

# Semi-Implicit Time Integration

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## 1 Governing Equations

We modify the original set of governing equations by setting  $h_i = 1 + \eta_i$ , and separating linear and non-linear terms. We will be treating linear terms implicitly, and non-linear terms explicitly.

### Momentum

$$\begin{aligned}\partial_t \vec{m}_1 = & -\frac{1}{Ro} \left( \hat{k} \times \vec{m}_1 \right) - \frac{1}{Fr^2} \nabla (\eta_1 + \eta_2) + \kappa \nabla^2 \vec{m}_1 \quad (\text{Linear}) \\ & - \nabla \cdot \left( \frac{1}{h_1} \vec{m}_1 \vec{m}_1 \right) - \frac{1}{Fr^2} \eta_1 \nabla (\eta_1 + \eta_2) - \frac{\vec{m}_1}{1 + \eta_1} \left( \beta \frac{\hat{Q}}{H} \hat{P}(Q) - \frac{T}{T_{RC}} (\eta_2 - \eta_1) \mathcal{H}(\eta_2 - \eta_1) \right) \quad (\text{Nonlinear})\end{aligned}$$

$$\begin{aligned}\partial_t \vec{m}_2 = & -\frac{1}{Ro} \left( \hat{k} \times \vec{m}_2 \right) - \frac{1}{Fr^2} \nabla (\eta_1 + \alpha \eta_2) + \kappa \nabla^2 \vec{m}_2 \quad (\text{Linear}) \\ & - \nabla \cdot \left( \frac{1}{1 + \eta_2} \vec{m}_2 \vec{m}_2 \right) - \frac{1}{Fr^2} \eta_1 \nabla (\eta_1 + \alpha \eta_2) - \frac{\vec{m}_2}{1 + \eta_2} \left( \beta \frac{\hat{Q}}{H} \hat{P}(Q) - \frac{T}{T_{RC}} (\eta_2 - \eta_1) \mathcal{H}(\eta_2 - \eta_1) \right) \quad (\text{Nonlinear})\end{aligned}$$

### Height/Mass

$$\begin{aligned}\partial_t \eta_1 = & -\nabla \cdot \vec{m}_1 + \kappa \nabla^2 \eta_1 \quad (\text{Linear}) \\ & - \left( \beta \frac{\hat{Q}}{H} \hat{P}(Q) - \frac{T}{T_{RC}} (\eta_2 - \eta_1) \mathcal{H}(\eta_2 - \eta_1) \right) \quad (\text{Nonlinear})\end{aligned}$$

$$\begin{aligned}\partial_t \eta_2 = & -\nabla \cdot \vec{m}_2 + \kappa \nabla^2 \eta_2 \quad (\text{Linear}) \\ & + \left( \beta \frac{\hat{Q}}{H} \hat{P}(Q) - \frac{T}{T_{RC}} (\eta_2 - \eta_1) \mathcal{H}(\eta_2 - \eta_1) \right) \quad (\text{Nonlinear})\end{aligned}$$

### Moisture

$$\begin{aligned}\partial_t Q = & \kappa \nabla^2 Q \quad (\text{Linear}) \\ & + \nabla \cdot (\vec{u}_1 Q) = \left( -1 + \frac{1}{\epsilon} \right) \hat{P}(Q) \quad (\text{Nonlinear})\end{aligned}$$

## 1.1 Basic Scheme

Consider a set of ODE's with linear  $L(x)$  and nonlinear  $N(x)$  terms.

$$\partial_t(\vec{x}) = L(\vec{x}) + N(\vec{x}) \quad (1)$$

The discretization is given by treating the nonlinear, explicit terms as the forcing terms in the linear, implicit scheme. Below is an example using the forward Euler time-stepping method.

$$\begin{aligned} \frac{1}{\Delta t} (\vec{x}^{(n+1)} - \vec{x}^{(n)}) &= L(\vec{x}^{(n+1)}) + N(\vec{x}^{(n+1)}) \\ \vec{x}^{(n+1)} &= \vec{x}^{(n)} + \Delta t (L(\vec{x}^{(n+1)}) + N(\vec{x}^{(n)})) \end{aligned}$$

Solving for  $\vec{x}^{(n+1)}$  includes the time integration step as well.

## 1.2 Linear Operator

We collect the linear terms and try to simplify the matrix linear operator. For now, we exclude the diffusion terms that were previously used to improve numerical stability, which eliminates linear terms in the  $q$  equation.

$$\partial_t \vec{x} = L\vec{x} = \left[ \begin{array}{ccc|ccc} 0 & \frac{1}{Ro}y & -\frac{1}{Fr^2}\partial_x & 0 & 0 & -\frac{1}{Fr^2}\partial_x \\ -\frac{1}{Ro}y & 0 & -\frac{1}{Fr^2}\partial_y & 0 & 0 & -\frac{1}{Fr^2}\partial_y \\ -\partial_x & -\partial_y & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\frac{1}{Fr}\partial_x & 0 & \frac{1}{Ro}y & -\frac{\alpha}{Fr^2}\partial_x \\ 0 & 0 & -\frac{1}{Fr}\partial_y & -\frac{1}{Ro}y & 0 & -\frac{\alpha}{Fr^2}\partial_y \\ 0 & 0 & 0 & -\partial_x & -\partial_y & 0 \end{array} \right] \begin{bmatrix} m_1 \\ n_1 \\ \eta_1 \\ m_2 \\ n_2 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \partial_t m_1 \\ \partial_t n_1 \\ \partial_t \eta_1 \\ \partial_t m_2 \\ \partial_t n_2 \\ \partial_t \eta_2 \end{bmatrix}$$

### 1.2.1 Option 1: FFT in x-direction

Since our domain has zonal periodicity, we FFT in the  $x$ -direction. Our original spatial domain from  $0^\circ$  to  $359.75^\circ$  longitudes is transformed in the  $k$ -wave number space. Furthermore, operator  $\partial(x)$  is now  $ik$ . Recall  $x[1441] = x[1]$ . So, our wave numbers should range from  $k = 1$  to  $k = 720$ .

$$\partial_t \vec{x} = \hat{L}_k \vec{x} = \left[ \begin{array}{ccc|ccc} 0 & \frac{1}{Ro}y & -\frac{1}{Fr^2}ki & 0 & 0 & -\frac{1}{Fr^2}ki \\ -\frac{1}{Ro}y & 0 & -\frac{1}{Fr^2}\partial_y & 0 & 0 & -\frac{1}{Fr^2}\partial_y \\ -ki & -\partial_y & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\frac{1}{Fr^2}ki & 0 & \frac{1}{Ro}y & -\frac{\alpha}{Fr^2}ki \\ 0 & 0 & -\frac{1}{Fr}\partial_y & -\frac{1}{Ro}y & 0 & -\frac{\alpha}{Fr^2}\partial_y \\ 0 & 0 & 0 & -ki & -\partial_y & 0 \end{array} \right] \begin{bmatrix} m_1 \\ n_1 \\ \eta_1 \\ m_2 \\ n_2 \\ \eta_2 \end{bmatrix} = \begin{bmatrix} \partial_t m_1 \\ \partial_t n_1 \\ \partial_t \eta_1 \\ \partial_t m_2 \\ \partial_t n_2 \\ \partial_t \eta_2 \end{bmatrix}$$

Goal: Find diagonalization of  $\hat{L}_k V = V D_k \Rightarrow \hat{L}_k = V D_k V^{-1} \Rightarrow D_k = V^{-1} \hat{L}_k V$ , such that the eigenvectors are independent from  $k$ .

If that's possible, then can solve for  $\vec{x}^{(n+1)}$ :

$$\begin{aligned}
\vec{x}^{(n+1)} &= \vec{x}^{(n)} + \Delta t \left( L_k(\vec{x}^{(n+1)}) + N(\vec{x}^{(n)}) \right) \\
(I - \Delta t L_k) \vec{x}^{(n+1)} &= \Delta t N(\vec{x}^{(n)}) + \vec{x}^{(n)} \\
V^{-1}(I - \Delta t V D_k V^{-1}) \vec{x}^{(n+1)} &= V^{-1} RHS \\
(I - \Delta t D_k)(V^{-1}) \vec{x}^{(n+1)} &= V^{-1} RHS \\
\vec{x}^{(n+1)} &= V(I - \Delta t D_k)^{-1} V^{-1} RHS
\end{aligned}$$

Multiplication of  $V$  and  $V^{-1}$  can be accelerated using the LU decomposition, and  $(I - \Delta t D_k)$  is a diagonal matrix, so it is easy to compute the its inverse.

### 1.2.2 Option 2: Decouple the two layers by using a transformation of the momenta into barotropic and baroclinic components

### 1.2.3 Option 3: Remove Coriolis terms from linear implicit scheme, and FFT in x-direction and DCT/DST in y-direction

- Justification for removing the Coriolis terms:
  - The high-frequency waves in our system are the small-scale gravity waves.
  - Earth's rotation has little effect on these waves
  - So, moving the Coriolis terms to the explicit part of the scheme should not have an effect on the high frequency gravity waves.
- Benefits for removing the Coriolis terms:
  - The removal of Coriolis terms allows us to take a Fourier transform in the y-direction as well.
  - With the Coriolis terms, we had  $\frac{1}{Ro} y \hat{k} \times \vec{m}_i$ . Multiplication in physical space becomes a convolution in the Fourier space. This is inconvenient, since this implies that the different modes are depend on each other.
  - The new linear system should be simpler.
  - The total system is now block diagonal.
- Boundary Conditions:
  - All variables are zonal-periodic.
  - Dirichlet BC: Meridional momenta ( $n_i$ ) are set to zero at the y-boundaries, since we assume that meridional momenta are conserved. Therefore, we use the sine transform.
  - Neumann BC: Change in zonal momenta ( $m_i$ ) and height fluctuations ( $\eta_i$ ) across the y-boundaries are zero (Why?). Therefore, we use the cosine transform.
  - Note that  $\partial_y$  is only every applied to  $\eta_i$ 's and  $n_i$ 's.
    - \*  $\partial_y \eta_i$ 's occur only in the RHS of  $\partial_t n_i$ . Since  $\hat{\eta}_{i,k_x,k_y}$  are cosines and  $\hat{n}_{i,k_x,k_y}$  are sines, this works out.  $\partial_y \rightarrow -k_y$ .
    - \*  $\partial_y n_i$ 's occur only in the RHS of  $\partial_t \eta_i$ . Since  $\hat{\eta}_{i,k_x,k_y}$  are cosines and  $\hat{n}_{i,k_x,k_y}$  are sines, this works out.  $\partial_y \rightarrow k_y$ .

For each  $k_x$  and  $k_y$  pair, we get the following system.

$$\partial_t \vec{x} = \hat{L}_{k_x, k_y} \vec{x} = \left[ \begin{array}{ccc|ccc} 0 & 0 & -\frac{1}{Fr^2} k_x i & 0 & 0 & -\frac{1}{Fr^2} k_x i \\ 0 & 0 & \frac{1}{Fr^2} k_y & 0 & 0 & \frac{1}{Fr^2} k_y \\ -k_x i & -k_y & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -\frac{1}{Fr^2} k_x i & 0 & 0 & -\frac{\alpha}{Fr^2} k_x i \\ 0 & 0 & \frac{1}{Fr^2} k_y & 0 & 0 & \frac{\alpha}{Fr^2} k_y \\ 0 & 0 & 0 & -k_x i & -k_y & 0 \end{array} \right] \begin{bmatrix} \hat{m}_{1, k_x, k_y} \\ \hat{n}_{1, k_x, k_y} \\ \hat{\eta}_{1, k_x, k_y} \\ \hat{m}_{2, k_x, k_y} \\ \hat{n}_{2, k_x, k_y} \\ \hat{\eta}_{2, k_x, k_y} \end{bmatrix} = \begin{bmatrix} \partial_t \hat{m}_{1, k_x, k_y} \\ \partial_t \hat{n}_{1, k_x, k_y} \\ \partial_t \hat{\eta}_{1, k_x, k_y} \\ \partial_t \hat{m}_{2, k_x, k_y} \\ \partial_t \hat{n}_{2, k_x, k_y} \\ \partial_t \hat{\eta}_{2, k_x, k_y} \end{bmatrix}$$

Define an operator  $\hat{L}$  such that it is a block diagonal matrix with all  $(k_x, k_y)$  pairs of  $\hat{L}_{k_x, k_y}$ 's. That is, we have:

$$\hat{L} = \begin{bmatrix} \hat{L}_{1,1} & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \hat{L}_{1, N_y} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \hat{L}_{2,1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \vdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & \hat{L}_{N_x, N_y} \end{bmatrix}$$

Let  $T$  and  $T^{-1}$  represent DFT in  $x$ - and DCT/DST in  $y$ -directions. We need to solve

$$\begin{aligned} (I - \Delta t L) x^{(n+1)} &= x^{(n)} + \Delta t N(x^{(n)}) \\ T(I - \Delta t L) x^{(n+1)} &= T \left[ x^{(n)} + \Delta t N(x^{(n)}) \right] \\ T(I - \Delta t L) T^{-1} T x^{(n+1)} &= T \left[ x^{(n)} + \Delta t N(x^{(n)}) \right] \\ (I - \Delta t T L T^{-1}) \hat{x}^{(n+1)} &= \hat{x}^{(n)} + \Delta t T N(x^{(n)}) \\ (I - \Delta t \hat{L}) \hat{x}^{(n+1)} &= \hat{x}^{(n)} + \Delta t T N(x^{(n)}) \end{aligned}$$

This leads to solving blocks of all wavenumber pairs simultaneously.

$$(I - \Delta t \hat{L}_{k_x, k_y}) \hat{x}_{k_x, k_y}^{(n+1)} = \hat{x}_{k_x, k_y}^{(n)} + \Delta t T N(x^{(n)})$$

The LU decomposition of  $\hat{L}_{k_x, k_y}$  is given below. Let  $A_{k_x, k_y} := I - \Delta t \hat{L}_{k_x, k_y}$ , and let  $a_{k_x, k_y} := \frac{(\Delta t)^2}{Fr^2} (k_x^2 + k_y^2)$ ,

and  $b_{k_x, k_y} := -\frac{1+a_{k_x, k_y}(1+a_{k_x, k_y}(\alpha-1)+\alpha)}{a_{k_x, k_y}} = -\left(a_{k_x, k_y}^{-1} + (1+a_{k_x, k_y}(\alpha-1)+\alpha)\right).$

$$A_{k_x, k_y} = P^\top \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -i\frac{k_x}{k_y} & 1 & 0 & 0 \\ 0 & 0 & 0 & i\Delta tk_x & 1 & 0 \\ i\Delta tk_x & \Delta tk_y & -\frac{1+a_{k_x, k_y}}{Fr^2\Delta tk_y} & 0 & 1+a_{k_x, k_y}^{-1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & i\frac{\Delta tk_x}{Fr^2} & 0 & 0 & i\frac{\Delta tk_x}{Fr^2} \\ 0 & 1 & -\frac{\Delta tk_y}{Fr^2} & 0 & 0 & -\frac{\Delta tk_y}{Fr^2} \\ 0 & 0 & -\frac{\Delta tk_y}{Fr^2} & 0 & 1 & -\frac{\alpha\Delta tk_y}{Fr^2} \\ 0 & 0 & 0 & 1 & i\frac{k_x}{k_y} & 0 \\ 0 & 0 & 0 & 0 & \frac{\Delta t(k_x^2+k_y^2)}{k_y} & 1 \\ 0 & 0 & 0 & 0 & 0 & b_{k_x, k_y} \end{bmatrix}$$

where  $P$  is the matrix associated with permutation  $[1, 2, 5, 4, 6, 3]$ (i.e. if  $A = [r_1; r_2; r_3; r_4; r_5; r_6]$ , then  $PA = [r_1; r_2; r_5; r_4; r_6; r_3]$ ).

We wish to solve  $A_{k_x, k_y} \hat{x}_{k_x, k_y}^{(n+1)} = v_{k_x, k_y}$  where  $v_{k_x, k_y} := T[x_{k_x, k_y}^{(n)} + \Delta t N(x_{k_x, k_y}^{(n)})]$ . Dropping the subscripts, we use the forward solve and backward solve to solve for  $\hat{x}^{(n+1)}$ .

1. Note that  $A = P^\top LU$ . Then,  $v = Ax = P^\top LUx$ , and  $LUx = Pv$ .
2. Let  $y := Ux$ . Solve for  $y$  using forward substitution. ( $Ly = \tilde{v}$ , where  $\tilde{v} = Pv$ .)
3. Solve for  $x$  using backward substitution.

Operations to do:

1.  $\mathbf{y}_1 \leftarrow \tilde{v}_1 = \mathbf{v}_1$  (No need to store this separately.)
2.  $\mathbf{y}_2 \leftarrow \tilde{v}_2 = \mathbf{v}_2$  (No need to store this separately.)
3.  $\mathbf{y}_3 \leftarrow \tilde{v}_3 = \mathbf{v}_5$  (No need to store this separately.)
4.  $\mathbf{y}_4 \leftarrow \tilde{v}_4 + i\frac{k_x}{k_y}\tilde{v}_3 = v_4 + i\frac{k_x}{k_y}\mathbf{v}_5$
5.  $\mathbf{y}_5 \leftarrow \tilde{v}_5 - i\Delta tk_x \mathbf{y}_4 = v_6 - i\Delta tk_x \mathbf{y}_4$
6.  $\mathbf{y}_6 \leftarrow \tilde{v}_6 - i\Delta tk_x \mathbf{y}_1 - \Delta tk_y \mathbf{y}_2 + \frac{1+a_{k_x, k_y}}{Fr^2\Delta tk_y} \mathbf{y}_3 - (1+a_{k_x, k_y}^{-1})\mathbf{y}_5$   
 $\mathbf{y}_6 \leftarrow \tilde{v}_6 - i\Delta tk_x \mathbf{v}_1 - \Delta tk_y \mathbf{v}_2 + \frac{1+a_{k_x, k_y}}{Fr^2\Delta tk_y} \mathbf{v}_5 - (1+a_{k_x, k_y}^{-1})\mathbf{y}_5$
7.  $\mathbf{x}_6 \leftarrow \frac{1}{b_{k_x, k_y}} \mathbf{y}_6$
8.  $\mathbf{x}_5 \leftarrow \frac{k_y}{\Delta t(k_x^2+k_y^2)} (\mathbf{y}_6 - \mathbf{x}_6)$
9.  $\mathbf{x}_4 \leftarrow \mathbf{y}_4 - i\frac{k_x}{k_y} \mathbf{x}_5$
10.  $\mathbf{x}_3 \leftarrow -\frac{Fr^2}{\Delta tk_y} (\mathbf{y}_3 + \frac{\alpha\Delta tk_y}{Fr^2} \mathbf{y}_6 - \mathbf{y}_5) = -\frac{Fr^2}{\Delta tk_y} (\mathbf{v}_5 + \frac{\alpha\Delta tk_y}{Fr^2} \mathbf{y}_6 - \mathbf{y}_5)$
11.  $\mathbf{x}_2 \leftarrow \mathbf{y}_2 + \frac{\Delta tk_y}{Fr^2} (\mathbf{y}_6 + \mathbf{y}_3) = \mathbf{v}_2 + \frac{\Delta tk_y}{Fr^2} (\mathbf{y}_6 + \mathbf{v}_5)$
12.  $\mathbf{x}_1 \leftarrow \mathbf{y}_1 - i\frac{\Delta tk_x}{Fr^2} (\mathbf{y}_6 + \mathbf{y}_3) = \mathbf{v}_1 - i\frac{\Delta tk_x}{Fr^2} (\mathbf{y}_6 + \mathbf{v}_5)$

Simplified:

1.  $\mathbf{y}_4 \leftarrow \tilde{v}_4 + i \frac{k_x}{k_y} \tilde{v}_3 = v_4 + i \frac{k_x}{k_y} \mathbf{v}_5$
2.  $\mathbf{y}_5 \leftarrow \tilde{v}_5 - i \Delta t k_x \mathbf{y}_4 = v_6 - i \Delta t k_x \mathbf{y}_4$
3.  $\mathbf{y}_6 \leftarrow \tilde{v}_6 - i \Delta t k_x \mathbf{v}_1 - \Delta t k_y \mathbf{v}_2 + \frac{1+a_{k_x,k_y}}{Fr^2 \Delta t k_y} \mathbf{v}_5 - (1+a_{k_x,k_y}^{-1}) \mathbf{y}_5$
4.  $\mathbf{x}_6 \leftarrow \frac{1}{b_{k_x,k_y}} \mathbf{y}_6$
5.  $\mathbf{x}_5 \leftarrow \frac{k_y}{\Delta t(k_x^2+k_y^2)} (\mathbf{y}_6 - \mathbf{x}_6)$
6.  $\mathbf{x}_4 \leftarrow \mathbf{y}_4 - i \frac{k_x}{k_y} \mathbf{x}_5$
7.  $\mathbf{x}_3 \leftarrow - - \frac{Fr^2}{\Delta t k_y} (\mathbf{v}_5 + \frac{\alpha \Delta t k_y}{Fr^2} \mathbf{y}_6 - \mathbf{y}_5)$
8.  $\mathbf{x}_2 \leftarrow \mathbf{v}_2 + \frac{\Delta t k_y}{Fr^2} (\mathbf{y}_6 + \mathbf{v}_5)$
9.  $\mathbf{x}_1 \leftarrow \mathbf{v}_1 - i \frac{\Delta t k_x}{Fr^2} (\mathbf{y}_6 + \mathbf{v}_5)$