩ, determine i_R and sketch both waveforms.
2. If $v_R = 100 \sin(\omega t + 30°)$ V and $R = 0.2$ MΩ, determine i_R and sketch both waveforms.

Answers
1. $i_R = 6 \cos \omega t$ mA. v_R and i_R are in phase.
2. $i_R = 0.5 \sin(\omega t + 30°)$ mA. v_R and i_R are in phase.

2.5 Inductance and Sinusoidal ac

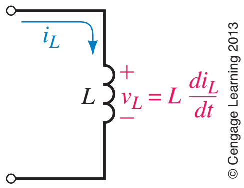


FIGURE 2–22 Voltage v_L is proportional to the rate of change of current i_L.

Phase Lag in an Inductive Circuit

As you saw in Chapter 13, for an ideal inductor, voltage v_L is proportional to the rate of change of current. Because of this, voltage and current are not in phase as they are for a resistive circuit. This can be shown with a bit of calculus. From Figure 2–22, $v_L = L di_L/dt$. For a sine wave of current, you get the following when you differentiate

$$v_L = L\frac{di_L}{dt} = L\frac{d}{dt}(I_m \sin \omega t) = \omega L I_m \cos \omega t = V_m \cos \omega t$$

Utilizing the trigonometric identity $\cos \omega t = \sin(\omega t + 90°)$, you can write this as

$$v_L = V_m \sin(\omega t + 90°) \tag{2–13}$$

where

$$V_m = \omega L I_m \tag{2–14}$$

Voltage and current waveforms are shown in Figure 2–23, and phasors in Figure 2–24. As you can see, *for a purely inductive circuit, current lags voltage by 90°* (i.e., ¼ cycle). Alternatively you can say that voltage leads current by 90°.

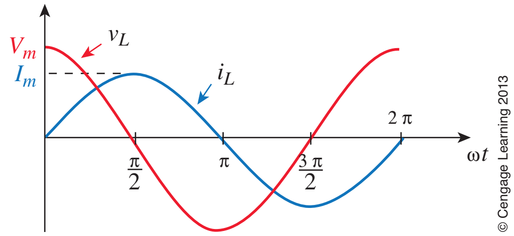


FIGURE 2–23 For inductance, current lags voltage by 90°. Here, i_L is reference.

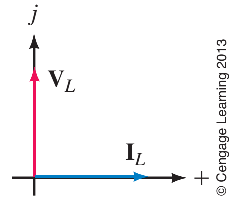
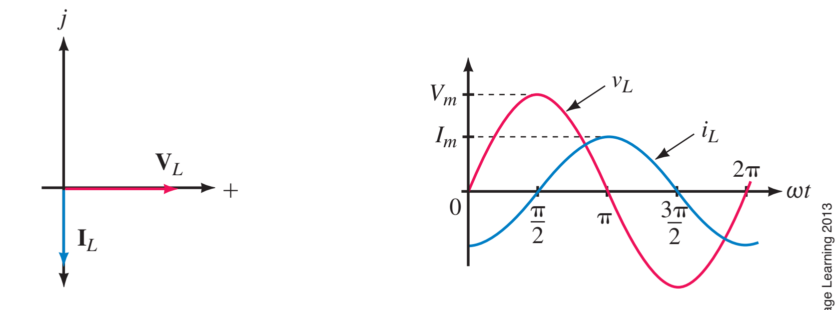


FIGURE 2–24 Phasors for the waveforms of Figure 2–23 showing the 90° lag of current.

Although we have shown that current lags voltage by 90° for the case of Figure 2–23, this relationship is true in general, that is, current always lags voltage by 90° regardless of the choice of reference. This is illustrated in Figure 2–25. Here, $\mathbf{V}_L$ is at 0° and $\mathbf{I}_L$ at $-90°$. Thus, voltage v_L will be a sine wave and current i_L a negative cosine wave, that is, $i_L = -I_m \cos \omega t$. Since i_L is a negative cosine wave, it can also be expressed as $i_L = I_m \sin(\omega t - 90°)$. The waveforms are shown in (b).



(a) Current $\mathbf{I}_L$ always lags voltage $\mathbf{V}_L$ by 90°

(b) Waveforms

FIGURE 2–25 Phasors and waveforms when $\mathbf{V}_L$ is used as reference.

Since current always lags voltage by 90° for a pure inductance, you can, if you know the phase of the voltage, determine the phase of the current, and vice versa. Thus, if v_L is known, i_L must lag it by 90°, while if i_L is known, v_L must lead it by 90°.

Inductive Reactance

From Equation 2–14, we see that the ratio V_m to I_m is

$$\frac{V_m}{I_m} = \omega L \qquad \textbf{(2–15)}$$

The ratio of V_m to I_m is defined as **inductive reactance** and is given the symbol X_L. Since the ratio of volts to amps is ohms, reactance has units of ohms. Thus,

$$X_L = \frac{V_m}{I_m} \quad (\Omega) \qquad \textbf{(2–2)}$$

Combining Equations 2–15 and 2–2 yields

$$X_L = \omega L \ (\Omega) \qquad \textbf{(2–17)}$$

where ω is in radians per second and L is in henries. *Reactance* $\mathrm{X_L}$ *represents the opposition that inductance presents to current for the sinusoidal ac case.*

We now have everything that we need to solve simple inductive circuits with sinusoidal excitation; that is, we know that current lags voltage by 90° and that their amplitudes are related by

$$I_m = \frac{V_m}{X_L} \qquad \textbf{(2–18)}$$

and

$$V_m = I_m X_L \qquad \textbf{(2–19)}$$

EXAMPLE 2–13

The voltage across a 0.2-H inductance is $v_L = 100 \sin(400t + 70°)$ V. Determine i_L and sketch it.

Solution $\omega = 400$ rad/s. Therefore, $X_L = \omega L = (400)(0.2) = 80\ \Omega$.

$$I_m = \frac{V_m}{X_L} = \frac{100\ \text{V}}{80\ \Omega} = 1.25\ \text{A}$$

The current lags the voltage by 90°. Therefore $i_L = 1.25 \sin(400t - 20°)$ A as indicated in Figure 2–26.

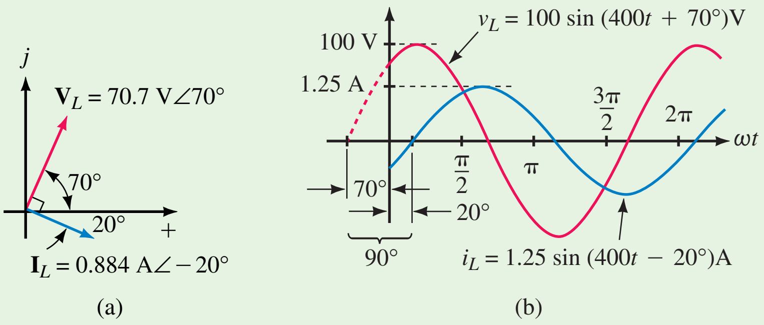


FIGURE 2–26 With voltage $\mathbf{V}_L$ at 70°, current $\mathbf{I}_L$ will be 90° later at –20°.

NOTES...

Remember to show phasors as rms values. However (by convention), we draw waveforms using peak amplitudes—V_m and I_m, and so on.

EXAMPLE 2–14

The current through a 0.01-H inductance is $i_L = 20 \sin(\omega t - 50°)$ A and $f =$ 60 Hz. Determine v_L.

Solution

$$\omega = 2\pi f = 2\pi(60) = 377 \text{ rad/s}$$
$$X_L = \omega L = (377)(0.01) = 3.77\ \Omega$$
$$V_m = I_m X_L = (20\text{ A})(3.77\ \Omega) = 75.4\text{ V}$$

Voltage leads current by 90°. Thus, $v_L = 75.4 \sin(377t + 40°)$ V as shown in Figure 2–27.

CircuitSim 2- 3

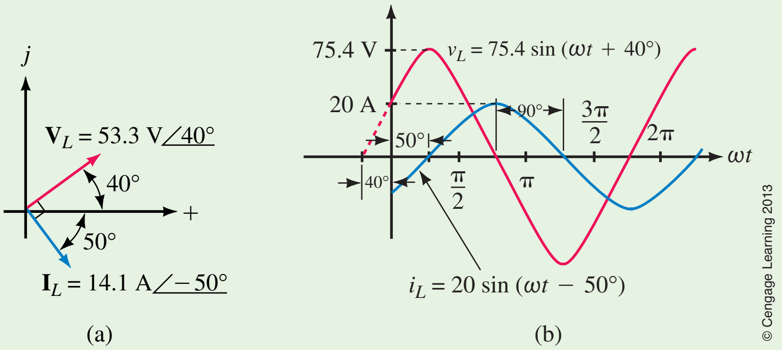


FIGURE 2–27

PRACTICE PROBLEMS 5

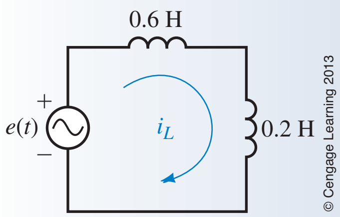
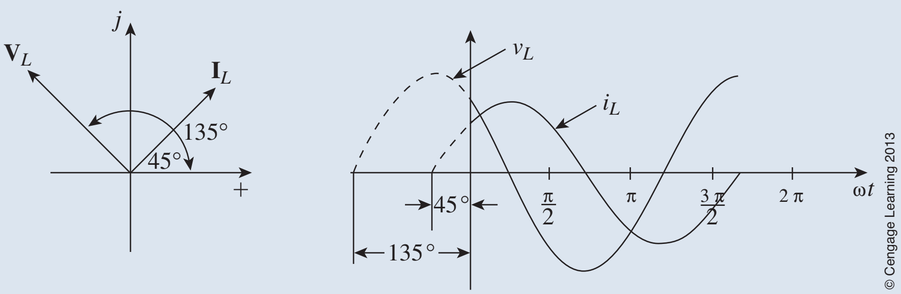


FIGURE 2 –28

1. Two inductances are connected in series (Figure 16–28). If $e = 100 \sin \omega t$ and $f = 10$ kHz, determine the current. Sketch voltage and current waveforms.
2. The current through a 0.5-H inductance is $i_L = 100 \sin(2400t + 45°)$ mA. Determine v_L and sketch voltage and current phasors and waveforms.

Answers

1. $i_L = 1.99 \sin(\omega t - 90°)$ mA. Waveforms same as Figure 2–25.
2. $v_L = 120 \sin(2400t + 135°\text{ V})$. See the following figure for waveforms.



Variation of Inductive Reactance with Frequency

Since $X_L = \omega L = 2\pi f L$, inductive reactance is directly proportional to frequency (Figure 2–29). Thus, if frequency is doubled, reactance doubles, while

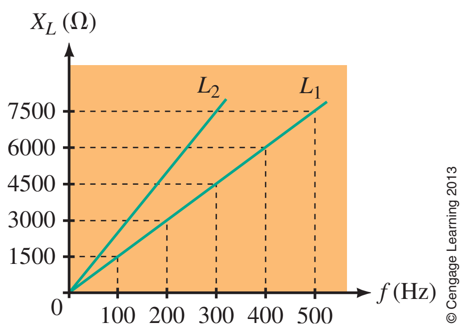


FIGURE 2–29 Variation of X_L with frequency. Note that $L_2 > L_1$.

if frequency is halved, reactance halves, and so on. In addition, X_L is directly proportional to inductance. Thus, if inductance is doubled, X_L is doubled, and so on. Note also that at $f = 0$, $X_L = 0\ \Omega$. This means that inductance looks like a short circuit to dc. (We already concluded this earlier in Chapter 13.)

PRACTICE PROBLEMS 6

A coil has 50 ohms inductive reactance. If both the inductance and the frequency are doubled, what is the new X_L?

Answer
200 Ω

2.6 Capacitance and Sinusoidal ac

Phase Lead in a Capacitive Circuit

For capacitance, current is proportional to the rate of change of voltage, that is, $i_C = C\,dv_C/dt$ [Figure 2–30(a)]. Thus, if v_C is a sine wave, you get upon substitution

$$i_C = C\frac{dv_C}{dt} = C\frac{d}{dt}(V_m \sin \omega t) = \omega C V_m \cos \omega t = I_m \cos \omega t$$

Using the appropriate trigonometric identity, this can be written as

$$i_C = I_m \sin(\omega t + 90°) \qquad \textbf{(2–20)}$$

where

$$I_m = \omega C V_m \qquad \textbf{(2–21)}$$

Waveforms are shown in Figure 2–30(b) and phasors in (c). As indicated, *for a purely capacitive circuit, current leads voltage by 90°*, or alternatively, voltage

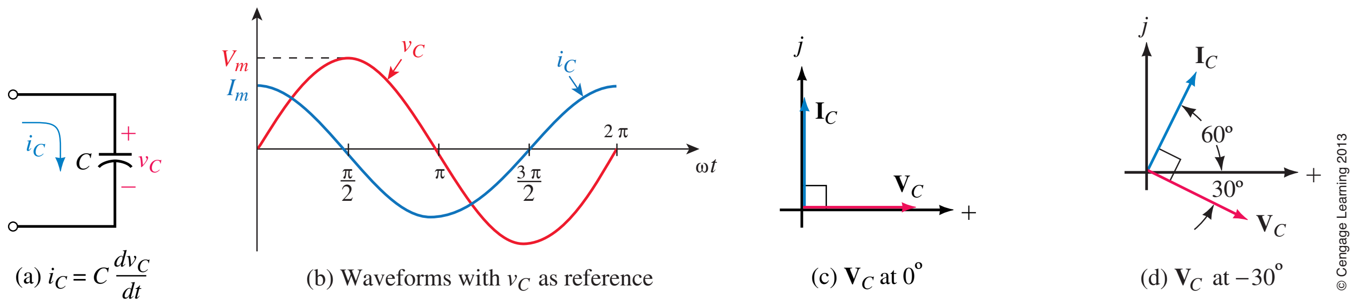


FIGURE 2–30 For capacitance, current always leads voltage by 90°.

lags current by 90°. This relationship is true regardless of reference. Thus, if the voltage is known, the current must lead by 90° while if the current is known, the voltage must lag by 90°. For example, if $\mathbf{I}_C$ is at 60° as in (d), $\mathbf{V}_C$ must be at −30°.

PRACTICE PROBLEMS 7

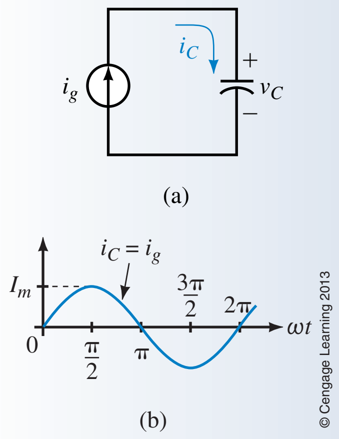


FIGURE 16–31

1. The current source of Figure 16–31(a) is a sine wave. Sketch phasors and capacitor voltage v_C.
2. Refer to the circuit of Figure 16–32(a):
   a. Sketch the phasors.
   b. Sketch capacitor current i_C.

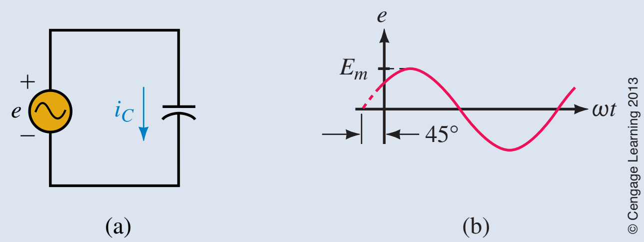


FIGURE 2–32

Answers

1. $\mathbf{I}_C$ is at 0°; $\mathbf{V}_C$ is at −90°; v_C is a negative cosine wave.
2. a. $\mathbf{V}_C$ is at 45° and $\mathbf{I}_C$ is at 135°.
   b. Waveforms are the same as for Problem 2, Practice Problems 5, except that voltage and current waveforms are interchanged.

Capacitive Reactance

Now consider the relationship between maximum capacitor voltage and current magnitudes. As we saw in Equation 2–21, they are related by $I_m = \omega C V_m$. Rearranging, we get $V_m/I_m = 1/\omega C$. The ratio of V_m to I_m is defined as **capacitive reactance** and is given the symbol X_C. That is,

$$X_C = \frac{V_m}{I_m} \quad (\Omega)$$

Since $V_m/I_m = 1/\omega C$, we also get

$$X_C = \frac{1}{\omega C} \quad (\Omega) \qquad \textbf{(2–22)}$$

where ω is in radians per second and C is in farads. *Reactance* X_C *represents the opposition that capacitance presents to current for the sinusoidal ac case.* It has units of ohms.

We now have everything that we need to solve simple capacitive circuits with sinusoidal excitation, that is, we know that current leads voltage by 90° and that

$$I_m = \frac{V_m}{X_C} \qquad \textbf{(2–23)}$$

and

$$V_m = I_m X_C \qquad \textbf{(2–24)}$$

EXAMPLE 2–15

The voltage across a 10-μF capacitance is $v_C = 100 \sin(\omega t - 40°)$ V and $f = 1000$ Hz. Determine i_C and sketch its waveform.

Solution

$$\omega = 2\pi f = 2\pi(1000 \text{ Hz}) = 6283 \text{ rad/s}$$

$$X_C = \frac{1}{\omega C} = \frac{1}{(6283)(10 \times 10^{-6})} = 15.92\ \Omega$$

$$I_m = \frac{V_m}{X_C} = \frac{100 \text{ V}}{15.92\ \Omega} = 6.28 \text{ A}$$

Since current leads voltage by 90°, $i_C = 6.28 \sin(6283t + 50°)$ A as indicated in Figure 2–33.

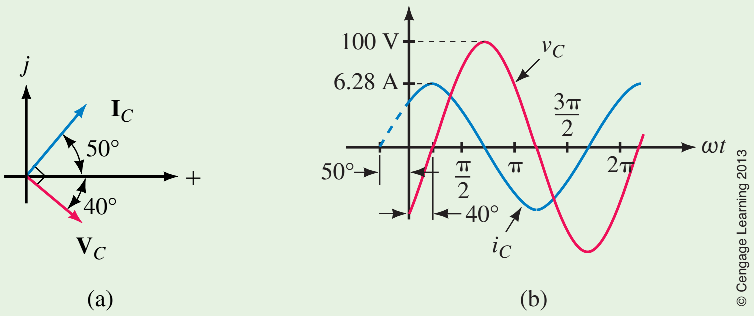


FIGURE 2–33 Phasors are not to scale with waveform.

CircuitSim 16-4

EXAMPLE 2–2

The current through a 0.1-μF capacitance is $i_C = 5 \sin(1000t + 120°)$ mA. Determine v_C.

Solution

$$X_C = \frac{1}{\omega C} = \frac{1}{(1000 \text{ rad/s})(0.1 \times 10^{-6}\ F)} = 10\ \text{k}\Omega$$

Thus, $V_m = I_m X_C = (5 \text{ mA})(10\ \text{k}\Omega) = 50$ V. Since voltage lags current by 90°, $v_C = 50 \sin(1000t + 30°)$ V. Waveforms and phasors are shown in Figure 2–34.

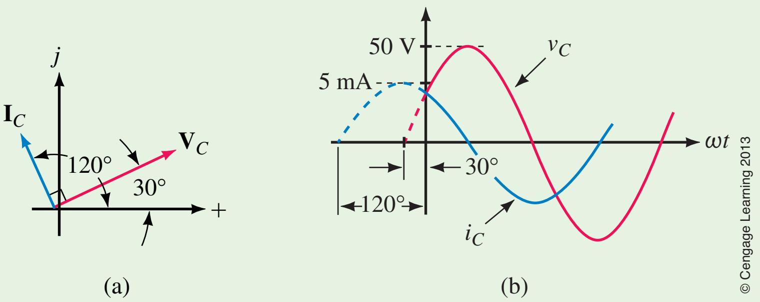


FIGURE 2–34 Phasors are not to scale with waveform.

PRACTICE PROBLEMS 8

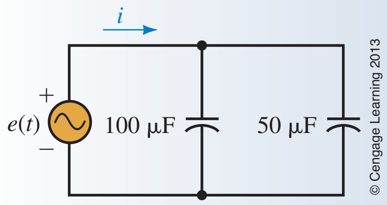


FIGURE 2 –35

Two capacitances are connected in parallel (Figure 2–35). If $e = 100 \sin \omega t$ V and $f = 10$ Hz, determine the source current. Sketch current and voltage phasors and waveforms.

Answer: $i = 0.942 \sin(62.8t + 90°) = 0.942 \cos 62.8t$ ASee Figure 2–30(b) and (c).

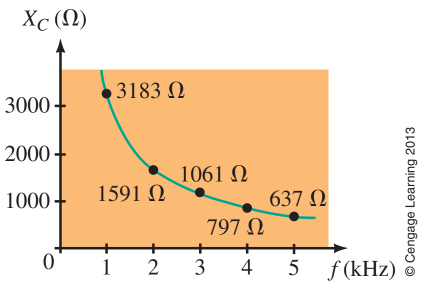


FIGURE 2–36 X_C varies inversely with frequency. Values shown are for $C = 0.05$ µF.

Variation of Capacitive Reactance with Frequency

Since $X_C = 1/\omega C = 1/2\pi fC$, the opposition that capacitance presents varies inversely with frequency. This means that the higher the frequency, the lower the reactance, and vice versa (Figure 2–36). At $f = 0$ (i.e., dc), capacitive reactance is infinite. This means that a capacitance looks like an open circuit to dc. (We already concluded this earlier in Chapter 10.) Note that X_C is also inversely proportional to capacitance. Thus, if capacitance is doubled, X_C is halved, and so on.

IN-PROCESS LEARNING CHECK 2

(Answers are at the end of the chapter.)

1. For a pure resistance, $v_R = 100 \sin(\omega t + 30°)$ V. If $R = 2\ \Omega$, what is the expression for i_R?
2. For a pure inductance, $v_L = 100 \sin(\omega t + 30°)$ V. If $X_L = 2\ \Omega$, what is the expression for i_L?
3. For a pure capacitance, $v_C = 100 \sin(\omega t + 30°)$ V. If $X_C = 2\Omega$, what is the expression for i_C?
4. If $f = 100$ Hz and $X_L = 400\ \Omega$, what is L?
5. If $f = 100$ Hz and $X_C = 400\ \Omega$, what is C?
6. For each of the phasor sets of Figure 16–37, identify whether the circuit is resistive, inductive, or capacitive. Justify your answers.

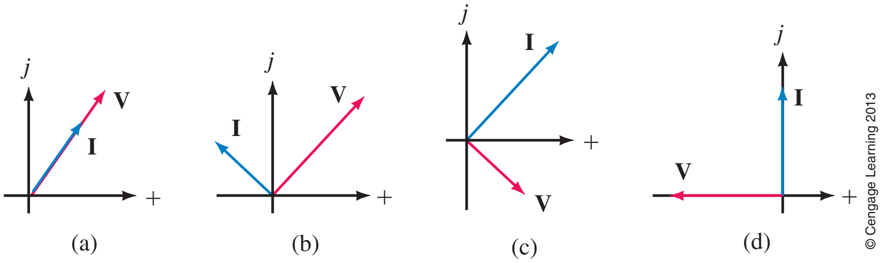


FIGURE 2–37

2.7 The Impedance Concept

During the learning process of Sections 2.5 and 2.6, we handled magnitude and phase analysis separately. However, this is not the way it is done in practice. In practice, we represent circuit elements by their impedance, and we determine magnitude and phase relationships in one step. Before we do this, however, we need to learn how to represent circuit elements as impedances.

Impedance

The opposition that a circuit element presents to current in the phasor domain is defined as its **impedance.** The impedance of the element of Figure 2–38, for example, is the ratio of its voltage phasor to its current phasor. Impedance is denoted by the boldface, uppercase letter **Z**. Thus,

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} \quad \text{(ohms)} \tag{2–25}$$

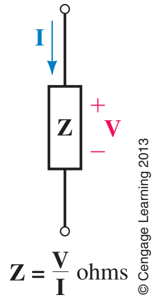


FIGURE 2–38 Impedance concept.

(This equation is sometimes referred to as Ohm's law for ac circuits.)

Since phasor voltages and currents are complex, **Z** is also complex. That is,

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V}{I}\angle\theta \tag{2–26}$$

where V and I are the rms magnitudes of **V** and **I,** respectively, and θ is the angle between them. From Equation 2–26,

$$\mathbf{Z} = Z\angle\theta \tag{2–27}$$

where $Z = V/I$. Since $V = 0.707V_m$ and $I = 0.707I_m$, Z can also be expressed as V_m/I_m. Once the impedance of a circuit is known, the current and voltage can be determined using

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} \tag{2–28}$$

and

$$\mathbf{V} = \mathbf{IZ} \tag{2–29}$$

Let us now determine impedance for the basic circuit elements R, L, and C.

Resistance

For a pure resistance (Figure 2–39), voltage and current are in phase. Thus, if voltage has an angle θ, current will have the same angle. For example, if $\mathbf{V}_R = V_R\angle\theta$, then $\mathbf{I} = I\angle\theta$. Substituting into Equation 2–25 yields:

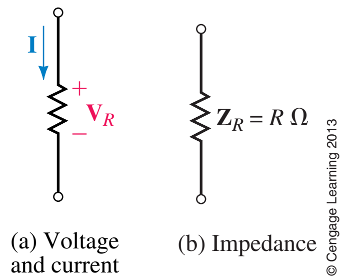


FIGURE 2–39 Impedance of a pure resistance.

$$\mathbf{Z}_R = \frac{\mathbf{V}_R}{\mathbf{I}} = \frac{V_R\angle\theta}{I\angle\theta} = \frac{V_R}{I}\angle 0° = R\angle 0° = R$$

Thus, the impedance of a resistor is just its resistance. That is,

$$\mathbf{Z}_R = R \tag{2–30}$$

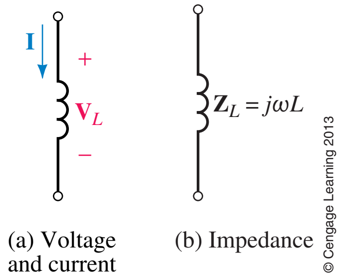
This agrees with what we know about resistive circuits, that the ratio of voltage to current is R and that the angle between them is 0°.

Inductance

For a pure inductance, current lags voltage by 90°. Assuming a 0° angle for voltage (we can assume any reference we want because we are interested only

NOTES...

Although **Z** is a complex number, it is not a phasor. (Phasors are complex numbers that are used to represent sinusoidally varying quantities such as voltage and current). However, **Z** does not represent anything that varies with time—thus it is not a phasor.



(a) Voltage and current (b) Impedance

FIGURE 2–40 Impedance of a pure inductance.

in the angle between $\mathbf{V}_L$ and $\mathbf{I}$), we can write $\mathbf{V}_L = V_L\angle 0°$ and $\mathbf{I} = I\angle -90°$. The impedance of a pure inductance (Figure 2–40) is therefore

$$\mathbf{Z}_L = \frac{\mathbf{V}_L}{\mathbf{I}} = \frac{V_L\angle 0°}{I\angle -90°} = \frac{V_L}{I}\angle 90° = \omega L\angle 90° = j\omega L$$

where we have used the fact that $V_L/I_L = \omega L$. Thus,

$$\mathbf{Z}_L = j\omega L = jX_L \tag{2–31}$$

since ωL is equal to X_L.

EXAMPLE 2–17

Consider again Example 2–13. Given $v_L = 100 \sin(400t + 70°)$ V and $L = 0.2$ H, determine i_L using the impedance concept.

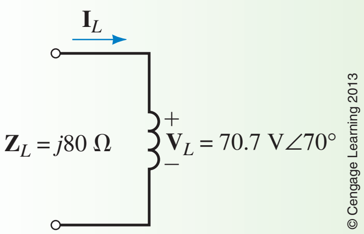


FIGURE 16–41

Solution See Figure 2–41. Use rms values. Thus,

$$\mathbf{V}_L = 70.7\ \text{V}\angle 70° \quad \text{and} \quad \omega = 400\ \text{rad/s}$$
$$\mathbf{Z}_L = j\omega L = j(400)(0.2) = j80\ \Omega$$
$$\mathbf{I}_L = \frac{\mathbf{V}_L}{\mathbf{Z}_L} = \frac{70.7\angle 70°}{j80} = \frac{70.7\angle 70°}{80\angle 90°} = 0.884\ \text{A}\angle -20°$$

In the time domain, $i_L = \sqrt{2}(0.884)\sin(400t - 20°) = 1.25 \sin(400t - 20°)$ A, which agrees with our previous solution, Example 2–13.

Capacitance

For a pure capacitance, current leads voltage by 90°. Its impedance (Figure 2–42) is therefore

$$\mathbf{Z}_C = \frac{\mathbf{V}_C}{\mathbf{I}} = \frac{V_C\angle 0°}{I\angle 90°} = \frac{V_C}{I}\angle -90° = \frac{1}{\omega C}\angle -90° = -j\frac{1}{\omega C} \quad \text{(ohms)}$$

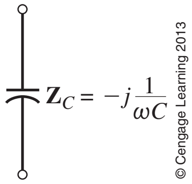


FIGURE 2–42 Impedance of a pure capacitance.

Thus,

$$\mathbf{Z}_C = -j\frac{1}{\omega C} = -jX_C \quad \text{(ohms)} \tag{2–32}$$

since $1/\omega C$ is equal to X_C.

EXAMPLE 2–18

Given $v_C = 100 \sin(\omega t - 40°)$ V, $f = 1000$ Hz, and $C = 10\ \mu$F, determine i_C in Figure 2–43.

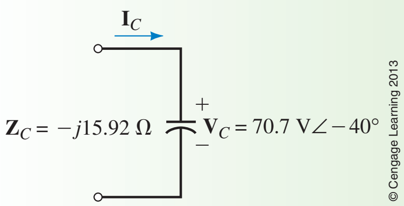


FIGURE 2 –43

Solution

$$\omega = 2\pi f = 2\pi(1000\ \text{Hz}) = 6283\ \text{rads/s}$$
$$\mathbf{V}_C = 70.7\ \text{V}\angle -40°$$

$$\mathbf{Z}_C = -j\frac{1}{\omega C} = -j\left(\frac{1}{6283 \times 10 \times 10^{-6}}\right) = -j15.92\ \Omega$$

$$\mathbf{I}_C = \frac{\mathbf{V}_C}{\mathbf{Z}_C} = \frac{70.7\angle -40°}{-j15.92} = \frac{70.7\angle -40°}{15.92\angle -90°} = 4.442\ \text{A}\angle 50°$$

In the time domain, $i_C = \sqrt{2}(4.442)\sin(6283t + 50°) = 6.28\sin(6283t + 50°)$ A, which agrees with our previous solution, in Example 2–15.

PRACTICE PROBLEMS 9

1. If $\mathbf{I}_L = 5\ \text{mA}\angle -60°$, $L = 2$ mH, and $f = 10$ kHz, what is $\mathbf{V}_L$?
2. A capacitor has a reactance of 50 Ω at 1200 Hz. If $v_C = 80 \sin 800t$ V, what is i_C?

Answers

1. 628 mV∠30°
2. $0.170 \sin(800t + 90°)$ A

A Look Ahead

The real power of the impedance method becomes apparent when you consider complex circuits with elements in series, parallel, and so on. This we do later, beginning in Chapter 18. Before we do this, however, there are some ideas on power that you need to know. These are considered in Chapter 17.

2.8 Computer Analysis of ac Circuits

MultiSim

PSPICE

Let us now use Multisim and PSpice to verify the assertion of Practice Problems 2 (Section 2.2) that $v(t) = 21.8 \sin(\omega t + 36.6°)$ is the sum of $e_1(t) = 10 \sin\omega t$ and $e_2(t) = 15 \sin(\omega t + 60°)$. To do this, we will build two circuits on the screen, one to generate the sum $e_1(t) + e_2(t)$, and the other to generate the equivalent voltage $v(t) = 21.8 \sin(\omega t + 36.6°)$, then compare the waveforms. Since the process is independent of frequency, we can choose any convenient frequency, say 500 Hz.

Multisim

Review Multisim Notes 1, 2, and 4, and then create the circuits of Figure 2–44 on your screen. (If you want, you can change the source designations to E1, E2, and E3 to match the original problem symbols. Double-click a source, select the *Label* tab, then change the designation.) Next, place junction dots to create nodes at the wire ends, then click *Options/Sheet Properties/Show All/OK* to display node numbers. Set source values, click *Simulate/Analyses/Transient Analysis*, set *TSTOP* to 0.002 and the *Minimum number of time points* to 200. Click the *Output* tab, add variables V(1), V(2), and V(2) − V(1) for display (see Notes 3 and 5), then click *Simulate*. Three waveforms appear. Add grid lines, then (optionally) turn off the select marks and change the background color to white.

NOTES...

Multisim

1. As noted in Chapter 15, Multisim source phase angles must be set up correctly. If you are unsure, click Options/Global Preferences then select the Simulation tab (or the Parts tab if you are a legacy user). Ensure that the Shift Right button is selected.
2. Multisim assigns node numbers according to the order in which you wire up a circuit; thus, the node numbers for your circuit may be different from those shown in Figure 2–44. For purposes of discussion, however, we will assume node numbers as in Figure 2–44.
3. Node numbers help you identify variables—for example, V(1) is the voltage (with respect to ground) at Node 1, V(2) is the voltage at Node 2, and so on. [Legacy users use $1 instead of V(1), etc.]
4. As described in Chapter 15, Multisim has two ac voltage source types. Be sure to use Signal_Voltage_Source sources in this example.
5. To create V(2)-V(1), use the Add Expression procedure found in the Output dialog box. If you have forgotten how, see Appendix A.

Results: Compare the results here to those of Figure 2–11 from earlier in the chapter. So far, the only waveforms plotted in Figure 2–44 are due to circuit (a). Click the *Show Legend* icon and note the legend at the bottom of the screen. Voltage V(1) represents the voltage at Node 1, which, as you can see, is source voltage e_2. Voltage V(2) represents the voltage at Node 2, which is the sum of e_1 and e_2. Thus, voltage V(2) − V(1), the difference between the two node voltages, is source voltage e_1. Waveforms are identical to those shown in Figure 2–11 (except that Multisim cannot plot the pre $t = 0$ portion of the traces—however, see Practice Problems 10).

Now add the waveform from circuit (b) as follows: click *Simulate/Analyses*, then *Transient Analysis*, select the *Output* tab, select V(3), click *Add*, then *Simulate*. At first glance, it appears that nothing has happened. However, your legend now shows that V(3), the voltage at Node 3, has been added. A closer look shows that the color of the "sum" trace has changed. What has happened is that since V(3) is identical to V(2), Multisim has simply graphed over the previously plotted trace. Since the two traces coincide perfectly, we conclude that the waveforms are identical, thus verifying our assertion of Practice Problems 2.

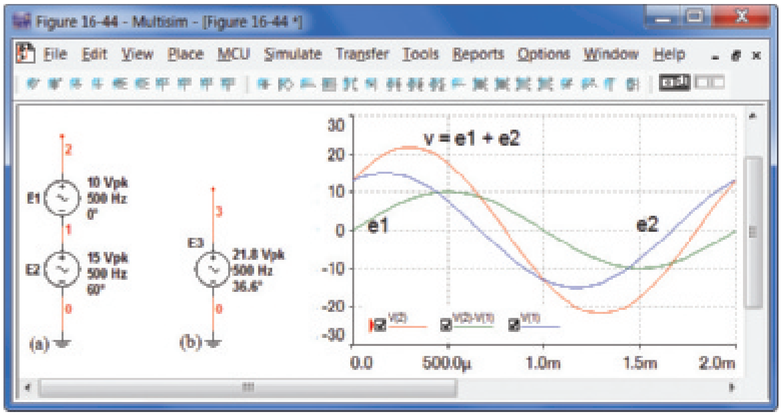


FIGURE 2–44 Using Multisim to demonstrate that $10 \sin \omega t + 15 \sin (\omega t + 60°) = 21.8 \sin (\omega t + 36.6°)$.

PRACTICE PROBLEMS 10

In the waveforms that you get from the preceding example, you cannot see the waveforms before $t = 0$. What if you want to see them? **Hint:** You can cheat a bit here. Note that the waveforms start to repeat at $t = 2$ ms on your plot. Mentally redefine this as $t = 0$, then rerun the simulation with *TSTART* set to 0.0015 and *TSTOP* to 0.004. Redefining the time scale in your mind as above, you can now see the pre $t = 0$ part of the solution.

PSpice

Create the circuit of Figure 2–45 using source *VSIN*. Double-click each parameter box in turn and key in values for voltage and frequency as indicated. Double-click the *V2* source symbol, and in its Property Editor, scroll to *PHASE* and type in **60**, click *Apply,* then close the Property Editor. Similarly, set the phase angle for source 3 to **36.6**. Click the *New Simulation Profile* icon, choose *Transient,* and set *TSTOP* to **2ms**, then click *OK*. Add differential voltage markers (probes) across source V1 and ordinary voltage markers elsewhere as indicated in Figure 2–45. (The differential markers are the ones with the +, − designations.) All markers will be gray, but they will turn color after simulation. Click the *Run* icon.

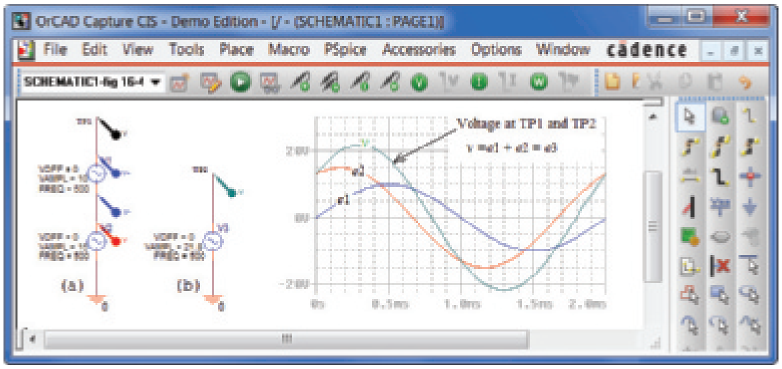


FIGURE 2–45 Using PSpice to demonstrate that $10 \sin \omega t + 15 \sin (\omega t + 60°) = 21.8 \sin (\omega t + 36.6°)$.

Results: Your graph colors may be different from those shown, but the curvesshould be the same. Compare waveforms to those of Figure 2–11 from earlier in the chapter. (Use the legend at the bottom of the screen to identify waveforms on your screen.) Note that results are identical. However, only three waveforms are visible in spite of the fact that we plotted four. The reason is that voltages at TP1 and TP2 are identical, and PSpice simply graphed one trace over the other. Since the two traces coincide perfectly, we conclude that the waveforms are identical, thus verifying our assertion of Practice Problems 2. (To confirm this, click the voltage probe at TP2 and delete it. You will see that voltage V3 has disappeared from your graph legend and the underlying trace, V2, remains.)

Another Example

PSpice makes it easy to study the response of circuits over a range of frequen-cies. This is illustrated in Example 2–19.

EXAMPLE 2–19

Compute and plot the reactance of a 12-μF capacitor over the range 10 Hz to 1000 Hz.

Solution PSpice has no command to compute reactance; however, you can calculate voltage and current over the desired frequency range, then plot their ratio. This gives reactance. Procedure: Create the circuit of Figure 2–46 on your screen. (Use source VAC here as it is the source to use for phasor analyses.) Note its default of 0V. Double-click the default value (not the symbol) and in the dialog box, enter **120V,** then click OK. Click the *New Simulations Profile* icon, enter a name, and then in the dialog box that opens, select *AC Sweep/Noise*. For the *Start Frequency,* key in **10Hz;** for the *End Frequency*, key **1kHz;** set AC Sweep type to *Logarithmic,* select *Decade,* and type **100** into the *Pts/Decade* (points per decade) box. Run the simulation and a set of empty axes appears. Click *Trace, Add Trace* and in the dialog box, click ***V1(C1),*** press the / key on the keyboard, then click ***I(C1)*** to yield the ratio V1(C1)/I(C1) (which is the capacitor's reactance). Click *OK* and PSpice will compute and plot the capacitor's reactance versus frequency. Compare its shape to Figure 2–36. Use the cursor to scale some values off the screen and verify each point using your calculator and $X_C = 1/\omega C$.

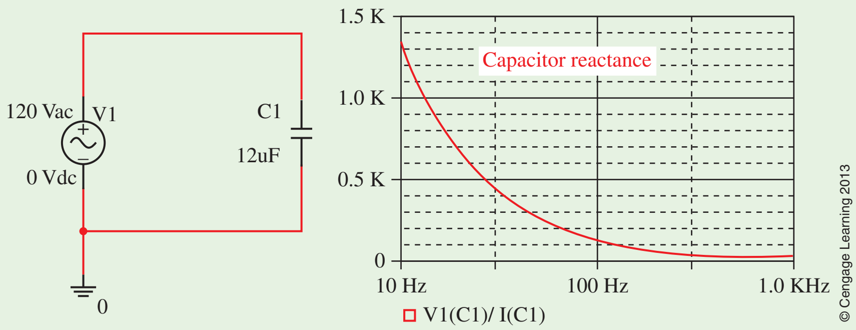
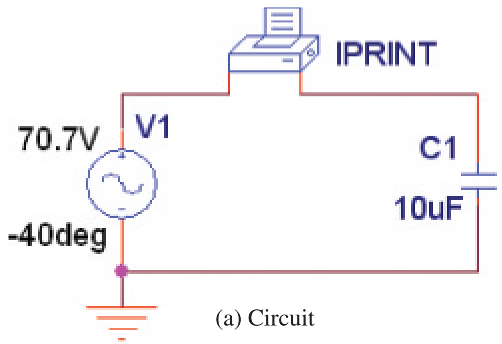


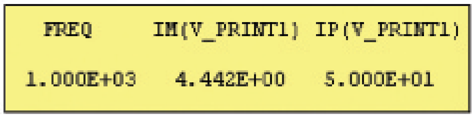
FIGURE 2–46 Computing reactance versus frequency for a 12-μF capacitor using PSpice.

Phasor Analysis

As a last example, we will show how to use PSpice to perform phasor analysis—that is, to solve problems with voltages and currents expressed in phasor form. To illustrate, consider again Example 2–18. Recall that $\mathbf{V}_C = 70.7\text{ V}\angle -40°$, $C = 10\ \mu\text{F}$, and $f = 1000$ Hz. Procedure: Create the circuit on the screen (Figure 2–47) using source *VAC* and component *IPRINT* (Note 1). Double-click the *VAC* symbol and in the Property Editor, set *ACMAG* to **70.7V** and *ACPHASE* to **−40,** click *Apply,* then close the editor—see Note 2. Double-click *IPRINT* and in the Property Editor, type **yes** into cells *AC, MAG,* and *PHASE*. Click *Apply* and close the editor. Click the *New Simulation Profile* icon, enter a name, select *AC Sweep/Noise,* set both *Start Frequency* and *End Frequency* to **1000Hz**, select *Linear,* then set *Total Points* to **1.** Run the simulation. When the simulation window opens, click *View, Output File,* then scroll to near the end of the file where you find the answers (see Note 3). The first number is the frequency (1000 Hz), the second number (*IM*) is the magnitude of the current (4.442 A), and the third (*IP*) is its phase (50 degrees). Thus, $\mathbf{I}_C = 4.442\text{ A}\angle 50°$ as we determined earlier in Example 2–18.



(a) Circuit



FREQ	IM(V_PRINT1)	IP(V_PRINT1)
1.000E+03	4.442E+00	5.000E+01

(b) Ammeter reading

FIGURE 2–47 Phasor analysis using PSpice. Component IPRINT is a software ammeter.

NOTES...

1. Component IPRINT is a software ammeter, found in the SPECIAL parts library. In this example, we configure it to display ac current in magnitude and phase angle format. Make sure that it is connected as shown inFigure 2–47, since if it is reversed, the phase angle of the measured current will be in error by 180°.
2. If you want to display the phase of the source voltage on the schematic asin Figure 2–47, double-click the source symbol and in the Property Editor click ACPHASE, Display, then select Value Only.
3. The results displayed by IPRINT are expressed in exponential format. Thus, frequency is shown as 1.000E+03, which is $1.000 \times 10^3 = 1000$ Hz, and so on.

PRACTICE PROBLEMS 11

Modify Example 2–19 to plot both capacitor current and reactance on the same graph. You will need to add a second Y-axis for the capacitor current. (See Appendix A if you need help.)

Problems

2.1 Complex Number Review

1. Convert each of the following to polar form:

   a. $5 + j12$ c. $-8 + j15$

   b. $9 - j6$ d. $-10 - j4$

2. Convert each of the following to rectangular form:
   - a. $6\angle 30°$
   - b. $14\angle 90°$
   - c. $16\angle 0°$
   - d. $6\angle 150°$
   - e. $20\angle -140°$
   - f. $-12\angle 30°$
   - g. $-15\angle -150°$
3. Plot each of the following on the complex plane:
   - a. $4 + j6$
   - b. $j4$
   - c. $6\angle -90°$
   - d. $10\angle 135°$
4. Simplify the following using powers of j:
   - a. $j(1 - j1)$
   - b. $(-j)(2 + j5)$
   - c. $j[j(1 + j6)]$
   - d. $(j4)(-j2 + 4)$
   - e. $(2 + j3)(3 - j4)$
5. Express your answer in rectangular form.
   - a. $(4 + j8) + (3 - j2)$
   - b. $(4 + j8) - (3 - j2)$
   - c. $(4.1 - j7.6) + 12\angle 20°$
   - d. $2.9\angle 25° - 7.3\angle -5°$
   - e. $9.2\angle -120° - (2.6 + j4.1)$
   - f. $\dfrac{1}{3+j4} + \dfrac{1}{8-j6}$
6. Express your answer in polar form.
   - a. $(37 + j9.8)(3.6 - j12.3)$
   - b. $(41.9\angle -80°)(2 + j2)$
   - c. $\dfrac{42 + j18.6}{19.1 - j4.8}$
   - d. $\dfrac{42.6 + j187.5}{11.2\angle 38°}$
7. Reduce each of the following to polar form:
   - a. $15 - j6 - \left[\dfrac{18\angle 40° + (12 + j8)}{11 + j11}\right]$
   - b. $\dfrac{21\angle 20° - j41}{36\angle 0° + (1 + j12) - 11\angle 40°}$
   - c. $\dfrac{18\angle 40° - 18\angle -40°}{7 + j12} - \dfrac{2 + j17 + 21\angle -60°}{4}$

2.2 Complex Numbers in ac Analysis

NOTES...

The answers given to Problems 8 through 11 assume that you are not using rms values since we do not start using rms until later in the chapter.

8. In the manner of Figure 16–10, represent each of the following as transformed sources—see Notes.
   - a. $e = 100 \sin(\omega t + 30°)$ V
   - b. $e = 15 \sin(\omega t - 20°)$ V
   - c. $e = 50 \sin(\omega t + 90°)$ V
   - d. $e = 50 \cos \omega t$ V
   - e. $e = 40 \sin(\omega t + 120°)$ V
   - f. $e = 80 \sin(\omega t - 70°)$ V
9. Determine the sinusoidal equivalent for each of the transformed sources of Figure 2–48—see Notes.

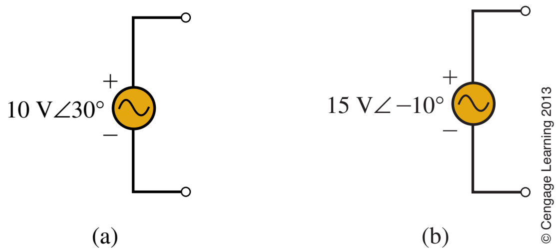


FIGURE 2–48

10. Given: $e_1 = 10 \sin(\omega t + 30°)$ V and $e_2 = 15 \sin(\omega t - 20°)$ V. Determinetheir sum $v = e_1 + e_2$ in the manner of Example 2–7, that is,
    a. Convert e_1 and e_2 to phasor form.
    b.D etermine $\mathbf{V} = \mathbf{E}_1 + \mathbf{E}_2$.
    c. Convert $\mathbf{V}$ to the time domain.
    d. Sketch e_1, e_2, and v as per Figure 2–13.
11. Repeat Problem 10 for $v = e_1 - e_2$.

Note: For the remaining problems and throughout the remainder of the book, express phasor quantities as rms values rather than as peak values.

12. Express the voltages and currents of Figure 2–49 as time domain andphasor domain quantities.

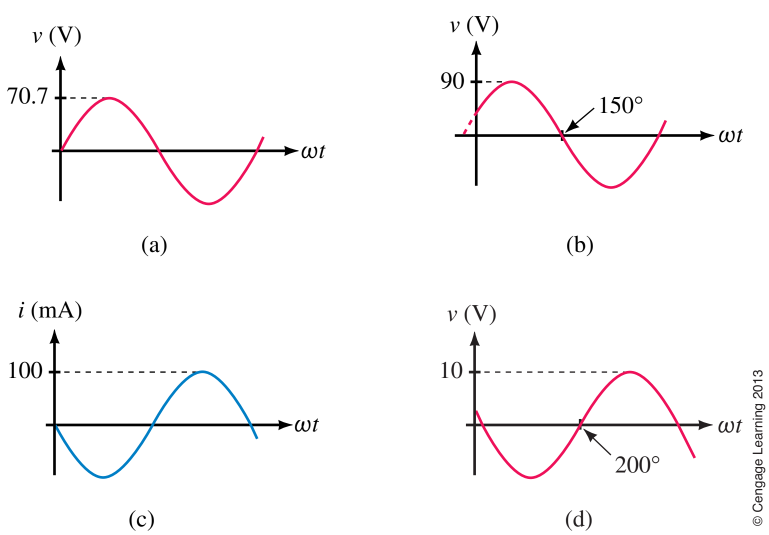


FIGURE 2–49

13. For Figure 16–50, $i_1 = 25 \sin(\omega t + 36°)$ mA and $i_2 = 40 \cos(\omega t - 10°)$ mA.
    a. Determine phasors $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_T$.
    b. Determine the equation for i_T in the time domain.
14. For Figure 16–50, $i_T = 50 \sin(\omega t + 60°)$ A and $i_2 = 20 \sin(\omega t - 30°)$ A.
    a. Determine phasors $\mathbf{I}_T$ and $\mathbf{I}_2$.
    b. Determine $\mathbf{I}_1$.
    c. From (b), determine the equation for i_1.
15. For Figure 16–18, $i_1 = 7 \sin \omega t$ mA, $i_2 = 4 \sin(\omega t - 90°)$ mA, and $i_3 = 6 \sin(\omega t + 90°)$ mA.
    a. Determine phasors $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_3$ and $\mathbf{I}_T$.
    b. Determine the equation for i_T in the time domain.
16. For Figure 16–51, $i_T = 38.08 \sin(\omega t - 21.8°)$ A, $i_1 = 35.36 \sin \omega t$ A, and $i_3 = 28.28 \sin(\omega t - 90°)$ A. Determine the equation for i_2.

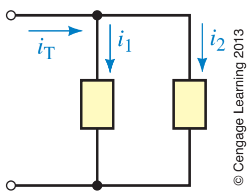


FIGURE 2–50

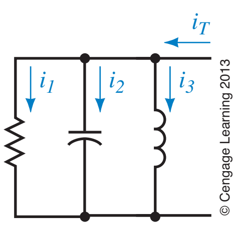


FIGURE 2–51

2.4 to 2.6

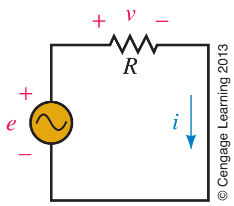


FIGURE 2–52

17. For Figure 16–52, $R = 12\ \Omega$. For each of the following, determine the unknown current or voltage and sketch.

    a. If $v = 120 \sin \omega t$ V, find i.

    b. If $v = 120 \sin(\omega t + 27°)$ V, find i.

    c. If $i = 17 \sin(\omega t - 56°)$ mA, find v.

    d. If $i = -17 \cos(\omega t - 67°)$ μA, find v.

18. Given $v = 120 \sin(\omega t + 52°)$ V and $i = 15 \sin(\omega t + 52°)$ mA, what is R?

19. Two resistors $R_1 = 10$ kΩ and $R_2 = 12.5$ kΩ are in series. If $i = 14.7 \sin(\omega t + 39°)$ mA,

    a. What are v_{R_1} and v_{R_2}?

    b. Compute $v_T = v_{R_1} + v_{R_2}$ and compare to v_T calculated from $v_T = i R_T$.

20. The voltage across a certain component is $v = 120 \sin(\omega t + 55°)$ V and its current is $-18 \cos(\omega t + 145°)$ mA. Show that the component is a resistor, and determine its value.

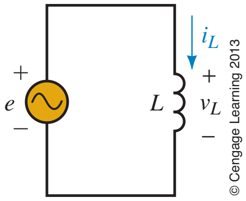


FIGURE 2–53

21. For Figure 16–53, $V_m = 10$ V and $I_m = 5$ A.

    a. If $v_L = 10 \sin(\omega t + 60°)$ V, what is i_L?

    b. If $v_L = 10 \sin(\omega t - 15°)$ V, what is i_L?

    c. If $i_L = 5 \cos(\omega t - 60°)$ A, what is v_L?

    d. If $i_L = 5 \sin(\omega t + 10°)$ A, what is v_L?

22. What is the reactance of a 0.5-H inductor at

    a. 60 Hz b. 1000 Hz c. 500 rad/s

23. For Figure 16–53, $e = 100 \sin \omega t$ and $L = 0.5$ H. Determine i_L at

    a. 60 Hz b. 1000 Hz c. 500 rad/s

24. For Figure 16–53, let $L = 200$ mH.

    a. If $v_L = 100 \sin 377t$ V, what is i_L?

    b. If $i_L = 10 \sin(2\pi \times 400t - 60°)$ mA, what is v_L?

25. For Figure 2–53, if

    a. $v_L = 40 \sin(\omega t + 30°)$ V, $i_L = 364 \sin(\omega t - 60°)$ mA, and $L = 2$ mH, what is f?

    b. $i_L = 250 \sin(\omega t + 40°)$ μA, $v_L = 40 \sin(\omega t + \theta)$ V, and $f = 500$ kHz, what are L and θ?

26. Repeat Problem 21 if the given voltages and currents are for a capacitor instead of an inductor.

27. What is the reactance of a 5-μF capacitor at

    a. 60 Hz b. 1000 Hz c. 500 rad/s

28. For Figure 2–54, $e = 100 \sin \omega t$ and $C = 5$ μF. Determine i_C at

    a. 60 Hz b. 1000 Hz c. 500 rad/s

29. For Figure 16–54, let $C = 50\ \mu\text{F}$.

    a. If $v_C = 100 \sin 377t$ V, what is i_C?

    b.I f $i_C = 10 \sin(2\pi \times 400t - 60°)$ mA, what is v_C?

30. For Figure 2–54, if

    a. $v_C = 362 \sin(\omega t - 33°)$ V, $i_C = 94 \sin(\omega t + 57°)$ mA, and $C = 2.2\ \mu$F, what is f?

    b. $i_C = 350 \sin(\omega t + 40°)$ mA, $v_C = 3.6 \sin(\omega t + \theta)$ V, and $f = 12$ kHz, what are C and θ?

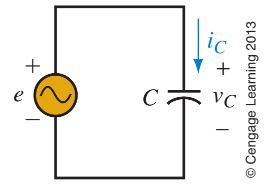


FIGURE 2–54

2.7 The Impedance Concept

31. Determine the impedance of each circuit element of Figure 16–55.

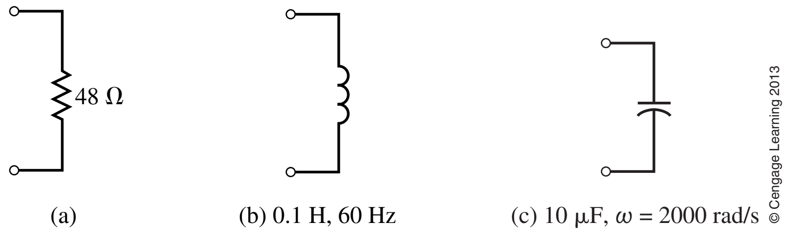


FIGURE 2–55

32. If $\mathbf{E} = 100\ \text{V}\angle 0°$ is applied across each of the circuit elements ofFigure 2–56:

    a. Determine each current in phasor form.

    b.E xpress each current in time domain form.

33. If the current through each circuit element of Figure 2–56 is $0.5\ \text{A}\angle 0°$:

    a. Determine each voltage in phasor form.

    b.E xpress each voltage in time domain form.

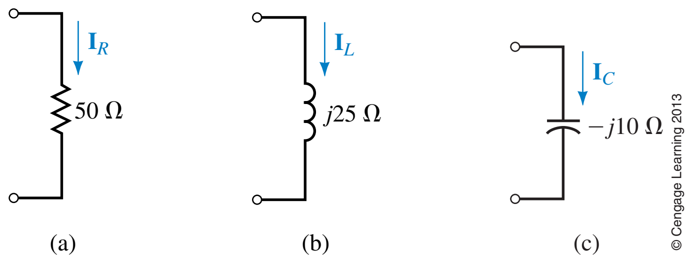


FIGURE 2–56

34. For each of the following, determine the impedance of the circuit element and state whether it is resistive, inductive, or capacitive.

    a. $\mathbf{V} = 240\ \text{V}\angle -30°$, $\mathbf{I} = 4\ \text{A}\angle -30°$.

    b. $\mathbf{V} = 40\ \text{V}\angle 30°$, $\mathbf{I} = 4\ \text{A}\angle -60°$.

    c. $\mathbf{V} = 60\ \text{V}\angle -30°$, $\mathbf{I} = 4\ \text{A}\angle 60°$.

    d. $\mathbf{V} = 140\ \text{V}\angle -30°$, $\mathbf{I} = 14\ \text{mA}\angle -120°$.

35. For each circuit of Figure 16–57, determine the unknown.

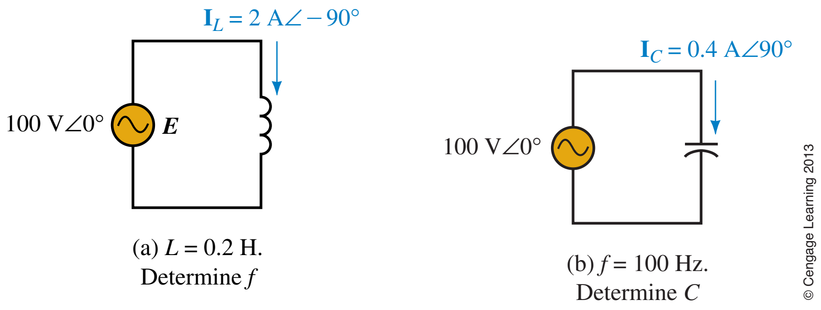


FIGURE 2–57

36. a. If $\mathbf{V}_L = 120\text{ V }\angle 67°$, $L = 600\ \mu\text{H}$, and $f = 10\text{ kHz}$, what is $\mathbf{I}_L$?
   b.I f $\mathbf{I}_L = 48\text{ mA }\angle -43°$, $L = 550\text{ mH}$, and $f = 700\text{ Hz}$, what is $\mathbf{V}_L$?
   c. If $\mathbf{V}_C = 50\text{ V }\angle -36°$, $C = 390\text{ pF}$, and $f = 470\text{ kHz}$, what is $\mathbf{I}_C$?
   d. If $\mathbf{I}_C = 95\text{ mA }\angle 87°$, $C = 6.5\text{ nF}$, and $f = 1.2\text{ MHz}$, what is $\mathbf{V}_C$?

2.8 Computer Analysis of ac Circuits

Multisim

37. Create the circuit of Figure 2–58 on your screen. (To access the power ac source, right-click *View* and ensure that *Power Source Components* is selected. Use your mouse to locate its icon on your screen, click and the source attaches to your pointer. Move it to your workspace screen and click to place. To locate the ammeter, click *View* and ensure that *Measurement Components* is selected. Use your mouse to locate its icon and place it on your screen.) Double-click the ammeter symbol and set Mode to *AC*. Click the ON/OFF power switch to energize the circuit. Compare the measured reading against the theoretical value.

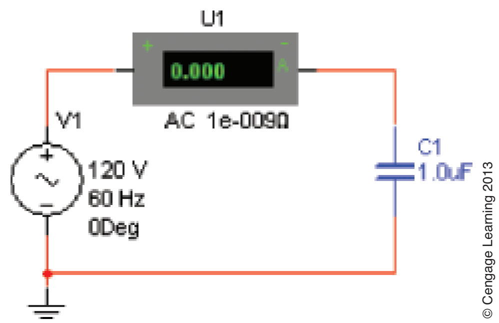


FIGURE 2–58 The source to use here is the power voltage source.

Multisim

38. Multisim can be used to plot capacitive reactance versus frequency. Create the Multisim equivalent of the PSpice circuit of Figure 2–46. Click *Simulate*, *Analyses, AC Analysis,* set *FSTART* to **10 Hz** and *FSTOP* to **1 kHz.** Set *Sweep type* as *Decade* and *Vertical scale* as *Linear*. Click *Output*, create the expression **V(1)/I(C1),** then click *Simulate*. You should get a magnitude graph similar to Figure 2–46 on your screen.

39. Create the circuit of Figure 2–53 on your screen. Use a source of 100 V$\angle 0°$, $L = 0.2$ H, and $f = 50$ Hz. Solve for current $\mathbf{I}_L$ (magnitude and angle). See note below.

PSPICE

40. Plot the reactance of a 2.387-H inductor versus frequency from 1 Hz to 500 Hz and compare to Figure 2–29. Use a linear X-axis scale.

PSPICE

41. For the circuit of Problem 39, plot current magnitude versus frequency from $f = 1$ Hz to $f = 20$ Hz. Measure the current at 10 Hz and verify with your calculator.

PSPICE

Note: PSpice does not permit source/inductor loops. To get around this, add a very small resistor in series, for example, $R = 0.00001\ \Omega$.

ANSWERS TO IN-PROCESS LEARNING CHECKS

IN-PROCESS LEARNING CHECK 1

1. a. $6\angle 90°$
   b. $4\angle -90°$
   c. $4.24\angle 45°$
   d. $7.21\angle -56.3°$
   e. $9.43\angle 122.0°$
   f. $2.24\angle -63.4°$
   g. $3.61\angle -123.7°$
2. a. $j4$
   b. $3 + j0$
   c. $-j2$
   d. $3.83 + j3.21$
   e. $-3 + j5.20$
   f. $2.35 - j0.855$
   g. $-1.64 - j0.599$
3. $12\angle 40°$
4. $88 - j7$; $-2 + j15$; $0.0113 + j\,0.0009$;$-0.0333 - j0.0312$
5. $288\angle -100°$; $2\angle 150°$
6. a. $14.70 + j2.17$
   b. $-8.94 + j7.28$
7. $18.0 \sin(\omega t - 56.3°)$
8. 7.07 A, 14.14 A, and 3.54 A for the branch currents, 12.7 A for total current.

IN-PROCESS LEARNING CHECK 2

1. $50 \sin(\omega t + 30°)$ A
2. $50 \sin(\omega t - 60°)$ A
3. $50 \sin(\omega t + 120°)$ A
4. 0.637 H
5. 3.98 μF
6. a. Voltage and current are in phase. Therefore, *R*.
   b.C urrent leads by 90°. Therefore, *C*.
   c. Current leads by 90°. Therefore, *C*.
   d. Current lags by 90°. Therefore, *L*.

KEY TERMS

Active Power
Apparent Power
Average Power
Eddy Currents
Effective Resistance
Instantaneous Power
Power Factor
Power Factor Correction
Power Triangle
Reactive Power
Real Power
Skin Effect
VAR (Volt-Amps Reactive)
Volt-Amperes (VA)
Wattmeter

OUTLINE

OBJECTIVES

After studying this chapter, you will be able to

- explain what is meant by active, reactive, and apparent power,
- compute the active power to a load,
- compute the reactive power to a load,
- compute the apparent power to a load,
- construct and use the power triangle to analyze power to complex loads,
- compute power factor,
- explain why equipment is rated in VA instead of watts,
- measure power in single-phase circuits,
- describe why effective resistance differs from geometric resistance,
- describe energy relations in ac circuits,
- use Multisim and PSpice to study instantaneous power.

3

POWER IN ac CIRCUITS

CHAPTER PREVIEW

In Chapter , you studied power in dc circuits. In this chapter, we turn our attention to power in ac circuits. In ac circuits, there are additional considerations that are not present with dc. In dc circuits, for example, the only power relationship you encounter is $P = VI$ watts or its alternate forms $P = I^2R$ and $P = V^2/R$. This power is referred to as *real power* or *active power* and is the power that does useful work such as light a lamp, power a heater, run an electric motor, and so on.

In ac circuits, you also encounter active power, but for ac circuits that contain reactive elements (i.e., inductance or capacitance), a second component of power also exists. This component, termed *reactive power,* represents energy that oscillates back and forth throughout the system. For example, during the buildup of current in an inductance, energy flows from the power source to the inductance to expand its magnetic field. When the magnetic field collapses, this energy is returned to the circuit. This movement of energy in and out of the inductance constitutes a flow of power. However, since it flows first in one direction, then in the other, it contributes nothing to the average flow of power from the source to the load. For this reason, reactive power is sometimes referred to as *wattless power.* (A similar situation exists regarding power flow to and from the electric field of a capacitor.)

For a circuit that contains resistive as well as reactive elements, some energy is dissipated while the remainder is shuttled back and forth between the reactive elements as described above; thus, both active and reactive components of power are present. This combination of real and reactive power is termed apparent power—see Notes.

NOTES...

Note that real power and reactive power are not separate "types" of power, they are merely different components of the total power.

In this chapter, we look at all three components of power. New ideas that emerge include the concept of power factor, the power triangle, the measurement of power in ac circuits, and the concept of effective resistance. ■

Putting It in Perspective

Henry Cavendish

CAVENDISH, AN ENGLISH CHEMIST and physicist born in 1731, is included here, not for what he did for the emerging electrical field, but for what he didn't do. A brilliant man, Cavendish was 50 years ahead of his time, and his experiments in electricity preceded and anticipated almost all the major discoveries that came about over the next half century (e.g., he discovered Coulomb's law before Coulomb did). However, Cavendish was interested in research and knowledge purely for its own sake and never bothered to publish most of what he learned, in effect depriving the world of his findings and holding back the development of the field of electricity by many years. Cavendish's work lay unknown for nearly a century before another great scientist, James Clerk Maxwell, had it published. Nowadays, Cavendish is better known for his work in the gravitational field than for his work in the electrical field. One of the amazing things he did was to determine the mass of the earth using the rather primitive technology of his day. ■

3.1 Introduction

At any given instant, the power to a load is equal to the product of voltage times current (Figure 3–1). This means that if voltage and current vary with time, so will power. This time-varying power is referred to as **instantaneous power** and is given the symbol $p(t)$ or just p. Thus,

$$p = vi \quad \text{(watts)} \tag{3–1}$$

Now consider the case of sinusoidal ac. Since voltage and current are positive at various times during their cycle and negative at others, instantaneous power may also be positive at some times and negative at others. This is illustrated in Figure 3–2, where we have multiplied voltage times current point by point to get the power waveform. For example, from $t = 0$ s to $t = t_1$, v and i are both positive; therefore, power is positive. At $t = t_1$, $v = 0$ V and thus $p = 0$ W. From t_1 to t_2, i is positive and v is negative; therefore, p is negative. From t_2 to t_3, both v and i are negative; therefore power is positive, and so on. As discussed in Chapter 4, Section 4.4, a positive value for p means that power transfer is in the direction of the reference arrow, while a negative value means that it is in the opposite direction. Thus, during positive parts of the power cycle, power flows from the source to the load, while during negative parts, it flows out of the load back into the circuit.

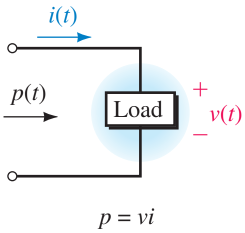


FIGURE 3–1 Voltage, current, and power references. When p is positive, power is in the direction of the reference arrow.

The waveform $p(t)$ of Figure 3–2 is the actual power waveform. We will now show that the key aspects of power flow embodied in this waveform can be described in terms of active power, reactive power, and apparent power.

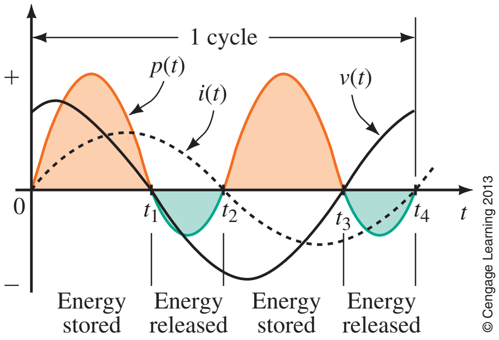


FIGURE 3–2 Instantaneous power in an ac circuit. Positive p represents power to the load; negative p represents power returned from the load.

Active Power

Since p in Figure 3–2 represents the power flowing to the load, its average will be the **average power** to the load. Denote this average by the letter P. If P is positive, then, on average, more power flows to the load than is returned from it. (If P is zero, all power sent to the load is returned.) Thus, if P has a positive value, it represents the power that is really dissipated by the load. For this reason, P is called **real power.** In modern terminology, real power is also called **active power.** Thus, *active power is the average value of the instantaneous power, and the terms real power, active power, and average power mean the same thing*. (We usually refer to it simply as power.) In this book, we use the terms interchangeably.

Reactive Power

Consider again Figure 3–2. During the intervals that p is negative, power is being returned from the load. (This can only happen if the load contains reactive elements: L or C.) The portion of power that flows into the load then back out is called **reactive power.** Since it first flows one way then the other, *its average value is zero;* thus, reactive power contributes nothing to the average power to the load.

Although reactive power does no useful work, it cannot be ignored. Extra current is required to create reactive power, and this current must be supplied by the source; this also means that conductors, circuit breakers, switches, transformers, and other equipment must be made physically larger to handle the extra current. This increases the cost of a system. (This is one of the reasons that reactive power is a major concern of power system engineers.)

As noted, the waveform of Figure 3–2 embodies both the real and reactive aspects of power. In this chapter, we learn how to separate them for purposes of analysis and measurement. We begin by looking at power to resistive, inductive, and capacitive circuit elements separately.

3.2 Power to a Resistive Load

First consider power to a purely resistive load (Figure 3–3). Here, current is in phase with voltage. Assume $i = I_m \sin \omega t$ and $v = V_m \sin \omega t$. Then,

$$p = vi = (V_m \sin \omega t)(I_m \sin \omega t) = V_m I_m \sin^2 \omega t$$

CircuitSim 3-1

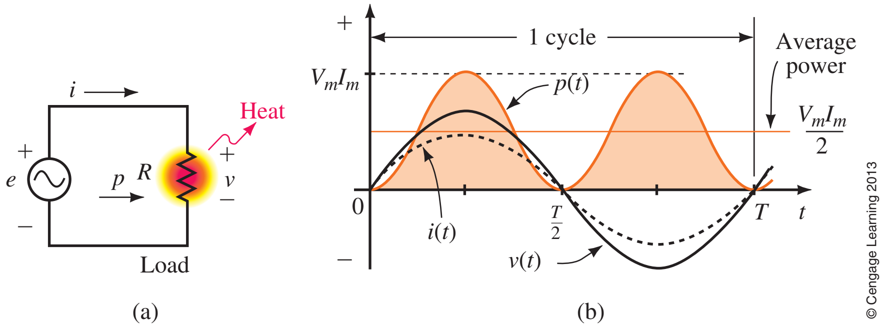


FIGURE 3–3 Power to a purely resistive load. The peak value of p is V_mI_m.

Therefore,

$$p = \frac{V_mI_m}{2}(1 - \cos 2\,\omega t) \tag{3–2}$$

where we have used the trigonometric identity $\sin^2 \omega t = ½\,(1 - \cos 2\,\omega t)$ from inside the front cover of the book.

A sketch of p versus time is shown in Figure 3–3(b). Note that p is always positive (except where it is momentarily zero). This means that power flows only from the source to the load. Since none is ever returned, all power delivered by the source is absorbed by the load. We therefore conclude that *power to a pure resistance consists of active power only.* Note also that the frequency of the power waveform is double that of the voltage and current waveforms. (This is confirmed by the double-frequency component 2ω in Equation 3–2.)

Average Power

Inspection of the power waveform of Figure 3–3 shows that its average value lies halfway between zero and its peak value of V_mI_m. That is,

$$P = V_mI_m/2$$

(You can also get the same result by averaging Equation 3–2 as we did in Chapter 15.) Since V (the magnitude of the rms value of voltage) is $V_m/\sqrt{2}$ and I (the magnitude of the rms value of current) is $I_m/\sqrt{2}$, this can be written as $P = VI$. Thus, average power to a purely resistive load is

$$P = VI \quad \text{(watts)} \tag{3–3}$$

Alternate forms are obtained by substituting $V = IR$ and $I = V/R$ into Equation 3–3. They are

$$P = I^2R \quad \text{(watts)} \tag{3–4}$$

$$= V^2/R \quad \text{(watts)} \tag{3–5}$$

Thus, the active power relationships for resistive circuits are the same for ac as for dc.

3.3 Power to an Inductive Load

For a purely inductive load as in Figure 3–4(a), current lags voltage by 90°. If we select current as reference, $i = I_m\sin \omega t$ and $v = V_m\sin(\omega t + 90°)$. A sketch of p versus time (obtained by multiplying v times i) then looks as

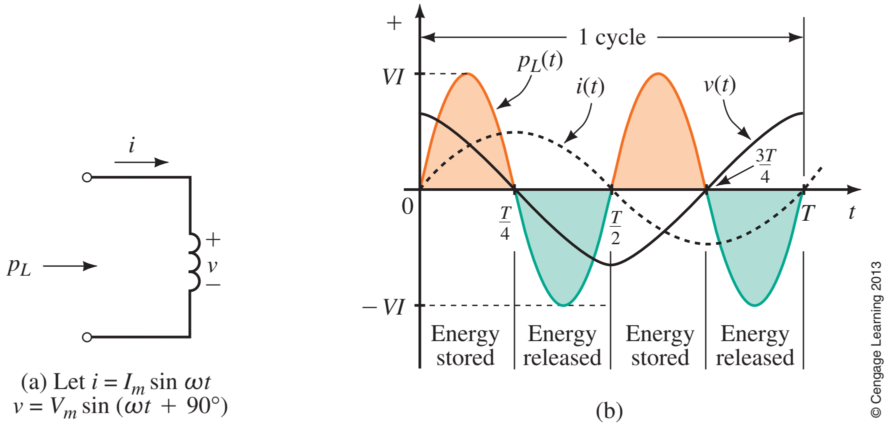


FIGURE 3–4 Power to a purely inductive load. Energy stored during each quarter-cycle is returned during the next quarter-cycle. Average power is zero.

shown in (b). Note that during the first quarter-cycle, p is positive and hence power flows to the inductance, while during the second quarter-cycle, p is negative and all power transferred to the inductance during the first quarter-cycle flows back out. Similarly for the third and fourth quarter-cycles. Thus, *the average power to an inductance over a full cycle is zero, that is, there are no power losses associated with a pure inductance.* Consequently, $P_L = 0$ W, and the only power flowing in the circuit is reactive power. This is true in general, that is, *the power that flows into and out of a pure inductance is reactive power only.*

To determine this power, consider again Equation 3–1.With $v = V_m\sin(\omega t + 90°)$ and $i = I_m\sin \omega t$, $p_L = vi$ becomes

$$p_L = V_m I_m \sin(\omega t + 90°)\sin \omega t$$

After some trigonometric manipulation, this reduces to

$$p_L = VI \sin 2\,\omega t \qquad \textbf{(3–6)}$$

where V and I are the magnitudes of the rms values of the voltage and current, respectively.

The product VI in Equation 3–6 is defined as reactive power and is given the symbol Q_L. Because it represents “power” that alternately flows into, then out of the inductance, Q_L contributes nothing to the average power to the load and, as noted earlier, is sometimes referred to as wattless power. As you will see in Section 3.8, however, reactive power is of major concern in the design and operation of electrical power systems.

Since Q_L is the product of voltage times current, its unit is the volt-amp (VA). To indicate that Q_L represents reactive volt-amps, an “R” is appended to yield a new unit, the **VAR (volt-amps reactive).** Thus,

$$Q_L = VI \quad \text{(VAR)} \qquad \textbf{(3–7)}$$

Substituting $V = IX_L$ and $I = V/X_L$ yields the following alternate forms:

$$Q_L = I^2X_L = \frac{V^2}{X_L} \quad \text{(VAR)} \qquad \textbf{(3–8)}$$

CircuitSim 3-2

By convention, Q_L is taken to be positive. Thus, if $I = 4$ A and $X_L = 2\ \Omega$, $Q_L = (4\text{ A})^2(2\ \Omega) = +32$ VAR. Note that the VAR (like the watt) is a scalar quantity with magnitude only and no angle.

3.4 Power to a Capacitive Load

For a purely capacitive load, current leads voltage by 90°. Taking current as reference, $i = I_m \sin \omega t$ and $v = V_m \sin(\omega t - 90°)$. Multiplication of v times i yields the power curve of Figure 3–5. Note that negative and positive loops of the power wave are identical; thus, over a cycle, the power returned to the circuit by the capacitance is exactly equal to that delivered to it by the source. This means that *the average power to a capacitance over a full cycle is zero, that is, there are no power losses associated with a pure capacitance.* Consequently, $P_C = 0$ W, and the only power flowing in the circuit is reactive power. This is true in general, that is, *the power that flows into and out of a pure capacitance is reactive power only.* This reactive power is given by

$$p_C = vi = V_m I_m \sin \omega t \sin(\omega t - 90°)$$

which reduces to

$$p_C = -VI \sin 2\omega t \tag{3–9}$$

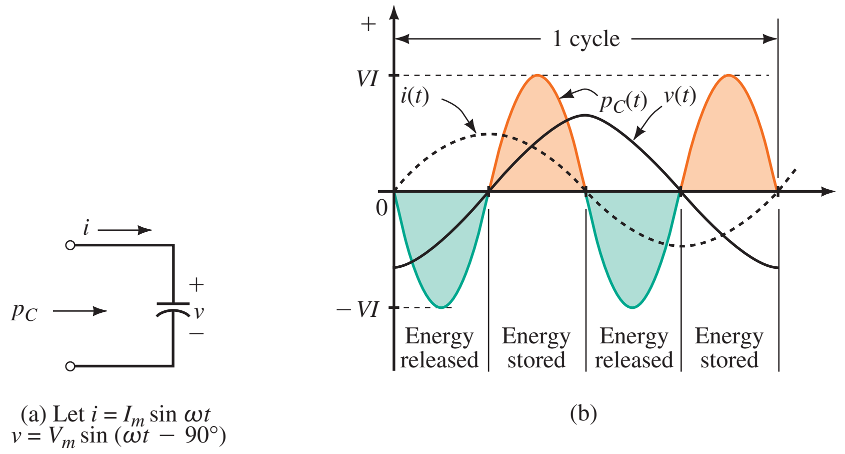


FIGURE 3–5 Power to a purely capacitive load. Average power is zero.

CircuitSim 3-3

where V and I are the magnitudes of the rms values of the voltage and current respectively. Now define the product VI as Q_C. This product represents reactive power. That is,

$$Q_C = VI \quad \text{(VAR)} \tag{3–10}$$

CircuitSim 3-4

Since $V = IX_C$ and $I = V/X_C$, Q_C can also be expressed as

$$Q_C = I^2 X_C = \frac{V^2}{X_C} \quad \text{(VAR)} \tag{3–11}$$

By convention, reactive power to capacitance is defined as negative. Thus, if $I = 4$ A and $X_C = 2\ \Omega$, then $I^2X_C = (4\text{ A})^2(2\ \Omega) = 32$ VAR. We can either explicitly include the minus sign as $Q_C = -32$ VAR or imply it by stating that Q represents capacitive VARs, that is, $Q_C = 32$ VAR (cap.).

EXAMPLE 3–1

For each circuit of Figure 3–6, determine real and reactive power.

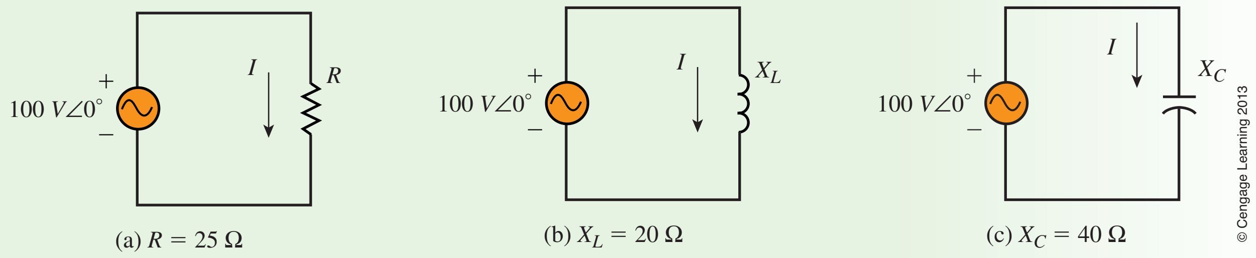


FIGURE 3–6

Solution Only voltage and current magnitudes are needed.

a. $I = 100\ \text{V}/25\ \Omega = 4\ \text{A}$. $P = VI = (100\ \text{V})(4\ \text{A}) = 400\ \text{W}$. $Q = 0\ \text{VAR}$.

b. $I = 100\ \text{V}/20\ \Omega = 5\ \text{A}$. $Q = VI = (100\ \text{V})(5\ \text{A}) = 500\ \text{VAR}$ (ind.). $P = 0\ \text{W}$.

c. $I = 100\ \text{V}/40\ \Omega = 2.5\ \text{A}$. $Q = VI = (100\ \text{V})(2.5\ \text{A}) = 250\ \text{VAR}$ (cap.). $P = 0\ \text{W}$.

The answer for (c) can also be expressed as $Q = -250\ \text{VAR}$.

PRACTICE PROBLEMS 1

1. If the power at some instant in Figure 17–1 is $p = -27\ \text{W}$, in what direction is the power at that instant?
2.F or a purely resistive load, v and i are in phase. Given $v = 10 \sin \omega t$ V and $i = 5 \sin \omega t$ A. Using graph paper, carefully plot v and i at 30° intervals. Now multiply the values of v and i at these points and plot the power. [The result should look like Figure 3–3(b).]
   a. From the graph, determine the peak power and average power.
   b.C ompute power using $P = VI$ and compare to the average value determined in (a).
3. Repeat Example 3–1 using Equations 3–4, 3–5, 3–8, and 3–11.

Answers

1. From the load to the source.
2. a. 50 W; 25 W
   b.S ame

3.5 Power in More Complex Circuits

The relationships described previously were developed using the load of Figure 3–1. However, they hold true for every element in a circuit, no matter how complex the circuit or how its elements are interconnected. Further, in any circuit, total real power P_T is found by summing real power to all circuit elements, while total reactive power Q_T is found by summing reactive power, taking into account that inductive Q is positive and capacitive Q is negative.

It is sometimes convenient to show power to circuit elements symbolically as illustrated in the next example.

EXAMPLE 3–2

For the *RL* circuit of Figure 3–7(a), $I = 5$ A. Determine *P* and *Q*.

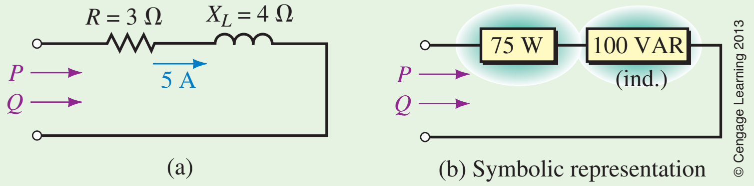


FIGURE 3–7 From the terminals, *P* and *Q* are the same for both (a) and (b).

Solution

$$P = I^2R = (5\text{ A})^2(3\ \Omega) = 75\text{ W}$$
$$Q = Q_L = I^2X_L = (5\text{ A})^2(4\ \Omega) = 100\text{ VAR (ind.)}$$

These can be represented symbolically as in Figure 3–7(b).

EXAMPLE 3–3

For the *RC* circuit of Figure 3–8(a), determine *P* and *Q*.

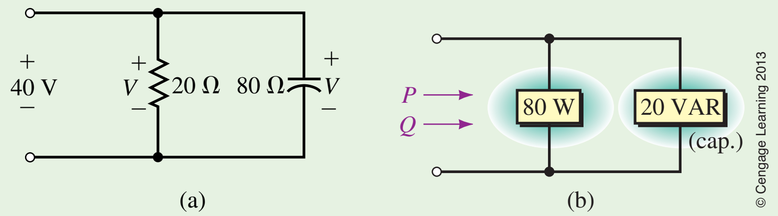


FIGURE 3–8 From the terminals, *P* and *Q* are the same for both (a) and (b).

Solution

$$P = V^2/R = (40\text{ V})^2/(20\ \Omega) = 80\text{ W}$$
$$Q = Q_C = V^2/X_C = (40\text{ V})^2/(80\ \Omega) = 20\text{ VAR (cap.)}$$

These can be represented symbolically as in Figure 3–8(b).

In terms of determining total *P* and *Q*, it does not matter how the circuit or system is connected or what electrical elements it contains. Elements can be connected in series, in parallel, or in series-parallel, for example, and the system can contain electric motors and the like, and total *P* is still found by summing the power to individual elements, while total *Q* is found by algebraically summing their reactive powers.

EXAMPLE 3–4

a. For Figure 17–9(a), compute P_T and Q_T.
b. Reduce the circuit to its simplest form.

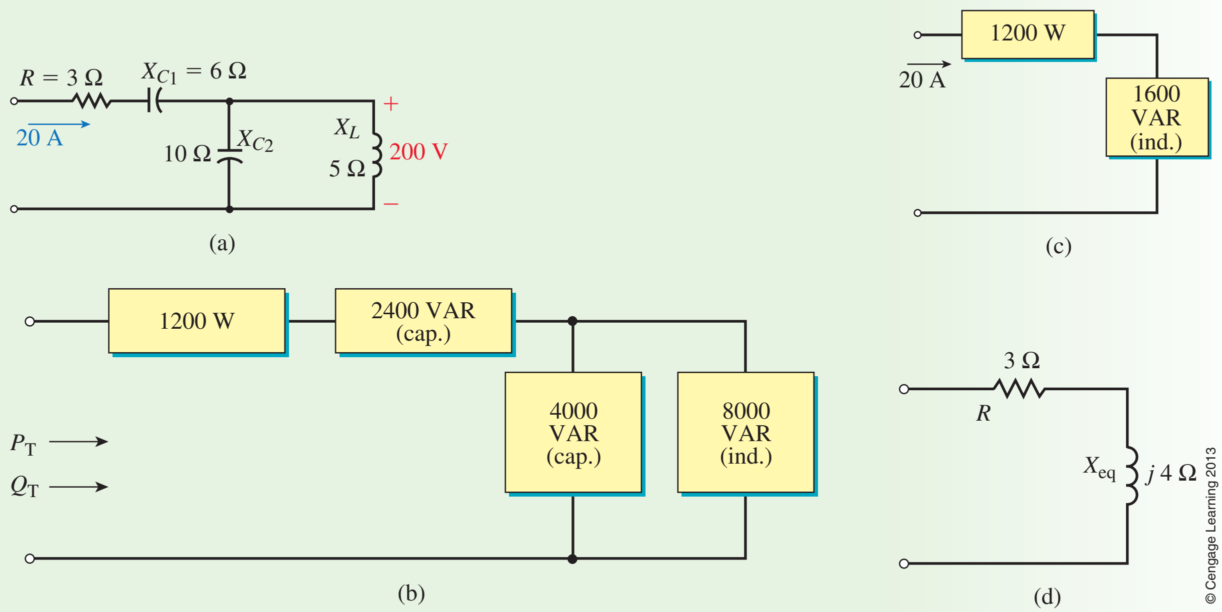


FIGURE 3–9

Solution

a. $P = I^2R = (20\text{ A})^2(3\ \Omega) = 1200\text{ W}$

$Q_{C_1} = I^2X_{C_1} = (20\text{ A})^2(6\ \Omega) = 2400\text{ VAR (cap.)}$

$$Q_{C_2} = \frac{V_2^2}{X_{C_2}} = \frac{(200\text{ V})^2}{(10\ \Omega)} = 4000\text{ VAR (cap.)}$$

$$Q_L = \frac{V_2^2}{X_L} = \frac{(200\text{ V})^2}{5\ \Omega} = 8000\text{ VAR (ind.)}$$

These are represented symbolically in part (b). $P_T = 1200$ W and $Q_T = -2400\text{ VAR} - 4000\text{ VAR} + 8000\text{ VAR} = 1600$ VAR. Thus, the load is net inductive as shown in (c).

b. $Q_T = I^2X_{eq}$. Thus, $X_{eq} = Q_T/I^2 = (1600\text{ VAR})/(20\text{ A})^2 = 4\ \Omega$. Circuit resistance remains unchanged. Thus, the equivalent is as shown in (d).

CircuitSim 17-5

PRACTICE PROBLEMS 2

For the circuit of Figure 3–10, $P_T = 1.9$ kW and $Q_T = 900$ VAR (ind.). Determine P_2 and Q_2.

Answer
300 W, 400 VAR (cap.)

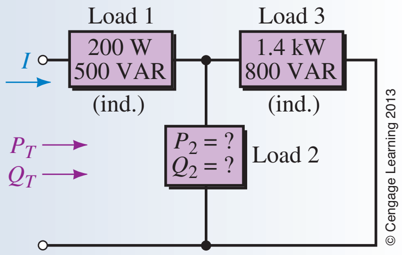


FIGURE 3 –10

3.6 Apparent Power

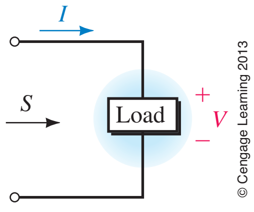


FIGURE 3–11 Apparent power $S = VI$.

When a load has voltage V across it and current I through it as in Figure 3–11, the power that appears to flow to it is VI. However, if the load contains both resistance and reactance, this product represents neither real power nor reactive power. Since VI appears to represent power, it is called **apparent power.** Apparent power is given the symbol $\boldsymbol{S}$ and has units of **volt-amperes (VA).** Thus,

$$S = VI \quad \text{(VA)} \tag{3–12}$$

where V and I are the magnitudes of the rms voltage and current, respectively. Since $V = IZ$ and $I = V/Z$, S can also be written as

$$S = I^2Z = V^2/Z \quad \text{(VA)} \tag{3–13}$$

For small equipment (such as found in electronics), VA is a convenient unit. However, for heavy power apparatus (Figure 3–12), it is too small and kVA (kilovolt-amps) is frequently used, where

$$S = \frac{VI}{1000} \quad \text{(kVA)} \tag{3–14}$$

In addition to its VA rating, it is common practice to rate electrical apparatus in terms of its operating voltage. Once you know these two, it is easy to determine rated current. For example, a piece of equipment rated 250 kVA, 4.16 kV has a rated current of $I = S/V = (250 \times 10^3 \text{ VA})/(4.16 \times 10^3 \text{ V}) = 60.1$ A.

FIGURE 3–12 Power apparatus is rated in apparent power. The transformer shown is a 167-kVA unit.

3.7 The Relationship between *P*, *Q*, and *S*

Until now, we have treated real, reactive, and apparent power separately. However, they are related by a very simple relationship through the power triangle.

The Power Triangle

Consider the series circuit of Figure 3–13(a). Let the current through the circuit be $\mathbf{I} = I\angle 0°$, with phasor representation (b). The voltages across the resistor and inductance are $\mathbf{V}_R$ and $\mathbf{V}_L$, respectively. As noted in Chapter 16, $\mathbf{V}_R$ is in phase with $\mathbf{I}$, while $\mathbf{V}_L$ leads it by 90°. Kirchhoff's voltage law applies for ac voltages in phasor form. Thus, $\mathbf{V} = \mathbf{V}_R + \mathbf{V}_L$ as indicated in (c).

The voltage triangle of (c) may be redrawn as in Figure 3–14(a) with magnitudes of V_R and V_L replaced by IR and IX_L, respectively. Now multiply all quantities by I. This yields sides of I^2R, I^2X_L, and hypotenuse VI as indicated in (b). Note that these represent P, Q, and S, respectively, as indicated in (c). This is called the **power triangle.** From the geometry of this triangle, you can see that

$$S = \sqrt{P^2 + Q_L^2} \qquad \textbf{(3–15)}$$

Alternatively, the relationship between P, Q, and S may be expressed as a complex number:

$$\mathbf{S} = P + jQ_L \qquad \textbf{(3–16a)}$$

or

$$\mathbf{S} = S\angle\theta \qquad \textbf{(3–16b)}$$

If the circuit is capacitive instead of inductive, Equation 3–16a becomes

$$\mathbf{S} = P - jQ_C \qquad \textbf{(3–3)}$$

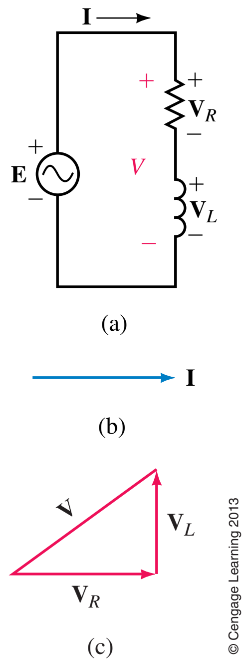


FIGURE 3–13 Steps in the development of the power triangle—continued in Figure 3–14.

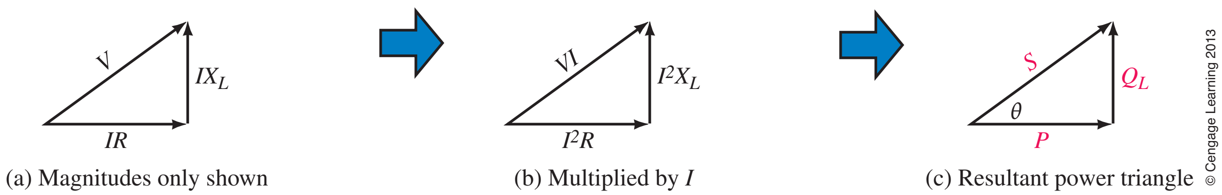


FIGURE 3–14 Steps in the development of the power triangle continued.

The power triangle in this case has a negative imaginary part as indicated in Figure 3–15.

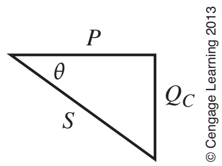


FIGURE 3–15 Power triangle for capacitive case.

The power relationships may be written in generalized forms as

$$\mathbf{S} = \mathbf{P} + \mathbf{Q} \qquad \textbf{(3–18)}$$

and

$$\mathbf{S} = \mathbf{V}\mathbf{I}^* \qquad \textbf{(3–19)}$$

where $\mathbf{P} = P\angle 0°$, $\mathbf{Q}_L = jQ_L$, $\mathbf{Q}_C = -jQ_C$, and $\mathbf{I}^*$ is the conjugate of current $\mathbf{I}$—see Note. These relationships hold true for all networks regardless of what they contain or how they are configured.

NOTES...

See Chapter 16, Section 16.1 for a discussion of complex conjugates.

When solving problems involving power, remember that P values can be added to get P_T, and Q values to get Q_T (where Q is positive for inductive elements and negative for capacitive). However, apparent power values cannot be added to get S_T, that is, $S_T \neq S_1 + S_2 + \cdots + S_N$. Instead, you must determine P_T and Q_T separately, then use the power triangle to obtain S_T—see Practice Problems 3, Problem 2.

EXAMPLE 3–5

The P and Q values for a circuit are shown in Figure 3–16(a).

a. Determine the power triangle.

b. Determine the magnitude of the current supplied by the source.

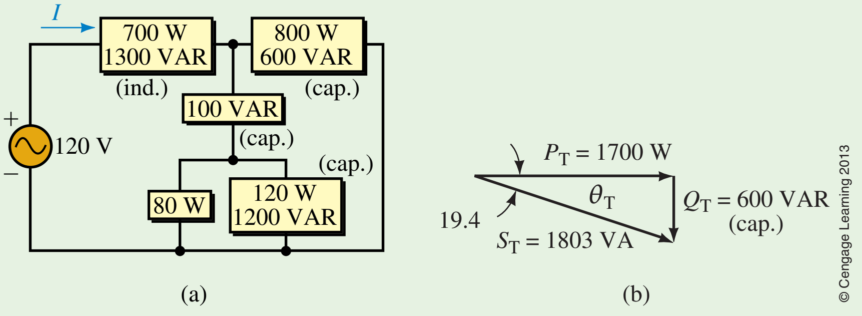


FIGURE 3–16

Solution

a. $P_T = 700 + 800 + 80 + 120 = 1700$ W

$Q_T = 1300 - 600 - 100 - 1200 = -600$ VAR $= 600$ VAR (cap.)

$\mathbf{S}_T = P_T + jQ_T = 1700 - j600 = 1803\angle -19.4°$ VA

The power triangle is as shown. The load is net capacitive.

b. $I = S_T/E = 1803$ VA/120 V $= 15.0$ A.

EXAMPLE 3–6

CircuitSim 3- 6

A generator supplies power to an electric heater, an inductive element, and a capacitor as in Figure 3–3(a).

a. Find P and Q for each load.

b.F ind total active and reactive power supplied by the generator.

c. Draw the power triangle for the combined loads and determine total apparent power.

d. Find the current supplied by the generator.

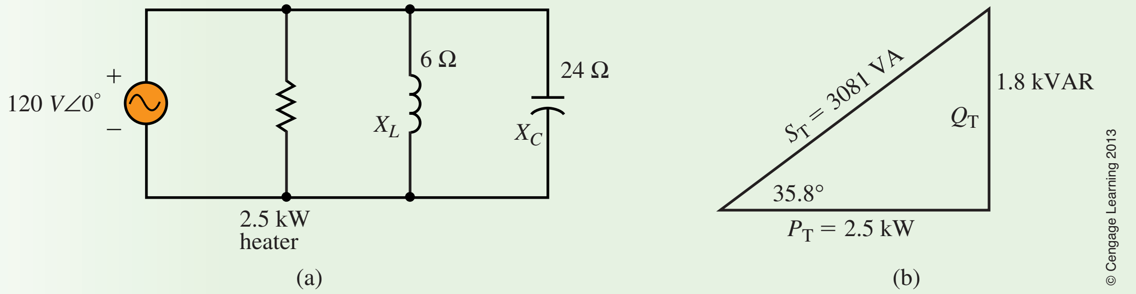


FIGURE 3 –17

Solution

a. The components of power are as follows:

Heater: $P_H = 2.5$ kW $\quad Q_H = 0$ VAR

Inductor: $P_L = 0$ W $\quad Q_L = \dfrac{V^2}{X_L} = \dfrac{(120 \text{ V})^2}{6\ \Omega} = 2.4$ kVAR (ind.)

Capacitor: $P_C = 0$ W $\quad Q_C = \dfrac{V^2}{X_C} = \dfrac{(120 \text{ V})^2}{24\ \Omega} = 600$ VAR (cap.)

b. $P_T = 2.5 \text{ kW} + 0 \text{ W} + 0 \text{ W} = 2.5$ kW

$Q_T = 0 \text{ VAR} + 2.4 \text{ kVAR} - 600 \text{ VAR} = 1.8$ kVAR (ind.)

c. The power triangle is sketched as Figure 17–7(b). Both the hypotenuse and the angle can be obtained easily using rectangular to polar conversion. $\mathbf{S}_T = P_T + jQ_T = 2500 + j1800 = 3081\angle 35.8°$. Thus, apparent power is $S_T = 3081$ VA.

d. Generator current is $I = \dfrac{S_T}{E} = \dfrac{3081 \text{ VA}}{120 \text{ V}} = 25.7$ A

Real and Reactive Power Equations

An examination of the power triangle of Figures 3–14 and 3–15 shows that P and Q may be expressed as

$$P = VI \cos\theta = S \cos\theta \quad \text{(W)} \tag{3–20}$$

and

$$Q = VI \sin\theta = S \sin\theta \quad \text{(VAR)} \tag{3–21}$$

where V and I are the magnitudes of the rms values of the voltage and current, respectively, and θ is the angle between them. P is always positive, while Q is positive for inductive circuits and negative for capacitive circuits. Thus, if $V = 120$ volts, $I = 50$ A, and $\theta = 30°$, $P = (120)(50)\cos 30° = 5196$ W and $Q = (120)(50)\sin 30° = 3000$ VAR.

PRACTICE PROBLEMS 3

1. A 208-V generator supplies power to a group of three loads. Load 1 has an apparent power of 500 VA with $\theta = 36.87°$ (i.e., it is net inductive). Load 2 has an apparent power of 1000 VA and is net capacitive with a power triangle angle of $-53.13°$. Load 3 is purely resistive with power $P_3 = 200$ W. Determine the power triangle for the combined loads and the generator current.
2. Using numerical values as indicated with Figure 17–10,
   a. Determine apparent power for each block, and the total apparent power to the circuit.
   b. Sum the apparent power for all blocks and show that it does not equal the actual apparent power to the circuit.

CircuitSim 3-7

Answers

1. $S_T = 1300$ VA, $\theta_T = -22.6°$, $I = 6.25$ A
2. a. 538.5 VA, 500 VA, 1612.5 VA, 2.10 KVA
   b. 2.651 kVA (Not the same.)

3.8 Power Factor

The quantity $\cos\theta$ in Equation 3–20 is defined as **power factor** and is given the symbol F_p. Thus,

$$F_p = \cos\theta \tag{3–22}$$

From Equation 3–20, we see that F_p may be computed as the ratio of real power to apparent power. Thus,

$$\cos\theta = P/S \tag{3–23}$$

Power factor is expressed as a number or as a percent. From Equation 3–23, it is apparent that the power factor cannot exceed 1.0 (or 100% if expressed in percent).

CircuitSim 3-8

The power factor angle θ is of interest. It can be found as

$$\theta = \cos^{-1}(P/S) \tag{3–24}$$

Angle θ is the angle between voltage and current. For a pure resistance, therefore, $\theta = 0°$. For a pure inductance, $\theta = 90°$; for a pure capacitance, $\theta = -90°$. For a circuit containing both resistance and inductance, θ will be somewhere between 0° and 90°; for a circuit containing both resistance and capacitance, θ will be somewhere between 0° and −90°.

Unity, Lagging, and Leading Power Factor

As indicated by Equation 3–23, a load's power factor shows how much of its apparent power is actually real power. For example, for a purely resistive circuit, $\theta = 0°$ and $F_p = \cos 0° = 1.0$. Therefore, $P = VI$ (watts) and all the load's apparent power is real power. This case ($F_p = 1$) is referred to as *unity* power factor.

For a load containing only resistance and inductance, the load current lags voltage. The power factor in this case is described as *lagging*. On the other hand, for a load containing only resistance and capacitance, current leads voltage and the power factor is described as *leading*. Thus, *an inductive circuit has a lagging power factor, while a capacitive circuit has a leading power factor.*

A load with a very poor power factor can draw excessive current. This is discussed next.

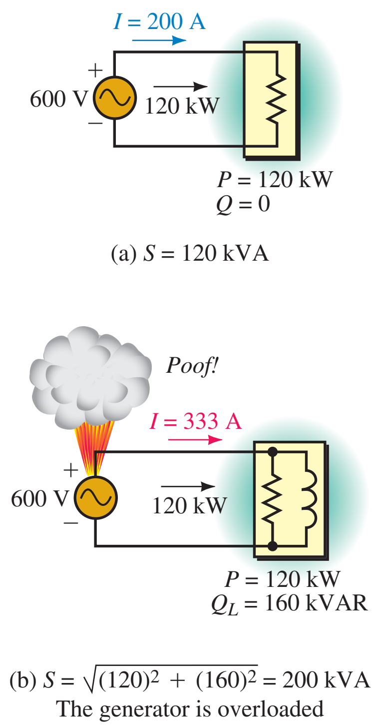


FIGURE 3–18 Illustrating why electrical apparatus is rated in VA instead of watts. Both loads dissipate 120 kW, but the current rating of generator (b) is exceeded because of the power factor of its load.

Why Equipment Is Rated in VA

As noted earlier, equipment is rated in terms of VA instead of watts. We now show why. Consider Figure 3–18. Assume that the generator is rated at 600 V, 120 kVA. This means that it is capable of supplying I = 120 kVA/600 V = 200 A. In (a), the generator is supplying a purely resistive load with 120 kW. Since $S = P$ for a purely resistive load, S = 120 kVA and the generator is supplying its rated current. In (b), the generator is supplying a load with P = 120 kW as before, but Q = 160 kVAR. Its apparent power is therefore S = 200 kVA, which means that the generator current is I = 200 kVA/600 V = 333.3 A. Even though it is supplying the same power as in (a), the generator is now greatly overloaded, and damage may result as indicated in (b).

This example illustrates clearly that rating a load or device in terms of power is a poor choice, as its current-carrying capability can be greatly exceeded (even though its power rating is not). Thus, *the size of electrical* *apparatus (generators, interconnecting wires, transformers, etc.) required to supply a load is governed, not by the load's power requirements, but rather by its VA requirements.*

Power Factor Correction

The problem shown in Figure 3–18 can be alleviated by cancelling some or all of the reactive component of power by adding reactance of the opposite type to the circuit. This is referred to as **power factor correction.** If you completely cancel the reactive component, the power factor angle is 0° and $F_p = 1$. This is referred to as unity power factor correction.

In practice, almost all loads, whether they be residential, industrial, or commercial, are inductive (due to the presence of motors, and other inductive elements); thus, you will probably never encounter a capacitive load that needs correcting. The upshot of this is that, in practice, virtually all power factor correction consists of adding a capacitor in order to cancel inductive effects. As is illustrated next, this capacitance is placed across the load and as near to it as practical.

EXAMPLE 3–7

For the overloaded generator problem of Figure 3–18(b), a capacitance with Q_C = 160 kVAR is added in parallel with the load as in Figure 3–19(a). Determine generator current *I*.

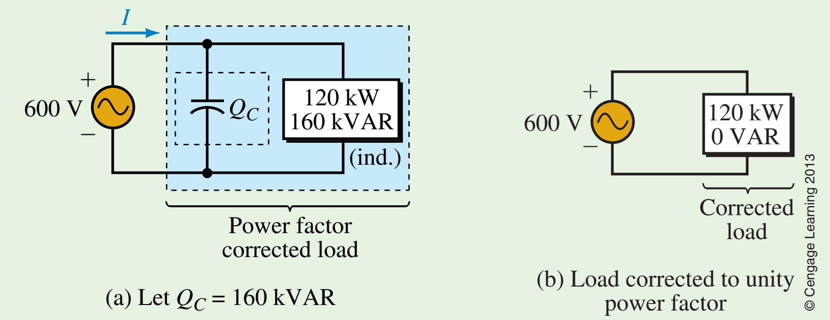


FIGURE 3–19 Power factor correction. The parallel capacitor greatly reduces source current.

Solution $Q_T = 160\text{ kVAR} - 160\text{ kVAR} = 0$. Therefore, $\mathbf{S}_T = 120\text{ kW} + j0\text{ kVAR}$. Thus, $S_T = 120$ kVA, and current drops from 333 A to $I = 120\text{ kVA}/600\text{ V} = 200$ A. Thus, the generator is no longer overloaded—see Notes.

NOTES...

Note carefully that, what has changed here is how much current the source has to supply and consequently, how much current the transmission and distribution lines between the source and capacitor carry. Since the load itself has not changed, it requires the same current as before and the wires between the capacitor and the load must carry this larger current. (This is why power system engineers place corrective capacitors as close to the load as practical.)

CircuitSim 17-9

Residential customers are not charged directly for VARs—that is, they pay their electrical bills based solely on the number of the kilowatt-hours they use. This is because all residential customers have essentially the same power factor, and the power factor effect is simply built into the rates that they pay. Industrial customers, on the other hand, often have widely differing power factors, and the utility may have to monitor their VARs (or their power factor), as well as their watts, in order to determine a suitable charge.

CircuitSim 3-10

To illustrate, assume that the loads of Figures 3–18(a) and (b) are two small industrial plants. If the utility based its charge solely on power, both customers would pay the same amount. However, it costs the utility more to supply customer (b) since larger conductors, larger transformers, larger switchgear, and so on are required to handle the larger current. For this reason, industrial customers may pay a penalty if their power factor drops below a prescribed value as set by the utility.

EXAMPLE 3–8

Suppose an industrial client is charged a penalty if the plant power factor drops below 0.85. The equivalent plant loads are as shown in Figure 3–20. The frequency is 60 Hz—see Note 3.

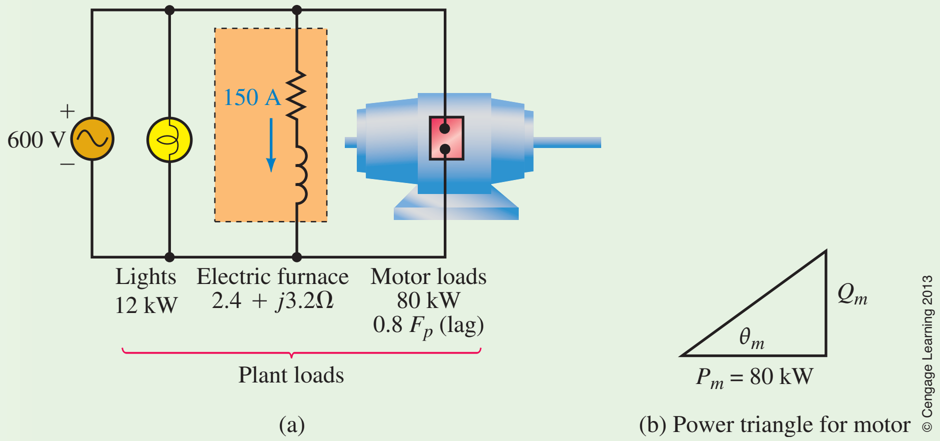


FIGURE 3–20

a. Determine P_T and Q_T.
b. Determine what value of capacitance (in microfarads) is required to bring the power factor up to 0.85.
c. Determine generator current before and after correction.

Solution

a. The components of power are as follows:

$$\text{Lights: } P = 12 \text{ kW}, \quad Q = 0 \text{ kVAR}$$

$$\text{Furnace: } P = I^2R = (150)^2(2.4) = 54 \text{ kW}$$

$$Q = I^2X = (150)^2(3.2) = 72 \text{ kVAR (ind.)}$$

Motor: $\theta_m = \cos^{-1}(0.8) = 36.9°$. Thus, from the motor power triangle,

$$Q_m = P_m \tan \theta_m = 80 \tan 36.9° = 60 \text{ kVAR (ind.)}$$

$$\text{Total: } P_T = 12 \text{ kW} + 54 \text{ kW} + 80 \text{ kW} = 146 \text{ kW}$$

$$Q_T = 0 + 72 \text{ kVAR} + 60 \text{ kVAR} = 132 \text{ kVAR (ind.)}$$

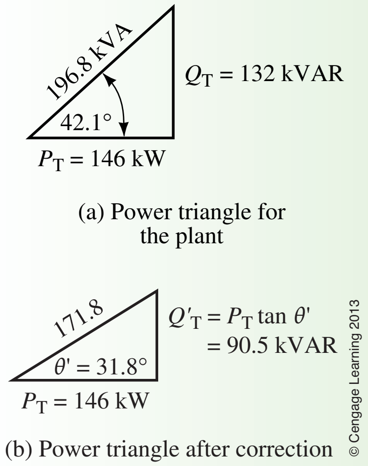


FIGURE 3 –21 Initial and final power triangles. Note that P_T does not change when we correct the power factor, since for the capacitor, $P = 0$ W.

b. The power triangle for the plant is shown in Figure 3–21(a). However, we must correct the power factor to 0.85. Thus, we need $\theta' = \cos^{-1}(0.85) = 31.8°$, where θ' is the power factor angle of the corrected load as indicated in Figure 3–21(b). The maximum reactive power that we can tolerate is thus $Q'_T = P_T \tan \theta' = 146 \tan 31.8° = 90.5$ kVAR.

Now consider Figure 3–22. $Q'_T = Q_C + 132$ kVAR, where $Q'_T = 90.5$ kVAR. Therefore, $Q_C = -41.5$ kVAR $= 41.5$ kVAR (cap.). But $Q_C = V^2/X_C$. Therefore, $X_C = V^2/Q_C = (600)^2/41.5$ kVAR $= 8.67$ Ω. But $X_C = 1/\omega C$. Thus, a capacitor of

$$C = \frac{1}{\omega X_C} = \frac{1}{(2\pi)(60)(8.67)} = 306 \ \mu\text{F}$$

will provide the required correction—see Notes.

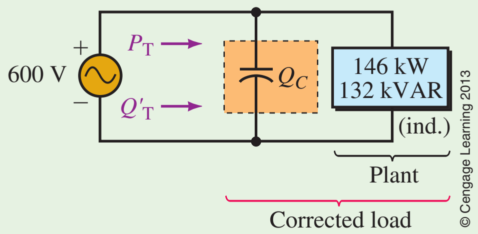


FIGURE 3–22

c. For the original circuit Figure 17–21(a), S_T = 196.8 kVA. Thus,

$$I = \frac{S_T}{E} = \frac{196.8 \text{ kVA}}{600 \text{ V}} = 328 \text{ A}$$

For the corrected circuit of Figure 3–21(b), S'_T = 171.8 kVA and

$$I = \frac{171.8 \text{ kVA}}{600 \text{ V}} = 286 \text{ A}$$

Thus, power factor correction has dropped the source current by 42 A.

NOTES...

1. Electric power companies use special utility grade capacitors for this job. The capacitor must be sized both for its value of capacitance and for its kVA rating.
2. Power factor correction capacitors are available in fixed values, or as switched banks with automatic controls that switch capacitors in or out, depending on load requirements.
3. Computer simulations for this example are found in Section 312.

PRACTICE PROBLEMS 4

1. By how many amps has power factor correction reduced the motor current of Figure 3–20?
2. Repeat Example 3–8, except correct the power factor to unity.
3. Due to plant expansion, 102 kW of purely resistive load is added to the plant of Figure 3–20. Determine whether power factor correction is needed to correct the expanded plant to 0.85 F_p, or better.

Answers

1. Zero—there is no change in motor current. It is the source current that is reduced.
2. 973 μF, 243 A. Other answers remain unchanged.
3. F_p = 0.88. No correction needed.

IN-PROCESS LEARNING CHECK 1

(Answers are at the end of the chapter.)

1. Sketch the power triangle for Figure 17–9(c). Using this triangle, determine the magnitude of the applied voltage.
2. For Figure 17–10, assume a source of E = 240 volts, P_2 = 300 W, and Q_2 = 400 VAR (cap.). What is the magnitude of the source current I?
3. What is the power factor of each of the circuits of Figures 17–7, 17–8, and 3–9? Indicate whether they are leading or lagging.
4. Consider the circuit of Figure 3–18(b). If P = 100 kW and Q_L = 80 kVAR, is the source overloaded, assuming it is capable of handling a 120-kVA load?

3.9 ac Power Measurement

To measure power in an ac circuit, you need a wattmeter. A **wattmeter** is a device that monitors current and voltage, and from these, determines power. Most modern units are digital instruments. For digital units, Figure 3–23, power is displayed on a numerical readout, while for analog instruments (see Note), power is indicated by a pointer on a scale much like the analog meter of Figure 2–22(b), Chapter 2. Note however, although their details differ, their method of use and connection in a circuit are the same—thus, the measurement techniques described herein apply to both.

To strengthen your understanding of the concept of power measurement, reexamine Figures 3–1, 3–2, and 3–3. Instantaneous load power is the product of load voltage times load current, and average power is the average of this product. One way to implement power measurement is therefore to create a meter with a current sensing circuit, a voltage sensing circuit, a multiplier circuit, and an averaging circuit. Figure 3–24 shows a simplified symbolic representation of such an instrument. Current is passed through its current coil (CC) to create a magnetic field proportional to the current, and a solid state sensor circuit connected across the load voltage reacts with this field to produce an output voltage proportional to the product of instantaneous voltage and current (i.e., proportional to instantaneous power). An averaging circuit averages this voltage and drives a display to indicate average power. (The scheme used by the meter of Figure 3–23 is actually considerably more sophisticated than this because it measures many things besides power—e.g., it measures VARs, VA, energy, and more. However, the basic idea is conceptually correct.)

FIGURE 3–23 Multifunction power/energy meter. It can measure active power (W), reactive power (VARs), apparent power (VA), power factor, energy, and more.

NOTES...

Older wattmeters, called electrodynamometers, may still be found in practice. They are electromechanical devices that utilize both a stationary coil connected in series with the load (the current coil) and a pivoted coil (the potential coil) connected across the load. The interaction of the magnetic fields of these two coils creates a torque, and the pointer, attached to the pivoted coil, takes up a position on the scale corresponding to average power—see Practice Problems 5, Problem 2.

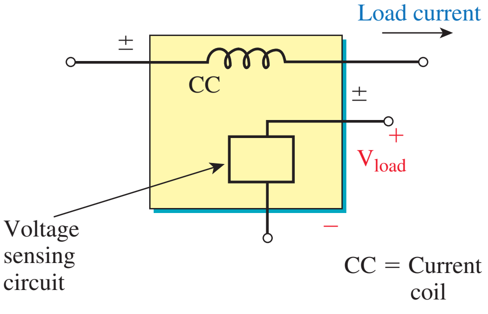


FIGURE 3–24 Conceptual representation of an electronic wattmeter.

Figure 3–25 shows how to connect a wattmeter into a circuit. Load current passes through its current coil circuit, and load voltage is impressed across its voltage sensing circuit. With this connection, the wattmeter computes and displays the product of the magnitude of the load voltage, the magnitude of the load current, and the cosine of the angle between them, that is, $V_{load} \cdot I_{load} \cdot \cos\theta_{load}$—see Notes. Thus, it measures load power. Note the ± marking on the terminals.

NOTES...

You need to be very clear on what angle to use in determining a wattmeter's reading. In terms of Figure 3–24, the angle to use is the angle between the voltage labelled V_{load} and the current labelled *Load current,* using the reference polarity for voltage and the reference direction for current shown. The diagram of Figure 3–25 helps clarify this.

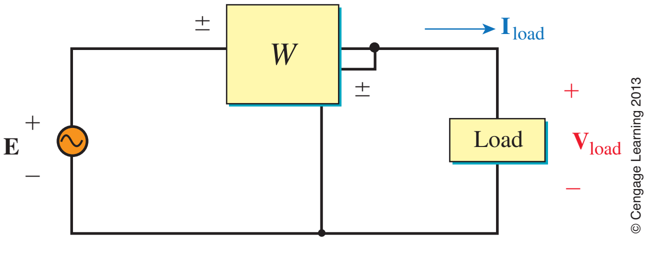


FIGURE 7–25 Connection of wattmeter.

The meter is connected so that load current enters the ± current terminal and the higher potential end of the load is connected to the ± voltage terminal. (On many meters, the ± voltage terminal is internally connected so that only three terminals are brought out as in Figure 3–26.)

When power is to be measured in a low power factor circuit, a low power factor wattmeter must be used. This is because, for low power factor loads, currents can be very high, even though the power is low. Thus, you can easily exceed the current rating of a standard wattmeter and damage it, even though the power indication on the meter is small.

EXAMPLE 3–9

For the circuit of Figure 3–25, what does the wattmeter indicate if

a. $\mathbf{V}_{load} = 100\text{ V}\angle 0°$ and $\mathbf{I}_{load} = 15\text{ A}\angle 60°$,
b. $\mathbf{V}_{load} = 100\text{ V}\angle 10°$ and $\mathbf{I}_{load} = 15\text{ A}\angle 30°$?

Solution

a. $\theta_{load} = 60°$. Thus, $P = (100)(15)\cos 60° = 750$ W,
b. $\theta_{load} = 10° - 30° = -20°$. Thus, $P = (100)(15)\cos(-20°) = 1410$ W.

Note: For (b), since $\cos(-20°) = \cos(+20°)$, it does not matter whether we include the minus sign in the calculation.

EXAMPLE 3–10

For Figure 3–26, determine the wattmeter reading.

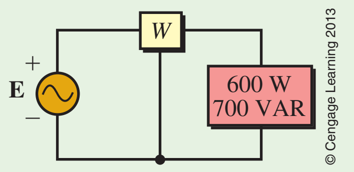


FIGURE 3–26 This wattmeter has its voltage side ± terminals connected internally.

Solution A wattmeter reads only active power. Thus, it indicates 600 W.

It should be noted that the wattmeter reads power only for circuit elements on the load side of the meter. In addition, if the load consists of several elements, it reads the total power.

CircuitSim 3-11

PRACTICE PROBLEMS 5

1. Determine the wattmeter reading for Figure 17–27.

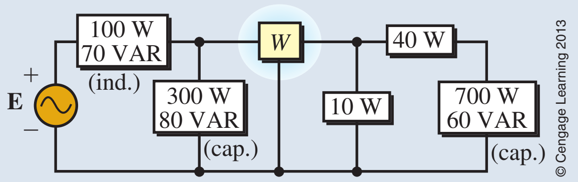


FIGURE 3–27

2. (Optional problem) Research the electrodynamometer wattmeter and prepare a short description complete with diagram showing its coils, pivoted pointer, scale, and so on. The Internet is a good source of information.

Answer
1. 750 W

3.10 Effective Resistance

Up to now, we have assumed that resistance is constant, independent of frequency. However, this is not entirely true. For a number of reasons, the resistance of a circuit to ac is greater than its resistance to dc. While this effect is small at low frequencies, it is very pronounced at high frequencies. AC resistance is known as **effective resistance.**

Before looking at why ac resistance is greater than dc resistance, we need to reexamine the concept of resistance itself. Recall from Chapter 3 that resistance was originally defined in the dc case as opposition to current, that is, $R = V/I$. (This is ohmic resistance.) Building on this, you learned in Chapter 4 that $P = I^2R$. It is this latter viewpoint that allows us to give meaning to ac resistance. That is, we define ac or effective resistance as

$$R_{\text{eff}} = \frac{P}{I^2} \quad (\Omega) \tag{3–25}$$

where P is dissipated power (as determined by a wattmeter). From this, you can see that anything that affects dissipated power affects resistance. For dc and low-frequency ac, both definitions for R, that is, $R = V/I$ and $R = P/I^2$, yield the same value. However, as frequency increases, other factors cause an increase in resistance. We will now consider some of these.

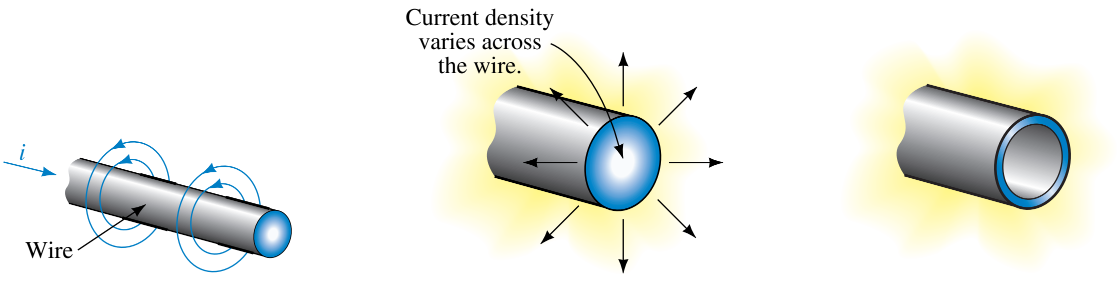
Eddy Currents and Hysteresis

The magnetic field surrounding a coil or other circuit-carrying ac current varies with time and thus induces voltages in nearby conductive material such as metal equipment cabinets, transformer cores, and so on. The resulting currents (called **eddy currents** because they flow in circular patterns like eddies in a brook) are unwanted and create power losses called eddy current losses. Since additional power must be supplied by the source to make up for these losses, P in Equation 3–25 increases, increasing the effective resistance of the coil.

If ferromagnetic material is also present, an additional power loss occurs due to hysteresis effects caused by the magnetic field alternately magnetizing the material in one direction, then the other. Hysteresis and eddy current losses are important even at low frequencies, such as the 60-Hz power system frequency. This is discussed in Chapter 23.

Skin Effect

AC currents create a time-varying magnetic field about a conductor, Figure 3–28(a). This varying field in turn induces voltage in the conductor. This voltage is of such a nature that it drives free electrons from the center of the wire to its periphery, Figure 3–28(b), resulting in a nonuniform distribution of current, with current density greatest near the periphery and smallest in the center. This phenomenon is known as the **skin effect.** Because the center of the wire carries little current, its cross-sectional area has effectively been reduced, thus increasing resistance. While skin effect is generally negligible at power line frequencies (except for conductors larger than several hundred thousand circular mils), it is so pronounced at microwave frequencies that the center of a wire carries almost no current. For this reason, hollow conductors are often used at very high frequencies instead of solid wires, as shown in Figure 3–28(c).



(a) Varying magnetic field induces voltage within the conductor.

(b) This voltage drives free electrons toward the periphery, leaving few electrons in the center.

(c) At high frequencies, the effect is so pronounced that hollow conductors may be used.

FIGURE 3–28 Skin effect in ac circuits.

Radiation Resistance

At high frequencies some of the energy supplied to a circuit escapes as radiated energy. For example, a radio transmitter supplies power to an antenna, where it is converted into radio waves and radiated into space. The resistance effect here is known as radiation resistance. This resistance is much higher than simple dc resistance. For example, a TV transmitting antenna may have a resistance of a fraction of an ohm to dc but several hundred ohms effective resistance at its operating frequency.

NOTES...

1. The resistance measured by an ohmmeter is dc resistance.
2. Many of the effects noted here will be treated in detail in your various electronics courses. We will not pursue them further.

3.11 Energy Relationships for ac

Recall that power and energy are related by the equation $p = dw/dt$. Thus, energy can be found by integration as

$$W = \int pdt = \int vidt \tag{3–26}$$

Inductance

For an inductance, $v = Ldi/dt$. Substituting this into Equation 3–26, cancelling dt, and rearranging terms yields

$$W_L = \int\left(L\frac{di}{dt}\right)idt = L\int idi \tag{3–27}$$

Recall from Figure 3–4(b) that energy flows into an inductor during time interval 0 to $T/4$ and is released during time interval $T/4$ to $T/2$. The process then repeats itself. The energy stored (and subsequently released) can thus be found by integrating power from $t = 0$ to $t = T/4$. Current at $t = 0$ is 0 and current at $t = T/4$ is I_m. Using these as our limits of integration, we find (see Note)

$$W_L = L\int_0^{I_m} idi = \frac{1}{2}LI_m^2 = LI^2 \quad \text{(J)} \tag{3–28}$$

where we have used $I = I_m/\sqrt{2}$ to express energy in terms of effective current.

NOTES...

The idea of stored energy has been encountered before—for example, Equation 3–28 appears in Chapter 13 as Equation 13–11 and Equation 3–30 appears in Chapter 10 as Equation 10–22.

Capacitance

For a capacitance, $i = Cdv/dt$. Substituting this into Equation 3–26 yields

$$W_C = \int v\left(C\frac{dv}{dt}\right)dt = C\int v\,dv \qquad \textbf{(3–29)}$$

Consider Figure 3–5(b). Energy stored can be found by integrating power from $T/4$ to $T/2$. The corresponding limits for voltage are 0 to V_m. Thus,

$$W_C = C\int_0^{V_m} v\,dv = \frac{1}{2}CV_m^2 = CV^2 \quad \text{(J)} \qquad \textbf{(3–30)}$$

where we have used $V = V_m/\sqrt{2}$.

3.12 Circuit Analysis Using Computers

The power relationships described in this chapter can be investigated easily using Multisim and PSpice. To illustrate, consider Examples 3–11 and 3–12.

EXAMPLE 3–11

For the circuit of Figure 3–1 (Section 3.1), assume an *RL* load. Typical waveforms are then as shown in Figure 3–2. Since current is used as reference, we will use a current source in our simulation. Additionally, because the analysis is independent of amplitude and frequency, we have arbitrarily chosen 5 A, 100 Hz for this source.

Multisim Solution

Read Note 1, then build the circuit of Figure 3–29(a) on your screen. Ensure that nodes are displayed—see Note 2. Select transient analysis, set *TSTOP* to **0.01**, select the *Output* tab, then add *I(L1)* and *V(1)* to the output display list—see Notes 3 and 4. To create the instantaneous power waveform, click *Add Expression*, double-click *V(1)*, double-click *, double-click *I(L1)*, click *OK*, then *Simulate*. The curves of (b) should appear—see Note 5. Compare the curves to those of Figure 3–2. Using a cursor, measure peak and trough values for the power waveform. Record values (for use in Extended Example 3–11).

NOTES...

1. To select the signal current source, right-click *Place* and ensure that *Signal Source Components* is selected. From its parts bin on your screen, select *Place ac Current Source.*
2. To display nodes, click *Options* then *Sheet Properties.* Under *Net names*, click *Show all.*
3. Node numbers depend on the order in which you wire up your circuit. Fo our discussion, we assume n de numbers as shown in Figure 3 –29(a).
4. If you have legacy software, use $1 instead of V(1), and I1#branch instead of I(L1).
5. If your display is jagged, increase the number of time points.

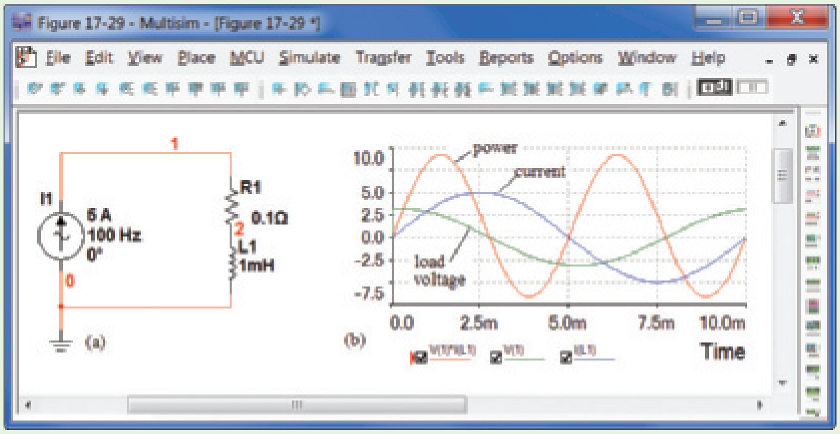


FIGURE 3–29 Investigating instantaneous power using Multisim.

Extended Example 3–11 Add a wattmeter to the circuit as in Figure 3–30. To locate the wattmeter, right-click *Place* and ensure that *Instruments* is selected. Use your mouse pointer to locate the wattmeter in the tool bin, position it on your screen as in Figure 3–30, then wire it in. Click the simulation ON/OFF switch to the ON position, double-click the wattmeter to open it, then read its value.

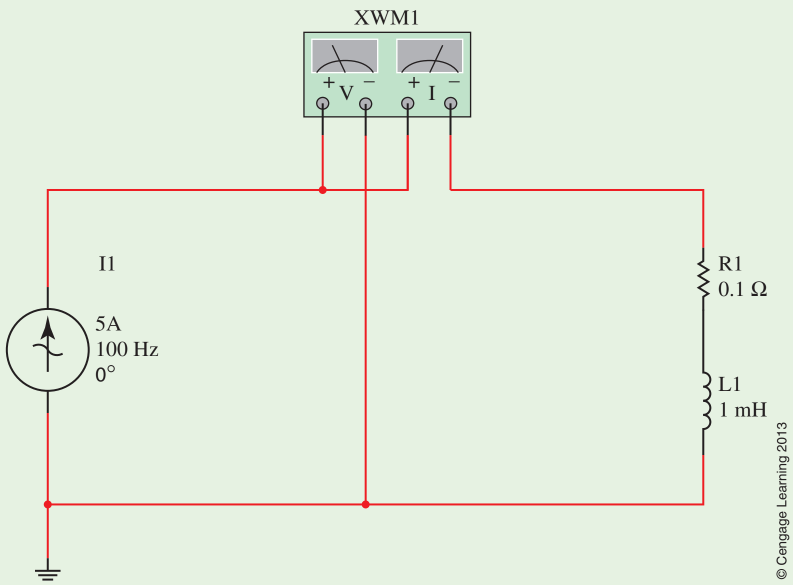


FIGURE 3–30 Verifying average power using a wattmeter.

Reconciliation of Results: First, consider the instantaneous power waveform. In the previous test, you should have measured peak and trough power values as 9.20 W and –6.70 W, respectively. Real power P, the average value of the power wave, is halfway between these two extremes. Thus, $P = (9.20 - 6.70)/2 = 1.25$ W. This agrees with the wattmeter reading. Now consider $P = I^2R$. RMS current is $I = (0.707)I_m = (0.707)(5\text{ A}) = 3.536$ A. This yields $P = (3.536)^2(0.1\Omega) = 1.25$ W. Thus, all results agree with theory.

PSpice Solution

Create the circuit on your screen as in Figure 3–31(a). Use current source *ISIN*—see Note 1. Double-click each small parameter box and enter values as

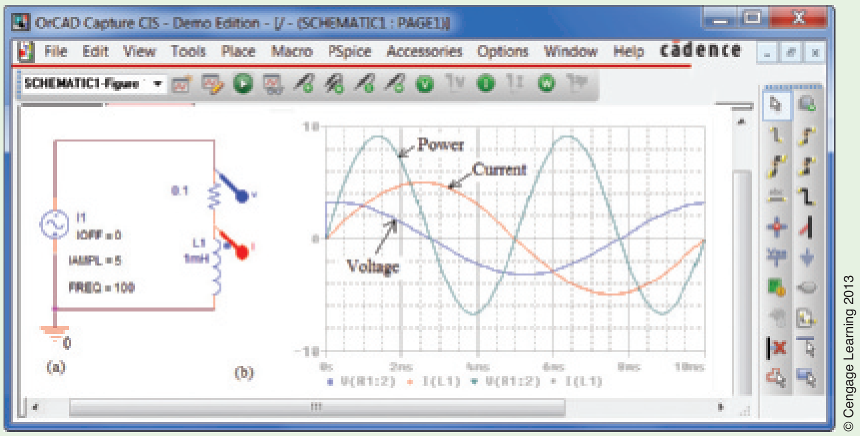


FIGURE 3–31 Investigating instantaneous power using PSpice.

NOTES...

1. PSpice represents current into devices, not out. For this reason, the source in Figure 3–31 must be rotated before positioning so that its end marked + is at the bottom.

2. Your trace names may be different. For purposes of this discussion, we use the names indicated in Figure 3–31. When you run this simulation, substitute your trace names.

shown. Now click the *New Simulation Profile* icon, type a name, choose *Transient,* set *TSTOP* to **10ms,** then click *OK*. Add voltage and current markers as shown, then run the simulation. Voltage and current waveforms as in Figure 3–31(b) appear. Note the legend below the trace—see Note 2. Voltage is identified as *V*(*R1:2*) and current as *I*(*L1*). To plot instantaneous power, click *Trace, Add Trace,* and when the dialog box opens, click *V*(*R1:2*), click the * symbol, click *I*(*L1*), then click *OK*. This creates and plots the power curve *V*(*R1:2*)**I*(*L1*). Compare the resulting power curve to Figure 3–2. Using a cursor, measure power at the peak and trough points of the waveform. (The peak should be about 9.3 W and the trough about −6.72 W.)

Reconciliation of Results: Your peak and trough measurements should have yielded approximately 9.3 W and −6.72 W, respectively. Real power P, the average value of the power wave, is halfway between these two extremes. Thus, $P = (9.3 - 6.72)/2 = 1.23$ W. Now consider $P = I^2R$. RMS current is $I = (0.707)I_m = (0.707)(5\text{ A}) = 3.536$ A. Thus, $P = (3.536)^2(0.1\ \Omega) = 1.25$ W. Agreement is good.

EXAMPLE 3–12

In this example, we investigate power factor correction. Consider Figure 3–20 from Example 3–8. The equivalent resistance of the lamp loads is $R_{Lamp} = V^2/P = (600\text{V})^2/12\,000 = 30\ \Omega$. The electric furnace is represented by a 2.4-Ω resistor in series with an inductance of $L = X_L/\omega = (3.2\ \Omega)/377 = 8.488$ mH. We also need the equivalent circuit for the motor loads. Based on the information given, it can be shown that the equivalent circuit is a resistance of $R_m = 2.88\ \Omega$ in series with an inductance of $L_m = 5.73$ mH. (You can check this out if you like.)

Multisim Solution

Build the circuit on your screen as shown as Figure 3–32. Use the ac power source (from the *Power Source Components* toolbar). To add the ammeter, right-click *Place*, ensure that *Measurement Components* is selected, then locate *Place Ammeter (Horizontal)* and position it on your screen. Double-click the ammeter and set its mode to *AC,* then click the Simulate ON/OFF switch to ON.

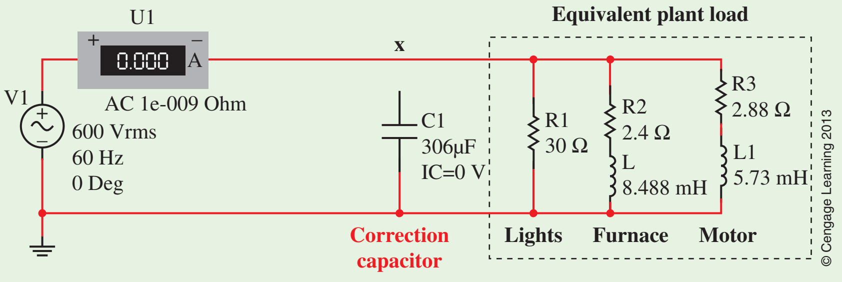


FIGURE 3–32 Investigating power factor correction using Multisim.

Note that source current is 328 A as we computed in Example 3–8, part (c). Now connect the capacitor to point *x* in your circuit and run the simulation again. Note that source current drops to 286 A as computed in Example 3–8.

PSpice

The PSpice circuit is shown as Figure 3–33. (Configure ammeter IPRINT as you did in Chapter 16.) Use power source *VAC*. Set simulation to *AC Sweep/Noise*

Analysis, set *Start Frequency* and *End Frequency* to **60** Hz, select *Linear,* set *Total Points* to **1,** click *OK,* then click the *Run* icon. When the simulation window opens, click *View/Output File,* then scroll until you find the answers indicated in Figure 3–33. As indicated, the magnitude of the current, IM(V_PRINT1), is $3.280 \times 10^2 = 328$ A, the same as we computed in Example 3–8. Now connect a 306-μF capacitor across the load between points *x-x,* then run the simulation again. Note that source current drops to 286 A as predicted in Example 3–8.

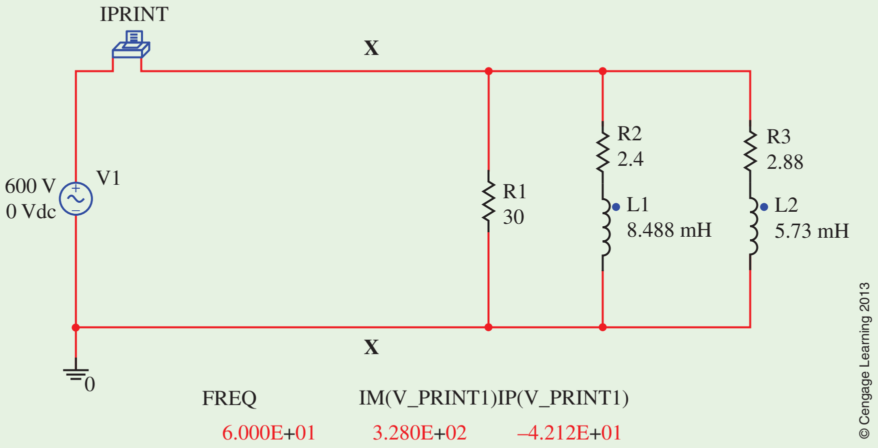


FIGURE 3–33 Investigating power factor correction using PSpice.

3.1–3.5 Problems

1. Note that the power curve of Figure 3–4 is sometimes positive and sometimes negative. What is the significance of this? Between $t = T/4$ and $t = T/2$, what is the direction of power flow?
2. What is real power? What is reactive power? Which power, real or reactive, has an average value of zero?3. A pair of electric heating elements is shown in Figure 3–34.

   a. Determine the active and reactive power to each. b. Determine the active and reactive power delivered by the source.4. For the circuit of Figure 3–35, determine the active and reactive power to the inductor.5. If the inductor of Figure 3–35 is replaced by a 40-μF capacitor and source frequency is 60 Hz, what is Q_C?

6. Find R and X_L for Figure 3–36.
7. For the circuit of Figure 3–37, $f = 100$ Hz. Find
   a. R b. X_C c. C
8. For the circuit of Figure 3–38, $f = 10$ Hz. Find
   a. P b. X_L c. L
9. For Figure 3–39, find X_C.
10. For Figure 3–40, $X_C = 42.5\ \Omega$. Find R, P, and Q.

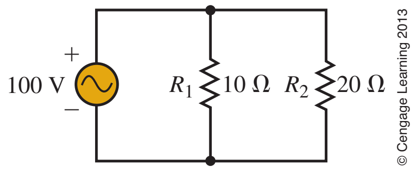


FIGURE 3–34

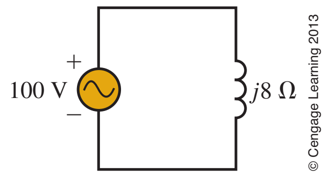


FIGURE 3–35

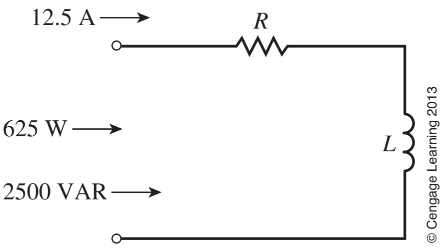


FIGURE 3–36

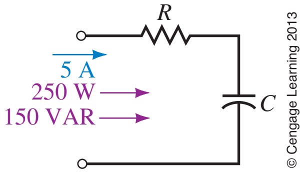


FIGURE 3–37

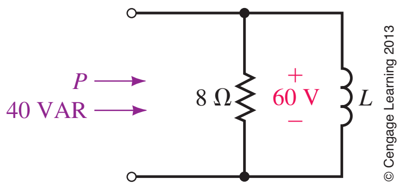


FIGURE 3–38

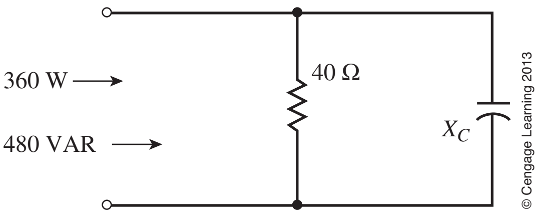


FIGURE 3–39

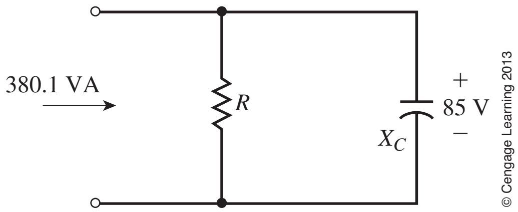


FIGURE 3–40

11. Find the total average power and the total reactive power supplied by the source for Figure 3–41.
12. If the source of Figure 3–41 is reversed, what is P_T and Q_T? What conclusion can you draw from this?

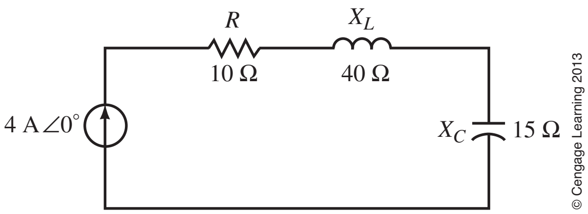


FIGURE 3–41

13. Refer to Figure 3–42. Find P_2 and Q_3. Is the element in Load 3 inductive or capacitive?

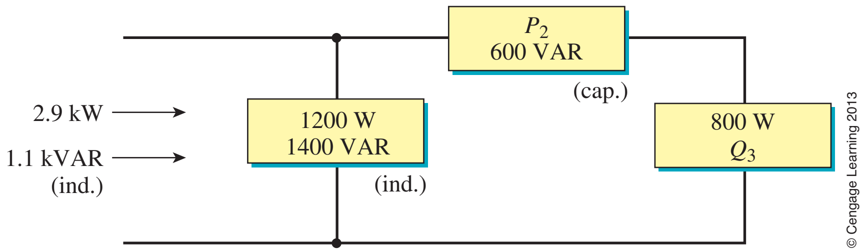


FIGURE 3–42

14. For Figure 3–43, determine P_T and Q_T.

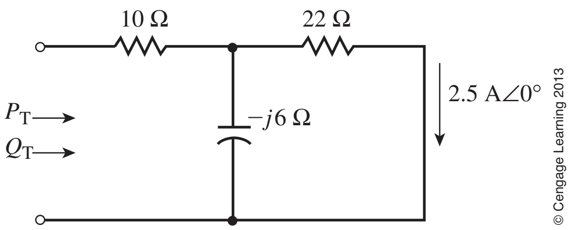


FIGURE 3–43

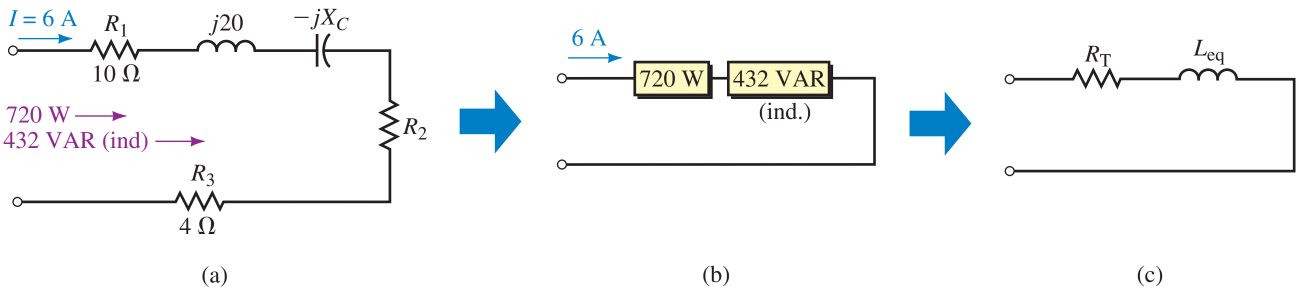


FIGURE 3–44

15. For Figure 17–44, ω = 10 rad/s. Determine
    a. R_T b. R_2 c. X_C d. L_{eq}
16. For Figure 3–45, determine the total P_T and Q_T.

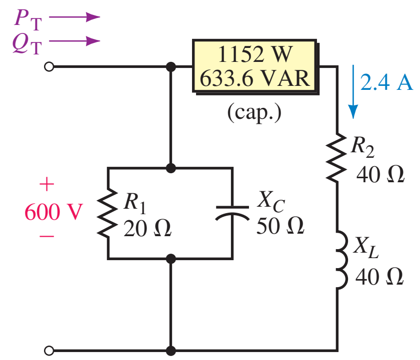


FIGURE 3–45

3.7 The Relationship between *P*, *Q*, and *S*

17. For the circuit of Figure 17–7, draw the power triangle and determine the apparent power.
18. Repeat Problem 17 for Figure 17–8.
19. Ignoring the wattmeter of Figure 17–27, determine the power triangle for the circuit as seen by the source.
20. For the circuit of Figure 17–46, what is the source current?
21. For Figure 3–47, the generator supplies 30 A. What is R?

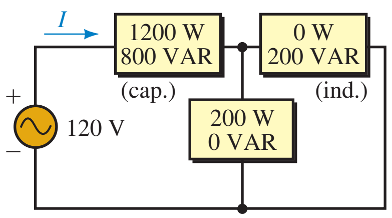


FIGURE 3–46

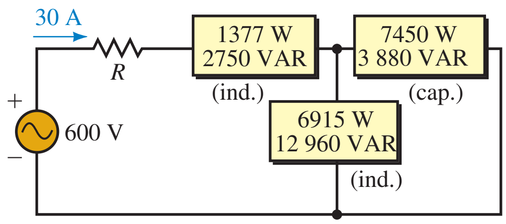


FIGURE 3–47

22. Suppose **V** = 100 V$\angle$60° and **I** = 10A$\angle$40°:
    a. What is θ, the angle between **V** and **I?**
    b.D etermine P from $P = VI \cos \theta$.
    c. Determine Q from $Q = VI \sin \theta$.
    d. Sketch the power triangle and from it, determine **S.**
    e. Show that **S** = **VI*** gives the same answer as (d).
23. For Figure 3–48, S_{gen} = 4835 VA. What is R?
24. Refer to the circuit of Figure 3–16:
    a. Determine the apparent power for each box.
    b.S um the apparent powers that you just computed. Why does the sum not equal S_T = 1803 VA as obtained in Example 3–5?

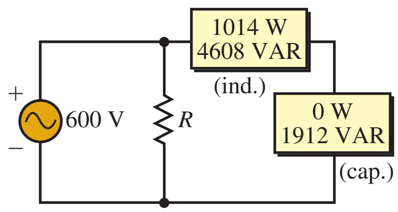


FIGURE 3–48

3.8 Power Factor

25. Refer to the circuit of Figure 17–49:
    a. Determine P_T, Q_T, and S_T.
    b. Determine whether the fuse will blow.
26. A motor with an efficiency of 87% supplies 10 hp to a load (Figure 17–50). Its power factor is 0.65 (lag).
    a. What is the power input to the motor?
    b. What is the reactive power to the motor?
    c. Draw the motor power triangle. What is the apparent power to the motor?

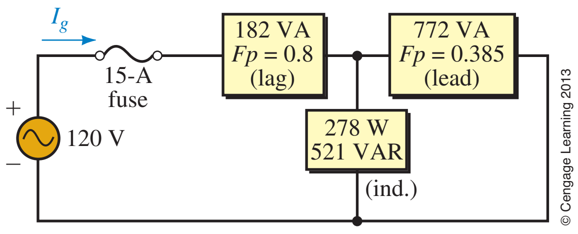


FIGURE 3–49

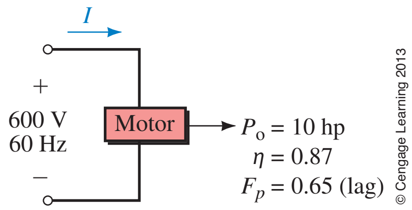


FIGURE 3–50

27. To correct the circuit power factor of Figure 17–50 to unity, a power factor correction capacitor is added.
    a. Show where the capacitor is connected.
    b.D etermine its value in microfarads.
28. Consider Figure 3–20. The motor is replaced with a new unit requiring $\mathbf{S}_m = (120 + j35)$ kVA. Everything else remains the same. Find the following:
    a. P_T b. Q_T c. S_T
    d. Determine how much kVAR capacitive correction is needed to correct to unity F_p.
29. A small electrical utility has a 600-V, 300-kVA capacity. It supplies a factory (Figure 3–51.) with the power triangle shown in Figure 3–51(b). This fully loads the utility. If a power factor correcting capacitor corrects the load to unity power factor, how much more power (at unity power factor) can theutility sell to other customers?

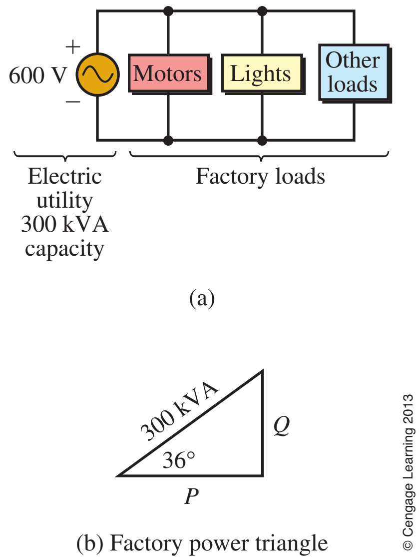


FIGURE 3–51

3.9 ac Power Measurement

30. a. Why does the wattmeter of Figure 3–52 indicate only 1200 watts?
    b.W here would the wattmeter have to be placed to measure power delivered by the source? Sketch the modified circuit.
    c. What would the wattmeter indicate in (b)?
31. Determine the wattmeter reading for Figure 3–53.

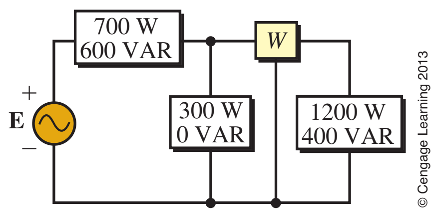


FIGURE 3–52

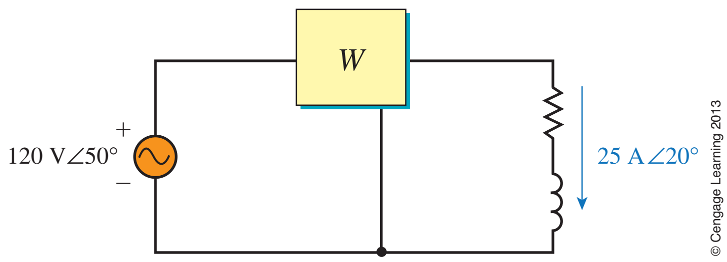


FIGURE 3–53

32. Determine the wattmeter reading for Figure 17–54.

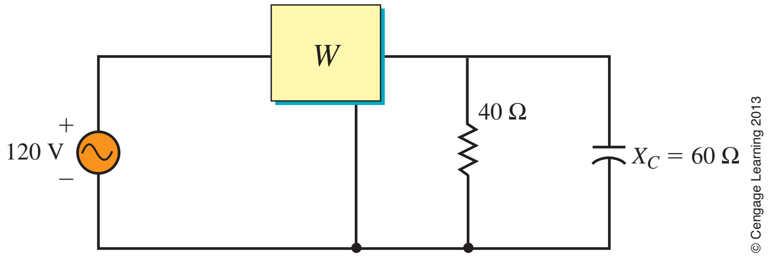


FIGURE 3–54

3.10 Effective Resistance

33. Measurements on an iron-core solenoid coil yield the following values: $V = 80$ V, $I = 400$ mA, $P = 25.6$ W, and $R = 140\ \Omega$. (The last measurement was taken with an ohmmeter.) What is the ac resistance of the solenoid coil?

3.12 Circuit Analysis Using Computers

34. Using Multisim, create the circuit of Figure 17–17 on your screen. (For the source, use the voltage source from Power Components.) Define the parallel *RLC* combination as the "load." Add an ammeter and a wattmeter to measure load current and load power. Using your calculator, compute the resistance of the heater, the inductance of the coil, and the capacitance of the capacitor, then set these values on your Multisim circuit. Run a simulation and measure load current, power, and power factor. Using the measured values, compute Q_T. Using the applied voltage and the measured current, compute apparent power. Compare these values of Q_T and S_T to the values determined in Example 3–6.

Multisim

35. A 10-μF capacitor has voltage $v = 10 \sin(\omega t - 90°)$ V. Given $f = 1000$ Hz, use Multisim or PSpice to investigate the power waveform and compare to Figure 3–5. (PSpice users, use VSIN.)

Multisim PSPICE

36. The voltage waveform of Figure 3–55 is applied to a 200-μF capacitor.

PSPICE

   a. Using your calculator and the principles of Chapter 10, determine the current through the capacitor and sketch. (Also sketch the voltage waveform on your graph.) Multiply the two waveforms to obtain a plot of $p(t)$. Compute power at its max and min points.

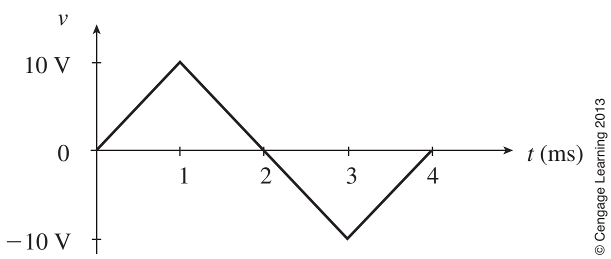


FIGURE 3–55

   b. Use PSpice to verify the results. Use voltage source VPWL. You have to describe the waveform to the source. It has a value of 0 V at $t = 0$, 10 V at $t = 1$ ms, -10 V at $t = 3$ ms, and 0 V at $t = 4$ ms. To set these, double-click the source symbol and enter values via the Property Editor as

NOTES...

PSpice represents current into devices. Thus, when you double-click a current source symbol (ISIN, IPWL, etc.) and specify a current waveform, you are specifying the current *into* the source. You must take this into consideration when you set up your current sources.

follows: **0** for T1, **0V** for V1, **1ms** for T2, **10V** for V2, and so on. Run the simulation and plot voltage, current, and power using the procedure we used to create Figure 3–31. Results should agree with those of part (a).

PSPICE

37. Repeat Question 36 for a current waveform identical to Figure 3–55 except that it oscillates between 2 A and −2 A applied to a 2-mH inductor. Use current source IPWL (see Note).

Multisim

38. Repeat Question 36 except use Multisim. To create the applied voltage waveform, use the Piecewise_Linear_Voltage source found in the Signal_Voltage_Sources components. Double-click it, select *Enter data points in table* and enter values to describe the waveform. Click *Repeat data during simulation,* then click *OK*. Choose transient analysis, set *TSTOP* to 0.008, click the *Output* tab, set items up for display, then click *Simulate*.

ANSWERS TO IN-PROCESS LEARNING CHECKS

IN-PROCESS LEARNING CHECK 1

1. 100 V

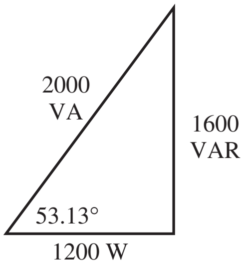


FIGURE 3–56

2. 8.76 A
3. Figure 17–7: 0.6 (lag); Figure 17–8: 0.97 (lead); Figure 3–9: 0.6 (lag)
4. Yes. (S = 128 kVA)

KEY TERMS

Admittance
Admittance Diagram
Capacitive Impedance
Cutoff (or Corner) Frequency
Impedance Diagram
Inductive Impedance
Leading and Lagging Power Factor
Resistive Circuit
Susceptance

OUTLINE

Ohm's Law for ac Circuits
ac Series Circuits
Kirchhoff's Voltage Law and the Voltage Divider Rule
ac Parallel Circuits
Kirchhoff's Current Law and the Current Divider Rule
Series-Parallel Circuits
Frequency Effects
Applications
Circuit Analysis Using Computers

OBJECTIVES

After studying this chapter, you will be able to

- apply Ohm's law to analyze simple series circuits,
- apply the voltage divider rule to determine the voltage across any element in a series circuit,
- apply Kirchhoff's voltage law to verify that the summation of voltages around a closed loop is equal to zero,
- apply Kirchhoff's current law to verify that the summation of currents entering a node is equal to the summation of currents leaving the same node,
- determine unknown voltage, current, and power for any series/parallel circuit,
- determine the series or parallel equivalent of any network consisting of a combination of resistors, inductors, and capacitors.

4

ac SERIES-PARALLEL CIRCUITS

CHAPTER PREVIEW

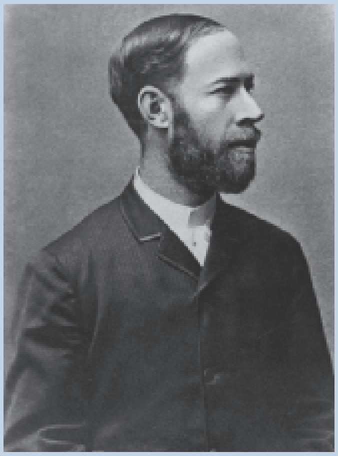
In this chapter we examine how simple circuits containing resistors, inductors, and capacitors behave when subjected to sinusoidal voltages and currents. Principally, we find that the rules and laws which were developed for dc circuits will apply equally well for ac circuits. The major difference between solving dc and ac circuits is that analysis of ac circuits requires using vector algebra.

In order to proceed successfully, it is suggested that the student spend time reviewing the important topics covered in dc analysis. These include Ohm's law, the voltage divider rule, Kirchhoff's voltage law, Kirchhoff's current law, and the current divider rule.

You will also find that a brief review of vector algebra will make your understanding of this chapter more productive. In particular, you should be able to add and subtract any number of vector quantities. ■

Putting It in Perspective

Heinrich Rudolph Hertz



Hulton Archive/Stringer/Hulton Archive/Getty Images

HEINRICH HERTZ WAS BORN IN HAMBURG, Germany, on February 22, 1857. He is known mainly for his research into the transmission of electromagnetic waves.

Hertz began his career as an assistant to Hermann von Helmholtz in the Berlin Institute physics laboratory. In 1885, he was appointed Professor of Physics at Karlsruhe Polytechnic, where he did much to verify James Clerk Maxwell's theories of electromagnetic waves.

In one of his experiments, Hertz discharged an induction coil with a rectangular loop of wire having a very small gap. When the coil discharged, a spark jumped across the gap. He then placed a second, identical coil close to the first, but with no electrical connection. When the spark jumped across the gap of the first coil, a smaller spark was also induced across the second coil. Today, more elaborate antennas use similar principles to transmit radio signals over vast distances. Through further research, Hertz was able to prove that electromagnetic waves have many of the characteristics of light: they have the same speed as light; they travel in straight lines; they can be reflected and refracted; and they can be polarized.

Hertz's experiments ultimately led to the development of radio communication by such electrical engineers as Guglielmo Marconi and Reginald Fessenden.

Heinrich Hertz died at the age of 36 on January 1, 1894. ■

4.1 Ohm's Law for ac Circuits

This section is a brief review of the relationship between voltage and current for resistors, inductors, and capacitors. Unlike in Chapter 16, all phasors are given as rms rather than as peak values. As you saw in Chapter 17, this approach simplifies the calculation of power.

Resistors

In Chapter 16, we saw that when a resistor is subjected to a sinusoidal voltage as shown in Figure 4–1, the resulting current is also sinusoidal and in phase with the voltage.

The sinusoidal voltage $v = V_m\sin(\omega t + \theta)$ may be written in phasor form as $\mathbf{V} = V\angle\theta$. Whereas the sinusoidal expression gives the instantaneous value of voltage for a waveform having an amplitude of V_m (volts peak), the phasor form has a magnitude which is the effective (or rms) value. The relationship between the magnitude of the phasor and the peak of the sinusoidal voltage is given as

$$V = \frac{V_m}{\sqrt{2}}$$

Because the resistance vector may be expressed as $\mathbf{Z}_R = R\angle 0°$, we evaluate the current phasor as follows:

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_R} = \frac{V\angle\theta}{R\angle 0°} = \frac{V}{R}\angle\theta = I\angle\theta$$

NOTES...

Although currents and voltages may be shown in either time domain (as sinusoidal quantities) or in phasor domain (as vectors), resistance and reactance are never shown as sinusoidal quantities. The reason for this is that whereas currents and voltages vary as functions of time, resistance and reactance do not.

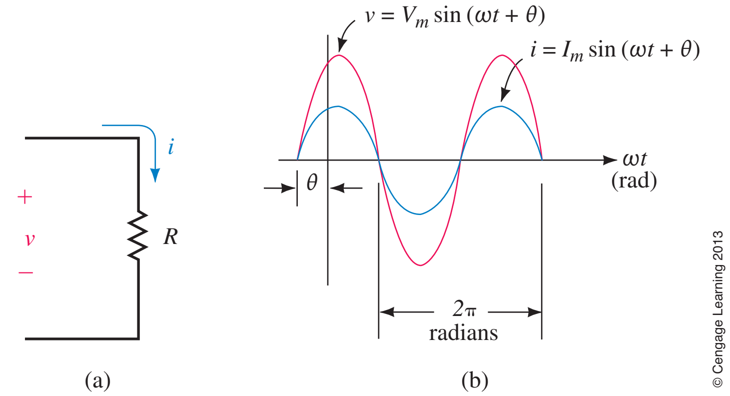


FIGURE 4–1 Sinusoidal voltage and current for a resistor.

If we wish to convert the current from phasor form to its sinusoidal equivalent in the time domain, we would have $i = I_m\sin(\omega t + \theta)$. Again, the relationship between the magnitude of the phasor and the peak value of the sinusoidal equivalent is given as

$$I = \frac{I_m}{\sqrt{2}}$$

The voltage and current phasors may be shown on a phasor diagram as in Figure 4–2.

Because one phasor is a current and the other is a voltage, the relative lengths of these phasors are purely arbitrary. Regardless of the angle θ, we see that the voltage across and the current through a resistor will always be in phase.

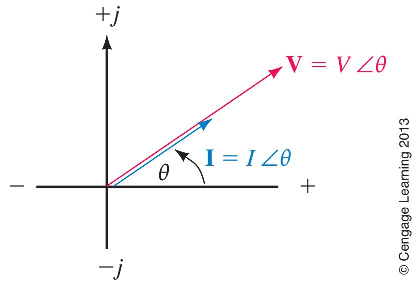


FIGURE 4–2 Voltage and current phasors for a resistor.

EXAMPLE 4–1

Refer to the resistor shown in Figure 4–3:

a. Find the sinusoidal current i using phasors.

b.S ketch the sinusoidal waveforms for v and i.

c. Sketch the phasor diagram of $\mathbf{V}$ and $\mathbf{I}$.

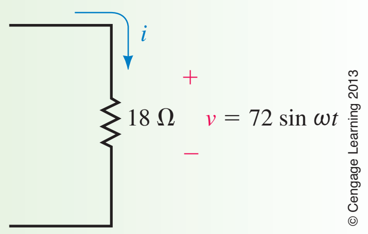


FIGURE 4 –3

Solution

a. The phasor form of the voltage is determined as follows:

$$v = 72 \sin \omega t \Leftrightarrow \mathbf{V} = 50.9 \text{ V}\angle 0°$$

From Ohm's law, the current phasor is determined to be

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_R} = \frac{50.9 \text{ V}\angle 0°}{18\ \Omega\angle 0°} = 2.83 \text{ A}\angle 0°$$

which results in the sinusoidal current waveform having an amplitude of

$$I_m = (\sqrt{2})(2.83 \text{ A}) = 4.0 \text{ A}$$

Therefore, the current i will be written as

$$i = 4 \sin \omega t$$

b. The voltage and current waveforms are shown in Figure 18–4.

c. Figure 18–5 shows the voltage and current phasors.

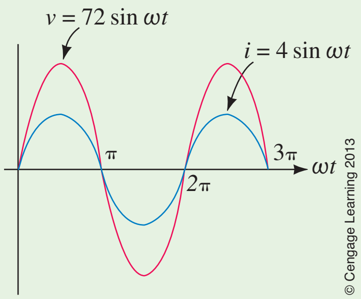


FIGURE 4–4

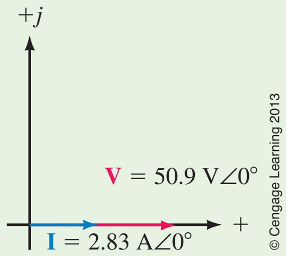


FIGURE 4–5

EXAMPLE 4–2

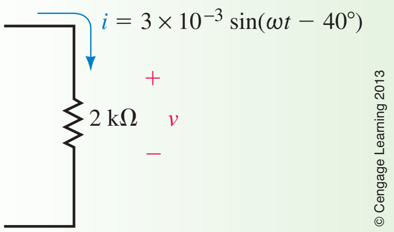


FIGURE 18–6

Refer to the resistor of Figure 4–6:

a. Use phasor algebra to find the sinusoidal voltage, v.

b. Sketch the sinusoidal waveforms for v and i.

c. Sketch a phasor diagram showing **V** and **I**.

Solution

a. The sinusoidal current has a phasor form as follows:

$$i = 3 \times 10^{-3} \sin(\omega t - 40°) \Leftrightarrow \mathbf{I} = 2.12 \text{ mA}\angle{-40°}$$

From Ohm's law, the voltage across the 2-kΩ resistor is determined as the phasor product

$$\begin{aligned}\mathbf{V} &= \mathbf{I}\mathbf{Z}_R \\ &= (2.12 \text{ mA}\angle{-40°})(2 \text{ k}\Omega\angle 0°) \\ &= 4.24 \text{ V}\angle{-40°}\end{aligned}$$

The amplitude of the sinusoidal voltage is

$$V_m = (\sqrt{2})(4.24 \text{ V}) = 6.0 \text{ V}$$

The voltage may now be written as

$$v = 6.0 \sin(\omega t - 40°)$$

b. Figure 4–7 shows the sinusoidal waveforms for v and i.

c. The corresponding phasors for the voltage and current are shown in Figure 4–8.

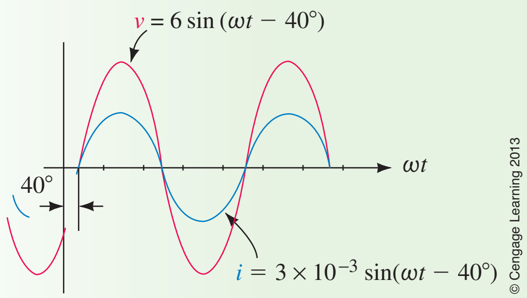


FIGURE 18–7

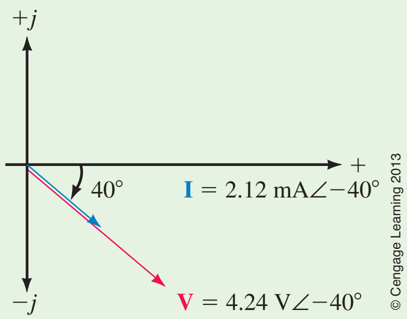


FIGURE 4–8

Inductors

When an inductor is subjected to a sinusoidal current, a sinusoidal voltage is induced across the inductor such that the voltage across the inductor leads the current waveform by exactly 90°. If we know the reactance of an inductor, then from Ohm's law the current in the inductor may be expressed in phasor form as

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_L} = \frac{V\angle\theta}{X_L\angle 90°} = \frac{V}{X_L}\angle(\theta - 90°)$$

In vector form, the reactance of the inductor is given as

$$\mathbf{Z}_L = X_L\angle 90°$$

where $X_L = \omega L = 2\pi f L$.

EXAMPLE 4–3

Consider the inductor shown in Figure 4–9:

a. Determine the sinusoidal expression for the current i using phasors.

b.S ketch the sinusoidal waveforms for v and i.

c. Sketch the phasor diagram showing **V** and **I**.

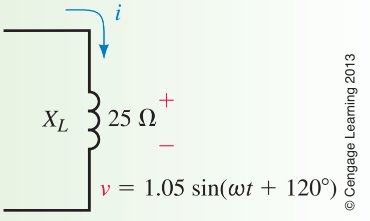


FIGURE 18–9

Solution

a. The phasor form of the voltage is determined as follows:

$$v = 1.05\sin(\omega t + 120°) \Leftrightarrow \mathbf{V} = 0.742\ \text{V}\ \angle 120°$$

From Ohm's law, the current phasor is determined to be

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_L} = \frac{0.742\ \text{V}\angle 120°}{25\ \Omega\angle 90°} = 29.7\ \text{mA}\angle 30°$$

The amplitude of the sinusoidal current is

$$I_m = (\sqrt{2})(29.7\ \text{mA}) = 42\ \text{mA}$$

The current i is now written as

$$i = 0.042\sin(\omega t + 30°)$$

b. Figure 18–10 shows the sinusoidal waveforms of the voltage and current.

c. The voltage and current phasors are shown in Figure 18–11.

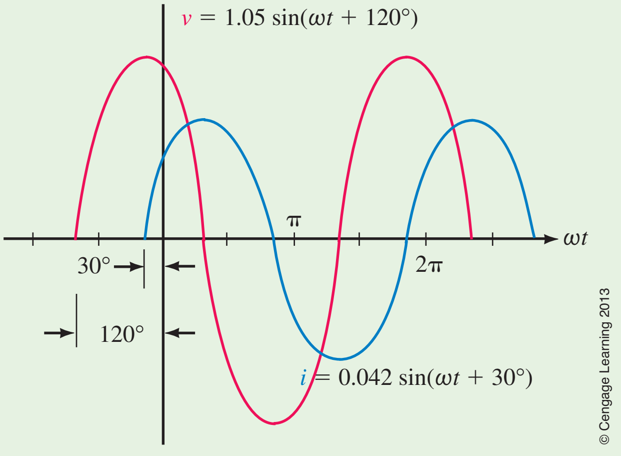


FIGURE 4–10 Sinusoidal voltage and current for an inductor.

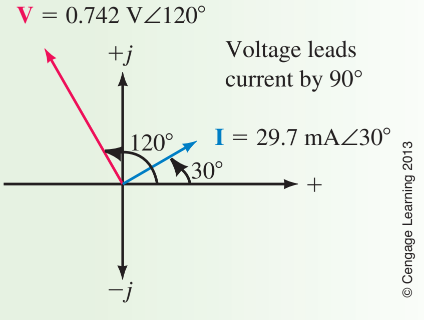


FIGURE 4– 11 Voltage and current phasors for an inductor.

Capacitors

When a capacitor is subjected to a sinusoidal voltage, a sinusoidal current results. The current through the capacitor leads the voltage by exactly 90°. If we know the reactance of a capacitor, then from Ohm's law the current in the capacitor expressed in phasor form is

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_C} = \frac{V\angle\theta}{X_C\angle -90^\circ} = \frac{V}{X_C}\angle(\theta + 90^\circ)$$

In vector form, the reactance of the capacitor is given as

$$\mathbf{Z}_C = X_C\angle -90^\circ$$

where

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

EXAMPLE 4–4

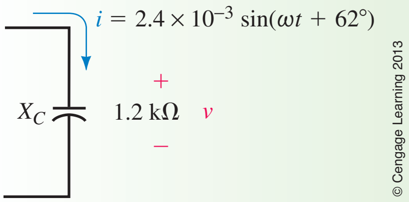


FIGURE 4–12

Consider the capacitor of Figure 4–12.

a. Find the voltage v across the capacitor.

b. Sketch the sinusoidal waveforms for v and i.

c. Sketch the phasor diagram showing $\mathbf{V}$ and $\mathbf{I}$.

Solution

a. Converting the sinusoidal current into its equivalent phasor form gives

$$i = 2.4 \times 10^{-3} \sin(\omega t + 62^\circ) \Leftrightarrow \mathbf{I} = 1.70 \text{ mA}\angle 62^\circ$$

From Ohm's law, the phasor voltage across the capacitor must be

$$\begin{aligned}\mathbf{V} &= \mathbf{I}\mathbf{Z}_C \\ &= (1.70 \text{ mA}\angle 62^\circ)(1.2 \text{ k}\Omega\angle -90^\circ) \\ &= 2.04 \text{ V}\angle -28^\circ\end{aligned}$$

The amplitude of the sinusoidal voltage is

$$V_m = (\sqrt{2})(2.04 \text{ V}) = 2.88 \text{ V}$$

The voltage v is now written as

$$v = 2.88 \sin(\omega t - 28^\circ)$$

b. Figure 4–13 shows the waveforms for v and i.

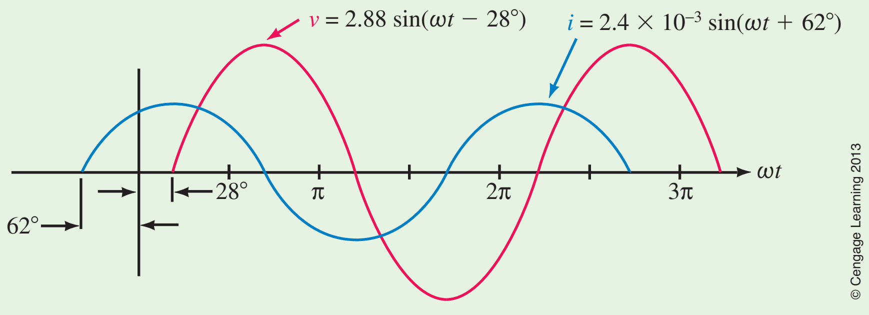


FIGURE 4–13 Sinusoidal voltage and current for a capacitor.

c. The corresponding phasor diagram for **V** and **I** is shown in Figure 4–14.

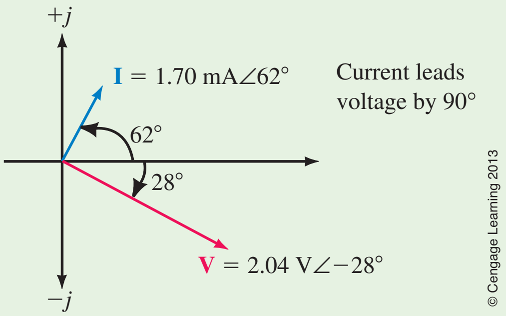


FIGURE 4–14 Voltage and current phasors for a capacitor.

The relationships between voltage and current, as illustrated in the previous three examples, will always hold for resistors, inductors, and capacitors.

IN-PROCESS LEARNING CHECK 1

(Answers are at the end of the chapter.)

1. What is the phase relationship between current and voltage for a resistor?
2. What is the phase relationship between current and voltage for a capacitor?
3. What is the phase relationship between current and voltage for an inductor?

PRACTICE PROBLEMS 1

A voltage source, $\mathbf{E} = 10\ \text{V}\angle 30°$, is applied to an inductive impedance of 50 Ω.

a. Solve for the phasor current, **I**.
b. Sketch the phasor diagram for **E** and **I**.
c. Write the sinusoidal expressions for e and i.
d. Sketch the sinusoidal expressions for e and i.

Answers
a. $\mathbf{I} = 0.2\ \text{A}\angle -60°$
c. $e = 14.1 \sin(\omega t + 30°)$
$i = 0.283 \sin(\omega t - 60°)$

PRACTICE PROBLEMS 2

A voltage source, $\mathbf{E} = 10\ \text{V}\angle 30°$, is applied to a capacitive impedance of 20 Ω.

a. Solve for the phasor current, **I**.
b. Sketch the phasor diagram for **E** and **I**.
c. Write the sinusoidal expressions for e and i.
d. Sketch the sinusoidal expressions for e and i.

Answers
a. $\mathbf{I} = 0.5\ \text{A}\angle 120°$
c. $e = 14.1 \sin(\omega t + 30°)$
$i = 0.707 \sin(\omega t + 120°)$

4.2 ac Series Circuits

When we examined dc circuits we saw that the current everywhere in a series circuit is always constant. The same applies when we have series elements with an ac source. Further, we had seen that the total resistance of a dc series circuit consisting of n resistors was determined as the summation

$$R_{\text{T}} = R_1 + R_2 + \cdots + R_n$$

When working with ac circuits we no longer work with only resistance but also with capacitive and inductive reactance. *Impedance is a term used to collectively determine how the resistance, capacitance, and inductance "impede" the current in a circuit.* The symbol for impedance is the letter Z and the unit is the ohm (Ω). Because impedance may be made up of any combination of resistances and reactances, it is written as a vector quantity $\mathbf{Z}$, where

$$\mathbf{Z} = Z\angle\theta \quad (\Omega)$$

Each impedance may be represented as a vector on the complex plane, such that the length of the vector is representative of the magnitude of the impedance. The diagram showing one or more impedances is referred to as an **impedance diagram**.

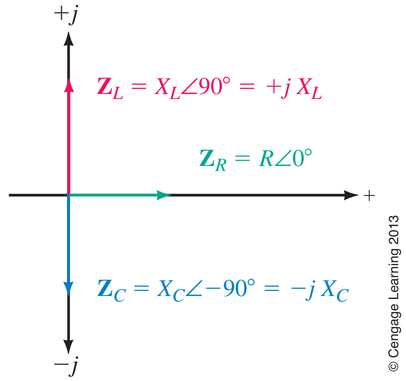


FIGURE 4–15

Resistive impedance $\mathbf{Z}_R$ is a vector having a magnitude of R along the positive real axis. Purely inductive impedance $\mathbf{Z}_L$ is a vector having a magnitude of X_L along the positive imaginary axis, while the purely capacitive impedance $\mathbf{Z}_C$ is a vector having a magnitude of X_C along the negative imaginary axis. Mathematically, each of the vector impedances is written as follows:

$$\mathbf{Z}_R = R\angle 0° = R + j0 = R$$
$$\mathbf{Z}_L = X_L\angle 90° = 0 + jX_L = jX_L$$
$$\mathbf{Z}_C = X_C\angle 90° = 0 - jX_C = -jX_C$$

An impedance diagram showing each of the preceding impedances is shown in Figure 4–15.

All impedance vectors will appear in either the first or the fourth quadrant, since the resistive impedance vector is always positive.

For a series ac circuit consisting of n impedances, as shown in Figure 4–16, the total impedance of the circuit is found as the vector sum

$$\mathbf{Z}_{\text{T}} = \mathbf{Z}_1 + \mathbf{Z}_2 + \cdots + \mathbf{Z}_n \qquad \textbf{(4–1)}$$

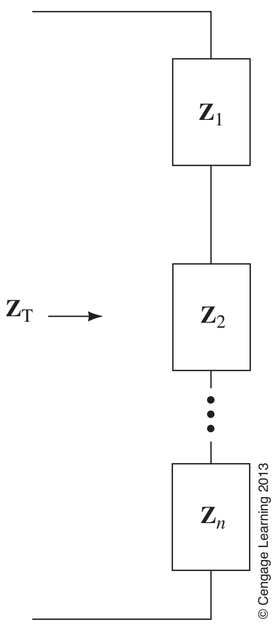


FIGURE 4–16

Consider the branch of Figure 4–17.
By applying Equation 4–1, we may determine the total impedance of the circuit as

$$\mathbf{Z}_{\text{T}} = (3\ \Omega + j0) + (0 + j4\ \Omega) = 3\ \Omega + j4\ \Omega$$
$$= 5\ \Omega\angle 53.13°$$

The preceding quantities are shown on an impedance diagram as in Figure 4–4.

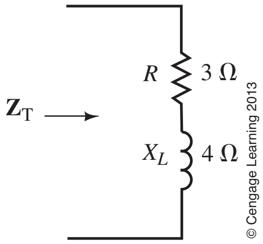


FIGURE 4–17

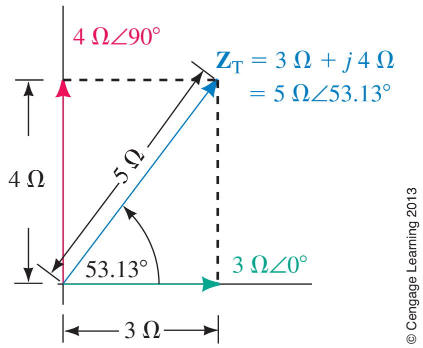


FIGURE 4–4

From Figure 4–4 we see that the total impedance of the series elements consists of a real component and an imaginary component. The corresponding total impedance vector may be written in either polar or rectangular form.

The rectangular form of an impedance is written as

$$\mathbf{Z} = R \pm jX$$

If we are given the polar form of the impedance, then we may determine the equivalent rectangular expression from

$$R = Z\cos\theta \tag{4–2}$$

and

$$X = Z\sin\theta \tag{4–3}$$

In the rectangular representation for impedance, the resistance term, R, is the total of all resistance looking into the network. The reactance term, X, is the difference between the total capacitive and inductive reactances. The sign for the imaginary term will be positive if the inductive reactance is greater than the capacitive reactance. In such a case, the impedance vector will appear in the first quadrant of the impedance diagram and is referred to as being an **inductive impedance.** If the capacitive reactance is larger, then the sign for the imaginary term will be negative. In such a case, the impedance vector will appear in the fourth quadrant of the impedance diagram, and the impedance is said to be **capacitive**.

The polar form of any impedance will be written in the form

$$\mathbf{Z} = Z\angle\theta$$

The value Z is the magnitude (in ohms) of the impedance vector $\mathbf{Z}$ and is determined as follows:

$$Z = \sqrt{R^2 + X^2}\quad (\Omega) \tag{4–4}$$

The corresponding angle of the impedance vector is determined as

$$\theta = \pm\tan^{-1}\left(\frac{X}{R}\right) \tag{4–5}$$

Whenever a capacitor and an inductor having equal reactances are placed in series, as shown in Figure 4–19, the equivalent circuit of the two components is a short circuit since the inductive reactance will be exactly balanced by the capacitive reactance.

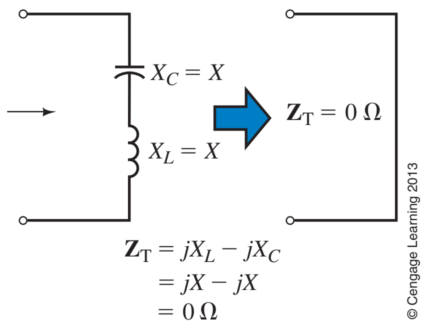


FIGURE 4–19

Any ac circuit having a total impedance with only a real component is referred to as a **resistive circuit.** In such a case, the impedance vector $\mathbf{Z}_T$ will be located along the positive real axis of the impedance diagram and the angle of the vector will be 0°. The condition under which series reactances are equal is referred to as "series resonance" and is examined in greater detail in Chapter 21.

If the impedance $\mathbf{Z}$ is written in polar form, then the angle θ will be positive for an inductive impedance and negative for a capacitive impedance. In the event that the circuit is purely reactive, the resulting angle θ will be either $+90°$ (inductive) or $-90°$ (capacitive). If we reexamine the impedance diagram of Figure 4–4, we conclude that the original circuit is inductive.

EXAMPLE 4–5

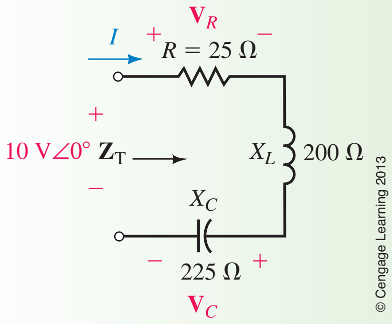


FIGURE 18–20

Consider the network of Figure 4–20.

a. Find $\mathbf{Z}_T$.
b. Sketch the impedance diagram for the network and indicate whether the total impedance of the circuit is inductive, capacitive, or resistive.
c. Use Ohm's law to determine $\mathbf{I}$, $\mathbf{V}_R$, and $\mathbf{V}_C$.

Solution

a. The total impedance is the vector sum

$$\begin{aligned}\mathbf{Z}_T &= 25\ \Omega + j200\ \Omega + (-j225\ \Omega)\\ &= 25\ \Omega - j25\ \Omega\\ &= 35.36\ \Omega\angle -45°\end{aligned}$$

b. The corresponding impedance diagram is shown in Figure 4–21.

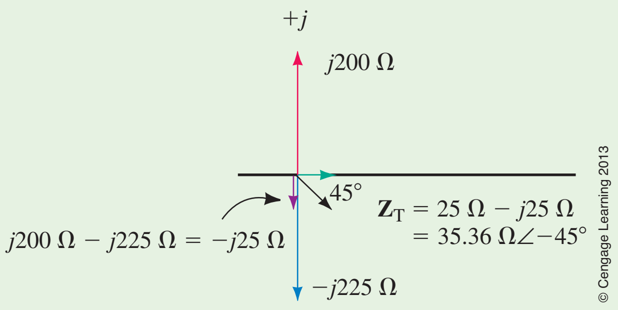


FIGURE 4–21

Because the total impedance has a negative reactance term ($-j25\ \Omega$), $\mathbf{Z}_T$ is capacitive.

c.
$$\mathbf{I} = \frac{10\ \text{V}\angle 0°}{35.36\ \Omega\angle -45°} = 0.283\ \text{A}\angle 45°$$
$$\mathbf{V}_R = (282.8\ \text{mA}\angle 45°)(25\ \Omega\angle 0°) = 7.07\ \text{V}\angle 45°$$
$$\mathbf{V}_C = (282.8\ \text{mA}\angle 45°)(225\ \Omega\angle -90°) = 63.6\ \text{V}\angle -45°$$

Notice that the magnitude of the voltage across the capacitor is many times larger than the source voltage applied to the circuit. This example illustrates that the voltages across reactive elements must be calculated to ensure that maximum ratings for the components are not exceeded.

EXAMPLE 4–6

Determine the impedance **Z** which must be within the indicated block of Figure 4–22 if the total impedance of the network is 13 $\Omega\angle 22.62°$.

Solution Converting the total impedance from polar to rectangular form, we get

$$\mathbf{Z}_\text{T} = 13\ \Omega\angle 22.62° \Leftrightarrow 12\ \Omega + j5\ \Omega$$

Now, we know that the total impedance is determined from the summation of the individual impedance vectors, namely

$$\mathbf{Z}_\text{T} = 2\ \Omega + j10\ \Omega + \mathbf{Z} = 12\ \Omega + j5\ \Omega$$

Therefore, the impedance **Z** is found as

$$\begin{aligned}\mathbf{Z} &= 12\ \Omega + j5\ \Omega - (2\ \Omega + j10\ \Omega)\\ &= 10\ \Omega - j5\ \Omega\\ &= 11.4\ \Omega\angle -26.57°\end{aligned}$$

In its most simple form, the impedance **Z** will consist of a series combination of a 10-Ω resistor and a capacitor having a reactance of 5 Ω. Figure 4–23 shows the elements that may be contained within **Z** to satisfy the given conditions.

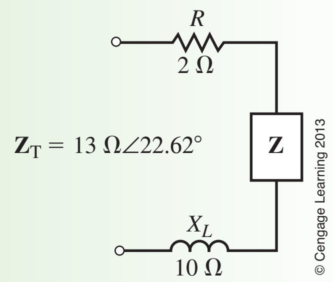


FIGURE 18 22

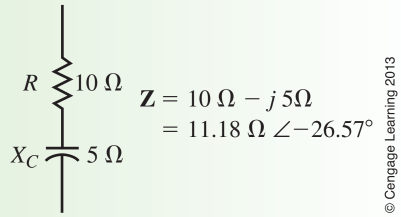


FIGURE 4 –23

EXAMPLE 4–7

Find the total impedance for the network of Figure 4–24. Sketch the impedance diagram showing $\mathbf{Z}_1$, $\mathbf{Z}_2$, and $\mathbf{Z}_\text{T}$.

Solution

$$\begin{aligned}\mathbf{Z}_\text{T} &= \mathbf{Z}_1 + \mathbf{Z}_2\\ &= (50\ \Omega + j90\ \Omega) + (30\ \Omega - j30\ \Omega)\\ &= (80\ \Omega + j60\ \Omega) = 100\ \Omega\angle 36.87°\end{aligned}$$

The polar forms of the vectors $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are as follows:

$$\mathbf{Z}_1 = 50\ \Omega + j90\ \Omega = 102.96\ \Omega\angle 60.95°$$

$$\mathbf{Z}_2 = 30\ \Omega - j30\ \Omega = 42.43\ \Omega\angle -45°$$

The resulting impedance diagram is shown in Figure 4–25.

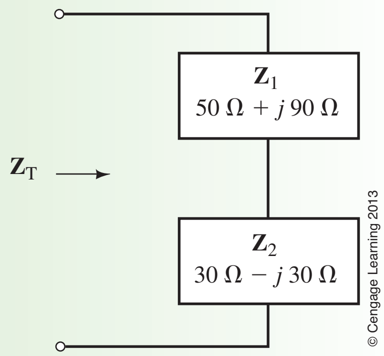


FIGURE 4 –24

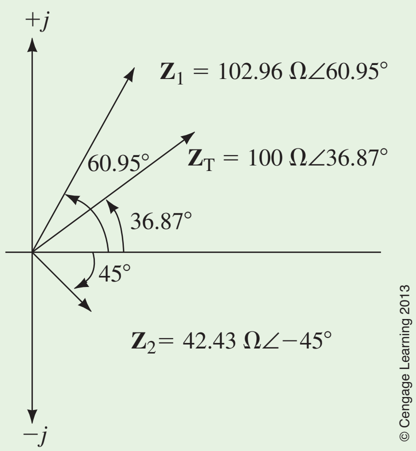


FIGURE 4–25

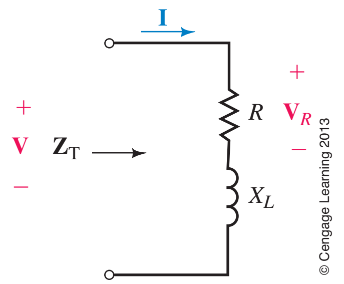


FIGURE 4–26

The phase angle θ for the impedance vector $\mathbf{Z} = Z\angle\theta$ provides the phase angle between the voltage **V** across **Z** and the current **I** through the impedance. For an inductive impedance the voltage will lead the current by θ. If the impedance is capacitive, then the voltage will lag the current by an amount equal to the magnitude of θ.

The phase angle θ is also useful for determining the average power dissipated by the circuit. In the simple series circuit shown in Figure 4–26, we know that only the resistor will dissipate power.

The average power dissipated by the resistor may be determined as follows:

$$P = V_R I = \frac{V_R^2}{R} = I^2 R \tag{4–6}$$

Notice that Equation 4–6 uses only the magnitudes of the voltage, current, and impedance vectors. *Power is never determined by using phasor products.*

Ohm's law provides the magnitude of the current phasor as

$$I = \frac{V}{Z}$$

Substituting this expression into Equation 4–6, we obtain the expression for power as

$$P = \frac{V^2}{Z^2}R = \frac{V^2}{Z}\left(\frac{R}{Z}\right) \tag{4–7}$$

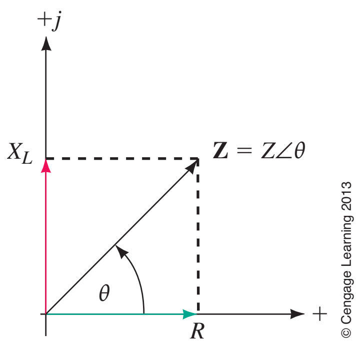


FIGURE 4–27

From the impedance diagram of Figure 4–27, we see that

$$\cos\theta = \frac{R}{Z}$$

The previous chapter had defined the power factor as $F_p = \cos\theta$, where θ is the angle between the voltage and current phasors. We now see that for a series circuit, the power factor of the circuit can be determined from the magnitudes of resistance and total impedance.

$$F_p = \cos\theta = \frac{R}{Z} \tag{4–8}$$

The power factor, F_p, is said to be **leading** if the current leads the voltage (capacitive circuit) and **lagging** if the current lags the voltage (inductive circuit).

Now substituting the expression for the power factor into Equation 4–7, we express power delivered to the circuit as

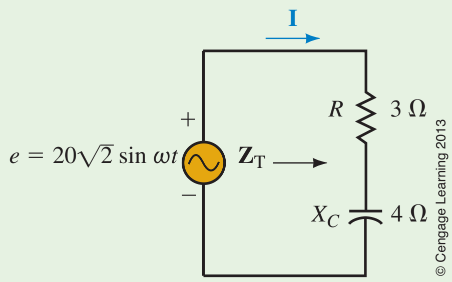
$$P = VI\cos\theta$$

Since $V = IZ$, power may be expressed as

$$P = VI\cos\theta = I^2 Z\cos\theta = \frac{V^2}{Z}\cos\theta \tag{4–9}$$

EXAMPLE 4–8

Refer to the circuit of Figure 4–28.



CircuitSim 18-1

CircuitSim 18-1A

FIGURE 4–28

a. Find the impedance $\mathbf{Z}_T$.
b. Calculate the power factor of the circuit.
c. Determine **I**.
d. Sketch the phasor diagram for **E** and **I**.
e. Find the average power delivered to the circuit by the voltage source.
f. Calculate the average power dissipated by both the resistor and the capacitor.

Solution

a. $\mathbf{Z}_T = 3\ \Omega - j4\ \Omega = 5\ \Omega\angle -53.13°$
b. $F_p = \cos\theta = 3\ \Omega/5\ \Omega = 0.6$ (leading)
c. The phasor form of the applied voltage is

$$\mathbf{E} = \frac{(\sqrt{2})(20\text{ V})}{\sqrt{2}}\angle 0° = 20\text{ V}\angle 0°$$

which gives a current of

$$\mathbf{I} = \frac{20\text{ V}\angle 0°}{5\ \Omega\angle -53.13°} = 4.0\text{ A}\angle 53.13°$$

d. The phasor diagram is shown in Figure 18–29.

From this phasor diagram, we see that the current phasor for the capacitive circuit leads the voltage phasor by 53.13°.

e. The average power delivered to the circuit by the voltage source is

$$P = (20\text{ V})(4\text{ A})\cos 53.13° = 48.0\text{ W}$$

f. The average power dissipated by the resistor and capacitor will be

$$P_R = (4\text{ A})^2(3\ \Omega)\cos 0° = 48\text{ W}$$
$$P_C = (4\text{ A})^2(4\ \Omega)\cos 90° = 0\text{ W} \quad \text{(as expected!)}$$

Notice that the power factor used in determining the power dissipated by each of the elements is the power factor for that element and not the total power factor for the circuit.

As expected, the summation of powers dissipated by the resistor and capacitor is equal to the total power delivered by the voltage source.

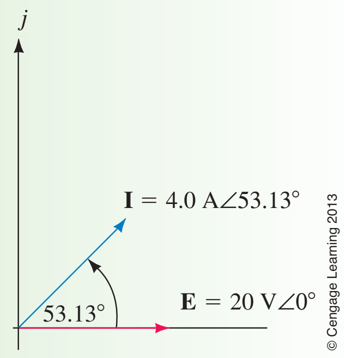


FIGURE 4 –29

PRACTICE PROBLEMS 3

A circuit consists of a voltage source $\mathbf{E} = 50\text{ V}\angle 25°$ in series with $L = 20$ mH, $C = 50\ \mu$F, and $R = 25\ \Omega$. The circuit operates at an angular frequency of 2 krad/s.

a. Determine the current phasor, **I**.

b. Solve for the power factor of the circuit.

c. Calculate the average power dissipated by the circuit and verify that this is equal to the average power delivered by the source.

d. Use Ohm's law to find $\mathbf{V}_R$, $\mathbf{V}_L$, and $\mathbf{V}_C$.

Answers

a. $\mathbf{I} = 1.28\text{ A}\angle -25.19°$

b. $F_p = 0.6402$ lagging

c. $P = 41.0$ W

d. $\mathbf{V}_R = 32.0\text{ V}\angle -25.19°$
$\mathbf{V}_C = 12.8\text{ V}\angle -115.19°$
$\mathbf{V}_L = 51.2\text{ V}\angle 64.81°$

4.3 Kirchhoff's Voltage Law and the Voltage Divider Rule

When a voltage is applied to impedances in series, as shown in Figure 4–30, Ohm's law may be used to determine the voltage across any impedance as

$$\mathbf{V}_x = \mathbf{I}\mathbf{Z}_x$$

The current in the circuit is

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_\mathrm{T}}$$

Now, by substitution we arrive at the voltage divider rule for any series combination of elements as

$$\mathbf{V}_x = \frac{\mathbf{Z}_x}{\mathbf{Z}_\mathrm{T}}\mathbf{E} \qquad \textbf{(4–10)}$$

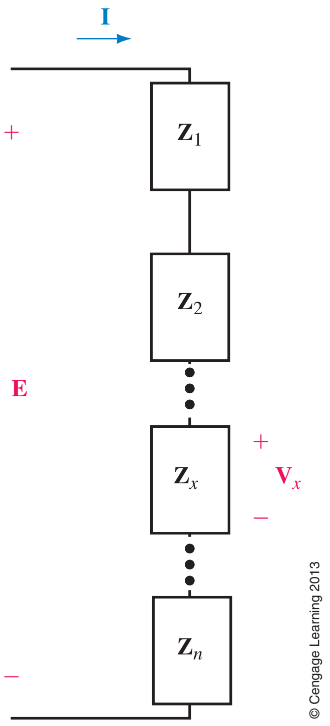


FIGURE 4–30

Equation 4–10 is very similar to the equation for the voltage divider rule in dc circuits. The fundamental differences in solving ac circuits are that we use impedances rather than resistances and that the voltages found are phasors. Because the voltage divider rule involves solving products and quotients of phasors, we generally use the polar form rather than the rectangular form of phasors.

Kirchhoff's voltage law must apply for all circuits whether they are dc or ac circuits. However, because ac circuits have voltages expressed in either sinusoidal or phasor form, Kirchhoff's voltage law for ac circuits may be stated as follows:

The phasor sum of voltage drops and voltage rises around a closed loop is equal to zero.

When adding phasor voltages, we find that the summation is generally done more easily in rectangular form than in the polar form.

EXAMPLE 4–9

Consider the circuit of Figure 4–31.

a. Find $\mathbf{Z}_T$.

b.D etermine the voltages $\mathbf{V}_R$ and $\mathbf{V}_L$ using the voltage divider rule.

c. Verify Kirchhoff's voltage law around the closed loop.

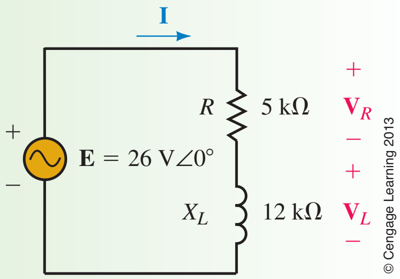


FIGURE 4 –31

Solution

a. $\mathbf{Z}_T = 5\text{ k}\Omega + j12\text{ k}\Omega = 13\text{ k}\Omega\angle 67.38°$

b. $$\mathbf{V}_R = \left(\frac{5\text{ k}\Omega\angle 0°}{13\text{ k}\Omega\angle 67.38°}\right)(26\text{ V}\angle 0°) = 10\text{ V}\angle -67.38°$$

$$\mathbf{V}_L = \left(\frac{12\text{ k}\Omega\angle 90°}{13\text{ k}\Omega\angle 67.38°}\right)(26\text{ V}\angle 0°) = 24\text{ V}\angle 22.62°$$

c. Kirchhoff's voltage law around the closed loop will give

$$26\text{ V}\angle 0° - 10\text{ V}\angle -67.38° - 24\text{ V}\angle 22.62° = 0$$
$$(26 + j0) - (3.846 - j9.231) - (22.154 + j9.231) = 0$$
$$(26 - 3.846 - 22.154) + j(0 + 9.231 - 9.231) = 0$$
$$0 + j0 = 0$$

EXAMPLE 4–10

Consider the circuit of Figure 4–32:

a. Calculate the sinusoidal voltages v_1 and v_2 using phasors and the voltage divider rule.

b.S ketch the phasor diagram showing $\mathbf{E}$, $\mathbf{V}_1$, and $\mathbf{V}_2$.

c. Sketch the sinusoidal waveforms of e, v_1, and v_2.

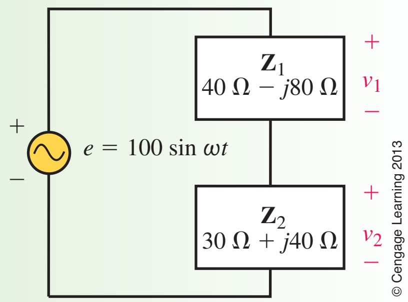


FIGURE 4 –32

Solution

a. The phasor form of the voltage source is determined as

$$e = 100 \sin \omega t \Leftrightarrow \mathbf{E} = 70.71\ \angle\text{V } 0°$$

Applying VDR, we get

$$\mathbf{V}_1 = \left(\frac{40\ \Omega - j80\ \Omega}{(40\ \Omega - j80\ \Omega) + (30\ \Omega + j40\ \Omega)}\right)(70.71\text{ V}\angle 0°)$$
$$= \left(\frac{89.44\ \Omega\angle -63.43°}{80.62\ \Omega\angle -29.74°}\right)(70.71\text{ V}\angle 0°)$$
$$= 78.4\text{ V}\angle -33.69°$$

and

$$\mathbf{V}_2 = \left(\frac{30\ \Omega + j40\ \Omega}{(40\ \Omega - j80\ \Omega) + (30\ \Omega + j40\ \Omega)}\right)(70.71\text{ V}\angle 0°)$$
$$= \left(\frac{50.00\ \Omega\angle 53.13°}{80.62\ \Omega\angle -29.74°}\right)(70.71\text{ V}\angle 0°)$$
$$= 43.9\text{ V}\angle 82.87°$$

The sinusoidal voltages are determined to be

$$v_1 = (\sqrt{2})(78.4)\sin(\omega t - 33.69°)$$
$$= 111 \sin(\omega t - 33.69°)$$

and

$$v_2 = (\sqrt{2})(43.9)\sin(\omega t + 82.87°)$$
$$= 62.0 \sin(\omega t + 82.87°)$$

b. The phasor diagram is shown in Figure 18–33.

c. The corresponding sinusoidal voltages are shown in Figure 18–34.

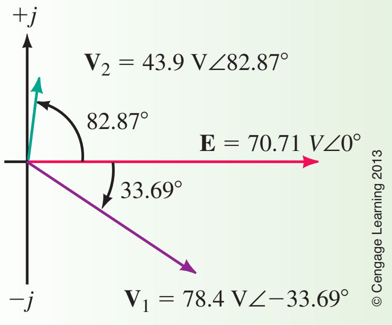


FIGURE 18–33

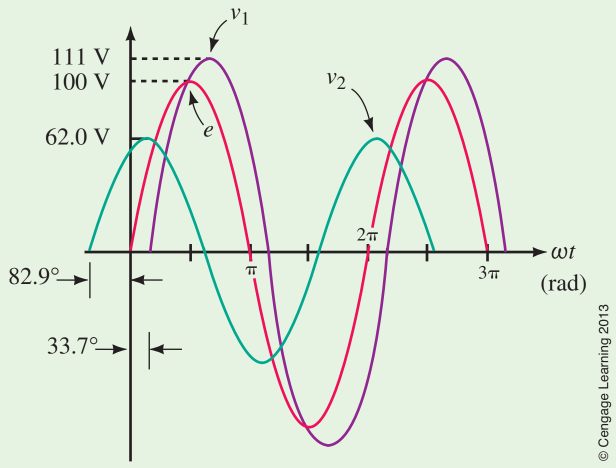


FIGURE 4–34

IN-PROCESS LEARNING CHECK 2

(Answers are at the end of the chapter.)

1. Express Kirchhoff's voltage law as it applies to ac circuits.
2. What is the fundamental difference between how Kirchhoff's voltage law is used in ac circuits and how it is used in dc circuits?

PRACTICE PROBLEMS 4

A circuit consists of a voltage source $\mathbf{E} = 50\text{ V}\angle 25°$ in series with $L = 20$ mH, $C = 50\ \mu$F, and $R = 25\ \Omega$. The circuit operates at an angular frequency of 2 krad/s.

a. Use the voltage divider rule to determine the voltage across each element in the circuit.

b. Verify that Kirchhoff's voltage law applies for the circuit.

Answers

a. $\mathbf{V}_L = 51.2\text{ V}\angle 64.81°$, $\mathbf{V}_C = 12.8\text{ V}\angle -115.19°$

$\mathbf{V}_R = 32.0\text{ V}\angle -25.19°$

b. $51.2\text{ V}\angle 64.81° + 12.8\text{ V}\angle -115.19° + 32.0\text{ V}\angle -25.19° = 50\text{ V}\angle 25°$

4.4 ac Parallel Circuits

The **admittance Y** of any impedance is defined as a vector quantity which is the reciprocal of the impedance **Z**.

Mathematically, admittance is expressed as

$$\mathbf{Y}_{\mathrm{T}} = \frac{1}{\mathbf{Z}_{\mathrm{T}}} = \frac{1}{Z_{\mathrm{T}}\angle\theta} = \left(\frac{1}{Z_{\mathrm{T}}}\right)\angle-\theta = Y_{\mathrm{T}}\angle-\theta \quad \text{(S)} \qquad \textbf{(4–11)}$$

where the unit of admittance is the siemens (S).

In particular, we have seen that the admittance of a resistor R is called conductance and is given the symbol $\mathbf{Y}_R$. If we consider resistance as a vector quantity, then the corresponding vector form of the conductance is

$$\mathbf{Y}_R = \frac{1}{R\angle 0°} = \frac{1}{R}\angle 0° = G\angle 0° = G + j0 \quad \text{(S)} \qquad \textbf{(4–12)}$$

If we determine the admittance of a purely reactive component X, the resultant admittance is called the **susceptance** of the component and is assigned the symbol B. The unit for susceptance is siemens (S). In order to distinguish between inductive susceptance and capacitive susceptance, we use the subscripts L and C, respectively. The vector forms of reactive admittance are given as follows:

$$\mathbf{Y}_L = \frac{1}{X_L\angle 90°} = \frac{1}{X_L}\angle-90° = B_L\angle-90° = 0 - jB_L \quad \text{(S)} \qquad \textbf{(4–13)}$$

$$\mathbf{Y}_C = \frac{1}{X_C\angle-90°} = \frac{1}{X_C}\angle 90° = B_C\angle 90° = 0 + jB_C \quad \text{(S)} \qquad \textbf{(4–14)}$$

In a manner similar to impedances, admittances may be represented on the complex plane in an **admittance diagram** as shown in Figure 4–35.

The lengths of the various vectors are proportional to the magnitudes of the corresponding admittances. The resistive admittance vector **G** is shown on the positive real axis, whereas the inductive and capacitive admittance vectors $\mathbf{Y}_L$ and $\mathbf{Y}_C$ are shown on the negative and positive imaginary axes, respectively.

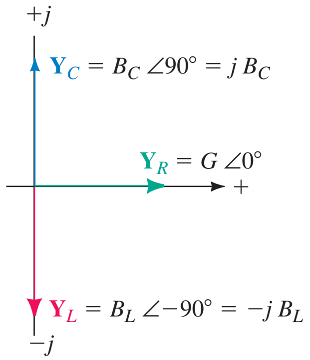


FIGURE 4–35 Admittance diagram showing conductance ($\mathbf{Y}_R$) and susceptance ($\mathbf{Y}_L$ and $\mathbf{Y}_C$).

EXAMPLE 4–11

Determine the admittances of the following impedances. Sketch the corresponding admittance diagram.

a. $R = 10\ \Omega$
b. $X_L = 20\ \Omega$
c. $X_C = 40\ \Omega$

Solution

a. $\mathbf{Y}_R = \dfrac{1}{R} = \dfrac{1}{10\ \Omega\angle 0°} = 100\ \text{mS}\angle 0°$

b. $\mathbf{Y}_L = \dfrac{1}{X_L} = \dfrac{1}{20\ \Omega\angle 90°} = 50\ \text{mS}\angle-90°$

c. $\mathbf{Y}_C = \dfrac{1}{X_C} = \dfrac{1}{40\ \Omega\angle-90°} = 25\ \text{mS}\angle 90°$

The admittance diagram is shown in Figure 4–36.

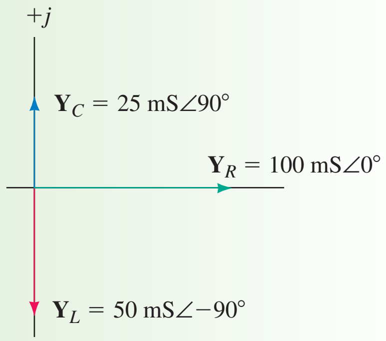


FIGURE 4 –36

For any network of n admittances as shown in Figure 4–37, the total admittance is the vector sum of the admittances of the network. Mathematically, the total admittance of a network is given as

$$\mathbf{Y}_T = \mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_n \quad \text{(S)} \tag{4–15}$$

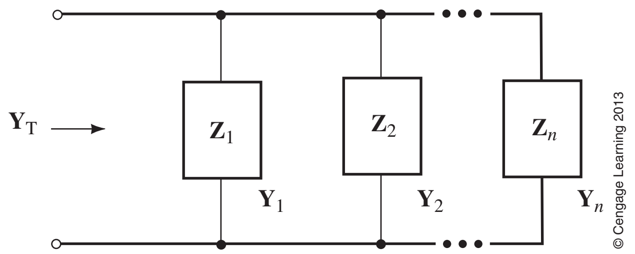


FIGURE 4–37

The resultant impedance of a parallel network of n impedances is determined to be

$$\mathbf{Z}_T = \frac{1}{\mathbf{Y}_T} = \frac{1}{\mathbf{Y}_1 + \mathbf{Y}_2 + \cdots + \mathbf{Y}_n}$$

$$\mathbf{Z}_T = \frac{1}{\dfrac{1}{\mathbf{Z}_1} + \dfrac{1}{\mathbf{Z}_2} + \cdots + \dfrac{1}{\mathbf{Z}_n}} \quad (\Omega) \tag{4–16}$$

EXAMPLE 4–12

Find the equivalent admittance and impedance of the network of Figure 4–38. Sketch the admittance diagram.

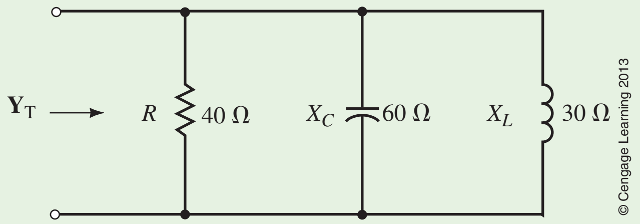


FIGURE 4–38

Solution The admittances of the various parallel elements are

$$\mathbf{Y}_1 = \frac{1}{40\ \Omega\angle 0°} = 25.0\ \text{mS}\angle 0° = 25.0\ \text{mS} + j0$$

$$\mathbf{Y}_2 = \frac{1}{60\ \Omega\angle -90°} = 16.\overline{6}\ \text{mS}\angle 90° = 0 + j16.\overline{6}\ \text{mS}$$

$$\mathbf{Y}_3 = \frac{1}{30\ \Omega\angle 90°} = 33.\overline{3}\ \text{mS}\angle -90° = 0 - j33.\overline{3}\ \text{mS}$$

The total admittance is determined as

$$\begin{aligned}\mathbf{Y}_T &= \mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3 \\ &= 25.0\ \text{mS} + j16.\overline{6}\ \text{mS} + (-j33.\overline{3}\ \text{mS}) \\ &= 25.0\ \text{mS} - j16.\overline{6}\ \text{mS} \\ &= 30.0\ \text{mS}\angle -33.69°\end{aligned}$$

This results in a total impedance for the network of

$$\begin{aligned}\mathbf{Z}_\mathrm{T} &= \frac{1}{\mathbf{Y}_\mathrm{T}}\\ &= \frac{1}{30.0\ \mathrm{mS}\angle -33.69^\circ}\\ &= 33.3\ \Omega\angle 33.69^\circ\end{aligned}$$

The admittance diagram is shown in Figure 4–39.

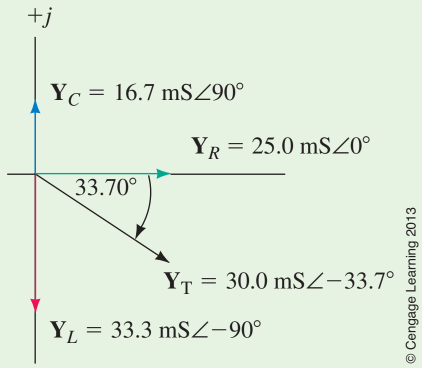


FIGURE 4–39

Two Impedances in Parallel

By applying Equation 4–14 for two impedances, we determine the equivalent impedance of two impedances as

$$\mathbf{Z}_\mathrm{T} = \frac{\mathbf{Z}_1\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2} \quad (\Omega) \tag{4–17}$$

From the preceding expression, we see that for two impedances in parallel, the equivalent impedance is determined as the product of the impedances over the sum. Although the expression for two impedances is very similar to the expression for two resistors in parallel, the difference is that the calculation of impedance involves the use of complex algebra.

EXAMPLE 4–13

Find the total impedance for the network shown in Figure 4–40.

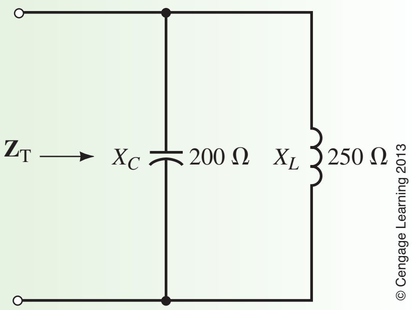


FIGURE 4 –40

Solution

$$\begin{aligned}\mathbf{Z}_\mathrm{T} &= \frac{(200\ \Omega\angle -90^\circ)(250\ \Omega\angle 90^\circ)}{-j200\ \Omega + j250\ \Omega}\\ &= \frac{50\ \mathrm{k}\Omega\angle 0^\circ}{50\angle 90^\circ} = 1\ \mathrm{k}\Omega\angle -90^\circ\end{aligned}$$

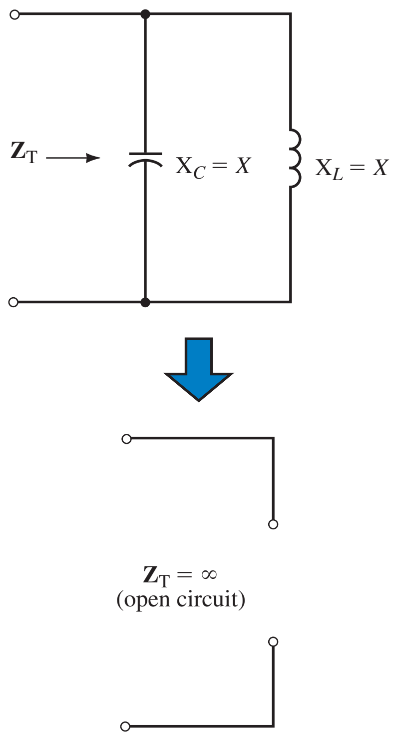


FIGURE 4–41

The previous example illustrates that unlike total parallel resistance, the total impedance of a combination of parallel reactances may be much larger than either of the individual impedances. Indeed, if we are given a parallel combination of equal inductive and capacitive reactances, the total impedance of the combination is equal to infinity (namely an open circuit). Consider the network of Figure 4–41.

The total impedance $\mathbf{Z}_T$ is found as

$$\mathbf{Z}_T = \frac{(X_L\angle 90°)(X_C\angle -90°)}{jX_L - jX_C} = \frac{X^2\angle 0°}{0\angle 0°} = \infty\angle 0°$$

Because the denominator of the preceding expression is equal to zero, the magnitude of the total impedance will be undefined ($Z = \infty$). The magnitude is undefined and the algebra yields a phase angle $\theta = 0°$, which indicates that the vector lies on the positive real axis of the impedance diagram.

Whenever a capacitor and an inductor having equal reactances are placed in parallel, the equivalent circuit of the two components is an open circuit.

The principle of equal parallel reactances will be studied in a later chapter dealing with “resonance.”

Three Impedances in Parallel

Equation 4–16 may be solved for three impedances to give the equivalent impedance as

$$\mathbf{Z}_T = \frac{\mathbf{Z}_1\mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3} \quad (\Omega) \qquad \textbf{(4–4)}$$

although this is less useful than the general equation.

EXAMPLE 4–14

Find the equivalent impedance of the network of Figure 4–42.

