Stanford University
Winter 2013-2014

Signal Processing and Linear Systems I

Lecture 13: Impulse trains, Periodic Signals, and Sampling

February 19, 2014

Fourier Transforms of Periodic Signals

So far we have used Fourier series to handle periodic signals, they do not have a Fourier transform in the usual sense (not finite energy).

We can generalize Fourier transform to such signals. Given a periodic signal $f(t)$ with period T_0, $f(t)$ has a Fourier Series.

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t}$$

where

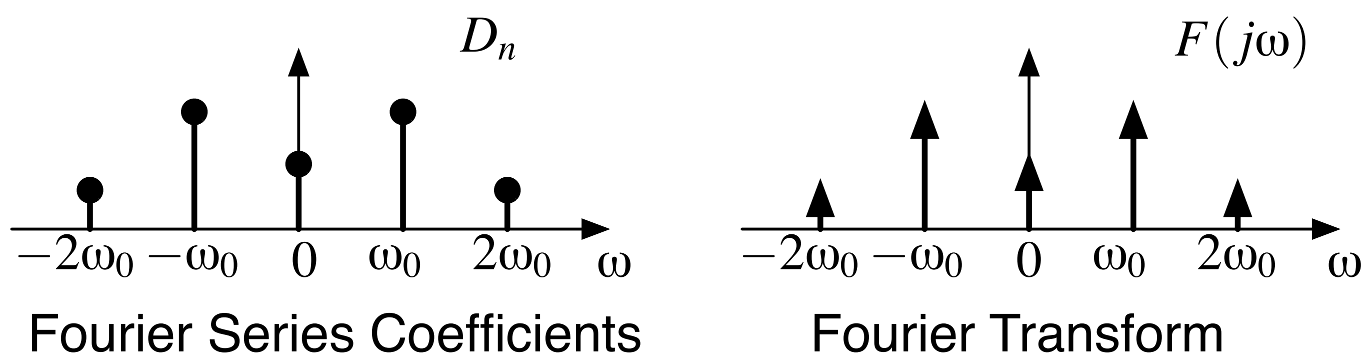
$$D_n = \frac{1}{T} \int_0^T f(t) e^{-jn\omega_0 t} \, dt$$

and $\omega_0 = 2\pi/T$.

Fourier series resembles an inverse Fourier transform of $f(t)$, but it is a $\sum$ and not an $\int$.

We can make the connection much clearer using the Fourier transform for complex exponentials, and extended linearity:

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \Leftrightarrow F(j\omega) = \sum_{n=-\infty}^{\infty} D_n 2\pi\delta(\omega - n\omega_0)$$



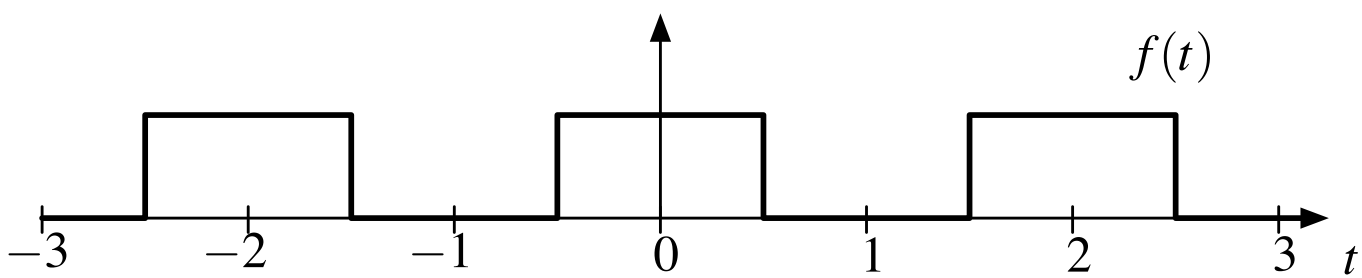
The Fourier series coefficients and Fourier transform are the same! (with a scale factor of 2π).

Example: Square Wave

Consider the square wave

$$f(t) = \sum_{n=-\infty}^{\infty} \mathrm{rect}(t - 2n)$$

This is the square pulse of width $T = 1$ defined on the interval of width $\tau = 2$ and then replicated infinitely often.



The Fourier series from before (Lecture 7, page 38) is

$$f(t) = \sum_{n=-\infty}^{\infty} D_n e^{j2\pi nt/\tau} = \sum_{n=-\infty}^{\infty} D_n e^{j\pi nt}$$

with Fourier coefficients

$$D_n = \frac{T}{\tau}\operatorname{sinc}\left(n\frac{T}{\tau}\right) = \frac{1}{2}\operatorname{sinc}(n/2)$$

so that

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2}\operatorname{sinc}(n/2)e^{j\pi nt}.$$

