Research Notes

Minah Yang

October 15, 2018

Summary

Currently the plan is to consider at least two, maybe three models. Since the dynamics are simple enough, we keep the equations fully non-linear.

- We consider a single layer model that includes the moisture dynamics based on the Craig and Mack paper. Although we are not sure if we can include subsidence drying (it probably does not fit the framework of the equations for the dry dynamics), it is crucial for the double-well potential for the moisture variable, and so we are including it at the moment.
- We consider a two-layer model with the Craig and Mack moisture dynamics, but we exclude subsidence drying, and we assum $Q_2 = 0$. The Laemberts paper use the conservation of moist enthalpy ME ($\frac{\mathrm{d}}{\mathrm{d}t}\left(\theta + \frac{L}{c_p}Q\right) = 0$) to derive equations for Q. This was appropriate when we had assumed that precipitation was the only moisture sink. However, convective moistening, and radiative cooling do affect ME, and it is no longer conserved. The equations for Q need to be derived again.
- At the suggestion of Hottovy and Stechman who assert that an active upper-layer moisture is neccessary, now we take the two-layer model and allow $Q_2 \neq 0$. We hope to find/derive entrainment of dry air from the upper layer into the bottom layer by allowing this. (Look at Flierl and Davis paper).

1 Single Layer Model

I looked at the Bouchut-Laembert paper for the single-layer equations.

$$\partial_t \vec{v} + (\vec{v} \cdot \nabla)\vec{v} = -g\nabla h - f\hat{z} \times \vec{v} \tag{1}$$

$$\partial_t h + \nabla \cdot (\vec{v}h) = -\gamma P = -\frac{(h - h_0)}{\tau} H(h_0 - h) \tag{2}$$

$$\partial_t Q + \nabla \cdot (\vec{v}Q) = ?? \tag{3}$$

Craig and Mack suggest a model where the change in moisture (CWV) with respect to time is dependent on three physical phenomena: subsidence drying, convective moistening, and horizontal transport (diffusive?).

$$
\begin{aligned}
\partial_t Q + \nabla \cdot (\vec{v} Q) &= S + C + T && (4)\\
S &= -\alpha Q && (5)\\
C &= \frac{1-\epsilon}{\epsilon} P && (6)\\
T &= K \nabla^2 Q && (7)\\
P &= a\left(\exp(b\frac{Q}{Q_s}) - 1\right) && (8)\\
\epsilon &= \hat{\beta}\frac{Q}{Q_s} && (9)
\end{aligned}
$$

The standard values for some of these parameters are as follows: $\hat{\beta} \approx 1.1$, $b \approx 11.4$. Details on the rest can be found in the Craig and Mack paper.

- Radiative cooling appears in this model as γP (layer height equation).

2 Two-layer, $Q_2 = 0$, no subsidence drying

2.1 Rederivation of Laembert 2.15

First I will re-derive 2.15 from Laembert's paper. Define the following operators:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} a &= \partial_t a + \nabla \cdot (\vec{u} a) + \partial_z (wa)\\
\Delta a &= \partial_t a + \nabla \cdot (\vec{u} a)
\end{aligned}
$$

where $\vec{u} = (u, v)$, just the horizontal components of velocity. Note that $\frac{\mathrm{d}}{\mathrm{d}t}$ is equivalent to the standard 3-D Lagrangian derivative as a result of incompressibility:

$$
\nabla \cdot \vec{u} + \partial_z w = 0 \tag{10}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} a &= \partial_t a + \nabla \cdot (\vec{u} a) + \partial_z (wa)\\
&= \partial_t a + (\nabla \cdot \vec{u}) a + \vec{u} \cdot \nabla a + (\partial_z w) a + w \partial_z a, \quad \text{(Product Rule)}\\
&= (\partial_t + \vec{u} \cdot \nabla + w \partial_z) a
\end{aligned}
$$

We assume that precipitation is the only moisture sink: $\frac{\mathrm{d}}{\mathrm{d}t} \int_0^h q \, \mathrm{d}z = -P$. We wish to derive an alternate formula for $\frac{\mathrm{d}}{\mathrm{d}t} \int_0^h q \, \mathrm{d}z$ by vertically integrating the expression for conservation of moist enthalpy: $\frac{\mathrm{d}}{\mathrm{d}t}(\theta + \frac{L}{c_p} q) = 0$. In addition, we have incompressibility:

We will also let upper case letters represent the vertically integrated versions of the lower case letters.