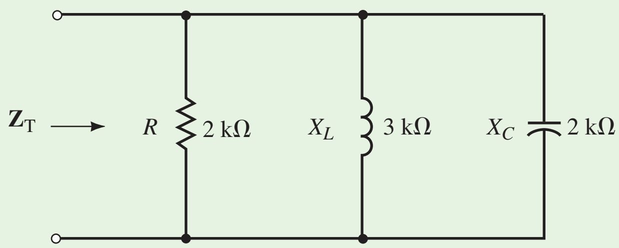


FIGURE 4–42

Solution

$$\mathbf{Z}_T = \frac{(2\text{ k}\Omega\angle 0°)(3\text{ k}\Omega\angle 90°)(2\text{ k}\Omega\angle -90°)}{(2\text{ k}\Omega\angle 0°)(3\text{ k}\Omega\angle 90°) + (2\text{ k}\Omega\angle 0°)(2\text{ k}\Omega\angle -90°) + (3\text{ k}\Omega\angle 90°)(2\text{ k}\Omega\angle -90°)}$$

$$= \frac{12 \times 10^9\ \Omega\angle 0°}{6 \times 10^6\angle 90° + 4 \times 10^6\angle -90° + 6 \times 10^6\angle 0°}$$

$$= \frac{12 \times 10^9\ \Omega\angle 0°}{6 \times 10^6 + j2 \times 10^6} = \frac{12 \times 10^9\ \Omega\angle 0°}{6.325 \times 10^6\angle 4.43°}$$

$$= 1.90\text{ k}\Omega\angle -4.43°$$

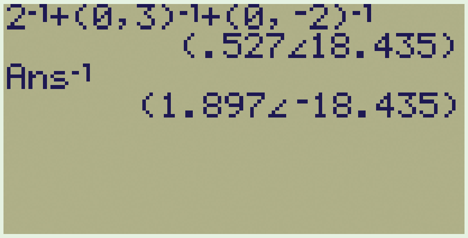
And so the equivalent impedance of the network is

$$Z_T = 1.80\text{ k}\Omega - j0.6\text{ k}\Omega$$

Calculator Hint: In Chapter 5, it was shown that the equivalent value of several resistors in parallel can be found using the x^{-1} key found on all scientific calculators. If your calculator is able to perform operations on complex numbers, a similar method can be used for solving the equivalent impedance of any circuit, even one with reactive elements such as in Figure 4–42. The following keystrokes show h ow $\mathbf{Z}_T$ is determined for a typical scientific calculator. Recognizing that all impedances are in kΩ, we can avoid needless keystrokes by simply entering the numerical values as follows:

2	x^{-1}	+	(	i	3	)	x^{-1}	+
(	+/−	i	2	)	x^{-1}	ENTER	x^{-1}	ENTER

Depending on your calculator, your display will appear similar to the following:



PRACTICE PROBLEMS 5

A circuit consists of a current source, $i = 0.030 \sin 500t$, in parallel with $L =$ 20 mH, $C = 50$ μF, and $R = 25$ Ω.

a. Determine the voltage $\mathbf{V}$ across the circuit.

b. Solve for the power factor of the circuit.

c. Calculate the average power dissipated by the circuit and verify that this is equal to the power delivered by the source.

d. Use Ohm's law to find the phasor quantities, $\mathbf{I}_R$, $\mathbf{I}_L$, and $\mathbf{I}_C$.

Answers

a. $\mathbf{V} = 0.250 \text{ V}\angle 61.93°$

b. $F_p = 0.4705$ lagging

c. $P_R = 2.49 \text{ mW} = P_T$

d. $\mathbf{I}_R = 9.98 \text{ mA}\angle 61.98°$

$\mathbf{I}_C = 6.24 \text{ mA}\angle 151.93°$

$\mathbf{I}_L = 25.0 \text{ mA}\angle -28.07°$

PRACTICE PROBLEMS 6

A circuit consists of a 2.5-A_{rms} current source connected in parallel with a resistor, an inductor, and a capacitor. The resistor has a value of 10 Ω and dissipates 40 W of power.

a. Calculate the values of X_L and X_C if $X_L = 3X_C$.

b. Determine the magnitudes of current through the inductor and the capacitor.

Answers
a. $X_L = 26.7\ \Omega$, $X_C = 8.89\ \Omega$
b. $I_L = 0.75$ A, $I_C = 2.25$ A

4.5 Kirchhoff's Current Law and the Current Divider Rule

The current divider rule for ac circuits has the same form as for dc circuits with the notable exception that currents are expressed as phasors. For a parallel network as shown in Figure 4–43, the current in any branch of the network may be determined using either admittance or impedance.

$$\mathbf{I}_x = \frac{\mathbf{Y}_x}{\mathbf{Y}_\mathrm{T}}\mathbf{I} \quad \text{or} \quad \mathbf{I}_x = \frac{\mathbf{Z}_\mathrm{T}}{\mathbf{Z}_x}\mathbf{I} \tag{4–19}$$

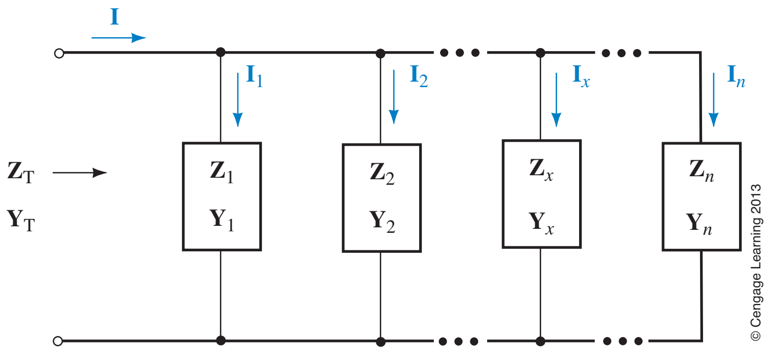


FIGURE 4–43

For two branches in parallel the current in either branch is determined from the impedances as

$$\mathbf{I}_1 = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I} \tag{4–20}$$

Also, as one would expect, Kirchhoff's current law (KCL) must apply to any node within an ac circuit. For such circuits, KCL may be stated as follows:

The summation of current phasors entering and leaving a node is equal to zero.

EXAMPLE 4–15

Calculate the current in each of the branches in the network of Figure 4–44.

Solution

$$\mathbf{I}_1 = \left(\frac{250\ \Omega\angle -90^\circ}{j200\ \Omega - j250\ \Omega}\right)(2\ \mathrm{A}\angle 0^\circ)$$

$$= \left(\frac{250\ \Omega\angle -90^\circ}{50\ \Omega\angle -90^\circ}\right)(2\ \mathrm{A}\angle 0^\circ) = 10\ \mathrm{A}\angle 0^\circ$$

and

$$\mathbf{I}_2 = \left(\frac{200\ \Omega\angle 90^\circ}{j200\ \Omega - j250\ \Omega}\right)(2\ \text{A}\angle 0^\circ)$$
$$= \left(\frac{200\ \Omega\angle 90^\circ}{50\ \Omega\angle -90^\circ}\right)(2\ \text{A}\angle 0^\circ) = 8\ \text{A}\angle 180^\circ$$

The preceding results illustrate that the currents in parallel reactive components may be significantly larger than the applied current. If the current through the component exceeds the maximum current rating of the element, severe damage may occur.

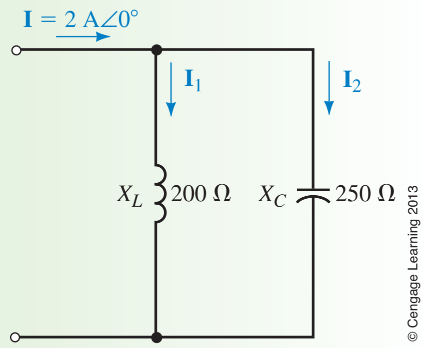


FIGURE 4 –44

EXAMPLE 4–16

Refer to the circuit of Figure 4–45:

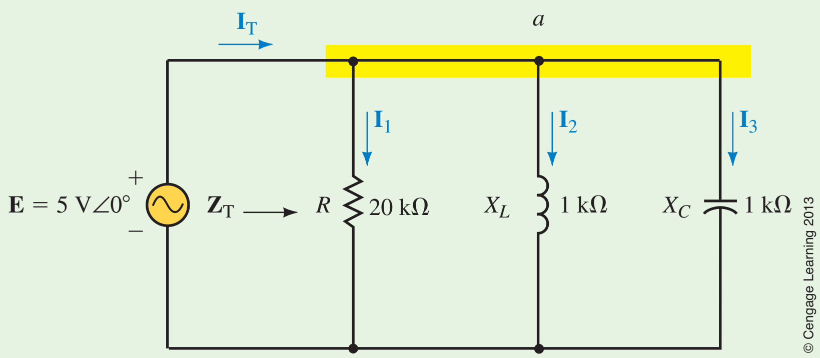


FIGURE 4–45

CircuitSim 18-2

a. Find the total impedance, $\mathbf{Z}_\text{T}$.

b.D etermine the supply current, $\mathbf{I}_\text{T}$.

c. Calculate $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$ using the current divider rule.

d. Verify Kirchhoff's current law at node *a*.

Solution

a. Because the inductive and capacitive reactances are in parallel and have the same value, we may replace the combination with an open circuit. Consequently, only the resistor R needs to be considered. As a result

$$\mathbf{Z}_\text{T} = 20\ \text{k}\Omega\angle 0^\circ$$

b.
$$\mathbf{I}_\text{T} = \frac{5\ \text{V}\angle 0^\circ}{20\ \text{k}\Omega\angle 0^\circ} = 250\ \mu\text{A}\angle 0^\circ$$

c.
$$\mathbf{I}_1 = \left(\frac{20\ \text{k}\Omega\angle 0^\circ}{20\ \text{k}\Omega\angle 0^\circ}\right)(250\ \mu\text{A}\angle 0^\circ) = 250\ \mu\text{A}\angle 0^\circ$$
$$\mathbf{I}_2 = \left(\frac{20\ \text{k}\Omega\angle 0^\circ}{1\ \text{k}\Omega\angle 90^\circ}\right)(250\ \mu\text{A}\angle 0^\circ) = 5.0\ \text{mA}\angle -90^\circ$$
$$\mathbf{I}_3 = \left(\frac{20\ \text{k}\Omega\angle 0^\circ}{1\ \text{k}\Omega\angle -90^\circ}\right)(250\ \mu\text{A}\angle 0^\circ) = 5.0\ \text{mA}\angle 90^\circ$$

d. Notice that the currents through the inductor and capacitor are 180° out of phase. By adding the current phasors in rectangular form, we have

$$\mathbf{I}_T = 250\ \mu\text{A} - j5.0\ \text{mA} + j5.0\ \text{mA} = 250\ \mu\text{A} + j0 = 250\ \mu\text{A}\angle 0^\circ$$

This result satisfies Kirchhoff's current law at the node.

IN-PROCESS LEARNING CHECK 3

(Answers are at the end of the chapter.)

1. Express Kirchhoff's current law as it applies to ac circuits.
2. What is the fundamental difference between how Kirchhoff's current law is applied to ac circuits and how it is applied to dc circuits?

PRACTICE PROBLEMS 7

a. Use the current divider rule to determine current through each branch in the circuit of Figure 4–46.

CircuitSim 18-3

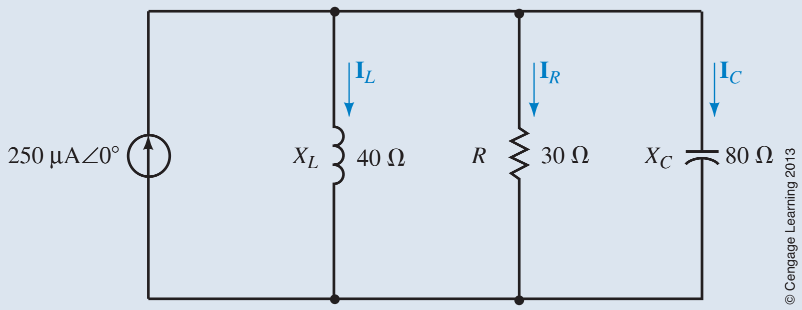


FIGURE 4–46

b. Verify that Kirchhoff's current law applies to the circuit of Figure 4–46.

Answers

a. $\mathbf{I}_L = 176\ \mu\text{A}\angle -69.44^\circ$

$\mathbf{I}_R = 234\ \mu\text{A}\angle 20.56^\circ$

$\mathbf{I}_C = 86.8\ \mu\text{A}\angle 110.56^\circ$

b. $\Sigma\mathbf{I}_{\text{out}} = \Sigma\mathbf{I}_{\text{in}} = 250\ \mu\text{A}$

4.6 Series-Parallel Circuits

We may now apply the analysis techniques of series and parallel circuits in solving more complicated circuits. As in dc circuits, the analysis of such circuits is simplified by starting with easily recognized combinations. If necessary, the original circuit may be redrawn to make further simplification more apparent. Regardless of the complexity of the circuits, we find that the fundamental rules and laws of circuit analysis must apply in all cases.

Consider the network of Figure 4–47.

We see that the impedances $\mathbf{Z}_2$ and $\mathbf{Z}_3$ are in series. The branch containing this combination is then seen to be in parallel with the impedance $\mathbf{Z}_1$.

The total impedance of the network is expressed as

$$\mathbf{Z}_T = \mathbf{Z}_1 \parallel (\mathbf{Z}_2 + \mathbf{Z}_3)$$

Solving for $\mathbf{Z}_T$ gives the following:

$$\begin{aligned}
\mathbf{Z}_T &= (2\ \Omega - j8\ \Omega)\|(2\ \Omega - j5\ \Omega + 6\ \Omega + j7\ \Omega) \\
&= (2\ \Omega - j8\ \Omega) \parallel (8\ \Omega + j2\ \Omega) \\
&= \frac{(2\ \Omega - j8\ \Omega)(8\ \Omega + j2\ \Omega)}{2\ \Omega - j8\ \Omega + 8\ \Omega + j2\ \Omega)} \\
&= \frac{(8.246\ \Omega\angle -75.96^\circ)(8.246\ \Omega\angle 14.04^\circ)}{11.66\ \Omega\angle -30.96^\circ} \\
&= 5.832\ \Omega\angle -30.96^\circ = 5.0\ \Omega - j3.0\ \Omega
\end{aligned}$$

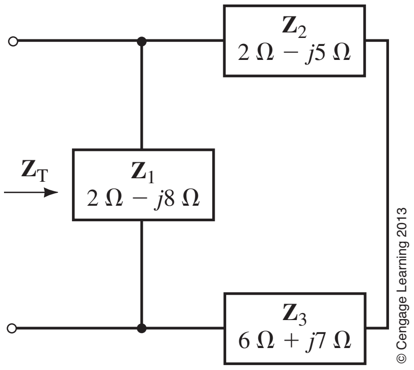


FIGURE 4–47

EXAMPLE 4–17

Determine the total impedance of the network of Figure 4–48. Express the impedance in both polar form and rectangular form.

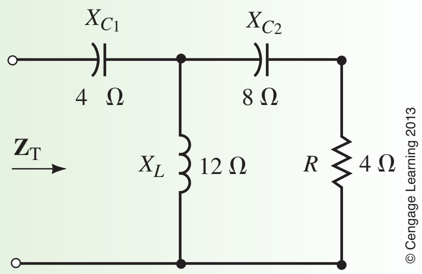
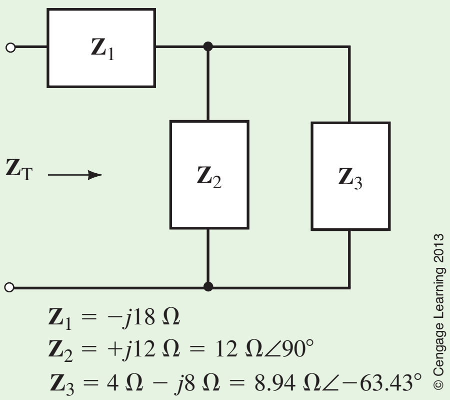


FIGURE 4 –48

Solution After redrawing and labelling the given circuit, we have the circuit shown in Figure 4–49.



$\mathbf{Z}_1 = -j18\ \Omega$
$\mathbf{Z}_2 = +j12\ \Omega = 12\ \Omega\angle 90^\circ$
$\mathbf{Z}_3 = 4\ \Omega - j8\ \Omega = 8.94\ \Omega\angle -63.43^\circ$

FIGURE 4–49

The total impedance is given as

$$\mathbf{Z}_T = \mathbf{Z}_1 + \mathbf{Z}_2 \parallel \mathbf{Z}_3$$

where

$\mathbf{Z}_1 = -j18\ \Omega = 4\Omega\angle -90^\circ \mathbf{Z}_2 = j12$
$\Omega = 12\Omega\angle 90^\circ$
$\mathbf{Z}_3 = 4\ \Omega - j8\ \Omega = 8.94\ \Omega\angle -63.43^\circ$

We determine the total impedance as

$$\begin{aligned}
\mathbf{Z}_T &= -j18\ \Omega + \left[\frac{(12\ \Omega\angle 90^\circ)(8.94\ \Omega\angle -63.43^\circ)}{j12\ \Omega + 4\ \Omega - j8\ \Omega}\right] \\
&= -j18\ \Omega + \left(\frac{107.3\ \Omega\angle 26.57^\circ}{5.66\angle 45^\circ}\right) \\
&= -j18\ \Omega + 19.0\ \Omega\angle -4.43^\circ \\
&= -j18\ \Omega + 4\ \Omega - j6\ \Omega \\
&= 4\ \Omega - j24\ \Omega = 30\ \Omega\angle -53.13^\circ
\end{aligned}$$

EXAMPLE 4–4

Consider the circuit of Figure 4–50:

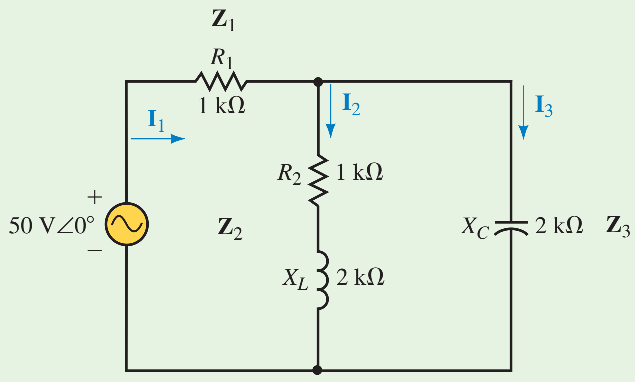


FIGURE 4–50

a. Find $\mathbf{Z}_T$.

b.D etermine the currents $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$.

c. Calculate the total power provided by the voltage source.

d. Determine the average powers P_1, P_2, and P_3 dissipated by each of the impedances. Verify that the average power delivered to the circuit is the same as the power dissipated by the impedances.

Solution

a. The total impedance is determined by the combination

$$\mathbf{Z}_T = \mathbf{Z}_1 + \mathbf{Z}_2 \parallel \mathbf{Z}_3$$

For the parallel combination we have

$$\mathbf{Z}_2 \parallel \mathbf{Z}_3 = \frac{(1\text{ k}\Omega + j2\text{ k}\Omega)(-j2\text{ k}\Omega)}{1\text{ k}\Omega + j2\text{ k}\Omega - j2\text{ k}\Omega}$$

$$= \frac{(2.236\text{ k}\Omega\angle 63.43^\circ)(2\text{ k}\Omega\angle -90^\circ)}{1\text{ k}\Omega\angle 0^\circ}$$

$$= 4.472\text{ k}\Omega\angle -26.57^\circ = 4.0\text{ k}\Omega - j2.0\text{ k}\Omega$$

b. And so the total impedance is

$$\mathbf{Z}_T = 5\text{ k}\Omega - j2\text{ k}\Omega = 5.385\text{ k}\Omega\angle -21.80^\circ$$

$$\mathbf{I}_1 = \frac{50\text{ V}\angle 0^\circ}{5.385\text{ k}\Omega\angle -21.80^\circ}$$

$$= 9.285\text{ mA}\angle 21.80^\circ$$

Applying the current divider rule, we get

$$\mathbf{I}_2 = \frac{(2\text{ k}\Omega\angle -90^\circ)(9.285\text{ mA}\angle 21.80^\circ)}{1\text{ k}\Omega + j2\text{ k}\Omega - j2\text{ k}\Omega}$$

$$= 4.57\text{ mA}\angle -68.20^\circ$$

and

$$\mathbf{I}_3 = \frac{(1\ \text{k}\Omega + j2\ \text{k}\Omega)(9.285\ \text{mA}\angle 21.80^\circ)}{1\ \text{k}\Omega + j2\ \text{k}\Omega - j2\ \text{k}\Omega}$$

$$= \frac{(2.236\ \text{k}\Omega\angle 63.43^\circ)(9.285\ \text{mA}\angle 21.80^\circ)}{1\ \text{k}\Omega\angle 0^\circ}$$

$$= 20.761\ \text{mA}\angle 85.23^\circ$$

c. $$P_T = (50\ \text{V})(9.285\ \text{mA})\cos 21.80^\circ = 431.0\ \text{mW}$$

d. Because only the resistors will dissipate power, we may use $P = I^2R$:

$$P_1 = (9.285\ \text{mA})^2(1\ \text{k}\Omega) = 86.2\ \text{mW}$$
$$P_2 = (4.57\ \text{mA})^2(1\ \text{k}\Omega) = 344.8\ \text{mW}$$

Alternatively, the power dissipated by $\mathbf{Z}_2$ may have been determined as $P = I^2Z\cos\theta$:

$$P_2 = (4.57\ \text{mA})^2(2.236\ \text{k}\Omega)\cos 63.43^\circ = 344.8\ \text{mW}$$Since $\mathbf{Z}_3$ is

purely capacitive, it will not dissipate any power:

$$P_3 = 0$$

By combining these powers, the total power dissipated is found:

$$P_T = 86.2\ \text{mW} + 344.8\ \text{mW} + 0 = 431.0\ \text{mW} \quad \text{(checks!)}$$

PRACTICE PROBLEMS 8

Refer to the circuit of Figure 4–51:

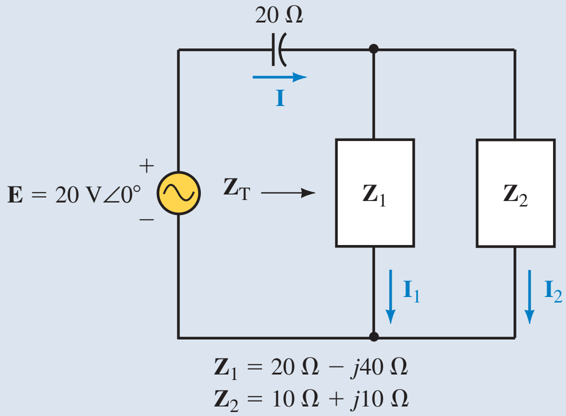


FIGURE 4–51

a. Calculate the total impedance, $\mathbf{Z}_T$.

b.F ind the current $\mathbf{I}$.

c. Use the current divider rule to find $\mathbf{I}_1$ and $\mathbf{I}_2$.

d. Determine the power factor for each impedance, $\mathbf{Z}_1$ and $\mathbf{Z}_2$.

e. Determine the power factor for the circuit.

f. Verify that the total power dissipated by impedances $\mathbf{Z}_1$ and $\mathbf{Z}_2$ is equal to the power delivered by the voltage source.

Answers

a. $\mathbf{Z}_T = 4.9\ \Omega\angle{-45°}$

b. $\mathbf{I} = 1.06\ \text{A}\angle 45°$

c. $\mathbf{I}_1 = 0.354\ \text{A}\angle 135°$, $\mathbf{I}_2 = 1.12\ \text{A}\angle 26.57°$

d. $F_{P(1)} = 0.4472$ leading, $F_{P(2)} = 0.7071$ lagging

e. $F_P = 0.7071$ leading

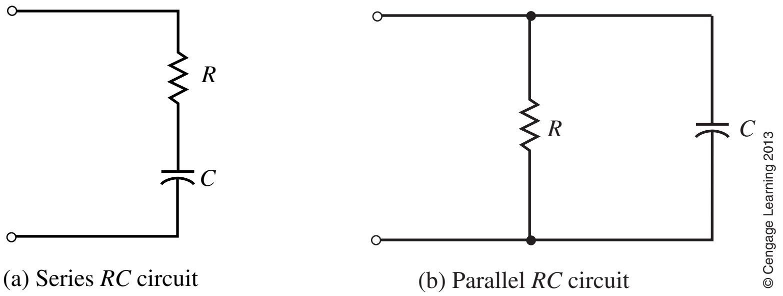
f. $P_T = 15.0\ \text{W}$, $P_1 = 2.50\ \text{W}$, $P_2 = 12.5\ \text{W}$
$P_1 + P_2 = 15.0\ \text{W} = P_T$

4.7 Frequency Effects

As we have already seen, the reactance of inductors and capacitors depends on frequency. Consequently, the total impedance of any network having reactive elements is also frequency dependent. Any such circuit would need to be analyzed separately at each frequency of interest. We will examine several fairly simple combinations of resistors, capacitors, and inductors to see how the various circuits operate at different frequencies. Some of the more important combinations will be examined in greater detail in later chapters that deal with resonance and filters.

RC Circuits

As the name implies, *RC* circuits consist of a resistor and a capacitor. The components of an *RC* circuit may be connected either in series or in parallel as shown in Figure 4–52.



(a) Series *RC* circuit (b) Parallel *RC* circuit

FIGURE 4–52

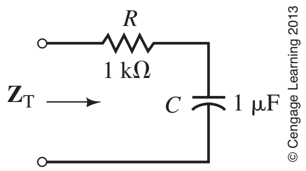


FIGURE 4–53

Consider the *RC* series circuit of Figure 4–53. Recall that the capacitive reactance, X_C, is given as

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

The total impedance of the circuit is a vector quantity expressed as

$$\mathbf{Z}_T = R - j\frac{1}{\omega C} = R + \frac{1}{j\omega C}$$

$$\mathbf{Z}_T = \frac{1 + j\omega RC}{j\omega C} \qquad \textbf{(4–21)}$$

If we define the **cutoff** or **corner frequency** for an *RC* circuit as

$$\omega_c = \frac{1}{RC} = \frac{1}{\tau} \quad \text{(rad/s)} \qquad \textbf{(4–22)}$$

or equivalently as

$$f_c = \frac{1}{2\pi RC} \quad \text{(Hz)} \tag{4–23}$$

then several important points become evident.

For $\omega \leq \omega_c/10$ (or $f \leq f_c/10$) Equation 4–21 can be expressed as

$$\mathbf{Z}_\text{T} \simeq \frac{1 + j0}{j\omega C} = \frac{1}{j\omega C}$$

and for $\omega \geq 10\omega_c$, the expression of Equation 4–21 can be simplified as

$$\mathbf{Z}_\text{T} \simeq \frac{0 + j\omega RC}{j\omega C} = R$$

Solving for the magnitude of the impedance at several angular frequencies, we have the results shown in Table 4–1.

If the magnitude of the impedance $\mathbf{Z}_\text{T}$ is plotted as a function of angular frequency ω, we get the graph of Figure 4–54. Notice that the abscissa and ordinate of the graph are not scaled linearly, but rather logarithmically. This allows for the display of results over a wide range of frequencies.

TABLE 4–1

Angular Frequency, ω (rad/s)	X_C (Ω)	Z_T (Ω)
0	∞	∞
1	1 M	1 M
10	100 k	100 k
100	10 k	10.05 k
200	5 k	5.099 k
500	2 k	2.236 k
1000	1 k	1.414 k
2000	500	1118
5000	200	1019
10 k	100	1005
100 k	10	1000

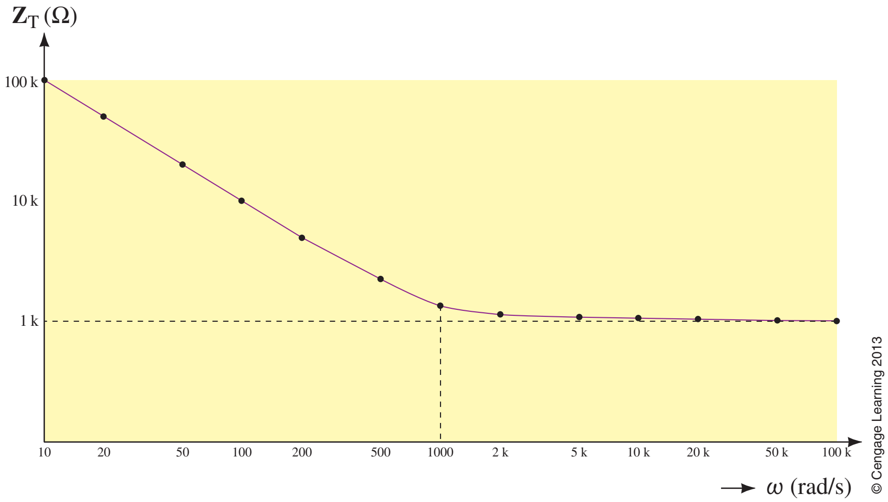


FIGURE 4–54 Impedance versus angular frequency for the network of Figure 4–53.

The graph illustrates that the reactance of a capacitor is very high (effectively an open circuit) at low frequencies. Consequently, the total impedance of the series circuit will also be very high at low frequencies. Second, we notice that as the frequency increases, the reactance decreases. Therefore, as the frequency gets higher, the capacitive reactance has a diminished effect in the circuit. At very high frequencies (typically for $\omega \geq 10\omega_c$), the impedance of the circuit will effectively be $R = 1\ \text{k}\Omega$.

Consider the parallel RC circuit of Figure 4–55. The total impedance, $\mathbf{Z}_\text{T}$, of the circuit is determined as

$$\mathbf{Z}_\text{T} = \frac{\mathbf{Z}_R\mathbf{Z}_C}{\mathbf{Z}_R + \mathbf{Z}_C} = \frac{R\left(\dfrac{1}{j\omega C}\right)}{R + \dfrac{1}{j\omega C}}$$

$$= \frac{\dfrac{R}{j\omega C}}{\dfrac{1 + j\omega RC}{j\omega C}}$$

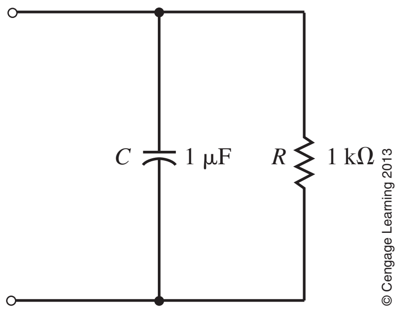


FIGURE 4–55

TABLE 4–2

Angular Frequency, ω (rad/s)	X_C (Ω)	Z_T (Ω)
0	∞	1000
1	1 M	1000
10	100 k	1000
100	10 k	995
200	5 k	981
500	2 k	894
1 k	1 k	707
2 k	500	447
5 k	200	196
10 k	100	99.5
100 k	10	10

which may be simplified as

$$\mathbf{Z}_T = \frac{R}{1 + j\omega RC} \tag{4–24}$$

As before, the cutoff frequency is given by Equation 4–22. Now, by examining the expression of Equation 4–24 for $\omega \leq \omega_c/10$, we have the following result:

$$\mathbf{Z}_T \simeq \frac{R}{1 + j0} = R$$

For $\omega \geq 10\omega_c$, we have

$$\mathbf{Z}_T \simeq \frac{R}{0 + j\omega RC} = \frac{1}{j\omega C}$$

If we solve for the impedance of the circuit in Figure 4–55 at various angular frequencies, we obtain the results of Table 4–2.

Plotting the magnitude of the impedance $\mathbf{Z}_T$ as a function of angular frequency ω, we get the graph of Figure 4–56. Notice that the abscissa and ordinate of the graph are again scaled logarithmically, allowing for the display of results over a wide range of frequencies.

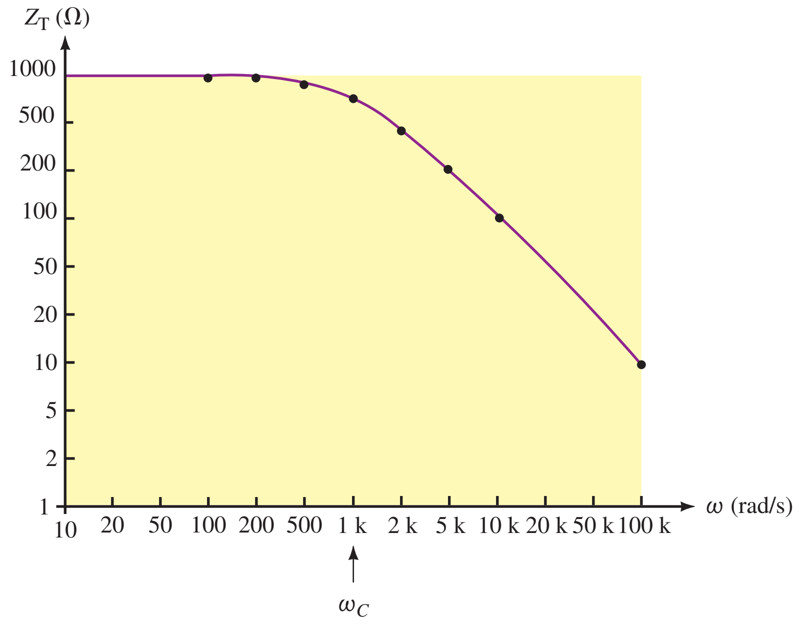


FIGURE 4–56 Impedance versus angular frequency for the network of Figure 4–55.

The results indicate that at dc ($f = 0$ Hz) the capacitor, which behaves as an open circuit, will result in a circuit impedance of $R = 1\ \text{k}\Omega$. As the frequency increases, the capacitor reactance approaches 0 Ω, resulting in a corresponding decrease in circuit impedance.

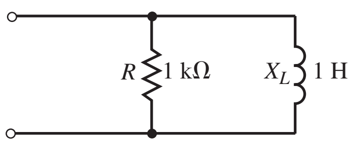


FIGURE 4–57

RL Circuits

RL circuits may be analyzed in a manner similar to the analysis of *RC* circuits. Consider the parallel *RL* circuit of Figure 4–57.

The total impedance of the parallel circuit is found as follows:

$$\mathbf{Z}_{\mathrm{T}} = \frac{\mathbf{Z}_R \mathbf{Z}_L}{\mathbf{Z}_R + \mathbf{Z}_L} = \frac{R(j\omega L)}{R + j\omega L}$$

$$\mathbf{Z}_{\mathrm{T}} = \frac{j\omega L}{1 + j\omega \dfrac{L}{R}} \tag{4–25}$$

If we define the *cutoff* or *corner frequency* for an *RL* circuit as

$$\omega_c = \frac{R}{L} = \frac{1}{\tau} \quad \text{(rad/s)} \tag{4–26}$$

or equivalently as

$$f_c = \frac{R}{2\pi L} \quad \text{(Hz)} \tag{4–27}$$

then several important points become evident.

For $\omega \leq \omega_c/10$ (or $f \leq f_c/10$) Equation 4–25 can be expressed as

$$\mathbf{Z}_{\mathrm{T}} \simeq \frac{j\omega L}{1 + j0} = j\omega L$$

The previous result indicates that for low frequencies, the inductor has a very small reactance, resulting in a total impedance which is essentially equal to the inductive reactance.

For $\omega \geq 10\omega_c$, the expression of Equation 4–25 can be simplified as

$$\mathbf{Z}_{\mathrm{T}} \simeq \frac{j\omega L}{0 + j\omega \dfrac{L}{R}} = R$$

The preceding results indicate that for high frequencies, the impedance of the circuit is essentially equal to the resistance, due to the very high impedance of the inductor.

Evaluating the impedance at several angular frequencies, we have the results of Table 4–3.

When the magnitude of the impedance $\mathbf{Z}_{\mathrm{T}}$ is plotted as a function of angular frequency ω, we get the graph of Figure 4–58.

TABLE 4–3

Angular Frequency, ω (rad/s)	X_L (Ω)	Z_{T} (Ω)
0	0	0
1	1	1
10	10	10
100	100	99.5
200	200	196
500	500	447
1 k	1 k	707
2 k	2 k	894
5 k	5 k	981
10 k	10 k	995
100 k	100 k	1000

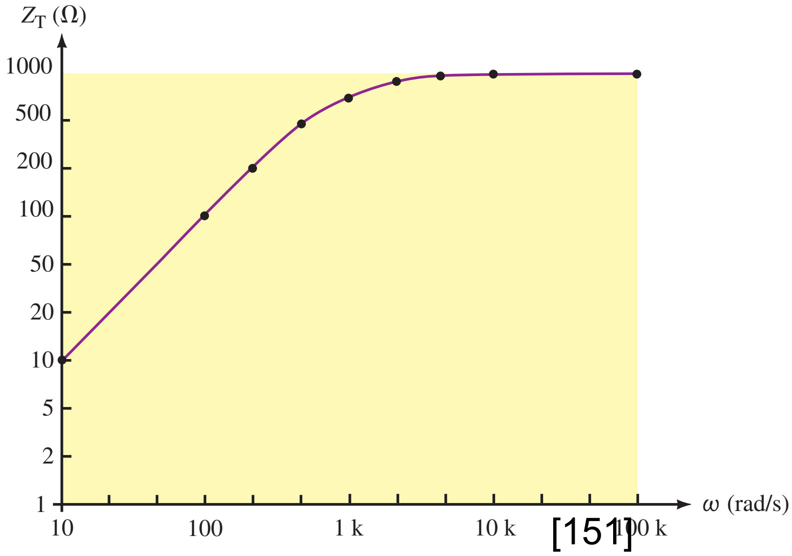


FIGURE 4–58 Impedance versus angular frequency for the network of Figure 4–57.

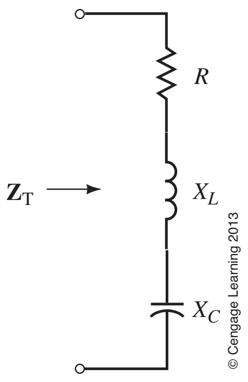


FIGURE 4–59

RLC Circuits

When numerous capacitive and inductive components are combined with resistors in series-parallel circuits, the total impedance $\mathbf{Z}_T$ of the circuit may rise and fall several times over the full range of frequencies. The analysis of such complex circuits is outside the scope of this textbook. However, for illustrative purposes we examine the simple series *RLC* circuit of Figure 4–59.

The impedance $\mathbf{Z}_T$ at any frequency will be determined as

$$\begin{aligned}\mathbf{Z}_T &= R + jX_L - jX_C \\ &= R + j(X_L - X_C)\end{aligned}$$

At very low frequencies, the inductor will appear as a very low impedance (effectively a short circuit), while the capacitor will appear as a very high impedance (effectively an open circuit). Because the capacitive reactance will be much larger than the inductive reactance, the circuit will have a very large capacitive reactance. This results in a very high circuit impedance, $\mathbf{Z}_T$.

As the frequency increases, the inductive reactance increases, while the capacitive reactance decreases. At some frequency, f_0, the inductor and the capacitor will have the same magnitude of reactance. At this frequency, the reactances cancel, resulting in a circuit impedance which is equal to the resistance value. As the frequency increases still further, the inductive reactance becomes larger than the capacitive reactance. The circuit becomes inductive and the magnitude of the total impedance of the circuit again rises. Figure 4–60 shows how the impedance of a series *RLC* circuit varies with frequency. The complete analysis of the series *RLC* circuit and the parallel *RLC* cir-cuit is left until we examine the principle of resonance in a later chapter.

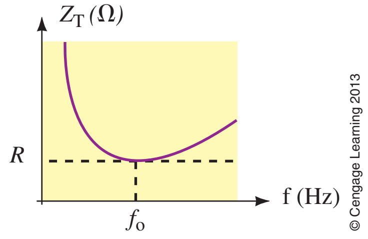


FIGURE 4–60

IN-PROCESS LEARNING CHECK 4

(Answers are at the end of the chapter.)

1. For a series network consisting of a resistor and a capacitor, what will be the impedance of the network at a frequency of 0 Hz (dc)? What will be the impedance of the network as the frequency approaches infinity?
2. For a parallel network consisting of a resistor and an inductor, what will be the impedance of the network at a frequency of 0 Hz (dc)? What will be the impedance of the network as the frequency approaches infinity?

PRACTICE PROBLEMS 9

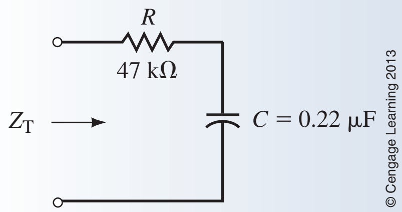


FIGURE 4 –61

Given the series *RC* network of Figure 4–61, calculate the cutoff frequencyin hertz and in radians per second. Sketch the frequency response of Z_T (magnitude) versus angular frequency ω for the network. Show the magnitude Z_T at $\omega_c/10$, ω_c, and $10\omega_c$.

Answers

$\omega_c = 96.7$ rad/s, $f_c = 15.4$ Hz

At 0.1 ω_c: $Z_T = 472$ kΩ, At ω_c: $Z_T = 66.5$ kΩ, At 10 ω_c: $Z_T = 47.2$ kΩ

Bridge Networks

5.6 Bridge Networks

Bridge circuits, similar to the network of Figure 5–36, are used extensively in electronics to measure the values of unknown components.

Recall from Chapter 8 that any bridge circuit is said to be balanced when the current through the branch between the two arms is zero. In a practical circuit, component values of very precise resistors are adjusted until the current through the central element (usually a sensitive galvanometer) is exactly equal

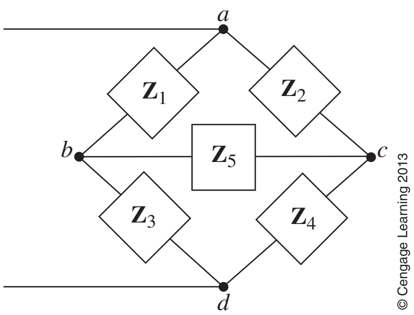


FIGURE 5–36

to zero. For ac circuits, the condition of a **balanced bridge** occurs when the impedance vectors of the various arms satisfy the following condition:

$$\frac{\mathbf{Z}_1}{\mathbf{Z}_3} = \frac{\mathbf{Z}_2}{\mathbf{Z}_4} \tag{5–9}$$

When a balanced bridge occurs in a circuit, the equivalent impedance of the bridge network is easily determined by removing the central impedance and replacing it with either an open or a short circuit. The resulting impedance of the bridge circuit is then found as either of the following:

$$\mathbf{Z}_\mathrm{T} = \mathbf{Z}_1 \| \mathbf{Z}_2 + \mathbf{Z}_3 \| \mathbf{Z}_4$$

or

$$\mathbf{Z}_\mathrm{T} = (\mathbf{Z}_1 + \mathbf{Z}_3) \| (\mathbf{Z}_2 + \mathbf{Z}_4)$$

If, on the other hand, the bridge is not balanced, then the total impedance must be determined by performing a Δ-to-Y conversion. Alternatively, the circuit may be analyzed by using either mesh analysis or nodal analysis.

EXAMPLE 5–13

Given that the circuit of Figure 5–37 is a balanced bridge.

a. Calculate the unknown impedance, $\mathbf{Z}_x$.
b. Determine the values of L_x and R_x if the circuit operates at a frequency of 1 kHz.

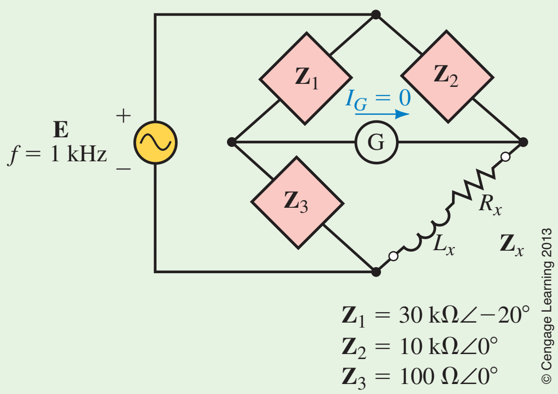


FIGURE 5–37

Solution

a. The expression for the unknown impedance is determined from Equation 5–9 as

$$\begin{aligned}\mathbf{Z}_x &= \frac{\mathbf{Z}_2\mathbf{Z}_3}{\mathbf{Z}_1} \\ &= \frac{(10\ \mathrm{k\Omega})(100\ \Omega)}{30\ \mathrm{k\Omega}\angle -20°} \\ &= 33.3\ \Omega\angle 20° \\ &= 31.3 + j11.4\ \Omega\end{aligned}$$

b. From the preceding result, we have

$$R_x = 31.3\ \Omega$$

and

$$L_x = \frac{X_L}{2\pi f} = \frac{11.4\ \Omega}{2\pi(1000\ \mathrm{Hz})} = 1.81\ \mathrm{mH}$$

We will now consider various forms of bridge circuits that are used in electronic circuits to determine the values of unknown inductors and capacitors. As in resistor bridges, the circuits use variable resistors together with very sensitive galvanometer movements to ensure a balanced condition for the bridge. However, rather than using a dc source to provide current in the circuit, the bridge circuits use ac sources operating at a known frequency (usually 1 kHz). Once the bridge is balanced, the value of unknown inductance or capacitance may be easily determined by obtaining the reading directly from the instrument. Most instruments using bridge circuitry will incorporate several different bridges to enable the measurement of various types of unknown impedances.

Maxwell Bridge

The **Maxwell bridge,** shown in Figure 5–38, is used to determine the induc-tance and series resistance of an inductor having a relatively large series resistance(in comparison to $X_L = \omega L$).

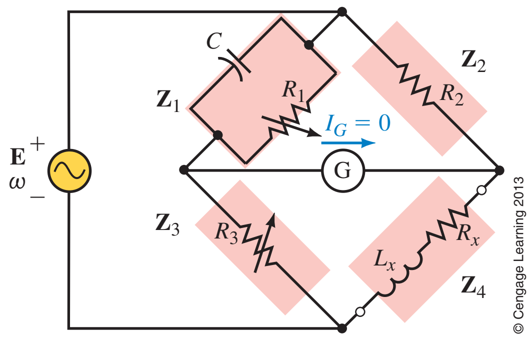


FIGURE 5–38 Maxwell bridge.

Resistors R_1 and R_3 are adjusted to provide the balanced condition (when the current through the galvanometer is zero: $I_G = 0$).

When the bridge is balanced, we know that the following condition must apply:

$$\frac{\mathbf{Z}_1}{\mathbf{Z}_2} = \frac{\mathbf{Z}_3}{\mathbf{Z}_4}$$

If we write the impedances using the rectangular forms, we obtain

$$\frac{\left[\dfrac{(R_1)\left(-j\dfrac{1}{\omega C}\right)}{R_1 - j\dfrac{1}{\omega C}}\right]}{R_2} = \frac{R_3}{R_x + j\omega L_x}$$

$$\frac{\left(-j\dfrac{R_1}{\omega C}\right)}{\left(\dfrac{\omega R_1 C - j1}{\omega C}\right)} = \frac{R_2 R_3}{R_x + j\omega L_x}$$

$$\frac{-jR_1}{\omega C R_1} - j = \frac{R_2 R_3}{R_x + j\omega L_x}$$

$$(-jR_1)(R_x + j\omega L_x) = R_2 R_3(\omega C R_1 - j)$$

$$\omega L_x R_1 - jR_1 R_x = \omega R_1 R_2 R_3 C - jR_2 R_3$$

Now, since two complex numbers can be equal only if their real parts are equal and if their imaginary parts are equal, we must have the following:

$$\omega L_x R_1 = \omega R_1 R_2 R_3 C$$

and

$$R_1 R_x = R_2 R_3$$

Simplifying these expressions, we get the following equations for a Maxwell bridge:

$$L_x = R_2 R_3 C \qquad \textbf{(5–10)}$$

and

$$R_x = \frac{R_2 R_3}{R_1} \qquad \textbf{(5–11)}$$

EXAMPLE 5–14

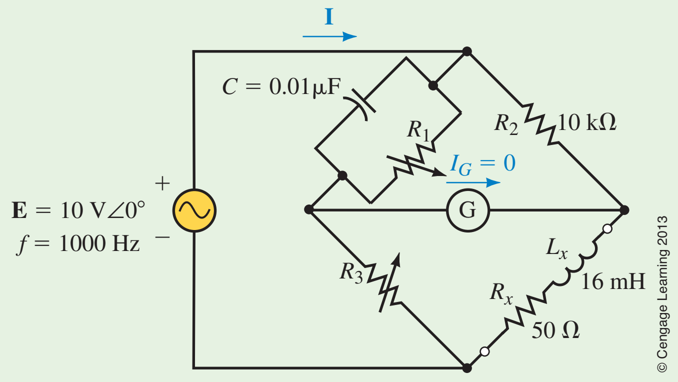


FIGURE 5–39

a. Determine the values of R_1 and R_3 so that the bridge of Figure 5–39 is balanced.

b. Calculate the current **I** when the bridge is balanced.

Solution

a. Rewriting Equations 5–10 and 5–11 and solving for the unknowns, we have

$$R_3 = \frac{L_x}{R_2 C} = \frac{16\text{ mH}}{(10\text{ k}\Omega)(0.01\ \mu\text{F})} = 160\ \Omega$$

and

$$R_1 = \frac{R_2 R_3}{R_x} = \frac{(10\text{ k}\Omega)(160\ \Omega)}{50\ \Omega} = 32\text{ k}\Omega$$

b. The total impedance is found as

$$\mathbf{Z}_\text{T} = (\mathbf{Z}_C \| \mathbf{R}_1 \| \mathbf{R}_2) + [\mathbf{R}_3 \| (\mathbf{R}_x + \mathbf{Z}_{Lx})]$$

$$\mathbf{Z}_\text{T} = (-j15.915\text{ k}\Omega) \| 32\text{ k}\Omega \| 10\text{ k}\Omega + [160\ \Omega \| (50\ \Omega + j100.5\ \Omega)]$$

$$= 6.87\text{ k}\Omega\angle -25.6^\circ + 77.2\ \Omega\angle 38.0^\circ$$

$$= 6.91\text{ k}\Omega\angle -25.0^\circ$$

The resulting circuit current is

$$\mathbf{I} = \frac{10\text{ V}\angle 0^\circ}{6.91\text{ k}\Omega\angle -25^\circ} = 1.45\text{ mA}\angle 25.0^\circ$$

Hay Bridge

In order to measure the inductance and series resistance of an inductor having a small series resistance, a **Hay bridge** is generally used. The Hay bridge is shown in Figure 5–40. By applying a method similar to that used to determine the values of the unknown inductance and resistance of the Maxwell bridge, we can show thatthe following equations for the Hay bridge apply:

$$L_x = \frac{R_2R_3C}{\omega^2{R_1}^2C^2 + 1} \tag{5–12}$$

and

$$R_x = \frac{\omega^2R_1R_2R_3C^2}{\omega^2{R_1}^2C^2 + 1} \tag{5–13}$$

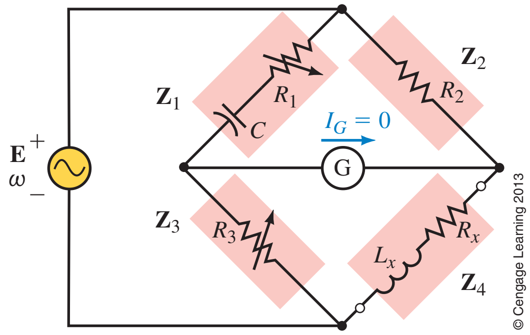


FIGURE 5–40 Hay bridge.

Schering Bridge

The **Schering bridge,** shown in Figure 5–41, is a circuit used to determine the value of unknown capacitance.

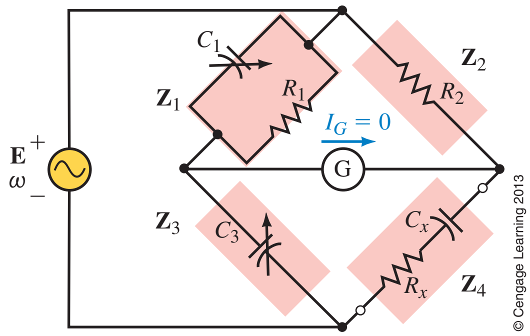


FIGURE 5–41 Schering bridge.

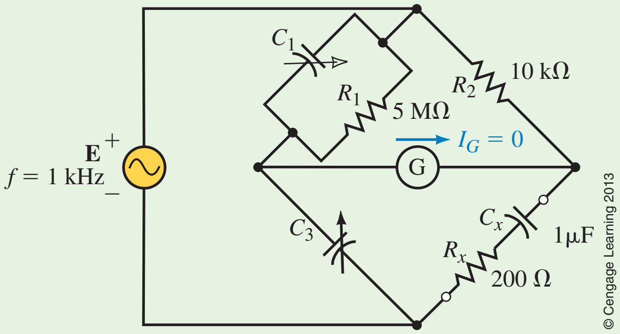
By solving for the balanced bridge condition, we have the following equations for the unknown quantities of the circuit:

$$C_x = \frac{R_1C_3}{R_2} \tag{5–14}$$

$$R_x = \frac{C_1R_2}{C_3} \tag{5–15}$$

EXAMPLE 5–15

Determine the values of C_1 and C_3 that will result in a balanced bridge for the circuit of Figure 5–42.



CircuitSim 5-2

FIGURE 5–42

Solution Rewriting Equations 5–14 and 5–15, we solve for the unknown capacitances as

$$C_3 = \frac{R_2 C_x}{R_1} = \frac{(10\ \text{k}\Omega)(1\ \mu\text{F})}{5\ \text{M}\Omega} = 0.002\ \mu\text{F}$$

and

$$C_1 = \frac{C_3 R_x}{R_2} = \frac{(0.002\ \mu\text{F})(200\ \Omega)}{10\ \text{k}\Omega} = 40\ \text{pF}$$

PRACTICE PROBLEMS 6

Determine the values of R_1 and R_3 so that the bridge of Figure 5–43 is balanced.

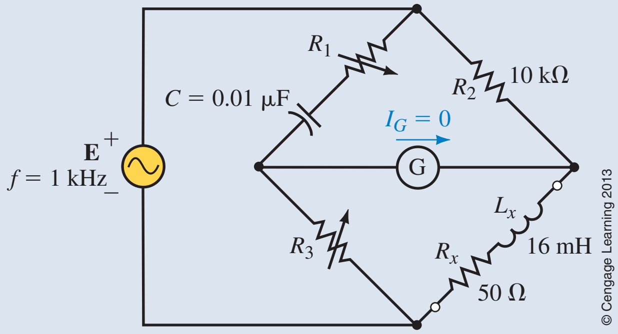


FIGURE 5–43

Answers
$R_1 = 7916\ \Omega$, $R_3 = 199.6\ \Omega$

5.7 Circuit Analysis Using Computers

In some of the examples in this chapter, we analyzed circuits that resulted in as many as three simultaneous linear equations. You have no doubt wondered if there is a less complicated way to solve these circuits without the need for using complex algebra. Computer programs are particularly useful for solving such ac circuits. Both Multisim and PSpice have individual strengths in the solution of ac circuits. As in previous examples, Multisim provides an excellent simulation of how measurements are taken in a lab. PSpice, on the other hand, provides voltage and current readings, complete with magnitude and phase angle. The following examples show how these programs are useful for examining the circuits in this chapter.

