Assignment 0:

Zahra Parham- 2122841

Probability Theory

1.

(a). $P(Both\ children\ are\ boys) = \dfrac{1}{2}$

(b). $P(color - blind) = \dfrac{1}{2} * 5 + \dfrac{1}{2} * 0.25 = 2.6\%$

$P(person\ be\ male \mid color - blind) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.05 * \frac{1}{2}}{0.02625}$ =0.94

2.

(a). $First: \quad 1 \leq y \leq \infty \rightarrow 1 \leq y^2 \leq \infty \rightarrow 0 \leq \dfrac{1}{y^2} \leq 1 \rightarrow -1 \leq -\dfrac{1}{y^2} \leq 0 \rightarrow 0 \leq 1 - \dfrac{1}{y^2} \leq 1 \rightarrow 0 \leq F_Y(y) \leq 1$

$Second: \ \lim_{x \to +1} 1 - \dfrac{1}{y^2} = 0$

$Third: \quad \lim_{x \to \infty} 1 - \dfrac{1}{y^2} = 1$

$Fourth: if \quad y_1 \leq y_2 \rightarrow y_1^2 \leq y_2^2 \rightarrow \dfrac{1}{y_2^2} \leq \dfrac{1}{y_1^2} \rightarrow 1 - \dfrac{1}{y_1^2} \leq 1 - \dfrac{1}{y_2^2} \rightarrow F_Y(y_1) \leq F_Y(y_2)$

so this function has all the conditions and it is a CDF.

(b). $f_Y(y) = \frac{\partial F_Y(y)}{\partial y} = \frac{\partial(1-\frac{1}{y^2})}{\partial y} = \frac{2}{y^3}$

(c). $F_Z(z) = P(Z \leq z) = P(10(Y-1) \leq z) \rightarrow^{P(Y \leq y) = 1 - \frac{1}{y^2}} P(Y \leq \frac{z}{10} + 1) = 1 - \frac{1}{(\frac{z}{10}+1)^2} \qquad \frac{11}{10} \leq z \leq \infty$

$f_Z(z) = F_Z(z)' = \frac{0.2}{(\frac{z}{10}+1)^3}$

3.

(a). it has to be two condition:

$first: f(x) \geq 0 \rightarrow e^{\frac{-x^2}{\beta^2}} \geq 0 \rightarrow^{x^2 \geq 0} x^2 e^{\frac{-x^2}{\beta^2}} \geq 0 \rightarrow^{\beta > 0} \frac{x^2 e^{\frac{-x^2}{\beta^2}}}{\beta^3 \sqrt{\pi}} \geq 0$

$second: \int_{-\infty}^{\infty} f_X(x)\, dx = 1$

(b). $E(x) = \int_{-\infty}^{\infty} f_X(x)\, dx \qquad Var[x] = E[(x - E(x)^2]$

4.

(a). $P(Y=3) = P(Y=3, X=1) + P(Y=3, X=2) + P(Y=3, X=3) = \frac{1}{6} + 0 + \frac{1}{6} = \frac{1}{3}$

$P(X=2) = P(Y=2, X=2) + P(Y=3, X=2) + P(Y=4, X=2) = \frac{1}{6} + 0 + \frac{1}{3} = \frac{1}{2}$

$0 = P_{XY}(2,3) \neq P_X(2) P_Y(3) = \frac{1}{2} * \frac{1}{3} = \frac{1}{3}$

So X and Y are dependent.

(b).

	u=1	u=2	u=3
$v=2$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
$v=3$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$
$v=4$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{12}$

Linear Algebra

5.

(a). $\|u\|_2 = (\sum_{n=1}^{d} \|u_i\|^2)^{\frac{1}{2}}$

(b). $u.v = u_1.v_1 + u_2.v_2 + ... + u_d.v_d$ also the dimension of the inner product is 1 (scalar)

(c). $Au = \begin{bmatrix} a_{11}u_1 + a_{12}u_2 + \cdots + a_{1d}u_d \\ a_{21}u_1 + a_{22}u_2 + \cdots + a_{2d}u_d \\ \vdots \\ a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nd}u_d \end{bmatrix}$

(d). $\|A\|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{d} |a_{ij}|^2}$

(e). $A^T A = d * d \quad and \qquad AA^T = n * n$

6.

consider X to be a vector in E_2 which will be found in E_1:

$\lambda_1 X = AX$
$\lambda_2 X = AX \rightarrow (\lambda_1 - \lambda_2)X = 0 \rightarrow^{\lambda_1 \neq \lambda_2} X = 0$

so we disproved the following statement.

7.

(a). Yes: A matrix is invertible if and only if it does not have zero as an eigenvalue since the determinant of a matrix is the product of its eigenvalues so, if one of the eigenvalues is 0, then the determinant of the matrix is also 0 and a square matrix is called invertible if and only if the value of its determinant is not equal to zero.

(b). No: A matrix is diagonalizable if and only if for each eigenvalue the dimension of the eigenspace is equal to the multiplicity of the eigenvalue. Meaning, if the matrices have distinct eigenvalues(multiplicity = 1), But here the multiplicity of eigenvalue =2 is not 1.

8.

(a). Depends: if it has distinct eigenvalues then it is invertible but still it is possible that it has repeating eigenvalues.

(b). Yes: If A is diagonalizable, there is a P such that P−1 exists and AP=PD (D is diagonal). Therefore, columns of P are linearly independent and they are eigenvectors of A. Therefore, A has n linearly independent eigenvectors.

(c). Yes: Since all the columns of S are independent so S is invertible.

(d). Depends: S is invertible and Invertibility and diagonalizability do not affect each other and are two completely different concepts.

Real Analysis

9.

The main rules I used:

Differentiation Rules	
Constant Rule	$\frac{d}{dx}[c]=0$
Power Rule	$\frac{d}{dx}x^n = nx^{n-1}$
Product Rule	$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x)+f(x)g'(x)$
Quotient Rule	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$
Chain Rule	$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$

Derivative	Domain
$(\sin x)' = \cos x$	$-\infty < x < \infty$
$(\cos x)' = -\sin x$	$-\infty < x < \infty$
$(\tan x)' = \frac{1}{\cos^2 x} = \sec^2 x$	$x \neq \frac{\pi}{2} + \pi n, n \in \mathbb{Z}$
$(\cot x)' = -\frac{1}{\sin^2 x} = -\csc^2 x$	$x \neq \pi n, n \in \mathbb{Z}$
$(\sec x)' = \tan x \sec x$	$x \neq \frac{\pi}{2} + \pi n, n \in \mathbb{Z}$
$(\csc x)' = -\cot x \csc x$	$x \neq \pi n, n \in \mathbb{Z}$

Derivatives of Logarithmic Functions

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\frac{d}{dx}\ln[f(x)] = \frac{1}{f(x)}f'(x)$$

$$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln a}$$

$$\frac{d}{dx}\log_a[f(x)] = \frac{1}{f(x)\ln a}f'(x)$$

(a). Product Rule : $f(x) = x^4 \ and \ \ g(x) = sinx^3 - cosx^2$

$f'(x) = 4x^3 \ \ and \ \ g'(x) = 3x^2cosx^3 + 2xsinx^2$

$y' = f'(x)g(x) + f(x)g'(x) = 4x^3(sinx^3 - cosx^2) + x^4(3x^2cosx^3 + 2xsinx^2$

(b). Power Rule/ Constant Rule: $y' = 48x^3 - 10x$

(c). Logarithmic function: $y = logx^2 \rightarrow y' = \frac{1}{f(x)ln}f'(x) = \frac{1}{x^2ln}(2x)$

10.

(a). $f(x) = x^3 - 6x^2 + 9x + 15 \rightarrow f'(x) = 3x^2 - 12x + 9$

let $f'(x) = 0 \rightarrow 3(x^2 - 4x + 3) = 0 \rightarrow x = 1, 3$

$f(1) = 19 \rightarrow maxima \qquad f(3) = -39 \rightarrow minima$

(b). Now let us find the second derivative

$f^{''}(x) = 6x - 12 \rightarrow f^{''}(1) = -6 < 0 \rightarrow maxima$

$f^{''}(3) = 6 \rightarrow minima$