Padè Approximation of Taylor Expansion of functions

asdf

September 27, 2019

These notes show in detail Example 1 of Bengt Fornberg's notes on Padé approximations of polynomials. A Padé approximation of order (M,N),

$$P_M^N(x) = \frac{\sum_{n=0}^N a_n x^n}{1 + \sum_{m=1}^M b_n x^n}, \tag{1}$$

generalizes a Taylor expansion of degree $M + N$.

Suppose that we have available to us an $M + N$ degree Taylor expansion of a function,

$$T_{N+M}(x) = \sum_{j=0}^{N+M} c_j x^n, \tag{2}$$

but want to approximate it with a rational function where the numerator has degree N and denominator has degree M.

We set $T_{N+M}(x)$ and $P_M^N(x)$ equal, but multiply both sides by $1 + \sum_{m=1}^M b_n x^n$.

$$\sum_{n=0}^N a_n x^n = \left(1 + \sum_{m=1}^M b_n x^n\right) \sum_{j=0}^{N+M} c_j x^n \tag{3}$$

1 Case 1: $N \leq M$

We match the LHS coefficients to the coefficients of the product of sums.

$$\begin{aligned}
a_0 &= c_0 \\
a_1 &= c_1 + b_1 c_0 \\
\vdots &= \quad \vdots \\
a_N &= c_N + b_1 c_{N-1} + \cdots + b_N c_0
\end{aligned}$$

Since the numerator is only an N-degree polynomial, the remaining M equations are formed as,

$$\begin{aligned}
0 &= c_{N+1} + b_1 c_N + \cdots + b_{N+1} c_0 \\
\vdots &= \quad \vdots \\
0 &= c_M + b_1 c_{M-1} + \cdots + b_M c_0 \\
0 &= c_{M+1} + b_1 c_M + \cdots + b_M c_1 \\
\vdots &= \quad \vdots \\
0 &= c_{M+N} + b_1 c_{M+N-1} + \cdots + b_M c_N
\end{aligned}$$

This system of M equations can be formulated as an inverse problem, since all the c's are known.

$$\begin{bmatrix}
c_N & c_{N-1} & \cdots & c_0 & 0 & \cdots & 0 \\
c_{N+1} & c_N & \cdots & c_1 & c_0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
c_{M-1} & c_{M-2} & \cdots & c_{M-N-1} & c_{M-N-2} & \cdots & c_0 \\
c_M & c_{M-1} & \cdots & c_{M-N} & c_{M-N-1} & \cdots & c_1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_{M+N-1} & c_{M+N-2} & \cdots & c_M & c_{M-1} & \cdots & c_{N-1}
\end{bmatrix}
\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{N+1} \\ b_{N+2} \\ \vdots \\ b_M \end{bmatrix}
=
\begin{bmatrix} -c_{N+1} \\ -c_{N+2} \\ \vdots \\ -c_M \\ -c_{M+1} \\ \vdots \\ -c_{M+N} \end{bmatrix}$$

Once we have solved for b's, we can easily solve for all the a coefficients.

2 Case 1: $N > M$

We match the LHS coefficients to the coefficients of the product of sums.

$$\begin{aligned}
a_0 &= c_0 \\
a_1 &= c_1 + b_1 c_0 \\
\vdots &= \quad \vdots \\
a_M &= c_M + b_1 c_{M-1} + \cdots + b_M c_0 \\
a_{M+1} &= c_{M+1} + b_1 c_M + \cdots b_M c_1 \\
a_N &= c_N + b_1 c_{N-1} + \cdots b_M c_{N-M}
\end{aligned}$$

Since the numerator is only an N-degree polynomial, the remaining M equations are formed as,

$$\begin{aligned}
0 &= c_{N+1} + b_1 c_N + \cdots + b_M c_{N+1-M} \\
0 &= c_{N+2} + b_1 c_{N+1} + \cdots + b_M c_{N+2-M} \\
\vdots &= \quad \vdots \\
0 &= c_M + b_1 c_{M-1} + \cdots + b_M c_0 \\
0 &= c_{M+1} + b_1 c_M + \cdots + b_M c_1 \\
\vdots &= \quad \vdots \\
0 &= c_{M+N} + b_1 c_{M+N-1} + \cdots + b_M c_N
\end{aligned}$$

This system of M equations can be formulated as an inverse problem, since all the c's are known.

$$\begin{bmatrix} c_N & c_{N-1} & \cdots & c_{N-(M-1)} \\ c_{N+1} & c_N & \cdots & c_{N-(M-2)} \\ \vdots & \vdots & \vdots & \vdots \\ c_{N+M-1} & c_{M+N-2} & \cdots & c_N \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix} = \begin{bmatrix} -c_{N+1} \\ -c_{N+2} \\ \vdots \\ -c_{M+N} \end{bmatrix}$$

Once we have solved for b's, we can easily solve for all the a coefficients.