"And where a mathematical reasoning can be had, it is as great a folly to make use of any other, as to grope for a thing in the dark, when you have a candle standing by you" – John Arbuthnot, *On The Laws of Chance*

Notes on notation:
$\in$ Element of
$\subseteq$ Subset of
$\subset$ Proper subset of
$\supseteq$ Superset of
$\supseteq$ Proper superset of
θ Empty set
$\cup$ Set union
$\cap$ Set intersection
$\times$ Cartesian product
2^A Powerset of set A
$|\ |$ Set cardinality
$\{\ \}$ Set
$<\ >$ Ordered tuple
( ) Generator
[ ] Class generated by a generator
$(A - B)$ "set difference" between A and B (or "relative complement of A with respect to B"). IOW, "the set of elements in A but not in B" See:
https://en.wikipedia.org/wiki/Complement_(set_theory)

Definition 3.1: A *diagnostic problem* P is a 4-tuple $<$D, M, C, M$^+$$>$ where $D = \{d_1, d_2, \ldots, d_n\}$ is a finite, non-empty set of objects, called disorders, $M = \{m_1, m_2, \ldots, m_n\}$ is a finite, non-empty set of objects called manifestations, and $C \subseteq D \times M$ is a relation with *domain*(C) = D and *range*(C) = M, called causation, and $M^+ \subseteq M$ is a distinguished subset of M which is said to be **present**.

Definition 3.2: For any element $d_i \in D$ and $m_j \in M$ in a diagnostic problem $<$D, M, C, M$^+$$>$, *effects*$(d_i) = \{m_j \mid \langle d_i, m_j \rangle \in C\}$, the set of objects directly caused by d_i, and *causes*$(m_j) = \{d_i \mid \langle d_i, m_j \rangle \in C\}$, the set of objects which can directly cause m_j

Definition 3.3: For any $D_I \subseteq D$ and $M_J \subseteq M$ in a diagnostic problem $<$D, M, C, M$^+$$>$, *effects*$(D_I) = \bigcup_{d_i \in D_I} effects(d_i)$, and *causes*$(M_J) = \bigcup_{m_j \in M_J} causes(m_j)$

Thus, for example, the effects of a set of disorders are just the union ("sum") of effects of individual disorders in the set.

Definition 3.4: The set $D_I \subseteq D$ is said to be a *cover* of $M_J \subseteq M$ if $M_J \subseteq effects(D_I)$

Definition 3.5: A set $E \subseteq D$ is said to be an *explanation* of M$^+$ for a problem P = <D, M, C, M$^+$>if E *covers* M$^+$ and E satisfies a given parsimony condition.

Definition 3.6:

1. A cover, D_I of M_J is said to be *minimum* if its cardinality is smallest among all covers of M_J.
2. A cover, D_I of M_J is said to be *irredundant* if none of its proper subsets is also a cover of M_J. It is said to be *redundant* otherwise.
3. A cover, D_I of M$^+$ is said to be *relevant* if it is a subset of $causes(M^+)$; it is *irrelevant* otherwise.

Definition 3.7: The *solution* to a diagnostic problem P = <D, M, C, M$^+$>designated $Sol(P)$ is the set of all explanations of M$^+$.

Lemma 3.1: Let P = <D, M, C, M$^+$>be the causal network for a diagnostic problem and $d_i \in D, m_j \in M, D_I, D_K \subseteq D, \text{and} M_J \subseteq M$, then:

(a) $effects(d_i) \neq \theta$, $causes(m_j) \neq \theta$

(b) $d_i \in causes(effects(d_i))$, $m_j \in effects(causes(m_j))$

(c) $D_I \subseteq causes(effects(D_I))$, $M_J \subseteq effects(causes(M_J))$

(d) $M = effects(D)$, $D = causes(M)$

(e) $d_i \in causes(m_j)$ iff $m_j \in effects(d_i)$

(f) $effects(D_I) - effects(D_K) \subseteq effects(D_I - D_K)$

Lemma 3.2: If P = <D, M, C, M$^+$>is the causal network for a diagnostic problem with $D_I \subseteq D$ and $M_J \subseteq M$, then $D_I \cap causes(M_J) = \theta$ iff $M_J \cap effects(D_I) = \theta$.

Lemma 3.3: If D_K is a cover of M_J in a diagnostic problem, then there exists a $D_I \subseteq D_K$ which is an irredundant cover of M_J.

Thereom 3.4: (Explanation Existence Theorem) There exists at least one explanation for M$^+$ for any diagnostic problem P = <D, M, C, M$^+$>
Follows from Lemma 3.1 (d) and Lemma 3.3

Lemma 3.5: A cover D_I of M_J is irredundant iff for every $d_i \in D_I$ there exists some $m_j \in M_J$ which is uniquely covered by d_i, i.e., $m_j \in effects(d_i)$ but $m_j \notin effects(D_I - \{d_i\})$

Lemma 3.6: If D_I is an irredundant cover of M_J then $|D_I| \leq |M_J|$. More specifically, if E is an explanation of M$^+$ for a diagnostic problem, then $|E| \leq |M^+|$.

Lemma 3.7: $E = \theta$ is the only explanation for $M^+ = \theta$.

Thereom 3.8: (Competing Disorders Theorem) Let E be an explanation for M$^+$, and let $M^+ \cap effects(d_1) \subseteq M^+ \cap effects(d_2)$ for some $d_1, d_2 \in D$. Then,

1. d_1 and d_2 are not both in E; and
2. if $d_1 \in E$, then there is another explanation E' for M$^+$ containing d_2 but not d_1, of equal or smaller cardinality.

Lemma 3.9: Let 2^D be the power set of D, and let S_{mc}, S_{ic}, S_{rc}, and S_c be sets of all minimum covers, all irredundant covers, all redundant covers, and all covers of M$^+$ respectively, for a diagnostic problem. Then $\theta \subseteq S_{mc} \subseteq S_{ic} \subseteq S_{rc} \subseteq S_c \subseteq 2^D$.

Lemma 3.10: TBD

Definition 3.8: TBD

Definition 3.9: TBD

Definition 3.10: Let $G_I = (g_1, g_2, \ldots, g_n)$ be a generator and let $H_1 \subseteq D$ where $H_1 \neq \theta$.
Then $Q = \{Q_k | Q_k$ is a generator $\}$ is a division of G_I by H_1 if for all k, $1 < k < n$, $Q_k = (\,q_{k1}, q_{k2}, \ldots, q_{kn})$ where

$$\begin{cases} g_j - H_1, \text{ if } j < k, \\ g_j \cap H_1, \text{ if } j = k, \\ g_j \text{ if } j > k \end{cases}$$

Definition 3.11:

Lemma 3.11:

Definition 3.12:

Lemma 3.12:

Definition 3.13:

Lemma 3.13:

Definition 3.14:

Lemma 3.14:

Definition 3.15:

Lemma 3.15:

Lemma 3.16:

Lemma 3.17:

Lemma 3.18:

Thereom 3.19:

Definition 3.16:

Lemma 3.20:

Definition 3.17:

Lemma 3.21:

Definition 3.18:

Lemma 3.22:

Thereom 3.23:

Definition 3.19:

Definition 3.20:

Lemma 3.24:

TBD
Thereom 3.25: