"And where a mathematical reasoning can be had, it is as great a folly to make use of any other, as to grope for a thing in the dark, when you have a candle standing by you" – John Arbuthnot, *On The Laws of Chance*

Notes on notation:
$\in$ Element of
$\subseteq$ Subset of
$\subset$ Proper subset of
$\supseteq$ Superset of
$\supseteq$ Proper superset of
θ Empty set
$\cup$ Set union
$\cap$ Set intersection
$\times$ Cartesian product
2^A Powerset of set A
| | Set cardinality
{ } Set
$< >$ Ordered tuple
( ) Generator
[ ] Class generated by a generator
$(A - B)$ "set difference" between A and B (or "relative complement of A with respect to B"). IOW, "the set of elements in A but not in B" See:
https://en.wikipedia.org/wiki/Complement_(set_theory)

Definition 3.1: A *diagnostic problem* P is a 4-tuple <D, M, C, M$^+$> where $D = \{d_1, d_2, \ldots, d_n\}$ is a finite, non-empty set of objects, called disorders, $M = \{m_1, m_2, \ldots, m_n\}$ is a finite, non-empty set of objects called manifestations, and $C \subseteq D \times M$ is a relation with *domain*(C) = D and *range*(C) = M, called causation, and $M^+ \subseteq M$ is a distinguished subset of M which is said to be **present**.

Definition 3.2: For any element $d_i \in D$ and $m_j \in M$ in a diagnostic problem <D, M, C, M$^+$>, *effects*$(d_i) = \{m_j \mid \langle d_i, m_j \rangle \in C\}$, the set of objects directly caused by d_i, and *causes*$(m_j) = \{d_i \mid \langle d_i, m_j \rangle \in C\}$, the set of objects which can directly cause m_j

Definition 3.3: For any $D_I \subseteq D$ and $M_J \subseteq M$ in a diagnostic problem <D, M, C, M$^+$>, *effects*$(D_I) = \bigcup_{d_i \in D_I} effects(d_i)$, and *causes*$(M_J) = \bigcup_{m_j \in M_J} causes(m_j)$

Thus, for example, the effects of a set of disorders are just the union ("sum") of effects of individual disorders in the set.

Definition of *COVER*

Definition 3.4: The set $D_I \subseteq D$ is said to be a *cover* of $M_J \subseteq M$ if $M_J \subseteq effects(D_I)$

In other words, M_J is *covered* by D_I if every manifestation in M_J is causally associated with some member(s) of D_I. Or, to put it another way, the set of disorders D_I is sufficient to explain every manifestation in M_J.

Definition 3.5: A set $E \subseteq D$ is said to be an *explanation* of M$^+$ for a problem P = <D, M, C, M$^+$>if E *covers* M$^+$ and E satisfies a given parsimony condition.

Definition of *IRREDUNDANT*

Definition 3.6:

1. A cover, D_I of M_J is said to be *minimum* if its cardinality is smallest among all covers of M_J.

2. A cover, D_I of M_J is said to be *irredundant* if none of its proper subsets is also a cover of M_J. It is said to be *redundant* otherwise.

3. A cover, D_I of M$^+$ is said to be *relevant* if it is a subset of $causes(M^+)$; it is *irrelevant* otherwise.

In other words, a cover is irredundant if you can't take away any of its members and have it remain a cover. There may be covers with smaller cardinality though, because they may be made up of disorders that - individually - explain more manifestations.

Similarly, a cover is relevant if all of its members are a cause of at least one of the manifestations. If a cover contains any disorders that explain none of the manifestations that are present, then it is called *irrelevant*.

Definition 3.7: The *solution* to a diagnostic problem P = <D, M, C, M$^+$>designated $Sol(P)$ is the set of all explanations of M$^+$.