For example, $Q = \int_0^h q \, \mathrm{d}z$. For simplification, I will write $me = \theta + \frac{L}{c_p} q$.

$$\Delta ME = \partial_t \int_0^h me \, \mathrm{d}z + \nabla \cdot \left(\vec{u} \int_0^h me \, \mathrm{d}z \right)$$

$$\partial_t \int_0^h me \, \mathrm{d}z = me(h) \partial_t h + \int_0^h \partial_t me \, \mathrm{d}z \quad \text{(Leibniz Rule)}$$

$$\begin{aligned}
\nabla \cdot \left(\vec{u} \int_0^h me \, \mathrm{d}z \right) &= (\partial_x u + \partial_y v) \int_0^h me \, \mathrm{d}z + \left(u \partial_x \int_0^h me \, \mathrm{d}z + v \partial_y \int_0^h me \, \mathrm{d}z \right) \quad \text{(Product Rule)} \\
&= (\nabla \cdot \vec{u}) \int_0^h me \, \mathrm{d}z + u \left(me(h) \partial_x h + \int_0^h \partial_x me \, \mathrm{d}z \right) + v \left(me(h) \partial_y h + \int_0^h \partial_y me \, \mathrm{d}z \right) \quad \text{(Leibniz Rule)} \\
&= (\nabla \cdot \vec{u}) \int_0^h me \, \mathrm{d}z + me(h) \, (\vec{u} \cdot \nabla h) + \vec{u} \cdot \int_0^h \nabla me \, \mathrm{d}z
\end{aligned}$$

Therefore, we result in:

$$\Delta ME = \Delta \int_0^h me \, \mathrm{d}z = me(h) \, (\partial_t h + \vec{u} \cdot \nabla h) + \int_0^h \partial_t me \, \mathrm{d}z + (\nabla \cdot \vec{u}) \int_0^h me \, \mathrm{d}z + \vec{u} \cdot \int_0^h \nabla me \, \mathrm{d}z \tag{11}$$

Due to conservation of moist enthalpy, we know:

$$\begin{aligned}
0 &= \int_0^h 0 \, \mathrm{d}z = \int_0^h \frac{\mathrm{d}}{\mathrm{d}t} me \, \mathrm{d}z \\
&= \int_0^h \partial_t me + \int_0^h \nabla \cdot (\vec{u} me) + \int_0^h \partial_z (w me) \, \mathrm{d}z \\
&= \int_0^h \partial_t me \, \mathrm{d}z + \int_0^h (\nabla \cdot \vec{u}) \, me \, \mathrm{d}z + \int_0^h \vec{u} \cdot \nabla me \, \mathrm{d}z + w(h) me(h) - w(0) me(0)
\end{aligned}$$

Since we are in a mixed layer, $\vec{u}$ and $\nabla \cdot \vec{u}$ are independent from z, and we also have $w(0) = 0$.

$$0 = \int_0^h \partial_t me \, \mathrm{d}z + (\nabla \cdot \vec{u}) \int_0^h me \, \mathrm{d}z + \vec{u} \cdot \int_0^h \nabla me \, \mathrm{d}z + w(h) me(h))$$

Subtracting this from 11, we result in:

$$\Delta ME = me(h) \, (\partial_t h + (\vec{u} \cdot \nabla h) - w(h)) \tag{12}$$

Vertical integration of the incompressibility condition (10) gives us another useful identity.

$$0 = \int_0^h 0 \, \mathrm{d}z = \int_0^h \nabla \cdot \vec{u} + \partial_z w \, \mathrm{d}z \tag{13}$$

$$= (\nabla \cdot \vec{u}) \int_0^h \mathrm{d}z + w(h) \quad \text{(mixed layer is homogenous)} \tag{14}$$

$$-w(h) = h (\nabla \cdot \vec{u}) \tag{15}$$

Using 15, product rule, and the equation for conservation of mass ($\Delta h = -W$) in 12, we result in:

$$\Delta ME = me(h) \, (\partial_t h + (\vec{u} \cdot \nabla h) - w(h)) \tag{16}$$

$$= me(h) \, (\partial_t h + \vec{u} \cdot \nabla h + h (\nabla \cdot \vec{u})) \tag{17}$$

$$= me(h) \Delta h \tag{18}$$

$$= me(h)(-W) \tag{19}$$

Now we invoke that temperature is constant($\equiv \theta_1$) throughout the mixed layer.

$$\begin{aligned}
\Delta ME = \Delta\left(\int_0^h \theta \,\mathrm{d}z + \frac{L}{c_p}Q\right) &= (\theta(h) + \frac{L}{c_p}q(h))(-W) \\
\Delta\left(h\theta_1 + \frac{L}{c_p}Q\right) &= (\theta(h) + \frac{L}{c_p}q(h))(-W) \\
\theta_1 \Delta h + \frac{L}{c_p}\Delta Q &= (\theta(h) + \frac{L}{c_p}q(h))(-W) \\
\frac{L}{c_p}\Delta Q &= (\theta(h) + \frac{L}{c_p}q(h) - \theta_1)(-W) \\
&= -\frac{L}{c_p}P
\end{aligned}$$

verbatim from Laemberts (2.14): By choosing a "dry" stable stratification of the atmosphere,

$$\theta_{i+1} = \theta(z_i) + \frac{L}{c_p}q(z_i) \approx \theta_i + \frac{L}{c_p}q(z_i) > \theta_i. \tag{20}$$

Therefore, we are able to write W and P being proportional:

$$\begin{aligned}
-W(\theta(h) + \frac{L}{c_p}q(h) - \theta_1) &= -\frac{L}{c_p}P \\
-W(\theta_2 - \theta_1) &= -\frac{L}{c_p}P \\
W &= \beta P \\
\beta &= \frac{L}{c_p}\frac{1}{\theta_2 - \theta_1}
\end{aligned}$$

2.2 Our actual model

Consider writing the convective moistening term into two parts: precipitation, and the resulting moistening. Let m represent moistening. We have $c = -p + m$. Following Craig & Mack's example yields:

$$c = \frac{1-\epsilon}{\epsilon}p = -p + \frac{1}{\epsilon}p \tag{21}$$

The negative sign on the p makes sense since we will be losing mosture. So we get:

$$m = \frac{1}{\epsilon}p \tag{22}$$

This is the primary difference between our model and Laembert's model– precipitation is NOT the only moisture sink. In other words, we have:

$$\Delta Q = -P + M \tag{23}$$

So, now we consider an equation for the moisture with the following physical phenomena: precipitation, moistening, horizontal transport.

$$\partial_t q = -p + m + t \tag{24}$$

We also add that radiative cooling is related to the temperature:

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta = rc \tag{25}$$

If rc and m were excluded, we would still achieve conservation of moist enthalpy. So we can write:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}me = \frac{\mathrm{d}}{\mathrm{d}t}\left(\theta + \frac{L}{c_p}q\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\theta + \frac{L}{c_p}\left(\partial_t q + \vec{u}\cdot\nabla q + w\partial_z q\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\theta + \frac{L}{c_p}\left(-p + m + t + \vec{u}\cdot\nabla Q + w\partial_z Q\right) \\
&= rc + \frac{L}{c_p}m \\
&= rc + \frac{L}{c_p}\left(\frac{1}{\epsilon}p\right)
\end{aligned}
$$

Similarly as before, we compute: $\frac{\mathrm{d}}{\mathrm{d}t}ME$, and add in our results from incompressibility and conservation of mass.

$$
\begin{aligned}
\partial_t h + \vec{u}\cdot\nabla h &= \partial_t h + \nabla(\vec{u}\cdot h) + w(h) \quad \text{(Product Rule and incompressibility)} \\
&= \Delta h + w(h) = -W + w(h) \quad \text{(Conservation of Mass)}
\end{aligned}
$$

$$
\Delta ME = \Delta\int_0^h me\,\mathrm{d}z = me(h)\left(-W + w(h)\right) + \int_0^h \partial_t me\,\mathrm{d}z + (\nabla\cdot\vec{u})\int_0^h me\,\mathrm{d}z + \vec{u}\cdot\int_0^h \nabla me\,\mathrm{d}z \tag{26}
$$

Although we no longer have conservation of moist enthalpy, we can still compute $\frac{\mathrm{d}}{\mathrm{d}t}me$, and vertically integrate it to get another expression for the integral terms in 26.

$$
\begin{aligned}
\int_0^h \frac{\mathrm{d}}{\mathrm{d}t}me\,\mathrm{d}z &= \int_0^h \partial_t me\,\mathrm{d}z + \int_0^h (\nabla\cdot\vec{u})me\,\mathrm{d}z + \int_0^h \vec{u}\cdot\nabla me\,\mathrm{d}z + \int_0^h \partial_z(wme)\,\mathrm{d}z \qquad (27) \\
&= \int_0^h \partial_t me\,\mathrm{d}z + (\nabla\cdot\vec{u})\int_0^h me\,\mathrm{d}z + \vec{u}\cdot\int_0^h \nabla me\,\mathrm{d}z + w(h)me(h) \qquad (28) \\
&= RC + \frac{L}{c_p}M \qquad (29)
\end{aligned}
$$

Combining 26 and 29 results in:

$$
\Delta ME = -Wme(h) + RC + \frac{L}{c_p}M \tag{30}
$$

We deconstruct ME into its components to get another relationship between P, W, and RC.

$$
\begin{aligned}
\Delta\left(\int_0^h \theta\,\mathrm{d}z + \frac{L}{c_p}Q\right) &= -Wme(h) + RC + \frac{L}{c_p}M \\
\theta_1(-W) + \frac{L}{c_p}\Delta Q &= -Wme(h) + RC + \frac{L}{c_p}M \\
W\left(me(h) - \theta_1\right) &= RC + \frac{L}{c_p}M - \frac{L}{c_p}\Delta Q \\
\Delta Q &= \frac{c_p}{L}\left(W\left(\theta_1 - me(h)\right) + RC + \frac{L}{c_p}M\right)
\end{aligned}
$$

Fitting this with our assumption from before (23) and the choice of a "dry" stable stratification of the atmosphere we result in:

$$
W = \beta\left(P + \frac{c_p}{L}RC\right) \tag{31}
$$

$$
\beta = \frac{L}{c_p}\frac{1}{\theta_2 - \theta_1} \tag{32}
$$

2.3 All Governing Equations

Velocity Equations:

$$\partial_t \vec{u_1} + (\vec{u_1} \cdot \nabla)\vec{u_1} + f\hat{k} \times \vec{u_1} = -g\nabla(h_1 + h_2) \tag{33}$$

$$\partial_t \vec{u_2} + (\vec{u_2} \cdot \nabla)\vec{u_2} + f\hat{k} \times \vec{u_2} = -g\nabla(h_1 + \alpha h_2) + \frac{\vec{u_1} - \vec{u_2}}{h_2}\beta\left(P + \frac{c_p}{L}RC\right) \tag{34}$$

Here, $\alpha = \frac{\theta_2}{\theta_1}$. We also assume that $W_2 \equiv 0$.

Conservation of Mass (height):

$$\Delta h_1 = -W_1 = -\beta\left(P + \frac{c_p}{L}RC\right) \tag{35}$$

$$\Delta h_2 = -W_2 + W_1 = \beta\left(P + \frac{c_p}{L}RC\right) \tag{36}$$

Moisture Dynamics:

$$\Delta Q = -P + M \tag{37}$$

$$= \left(-1 + \frac{1}{\epsilon}\right) P \tag{38}$$

2.4 Nondimensionalization

I will be using the following changes of variables. $i = 1, 2$ All of the first letters on the RHS are the constants, and the second letters are variables.

$$\vec{u}_i = U\vec{\xi}_i \tag{39}$$

$$h_i = H\zeta_i \tag{40}$$

$$Q = \hat{Q}K \tag{41}$$

$$x = \hat{L}X \tag{42}$$

$$y = \hat{L}Y \tag{43}$$

$$t = T\tau \tag{44}$$

$$P(Q) = P(\hat{Q}K) \tag{45}$$

$\hat{L}$	10^6 m
H	5000 m
U	5 ms^{-1}
$\hat{Q}$	50 mm = 0.05 m
T_{RC}	16 days = 1,382,400 s

Setting $U = V$ and $T = \frac{\hat{L}}{U}$, allow us to have the material derivative in terms of all of the new variables. Rewriting all of the nondimensionalized equations in terms of the old variables gives us the following set of equations. In addition, we will write the relationship between the dimensional and nondimensional precipitation functions as: $P(\hat{Q}Q) = \frac{\hat{Q}}{T}\hat{P}(Q)$.

2.4.1 Radiative Cooling

We formulate the radiative cooling function with the following:

$$\beta \frac{c_p}{L} RC = -\frac{h_2 - h_1}{T_{RC}} \mathcal{H}(h_2 - h_1) \tag{46}$$

where H is the heaviside function. ??$\frac{c_p}{L}$ has the units of temperature. ?? The heaviside function ensure that only radiative cooling (and not heating!) occurs. That is, when the upper layer is taller ($h_2 > h_1$), the whole term is negative. T_{RC} is introduced to scale the cooling phenomena. We will approximate it to be ≈ 16 days. With the nondimensionalized variables, we result in:

$$\beta \frac{c_p}{L} RC = -H \frac{h_2 - h_1}{T_{RC}} \mathcal{H}(h_2 - h_1). \tag{47}$$

Velocity equations:

$$\partial_t \vec{u_1} + (\vec{u_1} \cdot \nabla)\vec{u_1} + \frac{1}{Ro}(\hat{k} \times \vec{u_1}) = -\frac{1}{Fr^2} \nabla(h_1 + h_2) \tag{48}$$

$$\partial_t \vec{u_2} + (\vec{u_2} \cdot \nabla)\vec{u_2} + \frac{1}{Ro}(\hat{k} \times \vec{u_2}) = -\frac{1}{Fr^2} \nabla(h_1 + \alpha h_2) + \frac{\vec{u_1} - \vec{u_2}}{h_2} \left(\beta \frac{\hat{Q}}{H} \hat{P}(Q) - \frac{T}{T_{RC}}(h_2 - h_1)\mathcal{H}(h_2 - h_1) \right) \tag{49}$$

, where we have:

$$Ro = \frac{U}{\hat{L} f}$$

$$Fr = \frac{U}{\sqrt{gH}}$$

Conservation of Mass (height):

$$\partial_t h_1 + \nabla \cdot (\vec{u_1} h_1) = -\left(\beta \frac{\hat{Q}}{H} \hat{P}(Q) - \frac{T}{T_{RC}}(h_2 - h_1)\mathcal{H}(h_2 - h_1) \right) \tag{50}$$

$$\partial_t h_2 + \nabla \cdot (\vec{u_2} h_2) = \left(\beta \frac{\hat{Q}}{H} \hat{P}(Q) - \frac{T}{T_{RC}}(h_2 - h_1)\mathcal{H}(h_2 - h_1) \right) \tag{51}$$

Moisture Dynamics:

$$\partial_t Q + \nabla \cdot (\vec{u_1} Q) = \left(-1 + \frac{1}{\epsilon} \right) \hat{P}(Q) \tag{52}$$

We will be using several different types of functions for precipitation. Details in the nondimensionalization of the precipitation functions are in a separate document.

2.4.2 Coriolis Parameter, Rossby, and Froude Numbers

We approximate Coriolis parameter $f = 2\Omega \sin(\phi)$, where ϕ represents the latitude (° north or south from the equator). The arclength is our $y \Rightarrow r\phi = y \Rightarrow \phi = \frac{y}{r}$, where r is the radius of the Earth.

$\sin(\frac{y}{r}) \approx \frac{y}{r} = \hat{L}\frac{y}{r}$, where the second y is the new y, and Ω is the angular velocity of Earth's rotation. That is, $\frac{2\pi}{day}$. So,

$$f \approx \frac{4\pi \hat{L}}{24 * 3600} \frac{y}{6371000}$$

$$\frac{1}{Ro} = Tf = \frac{\hat{L} f}{U} \approx \frac{4\pi \hat{L}^2}{24 * 3600 * U * 6371000} y = \frac{\bar{\beta} \hat{L}^2}{U} \approx 4.56y$$

where $\bar{\beta} = \frac{4\pi}{\text{(day)(radius of earth)}}$.

$$\frac{1}{Fr^2} = \frac{gH}{U^2} \frac{ms^{-2}m}{(ms^{-1})^2} \approx 1960 \tag{53}$$

2.5 Parameter Values

We desire units in terms of: seconds, meters, and kilograms. Note that for water:

$$\frac{kg}{m^2} = \frac{L}{m^2} = \frac{1000mL}{m^2}$$
$$= \frac{1000cm^3}{(100cm)^2} = \frac{1}{10}cm = 1mm = 0.001m$$

We also approximate $\alpha = \frac{\theta_2}{\theta_1}$ via the relation $\sqrt{g'H} = \sqrt{g(\alpha-1)H} \approx 30\text{m}s^{-1}$ (speed of Kelvin wave). This yields $\alpha \approx 1 + \frac{900}{5000g} \approx 1.02$.

g	$9.80665m/s^2$
α	$\frac{\theta_2}{\theta_1} \approx 1.02$
β	≈ 750
ϵ	$\delta\frac{Q}{Qs}$
δ	≈ 1.1
b	≈ 11.4
radius of earth	$6,371,000$ m
P (kg m^{-2}day^{-1} = mm/day) P (m s^{-1}) α in Craig and Mack	$a(e^{b\frac{Q}{Q_s}}-1)$ $= \frac{a(t)}{86{,}400{,}000}(e^{b\frac{Q}{Q_s}}-1)$ $5 * 10^{-6}s^{-1}$
P$_{av}$	$= 8$ kg m^{-2} day^{-1} $= \frac{8}{86{,}400{,}000}$m s^{-1}
$a(t)$	$= \frac{P_{av}}{\frac{1}{A}\int(e^{b\frac{Q}{Q_s}}-1)}$

2.6 Conversion into Momentum Flux-form equations (instead of velocity)

We multiply h_1 to 48, $\vec{u_1}$ to 50 and add them together, and repeat with the upper layer. Setting $\vec{m_1} = h_1\vec{u_1}$, and $m_2 = h_2\vec{u_2}$ gives us:

$$\partial_t\vec{m_1} + \nabla\cdot\left(\frac{1}{h_1}\vec{m_1}\vec{m_1}\right) + \frac{1}{Ro}\left(\hat{k}\times\vec{m_1}\right) = -\frac{h_1}{Fr^2}\nabla\left(h_1+h_2\right) - \frac{\vec{m_1}}{h_1}\left(\beta\frac{\hat{Q}}{H}\hat{P}(Q) - \frac{T}{T_{RC}}(h_2-h_1)\mathcal{H}(h_2-h_1)\right) \tag{54}$$

$$\partial_t\vec{m_2} + \nabla\cdot\left(\frac{1}{h_2}\vec{m_2}\vec{m_2}\right) + \frac{1}{Ro}\left(\hat{k}\times\vec{m_2}\right) = -\frac{h_2}{Fr^2}\nabla\left(h_1+\alpha h_2\right) + \frac{\vec{m_1}}{h_1}\left(\beta\frac{\hat{Q}}{H}\hat{P}(Q) - \frac{T}{T_{RC}}(h_2-h_1)\mathcal{H}(h_2-h_1)\right) \tag{55}$$

Note that $\vec{a}\vec{a} = \vec{a}\vec{a}^\top$, and $\nabla\cdot(\vec{a}\vec{a}) = \nabla\cdot(\vec{a}\vec{a}^\top) = \left(\nabla^\top\vec{a}\vec{a}^\top\right)^\top$ (last tranpose used to result in a column vector.

UPDATE: added diffusion terms to q, h_1, and h_2.