A pair of glasses with a dark frame and light-colored lenses is resting on a piece of white paper. The background is a soft, out-of-focus yellow and orange gradient.

Data Structures

Chapter 1

1. Recursion
2. Performance Analysis
- 3. Asymptotic Analysis**
 - Revisit – Step Count
 - Asymptotic Analysis
 - Asymptotic Notations

Revisit – Step Count

- Why step count?
- It is to compare the **time complexities** of two programs that compute the same function and also to predict the **growth rate** in run time.
- **Example:** Let's compute the step count for three programs and compare their time complexities.
 1. $T_{\text{add}}(n)$ – adding two numbers
 2. $T_{\text{sum}}(n)$ – adding list of numbers
 3. $T_{\text{mtx}}(n)$ – adding two matrix

Revisit – Step Count n

Program add	step count
<pre>float add(int a, int b) { return a + b; }</pre>	1

Program sum of list	step count
<pre>float sum(float list[], int n) { float total = 0; int i; for (i=0; i<n; i++) total += list[i]; return total; }</pre>	1 n + 1 n 1

Program sum of matrix	step count
<pre>void add(int a[][MAX_SIZE], int b[][MAX_SIZE], int c[][MAX_SIZE], int rows, int cols) { for(int i=0; i<rows; i++) for(int j=0; j<cols; j++) c[i][j] = a[i][j] + b[i][j]; }</pre>	rows + 1 rows * (cols+1) rows * cols

Revisit – Step Count n

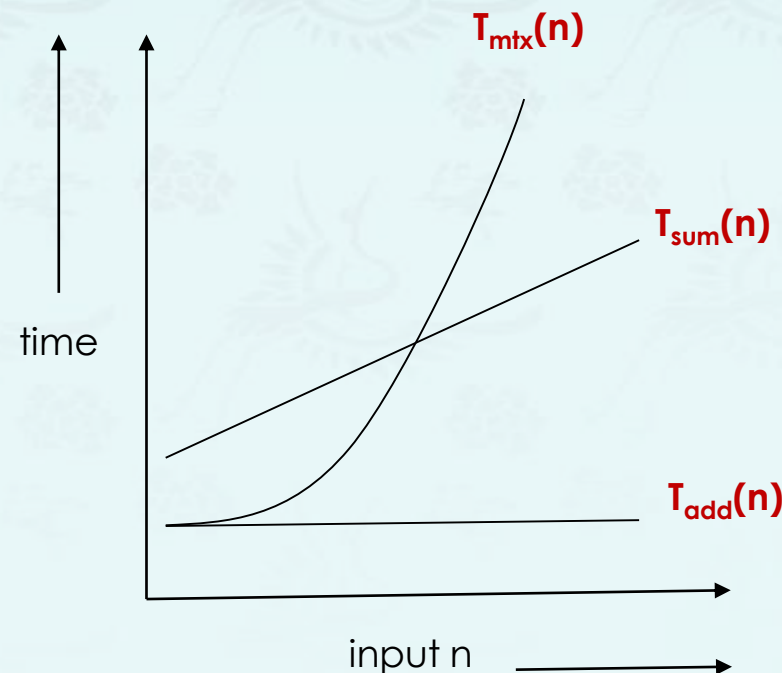
- $T_{add}(n) = 2$
- $T_{sum}(n) = 1 + 2(n + 1) + 2n + 1 = 4n + 4$
 $= c * n + c'$
- $T_{mtx}(n) = 2\ rows * cols + 2\ rows + 1$
 $= a * n^2 + b * n + c$

Revisit – Step Count n

- $T_{add}(n) = 2 \rightarrow O(1)$
- $T_{sum}(n) = 1 + 2(n + 1) + 2n + 1 = 4n + 4 \rightarrow O(n)$
 $= c * n + c'$
- $T_{mtx}(n) = 2\ rows * cols + 2\ rows + 1 \rightarrow O(n^2)$
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Revisit – Step Count n

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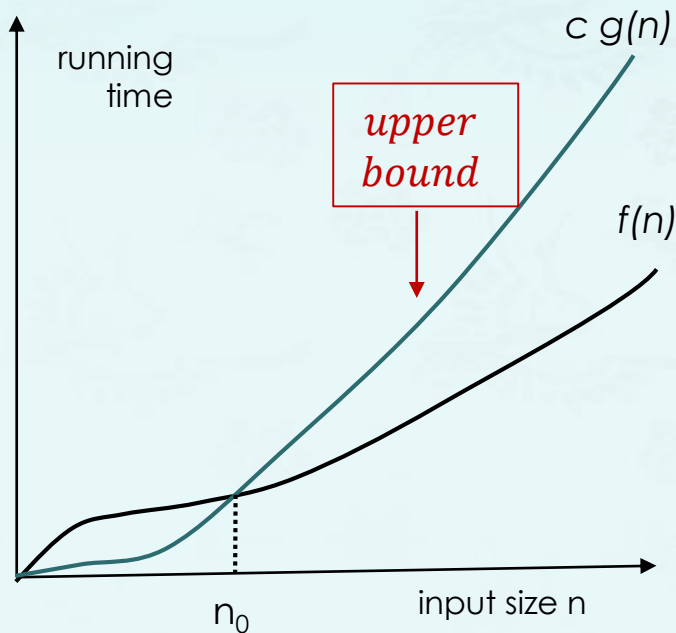


Asymptotic Analysis 점근적 분석과 표기법

- The "Big-Oh" Notation:
- Let $f(n)$ and $g(n)$ be functions mapping nonnegative integers to real numbers. We say that **$f(n)$ is $O(g(n))$** iff there are positive constants **c** and **n_0** such that
 - $f(n) \leq c g(n), \text{ for } n \geq n_0.$
 - Then it is pronounced as " $f(n)$ **is** big Oh of $g(n)$ or $f(n) = O(g(n))$ "

Asymptotic Analysis 점근적 분석과 표기법

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Example: Justify that the function **$8n - 2$ is $O(n)$** .
Given $f(n) = 8n - 2$, $g(n) = n$,
we need to find **c** and **n_0** such that
 $8n - 2 \leq c n$ for every integer $n \geq n_0$.

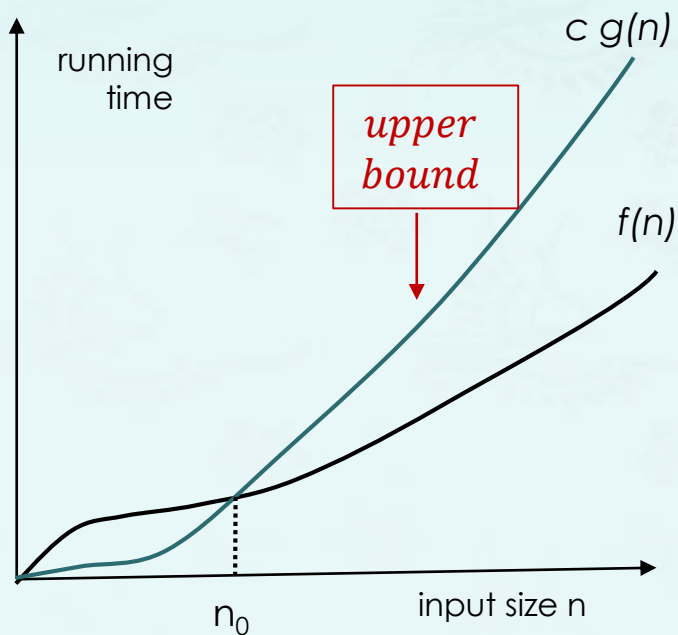
An easy choice among many is **$c = 8$ and $n_0 = 1$** .
Therefore, **$f(n)$ is $O(n)$** .

$g(n) = n$

Asymptotic Analysis 점근적 분석과 표기법

- **Example:** Find c and n_0 to justify that the function $7n + 5$ is $O(n)$.

$7n + 5$ is $O(n)$, we have to find c and n_0 such that
 $7n + 5 \leq c n$ for $n \geq n_0$



Asymptotic Analysis 점근적 분석과 표기법

- **Example:** Find c and n_0 to justify that the function $7n + 5$ is $O(n)$.

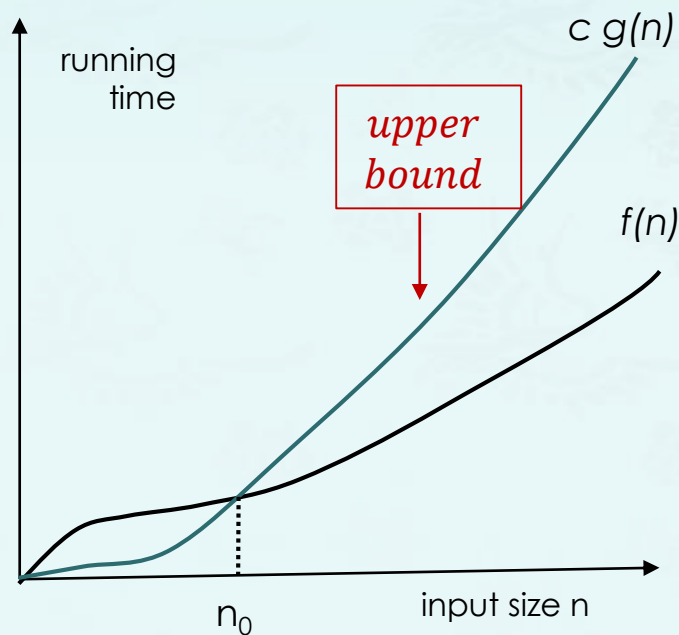
$7n + 5$ is $O(n)$, we must find c and n_0 such that

$$7n + 5 \leq c n \text{ for } n \geq n_0$$

$$7n + 5 \leq 7n + n$$

$$7n + 5 \leq 8n, \text{ for } n \geq n_0 = 5$$

Therefore, $7n + 5 \leq c n$ for $c = 8$ and $n_0 = 5$



Asymptotic Analysis 점근적 분석과 표기법

- **Example:** Find c and n_0 to justify that the function $7n + 5$ is $O(n)$.

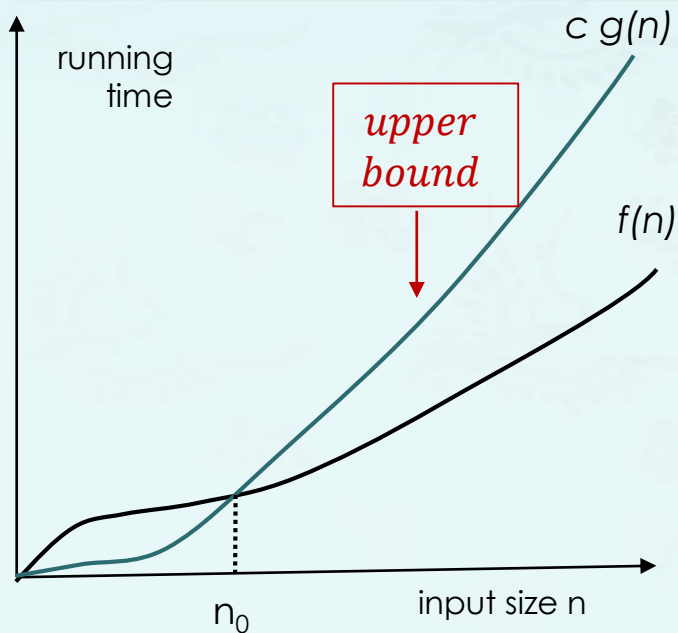
$7n + 5$ is $O(n)$, we must find c and n_0 such that

$$7n + 5 \leq c n \text{ for } n \geq n_0$$

$$7n + 5 \leq 7n + n$$

$$7n + 5 \leq 8n, \text{ for } n \geq n_0 = 5$$

Therefore, $7n + 5 \leq c n$ for $c = 8$ and $n_0 = 5$



$$7n + 5 \leq c n \quad \text{for } n \geq n_0$$

$$7n + 5 \leq 12n \quad \text{for } n \geq n_0 = 1$$

Therefore, $7n + 5 \leq c n$ for $c = 12$ and $n_0 = 1$

Asymptotic Analysis 점근적 분석과 표기법

- **Example:** Find c and n_0 to justify that the function $27n^2 + 16n$ is $O(n^2)$.

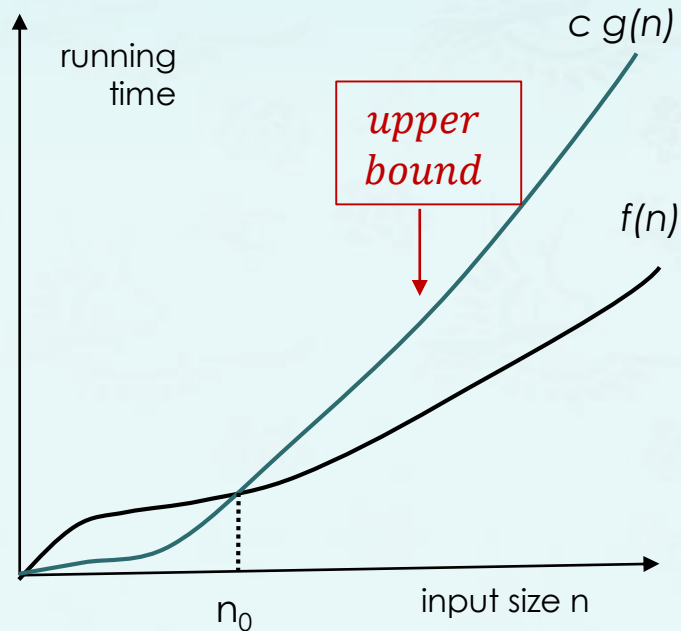
$27n^2 + 16n$ is $O(n^2)$, we must find c and n_0 such that

For $16n \leq n^2$

$$27n^2 + 16n \leq 27n^2 + n^2$$

$$27n^2 + 16n \leq 28n^2 \text{ for } n \geq n_0 = 16$$

Hence, $c = 28$ and $n_0 = 16$, Therefore, $f(n) = O(n^2)$.



$27n^2 + 16n$ is $O(n^2)$, we have to find c and n_0 such that

$$27n^2 + 16n \leq 43n^2$$

$$27n^2 + 16n \leq 43n^2 \text{ for } n \geq n_0 = 1$$

Hence, $c = 43$ and $n_0 = 1$, Therefore, $f(n) = O(n^2)$.

Asymptotic Analysis 점근적 분석과 표기법

- More Examples:

1) $3n + 2 =$ 

2) $3n + 3 =$ 

3) $100n + 6 =$ 

4) $10n^2 + 4n + 2 =$  5,

5) $6 * 2^n + n^2 =$ 

6) $3n + 3 =$ 

7) $10n^2 + 4n + 2 =$ 

❌ 8) $3n + 2 \neq O(1)$ as $3n + 2$ is **not** $\leq c$ for any c and all $n, n \geq n_0$.

❌ 9) $10n^2 + 4n + 2 \neq O(n)$

Asymptotic Analysis 점근적 분석과 표기법

- Preferred Big-Oh usage:
- Pick the tightest bound. If $f(N) = 5N$, then:
 - $f(N) = O(N^5)$
 - $f(N) = O(N^3)$
 - $f(N) = O(N \log N)$
 - **$f(N) = O(N)$** ← preferred or right!

Asymptotic Analysis 점근적 분석과 표기법

- **Preferred Big-Oh usage:**
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 - $f(N) = O(N^3)$
 - $f(N) = O(N \log N)$
 - **$f(N) = O(N)$** ← preferred or right!
- **Ignore constant factors and low order terms:**
 - $f(N) = O(N)$, *not* $f(N) = O(5N)$
 - $f(N) = O(N^3)$, *not* $f(N) = O(N^3 + N^2 + 15)$
- Wrong: $f(N) \leq O(g(N))$
- Wrong: $f(N) \geq O(g(N))$
- **Right:** **$f(N) = O(g(N))$**

Asymptotic Analysis 점근적 분석과 표기법

- Suppose two algorithms, A and B, solving the same problem have the running time of $O(n)$ and $O(n^2)$, respectively.
- Then algorithm A is **asymptotically better** than algorithm B.

※ $O(1) < O(\log n) < O(n) < O(n \log n) < O(n^2) < O(n^3) < O(2^n)$

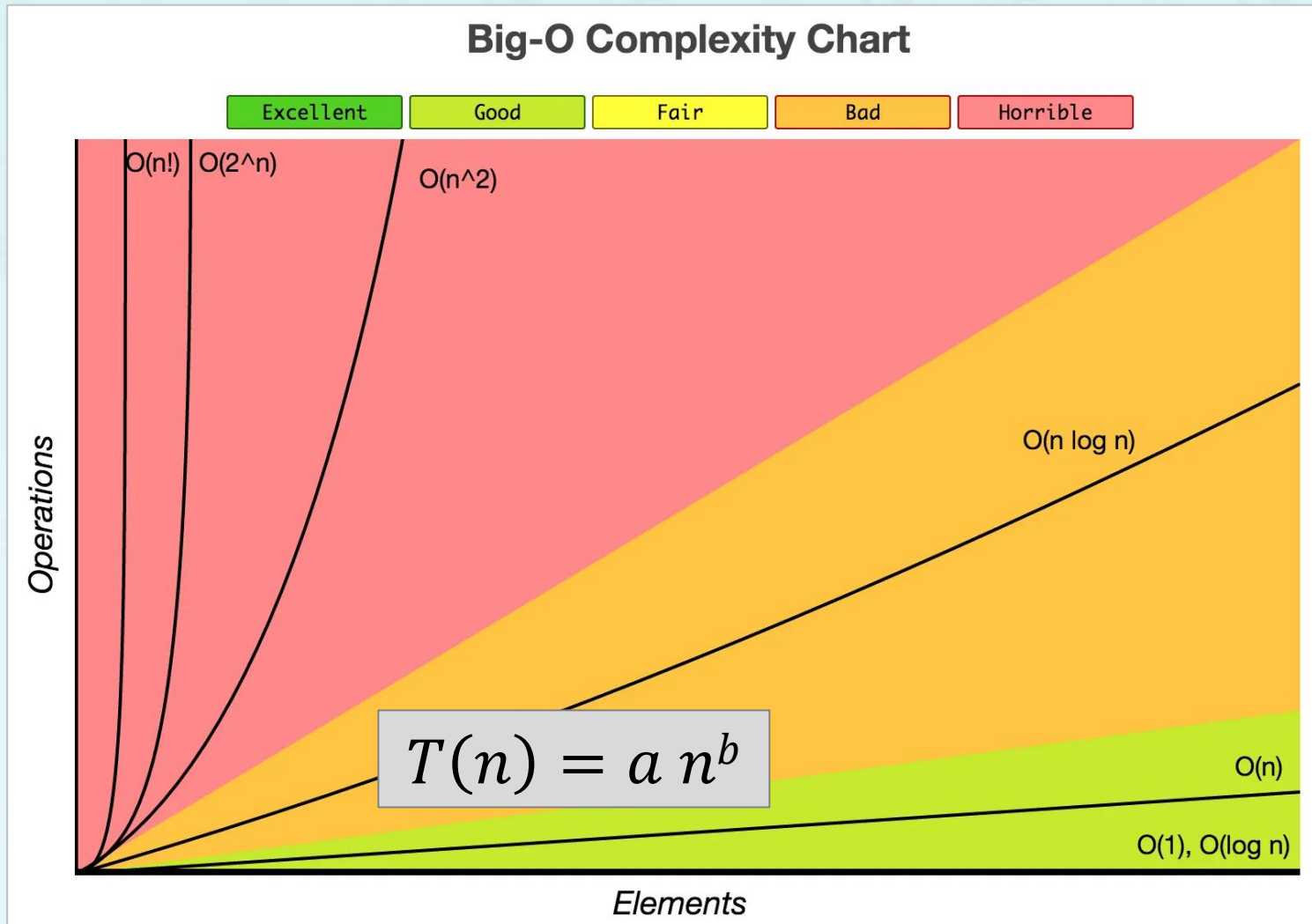
logarithmic linear linearithmic quadratic cubic exponential



$$T(n) = a n^b$$

Asymptotic Analysis 점근적 분석과 표기법

$$O(1) < O(\log n) < O(n) < O(n \log n) < O(n^2) < O(n^3) < O(2^n)$$



Asymptotic Analysis 점근적 분석과 표기법

[Omega] $f(n) = \Omega(g(n))$ iff there exist positive constants c and n_0 such that

$$f(n) \geq c g(n), \text{ for } n \geq n_0.$$

Asymptotic Analysis 점근적 분석과 표기법

[Omega] $f(n) = \Omega(g(n))$ iff there exist positive constants c and n_0 such that

$$f(n) \geq c g(n), \text{ for } n \geq n_0.$$

- **Example:** Let's suppose we have

$$f(n) = 5n^2 + 2n + 1$$

$$g(n) = n^2$$

For all $n \geq 0$, this $(2n + 1)$ will be ≥ 1 , **if** we have $c = 5$ and $n_0 = 0$.

Then, $5n^2 \leq f(n)$, for all $n \geq 0$

Therefore, we can say that the time complexity of $f(n)$ is $\Omega(n^2)$;

Asymptotic Analysis 점근적 분석과 표기법

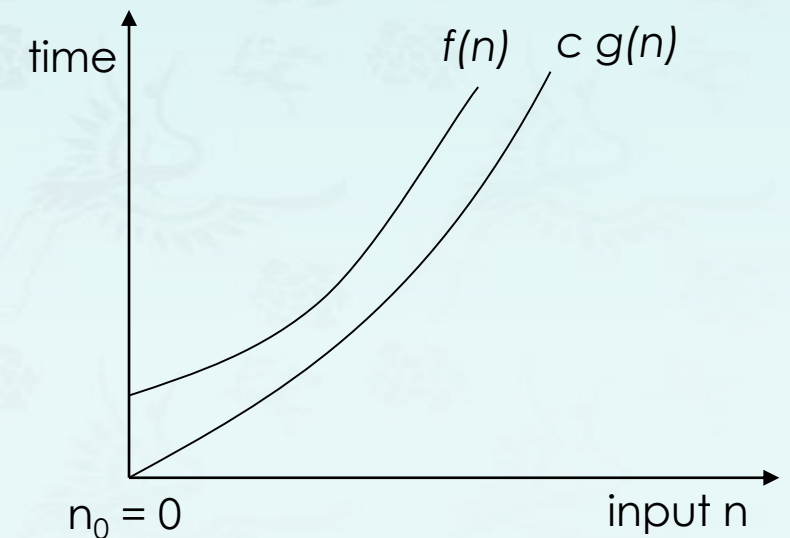
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- **Example:** Let's suppose we have

$$f(n) = 5n^2 + 2n + 1$$

$$g(n) = n^2$$



- **Omega** notation gives us the **lower bound** of the growth rate of a function.

Asymptotic Analysis 점근적 분석과 표기법

[Omega] $f(n) = \Omega(g(n))$ iff there exist positive constants c and n_0 such that

$$f(n) \geq c g(n), \text{ for } n \geq n_0$$

- **More Example:**

1) $3n + 2 = \Omega(n)$ since $3n + 2 \geq 3n$ for $n \geq 1$

2) $3n + 3 = \Omega(n)$ since $3n + 3 \geq 3n$ for $n \geq 1$

3) $100n + 6 = \Omega(n)$ since $100n + 6 \geq 100n$ for $n \geq 1$

4) $100n^2 + 4n + 2 = \Omega(n^2)$ since $100n^2 + 4n + 2 \geq n^2$ for $n \geq 1$

5) $6 * 2^n + n^2 = \Omega(2^n)$ since $6 * 2^n + n^2 \geq 2^n$ for $n \geq 1$

- **Omega** notation gives us the **lower bound** of the growth rate of a function.

Asymptotic Analysis 점근적 분석과 표기법

[Theta] $f(n) = \Theta(g(n))$ iff there exist positive constants c and n_0 such that

- $c_1g(n) \leq f(n) \leq c_2g(n), \text{ for } n \geq n_0.$

- **Example:** Let's suppose we have

$$f(n) = 5n^2 + 2n + 1$$

$$g(n) = n^2$$

- Then, we can choose $c_1 = 5, c_2 = 8$, and $n_0 = 1$; and our inequality will hold.
Therefore, we can say that the time complexity of

$$f(n) = 5n^2 + 2n + 1 = \Theta(n^2)$$

Asymptotic Analysis 점근적 분석과 표기법

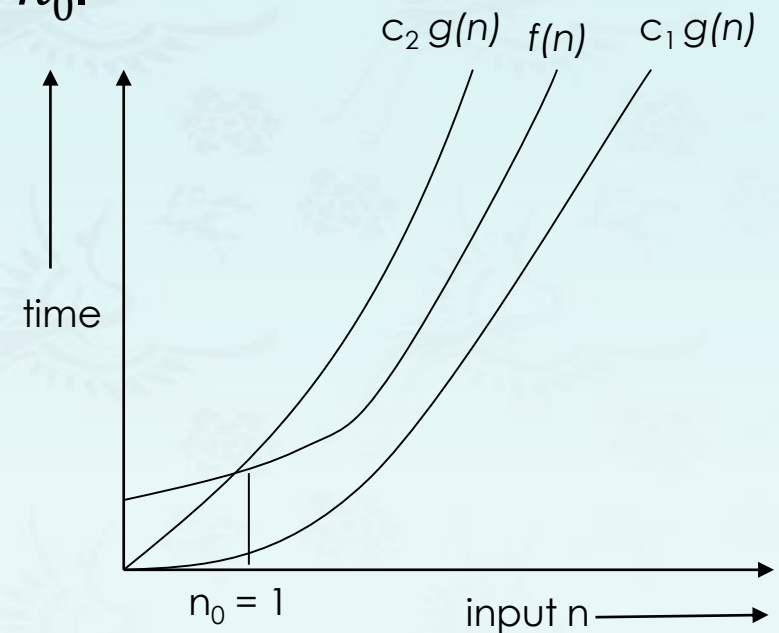
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$$c_1 g(n) \leq f(n) \leq c_2 g(n), \text{ for } n \geq n_0.$$

- **Example:** Let's suppose we have

$$f(n) = 5n^2 + 2n + 1$$

$$g(n) = n^2$$



- **Θ notation** best describes or give the best idea about the growth rate of the function because it gives us a **tight bound** unlike **O** and **Ω** which give us **upper bound** and **lower bound**, respectively.

Asymptotic Analysis 점근적 분석과 표기법

[Theta] $f(n) = \Theta(g(n))$ iff there exist positive constants c and n_0 such that

- $c_1g(n) \leq f(n) \leq c_2g(n), \text{ for } n \geq n_0.$

- **More Examples:**

1) $3n + 2 = \Theta(n)$

since $3n \leq 3n + 2 \leq 4n$ for all $n \geq 2, c_1 = 3, c_2 = 4, \text{ and } n_0 = 2$

2) $3n + 3 = \Theta(n)$

3) $10n^2 + 4n + 2 = \Theta(n^2)$

4) $6 * 2^n + n^2 = \Theta(2^n)$

5) $10 * \log n + 4 = \Theta(\log n)$

Asymptotic Analysis - Quiz

- Example: Running time estimates - empirical analysis
 - Personal computer executes 10^8 compares/second
 - Supercomputer executes 10^{12} compares/second

	Selection sort – $O(N^2)$			Merge sort – $O(N \log_2 N)$		
N	Million	10 million	Billion	Million	10 million	Billion
PC	2.8 hours	11.6 days		0.2 sec	2.3 sec	
Super Com	1 sec			Instant	Instant	Instant

$\log_{10} 2 \cong 0.3$
 $\log_2 10 \cong 3.3$
86,400sec/day

Use a reasonable or understandable time units.
Do not say, for example, "3660 days" nor "1220 seconds",
but 10.0 years or 20.3 min, respectively.
Use "**Instant**" if less than 0.1 sec.

※ **Bottom line**: Good algorithms are better than supercomputers.

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