Lemma 3.1: Let P = <D, M, C, M$^+$>be the causal network for a diagnostic problem and $d_i \in D, m_j \in M, D_I, D_K \subseteq D,$ and$M_J \subseteq M$, then:

(a) $effects(d_i) \neq \theta$, $causes(m_j) \neq \theta$

(b) $d_i \in causes(effects(d_i))$, $m_j \in effects(causes(m_j))$

(c) $D_I \subseteq causes(effects(D_I))$, $M_J \subseteq effects(causes(M_J))$

(d) $M = effects(D)$, $D = causes(M)$

(e) $d_i \in causes(m_j)$ iff $m_j \in effects(d_i)$

(f) $effects(D_I) - effects(D_K) \subseteq effects(D_I - D_K)$

Lemma 3.2: If P = <D, M, C, M$^+$>is the causal network for a diagnostic problem with $D_I \subseteq D$ and $M_J \subseteq M$, then $D_I \cap causes(M_J) = \theta$ iff $M_J \cap$

$effects(D_I) = \theta$.

Lemma 3.3: If D_K is a cover of M_J in a diagnostic problem, then there exists a $D_I \subseteq D_K$ which is an irredundant cover of M_J.

Thereom 3.4: (Explanation Existence Theorem) There exists at least one explanation for M$^+$ for any diagnostic problem P = <D, M, C, M$^+$>
Follows from Lemma 3.1 (d) and Lemma 3.3

Lemma 3.5: A cover D_I of M_J is irredundant iff for every $d_i \in D_I$ there exists some $m_j \in M_J$ which is uniquely covered by d_i, i.e., $m_j \in effects(d_i)$ but $m_j \notin effects(D_I - \{d_i\})$

Lemma 3.6: If D_I is an irredundant cover of M_J then $|D_I| \leq |M_J|$. More specifically, if E is an explanation of M$^+$ for a diagnostic problem, then $|E| \leq |M^+|$.

Lemma 3.7: $E = \theta$ is the only explanation for $M^+ = \theta$.

Thereom 3.8: (Competing Disorders Theorem) Let E be an explanation for M$^+$, and let $M^+ \cap effects(d_1) \subseteq M^+ \cap effects(d_2)$ for some $d_1, d_2 \in D$. Then,

1. d_1 and d_2 are not both in E; and
2. if $d_1 \in E$, then there is another explanation E' for M$^+$ containing d_2 but not d_1, of equal or smaller cardinality.

Lemma 3.9: Let 2^D be the power set of D, and let S_{mc}, S_{ic}, S_{rc}, and S_c be sets of all minimum covers, all irredundant covers, all redundant covers, and all covers of M$^+$ respectively, for a diagnostic problem. Then $\theta \subseteq S_{mc} \subseteq S_{ic} \subseteq S_{rc} \subseteq S_c \subseteq 2^D$.

Lemma 3.10: (Subsumption Property) - For a diagnostic problem P, let S_{ic} be the set of all irredundant covers of M$^+$. Then S_{ic} is the smallest set of covers such that for any $D_K \subseteq D$ covering M$^+$, there is a D_I in that set of covers with $D_I \subseteq D_K$.

Definition 3.8: Let $g_1, g_2, \ldots, g_n$ be non-empty, pairwise-disjoint subsets of D. Then $G_I = \{g_1, g_2, \ldots, g_n\}$ is a *generator*. The class generated by G_I, designated as $[G_I]$ is defined to be $[G_I] = \{\{d_1, d_2, \ldots, d_n\} \mid d_i \in g_i, 1 \leq i \leq n\}$.
Note: what does “pairwise disjoint” mean? See: `https://en.wikipedia.org/wiki/Disjoint_sets`
In mathematics, two sets are said to be disjoint sets if they have no element in common. Equivalently, disjoint sets are sets whose intersection is the empty set. For example, 1, 2, 3 and 4, 5, 6 are disjoint sets, while 1, 2, 3 and 3, 4, 5 are not.
This definition of disjoint sets can be extended to any family of sets. A family of sets is pairwise disjoint or mutually disjoint if every two different sets in the

family are disjoint. For example, the collection of sets 1, 2, 3, ... is pairwise disjoint.

Note: What do we mean by “class”? See: https://en.wikipedia.org/wiki/Class_(set_theory)

In set theory and its applications throughout mathematics, a class is a collection of sets (or sometimes other mathematical objects) that can be unambiguously defined by a property that all its members share. The precise definition of "class" depends on foundational context.

Examples:

The collection of all algebraic objects of a given type will usually be a proper class. Examples include the class of all groups, the class of all vector spaces, and many others.

The surreal numbers are a proper class of objects that have the properties of a field.

Within set theory, many collections of sets turn out to be proper classes. Examples include the class of all sets, the class of all ordinal numbers, and the class of all cardinal numbers.

Note: Definition of “Generator” in mathematics:
https://en.wikipedia.org/wiki/Generator_(mathematics)

Regarding the pairwise disjoint subsets that go into a generator: The members of any one such set are *competing possibilities* where one such possibility, combined with one possibility each from the other sets, makes up one discrete hypothesis. To put it another way, within each generator-member-set, (or competing-possibility-set) the members compete to explain some portion of the present manifestations in M^+.

Definition 3.9: $G = \{G_1, G_2, \ldots, G_N\}$ is a *generator-set* if each $G_I \in G$ is a generator and $[G_I] \cap [G_J] = \theta$ for $I \neq J$. The class generated by G is $[G] = \bigcup_{I=1}^{N} [G_I]$.

DIVISION

Using the disorders evoked by a newly discovered manifestation, a division operation selects from existing hypotheses those which cover both the old manifestations AND the new manifestation.

Definition 3.10: Let $G_I = (g_1, g_2, \ldots, g_n)$ be a generator and let $H_1 \subseteq D$ where $H_1 \neq \theta$.

Then $Q = \{Q_k \mid Q_k \text{ is a generator }\}$ is a division of G_I by H_1 if for all k,

$1 < k < n$, $Q_k = (\, q_{k1}, q_{k2}, \ldots, q_{kn})$ where

$$q_{kj} = \begin{cases} g_j - H_1, \text{ if } j < k, \\ g_j \cap H_1, \text{ if } j = k, \\ g_j, \text{ if } j > k \end{cases}$$

Informally, all generators resulting from a division can be considered to be calculated as follows:
For any given order of g_j's in G_I, the first generator in the division of G_I by H_1 is the same as G_I except g_1 is replaced by $g_1 \cap H_1$. In the second generator, g_1 is replaced by $g_i - H_1$, g_2 is replaced by $g_2 \cap H_1$, other g_j's are not changed, and so on. In the K^{th} generator, all g_j's prior to g_k are replaced by $g_j - H_1$, g_k is replaced by $g_k \cap H_1$, and all g_j's after g_k are unchanged.
It is clear from the foregoing definition that for any generator Q_k resulting from division of G_I by H_1, one set it contains - namely q_{kk}, is a subset of H_1. The set difference operations for $j < k$ are to make sure that the classes for different generators resulting from the division are disjoint and thus to ensure that Q is a generator-set.

Definition 3.11: Let G be a generator-set and $H_1 \subseteq D$ where $H_1 \neq \theta$. A divsion of G by H_1 is $div(G, H_1) = \bigcup_{G_I \in G} div(G_I, H_1)$

Lemma 3.11: Let G_I be a generator, G be a generator-set, and $H_1 \subseteq D$ where $H_1 \neq \theta$. Then:

(a) $div(G_I, H_1)$ is a generator-set with $[div(G_I, H_1)] = \{E \in [G_I] \mid E \cap H_1 \neq \theta\}$; and

(b) $div(G, H_1)$ is a generator-set with $[div(G, H_1)] = \{E \in [G] \mid E \cap H_1 \neq \theta\}$

Definition 3.12: Let $G_I = (g_1, g_2, \ldots, g_n)$ be a generator, G a generator-set, and $H_1 \subseteq D$ where $H_1 \neq \theta$. Then the *residual of division* of G_I by H_1 is

$$res(G_I, H_1) = \begin{cases} \{(g_1 - H_1, \ldots, g_n - H_1)\}, \text{ if } g_i - H_1 \neq \theta \text{ for all } i, 1 \leq i \leq n \\ \theta, \text{ otherwise} \end{cases}$$

and the *residual of division* of G by H_1 is
$res(G, H_1) = \bigcup_{G_I \in G} res(G_I, H_1)$

Lemma 3.12: For $G_I, G,$ and H_1, as defined in Definition 3.12,

(a) $res(G_I, H_1)$ is a generator set with $[res(G_I, H_1)] = \{E \in [G_I] | E \cap H_1 = \theta\}$; and

(b) $res(G, H_1)$ is a generator set with $[res(G, H_1)] = \{E \in [G] \mid E \cap H_1 = \theta\}$

Definition 3.13: Let G and Q be generator-sets, $G_I \in G$ and $Q_J \in Q$ be generators, and $q_j \in Q_J$. Then a division of G_I by Q_J is

$$div(G_I, Q_J) = \begin{cases} \{G_I\}, \text{ if } Q_J = \theta \\ div(div(G_I, q_j), Q_I - (q_j)), \text{ otherwise} \end{cases}$$

A divsion of G by Q_J is $div(G, Q_J) = \bigcup_{G_I \in G} div(G_I, Q_J)$

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Lemma 3.13: Let G, G_I, Q_J and q_j be as defined in Definition 3.13. Then:

(a) $div(G_I, Q_J)$ is a generator-set with $[div(G_I, Q_J)] = \{E \in [G_I] \mid \text{ there exists } E' \in [Q_J] \text{ where } E' \subseteq E\}$

(b) $div(G, Q_J)$ is a generator set with $[div(G, Q_J)] = \{E \in [G] \mid \text{ there exists } E' \in [Q_J] \text{ where } E' \subseteq E\}$

Definition 3.14: Let G and Q be generator-sets, $G_I \in G$ and $Q_J \in Q$ be generators, $q_j \in Q_J$. Then a *residual of division* of G_I by Q_J is

$$res(G_I, Q_J) = \begin{cases} \theta \text{ if } Q_J = \theta \\ res(G_I, q_j) \cup res(div(G_I, q_j), Q_J - (q_j)), \text{ otherwise} \end{cases}$$

A *residual of division* of G by Q_J is

$$res(G, Q_J) = \bigcup_{G_I \in G} res(G_I, Q_J)$$

And a *residual of division* of G by Q is

$$res(G, Q) = \begin{cases} G, \text{ if } Q = \theta, \\ res(res(G, Q_J), Q - \{Q_J\}), \text{ otherwise} \end{cases}$$

Lemma 3.14: Let G, Q, G_I, Q_J, and q_j be as defined in definition 3.14. Then:

(a) $res(G_I, Q_J)$ is a generator-set with $[res(G_I, Q_J)] = \{E \in [G_J] \mid \text{ there does not exist } E' \in [Q_J] \text{ where } E' \subseteq E\}$

(b) $res(G, Q_J)$ is a generator-set with $[res(G, Q_J)] = \{E \in [G] | \text{ there does not exist } E' \in [Q_J] \text{ where } E' \subseteq E\}$

(c) $res(G, Q)$ is a generator-set with: $[res(G, Q)] = \{E \in [G] : \text{ there does not exist } E' \text{ where } E' \subseteq E\}$

Definition 3.15: Let $G_I = (g_1, g_2, \ldots, g_n)$ be a generator, G a generator-set, and $H_1 \subseteq D$ where $H_1 \neq \theta$. Then the *augmented residual* of division of G_I by H_1 is

$$augres(G_I, H_1) = \begin{cases} \{(g_1 - H1, \ldots, g_n - H_1, A)\} \text{ if } g_i - H_1 \neq \theta, I \leq i \leq n, A \neq \theta \\ \theta, \text{ otherwise} \end{cases}$$

where $A = H_1 - \bigcup_{i=1}^{n} g_i$

The augmented residual of division of G by H_1 is

$$augres(G, H_1) = \bigcup_{G_I \in G} augres(G_I, H_1)$$

Lemma 3.15: Let G_I, G, and H_1 be as defined in Definition 3.15. Then $augres(G_I, H_1)$ and $augres(G, H_1)$ are generator-sets.

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Algorithm BIPARTITE

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Lemma 3.16:

Lemma 3.17:

Lemma 3.18:

Thereom 3.19:

Definition 3.16:

Lemma 3.20:

Definition 3.17:

Lemma 3.21:

Definition 3.18:

Lemma 3.22:

Thereom 3.23:

Definition 3.19:

Definition 3.20:

Lemma 3.24:

TBD
Thereom 3.25: