Semi-Implicit Time Integration

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1 Governing Equations

We modify the original set of governing equations by setting $h_i = H_i + \eta_i$, and separating linear and non-linear terms. We had used $H_1 = H_2 = 1$, but now will consider a much thicker lower layer and a thinner higher layer such that $\frac{H_2}{H_1} << 1$, and $H_1 + H_2 = 2$ to keep the same nondimensional constant as before. The goal of this adjustment is to get similar results as the single-layer models. We will be treating linear terms implicitly, and non-linear terms explicitly.

Momentum

$$\begin{aligned}
\partial_t \vec{m_1} =& -\frac{1}{Ro}\left(\hat{k} \times \vec{m_1}\right) - \frac{1}{Fr^2}\nabla\left(\eta_1 + \eta_2\right) + \kappa\nabla^2\vec{m_1} \quad \text{(Linear)} \\
& -\nabla\cdot\left(\frac{1}{H_1+\eta_1}\vec{m_1}\vec{m_1}\right) - \frac{1}{Fr^2}\eta_1\nabla\left(\eta_1+\eta_2\right) - \frac{\vec{m_1}}{H_1+\eta_1}\left(\beta\frac{\hat{Q}}{H}\hat{P}(Q) - \frac{T}{T_{RC}}(\eta_2-\eta_1)\mathcal{H}(\eta_2-\eta_1)\right) \quad \text{(Nonlinear)}
\end{aligned}$$

$$\begin{aligned}
\partial_t \vec{m_2} =& -\frac{1}{Ro}\left(\hat{k} \times \vec{m_2}\right) - \frac{1}{Fr^2}\nabla\left(\eta_1 + \alpha\eta_2\right) + \kappa\nabla^2\vec{m_2} \quad \text{(Linear)} \\
& -\nabla\cdot\left(\frac{1}{H_2+\eta_2}\vec{m_2}\vec{m_2}\right) - \frac{1}{Fr^2}\eta_1\nabla\left(\eta_1+\alpha\eta_2\right) + \frac{\vec{m_1}}{H_1+\eta_1}\left(\beta\frac{\hat{Q}}{H}\hat{P}(Q) - \frac{T}{T_{RC}}(\eta_2-\eta_1)\mathcal{H}(\eta_2-\eta_1)\right) \quad \text{(Nonlinear)}
\end{aligned}$$

Height/Mass

$$\begin{aligned}
\partial_t \eta_1 =& -\nabla\cdot\vec{m_1} + \kappa\nabla^2\eta_1 \quad \text{(Linear)} \\
& -\left(\beta\frac{\hat{Q}}{H}\hat{P}(Q) - \frac{T}{T_{RC}}(\eta_2-\eta_1)\mathcal{H}(\eta_2-\eta_1)\right) \quad \text{(Nonlinear)}
\end{aligned}$$

$$\begin{aligned}
\partial_t \eta_1 =& -\nabla\cdot\vec{m_2} + \kappa\nabla^2\eta_2 \quad \text{(Linear)} \\
& +\left(\beta\frac{\hat{Q}}{H}\hat{P}(Q) - \frac{T}{T_{RC}}(\eta_2-\eta_1)\mathcal{H}(\eta_2-\eta_1)\right) \quad \text{(Nonlinear)}
\end{aligned}$$

Moisture

$$\begin{aligned}
\partial_t Q =& \kappa\nabla^2 q \quad \text{(Linear)} \\
& +\nabla\cdot\left(\vec{u_1}Q\right) = \left(-1+\frac{1}{\epsilon}\right)\hat{P}(Q) \quad \text{(Nonlinear)}
\end{aligned}$$

1.1 Basic Scheme

Consider a set of ODE's with linear $L(x)$ and nonlinear $N(x)$ terms.

$$\partial_t(\vec{x}) = L(\vec{x}) + N(\vec{x}) \tag{1}$$

The discretization is given by treating the nonlinear, explicit terms as the forcing terms in the linear, implicit scheme. Below is an example using the forward Euler time-stepping method.

$$\begin{aligned}\frac{1}{h}\left(\vec{x}^{(n+1)} - \vec{x}^{(n)}\right) &= L(\vec{x}^{(n+1)}) + N(\vec{x}^{(n+1)}) \\ \vec{x}^{(n+1)} &= \vec{x}^{(n)} + h\left(L(\vec{x}^{(n+1)}) + N(\vec{x}^{(n)})\right)\end{aligned}$$

Solving for $\vec{x}^{(n+1)}$ includes the time integration step as well.

1.2 Linear Operator

We collect the linear terms and try to simplify the matrix linear operator. Let $\frac{\beta}{h}$ be the diffusion constant, where h is the time step size.

$$\partial_t\vec{x} = L\vec{x} = \left[\begin{array}{ccc|ccc} \frac{\beta}{h}(\partial_{xx}+\partial_{yy}) & \frac{1}{Ro}y & -\frac{1}{Fr^2}\partial_x & & 0 & -\frac{1}{Fr^2}\partial_x \\ -\frac{1}{Ro}y & \frac{\beta}{h}(\partial_{xx}+\partial_{yy}) & -\frac{1}{Fr^2}\partial_y & 0 & 0 & -\frac{1}{Fr^2}\partial_y \\ -\partial_x & -\partial_y & \frac{\beta}{h}(\partial_{xx}+\partial_{yy}) & 0 & 0 & 0 \\ \hline 0 & 0 & -\frac{1}{Fr}\partial_x & \frac{\beta}{h}(\partial_{xx}+\partial_{yy}) & \frac{1}{Ro}y & -\frac{\alpha}{Fr^2}\partial_x \\ 0 & 0 & -\frac{1}{Fr}\partial_y & -\frac{1}{Ro}y & \frac{\beta}{h}(\partial_{xx}+\partial_{yy}) & -\frac{\alpha}{Fr^2}\partial_y \\ 0 & 0 & 0 & -\partial_x & -\partial_y & \frac{\beta}{h}(\partial_{xx}+\partial_{yy}) \end{array}\right]\begin{bmatrix} m_1 \\ n_1 \\ \eta_1 \\ m_2 \\ n_2 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \partial_t m_1 \\ \partial_t n_1 \\ \partial_t \eta_1 \\ \partial_t m_2 \\ \partial_t n_2 \\ \partial_t \eta_2 \end{bmatrix}$$

1.2.1 Option 3: Remove Coriolis terms from linear implicit scheme, and FFT in x-direction and DCT/DST in y-direction

- Justification for removing the Coriolis terms:
  - The high-frequency waves in our system are the small-scale gravity waves.
  - Earth's rotation has little effect on these waves
  - So, moving the Coriolis terms to the explicit part of the scheme should not have an effect on the high frequency gravity waves.
- Benefits for removing the Coriolis terms:
  - The removal of Coriolis terms allows us to take a Fourier transform in the y-direction as well.
  - With the Coriolis terms, we had $\frac{1}{Ro}y\hat{k}\times\vec{m}_i$. Multiplication in physical space becomes a convolution in the Fourier space. This is inconvenient, since this implies that the different modes are depend on each other.
  - The new linear system should be simpler.
  - The total system is now block diagonal.
- Boundary Conditions:
  - All variables are zonal-periodic.

- Dirichlet BC: Meridional momenta (n_i) are set to zero at the y-boundaries, since we assume that meridional momenta are conserved. Therefore, we use the sine transform.
- Neumann BC: Change in zonal momenta (m_i) and height fluctuations (η_i) across the y-boundaries are zero (Why?). Therefore, we use the cosine transform.
- Note that ∂_y is only every applied to η_i's and n_i's.
  * $\partial_y \eta_i$'s occur only in the RHS of $\partial_t n_i$. Since $\hat{\eta}_{i,k_x,k_y}$ are cosines and $\hat{n}_{i,k_x,k_y}$ are sines, this works out. $\partial_y \to -k_y$.
  * $\partial_y n_i$'s occur only in the RHS of $\partial_t \eta_i$. Since $\hat{\eta}_{i,k_x,k_y}$ are cosines and $\hat{n}_{i,k_x,k_y}$ are sines, this works out. $\partial_y \to k_y$.

