

Double Diffusive Equations

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1 2D (see [1])

If $\mathbf{u} = \nabla \times \psi \hat{y}$, the 2D governing equations in (x, z) are

$$\frac{\tau}{Pr} (\partial_t \nabla^2 \psi + J[\psi, \nabla^2 \psi]) = \frac{1}{\tau} \left(\partial_x \tilde{T} - \frac{1}{R_\rho} \partial_x \tilde{S} \right) + \nabla^4 \psi, \quad (1)$$

$$\partial_t \tilde{T} + J[\psi, \tilde{T}] + \partial_x \psi = \frac{1}{\tau} \nabla^2 \tilde{T}, \quad (2)$$

$$\partial_t \tilde{S} + J[\psi, \tilde{S}] + \partial_x \psi = \frac{1}{\tau} \nabla^2 \tilde{S}, \quad (3)$$

where $J[f, g] = \partial_x f \partial_y g - \partial_y f \partial_x g$. Some relevant nondimensional quantities are:

$$Pr = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_S}{\kappa_T}, \quad Sc = \frac{\nu}{\kappa_S}, \quad R_\rho = \frac{\alpha_T \beta_T}{\alpha_S \beta_S}.$$

We are interested in $\tau \ll 1$ and $Sc \gg 1$, which results from $\nu \gg \kappa_S$ and $\kappa_S \ll \kappa_T$. Note that $\frac{\tau}{Pr} \equiv \frac{1}{Sc} \ll 1$.

Using

- $\nabla^2 = -(k^2 + m^2)$,
- $\partial_t = \lambda$,
- $\partial_x = ik$,
- $\nabla^4 = (k^2 + m^2)^2$,

we can rearrange the governing equations to form $\partial_t \mathbf{x} = \mathbf{Lx}^{(n+1)} + \mathbf{N}(\mathbf{x}^{(n)})$:

$$-\frac{(k^2 + m^2)}{Sc} \partial_t \hat{\psi} = +\frac{ik}{\tau} \hat{T}^{(n+1)} - \frac{ik}{\tau R_\rho} \hat{S}^{(n+1)} + (k^2 + m^2)^2 \hat{\psi}^{(n+1)} - \frac{1}{Sc} \mathcal{F}(J[\psi, \nabla^2 \psi]) \quad (4)$$

$$\partial_t \hat{T} = -ik \hat{\psi}^{(n+1)} - \frac{1}{\tau} (k^2 + m^2) \hat{T}^{(n+1)} - \mathcal{F}(J[\psi, \tilde{T}]) \quad (5)$$

$$\partial_t \hat{S} = -ik \hat{\psi}^{(n+1)} - \frac{1}{\tau} (k^2 + m^2) \hat{S}^{(n+1)} - \mathcal{F}(J[\psi, \tilde{S}]) \quad (6)$$

Equation (4) can be rearranged to

$$\begin{aligned}\partial_t \hat{\psi} = & -\frac{Sc}{(k^2 + m^2)} \left[\frac{ik}{\tau} \hat{T}^{(n+1)} - \frac{ik}{\tau R_\rho} \hat{S}^{(n+1)} + (k^2 + m^2)^2 \hat{\psi}^{(n+1)} \right] \\ & + \frac{1}{(k^2 + m^2)} \mathcal{F}(J[\psi, \nabla^2 \psi]).\end{aligned}$$

The linear operator in Fourier space is:

$$\mathbf{L} = \begin{bmatrix} -Sc(k^2 + m^2) & -\frac{ikSc}{(k^2 + m^2)\tau} & +\frac{ikSc}{(k^2 + m^2)\tau R_\rho} \\ -ik & -\frac{(k^2 + m^2)}{\tau} & 0 \\ -ik & 0 & -\frac{(k^2 + m^2)}{\tau} \end{bmatrix},$$

and the nonlinear function in physical space is

$$\mathbf{N} \left(\begin{bmatrix} \psi \\ \tilde{T} \\ \tilde{S} \end{bmatrix} \right) = \begin{bmatrix} \frac{1}{(k^2 + m^2)} J[\psi, \nabla^2 \psi] \\ -J[\psi, \tilde{T}] \\ -J[\psi, \tilde{S}] \end{bmatrix}$$