The Fourier transform is then

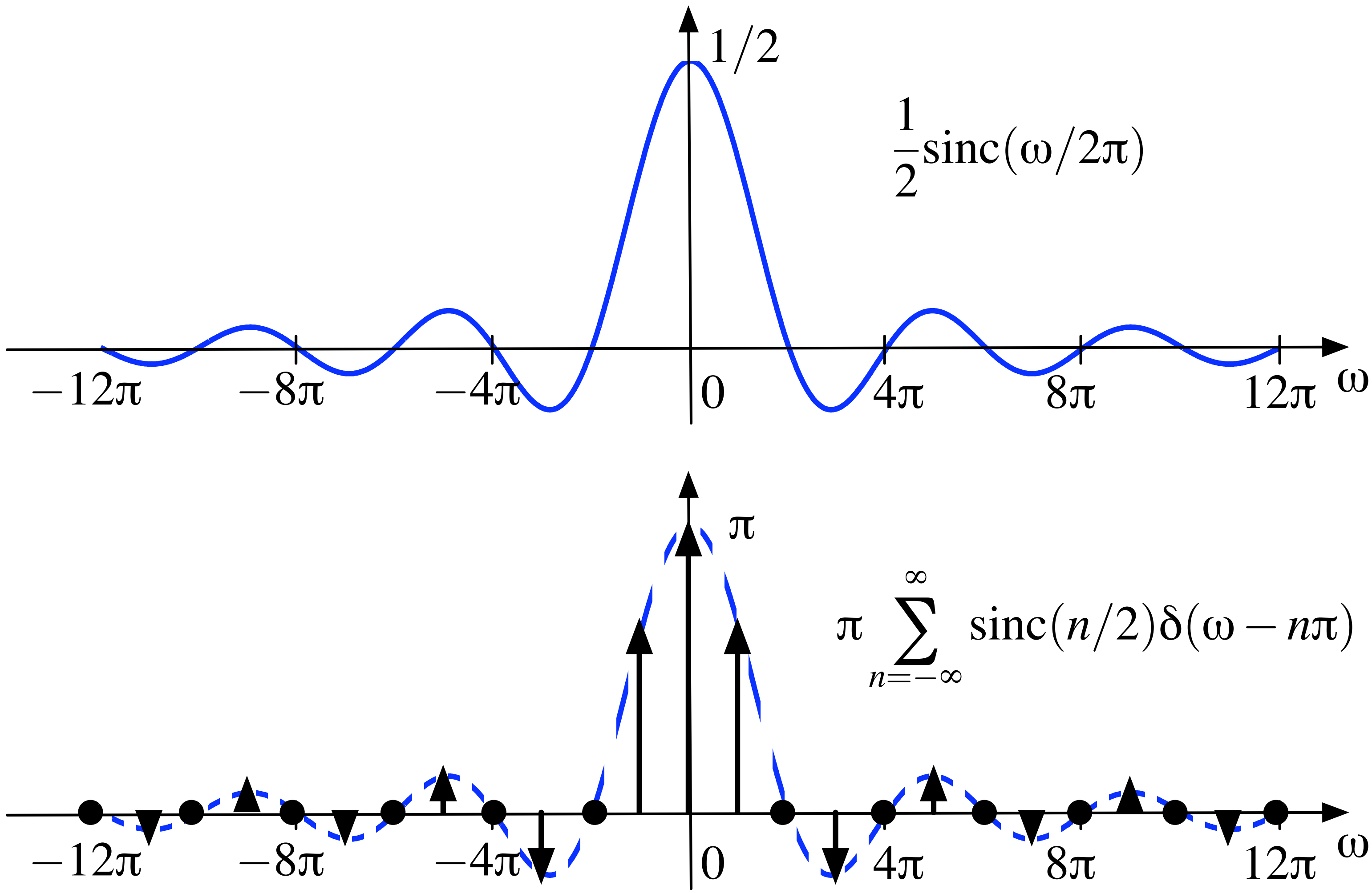
$$F(j\omega) \quad = \quad \sum_{n=-\infty}^{\infty} \frac{1}{2}\operatorname{sinc}(n/2)(2\pi\delta(\omega - n\pi))$$

$$= \pi \sum_{n=-\infty}^{\infty} \mathrm{sinc}(n/2)\delta(\omega - n\pi)$$

Note that this can also be written:

$$F(j\omega) = \pi \sum_{n=-\infty}^{\infty} \mathrm{sinc}(\omega/2\pi)\delta(\omega - n\pi).$$

This is the Fourier transform of the rect, multiplied by an array of evenly spaced δ's.

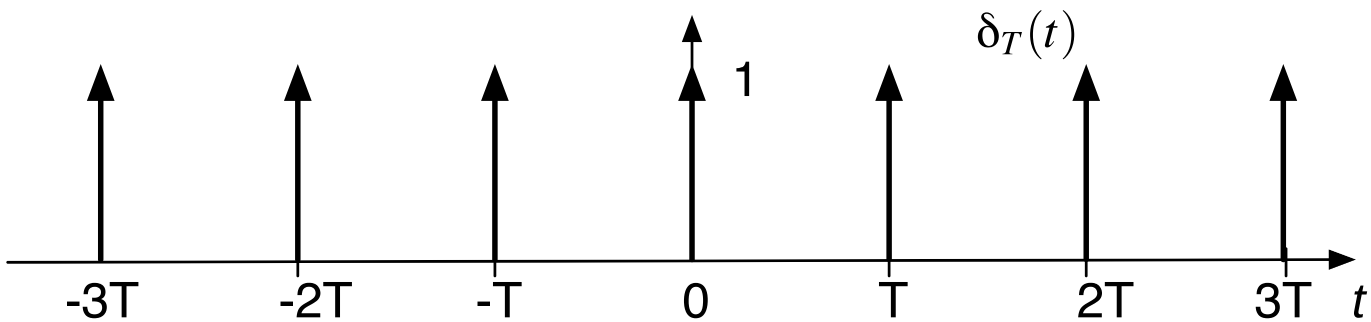

1/2
$\frac{1}{2}\mathrm{sinc}(\omega/2\pi)$
-12π
-8π
-4π
0
4π
8π
12π
ω
π
$\pi\sum_{n=-\infty}^{\infty}\mathrm{sinc}(n/2)\delta(\omega-n\pi)$
-12π
-8π
-4π
0
4π
8π
12π
ω

Impulse Trains – Sampling Functions

Define $\delta_T(t)$ to be a sequence of unit δ functions spaced by T,

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

which looks like



What do we get if we expand this function as a Fourier series over $-T/2$ to $T/2$?

The Fourier coefficients are

$$
\begin{aligned}
D_n &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-j2\pi nt/T}dt \\
&= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-j2\pi nt/T}dt \\
&= \frac{1}{T}.
\end{aligned}
$$

All of the coefficients are the same!

The Fourier series is then

$$
\begin{aligned}
f(t) &= \sum_{n=-\infty}^{\infty} \frac{1}{T}e^{j2\pi nt/T} \\
&= \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}.
\end{aligned}
$$

The Fourier transform of $\delta_T(t)$ is then

$$
\begin{aligned}
\mathcal{F}\left[\delta_T(t)\right] &= \frac{1}{T}\sum_{n=-\infty}^{\infty} 2\pi\delta(\omega - n\omega_0) \\
&= \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) \\
&= \omega_0 \delta_{\omega_0}(\omega)
\end{aligned}
$$

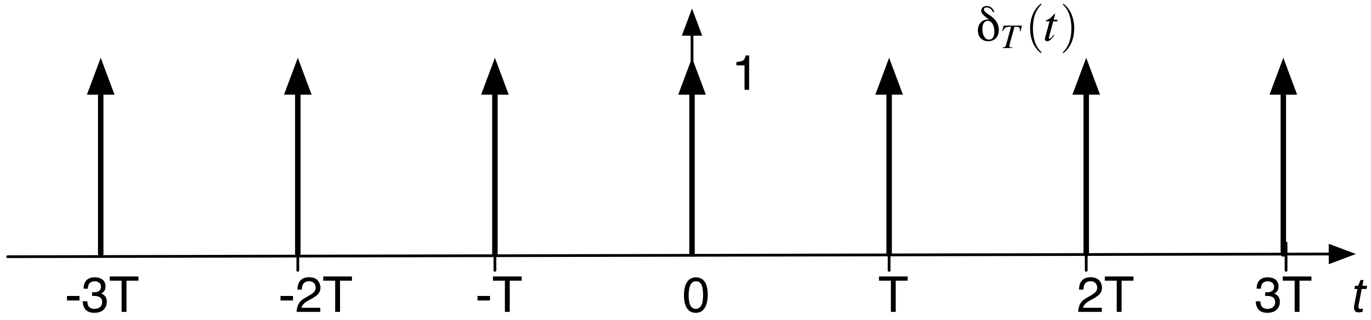
since $\omega_0 = 2\pi/T$.

We then have the transform pair

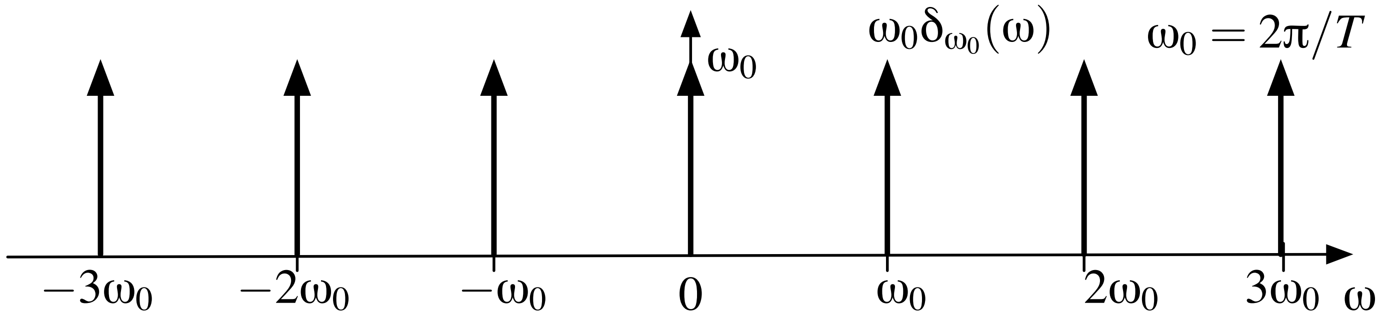
$$\delta_T(t) \Leftrightarrow \omega_0 \delta_{\omega_0}(\omega)$$

The Fourier transform of an array evenly spaced δ's is another array of evenly spaced δ's!

The delta train $\delta_T(t)$ is



and its Fourier transform $\omega_0\delta_{\omega_0}(t)$ is



These are *very* useful functions!

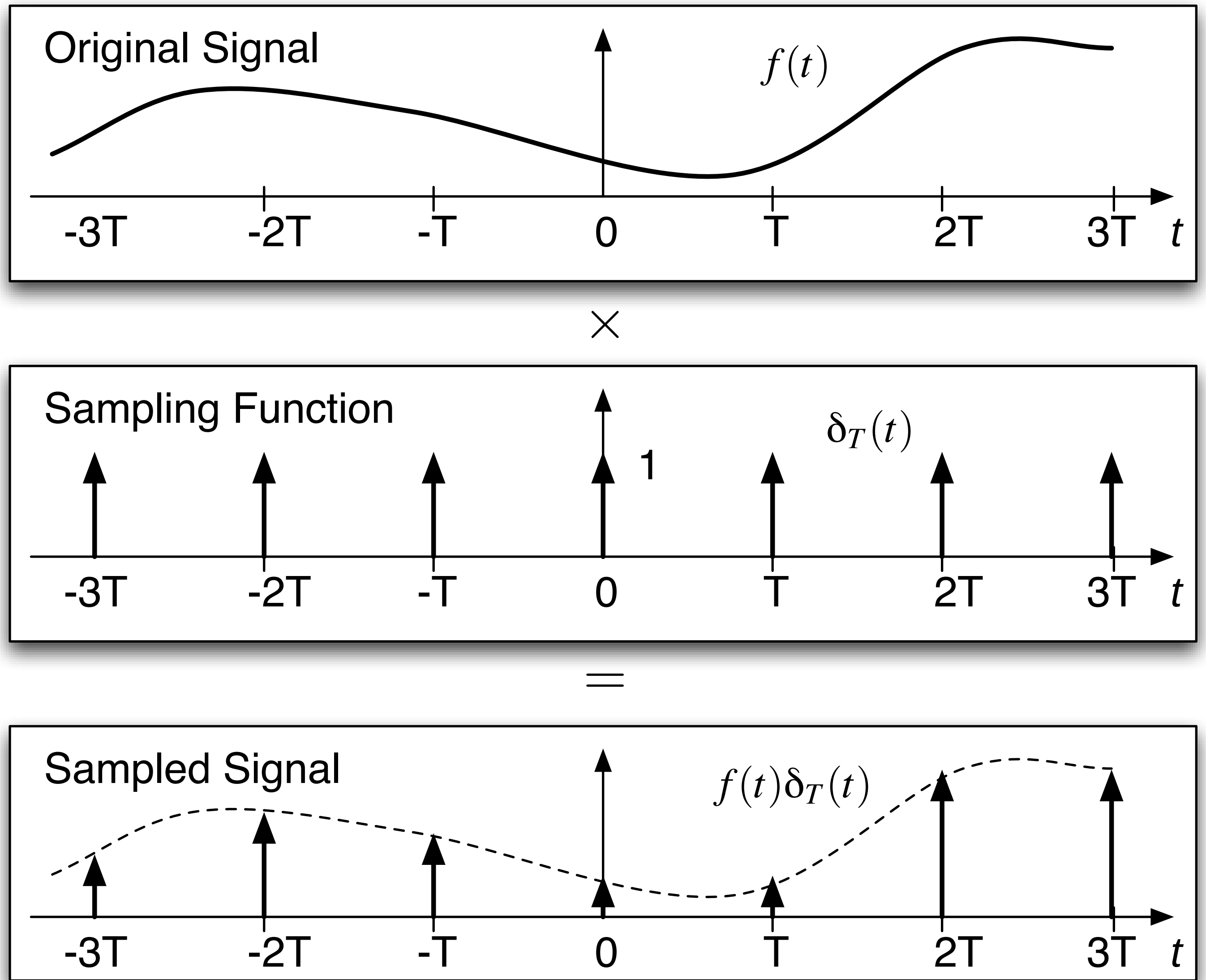
Ideal Sampling

One of the most important uses of $\delta_T(t)$ is to represent *sampling*.

