Double Diffusive Equations

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1 2D (see [1])

If $\mathbf{u} = \nabla \times \psi \hat{y}$, the 2D governing equations in (x, z) are

$$\frac{\tau}{Pr}\left(\partial_t \nabla^2 \psi + J[\psi, \nabla^2 \psi]\right) = \frac{1}{\tau}\left(\partial_x \tilde{T} - \frac{1}{R_\rho}\partial_x \tilde{S}\right) + \nabla^4 \psi, \tag{1}$$

$$\partial_t \tilde{T} + J[\psi, \tilde{T}] + \partial_x \psi = \frac{1}{\tau}\nabla^2 \tilde{T}, \tag{2}$$

$$\partial_t \tilde{S} + J[\psi, \tilde{S}] + \partial_x \psi = \nabla^2 \tilde{S}, \tag{3}$$

where $J[f, g] = \partial_x f \partial_y g - \partial_y f \partial_x g$. Some relevant nondimensional quantities are:

$$Pr = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_S}{\kappa_T}, \quad Sc = \frac{\nu}{\kappa_S}, \quad R_\rho = \frac{\alpha_T \beta_T}{\alpha_S \beta_S}.$$

We are interested in $\tau \ll 1$ and $Sc \gg 1$, which results from $\nu \gg \kappa_S$ and $\kappa_S \ll \kappa_T$. Note that $\frac{\tau}{Pr} \equiv \frac{1}{Sc} \ll 1$.

Using

- $\nabla^2 = -(k^2 + m^2)$,
- $\partial_t = \lambda$,
- $\partial_x = ik$,
- $\nabla^4 = (k^2 + m^2)^2$,

we can rearrange the governing equations to form $\partial_t \mathbf{x} = \mathbf{L}\mathbf{x}^{(n+1)} + \mathbf{N}(\mathbf{x}^{(n)})$:

$$-\frac{(k^2 + m^2)}{Sc}\partial_t \hat{\psi} = +\frac{\mathrm{i}k}{\tau}\hat{T}^{(n+1)} - \frac{\mathrm{i}k}{\tau R_\rho}\hat{S}^{(n+1)} + (k^2 + m^2)^2 \hat{\psi}^{(n+1)} - \frac{1}{Sc}\mathcal{F}(J[\psi, \nabla^2 \psi]) \tag{4}$$

$$\partial_t \hat{T} = -ik\hat{\psi}^{(n+1)} - \frac{1}{\tau}(k^2 + m^2)\hat{T}^{(n+1)} - \mathcal{F}(J[\psi, \tilde{T}]) \tag{5}$$

$$\partial_t \hat{S} = -ik\hat{\psi}^{(n+1)} - (k^2 + m^2)\hat{S}^{(n+1)} - \mathcal{F}(J[\psi, \tilde{S}]) \tag{6}$$

Equation (4) can be rearranged to

$$\partial_t \hat{\psi} = -\frac{Sc}{(k^2+m^2)}\left[\frac{\mathrm{i}k}{\tau}\hat{T}^{(n+1)} - \frac{\mathrm{i}k}{\tau R_\rho}\hat{S}^{(n+1)} + (k^2+m^2)^2\hat{\psi}^{(n+1)}\right] \\ + \frac{1}{(k^2+m^2)}\mathcal{F}(J[\psi, \nabla^2\psi]).$$

The linear operator in Fourier space is:

$$\mathbf{L} = \begin{bmatrix} -Sc(k^2+m^2) & -\frac{\mathrm{i}kSc}{(k^2+m^2)\tau} & +\frac{\mathrm{i}kSc}{(k^2+m^2)\tau R_\rho} \\ -\mathrm{i}k & -\frac{(k^2+m^2)}{\tau} & 0 \\ -\mathrm{i}k & 0 & -(k^2+m^2) \end{bmatrix},$$

and the nonlinear function in physical space is

$$\mathbf{N}\left(\begin{bmatrix} \psi \\ \tilde{T} \\ \tilde{S} \end{bmatrix}\right) = \begin{bmatrix} \frac{1}{(k^2+m^2)}J[\psi, \nabla^2\psi] \\ -J[\psi, \tilde{T}] \\ -J[\psi, \tilde{S}] \end{bmatrix}$$

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The IMEX scheme is given by $(\mathbf{I} - h\mathbf{L})\hat{\mathbf{x}}^{(n+1)} = h\mathcal{F}\left(\mathbf{N}(\mathbf{x}^{(n)})\right)$:

$$\begin{bmatrix} 1 + hSc(k^2+m^2) & +\frac{h\mathrm{i}kSc}{(k^2+m^2)\tau} & -\frac{h\mathrm{i}kSc}{(k^2+m^2)\tau R_\rho} \\ h\mathrm{i}k & 1 + \frac{h(k^2+m^2)}{\tau} & 0 \\ h\mathrm{i}k & 0 & 1 + h(k^2+m^2) \end{bmatrix} \begin{bmatrix} \hat{\psi}^{(n+1)} \\ \hat{T}^{(n+1)} \\ \hat{S}^{(n+1)} \end{bmatrix} = h\mathcal{F}\left(\begin{bmatrix} \frac{1}{(k^2+m^2)}J[\psi, \nabla^2\psi] \\ -J[\psi, \tilde{T}] \\ -J[\psi, \tilde{S}] \end{bmatrix}\right)$$

Figure 1 shows the analytic solution to the inverse of the 3-by-3 problem for $\mathbf{I} - h\mathbf{L}$. km is $k^2 + m^2$, and R is R_ρ.

In[23]:=

$$LL = \begin{pmatrix} 1 + h\,Sc\,km & \frac{h\,i\,k\,Sc}{\tau\,km} & -\frac{h\,i\,k\,Sc}{\tau\,R\,km} \\ h\,i\,k & 1 + h\,\frac{km}{\tau} & 0 \\ h\,i\,k & 0 & 1 + h\,km \end{pmatrix};$$

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(*a = 1+h Sc km;
b = h i k;
c =1+ h km / τ;
d = h i k Sc/(τ km);
e = 1+ h km*)
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$$LLL[a\_,\ b\_,\ c\_,\ d\_,\ e\_,\ R\_] := \begin{pmatrix} a & d & -d/R \\ b & c & 0 \\ b & 0 & e \end{pmatrix};$$

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α[a_, b_, c_, d_, e_, R_] := a c e R + b d (c - e R)
Inverse[LLL[a, b, c, d, e, R]] * α[a, b, c, d, e, R] // FullSi
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26]//MatrixForm=

$$\begin{pmatrix} c\,e\,R & -d\,e\,R & c\,d \\ -b\,e\,R & b\,d + a\,e\,R & -b\,d \\ -b\,c\,R & b\,d\,R & (a\,c - b\,d)\,R \end{pmatrix}$$

In[22]:=

$$LL - LLL\left[1 + h\,Sc\,km,\ h\,i\,k,\ 1 + h\,km / \tau,\ \frac{h\,i\,k\,Sc}{\tau\,km},\ 1 + h\,km,\ R\right]$$

22]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Figure 1: asdf

2 3D (see [2])

The momentum equation of the Boussinesq equations for a simple double-diffusive model is

$$\frac{1}{Pr}\left(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = -\nabla p + (T' - S')\hat{z} + \nabla^2 \mathbf{u}. \tag{7}$$

The stream function formulation of $\mathbf{u}$ via

$$\mathbf{u} = \nabla \times \varphi\hat{z} + \nabla \times \nabla \times \psi\hat{z} = [u, v, w]^\top \tag{8}$$

automatically enforces the incompressibility condition, $\nabla \cdot \mathbf{u} = 0$. We denote the vorticity as $\boldsymbol{\omega} \equiv \nabla \times \mathbf{u}$, and name its vertical component, $\xi = \hat{z}\cdot\boldsymbol{\omega}$.

$$\mathbf{u} = \begin{bmatrix} \partial_y \varphi + \partial_{xz}\psi \\ -\partial_x\varphi + \partial_{yz}\psi \\ -\nabla_\perp^2 \psi \end{bmatrix}, \qquad \boldsymbol{\omega} = \begin{bmatrix} \partial_{xz}\varphi - \nabla^2\partial_y\psi \\ \partial_{yz}\varphi + \nabla^2\partial_x\psi \\ -\nabla_\perp^2\varphi \end{bmatrix}$$

Applying $\hat{z}\cdot\nabla\times$ and $\hat{z}\cdot\nabla\times\nabla\times$ to eq. (7) yields two scalar equations,

$$\frac{1}{Pr}\left(\partial_t \nabla_\perp^2 \varphi + N_\varphi(\varphi,\psi)\right) = \nabla^2\nabla_\perp^2\varphi \tag{9}$$

$$\frac{1}{Pr}\left(\partial_t \nabla^2\nabla_\perp^2 \psi + N_\psi(\varphi,\psi)\right) = -\nabla_\perp^2(T' - S') + \nabla^4\nabla_\perp^2\psi, \tag{10}$$

where

$$\begin{aligned} N_\varphi(\varphi,\psi) &= (\boldsymbol{\omega}\cdot\nabla)w - (\mathbf{u}\cdot\nabla)\xi \\ N_\psi(\varphi,\psi) &= \hat{z}\cdot\nabla\times\nabla\times(\boldsymbol{\omega}\times\mathbf{u}), \end{aligned}$$

which has a much more complicated formulation in terms of just φ and ψ.

The only other modification needed for the other equations is for the operator $\mathbf{u}\cdot\nabla$.

We now show the entire set of equations for the finger case.

$$\frac{1}{Pr}\left(\partial_t \nabla_\perp^2 \varphi + N_\varphi(\varphi,\psi)\right) = \nabla^2\nabla_\perp^2\varphi \tag{11}$$

$$\frac{1}{Pr}\left(\partial_t \nabla^2\nabla_\perp^2 \psi + N_\psi(\varphi,\psi)\right) = -\nabla_\perp^2(T' - S') + \nabla^4\nabla_\perp^2\psi \tag{12}$$

$$(\partial_t + \mathbf{u}\cdot\nabla)\, T' - \nabla_\perp^2\psi = \nabla^2 T' \tag{13}$$

$$(\partial_t + \mathbf{u}\cdot\nabla)\, S' - \frac{1}{R_p}\nabla_\perp^2\psi = \tau\nabla^2 S' \tag{14}$$

Using the vector identity

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a}),$$

where $\mathbf{a} = \mathbf{b} = \mathbf{u}$, we result in

$$\begin{aligned} \nabla(\mathbf{u}\cdot\mathbf{u}) &= (\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{u}\times(\nabla\times\mathbf{u}) + \mathbf{u}\times(\nabla\times\mathbf{u}) \\ (\mathbf{u}\cdot\nabla)\mathbf{u} &= -\mathbf{u}\times\boldsymbol{\omega} + \frac{1}{2}\nabla(\|u\|_2) \\ &= \boldsymbol{\omega}\times\mathbf{u} + \frac{1}{2}\nabla(\|u\|_2). \end{aligned}$$

Now, we take the curl again.

$$\begin{aligned} \nabla \times ((\mathbf{u} \cdot \nabla)\mathbf{u}) &= \nabla \times ((\mathbf{u} \cdot \nabla)\mathbf{u}) \\ &= \nabla \times \left(\boldsymbol{\omega} \times \mathbf{u} + \frac{1}{2}\nabla(\|u\|_2) \right) \\ &= \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) \end{aligned}$$

The second term was dropped since the curl of a gradient is always zero. Using this vector identity

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a}\nabla \cdot \mathbf{b} - \mathbf{b}\nabla \cdot \mathbf{a}$$

with $\mathbf{a} = \boldsymbol{\omega}$ and $\mathbf{b} = \mathbf{u}$, gives us

$$\nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \boldsymbol{\omega}\nabla \cdot \mathbf{u} - \mathbf{u}\nabla \cdot \boldsymbol{\omega}$$

The third term drops because we assume $\nabla \cdot \mathbf{u} = 0$, and the last term drops since the gradient of a curl is always zero. Finally,

$$\begin{aligned} \hat{z} \cdot \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) &= \hat{z} \cdot ((\mathbf{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u}) \\ &= (\mathbf{u} \cdot \nabla)\xi - (\boldsymbol{\omega} \cdot \nabla)w \\ &= -N_\varphi(\varphi, \psi) \end{aligned}$$

References

[1] J. H. Xie, B. Miquel, K. Julien, and E. Knobloch, "A reduced model for salt-finger convection in the small diffusivity ratio limit," *Fluids*, vol. 2, no. 1, pp. 1–26, 2017.

[2] T. Radko, *Double-diffusive convection*, vol. 9780521880. Cambridge: Cambridge University Press, 2012.