MultiSim

PSPICE

EXAMPLE 5–16

Use Multisim to show that the bridge circuit of Figure 5–44 is balanced.

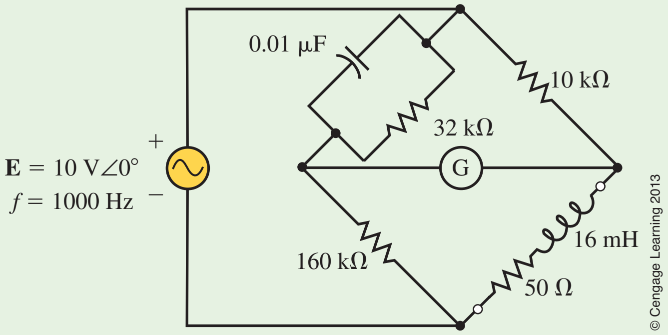


FIGURE 5–44

Solution Recall that a bridge circuit is balanced when the current through the branch between the two arms of the bridge is equal to zero. In this example, we will use a multimeter set on its ac ammeter range to verify the condition of the circuit. The ammeter is selected by clicking on **A,** and it is set to its ac range by clicking on the sinusoidal button. Figure 5–45 shows the circuit connections and the ammeter reading. The results correspond to the conditions that were previously analyzed in Example 5–14. (Note: When we use Multisim, the ammeter may not show exactly zero current in the balanced condition. This is due to the way the program does the calculations. Any current less than 5 μA is considered to be effectively zero.)

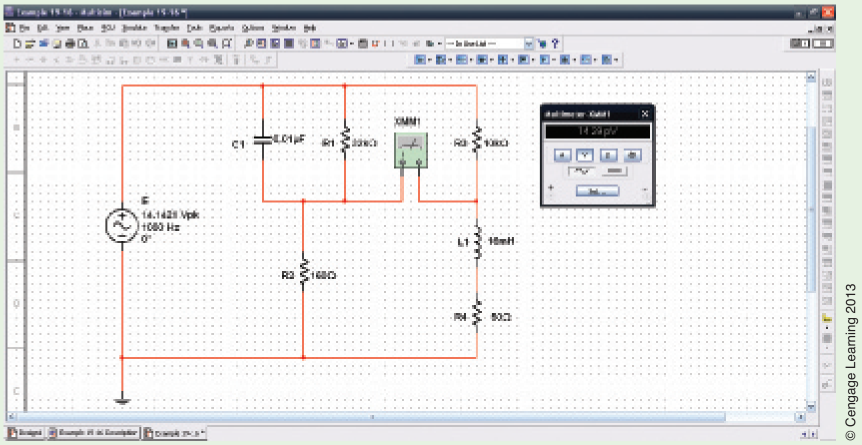


FIGURE 5–45

PRACTICE PROBLEMS 7

Use Multisim to verify that the results obtained in Example 5–15 result in a balanced bridge circuit. (Assume that the bridge is balanced if the galvanometer current is less than 5 μA.)

PSpice

EXAMPLE 5–17

Use PSpice to input the circuit of Figure 5–15. Assume that the circuit operates at a frequency of $\omega = 50$ rad/s ($f = 7.958$ Hz). Use PSpice to obtain a printout showing the currents through X_C, R_2, and X_L. Compare the results to those obtained in Example 5–6.

Solution Since the reactive components in Figure 5–15 were given as impedance,it is necessary to first determine the corresponding values in henries and farads.

$$L = \frac{4\ \Omega}{50\ \text{rad/s}} = 80\ \text{mH}$$

and

$$C = \frac{1}{(2\ \Omega)(50\ \text{rad/s})} = 10\ \text{mF}$$

Now we are ready to use OrCAD Capture CIS to input the circuit as shown in Figure 5–46. The basic steps are reviewed for you. Use the ac current source, ISRC from the SOURCE library and place one IPRINT part from the SPECIAL library. The resistor, inductor, and capacitor are selected from the ANALOG library, and the ground symbol is selected by using the Place ground tool.

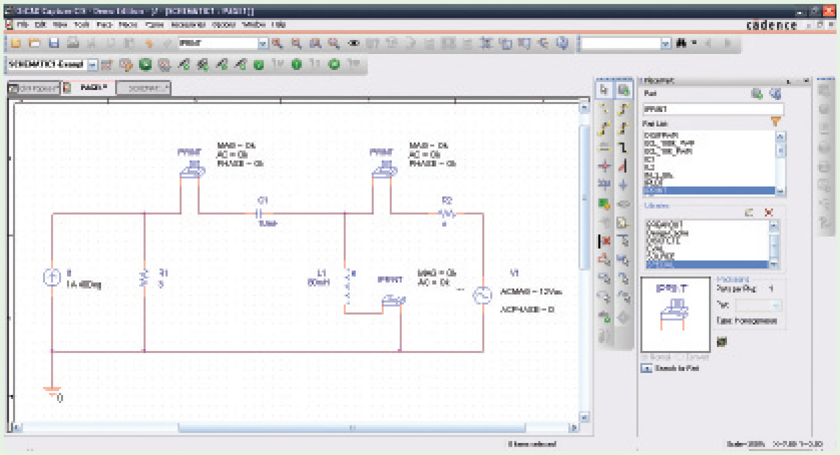


FIGURE 5–46

Change the value of the current source by double-clicking on the part and moving the horizontal scrollbar until you find the field titled AC. Enter **1A 40Deg** into this field. A space must be placed between the magnitude and phase angle. Click on *Apply*. In order for these values to be displayed on the schematic, you must click on the *Display* button and then *Value Only*. Click on OK to return to the properties editor and then close the editor by clicking on X.

KEY TERMS

Bandwidth
Damped Oscillations
Half-Power Frequencies
Parallel Resonant Circuit
Quality Factor
Resonant Circuit
Selectivity Curve
Series Resonant Circuit

OUTLINE

Series Resonance
Quality Factor, *Q*
Impedance of a Series Resonant Circuit
Power, Bandwidth, and Selectivity of a Series Resonant Circuit
Series-to-Parallel *RL* and *RC* Conversion
Parallel Resonance
Circuit Analysis Using Computers

OBJECTIVES

After studying this chapter, you will be able to

- determine the resonant frequency and bandwidth of a simple series or parallel circuit,
- determine the voltages, currents, and power of elements in a resonant circuit,
- sketch the impedance, current, and power response curves of a series resonant circuit,
- find the quality factor, *Q*, of a resonant circuit and use *Q* to determine the bandwidth for a given set of conditions,
- explain the dependence of bandwidth on the *L*/*C* ratio and on *R* for both a series and a parallel resonant circuit,
- design a resonant circuit for a given set of parameters,
- convert a series *RL* network into an equivalent parallel network for a given frequency.

6

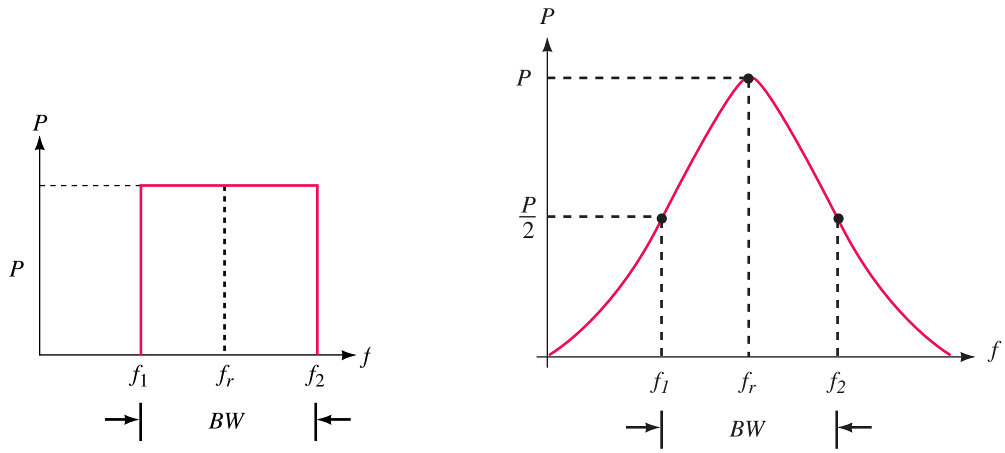
RESONANCE

CHAPTER PREVIEW

In this chapter, we build upon the knowledge obtained in previous chapters to observe how resonant circuits are able to pass a desired range of frequencies from a signal source to a load. In its most simple form, the **resonant circuit** consists of an inductor and a capacitor together with a voltage or current source. Although the circuit is simple, it is one of the most important circuits used in electronics. As an example, the resonant circuit, in one of its many forms, allows us to select a desired radio or television signal from the vast number of signals that are around us at any time. In order to obtain all the transmitted energy for a given radio station or television channel, we would like a circuit to have the frequency response shown in Figure 6–1(a). A circuit having an ideal frequency response would pass all frequency components in a band between f_1 and f_2, while rejecting all other frequencies. For a radio transmitter, the center frequency, f_r, would correspond to the carrier frequency of the station. The difference between the upper and lower frequencies that we would like to pass is called the **bandwidth.**

Whereas there are various configurations of resonant circuits, they all have several common characteristics. Resonant electronic circuits contain at least one inductor and one capacitor and have a bell-shaped response curve centered at some resonant frequency, f_r, as illustrated in Figure 6–1(b).

The response curve of Figure 6–1(b) indicates that power will be at a maximum at the resonant frequency, f_r. Varying the frequency in either direction results in a reduction of the power. The bandwidth of the resonant circuit is taken to be the difference between the half power points on the response curve of the filter.



(a) Ideal frequency response curve

(b) Actual response curve of a resonant circuit

FIGURE 6–1

If we were to apply variable-frequency sinusoidal signals to a circuit consisting of an inductor and capacitor, we would find that maximum energy will transfer back and forth between the two elements at the resonant frequency. In an ideal LC circuit (one containing no resistance), these oscillations would continue unabated even if the signal source were turned off. However, in the practical situation, all circuits have some resistance. As a result, the stored energy will eventually be dissipated by the resistance, resulting in **damped oscillations.** In a manner similar to pushing a child on a swing, the oscillations will continue indefinitely if a small amount of energy is applied to the circuit at exactly the right moment. This phenomenon illustrates the basis of how oscillator circuits operate and therefore provides us with another application of the resonant circuit.

In this chapter, we examine in detail the two main types of resonant circuits: the **series resonant circuit** and the **parallel resonant circuit.** ■

Putting It in Perspective

Edwin Howard Armstrong—Radio Reception

EDWIN ARMSTRONG WAS BORN in New York City on December 18, 1890. As a young man, he was keenly interested in experiments involving radio transmission and reception.

After earning a degree in electrical engineering at Columbia University, Armstrong used his theoretical background to explain and improve the operation of the triode vacuum tube, which had been invented by Lee de Forest. Edwin Armstrong was able to improve the sensitivity of receivers by using feedback to amplify a signal many times. By increasing the amount of signal feedback, Armstrong also designed and patented a circuit that used the vacuum tube as an oscillator.

Armstrong is best known for conceiving the concept of superheterodyning, in which a high frequency is lowered to a more usable intermediate frequency. Superheterodyning is still used in modern AM and FM receivers and in numerous other electronic circuits such as radar and communication equipment.

Edwin Armstrong was the inventor of FM transmission, which led to greatly improved fidelity in radio transmission.

Although Armstrong was a brilliant engineer, he was an uncompromising person who was involved in numerous lawsuits with Lee de Forest and the communications giant, RCA.

After spending nearly two million dollars in legal battles, Edwin Armstrong jumped to his death from his thirteenth-floor apartment window on January 31, 1954. ■

6.1 Series Resonance

A simple series resonant circuit is constructed by combining an ac source with an inductor, a capacitor, and optionally, a resistor as shown in Figure 6–2(a). By combining the generator resistance, R_G, with the series resistance, R_S, and the resistance of the inductor coil, R_{coil}, the circuit may be simplified as illustrated in Figure 6–2(b).

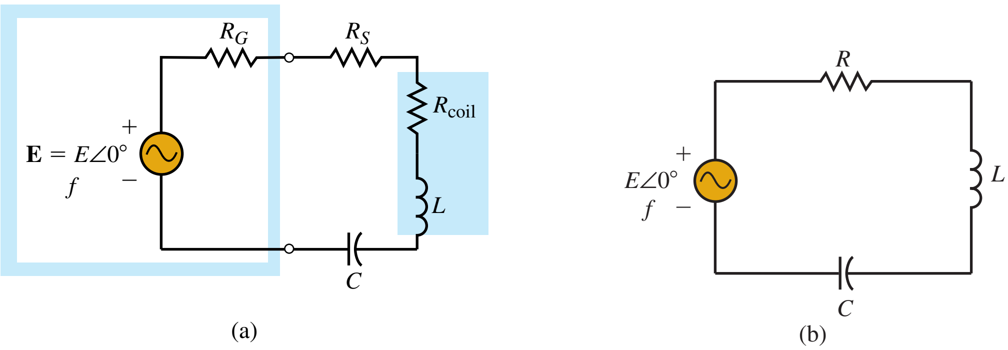


FIGURE 6–2

In this circuit, the total resistance is expressed as

$$R = R_G + R_S + R_{\text{coil}}$$

Because the circuit of Figure 6–2 is a series circuit, we calculate the total impedance as follows:

$$\begin{aligned}\mathbf{Z}_{\mathrm{T}} &= R + jX_L - jX_C \\ &= R + j(X_L - X_C)\end{aligned} \tag{6–1}$$

Resonance occurs when the reactance of the circuit is effectively eliminated, resulting in a total impedance that is purely resistive. We know that the reactances of the inductor and capacitor are given as follows:

$$X_L = \omega L = 2\pi f L \tag{6–2}$$

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C} \tag{6–3}$$

Examining Equation 6–1, we see that by setting the reactances of the capacitor and inductor equal to one another, the total impedance, $\mathbf{Z}_{\mathrm{T}}$, is purely resistive since the inductive reactance which is on the positive j axis cancels the capacitive reactance on the negative j axis. The total impedance of the series circuit at resonance is equal to the total circuit resistance, R. Hence, at resonance,

$$Z_{\mathrm{T}} = R \tag{6–4}$$

By letting the reactances be equal we are able to determine the series resonance frequency ω_{S} (in radians per second) as follows:

$$\begin{aligned}\omega L &= \frac{1}{\omega C} \\ \omega^2 &= \frac{1}{LC} \\ \omega_{\mathrm{S}} &= \frac{1}{\sqrt{LC}} \quad \text{(rad/s)}\end{aligned} \tag{6–5}$$

Since the calculation of the angular frequency, ω, in radians per second is easier than solving for frequency, f, in hertz, we generally express our resonant frequencies in the simpler form. Further calculations of voltage and current will usually be much easier by using ω rather than f. If, however, it becomes necessary to determine a frequency in hertz, recall that the relationship between ω and f is as follows:

$$\omega = 2\pi f \quad \text{(rad/s)} \tag{6–6}$$

Equation 6–6 is inserted into Equation 6–5 to give the resonant frequency as

$$f_{\mathrm{S}} = \frac{1}{2\pi\sqrt{LC}} \quad \text{(Hz)} \tag{6–7}$$

The subscript S in the preceding equations indicates that the frequency determined is the series resonant frequency.

At resonance, the total current in the circuit is determined from Ohm's law as

$$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_{\mathrm{T}}} = \frac{E\angle 0^\circ}{R\angle 0^\circ} = \frac{E}{R}\angle 0^\circ \tag{6–8}$$

By again applying Ohm's law, we find the voltage across each of the elements in the circuit as follows:

$$\mathbf{V}_R = IR\angle 0^\circ \tag{6–9}$$

$$\mathbf{V}_L = IX_L\angle 90^\circ \tag{6–10}$$

$$\mathbf{V}_C = IX_C\angle -90^\circ \tag{6–11}$$

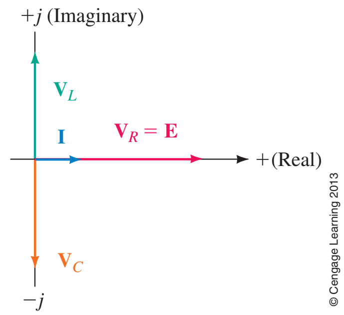


FIGURE 6–3

The phasor form of the voltages and current is shown in Figure 6–3. Notice that since the inductive and capacitive reactances have the same magnitude, the voltages across the elements must have the same magnitude but be 180° out of phase. We determine the average power dissipated by the resistor and the reactive powers of the inductor and capacitor as follows:

$$P_R = I^2R \quad \text{(W)}$$

$$Q_L = I^2X_L \quad \text{(VAR)}$$

$$Q_C = I^2X_C \quad \text{(VAR)}$$

These powers are illustrated graphically in Figure 6–4.

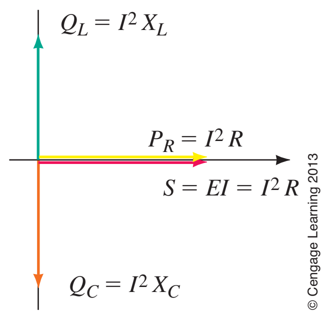


FIGURE 6–4

6.2 Quality Factor, Q

For any resonant circuit, we define the **quality factor,** Q, as the ratio of reactive power to average power, namely,

$$Q = \frac{\text{reactive power}}{\text{average power}} \tag{6–12}$$

Because the reactive power of the inductor is equal to the reactive power of the capacitor at resonance, we may express Q in terms of either reactive power. Consequently, the preceding expression is written as follows:

$$Q_S = \frac{I^2X_L}{I^2R}$$

and so we have

$$Q_S = \frac{X_L}{R} = \frac{\omega L}{R} \tag{6–13}$$

Quite often, the inductor of a given circuit will have a Q expressed in terms of its reactance and internal resistance, as follows:

$$Q_{\text{coil}} = \frac{X_L}{R_{\text{coil}}}$$

If an inductor with a specified Q_{coil} is included in a circuit, it is necessary to include its effects in the overall calculation of the total circuit Q.

We now examine how the Q of a circuit is used in determining other quantities of the circuit. By multiplying both the numerator and denominator of Equation 6–13 by the current, I, we have the following:

$$Q_S = \frac{IX_L}{IR} = \frac{V_L}{E} \quad \textbf{(6–14)}$$

Now, since the magnitude of the voltage across the capacitor is equal to the magnitude of the voltage across the inductor at resonance, we see that the voltages across the inductor and capacitor are related to the Q by the following expression:

$$V_C = V_L = Q_S E \quad \text{at resonance} \quad \textbf{(6–15)}$$

Note: Since the Q of a resonant circuit is generally significantly larger than 1, we see that the voltage across reactive elements can be many times greater than the applied source voltage. Therefore, it is always necessary to ensure that the reactive elements used in a resonant circuit are able to handle the expected voltages and currents.

EXAMPLE 6–1

Find the indicated quantities for the circuit of Figure 6–5.

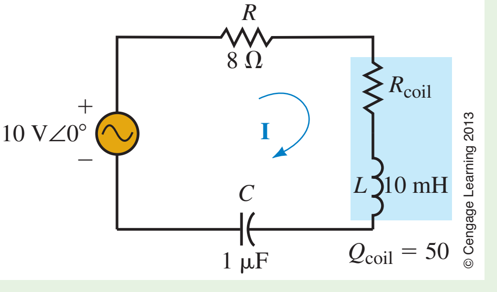


FIGURE 6–5

a. Resonant frequency expressed as ω(rad/s) and f(Hz).
b. Total impedance at resonance.
c. Current at resonance.
d. $\mathbf{V}_L$ and $\mathbf{V}_C$.
e. Reactive powers, Q_C and Q_L.
f. Quality factor of the circuit, Q_S.

Solution

a.
$$\omega_S = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(10\text{ mH})(1\mu\text{F})}} = 10\,000\text{ rad/s}$$

$$f_S = \frac{\omega}{2\pi} = 1592\text{ Hz}$$

b.
$$X_L = \omega L = (10\,000\text{ rad/s})(10\text{ mH}) = 100\ \Omega$$

$$R_{coil} = \frac{X_L}{Q_{coil}} = \frac{100\ \Omega}{50} = 2.00\ \Omega$$

$$R_T = R + R_{coil} = 10.0\ \Omega$$

$$\mathbf{Z}_T = 10\ \Omega\angle 0°$$

c. $$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}_T} = \frac{10\ \text{V}\angle 0°}{10\ \Omega\angle 0°} = 1.0\ \text{A}\angle 0°$$

d. $$\mathbf{V}_L = (100\ \Omega\angle 90°)(1.0\ \text{A}\angle 0°) = 100\ \text{V}\angle 90°$$

$$\mathbf{V}_C = (100\ \Omega\angle -90°)(1.0\ \text{A}\angle 0°) = 100\ \text{V}\angle -90°$$

Notice that the voltage across the reactive elements is 10 times greater than the applied signal voltage.

e. Although we use the symbol Q to designate both reactive power and the quality factor, the context of the question generally provides us with a clue as to which meaning to use.

$$Q_L = (1.0\ \text{A})^2(100\ \Omega) = 100\ \text{VAR}$$

$$Q_C = (1.0\ \text{A})^2(100\ \Omega) = 100\ \text{VAR}$$

f. $$Q_S = \frac{Q_L}{P} = \frac{100\ \text{VAR}}{10\ \text{W}} = 10$$

PRACTICE PROBLEMS 1

Consider the circuit of Figure 6–6:

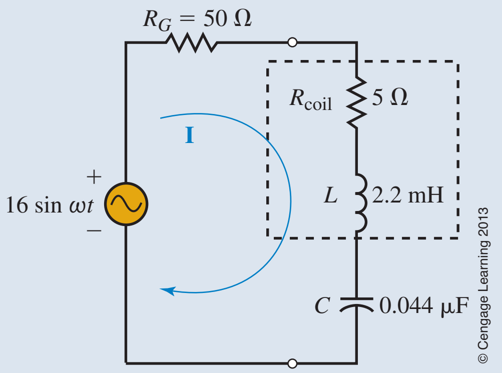


FIGURE 6–6

CircuitSim 6-1

a. Find the resonant frequency expressed as ω(rad/s) and f(Hz).

b.D etermine the total impedance at resonance.

c. Solve for **I**, $\mathbf{V}_L$, and $\mathbf{V}_C$ at resonance.

d. Calculate reactive powers Q_C and Q_L at resonance.

e. Find the quality factor, Q_s, of the circuit.

Answers

a. 102 krad/s, 16.2 kHz; b. 55.0 Ω∠0°

c. 0.206 A∠0°, 46.0 V∠90°, 46.0 V∠−90°; d. 9.46 VAR; e. 4.07

6.3 Impedance of a Series Resonant Circuit

In this section, we examine how the impedance of a series resonant circuit varies as a function of frequency. Because the impedances of inductors and capacitors are dependent upon frequency, the total impedance of a series resonant circuit must similarly vary with frequency. For algebraic simplicity, we use frequency expressed as ω in radians per second. If it becomes necessary to express the frequency in hertz, the conversion of Equation 6–6 is used.

The total impedance of a simple series resonant circuit is written as

$$\begin{aligned}\mathbf{Z}_{\mathrm{T}} &= R + j\omega L - j\frac{1}{\omega C} \\ &= R + j\left(\frac{\omega^2 LC - 1}{\omega C}\right)\end{aligned}$$

The magnitude and phase angle of the impedance vector, $\mathbf{Z}_{\mathrm{T}}$, are expressed as follows:

$$Z_{\mathrm{T}} = \sqrt{R^2 + \left(\frac{\omega^2 LC - 1}{\omega C}\right)^2} \qquad \textbf{(6–16)}$$

$$\theta = \tan^{-1}\left(\frac{\omega^2 LC - 1}{\omega RC}\right) \qquad \textbf{(6–17)}$$

Examining these equations for various values of frequency, we note that the following conditions will apply:

When $\omega = \omega_{\mathrm{S}}$:

$$Z_{\mathrm{T}} = R$$

and

$$\theta = \tan^{-1}0 = 0°$$

This result is consistent with the results obtained in the previous section.

When $\omega < \omega_{\mathrm{S}}$:

As we decrease ω from resonance, Z_{T} will get larger until $\omega = 0$. At this point, the magnitude of the impedance will be undefined, corresponding to an open circuit. As one might expect, the large impedance occurs because the capacitor behaves like an open circuit at dc.

The angle θ will occur between 0° and −90° since the numerator of the argument of the arctangent function will always be negative, corresponding to an angle in the fourth quadrant. Because the angle of the impedance has a negative sign, we conclude that the impedance must appear capacitive in this region.

When $\omega > \omega_{\mathrm{S}}$:

As ω is made larger than resonance, the impedance Z_{T} will increase due to the increasing reactance of the inductor.

For these values of ω, the angle θ will always be within 0° and +90° because both the numerator and the denominator of the arctangent function are positive. Because the angle of $\mathbf{Z}_{\mathrm{T}}$ occurs in the first quadrant, the impedance must be inductive.

Sketching the magnitude and phase angle of the impedance $\mathbf{Z}_{\mathrm{T}}$ as a function of angular frequency, we have the curves shown in Figure 6–7.

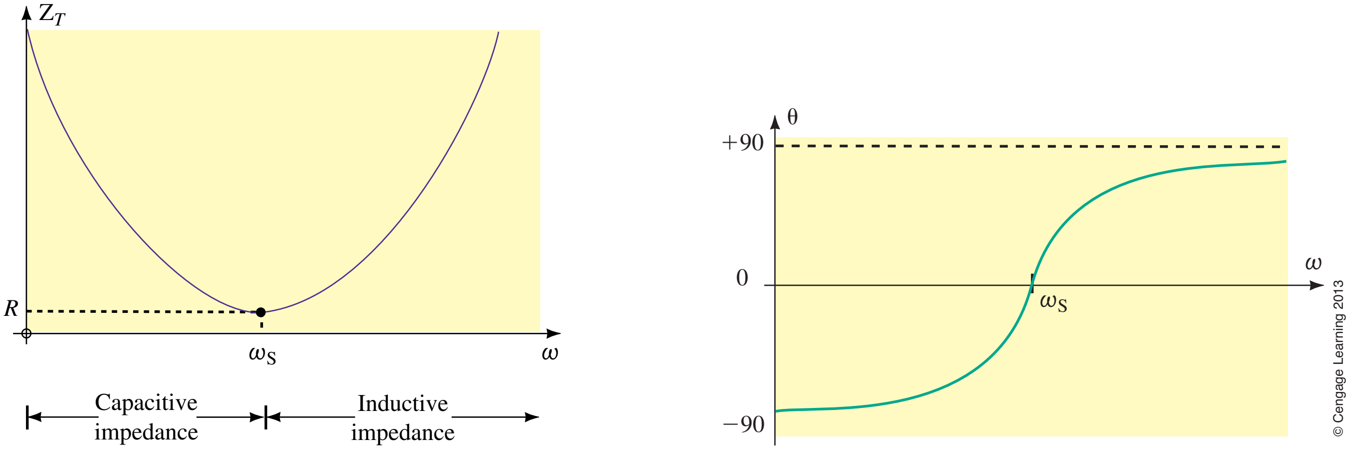


FIGURE 6–7 Impedance (magnitude and phase angle) versus angular frequency for a series resonant circuit.

6.4 Power, Bandwidth, and Selectivity of a Series Resonant Circuit

Due to the changing impedance of the circuit, we conclude that if a constant-amplitude voltage is applied to the series resonant circuit, the current and power of the circuit will not be constant at all frequencies. In this section, we examine how current and power are affected by changing the frequency of the voltage source.

Applying Ohm's law gives the magnitude of the current at resonance as follows:

$$I_{max} = \frac{E}{R} \tag{6–18}$$

For all other frequencies, the magnitude of the current will be less than I_{max} because the impedance is greater than at resonance. Indeed, when the frequency is zero (dc), the current will be zero since the capacitor is effectively an open circuit. On the other hand, at increasingly higher frequencies, the inductor begins to approximate an open circuit, once again causing the current in the circuit to approach zero. The current response curve for a typical series resonant circuit is shown in Figure 6–8.

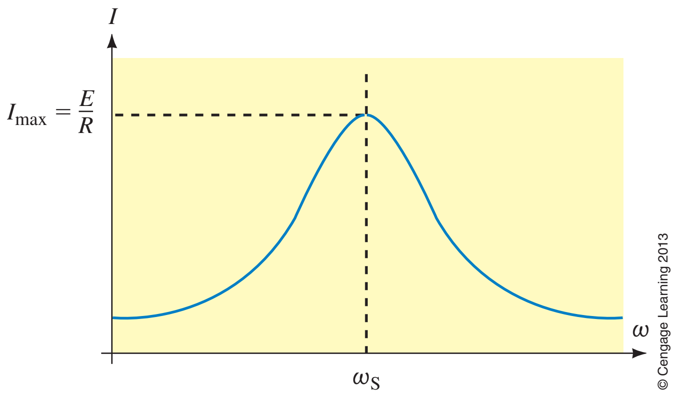


FIGURE 6–8 Current versus angular frequency for a series resonant circuit.

The total power dissipated by the circuit at any frequency is given as

$$P = I^2R \tag{6–19}$$

NOTES...

For Further Investigation

We see from Figure 6–9 that the selectivity curve is not perfectly symmetrical on both sides of the resonant frequency. As a result, ω_s is not exactly centered between the half-power frequencies. However, as Q increases, we find that the resonant frequency approaches the midpoint between ω_1 and ω_2. *In general, if* $Q > 10$, *then we assume that the resonant frequency is at the midpoint of half-power frequencies.* For additional information, refer to the Web site for this textbook. Go to www.cengagebrain.com and log in by entering your user name and password. Follow the links to *For Further Investigation* and select *The Importance of Selectivity in Broadcast Radio.*

Since the current is maximum at resonance, it follows that the power must similarly be maximum at resonance. The maximum power dissipated by the series resonant circuit is therefore given as

$$P_{\max} = I^2_{\max}R = \frac{E^2}{R} \tag{6–20}$$

The power response of a series resonant circuit has a bell-shaped curve called the **selectivity curve,** which is similar to the current response. Figure 6–9 illustrates the typical selectivity curve.

Examining Figure 6–9, we see that only frequencies around ω_s will permit significant amounts of power to be dissipated by the circuit. We define the bandwidth, BW, of the resonant circuit to be the difference between the frequencies at which the circuit delivers half of the maximum power. The frequencies ω_1 and ω_2 are called the **half-power frequencies,** the cutoff frequencies, or the band frequencies.

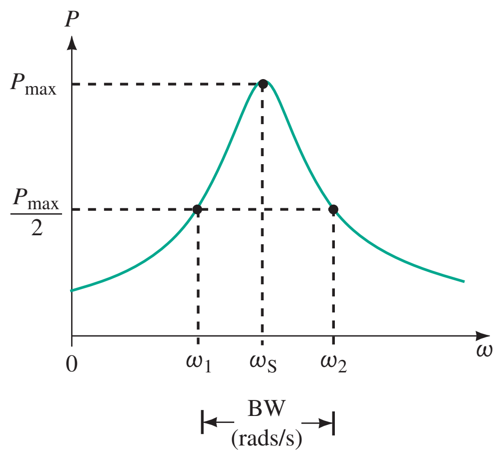


FIGURE 6–9 Selectivity curve.

If the bandwidth of a circuit is kept very narrow, the circuit is said to have a high selectivity, since it is highly selective to signals occurring within a very narrow range of frequencies. On the other hand, if the bandwidth of a circuit is large, the circuit is said to have a low selectivity.

The elements of a series resonant circuit determine not only the frequency at which the circuit is resonant, but also the shape (and hence the bandwidth) of the power response curve. Consider a circuit in which the resistance, R, and the resonant frequency, ω_s, are held constant. We find that by increasing the ratio of L/C, the sides of the power response curve become steeper. This in turn results in a decrease in the bandwidth. Inversely, decreasing the ratio of L/C causes the sides of the curve to become more gradual, resulting in an increased bandwidth. These characteristics are illustrated in Figure 6–10.

If, on the other hand, L and C are kept constant, we find that the bandwidth will decrease as R is decreased and will increase as R is increased. Figure 6–11 shows how the shape of the selectivity curve is dependent upon the value of resistance. A series circuit has the highest selectivity if the resistance of the circuit is kept to a minimum.

For the series resonant circuit the power at any frequency is determined as

$$P = I^2R$$
$$= \left(\frac{E}{Z_T}\right)^2 R$$

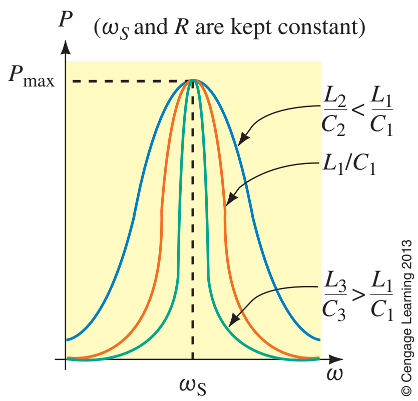


FIGURE 6–10

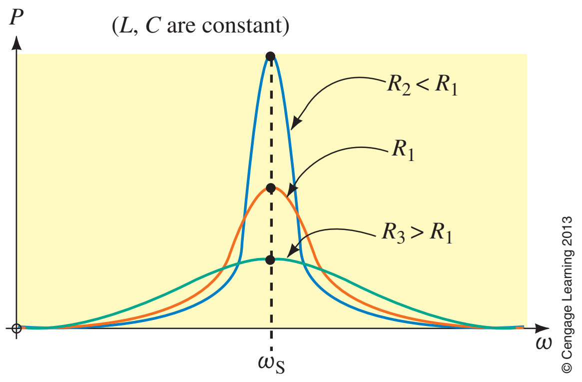


FIGURE 6–11

By substituting Equation 6–16 into the preceding expression, we arrive at the general expression for power as a function of frequency, ω:

$$P = \frac{E^2R}{R^2 + \left(\frac{\omega^2LC - 1}{\omega C}\right)^2} \qquad \textbf{(6–6)}$$

At the half-power frequencies, the power must be

$$P_{\text{hpf}} = \frac{E^2}{2R} \qquad \textbf{(6–22)}$$

Since the maximum current in the circuit is given as $I_{\max} = E/R$, we see that by manipulating the preceding expression, the magnitude of current at the half-power frequencies is

$$\begin{aligned} I_{\text{hpf}} &= \sqrt{\frac{P_{\text{hpf}}}{R}} = \sqrt{\frac{E^2}{2R^2}} = \sqrt{\frac{I^2_{\max}}{2}} \\ I_{\text{hpf}} &= \frac{I_{\max}}{\sqrt{2}} \end{aligned} \qquad \textbf{(6–23)}$$

The cutoff frequencies are found by evaluating the frequencies at which the power dissipated by the circuit is half of the maximum power. Combining Equations 6–6 and 6–22, we have the following:

$$\begin{aligned} \frac{E^2}{2R} &= \frac{E^2R}{R^2 + \left(\frac{\omega^2LC - 1}{\omega C}\right)^2} \\ 2R^2 &= R^2 + \left(\frac{\omega^2LC - 1}{\omega C}\right)^2 \\ \frac{\omega^2LC - 1}{\omega C} &= \pm R \\ \omega^2LC - 1 &= \pm\omega RC \quad \text{(at half-power)} \end{aligned} \qquad \textbf{(6–24)}$$

From the selectivity curve for a series circuit, we see that the two half-power points occur on both sides of the resonant angular frequency, ω_s.

When $\omega < \omega_S$, the term ω^2LC must be less than 1. In this case, the solution is determined as follows:

$$\omega^2LC - 1 = -\omega RC$$
$$\omega^2LC + \omega RC - 1 = 0$$

The solution of this quadratic equation gives the lower half-power frequency as

$$\omega_1 = \frac{-RC + \sqrt{(RC)^2 + 4LC}}{2LC}$$

or

$$\omega_1 = \frac{-R}{2L} + \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}} \qquad \textbf{(6–25)}$$

In a similar manner, for $\omega > \omega_S$, the upper half-power frequency is

$$\omega_2 = \frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}} \qquad \textbf{(6–26)}$$

Taking the difference between Equations 6–26 and 6–25, we find the bandwidth of the circuit as

$$\begin{aligned} \text{BW} &= \omega_2 - \omega_1 \\ &= \frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}} - \left(-\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}}\right) \end{aligned}$$

which gives

$$\text{BW} = \frac{R}{L} \quad \text{(rad/s)} \qquad \textbf{(6–27)}$$

If the preceding expression is multiplied by ω_S/ω_S we obtain

$$\text{BW} = \frac{\omega_S R}{\omega_S L}$$

and since $Q_S = \omega_S L/R$ we further simplify the bandwidth as

$$\text{BW} = \frac{\omega_S}{Q_S} \quad \text{(rad/s)} \qquad \textbf{(6–28)}$$

Because the bandwidth may alternately be expressed in hertz, the preceding expression is equivalent to having

$$\text{BW} = \frac{f_S}{Q_S} \quad \text{(Hz)} \qquad \textbf{(6–29)}$$

EXAMPLE 6–2

Refer to the circuit of Figure 6–12.

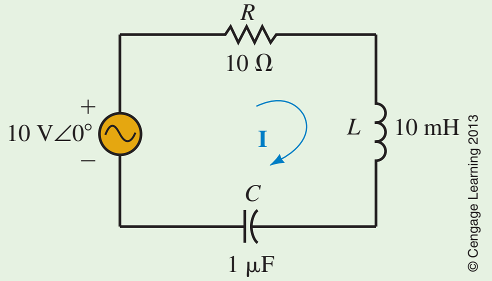


FIGURE 6–12

a. Determine the maximum power dissipated by the circuit.

b. Use the results obtained from Example 21–1 to determine the bandwidth of the resonant circuit and to arrive at the approximate half-power frequencies, ω_1 and ω_2.

c. Calculate the actual half-power frequencies, ω_1 and ω_2, from the given component values. Show two decimal places of precision.

d. Solve for the circuit current, I, and power dissipated at the lower half-power frequency, ω_1, found in part (c).

Solution

a. $$P_{\text{max}} = \frac{E^2}{R} = 10.0 \text{ W}$$

b. From Example 21–1, we had the following circuit characteristics:

$$Q_S = 10,\ \omega_S = 10 \text{ krad/s}$$

The bandwidth of the circuit is determined to be

$$\text{BW} = \omega_S Q_S = 1.0 \text{ krad/s}$$

If the resonant frequency were centered in the bandwidth, then the half-power frequencies occur at approximately

$$\omega_1 = 9.50 \text{ krad/s}$$

and

$$\omega_2 = 10.50 \text{ krad/s}$$

c. $$\begin{aligned}\omega_1 &= -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}} \\ &= -\frac{10\ \Omega}{(2)(10 \text{ mH})} + \sqrt{\frac{(10\ \Omega)^2}{(4)(10 \text{ mH})^2} + \frac{1}{(10 \text{ mH})(1\ \mu\text{F})}} \\ &= -500 + 10\,012.49 = 9512.49 \text{ rad/s} \quad (f_1 = 1514.0 \text{ Hz})\end{aligned}$$

$$\begin{aligned}\omega_2 &= \frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} + \frac{1}{LC}} \\ &= 500 + 10\,012.49 = 10\,512.49 \text{ rad/s} \quad (f_2 = 1673.1 \text{ Hz})\end{aligned}$$

Notice that the actual half-power frequencies are very nearly equal to the approximate values. For this reason, if $Q \geq 10$, it is often sufficient to calculate the cutoff frequencies by using the easier approach of part (b).

d. At $\omega_1 = 9.51249$ krad/s, the reactances are as follows:

$$X_L = \omega L = (9.51249 \text{ krad/s})(10 \text{ mH}) = 95.12\ \Omega$$

$$X_C = \frac{1}{\omega C} = \frac{1}{(9.51249 \text{ krad/s})(1\ \mu\text{F})} = 105.12\ \Omega$$

The current is now determined to be

$$\begin{aligned}\mathbf{I} &= \frac{10 \text{ V}\angle 0^\circ}{10\ \Omega + j95.12\ \Omega - j105.12\ \Omega} \\ &= \frac{10 \text{ V}\angle 0^\circ}{14.14\ \Omega\angle -45^\circ} \\ &= 0.707 \text{ A}\angle 45^\circ\end{aligned}$$

and the power is given as

$$P = I^2 R = (0.707 \text{ A})^2(10 \text{ V}) = 5.0 \text{ W}$$

As expected, we see that the power at the frequency ω_1 is indeed equal to half of the power dissipated by the circuit at resonance.

EXAMPLE 6–3

Refer to the circuit of Figure 6–13.

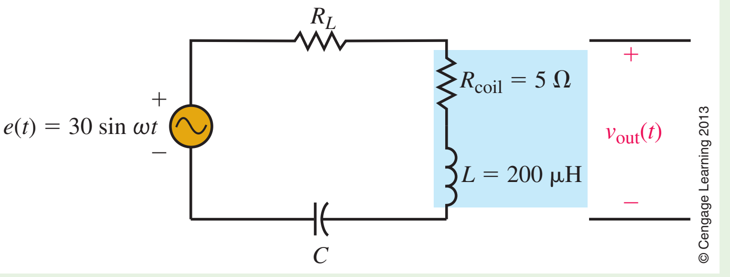


FIGURE 6–13

a. Calculate the values of R_L and C for the circuit to have a resonant frequency of 200 kHz and a bandwidth of 16 kHz.
b. Use the designed component values to determine the power dissipated by the circuit at resonance.
c. Solve for $v_{out}(t)$ at resonance.

Solution

a. Because the circuit is at resonance, we must have the following conditions:

$$Q_S = \frac{f_S}{\text{BW}} = \frac{200\text{ kHz}}{16\text{ kHz}} = 12.5$$

$$X_L = 2\pi fL = 2\pi(200\text{ kHz})(200\ \mu\text{H}) = 251.3\ \Omega$$

$$R = R_L + R_{coil} = \frac{X_L}{Q_s} = 20.1\ \Omega$$

and so R_L must be

$$R_L = 20.1\ \Omega - 5\ \Omega = 15.1\ \Omega$$

Since $X_C = X_L$, we determine the capacitance as

$$C = \frac{1}{2\pi f X_C} = \frac{1}{2\pi(200\text{ kHz})(251.3\ \Omega)} = 3.17\text{ nF } (\equiv 0.00317\ \mu\text{F})$$

b. The power at resonance is found from Equation 21–20 as

$$P_{max} = \frac{E^2}{R} = \frac{\left(\frac{30\text{ V}}{\sqrt{2}}\right)^2}{20.1\ \Omega} = 22.4\text{ W}$$

c. We see from the circuit of Figure 6–13 that the voltage $v_{out}(t)$ may be determined by applying the voltage divider rule to the circuit. However, we must first convert the source voltage from time domain into phasor domain as follows:

$$e(t) = 30 \sin \omega t \Leftrightarrow \mathbf{E} = 6.6 \text{ V}\angle 0°$$

Now, applying the voltage divider rule to the circuit, we have

$$\begin{aligned}\mathbf{V}_{out} &= \frac{(R_1 + j\omega L)}{R}\mathbf{E} \\ &= \frac{(5\ \Omega + j251.3\ \Omega)}{20.1\ \Omega} 6.6 \text{ V}\angle 0° \\ &= (251.4\ \Omega\angle 88.86°)(1.056 \text{ A}\angle 0°) \\ &= 265.5 \text{ V}\angle 88.86°\end{aligned}$$

which in time domain is given as

$$v_{out}(t) = 375 \sin(\omega t + 88.86°)$$

PRACTICE PROBLEMS 2

Refer to the circuit of Figure 6–14.

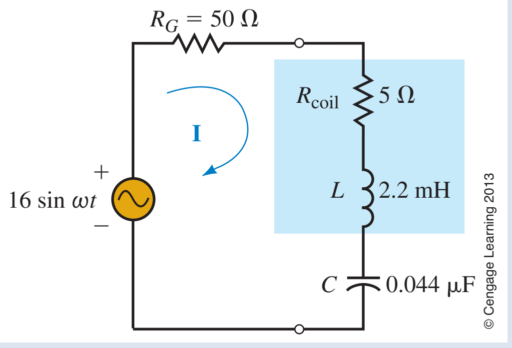


FIGURE 6–14

CircuitSim 6-2

a. Determine the maximum power dissipated by the circuit.
b. Use the results obtained from Practice Problems 1 to determine the bandwidth of the resonant circuit. Solve for the approximate values of the half-power frequencies, ω_1 and ω_2.
c. Calculate the actual half-power frequencies, ω_1 and ω_2, from the given component values. Compare your results to those obtained in part (b). Briefly explain why there is a discrepancy between the results.
d. Solve for the circuit current, **I**, and power dissipated at the lower half-power frequency, ω_1, found in part (c).

Answers

a. 2.33 W

b. BW = 25.0 krad/s (3.98 kHz), $\omega_1 \cong 89.1$ krad/s, $\omega_2 \cong 114.1$ krad/s

c. $\omega_1 = 89.9$ krad/s, $\omega_2 \cong 114.9$ krad/s. The approximation assumes that the power-frequency curve is symmetrical around ω_S, which is not quite true.

d. $\mathbf{I} = 0.145 \text{ A}\angle 45°$, $P = 1.16$ W

IN-PROCESS LEARNING CHECK 1

(Answers are at the end of the chapter.)

Refer to the series resonant circuit of Figure 6–15.

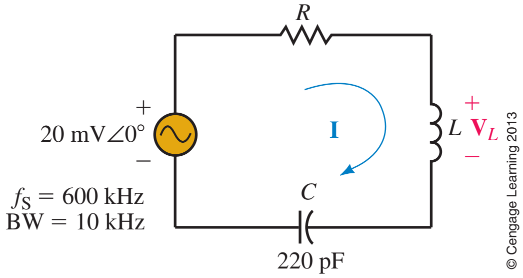


FIGURE 6–15

Suppose the circuit has a resonant frequency of 600 kHz and a bandwidth of 10 kHz.

a. Determine the value of inductor L in henries.
b. Calculate the value of resistor R in ohms.
c. Find $\mathbf{I}$, $\mathbf{V}_L$, and power, P, at resonance.
d. Find the approximate values of the half-power frequencies, f_1 and f_2.
e. Using the results of part (d), determine the current in the circuit at the lower half-power frequency, f_1, and show that the power dissipated by the resistor at this frequency is half the power dissipated at the resonant frequency.

IN-PROCESS LEARNING CHECK 2

(Answers are at the end of the chapter.)

Consider the series resonant circuit of Figure 6–16.

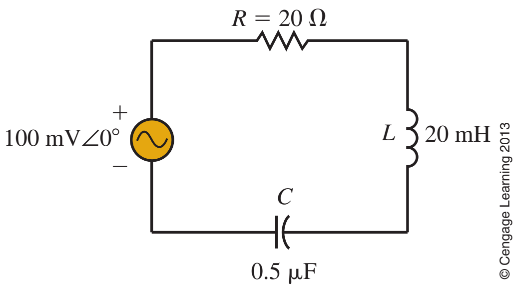


FIGURE 6–16

a. Solve for the resonant frequency of the circuit, ω_S, and calculate the power dissipated by the circuit at resonance.
b. Determine Q, BW, and the half-power frequencies, ω_1 and ω_2, in radians per second.
c. Sketch the selectivity curve of the circuit, showing P (in watts) versus ω (in radians per second).
d. Repeat parts (a) through (c) if the value of resistance is reduced to 10 Ω.
e. Explain briefly how selectivity depends upon the value of resistance in a series resonant circuit.

6.5 Series-to-Parallel *RL* and *RC* Conversion

As we have already seen, an inductor will always have some series resistance due to the length of wire used in the coil winding. Even though the resistance of the wire is generally small in comparison with the reactances in the circuit, this resistance may occasionally contribute tremendously to the overall circuit response of a parallel resonant circuit. We begin by converting the series *RL* network as shown in Figure 6–17 into an equivalent parallel *RL* network. It must be emphasized, however, that *the equivalence is only valid at a single frequency,* ω.

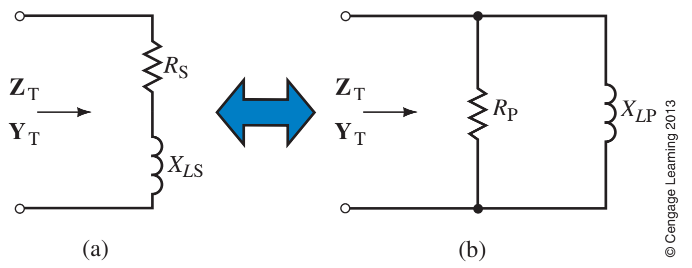


FIGURE 6–17

The networks of Figure 6–17 can be equivalent only if they each have the same input impedance, $\mathbf{Z}_T$ (and also the same input admittance, $\mathbf{Y}_T$).

The input impedance of the series network of Figure 6–17(a) is given as

$$\mathbf{Z}_T = R_S + jX_{LS}$$

which gives the input admittance as

$$\mathbf{Y}_T = \frac{1}{\mathbf{Z}_T} = \frac{1}{R_S + jX_{LS}}$$

Multiplying numerator and denominator by the complex conjugate, we have

$$\begin{aligned}\mathbf{Y}_T &= \frac{R_S - jX_{LS}}{(R_S + jX_{LS})(R_S - jX_{LS})} \\ &= \frac{R_S - jX_{LS}}{R_S^2 + X_{LS}^2} \\ &= \frac{R_S}{R_S^2 + X_{LS}^2} - j\frac{X_{LS}}{R_S^2 + X_{LS}^2}\end{aligned} \qquad \textbf{(6–30)}$$

From Figure 6–17(b), we see that the input admittance of the parallel network must be

$$\mathbf{Y}_T = G_P - jB_{LP}$$

which may also be written as

$$\mathbf{Y}_T = \frac{1}{R_P} - j\frac{1}{X_{LP}} \qquad \textbf{(6–31)}$$

The admittances of Equations 6–30 and 6–31 can only be equal if the realand the imaginary components are equal. As a result, we see that for a given frequency, the following equations enable us to convert a series *RL* network into its equivalent parallel network:

$$R_{\mathrm{P}} = \frac{R_{\mathrm{S}}^2 + X_{LS}^2}{R_{\mathrm{S}}} \tag{6–32}$$

$$X_{LP} = \frac{R_{\mathrm{S}}^2 + X_{LS}^2}{X_{LS}} \tag{6–33}$$

If we are given a parallel *RL* network, it is possible to show that the conversion to an equivalent series network is accomplished by applying the following equations:

$$R_{\mathrm{S}} = \frac{R_{\mathrm{P}} X_{LP}^2}{R_{\mathrm{P}}^2 + X_{LP}^2} \tag{6–34}$$

$$X_{LS} = \frac{R_{\mathrm{P}}^2 X_{LP}}{R_{\mathrm{P}}^2 + X_{LP}^2} \tag{6–35}$$

The derivation of the preceding equations is left as an exercise for the student. Equations 6–32 to 6–35 may be simplified by using the quality factor of the coil. Multiplying Equation 6–32 by $R_{\mathrm{S}}/R_{\mathrm{S}}$ and then using Equation 6–13, we have

$$R_{\mathrm{P}} = R_{\mathrm{S}} \frac{R_{\mathrm{S}}^2 + X_{LS}^2}{R_{\mathrm{S}}^2}$$

$$R_{\mathrm{P}} = R_S(1 + Q^2) \tag{6–36}$$

Similarly, Equation 6–33 is simplified as

$$X_{LP} = X_{LS} \frac{R_{\mathrm{S}}^2 + X_{LS}^2}{X_{LS}^2}$$

$$X_{LP} = X_{LS}\left(1 + \frac{1}{Q^2}\right) \tag{6–37}$$

The quality factor of the resulting parallel network must be the same as for the original series network because the reactive and the average powers must be the same. Using the parallel elements, the quality factor is expressed as

$$Q = \frac{X_{L\,\mathrm{S}}}{R_{\mathrm{S}}} = \frac{\left(\dfrac{R_{\mathrm{P}}^2 X_{LP}}{R_{\mathrm{P}}^2 + X_{LP}^2}\right)}{\left(\dfrac{R_{\mathrm{P}} X_{LP}^2}{R_{\mathrm{P}}^2 + X_{LP}^2}\right)}$$

$$= \frac{R_{\mathrm{P}}^2 X_{LP}}{R_{\mathrm{P}} X_{LP}^2}$$

$$Q = \frac{R_{\mathrm{P}}}{X_{LP}} \tag{6–38}$$

EXAMPLE 6–4

For the series network of Figure 6–18, find the Q of the coil at $\omega = 1000$ rad/s and convert the series *RL* network into its equivalent parallel network. Repeat these steps for $\omega = 10$ krad/s.

Solution

For $\omega = 1000$ rad/s,

$$X_L = \omega L = 20\ \Omega$$
$$Q = \frac{X_L}{R} = 2.0$$
$$R_P = R(1 + Q^2) = 50\ \Omega$$
$$X_{LP} = X_L\left(1 + \frac{1}{Q^2}\right) = 25\ \Omega$$

The resulting parallel network for $\omega = 1000$ rad/s is shown in Figure 6–19.

For $\omega = 10$ krad/s,

$$X_L = \omega L = 200\ \Omega$$
$$Q = \frac{X_L}{R} = 20$$
$$R_P = R(1 + Q^2) = 4010\ \Omega$$
$$X_{LP} = X_L\left(1 + \frac{1}{Q^2}\right) = 200.5\ \Omega$$

The resulting parallel network for $\omega = 10$ krad/s is shown in Figure 6–20.

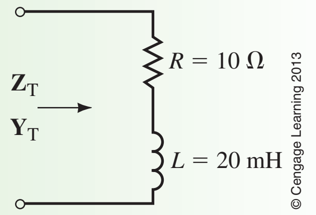


FIGURE 6–18

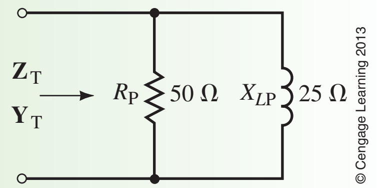


FIGURE 6–19

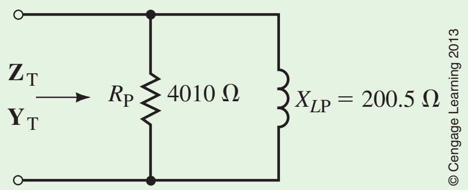


FIGURE 6–20

EXAMPLE 6–5

Find the Q of each of the networks of Figure 6–6 and determine the series equivalent for each.

Solution For the network of Figure 6–6(a),

$$Q = \frac{R_P}{X_{LP}} = \frac{10\ \text{k}\Omega}{250\ \Omega} = 40$$
$$R_S = \frac{R_P}{1 + Q^2} = \frac{10\ \text{k}\Omega}{1 + 40^2} = 6.25\ \Omega$$
$$X_{LS} = QR_S = (40)(6.25\ \Omega) = 250\ \Omega$$
$$L = \frac{X_L}{\omega} = \frac{250\ \Omega}{2500\ \text{rad/s}} = 0.1\ \text{H}$$

For the network of Figure 6–6(b),

$$Q = \frac{R_P}{X_{LP}} = \frac{2700\ \Omega}{900\ \Omega} = 3$$
$$R_S = \frac{R_P}{1 + Q^2} = \frac{2700\ \Omega}{1 + 3^2} = 270\ \Omega$$

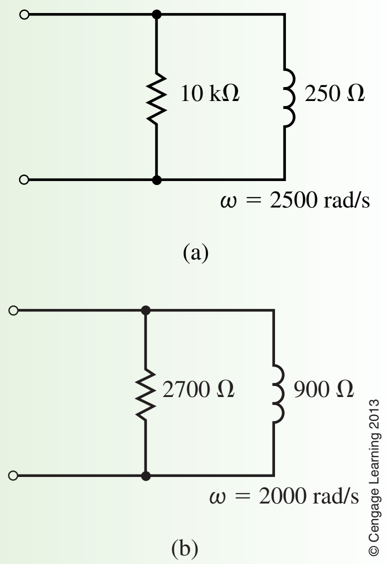


FIGURE 6 –21

$$X_{LS} = QR_S = (3)(270\ \Omega) = 810\ \Omega$$

$$L = \frac{X_L}{\omega} = \frac{810\ \Omega}{2000\ \text{rad/s}} = 0.405\ \text{H}$$

The resulting equivalent series networks are shown in Figure 6–22.

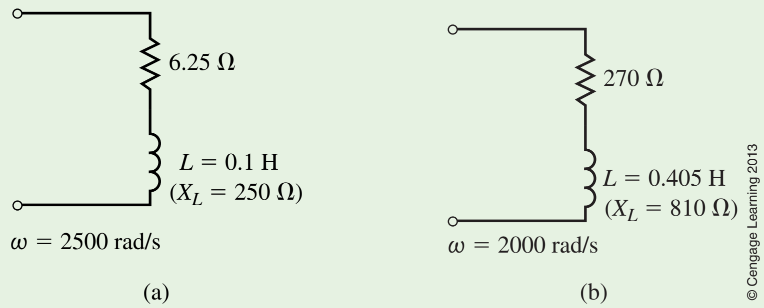


FIGURE 6–22

PRACTICE PROBLEMS 3

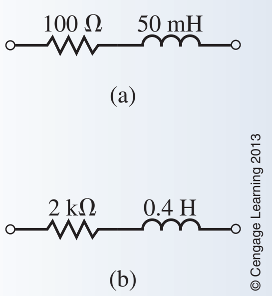


FIGURE 21–23

Refer to the networks of Figure 6–23.

a. Find the quality factors, Q, of the networks at $\omega_1 = 5$ krad/s.

b.U se the Q to find the equivalent parallel networks (resistance and reactance) at an angular frequency of $\omega_1 = 5$ krad/s.

c. Repeat parts (a) and (b) for an angular frequency of $\omega_2 = 25$ krad/s.

Answers

a. $Q_a = 2.5$		$Q_b = 1.0$		
b.N etwork a:		$R_P = 725\ \Omega$	$X_{LP} = 290\ \Omega$	
	Network b:	$R_P = 4\ \text{k}\Omega$	$X_{LP} = 4\ \text{k}\Omega$	
c. Network a:		$Q_a = 12.5$	$R_P = 15.725\ \text{k}\Omega$	$X_{LP} = 1.258\ \text{k}\Omega$
	Network b:	$Q_b = 5$	$R_P = 52\ \text{k}\Omega$	$X_{LP} = 10.4\ \text{k}\Omega$

The previous examples illustrate two important points which are valid if the Q of the network is large ($Q \geq 10$).

1. The resistance of the parallel network is approximately Q^2 larger than the resistance of the series network.
2. The inductive reactances of the series and parallel networks are approximately equal. Hence

$$R_P \cong Q^2 R_S \quad (Q \geq 10) \tag{6–39}$$

$$X_{LP} \cong X_{LS} \quad (Q \geq 10) \tag{6–40}$$

Although we have performed conversions between series and parallel *RL* circuits, it is easily shown that if the reactive element is a capacitor, the conversions apply equally well. In all cases, the equations are simply changed by replacing the terms X_{LS} and X_{LP} with X_{CS} and X_{CP}, respectively. The Q of the network is determined by the ratios

$$Q = \frac{X_{CS}}{R_S} = \frac{R_P}{X_{CP}} \tag{6–41}$$

PRACTICE PROBLEMS 4

Consider the networks of Figure 6–24.

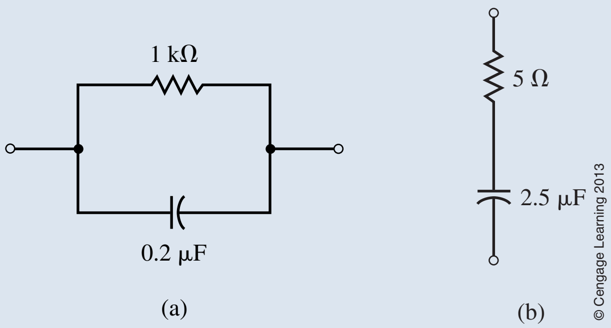


FIGURE 6–24

a. Find the Q of each network at a frequency of $f_1 = 1$ kHz.

b. Determine the series equivalent of the network in Figure 6–24(a) and the parallel equivalent of the network in Figure 6–24(b).

c. Repeat parts (a) and (b) for a frequency of $f_2 = 200$ kHz.

Answers

a.	$Q_a = 1.26$	$Q_b = 12.7$		
b.	Network a:	$R_S = 388\ \Omega$	$X_{CS} = 487\ \Omega$	
	Network b:	$R_P = 816\ \Omega$	$X_{CP} = 64.1\ \Omega$	
c.	Network a:	$Q_a = 251$	$R_S = 0.0158\ \Omega$	$X_{CS} = 3.98\ \Omega$
	Network b:	$Q_b = 0.0637$	$R_P = 5.02\ \Omega$	$X_{CP} = 78.9\ \Omega$

IN-PROCESS LEARNING CHECK 3

(Answers are at the end of the chapter.)

Refer to the networks of Figure 6–25.

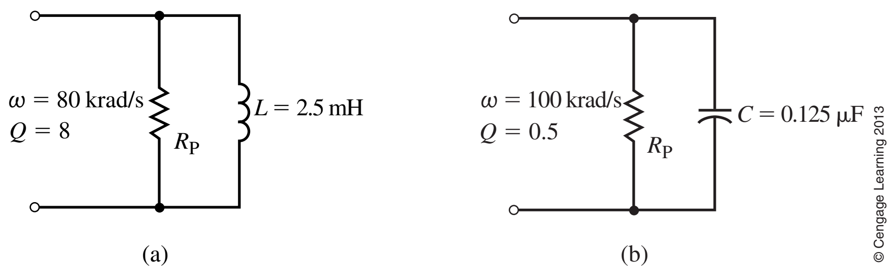


FIGURE 6–25

a. Determine the resistance, R_P, for each network.

b. Find the equivalent series network by using the quality factor for the given networks.

6.6 Parallel Resonance

A simple parallel resonant circuit is illustrated in Figure 6–26. The parallel resonant circuit is best analyzed using a constant-current source, unlike the series resonant circuit, which used a constant-voltage source.

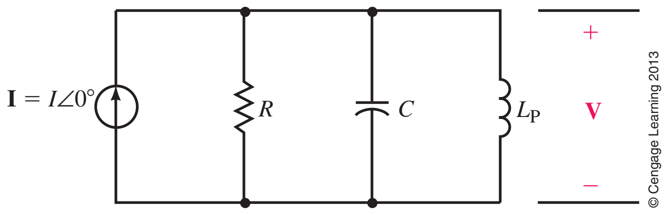


FIGURE 6–26 Simple parallel resonant circuit.

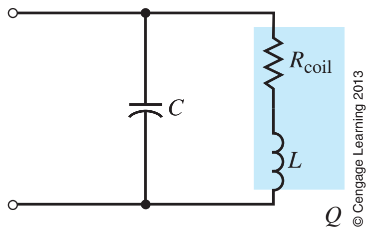


FIGURE 6–27

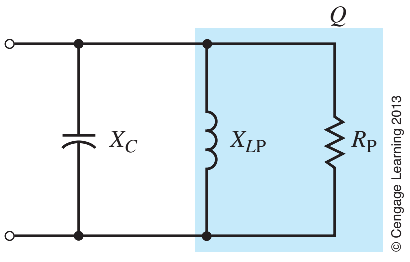


FIGURE 6–28

Consider the LC "tank" circuit shown in Figure 6–27. The tank circuit consists of a capacitor in parallel with an inductor. Due to its high Q and frequency response, the tank circuit is used extensively in communications equipment such as AM, FM, and television transmitters and receivers.

The circuit of Figure 6–27 is not exactly a parallel resonant circuit, since the resistance of the coil is in series with the inductance. In order to determine the frequency at which the circuit is purely resistive, we must first convert the series combination of resistance and inductance into an equivalent parallel network. The resulting circuit is shown in Figure 6–28.

At resonance, the capacitive and inductive reactances in the circuit of Figure 6–28 are equal. As we have observed previously, placing equal inductive and capacitive reactances in parallel effectively results in an open circuit at the given frequency. The input impedance of this network at resonance is therefore purely resistive and given as $Z_T = R_P$. We determine the resonant frequency of a tank circuit by first letting the reactances of the equivalent parallel circuit be equal:

$$X_C = X_{LP}$$

Now, using the component values of the tank circuit, we have

$$X_C = \frac{(R_{coil})^2 + X_L^2}{X_L}$$

$$\frac{1}{\omega C} = \frac{(R_{coil})^2 + (\omega L)^2}{\omega L}$$

$$\frac{L}{C} = (R_{coil})^2 + (\omega L)^2$$

which may be further reduced to

$$\omega = \sqrt{\frac{1}{LC} - \frac{R_{coil}^2}{L^2}}$$

Factoring $\sqrt{LC}$ from the denominator, we express the parallel resonant frequency as

$$\omega_P = \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{(R_{coil})^2 C}{L}} \qquad \textbf{(6–42)}$$

Notice that if $R_{coil}^2 < L/C$, then the term under the radical is approximately equal to 1.

Consequently, if $L/C \geq 100R_{coil}$, the parallel resonant frequency may be simplified as

$$\omega_P \cong \frac{1}{\sqrt{LC}} \qquad (\text{for } L/C \geq 100R_{coil}) \qquad \textbf{(6–43)}$$

NOTES...

For a high-Q circuit, ω_P can be approximated as $\omega_P = \frac{1}{\sqrt{LC}}$.

Recall that the quality factor, Q, of a circuit is defined as the ratio of reac-tive power to average power for a circuit at resonance. If we consider the par-allel resonant circuit of Figure 6–29, we make several important observations.

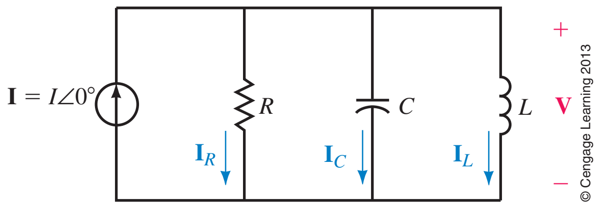


FIGURE 6–29

The inductor and capacitor reactances cancel, resulting in a circuit voltage simply determined by Ohm's law as

$$\mathbf{V} = \mathbf{I}R = IR\angle 0°$$

The frequency response of the impedance of the parallel circuit is shown in Figure 6–30.

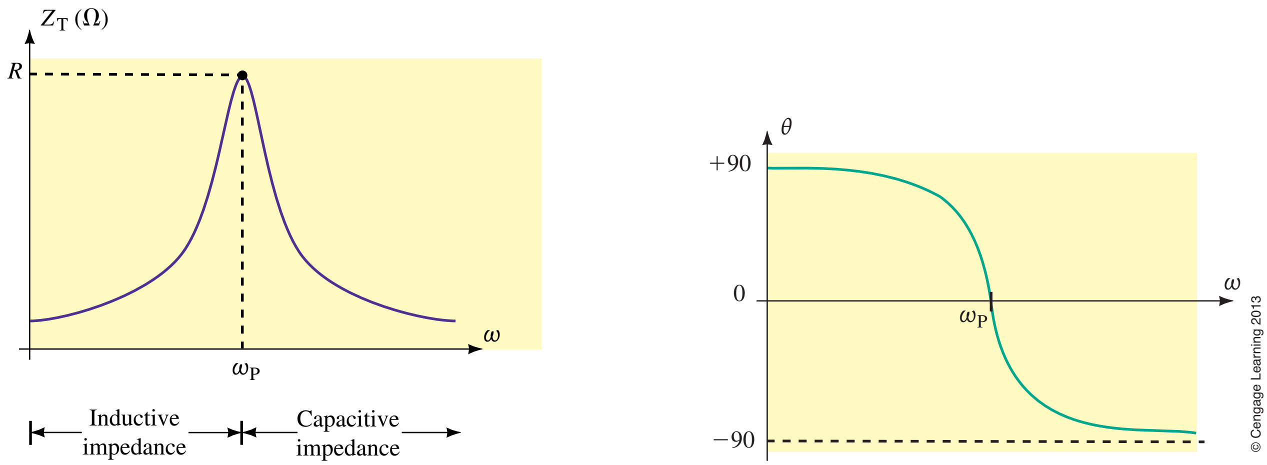


FIGURE 6–30 Impedance (magnitude and phase angle) versus angular frequency for a parallel resonant circuit.

Notice that the impedance of the entire circuit is maximum at resonance and minimum at the boundary conditions ($\omega = 0$ rad/s and $\omega \to \infty$). This result is exactly opposite to that observed in series resonant circuits, which have min-imum impedance at resonance. We also see that for parallel circuits, the imped-ance will appear inductive for frequencies less than the resonant frequency, ω_P. Inversely, the impedance is capacitive for frequencies greater than ω_P.

The Q of the parallel circuit is determined from the definition as

$$Q_P = \frac{\text{reactive power}}{\text{average power}} = \frac{V^2/X_L}{V^2/R} \qquad \textbf{(6–44)}$$

$$Q_P = \frac{R}{X_{LP}} = \frac{R}{X_C}$$

This is precisely the same result as that obtained when we converted an *RL* series network into its equivalent parallel network. If the resistance of the coil is the only resistance within a circuit, then the circuit Q will be equal to the Q of the coil. However, if the circuit has other sources of resistance, then the additional resistance will reduce the circuit Q.

For a parallel *RLC* resonant circuit, the currents in the various elements are found from Ohm's law as follows:

$$\mathbf{I}_R = \frac{\mathbf{V}}{\mathbf{R}} = \mathbf{I} \tag{6–45}$$

$$\begin{aligned}\mathbf{I}_L &= \frac{\mathbf{V}}{X_L\angle 90^\circ} \\ &= \frac{V}{R/Q_\text{P}}\angle -90^\circ \\ &= Q_\text{P} I\angle -90^\circ\end{aligned} \tag{6–46}$$

$$\begin{aligned}\mathbf{I}_C &= \frac{\mathbf{V}}{X_C\angle -90^\circ} \\ &= \frac{V}{R/Q_\text{P}}\angle 90^\circ \\ &= Q_\text{P} I\angle 90^\circ\end{aligned} \tag{6–47}$$

At resonance, the currents through the inductor and the capacitor have the same magnitudes but are 180° out of phase. Notice that the magnitude of current in the reactive elements at resonance is Q times greater than the applied source current. Because the Q of a parallel circuit may be very large, we see the importance of choosing elements that are able to handle the expected currents.

In a manner similar to that used in determining the bandwidth of a series resonant circuit, it may be shown that the half-power frequencies of a parallel resonant circuit are

$$\omega_1 = -\frac{1}{2RC} + \sqrt{\frac{1}{4R^2C^2} + \frac{1}{LC}} \quad \text{(rad/s)} \tag{6–48}$$

$$\omega_2 = \frac{1}{2RC} + \sqrt{\frac{1}{4R^2C^2} + \frac{1}{LC}} \quad \text{(rad/s)} \tag{6–49}$$

The bandwidth is therefore

$$\text{BW} = \omega_2 - \omega_1 = \frac{1}{RC} \quad \text{(rad/s)} \tag{6–50}$$

If the quality factor of the circuit $Q \geq 10$, then the selectivity curve is very nearly symmetrical around ω_P, resulting in half-power frequencies which are located at $\omega_\text{P} \pm \text{BW}/2$.

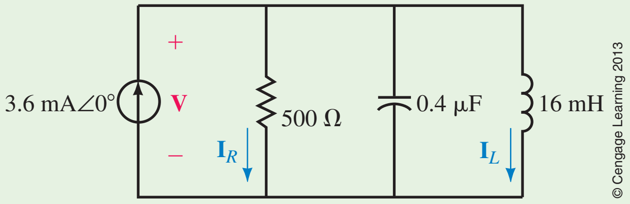
Multiplying Equation 6–50 by $\omega_\text{P}/\omega_\text{P}$ results in the following:

$$\begin{aligned}\text{BW} &= \frac{\omega_\text{P}}{R(\omega_\text{P} C)} = \frac{X_C}{R}\omega_\text{P} \\ \text{BW} &= \frac{\omega_\text{P}}{Q_\text{P}} \quad \text{(rad/s)}\end{aligned} \tag{6–51}$$

Notice that Equation 6–51 is the same for both series and parallel resonant circuits.

EXAMPLE 6–6

Consider the circuit shown in Figure 6–31.



CircuitSim 21-3

FIGURE 6–31

a. Determine the resonant frequencies, ω_P(rad/s) and f_P(Hz) of the tank circuit.
b. Find the Q of the circuit at resonance.
c. Calculate the voltage across the circuit at resonance.
d. Solve for currents through the inductor and the resistor at resonance.
e. Determine the bandwidth of the circuit in both radians per second and hertz.
f. Sketch the voltage response of the circuit, showing the voltage at the half-power frequencies.
g. Sketch the selectivity curve of the circuit showing P(watts) versus ω(rad/s).

Solution

a.
$$\omega_P = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(16\text{ mH})\,(0.4\ \mu\text{F})}} = 12.5\text{ krad/s}$$
$$f_P = \frac{\omega}{2\pi} = \frac{12.5\text{ krad/s}}{2\pi} = 1989\text{ Hz}$$

b.
$$Q_P = \frac{R_P}{\omega L} = \frac{500\ \Omega}{(12.5\text{ krad/s})\,(16\text{ mH})} = \frac{500\ \Omega}{200\ \Omega} = 2.5$$

c. At resonance, $\mathbf{V}_C = \mathbf{V}_L = \mathbf{V}_R$, and so
$$\mathbf{V} = \mathbf{IR} = (3.6\text{ mA}\angle 0^\circ)\,(500\ \Omega\angle 0^\circ) = 1.8\text{ V}\angle 0^\circ$$

d.
$$\mathbf{I}_L = \frac{\mathbf{V}_L}{\mathbf{Z}_L} = \frac{1.8\text{ V}\angle 0^\circ}{200\ \Omega\angle 90^\circ} = 9.0\text{ mA}\angle -90^\circ$$
$$\mathbf{I}_R = \mathbf{I} = 3.6\text{ mA}\angle 0^\circ$$

e.
$$\text{BW(rad/s)} = \frac{\omega_P}{Q_P} = \frac{12.5\text{ krad/s}}{2.5} = 5\text{ krad/s}$$
$$\text{BW(Hz)} = \frac{\text{BW(rad/s)}}{2\pi} = \frac{5\text{ krad/s}}{2\pi} = 795.8\text{ Hz}$$

f. The half-power frequencies are calculated from Equations 21–48 and 21–49 since the Q of the circuit is less than 10.
$$\begin{aligned}\omega_1 &= -\frac{1}{2RC} + \sqrt{\frac{1}{4R^2C^2} + \frac{1}{LC}} \\ &= -\frac{1}{0.0004} + \sqrt{\frac{1}{1.6\times 10^{-7}} + \frac{1}{6.4\times 10^{-9}}} \\ &= -2500 + 12\,748 \\ &= 10\,248\text{ rad/s}\end{aligned}$$

$$\omega_2 = \frac{1}{2RC} + \sqrt{\frac{1}{4R^2C^2} + \frac{1}{LC}}$$
$$= \frac{1}{0.0004} + \sqrt{\frac{1}{1.6 \times 10^{-7}} + \frac{1}{6.4 \times 10^{-9}}}$$
$$= 2500 + 12\,748$$
$$= 15\,248 \text{ rad/s}$$

The resulting voltage response curve is illustrated in Figure 6–32.

g. The power dissipated by the circuit at resonance is

$$P = \frac{V^2}{R} = \frac{(1.8 \text{ V})^2}{500\ \Omega} = 6.48 \text{ mW}$$

The selectivity curve is now easily sketched as shown in Figure 6–33.

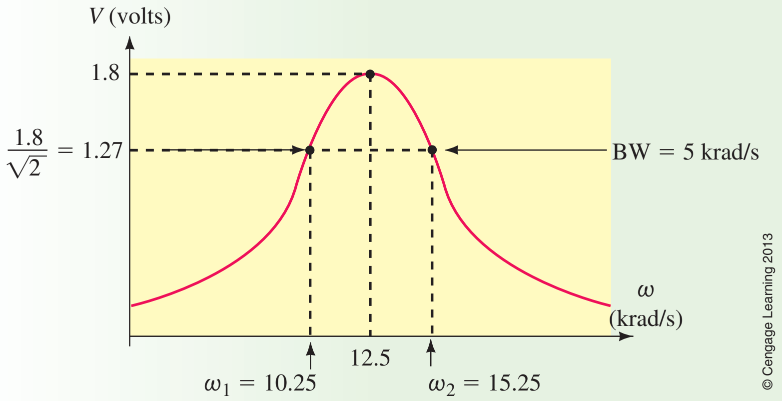


FIGURE 6–32

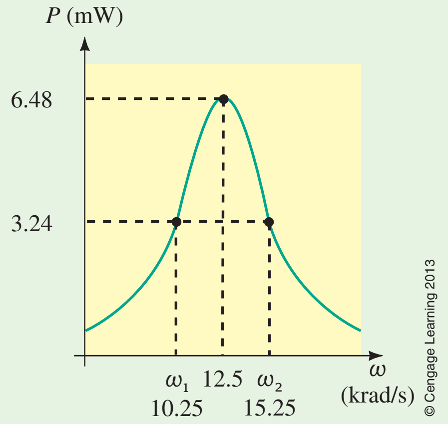


FIGURE 6–33

EXAMPLE 6–7

Consider the circuit of Figure 6–34.

CircuitSim 6-4

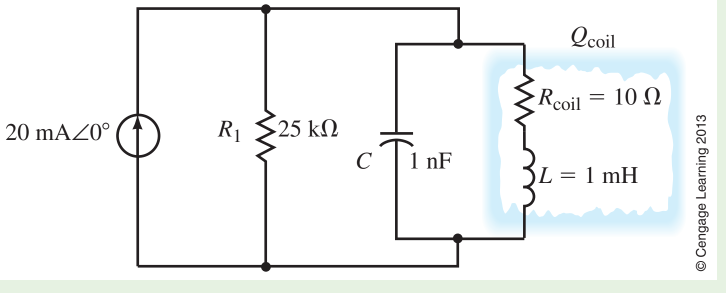


FIGURE 6–34

a. Calculate the resonant frequency, ω_P, of the tank circuit.

b. Find the Q of the coil at resonance.

c. Sketch the equivalent parallel circuit.

d. Determine the Q of the entire circuit at resonance.

e. Solve for the voltage across the capacitor at resonance.

f. Find the bandwidth of the circuit in radians per second.

g. Sketch the voltage response of the circuit showing the voltage at the half-power frequencies.

Solution

a. Since the ratio $L/C = 1000 \geq 100R_{\text{coil}}$, we use the approximation:

$$\omega_{\text{p}} = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(1\text{ mH})\,(1\text{ nF})}} = 1\text{ Mrad/s}$$

b. $$Q_{\text{coil}} = \frac{\omega L}{R_{\text{coil}}} = \frac{(1\text{ Mrad/s})\,(1\text{ mH})}{10\;\Omega} = 100$$

c. $$R_{\text{P}} \cong Q_{\text{coil}}{}^{2}R_{\text{coil}} = (100)^2(10\;\Omega) = 100\text{ k}\Omega$$

$$X_{LP} \cong X_L = \omega L = (1\text{ Mrad/s})\,(1\text{ mH}) = 1\text{ k}\Omega$$

The circuit of Figure 6–35 shows the circuit with the parallel equivalent of the inductor.

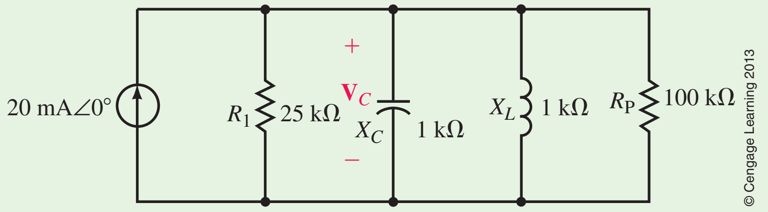


FIGURE 6–35

We see that the previous circuit may be further simplified by combining the parallel resistances:

$$R_{\text{eq}} = R_1 \| R_{\text{P}} = \frac{(25\text{ k}\Omega)(100\text{ k}\Omega)}{25\text{ k}\Omega + 100\text{ k}\Omega} = 20\text{ k}\Omega$$

The simplified equivalent circuit is shown in Figure 6–36.

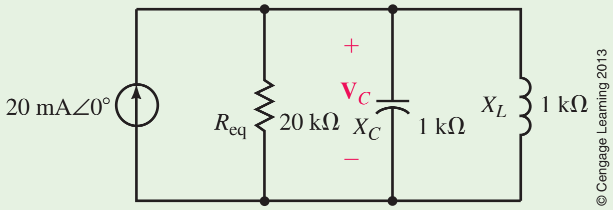


FIGURE 6–36

d. $$Q_{\text{P}} = \frac{R_{\text{eq}}}{X_L} = \frac{20\text{ k}\Omega}{1\text{ k}\Omega} = 20$$

e. At resonance,

$$\mathbf{V}_C = \mathbf{I}R_{\text{eq}} = (20\text{ mA}\angle 0^\circ)\,(20\text{ k}\Omega) = 400\text{ V}\angle 0^\circ$$

f. $$\text{BW} = \frac{\omega_{\text{P}}}{Q} = \frac{1\text{ Mrad/s}}{20} = 50\text{ krad/s}$$

g. The voltage response curve is shown in Figure 6–37. Since the circuit $Q \geq 10$, the half-power frequencies will occur at the following angular frequencies:

$$\omega_1 \cong \omega_P - \frac{BW}{2} = 1.0 \text{ Mrad/s} - \frac{50 \text{ krad/s}}{2} = 0.975 \text{ Mrad/s}$$

and

$$\omega_2 \cong \omega_P + \frac{BW}{2} = 1.0 \text{ Mrad/s} + \frac{50 \text{ krad/s}}{2} = 1.025 \text{ Mrad/s}$$

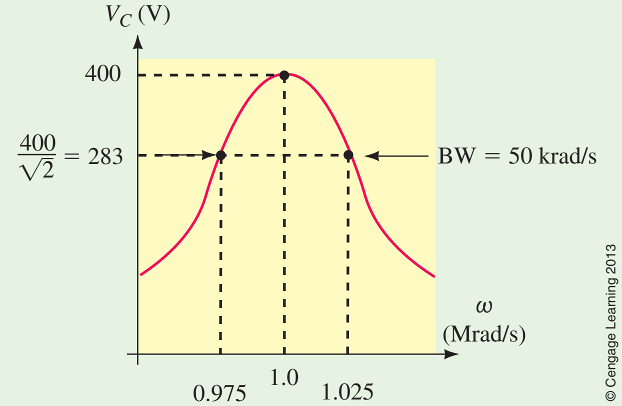


FIGURE 6–37

EXAMPLE 6–8

Determine the values of R_1 and C for the resonant tank circuit of Figure 6–38 so that the given conditions are met.

$L = 10$ mH, $R_{coil} = 30\ \Omega$

$f_P = 58$ kHz

BW = 1 kHz

Solve for the current, $\mathbf{I}_L$, through the inductor.

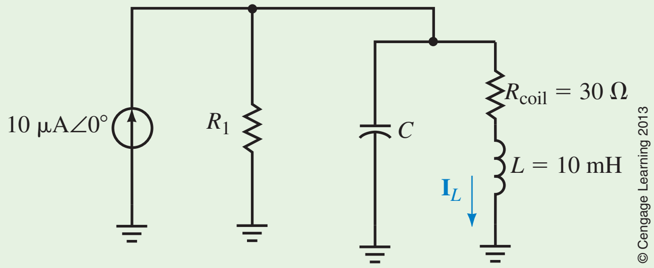


FIGURE 6–38

Solution

$$Q_P = \frac{f_P}{BW(Hz)} = \frac{58 \text{ kHz}}{1 \text{ kHz}} = 58$$

Now, because the frequency expressed in radians per second is more useful than hertz, we convert f_P to ω_P:

$$\omega_P = 2\pi f_P = (2\pi)(58 \text{ kHz}) = 364.4 \text{ krad/s}$$

The capacitance is determined from Equation 6–43 as

$$C = \frac{1}{\omega_P^2 L} = \frac{1}{(364.4\ \text{krad/s})^2(10\ \text{mH})} = 753\ \text{pF}$$

Solving for the Q of the coil permits us to easily convert the series RL network into its equivalent parallel network.

$$Q_{\text{coil}} = \frac{\omega_P L}{R_{\text{coil}}}$$

$$= \frac{(364.4\ \text{krad/s})(10\ \text{mH})}{30\ \Omega}$$

$$= \frac{3.644\ \text{k}\Omega}{30\ \Omega} = 121.5$$

$$R_P \cong Q_{\text{coil}}^2 R_{\text{coil}} = (121.5)^2(30\ \Omega) = 443\ \text{k}\Omega$$
$$X_{LP} \cong X_L = 3644\ \Omega$$

The resulting equivalent parallel circuit is shown in Figure 6–39.

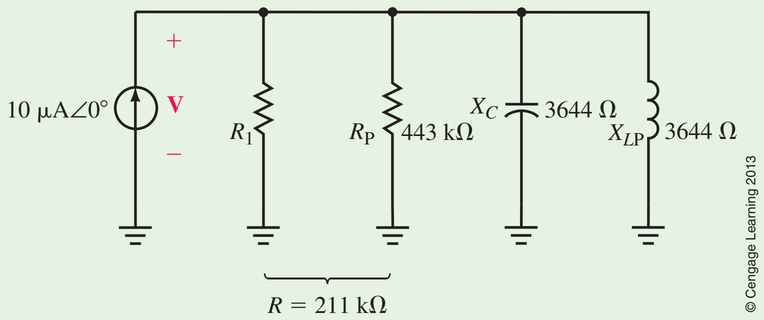


FIGURE 6–39

The quality factor Q_P is used to determine the total resistance of the circuit as

$$R = Q_P X_C = (58)(3.644\ \text{k}\Omega) = 211\ \text{k}\Omega$$

But

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_P}$$

$$\frac{1}{R_1} = \frac{1}{R} - \frac{1}{R_P} = \frac{1}{211\ \text{k}\Omega} - \frac{1}{443\ \text{k}\Omega} = 2.47\ \mu\text{S}$$

And so

$$R_1 = 405\ \text{k}\Omega$$

The voltage across the circuit is determined to be

$$\mathbf{V} = \mathbf{I}R = (10\ \mu\text{A}\angle 0^\circ)(211\ \text{k}\Omega) = 2.11\ \text{V}\angle 0^\circ$$

and the current through the inductor is

$$\mathbf{I}_L = \frac{\mathbf{V}}{R_{\text{coil}} + jX_L}$$
$$= \frac{2.11\ \text{V}\angle 0^\circ}{30 + j3644\ \Omega} = \frac{2.11\ \text{V}\angle 0^\circ}{3644\ \Omega\angle 89.95^\circ} = 579\ \mu\text{A}\angle -89.95^\circ$$

PRACTICE PROBLEMS 5

Refer to the circuit of Figure 6–40.

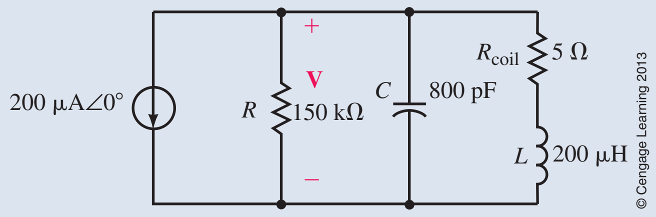


FIGURE 6–40

a. Determine the resonant frequency and express it in radians per second and in hertz.
b. Calculate the quality factor of the circuit.
c. Solve for the bandwidth.
d. Determine the voltage **V** at resonance.

Answers
a. 2.5 Mrad/s (398 kHz)
b. 75
c. 33.3 krad/s (5.31 kHz)
d. 7.5 V∠180°

IN-PROCESS LEARNING CHECK 4

(Answers are at the end of the chapter.)

Refer to the parallel resonant circuit of Figure 6–41.

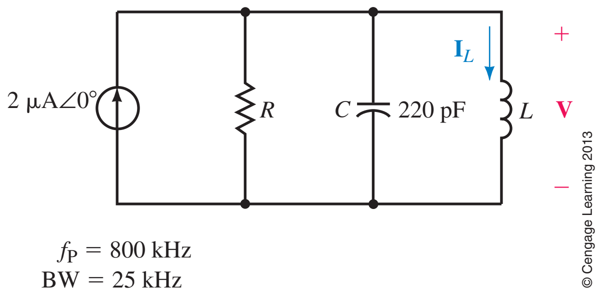


FIGURE 6–41

Suppose the circuit has a resonant frequency of 800 kHz and a bandwidth of 25 kHz.

a. Determine the value of the inductor, L, in henries.
b. Calculate the value of the resistance, R, in ohms.
c. Find **V**, $\mathbf{I}_L$, and power, P, at resonance.
d. Find the approximate values of the half-power frequencies, f_1 and f_2.
e. Determine the voltage across the circuit at the lower half-power frequency, f_1, and show that the power dissipated by the resistor at this frequency is half the power dissipated at the resonant frequency.

7

Simple Harmonic Motion

At first sight the eight physical systems in Figure 7.1 appear to have little in common.

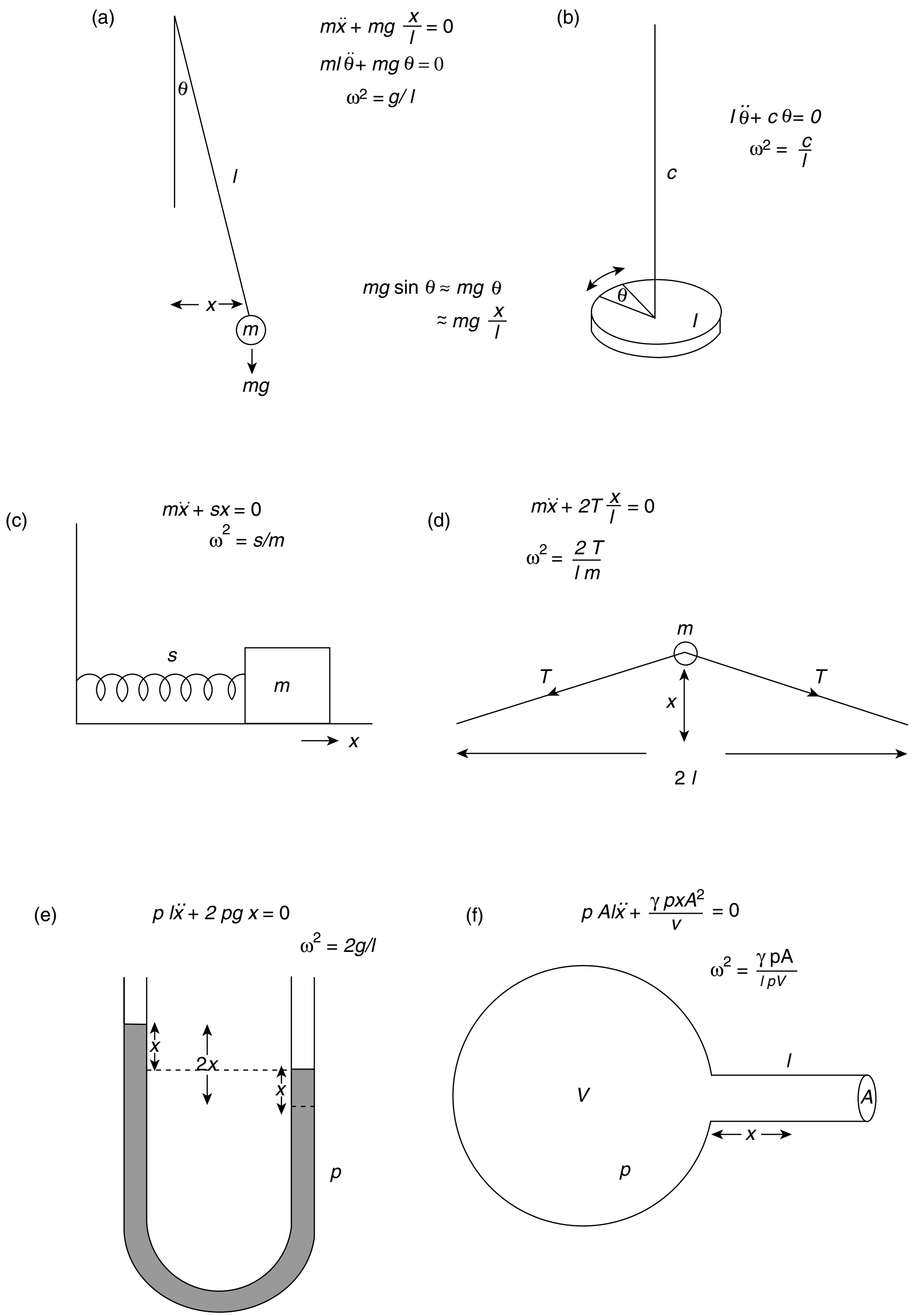
7.1(a) is a simple pendulum, a mass m swinging at the end of a light rigid rod of length l.
7.1(b) is a flat disc supported by a rigid wire through its centre and oscillating through small angles in the plane of its circumference.
7.1(c) is a mass fixed to a wall via a spring of stiffness s sliding to and fro in the x direction on a frictionless plane.
7.1(d) is a mass m at the centre of a light string of length $2l$ fixed at both ends under a constant tension T. The mass vibrates in the plane of the paper.
7.1(e) is a frictionless U-tube of constant cross-sectional area containing a length l of liquid, density ρ, oscillating about its equilibrium position of equal levels in each limb.
7.1(f) is an open flask of volume V and a neck of length l and constant cross-sectional area A in which the air of density ρ vibrates as sound passes across the neck.
7.1(g) is a hydrometer, a body of mass m floating in a liquid of density ρ with a neck of constant cross-sectional area cutting the liquid surface. When depressed slightly from its equilibrium position it performs small vertical oscillations.
7.1(h) is an electrical circuit, an inductance L connected across a capacitance C carrying a charge q.

All of these systems are simple harmonic oscillators which, when slightly disturbed from their equilibrium or rest postion, will oscillate with simple harmonic motion. This is the most fundamental vibration of a single particle or one-dimensional system. A small displacement x from its equilibrium position sets up a restoring force which is proportional to x acting in a direction towards the equilibrium position.

Thus, this restoring force F may be written

$$F = -sx$$

where s, the constant of proportionality, is called the stiffness and the negative sign shows that the force is acting against the direction of increasing displacement and back towards


(a)
mẍ + mg x/l = 0
mlθ̈ + mg θ = 0
ω² = g/l
mg sin θ ≈ mg θ
≈ mg x/l
θ
l
x
m
mg
(b)
Iθ̈ + cθ = 0
ω² = c/I
c
θ
I
(c)
mẍ + sx = 0
ω² = s/m
s
m
x
(d)
mẍ + 2T x/l = 0
ω² = 2T/lm
m
T
T
x
2l
(e)
ρlẍ + 2ρgx = 0
ω² = 2g/l
x
2x
x
ρ
(f)
ρAlẍ + γpxA²/V = 0
ω² = γpA/lρV
l
V
A
x
ρ

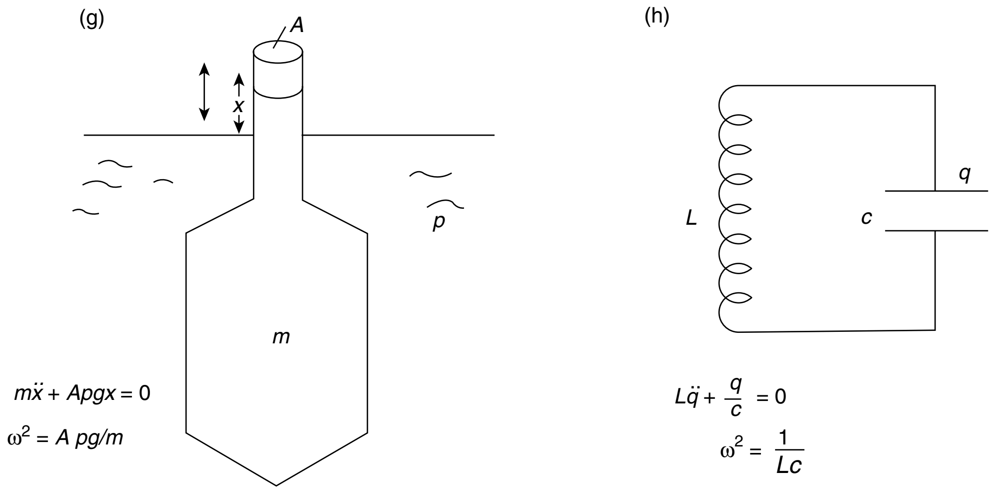


Figure 7.1 Simple harmonic oscillators with their equations of motion and angular frequencies ω of oscillation. (a) A simple pendulum. (b) A torsional pendulum. (c) A mass on a frictionless plane connected by a spring to a wall. (d) A mass at the centre of a string under constant tension T. (e) A fixed length of non-viscous liquid in a U-tube of constant cross-section. (f) An acoustic Helmholtz resonator. (g) A hydrometer mass m in a liquid of density ρ. (h) An electrical $L\ C$ resonant circuit

the equilibrium position. A constant value of the stiffness restricts the displacement x to small values (this is Hooke's Law of Elasticity). The stiffness s is obviously the restoring force per unit distance (or displacement) and has the dimensions

$$\frac{\text{force}}{\text{distance}} \equiv \frac{MLT^{-2}}{L}$$

The equation of motion of such a disturbed system is given by the dynamic balance between the forces acting on the system, which by Newton's Law is

$$\text{mass times acceleration} = \text{restoring force}$$

or

$$m\ddot{x} = -sx$$

where the acceleration

$$\ddot{x} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$$

This gives

$$m\ddot{x} + sx = 0$$

or

$$\ddot{x} + \frac{s}{m}x = 0$$

where the dimensions of

$$\frac{s}{m} \text{ are } \frac{MLT^{-2}}{ML} = T^{-2} = \nu^2$$

Here T is a time, or period of oscillation, the reciprocal of ν which is the frequency with which the system oscillates.

However, when we solve the equation of motion we shall find that the behaviour of x with time has a sinusoidal or cosinusoidal dependence, and it will prove more appropriate to consider, not ν, but the angular frequency $\omega = 2\pi\nu$ so that the period

$$T = \frac{1}{\nu} = 2\pi\sqrt{\frac{m}{s}}$$

where s/m is now written as ω^2. Thus the equation of simple harmonic motion

$$\ddot{x} + \frac{s}{m}x = 0$$

becomes

$$\boxed{\ddot{x} + \omega^2 x = 0} \tag{7.1}$$

(Problem 7.1)

Displacement in Simple Harmonic Motion

The behaviour of a simple harmonic oscillator is expressed in terms of its displacement x from equilibrium, its velocity $\dot{x}$, and its acceleration $\ddot{x}$ at any given time. If we try the solution

$$x = A\cos\omega t$$

where A is a constant with the same dimensions as x, we shall find that it satisfies the equation of motion

$$\ddot{x} + \omega^2 x = 0$$

for

$$\dot{x} = -A\omega\sin\omega t$$

and

$$\ddot{x} = -A\omega^2\cos\omega t = -\omega^2 x$$

Another solution

$$x = B \sin \omega t$$

is equally valid, where B has the same dimensions as A, for then

$$\dot{x} = B\omega \cos \omega t$$

and

$$\ddot{x} = -B\omega^2 \sin \omega t = -\omega^2 \mathrm{x}$$

The complete or general solution of equation (7.1) is given by the addition or superposition of both values for x so we have

$$x = A \cos \omega t + B \sin \omega t \tag{7.2}$$

with

$$\ddot{x} = -\omega^2(A \cos \omega t + B \sin \omega t) = -\omega^2 x$$

where A and B are determined by the values of x and $\dot{x}$ at a specified time. If we rewrite the constants as

$$A = a \sin \phi \quad \text{and} \quad B = a \cos \phi$$

where ϕ is a constant angle, then

$$A^2 + B^2 = a^2(\sin^2\phi + \cos^2\phi) = a^2$$

so that

$$a = \sqrt{A^2 + B^2}$$

and

$$\begin{aligned} x &= a \sin \phi \cos \omega t + a \cos \phi \sin \omega t \\ &= a \sin (\omega t + \phi) \end{aligned}$$

The maximum value of sin $(\omega t + \phi)$ is unity so the constant a is the maximum value of x, known as the amplitude of displacement. The limiting values of $\sin(\omega t + \phi)$ are ± 1 so the system will oscillate between the values of $x = \pm a$ and we shall see that the magnitude of a is determined by the total energy of the oscillator.

The angle ϕ is called the 'phase constant' for the following reason. Simple harmonic motion is often introduced by reference to 'circular motion' because each possible value of the displacement x can be represented by the projection of a radius vector of constant length a on the diameter of the circle traced by the tip of the vector as it rotates in a positive

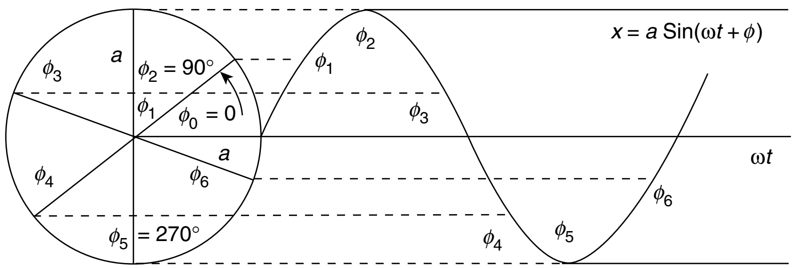


Figure 7.2 Sinusoidal displacement of simple harmonic oscillator with time, showing variation of starting point in cycle in terms of phase angle ϕ

anticlockwise direction with a constant angular velocity ω. Each rotation, as the radius vector sweeps through a phase angle of 2π rad, therefore corresponds to a complete vibration of the oscillator. In the solution

$$x = a \sin(\omega t + \phi)$$

the phase constant ϕ, measured in radians, defines the position in the cycle of oscillation at the time $t = 0$, so that the position in the cycle from which the oscillator started to move is

$$x = a \sin \phi$$

The solution

$$x = a \sin \omega t$$

defines the displacement only of that system which starts from the origin $x = 0$ at time $t = 0$ but the inclusion of ϕ in the solution

$$x = a \sin(\omega t + \phi)$$

where ϕ may take all values between zero and 2π allows the motion to be defined from any starting point in the cycle. This is illustrated in Figure 1.2 for various values of ϕ.

(Problems 7.2, 7.3, 7.4, 7.5)

Velocity and Acceleration in Simple Harmonic Motion

The values of the velocity and acceleration in simple harmonic motion for

$$x = a \sin(\omega t + \phi)$$

are given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = a\omega \cos(\omega t + \phi)$$

and

$$\frac{d^2x}{dt^2} = \ddot{x} = -a\omega^2 \sin(\omega t + \phi)$$

The maximum value of the velocity $a\omega$ is called the velocity *amplitude* and the *acceleration amplitude* is given by $a\omega^2$.

From Figure 1.2 we see that a positive phase angle of $\pi/2$ rad converts a sine into a cosine curve. Thus the velocity

$$\dot{x} = a\omega \cos(\omega t + \phi)$$

leads the displacement

$$x = a \sin(\omega t + \phi)$$

by a phase angle of $\pi/2$ rad and its maxima and minima are always a quarter of a cycle ahead of those of the displacement; the velocity is a maximum when the displacement is zero and is zero at maximum displacement. The acceleration is 'anti-phase' (π rad) with respect to the displacement, being maximum positive when the displacement is maximum negative and vice versa. These features are shown in Figure7.3.

Often, the relative displacement or motion between two oscillators having the same frequency and amplitude may be considered in terms of their phase difference $\phi_1 - \phi_2$ which can have any value because one system may have started several cycles before the other and each complete cycle of vibration represents a change in the phase angle of $\phi = 2\pi$. When the motions of the two systems are diametrically opposed; that is, one has

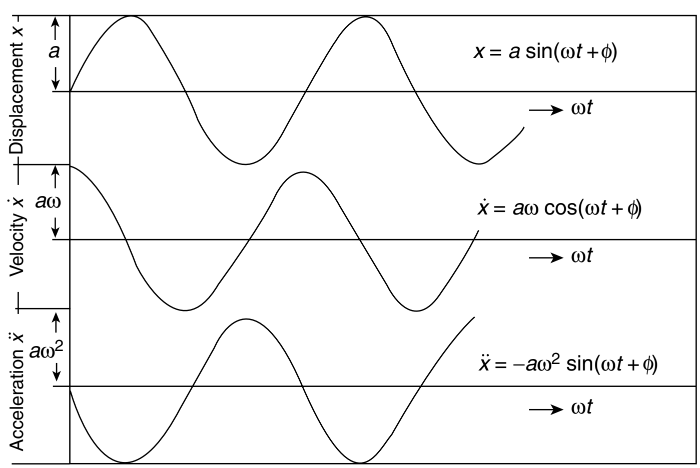


Figure7.3 Variation with time of displacement, velocity and acceleration in simple harmonic motion. Displacement lags velocity by $\pi/2$ rad and is π rad out of phase with the acceleration. The initial phase constant ϕ is taken as zero

$x = +a$ whilst the other is at $x = -a$, the systems are 'anti-phase' and the total phase difference

$$\phi_1 - \phi_2 = n\pi \text{ rad}$$

where n is an *odd* integer. Identical systems 'in phase' have

$$\phi_1 - \phi_2 = 2n\pi \text{ rad}$$

where n is any integer. They have exactly equal values of displacement, velocity and acceleration at any instant.

(Problems 7.6, 7.7,7.8, 7.9)

Non-linearity

If the stiffness s is constant, then the restoring force $F = -sx$, when plotted versus x, will produce a straight line and the system is said to be linear. The displacement of a linear simple harmonic motion system follows a sine or cosine behaviour. Non-linearity results when the stiffness s is not constant but varies with displacement x (see the beginning of Chapter 14).

Energy of a Simple Harmonic Oscillator

The fact that the velocity is zero at maximum displacement in simple harmonic motion and is a maximum at zero displacement illustrates the important concept of an exchange between kinetic and potential energy. In an ideal case the total energy remains constant but this is never realized in practice. If no energy is dissipated then all the potential energy becomes kinetic energy and vice versa, so that the values of (a) the total energy at any time, (b) the maximum potential energy and (c) the maximum kinetic energy will all be equal; that is

$$E_{\text{total}} = \text{KE} + \text{PE} = \text{KE}_{\text{max}} = \text{PE}_{\text{max}}$$

The solution $x = a \sin(\omega t + \phi)$ implies that the total energy remains constant because the amplitude of displacement $x = \pm a$ is regained every half cycle at the position of maximum potential energy; when energy is lost the amplitude gradually decays as we shall see later in Chapter 2. The potential energy is found by summing all the small elements of work $sx.\, dx$ (force sx times distance dx) *done by the system against the restoring force* over the range zero to x where $x = 0$ gives zero potential energy.

Thus the potential energy $= \int_0^x sx \cdot \mathrm{d}x = \frac{1}{2}sx^2$

The kinetic energy is given by $\frac{1}{2}m\dot{x}^2$ so that the total energy

$$\boxed{E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}sx^2}$$

Since E is constant we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} = (m\ddot{x} + sx)\dot{x} = 0$$

giving again the equation of motion

$$m\ddot{x} + sx = 0$$

The maximum potential energy occurs at $x = \pm a$ and is therefore

$$\mathrm{PE}_{\max} = \tfrac{1}{2} sa^2$$

The maximum kinetic energy is

$$\begin{aligned}\mathrm{KE}_{\max} = (\tfrac{1}{2}m\dot{x}^2)_{\max} &= \tfrac{1}{2}ma^2\omega^2[\cos^2(\omega t + \phi)]_{\max}\\ &= \tfrac{1}{2}ma^2\omega^2\end{aligned}$$

when the cosine factor is unity.

But $m\omega^2 = s$ so the maximum values of the potential and kinetic energies are equal, showing that the energy exchange is complete.

The total energy at any instant of time or value of x is

$$\begin{aligned}E &= \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}sx^2\\ &= \tfrac{1}{2}ma^2\omega^2[\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)]\\ &= \tfrac{1}{2}ma^2\omega^2\\ &= \tfrac{1}{2}sa^2\end{aligned}$$

as we should expect.

Figure 1.4 shows the distribution of energy versus displacement for simple harmonic motion. Note that the potential energy curve

$$\mathrm{PE} = \tfrac{1}{2}\, sx^2 = \tfrac{1}{2}\, ma^2\omega^2 \sin^2(\omega t + \phi)$$

is parabolic with respect to x and is symmetric about $x = 0$, so that energy is stored in the oscillator both when x is positive and when it is negative, e.g. a spring stores energy whether compressed or extended, as does a gas in compression or rarefaction. The kinetic energy curve

$$\mathrm{KE} = \tfrac{1}{2}m\dot{x}^2 = \tfrac{1}{2}ma^2\omega^2 \cos^2(\omega t + \phi)$$

is parabolic with respect to both x and $\dot{x}$. The inversion of one curve with respect to the other displays the $\pi/2$ phase difference between the displacement (related to the potential energy) and the velocity (related to the kinetic energy).

For any value of the displacement x the sum of the ordinates of both curves equals the total constant energy E.

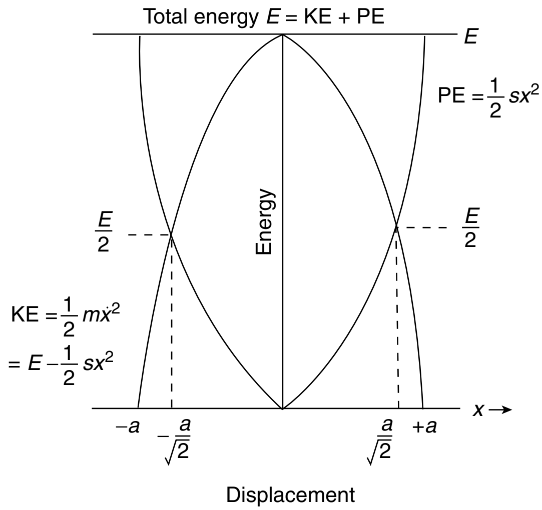


Figure 7.4 Parabolic representation of potential energy and kinetic energy of simple harmonic motion versus displacement. Inversion of one curve with respect to the other shows a 90° phase difference. At any displacement value the sum of the ordinates of the curves equals the total constant energy E

(Problems7.10,7.11, 7.12)

Simple Harmonic Oscillations in an Electrical System

So far we have discussed the simple harmonic motion of the mechanical and fluid systems of Figure 7.1, chiefly in terms of the inertial mass stretching the weightless spring of stiffness s. The stiffness s of a spring defines the difficulty of stretching; the reciprocal of the stiffness, the compliance C (where $s = 1/C$) defines the ease with which the spring is stretched and potential energy stored. This notation of compliance C is useful when discussing the simple harmonic oscillations of the electrical circuit of Figure 7.1(h) and Figure 7.5, where an inductance L is connected across the plates of a capacitance C. The force equation of the mechanical and fluid examples now becomes the voltage equation

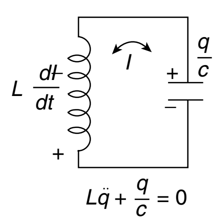


Figure 7.5 Electrical system which oscillates simple harmonically. The sum of the voltages around the circuit is given by Kirchhoff's law as $L\,\mathrm{d}I/\mathrm{d}t + q/C = 0$

(balance of voltages) of the electrical circuit, but the form and solution of the equations and the oscillatory behaviour of the systems are identical.

In the absence of resistance the energy of the electrical system remains constant and is exchanged between the *magnetic* field energy stored in the inductance and the *electric* field energy stored between the plates of the capacitance. At any instant, the voltage across the inductance is

$$V = -L\frac{\mathrm{d}I}{\mathrm{d}t} = -L\frac{\mathrm{d}^2 q}{\mathrm{d}t^2}$$

where I is the current flowing and q is the charge on the capacitor, the negative sign showing that the voltage opposes the increase of current. This equals the voltage q/C across the capacitance so that

$$L\ddot{q} + q/C = 0 \qquad \text{(Kirchhoff's Law)}$$

or

$$\ddot{q} + \omega^2 q = 0$$

where

$$\omega^2 = \frac{1}{LC}$$

The energy stored in the magnetic field or inductive part of the circuit throughout the cycle, as the current increases from 0 to I, is formed by integrating the power at any instant with respect to time; that is

$$E_{\mathrm{L}} = \int VI \cdot \mathrm{d}t$$

(where V is the magnitude of the voltage across the inductance).

So

$$E_{\mathrm{L}} = \int VI\,\mathrm{d}t = \int L\frac{\mathrm{d}I}{\mathrm{d}t}I\,\mathrm{d}t = \int_0^I LI\,\mathrm{d}I$$
$$= \tfrac{1}{2}LI^2 = \tfrac{1}{2}L\dot{q}^2$$

The potential energy stored mechanically by the spring is now stored electrostatically by the capacitance and equals

$$\tfrac{1}{2}CV^2 = \frac{q^2}{2C}$$

Comparison between the equations for the mechanical and electrical oscillators

$$\text{mechanical (force)} \rightarrow m\ddot{x} + sx = 0$$
$$\text{electrical (voltage)} \rightarrow L\ddot{q} + \frac{q}{C} = 0$$
$$\text{mechanical (energy)} \rightarrow \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}sx^2 = E$$
$$\text{electrical (energy)} \rightarrow \frac{1}{2}L\dot{q}^2 + \frac{1}{2}\frac{q^2}{C} = E$$

shows that magnetic field inertia (defined by the inductance L) controls the rate of change of current for a given voltage in a circuit in exactly the same way as the inertial mass controls the change of velocity for a given force. Magnetic inertial or inductive behaviour arises from the tendency of the magnetic flux threading a circuit to remain constant and reaction to any change in its value generates a voltage and hence a current which flows to oppose the change of flux. This is the physical basis of Fleming's right-hand rule.

Superposition of Two Simple Harmonic Vibrations in One Dimension

(1) Vibrations Having Equal Frequencies

In the following chapters we shall meet physical situations which involve the superposition of two or more simple harmonic vibrations on the same system.

We have already seen how the displacement in simple harmonic motion may be represented in magnitude and phase by a constant length vector rotating in the positive (anticlockwise) sense with a constant angular velocity ω. To find the resulting motion of a system which moves in the x direction under the simultaneous effect of two simple harmonic oscillations of equal angular frequencies but of different amplitudes and phases, we can represent each simple harmonic motion by its appropriate vector and carry out a vector addition.

If the displacement of the first motion is given by

$$x_1 = a_1 \cos(\omega t + \phi_1)$$

and that of the second by

$$x_2 = a_2 \cos(\omega t + \phi_2)$$

then Figure7.6 shows that the resulting displacement amplitude R is given by

$$\begin{aligned} R^2 &= (a_1 + a_2 \cos\delta)^2 + (a_2 \sin\delta)^2 \\ &= a_1^2 + a_2^2 + 2a_1a_2 \cos\delta \end{aligned}$$

where $\delta = \phi_2 - \phi_1$ is constant.

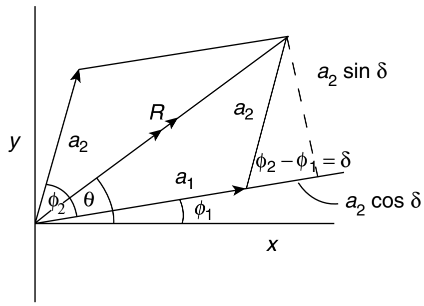


Figure 7.6 Addition of vectors, each representing simple harmonic motion along the x axis at angular frequency ω to give a resulting simple harmonic motion displacement $x = R\cos(\omega t + \theta)$ --- here shown for $t = 0$

The phase constant θ of R is given by

$$\tan\theta = \frac{a_1 \sin\phi_1 + a_2 \sin\phi_2}{a_1 \cos\phi_1 + a_2 \cos\phi_2}$$

so the resulting simple harmonic motion has a displacement

$$x = R\cos(\omega t + \theta)$$

an oscillation of the same frequency ω but having an amplitude R and a phase constant θ.

(Problem 7.13)

(2) Vibrations Having Different Frequencies

Suppose we now consider what happens when two vibrations of equal amplitudes but different frequencies are superposed. If we express them as

$$x_1 = a\sin\omega_1 t$$

and

$$x_2 = a\sin\omega_2 t$$

where

$$\omega_2 > \omega_1$$

then the resulting displacement is given by

$$x = x_1 + x_2 = a(\sin\omega_1 t + \sin\omega_2 t)$$
$$= 2a\sin\frac{(\omega_1+\omega_2)t}{2}\cos\frac{(\omega_2-\omega_1)t}{2}$$

This expression is illustrated in Figure 7.7. It represents a sinusoidal oscillation at the average frequency $(\omega_1+\omega_2)/2$ having a displacement amplitude of $2a$ which modulates; that is, varies between $2a$ and zero under the influence of the cosine term of a much slower frequency equal to half the difference $(\omega_2-\omega_1)/2$ between the original frequencies.

When ω_1 and ω_2 are almost equal the sine term has a frequency very close to both ω_1 and ω_2 whilst the cosine envelope modulates the amplitude $2a$ at a frequency $(\omega_2-\omega_1)/2$ which is very slow.

Acoustically this growth and decay of the amplitude is registered as 'beats' of strong reinforcement when two sounds of almost equal frequency are heard. The frequency of the 'beats' is $(\omega_2-\omega_1)$, the difference between the separate frequencies (not half the difference) because the maximum amplitude of $2a$ occurs twice in every period associated with the frequency $(\omega_2-\omega_1)/2$.

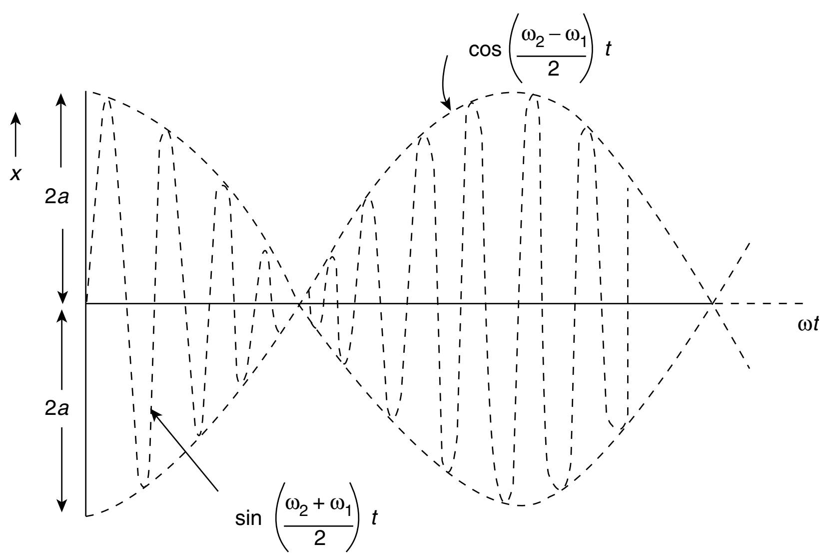


Figure 7.7 Superposition of two simple harmonic displacements $x_1 = a\sin\omega_1 t$ and $x_2 = a\sin\omega_2 t$ when $\omega_2 > \omega_1$. The slow cos $[(\omega_2-\omega_1)/2]t$ envelope modulates the sin $[(\omega_2+\omega_1)/2]t$ curve between the values $x = \pm 2a$

Superposition of Two Perpendicular Simple Harmonic Vibrations

(1) Vibrations Having Equal Frequencies

Suppose that a particle moves under the simultaneous influence of two simple harmonic vibrations of equal frequency, one along the x axis, the other along the perpendicular y axis. What is its subsequent motion?

This displacements may be written

$$x = a_1 \sin(\omega t + \phi_1)$$
$$y = a_2 \sin(\omega t + \phi_2)$$

and the path followed by the particle is formed by eliminating the time t from these equations to leave an expression involving only x and y and the constants ϕ_1 and ϕ_2.

Expanding the arguments of the sines we have

$$\frac{x}{a_1} = \sin\omega t \cos\phi_1 + \cos\omega t \sin\phi_1$$

and

$$\frac{y}{a_2} = \sin\omega t \cos\phi_2 + \cos\omega t \sin\phi_2$$

If we carry out the process

$$\left(\frac{x}{a_1}\sin\phi_2 - \frac{y}{a_2}\sin\phi_1\right)^2 + \left(\frac{y}{a_2}\cos\phi_1 - \frac{x}{a_1}\cos\phi_2\right)^2$$

this will yield

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1 a_2}\cos(\phi_2 - \phi_1) = \sin^2(\phi_2 - \phi_1) \tag{7.3}$$

which is the general equation for an ellipse.

In the most general case the axes of the ellipse are inclined to the x and y axes, but these become the principal axes when the phase difference

$$\phi_2 - \phi_1 = \frac{\pi}{2}$$

Equation (1.3) then takes the familiar form

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} = 1$$

that is, an ellipse with semi-axes a_1 and a_2.

If $a_1 = a_2 = a$ this becomes the circle

$$x^2 + y^2 = a^2$$

When

$$\phi_2 - \phi_1 = 0, 2\pi, 4\pi, \text{ etc.}$$

the equation simplifies to

$$y = \frac{a_2}{a_1}x$$

which is a straight line through the origin of slope a_2/a_1.

Again for $\phi_2 - \phi_1 = \pi$, 3π, 5π, etc., we obtain

$$y = -\frac{a_2}{a_1}x$$

a straight line through the origin of equal but opposite slope.

The paths traced out by the particle for various values of $\delta = \phi_2 - \phi_1$ are shown in Figure 7.8 and are most easily demonstrated on a cathode ray oscilloscope.

When

$$\phi_2 - \phi_1 = 0,\ \pi,\ 2\pi,\ \text{etc.}$$

and the ellipse degenerates into a straight line, the resulting vibration lies wholly in one plane and the oscillations are said to be *plane polarized*.

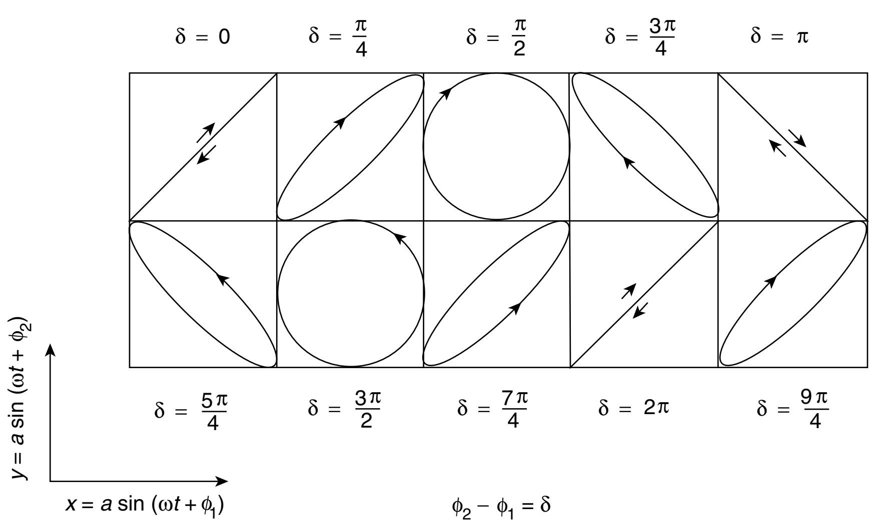


Figure 7.8 Paths traced by a system vibrating simultaneously in two perpendicular directions with simple harmonic motions of equal frequency. The phase angle δ is the angle by which the y motion leads the x motion

Convention defines the plane of polarization as that plane perpendicular to the plane containing the vibrations. Similarly the other values of

$$\phi_2 - \phi_1$$

yield *circular* or *elliptic* polarization where the tip of the vector resultant traces out the appropriate conic section.

(Problems 7.14, 7.15, 7.16)

*Polarization

Polarization is a fundamental topic in optics and arises from the superposition of two perpendicular simple harmonic optical vibrations. We shall see in Chapter 8 that when a light wave is plane polarized its electrical field oscillation lies within a single plane and traces a sinusoidal curve along the direction of wave motion. Substances such as quartz and calcite are capable of splitting light into two waves whose planes of polarization are perpendicular to each other. Except in a specified direction, known as the optic axis, these waves have different velocities. One wave, the ordinary or *O* wave, travels at the same velocity in all directions and its electric field vibrations are always perpendicular to the optic axis. The extraordinary or *E* wave has a velocity which is direction-dependent. Both ordinary and extraordinary light have their own refractive indices, and thus quartz and calcite are known as doubly refracting materials. When the ordinary light is faster, as in quartz, a crystal of the substance is defined as positive, but in calcite the extraordinary light is faster and its crystal is negative. The surfaces, spheres and ellipsoids, which are the loci of the values of the wave velocities in any direction are shown in Figure 7.9(a), and for a

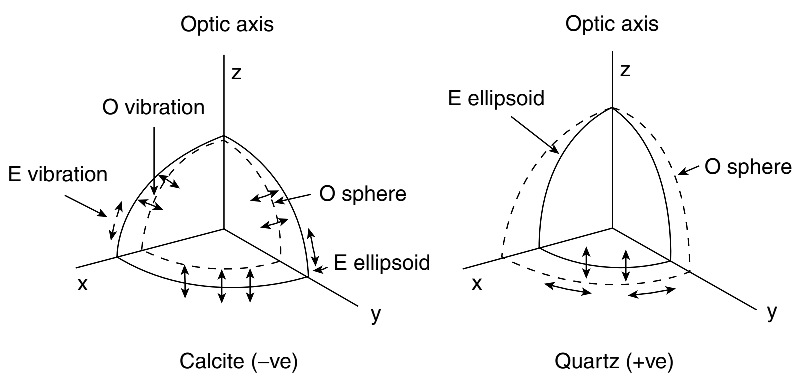


Figure 7.9a Ordinary (spherical) and extraordinary (elliposoidal) wave surfaces in doubly refracting calcite and quartz. In calcite the *E* wave is faster than the *O* wave, except along the optic axis. In quartz the *O* wave is faster. The *O* vibrations are always perpendicular to the optic axis, and the *O* and *E* vibrations are always tangential to their wave surfaces

*This section may be omitted at a first reading.

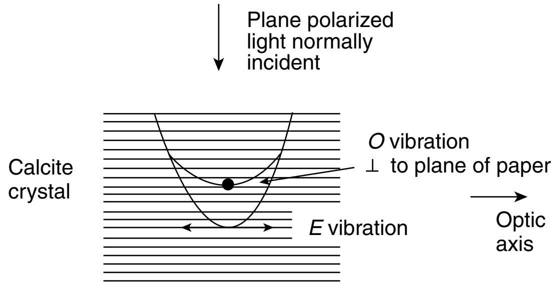


Figure 7.9b Plane polarized light normally incident on a calcite crystal face cut parallel to its optic axis. The advance of the *E* wave over the *O* wave is equivalent to a gain in phase

given direction the electric field vibrations of the separate waves are tangential to the surface of the sphere or ellipsoid as shown. Figure 7.9(b) shows plane polarized light normally incident on a calcite crystal cut parallel to its optic axis. Within the crystal the faster *E* wave has vibrations parallel to the optic axis, while the *O* wave vibrations are perpendicular to the plane of the paper. The velocity difference results in a phase gain of the *E* vibration over the *O* vibration which increases with the thickness of the crystal. Figure 7.9(c) shows plane polarized light normally incident on the crystal of Figure 7.9(b) with its vibration at an angle of 45° of the optic axis. The crystal splits the vibration into

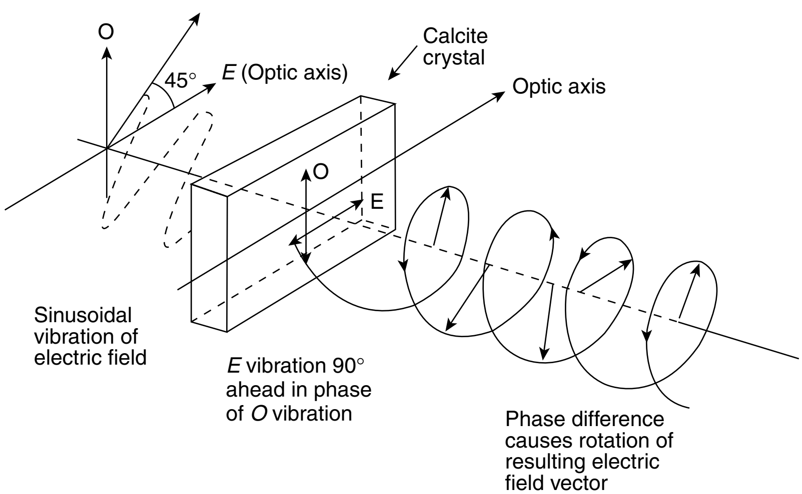


Figure 7.9c The crystal of Fig. 1.9c is thick enough to produce a phase gain of $\pi/2$ rad in the *E* wave over the *O* wave. Wave recombination on leaving the crystal produces circularly polarized light

equal E and O components, and for a given thickness the E wave emerges with a phase gain of 90° over the O component. Recombination of the two vibrations produces circularly polarized light, of which the electric field vector now traces a helix in the anticlockwise direction as shown.

(2) Vibrations Having Different Frequencies (Lissajous Figures)

When the frequencies of the two perpendicular simple harmonic vibrations are not equal the resulting motion becomes more complicated. The patterns which are traced are called Lissajous figures and examples of these are shown in Figure 7.10 where the axial frequencies bear the simple ratios shown and

$$\begin{aligned}\delta = \phi_2 - \phi_1 &= 0 \text{ (on the left)}\\ &= \frac{\pi}{2} \text{ (on the right)}\end{aligned}$$

If the amplitudes of the vibrations are respectively a and b the resulting Lissajous figure will always be contained within the rectangle of sides $2a$ and $2b$. The sides of the rectangle will be tangential to the curve at a number of points and the ratio of the numbers of these tangential points along the x axis to those along the y axis is the inverse of the ratio of the corresponding frequencies (as indicated in Figure 7.10).

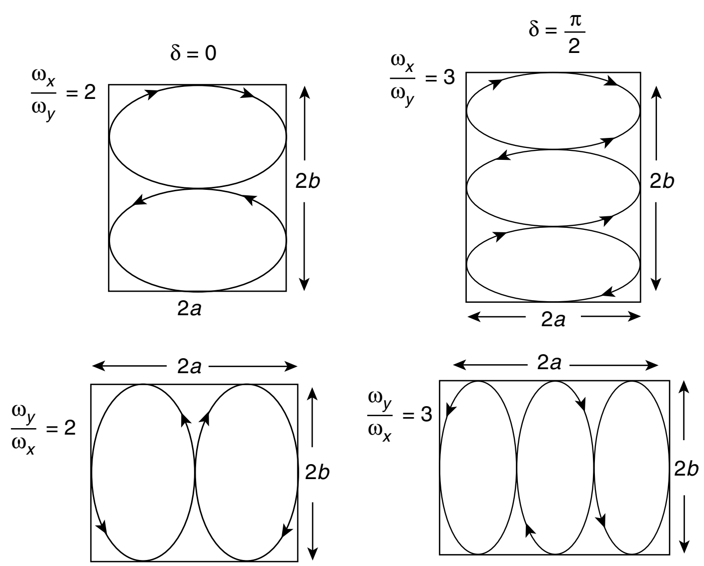


Figure 7.10 Simple Lissajous figures produced by perpendicular simple harmonic motions of different angular frequencies

Superposition of a Large Number n of Simple Harmonic Vibrations of Equal Amplitude a and Equal Successive Phase Difference δ

Figure 7.11 shows the addition of n vectors of equal length a, each representing a simple harmonic vibration with a constant phase difference δ from its neighbour. Two general physical situations are characterized by such a superposition. The first is met in Chapter ... as a wave group problem where the phase difference δ arises from a small *frequency difference*, $\delta\omega$, between consecutive components. The second appears in Chapter ··· where the intensity of optical interference and diffraction patterns are considered. There, the superposed harmonic vibrations will have the same frequency but each component will have a constant phase difference from its neighbour because of the extra *distance* it has travelled.

The figure displays the mathematical expression

$$R\cos(\omega t+\alpha)=a\cos\omega t+a\cos(\omega t+\delta)+a\cos(\omega t+2\delta)$$
$$+\cdots+a\cos(\omega t+[n-1]\delta)$$

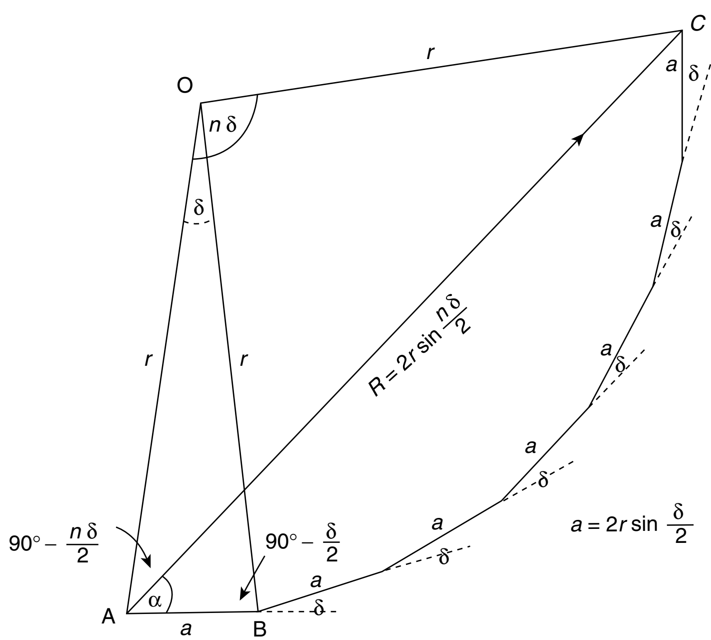


Figure 7.11 Vector superposition of a large number n of simple harmonic vibrations of equal amplitude a and equal successive phase difference δ. The amplitude of the resultant

$$R=2r\sin\frac{n\delta}{2}=a\frac{\sin n\delta/2}{\sin\delta/2}$$

and its phase with respect to the first contribution is given by

$$\alpha=(n-1)\delta/2$$

where R is the magnitude of the resultant and α is its phase difference with respect to the first component $a\cos\omega t$.

Geometrically we see that each length

$$a = 2r\sin\frac{\delta}{2}$$

where r is the radius of the circle enclosing the (incomplete) polygon.

From the isosceles triangle OAC the magnitude of the resultant

$$R = 2r\sin\frac{n\delta}{2} = a\frac{\sin n\delta/2}{\sin\delta/2}$$

and its phase angle is seen to be

$$\alpha = \mathrm{O\hat{A}B} - \mathrm{O\hat{A}C}$$

In the isosceles triangle OAC

$$\mathrm{\hat{O}AC} = 90^\circ - \frac{n\delta}{2}$$

and in the isosceles triangle OAB

$$\mathrm{O\hat{A}B} = 90^\circ - \frac{\delta}{2}$$

so

$$\alpha = \left(90^\circ - \frac{\delta}{2}\right) - \left(90^\circ - \frac{n\delta}{2}\right) = (n-1)\frac{\delta}{2}$$

that is, half the phase difference between the first and the last contributions. Hence the resultant

$$R\cos(\omega t + \alpha) = a\frac{\sin n\delta/2}{\sin\delta/2}\cos\left[\omega t + (n-1)\frac{\delta}{2}\right]$$

We shall obtain the same result later in this chapter as an example on the use of exponential notation.

For the moment let us examine the behaviour of the magnitude of the resultant

$$R = a\frac{\sin n\delta/2}{\sin\delta/2}$$

which is not constant but depends on the value of δ. When n is very large δ is very small and the polygon becomes an arc of the circle centre O, of length $na = A$, with R as the chord. Then

$$\alpha = (n-1)\frac{\delta}{2} \approx \frac{n\delta}{2}$$

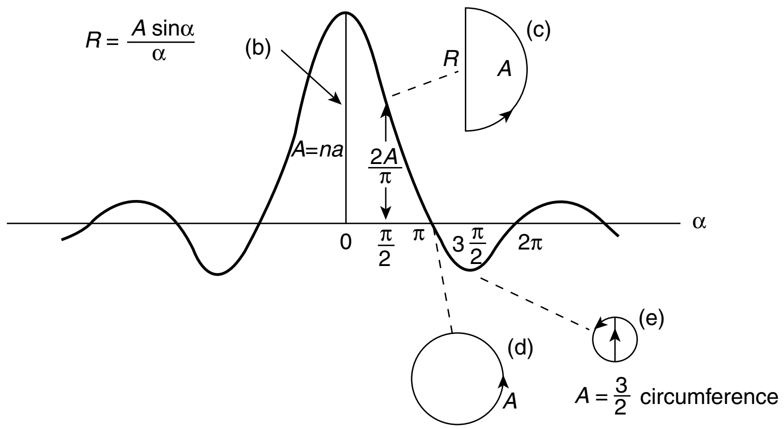


Figure 7.12 (a) Graph of A sin α/α versus α, showing the magnitude of the resultants for (b) $\alpha = 0$; (c) $\alpha = \pi/2$; (d) $\alpha = \pi$ and (e) $\alpha = 3\pi/2$

and

$$\sin\frac{\delta}{2} \to \frac{\delta}{2} \approx \frac{\alpha}{n}$$

Hence, in this limit,

$$R = a\frac{\sin n\delta/2}{\sin\delta/2} = a\frac{\sin\alpha}{\alpha/n} = na\frac{\sin\alpha}{\alpha} = \frac{A\sin\alpha}{\alpha}$$

The behaviour of $A\ \sin\alpha/\alpha$ versus α is shown in Figure 7.12. The pattern is symmetric about the value $\alpha = 0$ and is zero whenever sin $\alpha = 0$ except at $\alpha \to 0$ that is, when sin $\alpha/\alpha \to 1$. When $\alpha = 0$, $\delta = 0$ and the resultant of the n vectors is the straight line of length A, Figure 7.12(b). As δ increases A becomes the arc of a circle until at $\alpha = \pi/2$ the first and last contributions are out of phase $(2\alpha = \pi)$ and the arc A has become a semicircle of which the diameter is the resultant R Figure 7.12(c). A further increase in δ increases α and curls the constant length A into the circumference of a circle $(\alpha = \pi)$ with a zero resultant, Figure 7.12(d). At $\alpha = 3\pi/2$, Figure 7.12(e) the length A is now 3/2 times the circumference of a circle whose diameter is the amplitude of the first minimum.

*Superposition of *n* Equal SHM Vectors of Length *a* with Random Phase

When the phase difference between the successive vectors of the last section may take random values ϕ between zero and 2π (measured from the x axis) the vector superposition and resultant R may be represented by Figure 7.13.

*This section may be omitted at a first reading.

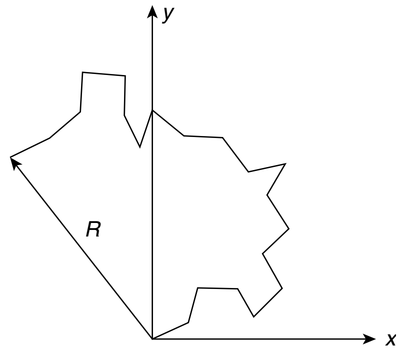


Figure 7.13 The resultant $R = \sqrt{n}a$ of n vectors, each of length a, having random phase. This result is important in optical incoherence and in energy loss from waves from random dissipation processes

The components of R on the x and y axes are given by

$$\begin{aligned} R_x &= a\cos\phi_1 + a\cos\phi_2 + a\cos\phi_3 \ldots a\cos\phi_n \\ &= a\sum_{i=1}^{n}\cos\phi_i \end{aligned}$$

and

$$R_y = a\sum_{i=1}^{n}\sin\phi_i$$

where

$$R^2 = R_x^2 + R_y^2$$

Now

$$R_x^2 = a^2\left(\sum_{i=1}^{n}\cos\phi_i\right)^2 = a^2\left[\sum_{i=1}^{n}\cos^2\phi_i + \sum_{\substack{i=1\\ i\neq j}}^{n}\cos\phi_i\sum_{j=1}^{n}\cos\phi_j\right]$$

In the typical term $2\cos\phi_i\cos\phi_j$ of the double summation, $\cos\phi_i$ and $\cos\phi_j$ have random values between ± 1 and the averaged sum of sets of these products is effectively zero.

The summation

$$\sum_{i=1}^{n}\cos^2\phi_i = n\overline{\cos^2\phi}$$

that is, the number of terms n times the average value $\overline{\cos^2\phi}$ which is the integrated value of $\cos^2\phi$ over the interval zero to 2π divided by the total interval 2π, or

$$\overline{\cos^2\phi} = \frac{1}{2\pi}\int_0^{2\pi} \cos^2\phi \, d\phi = \frac{1}{2} = \overline{\sin^2\phi}$$

So

$$R_x^2 = a^2 \sum_{i=1}^{n} \cos^2\phi_i = na^2\overline{\cos^2\phi_i} = \frac{na^2}{2}$$

and

$$R_y^2 = a^2 \sum_{i=1}^{n} \sin^2\phi_i = na^2\overline{\sin^2\phi_i} = \frac{na^2}{2}$$

giving

$$R^2 = R_x^2 + R_y^2 = na^2$$

or

$$R = \sqrt{n}a$$

Thus, the amplitude R of a system subjected to n equal simple harmonic motions of amplitude a with random phases in only $\sqrt{n}a$ whereas, if the motions were all in phase R would equal na.

Such a result illustrates a very important principle of random behaviour.

(Problem 7.17)

Applications

Incoherent Sources in Optics The result above is directly applicable to the problem of coherence in optics. Light sources which are in phase are said to be coherent and this condition is essential for producing optical interference effects experimentally. If the amplitude of a light source is given by the quantity a its intensity is proportional to a^2, n coherent sources have a resulting amplitude na and a total intensity n^2a^2. Incoherent sources have random phases, n such sources each of amplitude a have a resulting amplitude $\sqrt{n}a$ and a total intensity of na^2.

Random Processes and Energy Absorption From our present point of view the importance of random behaviour is the contribution it makes to energy loss or absorption from waves moving through a medium. We shall meet this in all the waves we discuss.

Random processes, for example collisions between particles, in Brownian motion, are of great significance in physics. Diffusion, viscosity or frictional resistance and thermal conductivity are all the result of random collision processes. These energy dissipating phenomena represent the transport of mass, momentum and energy, and change only in the direction of increasing disorder. They are known as 'thermodynamically irreversible' processes and are associated with the increase of entropy. Heat, for example, can flow only from a body at a higher temperature to one at a lower temperature. Using the earlier analysis where the length a is no longer a simple harmonic amplitude but is now the average distance a particle travels between random collisions (its mean free path), we see that after n such collisions (with, on average, equal time intervals between collisions) the particle will, on average, have travelled only a distance $\sqrt{n}a$ from its position at time $t = 0$, so that the distance travelled varies only with the square root of the time elapsed instead of being directly proportional to it. This is a feature of all random processes.

Not all the particles of the system will have travelled a distance $\sqrt{n}a$ but this distance is the most probable and represents a statistical average.

Random behaviour is described by the diffusion equation (see the last section of Chapter...) and a constant coefficient called the diffusivity of the process will always arise. The dimensions of a diffusivity are always length^2/time and must be interpreted in terms of a characteristic distance of the process which varies only with the square root of time.

Some Useful Mathematics

The Exponential Series

By a 'natural process' of growth or decay we mean a process in which a quantity changes by a constant fraction of itself in a given interval of space or time. A 5% per annum compound interest represents a natural growth law; attenuation processes in physics usually describe natural decay.

The law is expressed differentially as

$$\frac{\mathrm{d}N}{N} = \pm\alpha\,\mathrm{d}x \quad \text{or} \quad \frac{\mathrm{d}N}{N} = \pm\alpha\,\mathrm{d}t$$

where N is the changing quantity, α is a constant and the positive and negative signs represent growth and decay respectively. The derivatives $\mathrm{d}N/\mathrm{d}x$ or $\mathrm{d}N/\mathrm{d}t$ are therefore proportional to the value of N at which the derivative is measured.

Integration yields $N = N_0 e^{\pm\alpha x}$ or $N = N_0 e^{\pm\alpha t}$ where N_0 is the value at x or $t = 0$ and e is the exponential or the base of natural logarithms. The exponential series is defined as

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

and is shown graphically for positive and negative x in Figure 7.14. It is important to note that whatever the form of the index of the logarithmic base e, it is the power to which the

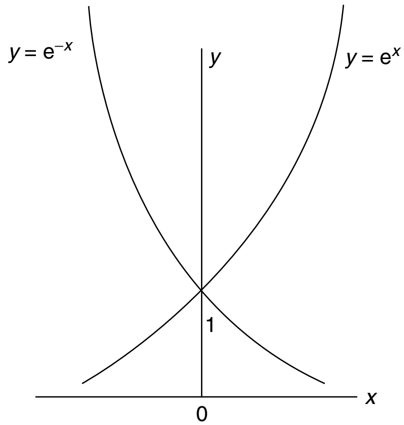


Figure 7.14 The behaviour of the exponential series $y = \mathrm{e}^x$ and $y = e^{-x}$

base is raised, and is therefore always non-dimensional. Thus $e^{\alpha x}$ is non-dimensional and α must have the dimensions of x^{-1}. Writing

$$\mathrm{e}^{\alpha x} = 1 + \alpha x + \frac{(\alpha x)^2}{2!} + \frac{(\alpha x)^3}{3!} + \cdots$$

it follows immediately that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{\alpha x}) &= \alpha + \frac{2\alpha^2}{2!}x + \frac{3\alpha^3}{3!}x^2 + \cdots \\ &= \alpha\left[1 + \alpha x + \frac{(\alpha x)^2}{2!} + \frac{(\alpha x)^3}{3!}\right) + \cdots\right] \\ &= \alpha \mathrm{e}^{\alpha x} \end{aligned}$$

Similarly

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\mathrm{e}^{\alpha x}) = \alpha^2 \mathrm{e}^{\alpha x}$$

In Chapter ... we shall use $\mathrm{d}(\mathrm{e}^{\alpha t})/\mathrm{d}t = \alpha\, \mathrm{e}^{\alpha t}$ and $\mathrm{d}^2\ (\mathrm{e}^{\alpha t})/\mathrm{d}t^2 = \alpha^2\, \mathrm{e}^{\alpha t}$ on a number of occasions.

By taking logarithms it is easily shown that $\mathrm{e}^x\, \mathrm{e}^y = \mathrm{e}^{x+y}$ since $\log_\mathrm{e}\ (\mathrm{e}^x\, \mathrm{e}^y) = \log_\mathrm{e} \mathrm{e}^x + \log_\mathrm{e} \mathrm{e}^y = x + y$.

The Notation $\mathrm{i} = \sqrt{-1}$

The combination of the exponential series with the complex number notation $\mathrm{i} = \sqrt{-1}$ is particularly convenient in physics. Here we shall show the mathematical convenience in expressing sine or cosine (oscillatory) behaviour in the form $\mathrm{e}^{\mathrm{i}x} = \cos x + \mathrm{i} \sin x$.

In Chapter... we shall see the additional merit of i in its role of vector operator.
The series representation of sin x is written

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$$

and that of cos x is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \cdots$$

Since

$$\mathrm{i} = \sqrt{-1}, \mathrm{i}^2 = -1, \mathrm{i}^3 = -\mathrm{i}$$

etc. we have

$$\begin{aligned} \mathrm{e}^{\mathrm{i}x} &= 1 + \mathrm{i}x + \frac{(\mathrm{i}x)^2}{2!} + \frac{(\mathrm{i}x)^3}{3!} + \frac{(\mathrm{i}x)^4}{4!} + \cdots \\ &= 1 + \mathrm{i}x - \frac{x^2}{2!} - \frac{\mathrm{i}x^3}{3!} + \frac{x^4}{4!} + \cdots \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \mathrm{i}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) \\ &= \cos x + \mathrm{i} \sin x \end{aligned}$$

We also see that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{\mathrm{i}x}) = \mathrm{i}\,\mathrm{e}^{\mathrm{i}x} = \mathrm{i}\cos x - \sin x$$

Often we shall represent a sine or cosine oscillation by the form $\mathrm{e}^{\mathrm{i}x}$ and recover the original form by taking that part of the solution preceded by i in the case of the sine, and the real part of the solution in the case of the cosine.

Examples

(1) In simple harmonic motion ($\ddot{x} + \omega^2 x = 0$) let us try the solution $x = a\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{e}^{\mathrm{i}\phi}$, where a is a constant length, and ϕ (and therefore $\mathrm{e}^{\mathrm{i}\phi}$) is a constant.

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \dot{x} = \mathrm{i}\omega a\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{e}^{\mathrm{i}\phi} = \mathrm{i}\omega x \\ \frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= \ddot{x} = \mathrm{i}^2\omega^2 a\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{e}^{\mathrm{i}\phi} = -\omega^2 x \end{aligned}$$

Therefore

$$\begin{aligned} x &= a\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{e}^{\mathrm{i}\phi} = a\,\mathrm{e}^{\mathrm{i}(\omega t + \phi)} \\ &= a\cos(\omega t + \phi) + \mathrm{i}\,a\sin(\omega t + \phi) \end{aligned}$$

is a complete solution of $\ddot{x} + \omega^2 x = 0$.

we used the sine form of the solution; the cosine form is equally valid and merely involves an advance of $\pi/2$ in the phase ϕ.

(2)

$$\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x} = 2\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) = 2\cos x$$
$$\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x} = 2\mathrm{i}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = 2\mathrm{i}\sin x$$

(3) On p.... we used a geometrical method to show that the resultant of the superposed harmonic vibrations

$$a\cos\omega t + a\cos(\omega t + \delta) + a\cos(\omega t + 2\delta) + \cdots + a\cos(\omega t + [n-1]\delta)$$
$$= a\frac{\sin n\delta/2}{\sin\delta/2}\cos\left\{\omega t + \left(\frac{n-1}{2}\right)\delta\right\}$$

We can derive the same result using the complex exponential notation and *taking the real part* of the series expressed as the geometrical progression

$$a\,\mathrm{e}^{\mathrm{i}\omega t} + a\,\mathrm{e}^{\mathrm{i}(\omega t+\delta)} + a\,\mathrm{e}^{\mathrm{i}(\omega t+2\delta)} + \cdots + a\,\mathrm{e}^{\mathrm{i}[\omega t+(n-1)\delta]}$$
$$= a\,\mathrm{e}^{\mathrm{i}\omega t}(1 + z + z^2 + \cdots + z^{(n-1)})$$

where $z = \mathrm{e}^{\mathrm{i}\delta}$.

Writing

$$S(z) = 1 + z + z^2 + \cdots + z^{n-1}$$

and

$$z[S(z)] = z + z^2 + \cdots + z^n$$

we have

$$S(z) = \frac{1-z^n}{1-z} = \frac{1-\mathrm{e}^{\mathrm{i}n\delta}}{1-\mathrm{e}^{\mathrm{i}\delta}}$$

So

$$a\,\mathrm{e}^{\mathrm{i}\omega t}S(z) = a\,\mathrm{e}^{\mathrm{i}\omega t}\frac{1-\mathrm{e}^{\mathrm{i}n\delta}}{1-\mathrm{e}^{\mathrm{i}\delta}}$$
$$= a\,\mathrm{e}^{\mathrm{i}\omega t}\frac{\mathrm{e}^{\mathrm{i}n\delta/2}(\mathrm{e}^{-\mathrm{i}n\delta/2} - \mathrm{e}^{\mathrm{i}n\delta/2})}{\mathrm{e}^{\mathrm{i}\delta/2}(\mathrm{e}^{-\mathrm{i}\delta/2} - \mathrm{e}^{\mathrm{i}\delta/2})}$$
$$= a\,\mathrm{e}^{\mathrm{i}[\omega t + (\frac{n-1}{2})\delta]}\frac{\sin n\delta/2}{\sin\delta/2}$$

with the real part

$$= a\cos\left[\omega t + \left(\frac{n-1}{2}\right)\delta\right]\frac{\sin n\delta/2}{\sin\delta/2}$$

which recovers the original cosine term from the complex exponential notation.

(Problem 7.18)

(4) Suppose we represent a harmonic oscillation by the complex exponential form

$$z = a\,\mathrm{e}^{\mathrm{i}\omega t}$$

where a is the amplitude. Replacing i by $-$i defines the *complex conjugate*

$$z^* = a\,\mathrm{e}^{-\mathrm{i}\omega t}$$

The use of this conjugate is discussed more fully in Chapter... but here we can note that the product of a complex quantity and its conjugate is always equal to the square of the amplitude for

$$\begin{aligned} zz^* &= a^2\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{e}^{-\mathrm{i}\omega t} = a^2\,\mathrm{e}^{(\mathrm{i}-\mathrm{i})\omega t} = a^2\,\mathrm{e}^0 \\ &= a^2 \end{aligned}$$

(Problem 7.19)

Problem 7.1
The equation of motion

$$m\ddot{x} = -sx \quad \text{with} \quad \omega^2 = \frac{s}{m}$$

applies directly to the system in Figure 7.1(c).

If the pendulum bob of Figure 7.1(a) is displaced a small distance x show that the stiffness (restoring force per unit distance) is mg/l and that $\omega^2 = g/l$ where g is the acceleration due to gravity. Now use the small angular displacement θ instead of x and show that ω is the same.

In Figure 7.1(b) the angular oscillations are rotational so the mass is replaced by the moment of inertia I of the disc and the stiffness by the restoring couple of the wire which is C rad^{-1} of angular displacement. Show that $\omega^2 = C/I$.

In Figure 7.1(d) show that the stiffness is $2T/l$ and that $\omega^2 = 2T/lm$.

In Figure 7.1(e) show that the stiffness of the system in $2\rho Ag$, where A is the area of cross section and that $\omega^2 = 2g/l$ where g is the acceleration due to gravity.

In Figure 7.1(f) only the gas in the flask neck oscillates, behaving as a piston of mass ρAl. If the pressure changes are calculated from the equation of state use the adiabatic relation pV^{γ} = constant and take logarithms to show that the pressure change in the flask is

$$\mathrm{d}p = -\gamma p \frac{\mathrm{d}V}{V} = -\gamma p \frac{Ax}{V},$$

where x is the gas displacement in the neck. Hence show that $\omega^2 = \gamma pA/l\rho V$. Note that γp is the stiffness of a gas ().

In Figure 7.1(g), if the cross-sectional area of the neck is A and the hydrometer is a distance x above its normal floating level, the restoring force depends on the volume of liquid displaced (Archimedes' principle). Show that $\omega^2 = g\rho A/m$.

Check the dimensions of ω^2 for each case.

Problem 7.2

Show by the choice of appropriate values for A and B in equation (7.2) that equally valid solutions for x are

$$\begin{aligned} x &= a\cos(\omega t + \phi) \\ x &= a\sin(\omega t - \phi) \\ x &= a\cos(\omega t - \phi) \end{aligned}$$

and check that these solutions satisfy the equation

$$\ddot{x} + \omega^2 x = 0$$

Problem 7.3

The pendulum in Figure 1.1(a) swings with a displacement amplitude a. If its starting point from rest is

$$\begin{aligned} &\text{(a)} \quad x = a \\ &\text{(b)} \quad x = -a \end{aligned}$$

find the different values of the phase constant ϕ for the solutions

$$\begin{aligned} x &= a\sin(\omega t + \phi) \\ x &= a\cos(\omega t + \phi) \\ x &= a\sin(\omega t - \phi) \\ x &= a\cos(\omega t - \phi) \end{aligned}$$

For each of the different values of ϕ, find the values of ωt at which the pendulum swings through the positions

$$\begin{aligned} x &= +a/\sqrt{2} \\ x &= a/2 \end{aligned}$$

and

$$x = 0$$

for the first time after release from

$$x = \pm a$$

Problem 7.4

When the electron in a hydrogen atom bound to the nucleus moves a small distance from its equilibrium position, a restoring force per unit distance is given by

$$s = e^2/4\pi\epsilon_0 r^2$$

where $r = 0.05$ nm may be taken as the radius of the atom. Show that the electron can oscillate with a simple harmonic motion with

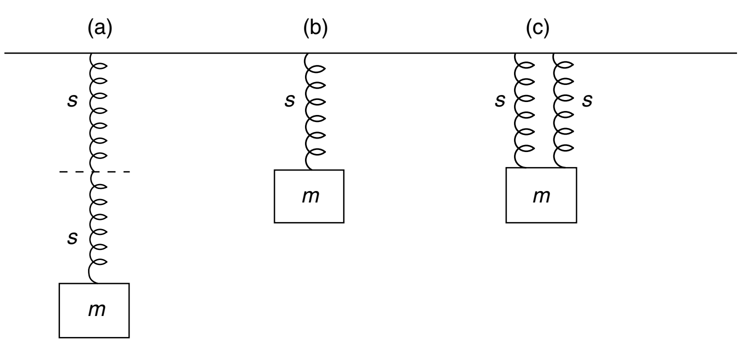
$$\omega_0 \approx 4.5 \times 10^{-16}\,\mathrm{rad\,s^{-1}}$$

If the electron is forced to vibrate at this frequency, in which region of the electromagnetic spectrum would its radiation be found?

$$e = 1.6 \times 10^{-19}\,\mathrm{C},\ \text{electron mass } m_e = 9.1 \times 10^{-31}\,\mathrm{kg}$$
$$\epsilon_0 = 8.85 \times 10^{-12}\,\mathrm{N^{-1}\,m^{-2}\,C^2}$$

Problem 7.5

Show that the values of ω^2 for the three simple harmonic oscillations (a), (b), (c) in the diagram are in the ratio 1 : 2 : 4.



Problem 7.6

The displacement of a simple harmonic oscillator is given by

$$x = a \sin(\omega t + \phi)$$

If the oscillation started at time $t = 0$ from a position x_0 with a velocity $\dot{x} = v_0$ show that

$$\tan \phi = \omega x_0 / v_0$$

and

$$a = (x_0^2 + v_0^2/\omega^2)^{1/2}$$

Problem 7.7

A particle oscillates with simple harmonic motion along the x axis with a displacement amplitude a and spends a time $\mathrm{d}t$ in moving from x to $x + \mathrm{d}x$. Show that the probability of finding it between x and $x + \mathrm{d}x$ is given by

$$\frac{\mathrm{d}x}{\pi(a^2 - x^2)^{1/2}}$$

(in wave mechanics such a probability is not zero for $x > a$).

Problem 7.8

Many identical simple harmonic oscillators are equally spaced along the x axis of a medium and a photograph shows that the locus of their displacements in the y direction is a sine curve. If the distance λ separates oscillators which differ in phase by 2π radians, what is the phase difference between two oscillators a distance x apart?

Problem 7.9

A mass stands on a platform which vibrates simple harmonically in a vertical direction at a frequency of 5 Hz. Show that the mass loses contact with the platform when the displacement exceeds 10^{-2}m.

Problem 7.10

A mass M is suspended at the end of a spring of length l and stiffness s. If the mass of the spring is m and the velocity of an element $\mathrm{d}y$ of its length is proportional to its distance y from the fixed end of the spring, show that the kinetic energy of this element is

$$\frac{1}{2}\left(\frac{m}{l}\mathrm{d}y\right)\left(\frac{y}{l}v\right)^2$$

where v is the velocity of the suspended mass M. Hence, by integrating over the length of the spring, show that its total kinetic energy is $\frac{1}{6}mv^2$ and, from the total energy of the oscillating system, show that the frequency of oscillation is given by

$$\omega^2 = \frac{s}{M + m/3}$$

Problem 7.11

The general form for the energy of a simple harmonic oscillator is

$$E = \tfrac{1}{2}\text{mass (velocity)}^2 + \tfrac{1}{2}\text{stiffness (displacement)}^2$$

Set up the energy equations for the oscillators in Figure 1.1(a), (b), (c), (d), (e), (f) and (g), and use the expression

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0$$

to derive the equation of motion in each case.

Problem 7.12
The displacement of a simple harmonic oscillator is given by $x = a \sin \omega t$. If the values of the displacement x and the velocity $\dot{x}$ are plotted on perpendicular axes, eliminate t to show that the locus of the points $(x, \dot{x})$ is an ellipse. Show that this ellipse represents a path of constant energy.

Problem 7.13
In Chapter... the intensity of the pattern when light from two slits interferes (Young's experiment) will be seen to depend on the superposition of two simple harmonic oscillations of equal amplitude a and phase difference δ. Show that the intensity

$$I = R^2 \propto 4a^2 \cos^2 \delta/2$$

Between what values does the intensity vary?

Problem 7.14
Carry out the process indicated in the text to derive equation (7.3).

Problem 7.15
The co-ordinates of the displacement of a particle of mass m are given by

$$x = a \sin \omega t$$
$$y = b \cos \omega t$$

Eliminate t to show that the particle follows an elliptical path and show by adding its kinetic and potential energy at any position x, y that the ellipse is a path of constant energy equal to the sum of the separate energies of the simple harmonic vibrations.

Prove that the quantity $m(x\dot{y} - y\dot{x})$ is also constant. What does this quantity represent?

Problem 7.16
Two simple harmonic motions of the same frequency vibrate in directions perpendicular to each other along the x and y axes. A phase difference

$$\delta = \phi_2 - \phi_1$$

exists between them such that the principal axes of the resulting elliptical trace are inclined at an angle to the x and y axes. Show that the measurement of two separate values of x (or y) is sufficient to determine the phase difference.
(Hint: use equation (7.3) and measure$y$(max), and y for $(x = 0.)$

Problem 7.17
Take a random group of $n > 7$ values of ϕ in the range $0 \leq \phi \leq \pi$ and form the product

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} \cos \phi_i \sum_{j=1}^{n} \cos \phi_j$$

Show that the average value obtained for several such groups is negligible with respect to $n/2$.

Problem 7.18

Use the method of example (3) (p....) to show that

$$a\sin\omega t + a\sin(\omega t+\delta) + a\sin(\omega t+2\delta) + \cdots + a\sin[\omega t+(n-1)\delta]$$
$$= a\sin\left[\omega t + \frac{(n-1)}{2}\delta\right]\frac{\sin n\delta/2}{\sin\delta/2}$$

Problem 7.19

If we represent the sum of the series

$$a\cos\omega t + a\cos(\omega t+\delta) + a\cos(\omega t+2\delta) + \cdots + a\cos[\omega t+(n-1)\delta]$$

by the complex exponential form

$$z = a\,\mathrm{e}^{\mathrm{i}\omega t}(1 + \mathrm{e}^{\mathrm{i}\delta} + \mathrm{e}^{\mathrm{i}2\delta} + \cdots + \mathrm{e}^{\mathrm{i}(n-1)\delta})$$

show that

$$zz^* = a^2\frac{\sin^2 n\delta/2}{\sin^2\delta/2}$$

Summary of Important Results

Simple Harmonic Oscillator (mass m, stiffness s, amplitude a)
Equation of motion $\ddot{x} + \omega^2 x = 0$ where $\omega^2 = s/m$
Displacement $x = a\sin(\omega t + \phi)$
Energy $= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}sx^2 = \frac{1}{2}m\omega^2 a^2 = \frac{1}{2}sa^2 =$ constant

Superposition (Amplitude and Phase) of two SHMs
One-dimensional

Equal ω, different amplitudes, phase difference δ, resultant R where $R^2 = a_1^2 + a_2^2 + 2a_1a_2\cos\delta$
Different ω, equal amplitude,

$$x = x_1 + x_2 = a(\sin\omega_1 t + \sin\omega_2 t)$$
$$= 2a\sin\frac{(\omega_1+\omega_2)t}{2}\cos\frac{(\omega_2-\omega_1)t}{2}$$

Two-dimensional: perpendicular axes
Equal ω, different amplitude—giving general conic section

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - \frac{2xy}{a_1a_2}\cos(\phi_2-\phi_1) = \sin^2(\phi_2-\phi_1)$$

(basis of optical polarization)

References

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