Quant Nugget 2

Linear vs. Compounded Returns

Common Pitfalls in Portfolio Management[1]

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Linear and compounded returns are at times used interchangeably: this practice has dangerous repercussions on risk and portfolio management.

1 The ingredients

Buy-side practitioners measure risk and reward in terms of returns for two reasons. First, returns provide a normalized measure of performance that can be compared across securities and asset classes. Second, in notable markets returns can be assumed to be invariants, i.e. to behave identically and independently across time: therefore, in these markets, returns can be used to infer the future distribution of performance from past history.

Two different notions of return are used in the industry: linear and compounded return. To introduce them, consider a generic security that has value V_t at time t and let us assume that any cashflow generated by that security is reinvested in the same security, so as to modify its value.

The *linear return* is defined as

$$L_t \equiv \frac{V_{t+1}}{V_t} - 1. \tag{1}$$

The linear return has the remarkable property that it *aggregates across securities*. Indeed, if we denote by $w_1, \ldots, w_n$ the weights of a set of securities in a portfolio P, and if we denote by $L_{t,1}, \ldots, L_{t,n}$ the linear returns of those securities, then the linear return of the portfolio is the weighted average of the linear returns of the securities

$$L_{t,P} = w_1 L_{t,1} + \cdots + w_n L_{t,n}. \tag{2}$$

Given this property, risk and portfolio managers use linear returns for risk analysis, performance attribution, and portfolio optimization.

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On the other hand, the *compounded return* is defined as

$$C_t \equiv \ln\left(\frac{V_{t+1}}{V_t}\right). \tag{3}$$

The compounded return *aggregates across time*: the compounded return over k periods is the sum of the intermediate 1-period compounded returns

$$C_t^k \equiv \ln\left(\frac{V_{t+k}}{V_t}\right) = C_t + C_{t+1} \cdots + C_{t+k-1}. \tag{4}$$

Given this property, *if* the compounded returns are invariants, it is easy to estimate their 1-period distribution and to project this distribution to a generic horizon k steps into the future, with a technique called Fourier transform.

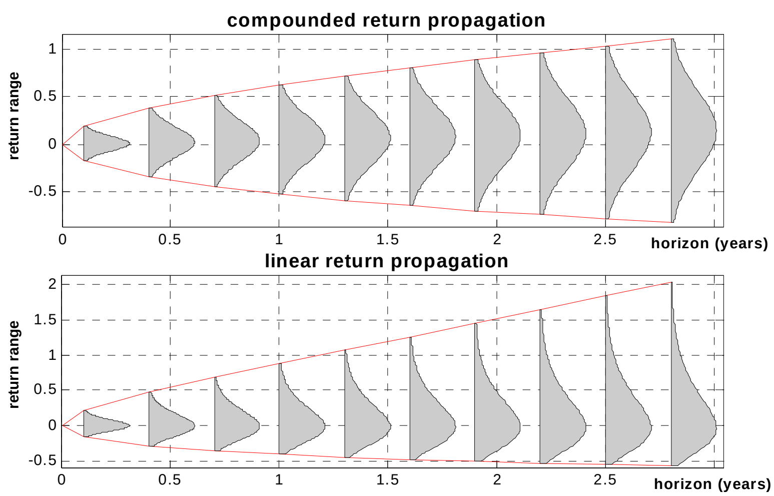

Figure 1: Linear and compounded return difference increases with investment horizon

If we look at asset values V_t and V_{t+1} at two different times to determine the linear return (1) and the compounded return (3), we obtain two numerical values: these values are very similar if V_t and V_{t+1} are close, but they are very different if V_t and V_{t+1} grow apart. Therefore, when the securities are not too volatile and the time step of the return is short, the distribution of linear and compounded returns is similar; but as the time step grows, so does the difference between the two distributions.

For instance, in Figure 1 we illustrate this phenomenon in the case of a typical stock. The top plot displays the propagation of the compounded return, which follows the square-root rule: mean and variance grow linearly with the horizon, and volatility grows as the square root of the horizon. The bottom plot illustrates the distorted propagation of the linear return. To download the fully commented code used to generate this example refer to Meucci (2009).

2 The pitfall

Given their similarities in the short run, linear and compounded returns are at times used interchangeably, but this practice can elicit dangerous consequences. Consider an allocation process in a given market of n securities. It is not uncommon to witness the following steps

a) Take the price series $V_t, V_{t-1}, V_{t-2}, \ldots$, say weekly, for all the n securities

b) Estimate the $n \times n$ covariance matrix $\boldsymbol{\Sigma}$ of the compounded returns, say by exponential smoothing

c) Estimate the n means $\boldsymbol{\mu}$ of the compounded returns, say by blending historical analysis, equilibrium assumptions, and analysts' views.

d) Determine the investment horizon, say k = fifty weeks

e) Project means and covariances to the horizon

$$\boldsymbol{\mu}^k \equiv k\boldsymbol{\mu}, \quad \boldsymbol{\Sigma}^k \equiv k\boldsymbol{\Sigma} \tag{5}$$

f) Compute the mean-variance efficient allocations, also known as the efficient frontier

$$\mathbf{w}_\lambda \equiv \underset{\mathbf{w} \text{ sat } \mathcal{C}}{\operatorname{argmax}} \left\{ \mathbf{w}'\boldsymbol{\mu}^k - \lambda \mathbf{w}'\boldsymbol{\Sigma}^k \mathbf{w} \right\}, \tag{6}$$

where $\mathbf{w}'$ denotes the transpose of $\mathbf{w}$ and "$\mathbf{w}$ sat $\mathcal{C}$" denotes that the n weights $\mathbf{w}$ must satisfy a set of investment constraints $\mathcal{C}$, such as the budget constraint, long-only or sector boundaries, maximum number of securities, etc.