For each k_x and k_y pair, we get the following system.

$$\partial_t \vec{x} = \hat{L}_{k_x,k_y} \vec{x}$$

$$= \left[\begin{array}{ccc|ccc} -\frac{\beta}{h}(k_x^2+k_y^2) & 0 & -\frac{1}{Fr^2}k_x i & 0 & 0 & -\frac{1}{Fr^2}k_x i \\ 0 & -\frac{\beta}{h}(k_x^2+k_y^2) & \frac{1}{Fr^2}k_y & 0 & 0 & \frac{1}{Fr^2}k_y \\ -k_x i & -k_y & -\frac{\beta}{h}(k_x^2+k_y^2) & 0 & 0 & 0 \\ \hline 0 & 0 & -\frac{1}{Fr^2}k_x i & -\frac{\beta}{h}(k_x^2+k_y^2) & 0 & -\frac{\alpha}{Fr^2}k_x i \\ 0 & 0 & \frac{1}{Fr^2}k_y & 0 & -\frac{\beta}{h}(k_x^2+k_y^2) & \frac{\alpha}{Fr^2}k_y \\ 0 & 0 & 0 & -k_x i & -k_y & -\frac{\beta}{h}(k_x^2+k_y^2) \end{array}\right] \begin{bmatrix} \hat{m}_{1,k_x,k_y} \\ \hat{n}_{1,k_x,k_y} \\ \hat{\eta}_{1,k_x,k_y} \\ \hat{m}_{2,k_x,k_y} \\ \hat{n}_{2,k_x,k_y} \\ \hat{\eta}_{2,k_x,k_y} \end{bmatrix} = \begin{bmatrix} \partial_t \hat{m}_{1,k_x,k_y} \\ \partial_t \hat{n}_{1,k_x,k_y} \\ \partial_t \hat{\eta}_{1,k_x,k_y} \\ \partial_t \hat{m}_{2,k_x,k_y} \\ \partial_t \hat{n}_{2,k_x,k_y} \\ \partial_t \hat{\eta}_{2,k_x,k_y} \end{bmatrix}$$

Define an operator $\hat{L}$ such that it is a block diagonal matrix with all (k_x, k_y) pairs of $\hat{L}_{k_x,k_y}$'s. That is, we have:

$$\hat{L} = \begin{bmatrix} \hat{L}_{1,1} & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \hat{L}_{1,N_y} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \hat{L}_{2,1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & \hat{L}_{N_x,N_y} \end{bmatrix}$$

Let T and T^{-1} represent DFT in $x-$ and DCT/DST in $y-$directions, and their inverse transformations. We need to solve

$$\begin{aligned} (I - hL)x^{(n+1)} &= x^{(n)} + hN(x^{(n)}) \\ T(I - hL)x^{(n+1)} &= T\left[x^{(n)} + hN(x^{(n)})\right] \\ T(I - hL)T^{-1}Tx^{(n+1)} &= T\left[x^{(n)} + hN(x^{(n)})\right] \\ (I - hTLT^{-1})\hat{x}^{(n+1)} &= \hat{x}^{(n)} + hTN(x^{(n)}) \\ (I - h\hat{L})\hat{x}^{(n+1)} &= \hat{x}^{(n)} + hTN(x^{(n)}) \end{aligned}$$

This leads to solving blocks of all wavenumber pairs simultaneously.

$$(I - h\hat{L}_{k_x,k_y})\hat{x}^{(n+1)}_{k_x,k_y} = \hat{x}^{(n)}_{k_x,k_y} + hTN(x^{(n)})$$

Let $A_{k_x,k_y} := I - h\hat{L}_{k_x,k_y}$. Then, the next time step can be computed as an inverse problem with the operator, A_{k_x,k_y} for all (k_x, k_y) pairs.

$$x^{(n+1)}_{k_x,k_y} = T^{-1}\hat{x}^{(n+1)}_{k_x,k_y} = T^{-1}A^{-1}_{k_x,k_y}RHS$$

We manually compute the LU decomposition of A_{k_x,k_y} by using Gaussian elimination.

$$A_{k_x,k_y} = I - h\hat{L}_{k_x,k_y}$$

$$= \begin{bmatrix} 1+\beta(k_x^2+k_y^2) & 0 & h\frac{1}{Fr^2}k_x i & 0 & 0 & h\frac{1}{Fr^2}k_x i \\ 0 & 1+\beta(k_x^2+k_y^2) & -h\frac{1}{Fr^2}k_y & 0 & 0 & -h\frac{1}{Fr^2}k_y \\ hk_x i & hk_y & 1+\beta(k_x^2+k_y^2) & 0 & 0 & 0 \\ 0 & 0 & h\frac{1}{Fr^2}k_x i & 1+\beta(k_x^2+k_y^2) & 0 & h\frac{\alpha}{Fr^2}k_x i \\ 0 & 0 & -h\frac{1}{Fr^2}k_y & 0 & 1+\beta(k_x^2+k_y^2) & -h\frac{\alpha}{Fr^2}k_y \\ 0 & 0 & 0 & hk_x i & hk_y & 1+\beta(k_x^2+k_y^2) \end{bmatrix}$$

The following steps transform A into an upper triangular form. It essentially performs $U = L^{-1}A$.

The main procedure is as follows: $LUx = Ax = b$, where b denotes whatever RHS we have. Letting $y = Ux$ gives us $Ly = b$. Solve for y then solve for x. Apply L^{-1} to b, Then, we are left with $Ux = L^{-1}b$. Use backward substitution to solve for x.

Let R_i denote the i^{th} row of A. The following functions are defined to somewhat optimize memory vs. computation.

$$\begin{aligned} a_0(\vec{k}) &:= \frac{h^2}{Fr}\|\vec{k}\|_2^2 \\ a(\vec{k} &:= (1+\beta\|\vec{k}\|_2^2)^{-1} \\ c(\vec{k}) &:= (1+\beta\|\vec{k}\|_2^2)^2 \\ b(\vec{k}) &:= a(\vec{k})(a_0(\vec{k}) + c(\vec{k})) \\ g(\vec{k}) &:= \frac{a_0(\vec{k})}{a_0(\vec{k}) + c(\vec{k})} \\ d(\vec{k}) &:= 1 + (\alpha-1)a(\vec{k})^2 a_0(\vec{k}) + g(\vec{k}) \\ f(\vec{k}) &:= \frac{a(\vec{k})}{a_0(\vec{k}) + c(\vec{k})}\Big((\alpha-1)a_0(\vec{k}) + \alpha c(\vec{k})\Big) \end{aligned}$$

1. $R_1 \leftarrow a(\vec{k})R_1$
2. $R_2 \leftarrow a(\vec{k})R_2$
3. $R_3 \leftarrow \frac{1}{b(\vec{k})}[R_3 - h\,(ik_xR_1 + k_yR_2)]$
4. $R_4 \leftarrow a(\vec{k})[R_4 - \frac{ihk_x}{Fr^2}R_3]$

5. $R_5 \leftarrow a(\vec{k})[R_5 + \frac{hk_y}{Fr^2} R_3]$

6. $R_6 \leftarrow a(\vec{k})[R_6 - h\,(ik_x R_4 + k_y R_5)]$

The resulting matrix is:

$$U = L^{-1}A = \begin{bmatrix} 1 & 0 & i\frac{h}{Fr^2}k_x a(\vec{k}) & 0 & 0 & i\frac{h}{Fr^2}k_x a(\vec{k}) \\ 0 & 1 & -\frac{h}{Fr^2}k_y a(\vec{k}) & 0 & 0 & -\frac{h}{Fr^2}k_y a(\vec{k}) \\ 0 & 0 & 1 & 0 & 0 & g(\vec{k}) \\ 0 & 0 & 0 & 1 & 0 & i\frac{h}{Fr^2}k_x f(\vec{k}) \\ 0 & 0 & 0 & 0 & 1 & -\frac{h}{Fr^2}k_y f(\vec{k}) \\ 0 & 0 & 0 & 0 & 0 & d(\vec{k}) \end{bmatrix}$$

The total set of operations is given below. Recall: $A\mathbf{x} = \mathbf{b} = LU\mathbf{y}$

1. $\mathbf{y_1} \leftarrow a\mathbf{b_1}$ (No need to store separately.)
2. $\mathbf{y_2} \leftarrow a\mathbf{b_2}$ (No need to store separately.)
3. $\mathbf{y_3} \leftarrow \frac{1}{b}\left(\mathbf{b_3} - h\left[ik_x\mathbf{y_1} + k_y\mathbf{y_2}\right]\right)$
   $\mathbf{y_3} \leftarrow \frac{a}{b}\left(\mathbf{b_3} - h\left[ik_x\mathbf{b_1} + k_y\mathbf{b_2}\right]\right)$
4. $\mathbf{y_4} \leftarrow a(\mathbf{b_4} - i\frac{h}{Fr^2}k_x\mathbf{y_3})$
5. $\mathbf{y_5} \leftarrow a(\mathbf{b_5} + \frac{h}{Fr^2}k_y\mathbf{y_3})$
6. $\mathbf{y_6} \leftarrow a(\mathbf{b_6} - h\left[ik_x\mathbf{y_4} + k_y\mathbf{y_5}\right])$
7. $\mathbf{x_6} \leftarrow \frac{1}{d}\mathbf{y_6}$
8. $\mathbf{x_5} \leftarrow \mathbf{y_5} + \frac{h}{Fr^2}k_y f\mathbf{x_6}$
9. $\mathbf{x_4} \leftarrow \mathbf{y_4} - i\frac{h}{Fr^2}k_x f\mathbf{x_6}$
10. $\mathbf{x_3} \leftarrow \mathbf{y_3} - g\mathbf{x_6}$
11. $\mathbf{x_2} \leftarrow \mathbf{y_2} + \frac{h}{Fr^2}k_y(\mathbf{x_3} + \mathbf{x_6})$
12. $\mathbf{x_1} \leftarrow \mathbf{y_1} - i\frac{h}{Fr^2}k_x(\mathbf{x_3} + \mathbf{x_6})$

Simplified:

1. $\mathbf{y_3} \leftarrow \mathbf{y_3} \leftarrow \frac{a}{b}\left(\mathbf{b_3} - h\left[ik_x\mathbf{b_1} + k_y\mathbf{b_2}\right]\right)$
2. $\mathbf{y_4} \leftarrow a(\mathbf{b_4} - i\frac{h}{Fr^2}k_x\mathbf{y_3})$
3. $\mathbf{y_5} \leftarrow a(\mathbf{b_5} + \frac{h}{Fr^2}k_y\mathbf{y_3})$
4. $\mathbf{x_6} \leftarrow \frac{a}{d}(\mathbf{b_6} - h\left[ik_x\mathbf{y_4} + k_y\mathbf{y_5}\right])$
5. $\mathbf{x_5} \leftarrow \mathbf{y_5} + \frac{h}{Fr^2}k_y f\mathbf{x_6}$
6. $\mathbf{x_4} \leftarrow \mathbf{y_4} - i\frac{h}{Fr^2}k_x f\mathbf{x_6}$
7. $\mathbf{x_3} \leftarrow \mathbf{y_3} - g\mathbf{x_6}$

8. $\mathbf{x_2} \leftarrow \mathbf{y_2} + \frac{h}{Fr^2} k_y (\mathbf{x_3} + \mathbf{x_6})$

9. $\mathbf{x_1} \leftarrow \mathbf{y_1} - i \frac{h}{Fr^2} k_x (\mathbf{x_3} + \mathbf{x_6})$

Some other speed-ups include storing ($i \frac{h}{Fr^2} k_x$ and $\frac{h}{Fr^2} k_y$) instead of k_x and k_y, and storing b and d as their reciprocals to have more multiplications instead of divisions. Note that you only need the following matrices: $i \frac{h}{Fr^2} k_x, \frac{h}{Fr^2} k_y, a, b^{-1}, d^{-1}, f, g$.