Coding Details

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Governing Equations

Momentum

$$\partial_t \vec{m_1} + \nabla \cdot \left(\frac{1}{h_1} \vec{m_1} \vec{m_1} \right) + \frac{1}{Ro} \left(\hat{k} \times \vec{m_1} \right) = -\frac{1}{Fr^2} h_1 \nabla \left(h_1 + h_2 \right) - \frac{\vec{m_1}}{h_1} \left(\beta \frac{\hat{Q}}{H} \hat{P}(Q) - \frac{T}{T_{RC}} (h_2 - h_1) \mathcal{H}(h_2 - h_1) \right) \tag{1}$$

$$\partial_t \vec{m_2} + \nabla \cdot \left(\frac{1}{h_2} \vec{m_2} \vec{m_2} \right) + \frac{1}{Ro} \left(\hat{k} \times \vec{m_2} \right) = -\frac{1}{Fr^2} h_1 \nabla \left(h_1 + \alpha h_2 \right) + \frac{\vec{m_1}}{h_1} \left(\beta \frac{\hat{Q}}{H} \hat{P}(Q) - \frac{T}{T_{RC}} (h_2 - h_1) \mathcal{H}(h_2 - h_1) \right) \tag{2}$$

Height/Mass

$$\partial_t h_1 + \nabla \cdot (\vec{u_1} h_1) = -\left(\beta \frac{\hat{Q}}{H} \hat{P}(Q) - \frac{T}{T_{RC}} (h_2 - h_1) \mathcal{H}(h_2 - h_1) \right) + \kappa \nabla^2 h_1 \tag{3}$$

$$\partial_t h_2 + \nabla \cdot (\vec{u_2} h_2) = \left(\beta \frac{\hat{Q}}{H} \hat{P}(Q) - \frac{T}{T_{RC}} (h_2 - h_1) \mathcal{H}(h_2 - h_1) \right) + \kappa \nabla^2 h_2 \tag{4}$$

Moisture

$$\partial_t Q + \nabla \cdot (\vec{u_1} Q) = \left(-1 + \frac{1}{\epsilon} \right) \hat{P}(Q) + \kappa \nabla^2 q \tag{5}$$

Nondimensionalization Constants and Parameters

Nondimensionalization Constants

$\hat{L}$	10^6 m
H	5000 m
U	5 ms^{-1}
$T = \frac{L}{U}$	2×10^5s
$\hat{Q}$	50 mm $=$ 0.05 m
T_{RC}	16 days $= 1,382,400$ s

We approximate the Coriolis parameter $f = 2\Omega\sin(\phi)$, where ϕ represents the latitude ($^\circ$ north or south from the equator). The arclength is our $y \Rightarrow r\phi = y \Rightarrow \phi = \frac{y}{RE}$, where RE is the radius of the Earth.

$\sin(\frac{y}{r}) \approx \frac{y}{r} = \hat{L}\frac{y}{r}$, where the second y is the new y, and Ω is the angular velocity of Earth's rotation. That is, $\frac{2\pi}{day}$. So,

$$f = 2\Omega\sin(\phi) \approx 2 \times \frac{2\pi}{\text{day in s}}\hat{L}\phi = \frac{4\pi}{3600 \times 24s}\hat{L}\phi = \frac{4\pi}{3600 \times 24s}\hat{L}\frac{y}{RE}$$

$$\begin{aligned}\frac{1}{Ro} = Tf &\approx \frac{2 \times 10^5 s \times 4\pi\hat{L}}{24 \times 3600s}\phi = \frac{10^3 \times \pi\hat{L}}{3 \times 36}\phi \approx 2.909 \times 10^7 \text{m}\phi \\ &\approx \frac{2 \times 10^5 s \times 4\pi\hat{L}}{24 \times 3600s}\frac{y}{6371000\text{m}} = \frac{10^3 \times \pi \times 10^6\text{m}}{3 \times 36}\frac{y}{6371000\text{m}} \approx 4.57y\end{aligned}$$

$$\frac{1}{Fr^2} = \frac{gH}{U^2}\frac{ms^{-2}m}{(ms^{-1})^2} \approx 1960 \tag{6}$$

Parameters

We desire units in terms of: seconds, meters, and kilograms. Note that for water:

$$\begin{aligned}\frac{kg}{m^2} &= \frac{L}{m^2} = \frac{1000mL}{m^2} \\ &= \frac{1000cm^3}{(100cm)^2} = \frac{1}{10}cm = 1mm = 0.001m\end{aligned}$$

