Integrating Factor Methods

September 12, 2019

We study ODEs of type

$$\frac{\mathrm{d}}{\mathrm{d}t} y = Ly + N(y), \tag{1}$$

where y may be a vector, L is a linear operator, and N is a nonlinear operator.

1 IF Methods

One form of the Integrating Factor method makes the change of variables $v(t) = e^{(t_{n-1}-t)L} y(t)$, which results in a new form.

$$\frac{\mathrm{d}}{\mathrm{d}t} v = f(v,t) = e^{(t_{n-1}-t)L} N(e^{(t-t_{n-1})L} v) \tag{2}$$

Note that while t_{n-1} can be replaced by any constant, this choice helps simplify the discretization of Equation 2. For example, at $t = t_{n-1}$, we have exactly $y(t_{n-1}) = v(t_{n-1})$. We will simplify the above notation as $y(t_n) = y_n$ and $v(t_n) = v_n$.

This class of methods apply explicit methods to Equation 2, then returns to the original variables.

IF-Euler Applying explicit Euler to Equation 2 results in,

$$v_n = v_{n-1} + hf(v_{n-1,t_{n-1}}) = v_{n-1} + hN(v_{n-1}).$$

Returning to the original variables gives us the full IF-Euler discretization.

$$y_n = e^{hL} v_n = e^{hL}(y_{n-1} + hN(y_{n-1})) \tag{3}$$

IF-RK2 (Heun/Trapezoidal) This is one of the simplest non-trivial Runge-Kutta method.

$$\begin{aligned} k_1 &= f(v_{n-1}, t_{n-1}) \\ k_2 &= f(v_{n-1} + hk_1, t_{n-1} + h) \\ v_n &= v_{n-1} + h\left(\frac{1}{2}k_1 + \frac{1}{2}k_2\right) \end{aligned}$$

Converting back to original variables:

$$
\begin{aligned}
k_1 &= f(v_{n-1}, t_{n-1}) = N(y_{n-1}) \\
k_2 &= f(v_{n-1} + hk_1, t_{n-1} + h) = e^{-hL} N(e^{hL}(y_{n-1} + hk_1)) \\
y_n &= e^{hL} v_n = e^{hL} \left(y_{n-1} + h \left(\frac{1}{2} k_1 + \frac{1}{2} k_2 \right) \right) \\
&= e^{hL} \left(y_{n-1} + \frac{h}{2} N(y_{n-1}) \right) + \frac{h}{2} N(e^{hL}(y_{n-1} + hN(y_{n-1})))
\end{aligned}
$$

IF-RK4 (Classic) This is the classic 4th order explicit Runge-Kutta method.

$$
\begin{aligned}
k_1 &= f(v_{n-1}, t_{n-1}) \\
k_2 &= f(v_{n-1} + \frac{h}{2} k_1, t_{n-1} + \frac{h}{2}) \\
k_3 &= f(v_{n-1} + \frac{h}{2} k_2, t_{n-1} + \frac{h}{2}) \\
k_4 &= f(v_{n-1} + hk_3, t_{n-1} + h) \\
v_n &= v_{n-1} + h(\frac{1}{6} k_1 + \frac{1}{3} k_2 + \frac{1}{3} k_3 + \frac{1}{6} k_4)
\end{aligned}
$$

Converting back to original variables:

$$
\begin{aligned}
k_1 &= f(v_{n-1}, t_{n-1}) = N(y_{n-1}) \\
k_2 &= f(v_{n-1} + \frac{h}{2} k_1, t_{n-1} + \frac{h}{2}) = e^{-\frac{1}{2}hL} N\left(e^{\frac{1}{2}hL}(y_{n-1} + \frac{h}{2} k_1) \right) \\
k_3 &= f(v_{n-1} + \frac{h}{2} k_2, t_{n-1} + \frac{h}{2}) = e^{-\frac{1}{2}hL} N\left(e^{\frac{1}{2}hL}(y_{n-1} + \frac{h}{2} k_2) \right) \\
k_4 &= f(v_{n-1} + hk_3, t_{n-1} + h) = e^{-hL} N\left(e^{hL}(y_{n-1} + hk_3) \right) \\
y_n &= e^{hL} v_n = e^{hL} \left(y_{n-1} + h(\frac{1}{6} k_1 + \frac{1}{3} k_2 + \frac{1}{3} k_3 + \frac{1}{6} k_4) \right)
\end{aligned}
$$