The above approach a)-f) is *incorrect*. First, (5) is a multivariate version of the square-root rule, see Meucci (2010). The square-root rule only applies under the assumption that the compounded returns are invariants, i.e. they behave identically and independently across time. Therefore (5) is approximately true for stocks, see the top plot in Figure 1, but it is *not* true for bonds and definitely *not* true for options.

Second, even for stocks, the optimization (6) is ill-posed. In particular $\mathbf{w}'\boldsymbol{\mu}^k$ is *not* the mean of the portfolio return (compounded or linear) over the horizon and $\mathbf{w}'\boldsymbol{\Sigma}^k\mathbf{w}$ is *not* its variance.

Following the above steps a)-f) would lead to suboptimal allocations. Among other consequences, the efficient frontier would not depend on the investment horizon!

3 The solution

The correct approach is to estimate the horizon means and covariances $\mathbf{m}^k$ and $\mathbf{S}^k$ of the *linear* returns and use these to compute the mean-variance efficient portfolios

$$\mathbf{w}_\lambda \equiv \underset{\mathbf{w} \text{ sat } \mathcal{C}}{\operatorname{argmax}} \left\{ \mathbf{w}'\mathbf{m}^k - \lambda \mathbf{w}'\mathbf{S}^k \mathbf{w} \right\}. \tag{7}$$

Indeed, from (2) the term $\mathbf{w}'\mathbf{m}^k$ *is* the mean of the portfolio linear return over the horizon and the term $\mathbf{w}'\mathbf{S}^k\mathbf{w}$ *is* its variance. To estimate $\mathbf{m}^k$ and $\mathbf{S}^k$ we must follow this process: search the suitable market invariants; estimate their distribution; project this distribution to the horizon; map this horizon distribution into the linear returns at the horizon; and finally extract from the whole distribution the means $\mathbf{m}^k$ and the covariances $\mathbf{S}^k$.

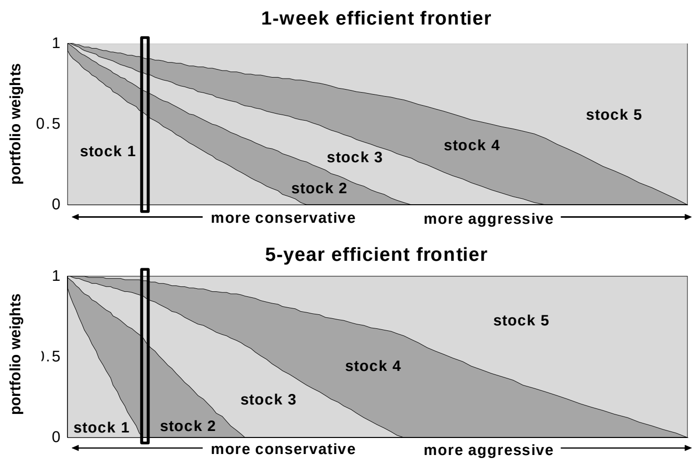

Figure 2: Efficient frontier depends on investment horizon

For instance, for stocks the compounded returns $\mathbf{C}_t$ can be assumed in first approximation to be invariants. Once the, say, weekly distribution is estimated, the projection to the, say, yearly distribution $\mathbf{C}_t^k$ is achieved through (4). The pricing, i.e. the mapping from invariants to linear returns then reads $\mathbf{L}_t^k = e^{\mathbf{C}_t^k} - 1$. Finally, $\mathbf{m}^k$ and $\mathbf{S}^k$ can be extracted either analytically or numerically from the distribution of $\mathbf{L}_t^k$.

To illustrate, we consider a simplified case of five stocks, with increasing annualized volatility ranging from 5% to 40% and an average correlation of 40%. We assume that the constraints are a budget constraint, i.e. weights summing to one, and a long-only constraint, i.e. non-negative weights. In Figure 2 we display the efficient portfolios for investment horizons of one week and five years. As expected, for a given level of risk-aversion, the optimal short-term portfolio (the vertical slit in the upper plot) concentrates on the first, low-volatility stock, with little exposure to the more aggressive stocks. On the other hand, the optimal long-term portfolio (the vertical slit in the lower plot) shifts and diversifies the allocation toward the aggressive stocks. This would *not* be the case if we followed the steps a)-f) above: the two plots would coincide! To download the fully commented code used to generate this example refer to Meucci (2009).

The steps in the above "fix", namely invariants estimation, projection, pricing, mean and covariance calculation constitute the foundation to build any risk platform. Following those steps becomes even more inevitable when we consider different asset classes, such as options. We will consider this case in an upcoming Quant Nugget.

References

Meucci, A., 2009, Exercises in advanced risk and portfolio management - with step-by-step solutions and fully documented code, *Free E-Book* available at http://ssrn.com/abstract=1447443.

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