We also approximate $\alpha = \frac{\theta_2}{\theta_1}$ via the relation $\sqrt{g'H} = \sqrt{g(\alpha-1)H} \approx 30\text{m}s^{-1}$ (speed of Kelvin wave). This yields $\alpha \approx 1 + \frac{900}{5000g} \approx 1.02$.

g	$9.80665m/s^2$
α	$\frac{\theta_2}{\theta_1} \approx 1.02$
β	≈ 750
ϵ	$\delta\frac{Q}{Qs}$
δ	≈ 1.1
b	≈ 11.4
radius of earth	$6,371,000$ m
P (kg m^{-2}day^{-1} = mm/day) $a(t) \approx$P$_{av}$ P (m s^{-1})	$a(t)(e^{b\frac{Q}{Qs}} - 1)$ $= 8$ mm/day $= \frac{8}{86{,}400{,}000}$m s^{-1} $= \frac{8}{86{,}400{,}000}(e^{b\frac{Q}{Qs}} - 1)$

Precipitation Nondimensionalization

We will write the relationship between the dimensional and nondimensional precipitation functions as: $P(\hat{Q}K) = \frac{\hat{Q}}{T}\hat{P}(K)$. I will show derivations of the nondimensional precipitation functions for the Betts-Miller Parametrization and the model proposed by Craig and Mack in (Cite).

Betts-Miller Parametrization Nondimensionalization

We start with :

$$P(Q) = \frac{Q - Q_s}{\tau_q}\mathcal{H}(Q - Qs)$$

where $\mathcal{H}(\cdot)$ denotes the heaviside function.

Inserting the change of variables $Q = \hat{Q}K$ yields:

$$\begin{aligned}
P(\hat{Q}K) &= \frac{\hat{Q}(K - K_s)}{\tau_q}\mathcal{H}(\hat{Q}(K - K_s)) \\
&= \frac{\hat{Q}}{T}\hat{P}(K) \\
\hat{P}(K) &:= \frac{T}{\tau_q}(K - K_s)\mathcal{H}(K - K_s)
\end{aligned}$$

Craig and Mack Precipitation Model Nondimensionalization

The precipitation model is the following:

$$P(Q) = a(t)\left(\exp(b\frac{Q}{Qs}) - 1\right), \tag{7}$$

where

$$a(t) = \frac{P_{ave}}{\frac{1}{A}\int\left(\exp(b\frac{Q}{Qs}) - 1\right)\,\mathrm{d}A} \tag{8}$$

This $a(t)$ is used to enforce a constant total amount of precipitation over the area at each time-step. We disregard this, and simply replace $a(t)$ with P_{ave} divided by some large number PP such as 15,000. The quantity 8 kg m^{-2} day^{-1} is a reasonable estimate for P_{ave} in radiative-convective equilibrium(C&M). Recall that kg m^{-2} day^{-1} = mm day^{-1} = $\frac{1}{86{,}400{,}000}$ m s^{-1}.

$$\begin{aligned}
\hat{Q} &= 0.05\mathrm{m} \\
T &= \frac{\hat{L}}{U} = \frac{10^6\mathrm{m}}{5\mathrm{ms}^{-1}} = 2\times 10^5\mathrm{s} \\
P(\hat{Q}K) &= a(t)\left(\exp(b\frac{\hat{Q}K}{\hat{Q}K_s}) - 1\right)\mathrm{ms}^{-1} \\
&\approx \frac{P_{ave}}{PP}\left(\exp(b\frac{K}{K_s}) - 1\right) \\
&= \frac{8}{86,400,000\times PP}\left(\exp(b\frac{K}{K_s}) - 1\right)\mathrm{ms}^{-1} \\
&= \frac{\hat{Q}}{T\times PP}\hat{P}(K) \\
\hat{P}(K) &:= \frac{8\mathrm{m}\div\hat{Q}}{86,400,000\mathrm{s}\div T\times PP}\left(\exp(b\frac{K}{K_s}) - 1\right) \\
&= \frac{10}{27\times PP}\left(\exp(b\frac{K}{K_s}) - 1\right), \quad \text{for our specified nondimensional constants.}
\end{aligned}$$

Grid, Discretizations, and Boundary Conditions

Grid

We are using the A-Grid where all of the dependent variables live on the same grids.

- Note that $q_{i,j}$ is represented by $q[j,i]$ in my code.
- I am using m to denote the zonal component of momentum, and n to denote the meridional component of momentum. That is, $\vec{m}_i = (m_i, n_i)^\top$ for $i = 1, 2$.
- RE is the radius of earth. ($\approx 6,371,000$)
- We convert latitudes/longitudes(in °(degrees)) to distances in meters by multiplying by $\frac{\pi}{180}\times$RE. As a result, $\delta_x = 0.25 \times \frac{\pi}{180}\times$RE.
- j ranges from 1 to 160. $y[1] = -19.875 \times \frac{\pi}{180}\times$RE, and $y[160] = 19.875 \times \frac{\pi}{180}\times$RE. As a result, $\delta_y = 0.25 \times \frac{\pi}{180}\times$RE.
- We nondimensionalize by dividing by $\hat{L}$.

Discretizations

$$\nabla \cdot \left(\frac{1}{h}\vec{m}\vec{m}\right) = \left(\begin{bmatrix}\partial_x & \partial_y\end{bmatrix}\begin{bmatrix}\frac{1}{h}mm & \frac{1}{h}mn \\ \frac{1}{h}nm & \frac{1}{h}nn\end{bmatrix}^\top\right)^\top = \begin{bmatrix}\partial_x\left(\frac{1}{h}m^2\right) + \partial_y\left(\frac{1}{h}mn\right) \\ \partial_x\left(\frac{1}{h}nm\right) + \partial_y\left(\frac{1}{h}n^2\right)\end{bmatrix} = \begin{bmatrix}\partial_x(um) + \partial_y(vm) \\ \partial_x(un) + \partial_y(vn)\end{bmatrix}$$

Term	Discretization
$\partial_x(uq)$	$\approx \frac{1}{4*\delta_x}\left[(u_{i,j} + u_{i+1,j})(q_{i,j} + q_{i+1,j}) - (u_{i-1,j} + u_{i,j})(q_{i-1,j} + q_{i,j})\right]$
$\partial_y(vq)$	$\approx \frac{1}{4*\delta_y}\left[(v_{i,j} + v_{i,j+1})(q_{i,j} + q_{i,j+1}) - (v_{i,j-1} + v_{i,j})(q_{i,j-1} + q_{i,j})\right]$
$\partial_x(v^2)$	$\approx \frac{1}{4*\delta_x}\left[v_{i+1,j}^2 + 2v_{i,j}(v_{i+1,j} - v_{i-1,j}) - v_{i-1,j}^2\right]$
$\partial_y(v^2)$	$\approx \frac{1}{4*\delta_x}\left[v_{i,j+1}^2 + 2v_{i,j}(v_{i,j+1} - v_{i,j-1}) - v_{i,j-1}^2\right]$
$\nabla^2 q$	$\approx \frac{1}{\delta_x^2}(q_{j,i+1} - 2q_{j,i} + q_{j,i-1}) + \frac{1}{\delta_y^2}(q_{j+1,i} - 2q_{j,i} + q_{j-1,i})$

Boundary Conditions

Note that we only approach the y-boundaries for the terms that include $\partial_y(quantity)$. Recall that I have denoted $\vec{m}_i = (m_i, n_i)^\top$.

- Recall that jj ranges from 1 to 160., and $jj = 1$ represents latitude $-19.875°$, and $jj = 160$ represents latitude $19.875°$. With the convention that the increment of jj by 1 increases the latitude by $0.25°$, the boundaries at $\pm 20°$ are at $jj = 0.5$ and $jj = 160.5$
- (???) We assume that meridional momentum is conserved, and enforce this by setting the meridional momentums $n_i \equiv 0$ at the boundaries, $\pm 20°$ $(jj = 0.5, 160.5)$. In other words, we set $n_i[\text{"}0.5\text{"}, ii] = n_i[\text{"}160.5\text{"}, ii] = 0$. This appears whenever we have ∂_y of a term that has n_1 or n_2 in it.

For all other variables, we just use a first-order (forward or backward)FD for now, keeping in mind that we can force a one-sided second-rder FD later on.

- - Forward (order 1): $\partial_y(q(x,y))|_{y=-19.875 \Rightarrow j=1} \approx \frac{1}{\delta_y}$ (q[2]-q[1])
- Forward (order 2): $\partial_y(q(x,y))|_{y=-19.875 \Rightarrow j=1} \approx \frac{1}{2*\delta_y}$ (-q[3]+4q[2]-3q[1])
- Backward (order 1): $\partial_y(q(x,y))|_{y=19.875 \Rightarrow j=160} \approx \frac{1}{2*\delta_y}$ (q[160]-q[159])
- Backward (order 2): $\partial_y(q(x,y))|_{y=19.875 \Rightarrow j=160} \approx \frac{1}{\delta_y}$ (3q[160]-4q[159]+q[158])