2 Resonant Triad

We study the Resonant Triad system ODEs given by Equations 4 to 6, which can be summarized in vector form, Equation 7.

$$
\frac{\mathrm{d}}{\mathrm{d}t} z_1 = i\omega_1 z_1 + \epsilon C_1 z_2^* z_3^* \tag{4}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t} z_2 = i\omega_1 z_2 + \epsilon C_2 z_1^* z_3^* \tag{5}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t} z_3 = i\omega_1 z_3 + \epsilon C_3 z_1^* z_2^* \tag{6}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t} \vec{z} = Lz + N(z) \tag{7}
$$

The linear operator for Equations 7 is given below.

$$L = \begin{bmatrix} i\omega_1 & 0 & 0 \\ 0 & i\omega_2 & 0 \\ 0 & 0 & i\omega_3 \end{bmatrix} \tag{8}$$

Since L is a diagonal matrix, its exponential is easy to calculate.

$$e^{hL} = \begin{bmatrix} e^{ih\omega_1} & 0 & 0 \\ 0 & e^{ih\omega_2} & 0 \\ 0 & 0 & e^{ih\omega_3} \end{bmatrix} \tag{9}$$

The nonlinear function is then,

$$N(\vec{z}) = \epsilon \begin{bmatrix} C_1 z_2^* z_3^* \\ C_2 z_1^* z_3^* \\ C_3 z_1^* z_2^* \end{bmatrix} \tag{10}$$

Let's apply the IF change of variables with $v(t) = e^{(t_{n-1}-t)L} z(t)$. Then, the ODE becomes,

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{v} = f(\vec{v}, t_{n-1} + \tau) = e^{-\tau L} N(e^{\tau L} v). \tag{11}$$

It turns out that $f(\vec{v}, t_{n-1} + \tau)$ can actually be represented just as a function of v. Some algebra to show this:

$$\begin{aligned} f(\vec{v}, t_{n-1} + \tau) &= e^{-\tau L} N(e^{\tau L} \vec{v}) \\ &= e^{-\tau L} N\left(\begin{bmatrix} e^{i\tau\omega_1} & 0 & 0 \\ 0 & e^{i\tau\omega_2} & 0 \\ 0 & 0 & e^{i\tau\omega_3} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right) = e^{-\tau L} N\left(\begin{bmatrix} e^{i\tau\omega_1} v_1 \\ e^{i\tau\omega_2} v_2 \\ e^{i\tau\omega_3} v_3 \end{bmatrix} \right) \\ &= e^{-\tau L} \epsilon \begin{bmatrix} C_1 e^{-i\tau(\omega_2+\omega_3)} v_2^* v_3^* \\ C_2 e^{-i\tau(\omega_1+\omega_3)} v_1^* v_3^* \\ C_1 e^{-i\tau(\omega_1+\omega_2)} v_1^* v_2^* \end{bmatrix} = e^{-\tau L} \epsilon \begin{bmatrix} C_1 e^{i\tau(\omega_1)} v_2^* v_3^* \\ C_2 e^{i\tau(\omega_2)} v_1^* v_3^* \\ C_3 e^{i\tau(\omega_3)} v_1^* v_2^* \end{bmatrix}, \text{ since } \omega_1 + \omega_2 + \omega_3 = 0, \\ &= e^{-\tau L} \begin{bmatrix} e^{i\tau\omega_1} & 0 & 0 \\ 0 & e^{i\tau\omega_2} & 0 \\ 0 & 0 & e^{i\tau\omega_3} \end{bmatrix} \epsilon \begin{bmatrix} C_1 v_2^* v_3^* \\ C_2 v_1^* v_3^* \\ C_3 v_1^* v_2^* \end{bmatrix} = e^{-\tau L} e^{\tau L} N(\vec{v}) = N(\vec{v}) \end{aligned}$$

So we actually end up with an autonomous ODEs in the new variable, $\vec{v}$ as well. Let's re-do the discretizations by using $\frac{\mathrm{d}}{\mathrm{d}t}\vec{v} = f(\vec{v}, t) = N(\vec{v})$.

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Returning to the original variables gives us the full IF-Euler discretization.

$$y_n = e^{hL} v_n = e^{hL}(y_{n-1} + hN(y_{n-1})) \tag{12}$$

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Converting back to original variables:

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k_4 &= N(v_{n-1} + hk_3) \\
v_n &= v_{n-1} + h(\frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3 + \frac{1}{6}k_4)
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\end{aligned}$$