If $f(t)$ is a signal, then

$$\begin{aligned} f(t)\delta_T(t) &= f(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) \\ &= \sum_{n=-\infty}^{\infty} f(t)\delta(t-nT) \\ &= \sum_{n=-\infty}^{\infty} f(nT)\delta(t-nT) \end{aligned}$$

where we have used the fact that $f(t)\delta(t-T) = f(T)\delta(t-T)$ for the last step.

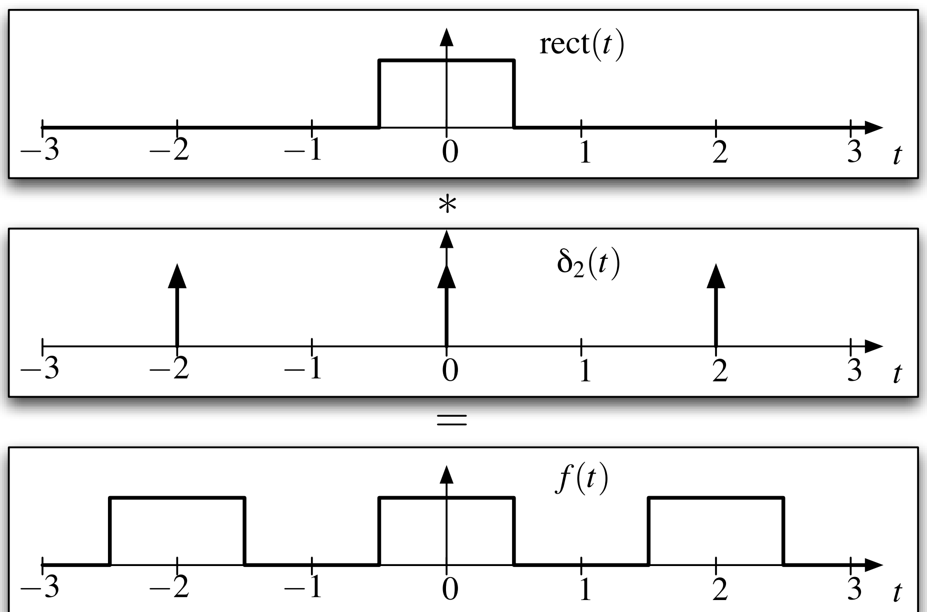

Original Signal
$f(t)$
-3T
-2T
-T
0
T
2T
3T
t
$\times$
Sampling Function
$\delta_T(t)$
1
-3T
-2T
-T
0
T
2T
3T
t
=
Sampled Signal
$f(t)\delta_T(t)$
-3T
-2T
-T
0
T
2T
3T
t

Periodic Signals

We can write the periodic square wave function from earlier today as a convolution of a $\text{rect}(t)$ with an impulse train of spacing 2:

$$f(t) = \text{rect}(t) * \delta_2(t)$$

which is illustrated below



Using the convolution theorem, the Fourier transform of $f(t)$ is

$$\begin{aligned}
\mathcal{F}[f(t)] &= \mathcal{F}[\mathrm{rect}(t)]\,\mathcal{F}[\delta_2(t)] \\
&= \mathrm{sinc}(\omega/2\pi)\omega_0\delta_{\omega_0}(\omega) \\
&= \pi\,\mathrm{sinc}(\omega/2\pi)\delta_\pi(\omega)
\end{aligned}$$

where $\omega_0 = 2\pi/T = 2\pi/2 = \pi$.

To get this into the form we found earlier, expand $\delta_\pi(\omega)$

$$\begin{aligned}
\mathcal{F}[f(t)] &= \pi\,\mathrm{sinc}(\omega/2\pi)\sum_{n=-\infty}^{\infty}\delta(\omega-\pi n) \\
&= \pi\sum_{n=-\infty}^{\infty}\mathrm{sinc}(\omega/2\pi)\delta(\omega-\pi n) \\
&= \pi\sum_{n=-\infty}^{\infty}\mathrm{sinc}(n/2)\delta(\omega-\pi n)
\end{aligned}$$

which is the same thing we obtained before.

In general, if $f_1(t)$ is one cycle of a periodic function with period T, then $f(t)$ is

$$f(t) = f_1(t) * \delta_T(t)$$

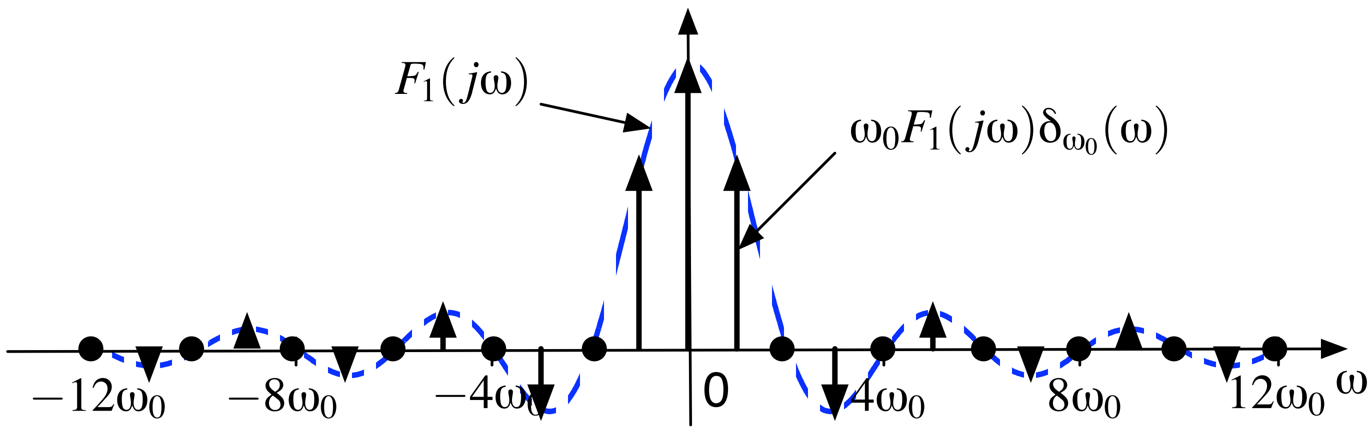
and the Fourier transform is

$$F(j\omega) = \omega_0 F_1(j\omega)\delta_{\omega_0}(\omega)$$

Recall that multiplying by $\delta_{\omega_0}(\omega)$ *samples* $F_1(j\omega)$ at multiples of ω_0.

The Fourier transform of the periodic signal is the sampled Fourier transform of one period

This is what we saw the periodic rect signal example:



Sampling Theorem

What is the Fourier transform of a signal that has been sampled in the time domain? If $f(t)$ is a signal, then $f(t)\delta_T(t)$ is the sampled signal,

$$\bar{f}(t) = f(t)\delta_T(t)$$

and it Fourier transform is

$$\begin{aligned}
\bar{F}(j\omega) &= \mathcal{F}\left[f(t)\delta_T(t)\right] \\
&= \frac{1}{2\pi}\mathcal{F}\left[f(t)\right] * \mathcal{F}\left[\delta_T(t)\right] \\
&= \frac{1}{2\pi}F(j\omega) * (\omega_0\delta_{\omega_0}(\omega)) \\
&= \frac{1}{T}F(j\omega) * \delta_{\omega_0}(\omega)
\end{aligned}$$

If we expand $\delta_{\omega_0}(\omega)$,

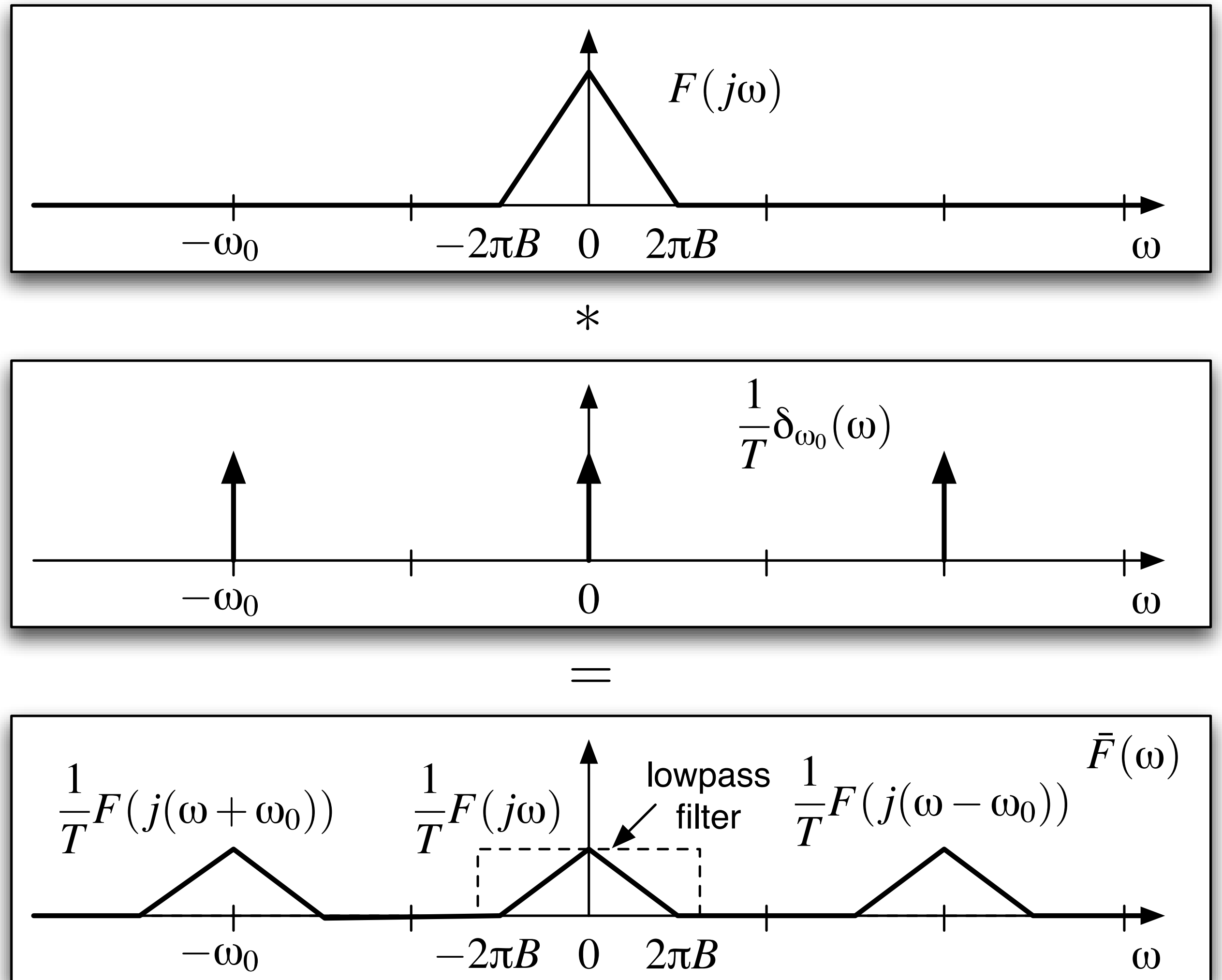
$$\begin{aligned}\bar{F}(j\omega) &= \frac{1}{T}F(j\omega) * \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0) \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty} F(j(\omega - n\omega_0))\end{aligned}$$

The spectrum of the sampled signal consists of shifted replicas of the original spectrum, scaled by $1/T$.

What this looks like exactly depends on the sampling frequency ω_0.

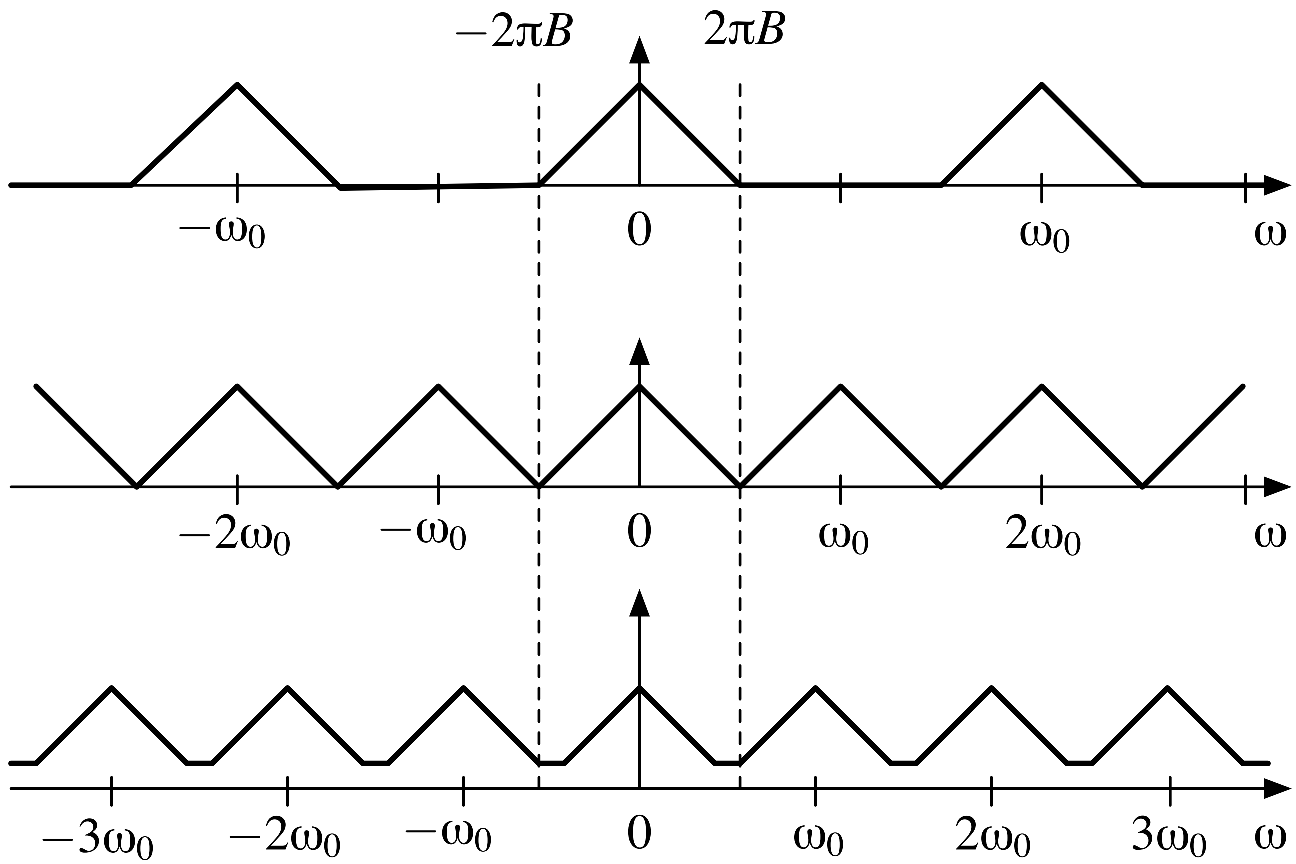
We'll define the bandwidth of $f(t)$ to be $\pm B$ Hz. The following plot shows the case where $2\pi B \ll \omega_0/2$

This is the case where the signal bandwidth is much less than the sampling rate.



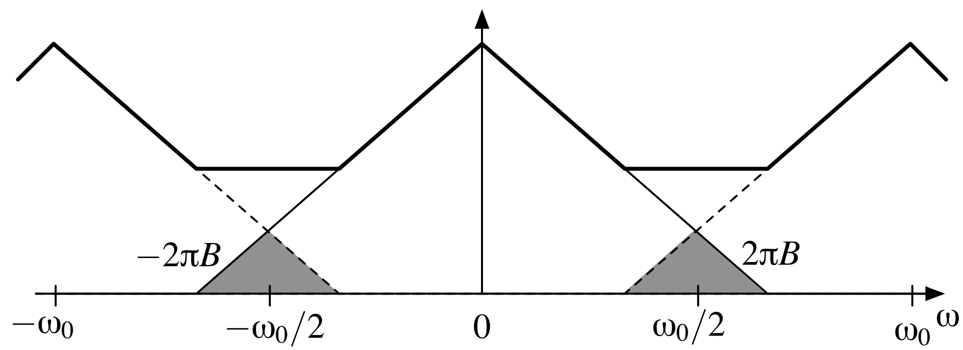
If we lowpass filter $\bar{F}(j\omega)$, then we can perfectly recover $F(j\omega)$, and $f(t)$!

As the sampling frequency ω_0 decreases (sampling period T increases) the spectral replicas get closer:



Eventually the replicas overlap, and $F(j\omega)$ cannot be recovered.

The overlap is called *aliasing* because the low frequencies of one band appear (alias) as high frequencies of the next band. High frequencies from one band also *alias* as low frequencies of the next band.



No aliasing occurs only if $2\pi B < \omega_0/2$, or

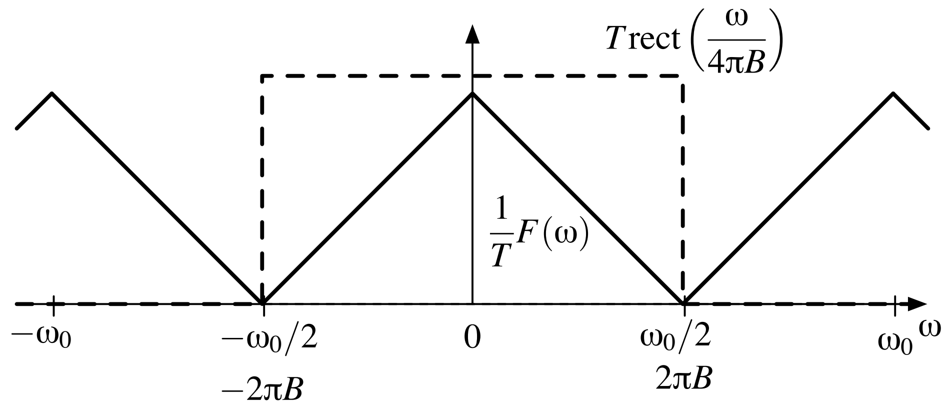
$$2B < \omega_0/2\pi = \frac{2\pi}{T}\frac{1}{2\pi} = \frac{1}{T} \text{ Hz}$$

The signal can be recovered exactly only if the signal bandwidth $2B$ Hz is less than or equal to the sampling rate $1/T$ Hz.

The sampling rate $2B$ is the *Nyquist rate* for $f(t)$, the lowest rate that allows $f(t)$ to be perfectly recovered. T is the *Nyquist interval*.

Signal Reconstruction: the Interpolation Formula

The ideal lowpass filter for $2B = 1/T$, sampling at the Nyquist rate, is



The lowpass filter has the Fourier transform

$$H(j\omega) = T\text{rect}\left(\frac{\omega}{4\pi B}\right)$$

Using the fact that sinc$(t) \Leftrightarrow$ rect$(\omega/2\pi)$ and the scaling theorem,

$$h(t) = 2BT\text{sinc}(2Bt) = \text{sinc}(2Bt)$$

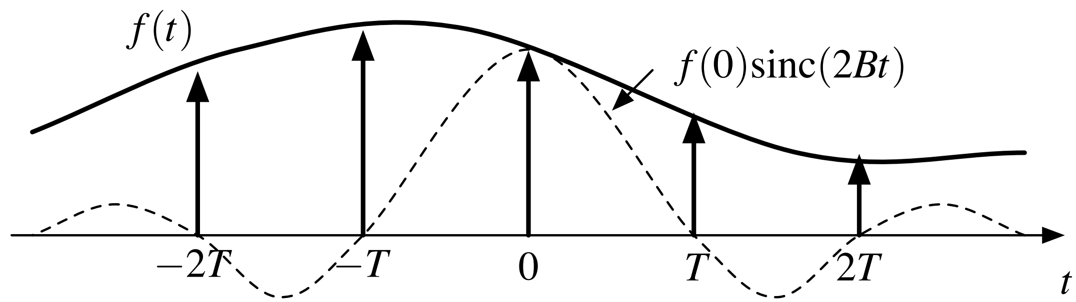
since we are assuming $2B = 1/T$ for sampling at the Nyquist rate.

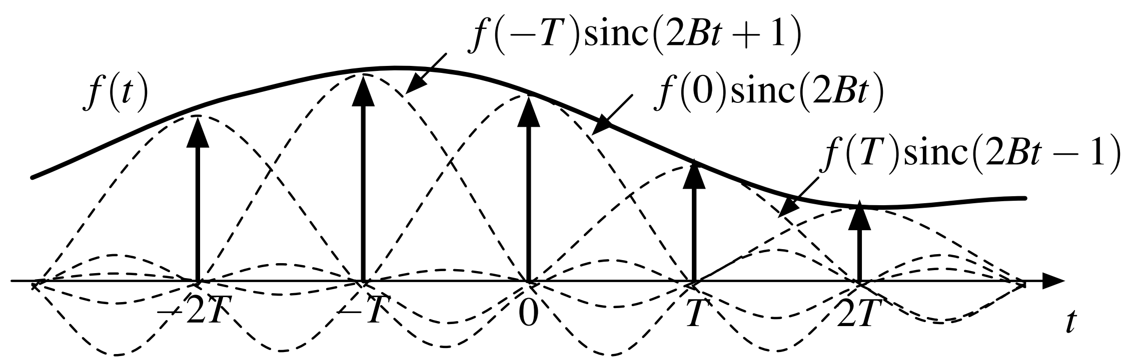
The reconstructed signal is

$$
\begin{aligned}
\bar{f}(t) * h(t) &= \left(\sum_{n=-\infty}^{\infty} f(nT)\delta(t - nT) \right) * h(t) \\
&= \sum_{n=-\infty}^{\infty} f(nT)h(t - nT) \\
&= \sum_{n=-\infty}^{\infty} f(nT) \text{ sinc}(2B(t - nT)) \\
&= \sum_{n=-\infty}^{\infty} f(nT) \text{ sinc}(2Bt - n)
\end{aligned}
$$

This is the *Whittaker-Kotelnikov-Shannon sampling theorem*.

The interpolated signal is a sum of shifted sincs, weighted by the samples $f(nT)$, which looks like:





The sinc shifted to nT is 1 at nT, and zeros at all other samples.

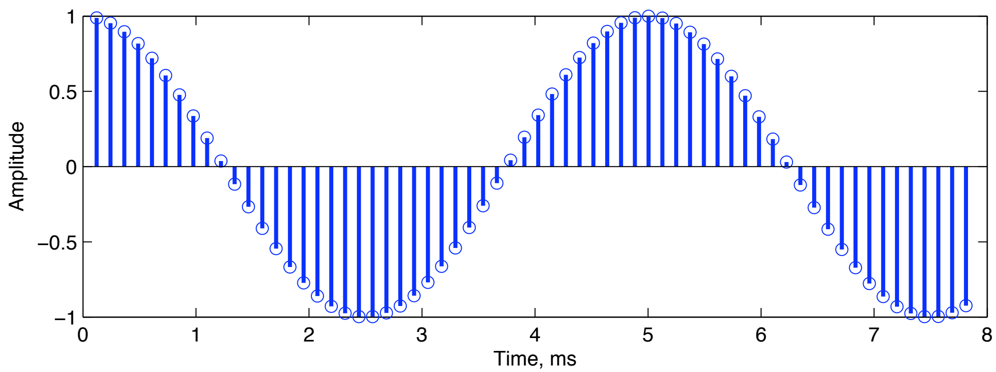
The sum of the weighted shifted sincs will agree with all of the samples!

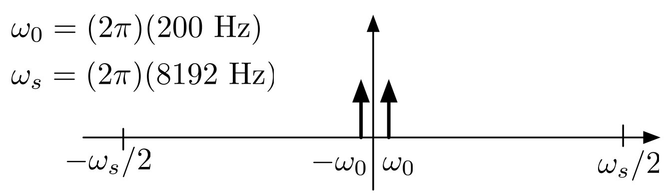
Sampled Sinusoids

To see how remarkable this is, we need to look at some sampled sinusoids.

