Semi-Implicit Time Integration

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1 Governing Equations

We modify the original set of governing equations by setting $h_i = 1 + \eta_i$, and separating linear and non-linear terms. We will be treating linear terms implicitly, and non-linear terms explicitly.

Momentum

$$\partial_t \vec{m_1} = -\frac{1}{Ro}\left(\hat{k} \times \vec{m_1}\right) - \frac{1}{Fr^2}\nabla\left(\eta_1 + \eta_2\right) + \kappa\nabla^2\vec{m_1} \quad \text{(Linear)}$$

$$-\nabla\cdot\left(\frac{1}{h_1}\vec{m_1}\vec{m_1}\right) - \frac{1}{Fr^2}\eta_1\nabla\left(\eta_1 + \eta_2\right) - \frac{\vec{m_1}}{1+\eta_1}\left(\beta\frac{\hat{Q}}{H}\hat{P}(Q) - \frac{T}{T_{RC}}(\eta_2 - \eta_1)\mathcal{H}(\eta_2 - \eta_1)\right) \quad \text{(Nonlinear)}$$

$$\partial_t \vec{m_2} = -\frac{1}{Ro}\left(\hat{k} \times \vec{m_2}\right) - \frac{1}{Fr^2}\nabla\left(\eta_1 + \alpha\eta_2\right) + \kappa\nabla^2\vec{m_2} \quad \text{(Linear)}$$

$$-\nabla\cdot\left(\frac{1}{1+\eta_2}\vec{m_2}\vec{m_2}\right) - \frac{1}{Fr^2}\eta_1\nabla\left(\eta_1 + \alpha\eta_2\right) - \frac{\vec{m_2}}{1+\eta_2}\left(\beta\frac{\hat{Q}}{H}\hat{P}(Q) - \frac{T}{T_{RC}}(\eta_2 - \eta_1)\mathcal{H}(\eta_2 - \eta_1)\right) \quad \text{(Nonlinear)}$$

Height/Mass

$$\partial_t \eta_1 = -\nabla\cdot\vec{m_1} + \kappa\nabla^2\eta_1 \quad \text{(Linear)}$$

$$-\left(\beta\frac{\hat{Q}}{H}\hat{P}(Q) - \frac{T}{T_{RC}}(\eta_2 - \eta_1)\mathcal{H}(\eta_2 - \eta_1)\right) \quad \text{(Nonlinear)}$$

$$\partial_t \eta_1 = -\nabla\cdot\vec{m_2} + \kappa\nabla^2\eta_2 \quad \text{(Linear)}$$

$$+\left(\beta\frac{\hat{Q}}{H}\hat{P}(Q) - \frac{T}{T_{RC}}(\eta_2 - \eta_1)\mathcal{H}(\eta_2 - \eta_1)\right) \quad \text{(Nonlinear)}$$

Moisture

$$\partial_t Q = \kappa\nabla^2 q \quad \text{(Linear)}$$

$$+\nabla\cdot(\vec{u_1}Q) = \left(-1 + \frac{1}{\epsilon}\right)\hat{P}(Q) \quad \text{(Nonlinear)}$$

1.1 Basic Scheme

Consider a set of ODE's with linear $L(x)$ and nonlinear $N(x)$ terms.

$$\partial_t(\vec{x}) = L(\vec{x}) + N(\vec{x}) \tag{1}$$

The discretization is given by treating the nonlinear, explicit terms as the forcing terms in the linear, implicit scheme. Below is an example using the forward Euler time-stepping method.

$$\frac{1}{h}\left(\vec{x}^{(n+1)} - \vec{x}^{(n)}\right) = L(\vec{x}^{(n+1)}) + N(\vec{x}^{(n+1)})$$

$$\vec{x}^{(n+1)} = \vec{x}^{(n)} + h\left(L(\vec{x}^{(n+1)}) + N(\vec{x}^{(n)})\right)$$

Solving for $\vec{x}^{(n+1)}$ includes the time integration step as well.

1.2 Linear Operator

We collect the linear terms and try to simplify the matrix linear operator. For now, we exclude the diffusion terms that were previously used to improve numerical stability, which eliminates linear terms in the q equation.

$$\partial_t \vec{x} = L\vec{x} = \left[\begin{array}{ccc|ccc} 0 & \frac{1}{Ro}y & -\frac{1}{Fr^2}\partial_x & 0 & 0 & -\frac{1}{Fr^2}\partial_x \\ -\frac{1}{Ro}y & 0 & -\frac{1}{Fr^2}\partial_y & 0 & 0 & -\frac{1}{Fr^2}\partial_y \\ -\partial_x & -\partial_y & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\frac{1}{Fr}\partial_x & 0 & \frac{1}{Ro}y & -\frac{\alpha}{Fr^2}\partial_x \\ 0 & 0 & -\frac{1}{Fr}\partial_y & -\frac{1}{Ro}y & 0 & -\frac{\alpha}{Fr^2}\partial_y \\ 0 & 0 & 0 & -\partial_x & -\partial_y & 0 \end{array}\right] \left[\begin{array}{c} m_1 \\ n_1 \\ \eta_1 \\ m_2 \\ n_2 \\ \eta_2 \end{array}\right] = \left[\begin{array}{c} \partial_t m_1 \\ \partial_t n_1 \\ \partial_t \eta_1 \\ \partial_t m_2 \\ \partial_t n_2 \\ \partial_t \eta_2 \end{array}\right]$$

1.2.1 Option 1: FFT in x-direction

Since our domain has zonal periodicity, we FFT in the x-direction. Our original spatial domain from $0°$ to $359.75°$ longitudes is transformed in the $k-$wave number space. Furthermore, operator $\partial_{(x)}$ is now ik. Recall $x[1441] = x[1]$. So, our wave numbers should range from $k = 1$ to $k = 720$.

$$\partial_t \vec{x} = \hat{L}_k \vec{x} = \left[\begin{array}{ccc|ccc} 0 & \frac{1}{Ro}y & -\frac{1}{Fr^2}ki & 0 & 0 & -\frac{1}{Fr^2}ki \\ -\frac{1}{Ro}y & 0 & -\frac{1}{Fr^2}\partial_y & 0 & 0 & -\frac{1}{Fr^2}\partial_y \\ -ki & -\partial_y & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\frac{1}{Fr^2}ki & 0 & \frac{1}{Ro}y & -\frac{\alpha}{Fr^2}ki \\ 0 & 0 & -\frac{1}{Fr}\partial_y & -\frac{1}{Ro}y & 0 & -\frac{\alpha}{Fr^2}\partial_y \\ 0 & 0 & 0 & -ki & -\partial_y & 0 \end{array}\right] \left[\begin{array}{c} m_1 \\ n_1 \\ \eta_1 \\ m_2 \\ n_2 \\ \eta_2 \end{array}\right] = \left[\begin{array}{c} \partial_t m_1 \\ \partial_t n_1 \\ \partial_t \eta_1 \\ \partial_t m_2 \\ \partial_t n_2 \\ \partial_t \eta_2 \end{array}\right]$$

Goal: Find diagonalization of $\hat{L}_k V = V D_k \Rightarrow \hat{L}_k = V D_k V^{-1} \Rightarrow D_k = V^{-1}\hat{L}_k V$, such that the eigenvectors are independent from k.

If that's possible, then can solve for $\vec{x}^{(n+1)}$:

$$\vec{x}^{(n+1)} = \vec{x}^{(n)} + h\left(L_k(\vec{x}^{(n+1)}) + N(\vec{x}^{(n)})\right)$$
$$(I - hL_k)\vec{x}^{(n+1)} = hN(\vec{x}^{(n)}) + \vec{x}^{(n)}$$
$$V^{-1}(I - hVD_kV^{-1})\vec{x}^{(n+1)} = V^{-1}RHS$$
$$(I - hD_k)(V^{-1})\vec{x}^{(n+1)} = V^{-1}RHS$$
$$\vec{x}^{(n+1)} = V(I - hD_k)^{-1}V^{-1}RHS$$

Multiplication of V and V^{-1} can be accelerated using the LU decomposition, and $(I - hD_k)$ is a diagonal matrix, so it is easy to compute the its inverse.

1.2.2 Option 2: Decouple the two layers by using a transformation of the momenta into barotropic and baroclinic components

1.2.3 Option 3: Remove Coriolis terms from linear implicit scheme, and FFT in x-direction and DCT/DST in y-direction

- Justification for removing the Coriolis terms:

    - The high-frequency waves in our system are the small-scale gravity waves.
    - Earth's rotation has little effect on these waves
    - So, moving the Coriolis terms to the explicit part of the scheme should not have an effect on the high frequency gravity waves.

- Benefits for removing the Coriolis terms:

    - The removal of Coriolis terms allows us to take a Fourier transform in the y-direction as well.
    - With the Coriolis terms, we had $\frac{1}{Ro}y\hat{k}\times\vec{m}_i$. Multiplication in physical space becomes a convolution in the Fourier space. This is inconvenient, since this implies that the different modes are depend on each other.
    - The new linear system should be simpler.
    - The total system is now block diagonal.

- Boundary Conditions:

    - All variables are zonal-periodic.
    - Dirichlet BC: Meridional momenta (n_i) are set to zero at the y-boundaries, since we assume that meridional momenta are conserved. Therefore, we use the sine transform.
    - Neumann BC: Change in zonal momenta (m_i) and height fluctuations (η_i) across the y-boundaries are zero (Why?). Therefore, we use the cosine transform.
    - Note that ∂_y is only every applied to η_i's and n_i's.
        * $\partial_y\eta_i$'s occur only in the RHS of $\partial_t n_i$. Since $\hat{\eta}_{i,k_x,k_y}$ are cosines and $\hat{n}_{i,k_x,k_y}$ are sines, this works out. $\partial_y \to -k_y$.
        * $\partial_y n_i$'s occur only in the RHS of $\partial_t \eta_i$. Since $\hat{\eta}_{i,k_x,k_y}$ are cosines and $\hat{n}_{i,k_x,k_y}$ are sines, this works out. $\partial_y \to k_y$.

For each k_x and k_y pair, we get the following system.

$$\partial_t \vec{x} = \hat{L}_{k_x,k_y}\vec{x} = \left[\begin{array}{ccc|ccc} 0 & 0 & -\frac{1}{Fr^2}k_x i & 0 & 0 & -\frac{1}{Fr^2}k_x i \\[2mm] 0 & 0 & \frac{1}{Fr^2}k_y & 0 & 0 & \frac{1}{Fr^2}k_y \\[2mm] -k_x i & -k_y & 0 & 0 & 0 & 0 \\[2mm] \hline 0 & 0 & -\frac{1}{Fr^2}k_x i & 0 & 0 & -\frac{\alpha}{Fr^2}k_x i \\[2mm] 0 & 0 & \frac{1}{Fr^2}k_y & 0 & 0 & \frac{\alpha}{Fr^2}k_y \\[2mm] 0 & 0 & 0 & -k_x i & -k_y & 0 \end{array}\right] \left[\begin{array}{c} \hat{m}_{1,k_x,k_y} \\[2mm] \hat{n}_{1,k_x,k_y} \\[2mm] \hat{\eta}_{1,k_x,k_y} \\[2mm] \hat{m}_{2,k_x,k_y} \\[2mm] \hat{n}_{2,k_x,k_y} \\[2mm] \hat{\eta}_{2,k_x,k_y} \end{array}\right] = \left[\begin{array}{c} \partial_t \hat{m}_{1,k_x,k_y} \\[2mm] \partial_t \hat{n}_{1,k_x,k_y} \\[2mm] \partial_t \hat{\eta}_{1,k_x,k_y} \\[2mm] \partial_t \hat{m}_{2,k_x,k_y} \\[2mm] \partial_t \hat{n}_{2,k_x,k_y} \\[2mm] \partial_t \hat{\eta}_{2,k_x,k_y} \end{array}\right]$$

Define an operator $\hat{L}$ such that it is a block diagonal matrix with all (k_x, k_y) pairs of $\hat{L}_{k_x,k_y}$'s. That is, we have:

$$\hat{L} = \left[\begin{array}{ccccccc} \hat{L}_{1,1} & 0 & 0 & 0 & 0 & \cdots & 0 \\[2mm] 0 & \ddots & 0 & 0 & 0 & \cdots & 0 \\[2mm] 0 & 0 & \hat{L}_{1,N_y} & 0 & 0 & \cdots & 0 \\[2mm] 0 & 0 & 0 & \hat{L}_{2,1} & 0 & \cdots & 0 \\[2mm] 0 & 0 & 0 & 0 & \ddots & \vdots & 0 \\[2mm] 0 & 0 & 0 & 0 & 0 & \cdots & \hat{L}_{N_x,N_y} \end{array}\right]$$

Let T and T^{-1} represent DFT in $x-$ and DCT/DST in $y-$directions. We need to solve

$$(I - hL)x^{(n+1)} = x^{(n)} + hN(x^{(n)})$$
$$T(I - hL)x^{(n+1)} = T\left[x^{(n)} + hN(x^{(n)})\right]$$
$$T(I - hL)T^{-1}Tx^{(n+1)} = T\left[x^{(n)} + hN(x^{(n)})\right]$$
$$(I - hTLT^{-1})\hat{x}^{(n+1)} = \hat{x}^{(n)} + hTN(x^{(n)})$$
$$(I - h\hat{L})\hat{x}^{(n+1)} = \hat{x}^{(n)} + hTN(x^{(n)})$$

This leads to solving blocks of all wavenumber pairs simultaneously.

$$(I - h\hat{L}_{k_x,k_y})\hat{x}_{k_x,k_y}^{(n+1)} = \hat{x}_{k_x,k_y}^{(n)} + hTN(x^{(n)})$$

We manually compute the LU decomposition by using Gaussian elimination.

$$A_{kx,ky} = I - h\hat{L}_{k_x,k_y} = \left[\begin{array}{cccccc} 1 & 0 & h\frac{1}{Fr^2}k_x i & 0 & 0 & h\frac{1}{Fr^2}k_x i \\[2mm] 0 & 1 & -h\frac{1}{Fr^2}k_y & 0 & 0 & -h\frac{1}{Fr^2}k_y \\[2mm] hk_x i & hk_y & 1 & 0 & 0 & 0 \\[2mm] 0 & 0 & h\frac{1}{Fr^2}k_x i & 1 & 0 & h\frac{\alpha}{Fr^2}k_x i \\[2mm] 0 & 0 & -h\frac{1}{Fr^2}k_y & 0 & 1 & -h\frac{\alpha}{Fr^2}k_y \\[2mm] 0 & 0 & 0 & hk_x i & hk_y & 1 \end{array}\right]$$

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The following steps transform A into an upper triangular form. It essentially performs $U = L^{-1}A$.

The main procedure is as follows: $LUx = Ax = b$, where b denotes whatever RHS we have. Leting $y = Ux$ gives us $Ly = b$. Solve for y then solve for x. Apply L^{-1} to b, Then, we are left with $Ux = L^{-1}b$. Use backward substitution to solve for x.

Let R_i denote the ith row of A. Let $a_{k_x,k_y} := \frac{h^2}{Fr^2}(k_x^2 + k_y^2)$, and let $b_{k_x,k_y} := a_{k_x,k_y}(\alpha - 1) + \alpha$

1. $R_3 \leftarrow \frac{1}{1+a_{k_x,k_y}}\left(R_3 - h\left[ik_xR_1 + k_yR_2\right]\right)$

2. $R_4 \leftarrow R_4 - i\frac{h}{Fr^2}k_xR_3$

3. $R_5 \leftarrow R_5 + \frac{h}{Fr^2}k_yR_3$

4. $R_6 \leftarrow R_6 - h\left[ik_xR_4 + k_yR_5\right]$

The resulting matrix is:

$$U = L^{-1}A = \begin{bmatrix} 1 & 0 & i\frac{h}{Fr^2}k_x & 0 & 0 & i\frac{h}{Fr^2}k_x \\[2mm] 0 & 1 & -\frac{h}{Fr^2}k_y & 0 & 0 & -\frac{h}{Fr^2}k_y \\[2mm] 0 & 0 & 1 & 0 & 0 & 1 - \frac{1}{1+a_{k_x,k_y}} \\[2mm] 0 & 0 & 0 & 1 & 0 & i\frac{h}{Fr^2}k_x\frac{b_{k_x,k_y}}{1+a_{k_x,k_y}} \\[2mm] 0 & 0 & 0 & 0 & 1 & -\frac{h}{Fr^2}k_y\frac{b_{k_x,k_y}}{1+a_{k_x,k_y}} \\[2mm] 0 & 0 & 0 & 0 & 0 & \frac{1+a_{k_x,k_y}(1+b_{k_x,k_y})}{1+a_{k_x,k_y}} \end{bmatrix}$$

The total set of operations is given below.

1. $\mathbf{y_1} \leftarrow \mathbf{b_1}$ (No need to store this separately.)

2. $\mathbf{y_2} \leftarrow \mathbf{b_2}$ (No need to store this separately.)

3. $\mathbf{y_3} \leftarrow \frac{1}{1+a_{k_x,k_y}}\left(\mathbf{b_3} - h\left[ik_x\mathbf{y_1} + k_y\mathbf{y_2}\right]\right)$
   $\mathbf{y_3} \leftarrow \frac{1}{1+a_{k_x,k_y}}\left(\mathbf{b_3} - h\left[ik_x\mathbf{b_1} + k_y\mathbf{b_2}\right]\right)$

4. $\mathbf{y_4} \leftarrow \mathbf{b_4} - i\frac{h}{Fr^2}k_x\mathbf{y_3}$

5. $\mathbf{y_5} \leftarrow \mathbf{b_5} + \frac{h}{Fr^2}k_y\mathbf{y_3}$

6. $\mathbf{y_6} \leftarrow \mathbf{b_6} - h\left[ik_x\mathbf{y_4} + k_y\mathbf{y_5}\right]$

7. $\mathbf{x_6} \leftarrow \frac{1+a_{k_x,k_y}}{1+a_{k_x,k_y}(1+b_{k_x,k_y})}\mathbf{y_6}$

8. $\mathbf{x_5} \leftarrow \mathbf{y_5} + \frac{h}{Fr^2}k_y\frac{b_{k_x,k_y}}{1+a_{k_x,k_y}}\mathbf{x_6}$

9. $\mathbf{x_4} \leftarrow \mathbf{y_4} - i\frac{h}{Fr^2}k_x\frac{b_{k_x,k_y}}{1+a_{k_x,k_y}}\mathbf{x_6}$

10. $\mathbf{x_3} \leftarrow \mathbf{y_3} - \left(1 - \frac{1}{1+a_{k_x,k_y}}\right)\mathbf{x_6}$

11. $\mathbf{x_2} \leftarrow \mathbf{y_2} + \frac{h}{Fr^2}k_y\left(\mathbf{x_3} + \mathbf{x_6}\right)$

12. $\mathbf{x_1} \leftarrow \mathbf{y_1} - i\frac{h}{Fr^2}k_x\left(\mathbf{x_3} + \mathbf{x_6}\right)$

Simplified:

1. $\mathbf{y_3} \leftarrow \frac{1}{1+a_{k_x,k_y}} \left(\mathbf{b_3} - h\left[ik_x\mathbf{b_1} + k_y\mathbf{b_2}\right]\right)$

2. $\mathbf{y_4} \leftarrow \mathbf{b_4} - i\frac{h}{Fr^2}k_x\mathbf{y_3}$

3. $\mathbf{y_5} \leftarrow \mathbf{b_5} + \frac{h}{Fr^2}k_y\mathbf{y_3}$

4. $\mathbf{y_6} \leftarrow \mathbf{b_6} - h\left[ik_x\mathbf{y_4} + k_y\mathbf{y_5}\right]$

5. $\mathbf{x_6} \leftarrow \frac{1+a_{k_x,k_y}}{1+a_{k_x,k_y}\left(1+b_{k_x,k_y}\right)}\mathbf{y_6}$

6. $\mathbf{x_5} \leftarrow \mathbf{y_5} + \frac{h}{Fr^2}k_y\frac{b_{k_x,k_y}}{1+a_{k_x,k_y}}\mathbf{x_6}$

7. $\mathbf{x_4} \leftarrow \mathbf{y_4} - i\frac{h}{Fr^2}k_x\frac{b_{k_x,k_y}}{1+a_{k_x,k_y}}\mathbf{x_6}$

8. $\mathbf{x_3} \leftarrow \mathbf{y_3} - \left(1 - \frac{1}{1+a_{k_x,k_y}}\right)\mathbf{x_6}$

9. $\mathbf{x_2} \leftarrow \mathbf{b_2} + \frac{h}{Fr^2}k_y(\mathbf{x_3} + \mathbf{x_6})$

10. $\mathbf{x_1} \leftarrow \mathbf{b_1} - i\frac{h}{Fr^2}k_x(\mathbf{x_3} + \mathbf{x_6})$