INF8245E(Fall 2021) : Machine Learning — Given on: Aug 30, 08 am

Assignment #0 — Due on : Sept 13, 10 pm

- The goal of this assignment is a self-assessment on fundamental problems from Probability Theory, Linear Algebra and Real Analysis. This assignment will NOT be graded.
- This is an individual assignment. Collaborations and discussions with others regarding the problems and solutions are strictly prohibited.
- You have to submit the `pdf` copy of the assignment on gradescope before the deadline. If you handwrite your solutions, you need to scan the pages, merge them to a single `pdf` file and submit.
- Your future assignments will not be graded if you fail to submit assignment-0.
- **Honour Code**: Please write out the following, digitally sign it or type your name under it and return it along with your assignment.
  By enrolling for INF8245E Machine Learning course, I agree that all the work submitted will be mine and original, and will not be plagiarized. Unless otherwise specifically stated by the instructor or TAs, I will not collaborate with anyone on my assignments or tests. I understand that any violation of this honor code will be strictly dealt with.

Probability Theory

1. (a) The Smiths have two children. At least one of them is a boy. What is the probability that both children are boys?

   (b) Suppose that 5% of men and 0.25% of women are color-blind. A person is chosen at random and that person is color-blind. What is the probability that the person is male? Assume males and females to be in equal numbers.

2. Abdel, a TA for INF8245E, will give the students a minimum mark of 1. The highest mark he would give, depending on his mood, will be based on this distribution function

$$F_Y(y) = P(Y \leq y) = 1 - \frac{1}{y^2}, 1 \leq y < \infty$$

   (a) Verify that $F_Y(y)$ is a cdf.

   (b) Find $f_Y(y)$, the pdf of Y.

   (c) Arjun, however, gives a minimum mark of 0, and uses a different marking scale that is 1/10th of Abdel's, and the highest mark becomes $Z = 10(Y - 1)$. Find $F_Z(z)$ and $f_Z(z)$.

3. Let X have the pdf

$$f(x) = \frac{4}{\beta^3\sqrt{\pi}} x^2 e^{-x^2/\beta^2}, 0 < x < \infty, \beta > 0$$

   (a) Verify that f(x) is a pdf.
   (b) Find E(X) and Var(X).

4. The random pair (X, Y) has the distribution:

	X=1	X=2	X=3
Y = 2	1/12	1/6	1/12
Y = 3	1/6	0	1/6
Y = 4	0	1/3	0

   (a) Show that X and Y are dependent.
   (b) Give a probability table for random variables U and V that have the same marginals as X and Y but are independent.

Linear Algebra

5. Let $u, v \in \mathbb{R}^d$ be two vectors of dimension d and let $A \in \mathbb{R}^{n\times d}$ be a $n \times d$ matrix. Answer the following questions.

   (a) Give a formula for the euclidean norm (L_2 norm) of the vector u. (You can give either a closed-form formula, or an expanded one in terms of the components of the vector, $u_1, u_2, \cdots, u_d$.)
   (b) Give a formula for the dot product (also called inner product) of the vectors u and v. (Again, you can give either a closed-form formula, or an expanded one. For the expanded formula, feel free to use the summation notation Σ to reduce clutter.) What is the dimension of the inner product? (scalar, vector, or matrix?)
   (c) Give a formula for the matrix-vector product of the matrix A and the vector u. This product is written as Au. The formula that you write, should be in terms of the components of the corresponding matrix and vector.
   (d) Write the Frobenius norm of the matrix A in terms of its components. Notice that you can easily do this by using the summation notation $\sum$.
   (e) What are the dimensions of A^TA and AA^T?

6. Let E_1 be a set of eigenvectors with eigenvalue λ_1 and E_2 be a set of eigenvectors with eigenvalue $\lambda_2 \neq \lambda_1$.
Prove or disprove the following statement -
There exists a vector in E_2 which can be expressed as a linear combination of vectors in E_1.

7. You are given that the eigenvalues of a matrix A are 3, 2 and 2. What can you tell about the following statements?
   (a) A is invertible
   (b) A is diagonalizable

   Your answer can be "yes","no" or "depends". For each one, give arguments and/or examples to support your answer.

8. For a matrix A of size n $\times$ n you are told that all its n eigen vectors are independent. Let S denote the matrix whose columns are the n eigen vectors of A.
   (a) Is A invertible?
   (b) Is A diagonalizable?
   (c) If S invertible?
   (d) Is S diagonalizable?

   Your answer can be "yes","no" or "depends". For each one, give arguments and/or examples to support your answer.

Real Analysis

9. Compute the first order differentiation for the following functions. Show the steps and clearly mention the differentiation rule used to arrive at that step.
   (a) $\mathrm{y} = \mathrm{x}^4(\mathrm{sinx}^3 - \mathrm{cosx}^2)$. Find $\frac{dy}{dx}$.
   (b) $\mathrm{y} = 12\mathrm{x}^4 - 5\mathrm{x}^2 + 15$. Find $\frac{dy}{dx}$.
   (c) $\mathrm{y} = \log \mathrm{p},\ \mathrm{p} = \mathrm{x}^2$. Find $\frac{\partial\ y}{\partial x}$.

10. Compute the minima and maxima of the following function.
    (a) $f(x) = x^3 - 6x^2 + 9x + 15$
    (b) Now, use first and second order derivatives to identify which of the critical points are minima and maxima without directly evaluating $f(x)$, the value of the function at the critical points.