The IMEX scheme is given by $(\mathbf{I} - h\mathbf{L})\hat{\mathbf{x}}^{(n+1)} = h\mathcal{F}(\mathbf{N}(\mathbf{x}^{(n)}))$:

$$\begin{bmatrix} 1 + hSc(k^2 + m^2) & +\frac{hikSc}{(k^2 + m^2)\tau} & -\frac{hikSc}{(k^2 + m^2)\tau R_\rho} \\ hik & 1 + \frac{h(k^2 + m^2)}{\tau} & 0 \\ hik & 0 & 1 + \frac{h(k^2 + m^2)}{\tau} \end{bmatrix} \begin{bmatrix} \hat{\psi}^{(n+1)} \\ \hat{T}^{(n+1)} \\ \hat{S}^{(n+1)} \end{bmatrix} = h\mathcal{F} \left(\begin{bmatrix} \frac{1}{(k^2 + m^2)} J[\psi, \nabla^2 \psi] \\ -J[\psi, \tilde{T}] \\ -J[\psi, \tilde{S}] \end{bmatrix} \right)$$

Figure 1 shows the analytic solution to the inverse of the 3-by-3 problem for $\mathbf{I} - h\mathbf{L}$. km is $k^2 + m^2$, and R is R_ρ .

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In[1]:= LL =  $\begin{pmatrix} 1 + \frac{h S c k m}{\tau} & \frac{h i k S c}{\tau k m} & -\frac{h i k S c}{\tau R k m} \\ \frac{h i k}{\tau} & 1 + \frac{h k m}{\tau} & 0 \\ \frac{h i k}{\tau} & 0 & 1 + \frac{h k m}{\tau} \end{pmatrix};$ 

(*a = 1+h S c k m;
b = h i k;
c = 1+ h k m /  $\tau$ ;
d =  $\frac{h i k S c}{\tau k m}$ ;*)

LLL[a_, b_, c_, d_, R_] :=  $\begin{pmatrix} a & d & -d / R \\ b & c & 0 \\ b & 0 & c \end{pmatrix};$ 

 $\alpha[a_, b_, c_, d_, R_] := (b c d + a c^2 R - b c d R)$ 
Inverse[LLL[a, b, c, d, R]] *  $\alpha[a, b, c, d, R]$  // FullSimplify

Out[4]/MatrixForm=
 $\begin{pmatrix} c^2 R & -c d R & c d \\ -b c R & b d + a c R & -b d \\ -b c R & b d R & (a c - b d) R \end{pmatrix}$ 

In[5]:= LL - LLL[1 + h S c k m, h i k, 1 + h k m /  $\tau$ ,  $\frac{h i k S c}{\tau k m}$ , R] /

Out[5]/MatrixForm=
 $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ 

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Figure 1: asdf

2 3D (see [2])

The momentum equation of the Boussinesq equations for a simple double-diffusive model is

$$\frac{1}{Pr} (\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u}) = -\nabla p + (T' - S') \hat{z} + \nabla^2 \mathbf{u}. \quad (7)$$

The stream function formulation of \mathbf{u} via

$$\mathbf{u} = \nabla \times \varphi \hat{z} + \nabla \times \nabla \times \psi \hat{z} = [u, v, w]^\top \quad (8)$$

automatically enforces the incompressibility condition, $\nabla \cdot \mathbf{u} = 0$. We denote the vorticity as $\boldsymbol{\omega} \equiv \nabla \times \mathbf{u}$, and name its vertical component, $\xi = \hat{z} \cdot \boldsymbol{\omega}$.

$$\mathbf{u} = \begin{bmatrix} \partial_y \varphi + \partial_{xz} \psi \\ -\partial_x \varphi + \partial_{yz} \psi \\ -\nabla_\perp^2 \psi \end{bmatrix}, \quad \boldsymbol{\omega} = \begin{bmatrix} \partial_{xz} \varphi - \nabla^2 \partial_y \psi \\ \partial_{yz} \varphi + \nabla^2 \partial_x \psi \\ -\nabla_\perp^2 \varphi \end{bmatrix}$$

Applying $\hat{z} \cdot \nabla \times$ and $\hat{z} \cdot \nabla \times \nabla \times$ to eq. (7) yields two scalar equations,

$$\frac{1}{Pr} (\partial_t \nabla_\perp^2 \varphi + N_\varphi(\varphi, \psi)) = \nabla^2 \nabla_\perp^2 \varphi \quad (9)$$

$$\frac{1}{Pr} (\partial_t \nabla^2 \nabla_\perp^2 \psi + N_\psi(\varphi, \psi)) = -\nabla_\perp^2 (T' - S') + \nabla^4 \nabla_\perp^2 \psi, \quad (10)$$

where

$$\begin{aligned} N_\varphi(\varphi, \psi) &= (\boldsymbol{\omega} \cdot \nabla) \varphi - (\mathbf{u} \cdot \nabla) \xi \\ N_\psi(\varphi, \psi) &= \hat{z} \cdot \nabla \times \nabla \times (\boldsymbol{\omega} \times \mathbf{u}), \end{aligned}$$

which has a much more complicated formulation in terms of just φ and ψ .

The only other modification needed for the other equations is for the operator $\mathbf{u} \cdot \nabla$.

We now show the entire set of equations for the finger case.

$$\frac{1}{Pr} (\partial_t \nabla_\perp^2 \varphi + N_\varphi(\varphi, \psi)) = \nabla^2 \nabla_\perp^2 \varphi \quad (11)$$

$$\frac{1}{Pr} (\partial_t \nabla^2 \nabla_\perp^2 \psi + N_\psi(\varphi, \psi)) = -\nabla_\perp^2 (T' - S') + \nabla^4 \nabla_\perp^2 \psi \quad (12)$$

$$(\partial_t + \mathbf{u} \cdot \nabla) T' - \nabla_\perp^2 \psi = \nabla^2 T' \quad (13)$$

$$(\partial_t + \mathbf{u} \cdot \nabla) S' - \frac{1}{R_p} \nabla_\perp^2 \psi = \tau \nabla^2 S' \quad (14)$$

Using the vector identity

$$\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}),$$

where $\mathbf{a} = \mathbf{b} = \mathbf{u}$, we result in

$$\begin{aligned} \nabla(\mathbf{u} \cdot \mathbf{u}) &= (\mathbf{u} \cdot \nabla) \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \mathbf{u} \times (\nabla \times \mathbf{u}) + \mathbf{u} \times (\nabla \times \mathbf{u}) \\ (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\mathbf{u} \times \boldsymbol{\omega} + \frac{1}{2} \nabla(\|\mathbf{u}\|_2^2) \\ &= \boldsymbol{\omega} \times \mathbf{u} + \frac{1}{2} \nabla(\|\mathbf{u}\|_2^2). \end{aligned}$$

Now, we take the curl again.

$$\begin{aligned}
\nabla \times ((\mathbf{u} \cdot \nabla)\mathbf{u}) &= \nabla \times ((\mathbf{u} \cdot \nabla)\mathbf{u}) \\
&= \nabla \times \left(\boldsymbol{\omega} \times \mathbf{u} + \frac{1}{2} \nabla(\|\mathbf{u}\|_2^2) \right) \\
&= \nabla \times (\boldsymbol{\omega} \times \mathbf{u})
\end{aligned}$$

The second term was dropped since the curl of a gradient is always zero. Using this vector identity

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a} \nabla \cdot \mathbf{b} - \mathbf{b} \nabla \cdot \mathbf{a}$$

with $\mathbf{a} = \boldsymbol{\omega}$ and $\mathbf{b} = \mathbf{u}$, gives us

$$\nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \boldsymbol{\omega} \nabla \cdot \mathbf{u} - \mathbf{u} \nabla \cdot \boldsymbol{\omega}$$

The third term drops because we assume $\nabla \cdot \mathbf{u} = 0$, and the last term drops since the gradient of a curl is always zero. Finally,

$$\begin{aligned}
\hat{z} \cdot \nabla \times (\boldsymbol{\omega} \times \mathbf{u}) &= \hat{z} \cdot ((\mathbf{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u}) \\
&= (\mathbf{u} \cdot \nabla)\xi - (\boldsymbol{\omega} \cdot \nabla)w \\
&= -N_\varphi(\varphi, \psi)
\end{aligned}$$

References

- [1] J. H. Xie, B. Miquel, K. Julien, and E. Knobloch, “A reduced model for salt-finger convection in the small diffusivity ratio limit,” *Fluids*, vol. 2, no. 1, pp. 1–26, 2017.
- [2] T. Radko, *Double-diffusive convection*, vol. 9780521880. Cambridge: Cambridge University Press, 2012.