Question: **addressed FSSP Problem (6.15)**

This problem is from Methods and Models in Mathematical Programming by MirHassani S.A., Hooshmand F.

Sets and indices and parameters are like the book.

The decision variables are as follows:

$\delta_{i,j,j'}$ $\forall i \in I$, $\forall j,j' \in J$, $j \neq j'$	:	Binary variable that if job j ' is done immediately after job j on machine i is one otherwise zero
$\alpha_{i,j}$ $\forall i \in I$, $\forall j \in J$	:	A binary variable that is one if the job j is done in machine i , otherwise it is zero.
$\beta_{i,j}$ $\forall i \in I$, $\forall j \in J$	:	The binary variable that if in machine i work j is done in the last position is one otherwise zero.
$x_{i,j}$ $\forall i \in I$, $\forall j \in J$	:	The continuous and nonnegative variable represents the start time j on machine i.

Now we define the model as follows:

$$Min \ Max_{j \in J} \ (x_{m,j} + t_{m,j}) \qquad (1)$$

$S.T.$

$$x_{m,j} + t_{m,j} \le d_j \qquad \forall j \in J \qquad (2)$$

$$x_{i,j} + t_{i,j} \le x_{i+1,j} \qquad \forall j \in J \ , \ i \in I \ , \ i < \qquad (3)$$

$$if \quad \delta_{i,j,j'} = 1 \quad then \quad x_{i,j} + t_{i,j} + p_{i,j,j'} \le x_{i,j'} \qquad \forall j,j' \in J \ , \ j \neq j' \ , \ \forall i \in I \qquad (4)$$

$$\alpha_{i,j} = 1 \quad \vee \quad \exists \ j' \in J \ \ \delta_{i,j',j} = 1 \qquad \forall j \in J \ , \ \forall i \in I \qquad (5)$$

$$\beta_{i,j} = 1 \quad \vee \quad \exists \ j' \in J \ \ \delta_{i,j,j'} = 1 \qquad \forall j \in J \ , \ \forall i \in I \qquad (6)$$

$$\sum_{j \in J} \alpha_{i,j} = 1 \qquad \forall i \in I \qquad (7)$$

$$\sum_{j \in J} \beta_{i,j} = 1 \qquad \forall i \in I \qquad (8)$$

$$\delta_{i,j,j'} \in \{0,1\} \qquad \forall i \in I \ , \ \forall j,j' \in J \ , \ j \neq j' \qquad (9)$$

$$\alpha_{i,j} \ , \ \beta_{i,j} \in \{0,1\} \qquad \forall i \in I \ , \ \forall j \in J \qquad (10)$$

$$x_{i,j} \ge 0 \qquad \forall i \in I \ , \ \forall j \in J \qquad (11)$$

Now I will linearize the above constraints:

We underline adverb (4) as follows:

$$x_{i,j} + t_{i,j} + p_{i,j,j'} \leq x_{i,j'} + M\left(1 - \delta_{i,j,j'}\right) \qquad \forall j, j' \in J,\ j \neq j',\quad \forall i \in I$$

and for M we consider:

$$x_{i,j} + t_{i,j} + p_{i,j,j'} - x_{i,j'} \leq M$$

In the GAMS code, I set the value of M to 100, which is a valid upper bound.

Now I linearize adverb (5) as follows:

$$\alpha_{i,j} = 1 \quad \vee \quad \exists\, j' \in J \ \ \delta_{i,j',j} = 1 \qquad \forall j, j' \in J,\ j \neq j',\quad \forall i \in I$$

$$\left(if\ \rho^{(1)}{}_{i,j} = 1\ \ then\ \alpha_{i,j} = 1\right)\ \ and\ \ \left(if\ \rho^{(1)}{}_{i,j} = 0\ \ then\ \ \exists\, j' \in J\ \ \delta_{i,j',j} = 1\right)$$

after linearization we have:

$$\alpha_{i,j} \geq 1 - M\left(1 - \rho^{(1)}{}_{i,j}\right)$$

$$\sum_{j' \in J,\ j' \neq j} \delta_{i,j',j} \geq 1 - M\rho^{(1)}{}_{i,j}$$

$$\sum_{j' \in J,\ j' \neq j} \delta_{i,j',j} \leq 1 + M\rho^{(1)}{}_{i,j}$$

$$\left(if\ \rho^{(2)}{}_{i,j} = 1\ \ then\ \beta_{i,j} = 1\right)\ \ and\ \ \left(if\ \rho^{(2)}{}_{i,j} = 0\ \ then\ \ \exists\, j' \in J\ \ \delta_{i,j,j'} = 1\right)$$

$$\beta_{i,j} \geq 1 - M\left(1 - \rho^{(2)}{}_{i,j}\right)$$

$$\sum_{j' \in J,\ j' \neq j} \delta_{i,j,j'} \geq 1 - M\rho^{(2)}{}_{i,j}$$

$$\sum_{j' \in J,\ j' \neq j} \delta_{i,j,j'} \leq 1 + M\rho^{(2)}{}_{i,j}$$

$$\rho^{(1)} + \rho^{(2)} \leq 1$$

Model in a linear way:

$$Min\ \ w \qquad (1)$$

$$S.T.$$

$$w \geq x_{m,j} + t_{m,j} \qquad \forall j \in J \qquad (2)$$

$$x_{m,j} + t_{m,j} \leq d_j \qquad \forall j \in J \qquad (3)$$

$$x_{i,j} + t_{i,j} \leq x_{i+1,j} \qquad \forall j \in J,\ \ i \in I,\ \ i < m \qquad (4)$$

$$x_{i,j} + t_{i,j} + p_{i,j,j'} \leq x_{i,j'} + M\left(1 - \delta_{i,j,j'}\right) \qquad \forall j \in J,\ \ \forall i \in I \qquad (5)$$

$$\alpha_{i,j} \geq 1 - M\left(1 - \rho^{(1)}{}_{i,j}\right) \qquad \forall j \in J \;,\; \forall i \in I \tag{6}$$

$$\sum_{j' \in J\,,\; j' \neq j} \delta_{i,j',j} \geq 1 - M\rho^{(1)}{}_{i,j} \qquad \forall j \in J \;,\; \forall i \in I \tag{7}$$

$$\sum_{j' \in J\,,\; j' \neq j} \delta_{i,j',j} \leq 1 + M\rho^{(1)}{}_{i,j} \qquad \forall j \in J \;,\; \forall i \in I \tag{8}$$

$$\beta_{i,j} \geq 1 - M\left(1 - \rho^{(2)}{}_{i,j}\right) \qquad \forall j \in J \;,\; \forall i \in I \tag{9}$$

$$\sum_{j' \in J\,,\; j' \neq j} \delta_{i,j,j'} \geq 1 - M\rho^{(2)}{}_{i,j} \qquad \forall j \in J \;,\; \forall i \in I \tag{10}$$

$$\sum_{j' \in J\,,\; j' \neq j} \delta_{i,j,j'} \leq 1 + M\rho^{(2)}{}_{i,j} \qquad \forall j \in J \;,\; \forall i \in I \tag{11}$$

$$\rho^{(1)}{}_{i,j} + \rho^{(2)}{}_{i,j} \leq 1 \qquad \forall j \in J \;,\; \forall i \in I \tag{12}$$

$$\sum_{j \in J} \alpha_{i,j} = 1 \qquad \forall i \in I \tag{13}$$

$$\sum_{j \in J} \beta_{i,j} = 1 \qquad \forall i \in I \tag{14}$$

$$\delta_{i,j,j'} \in \{0,1\} \qquad \forall i \in I \;,\; \forall j,j' \in J \;,\; j \neq j' \tag{15}$$

$$\alpha_{i,j} \;,\; \beta_{i,j} \in \{0,1\} \qquad \forall i \in I \;,\; \forall j \in J \tag{16}$$

$$x_{i,j} \geq 0 \qquad \forall i \in I \;,\; \forall j \in J \tag{17}$$

$$w \geq 0 \tag{18}$$

$$\rho^{(2)}{}_{i,j} \;,\; \rho^{(1)}{}_{i,j} \in \{0,1\} \qquad \forall j \in J \;,\; \forall i \in I$$

	machine 1	**machine 2**	**machine 3**
$x = 0$	job 7	-	-
$x = 1$	job 9	job 7	-
$x = 2$	-	-	-
$x = 3$	job 2	-	-
$x = 4$	-	-	-
$x = 5$	-	-	-
$x = 6$	-	job 9	job 7
$x = 7$	-	-	-
$x = 8$	job 1	-	-
$x = 9$	-	job 2	-
$x = 10$	job 5	job 1	job 2
$x = 11$	-	-	job 9
$x = 12$	job 8	-	-
$x = 13$	-	-	-
$x = 14$	-	-	-
$x = 15$	job 3	job 8	-

$x = 16$	-	-	job 1
$x = 17$	-	-	-
$x = 18$	-	-	-
$x = 19$	job 4	job 3	-
$x = 20$	job 6	job 5	job 3
$x = 21$	-	-	-
$x = 22$	-	job 4	job 5
$x = 23$	-	-	-
$x = 24$	-	-	-
$x = 25$	-	job 6	-
$x = 26$	-	-	job 4
$x = 27$	-	-	-
$x = 28$	-	-	job 8
$x = 29$	-	-	job 8

Arrangement of work on the machine 1:

$$9 - 2 - 1 - 5 - 8 - 3 - 4 - 6$$

Arrangement of work on the machine 2:

$$7 - 9 - 2 - 1 - 8 - 3 - 5 - 4 - 6$$

Arrangement of work on the machine 3:

$$7 - 2 - 9 - 1 - 3 - 5 - 4 - 8 - 6$$

$$w^* = 32$$

```
----     87 VARIABLE z.L                    =       32.000

----     87 VARIABLE x.L

            1           2           3           4           5           6

1       8.000       3.000      15.000      19.000      10.000      20.000
2      10.000       9.000      19.000      22.000      20.000      25.000
3      16.000      10.000      20.000      26.000      22.000      29.000

+           7           8           9

1                  12.000       1.000
2       1.000      15.000       6.000
3       6.000      28.000      11.000
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```
----          87 VARIABLE a.L

                   7

1            1.000
2            1.000
3            1.000

----          87 VARIABLE b.L

                   6

1            1.000
2            1.000
3            1.000
```

```
----     87 VARIABLE y.L

              1           2           3           4           5           6

1.1                                                       1.000
1.2       1.000
1.3                                           1.000
1.4                                                                   1.000
1.8                               1.000
1.9                   1.000
2.2       1.000
2.3                                                       1.000
2.4                                                                   1.000
2.5                                           1.000
2.8                               1.000
2.9                   1.000
3.1                               1.000
3.3                                                       1.000
3.5                                           1.000
3.7                   1.000
3.8                                                                   1.000
3.9       1.000

 +            8           9

1.5       1.000
1.7                   1.000
2.1       1.000
2.7                   1.000
3.2                   1.000
3.4       1.000
```