A bandlimited signal with a bandwidth $2B$ can be reconstructed perfectly from its samples providing the sampling rate $1/T > 2B$,

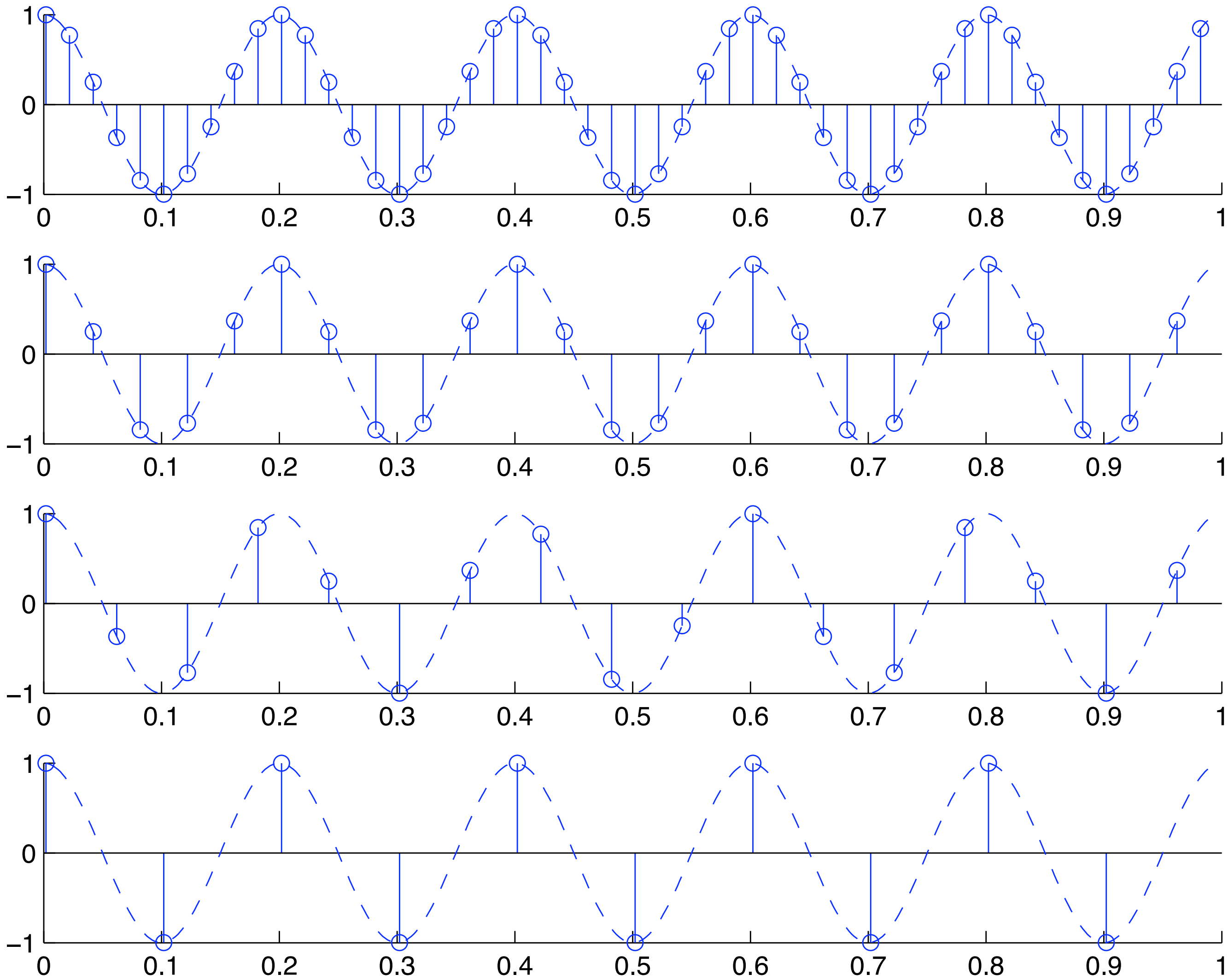
Typically, we think of sampled sinusoids as looking like



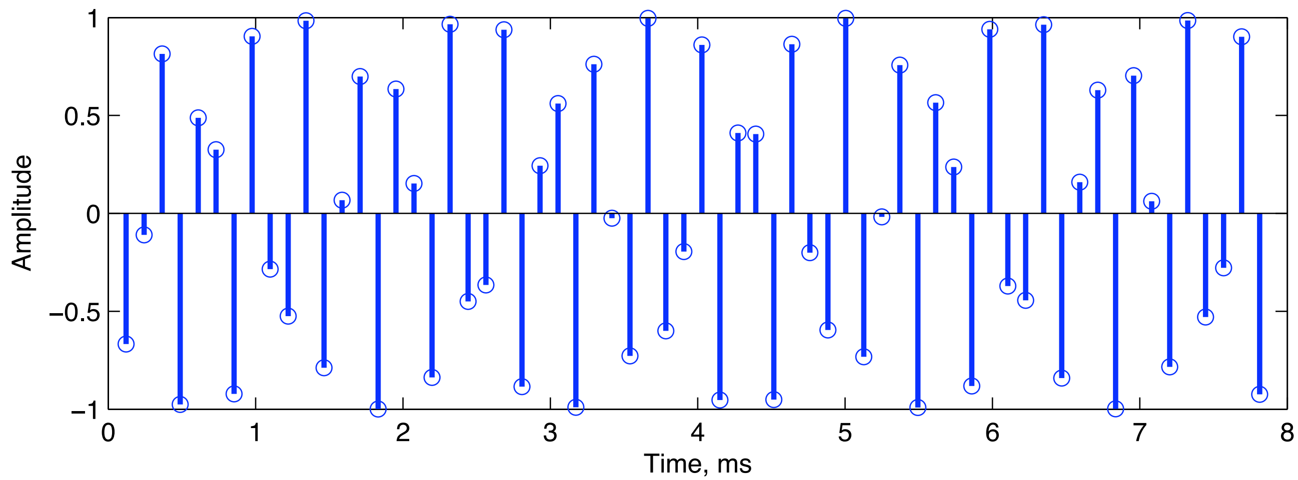


At this sampling rate, it is easy to believe that you can reconstruct the sinusoid from its samples.

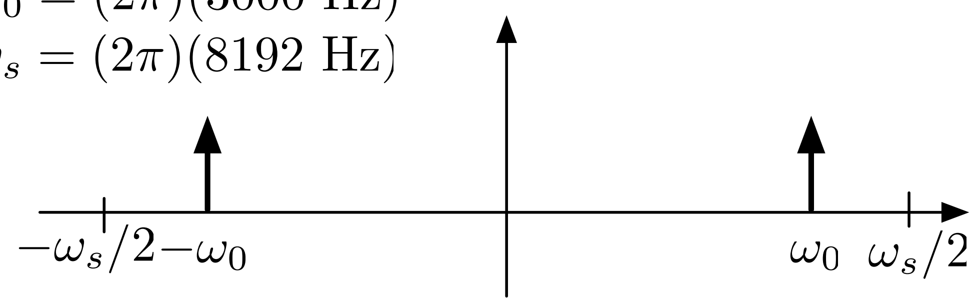
As the sampling rate decreases the reconstruction task looks harder,



Most sampled sinusoids are not very recognizable!



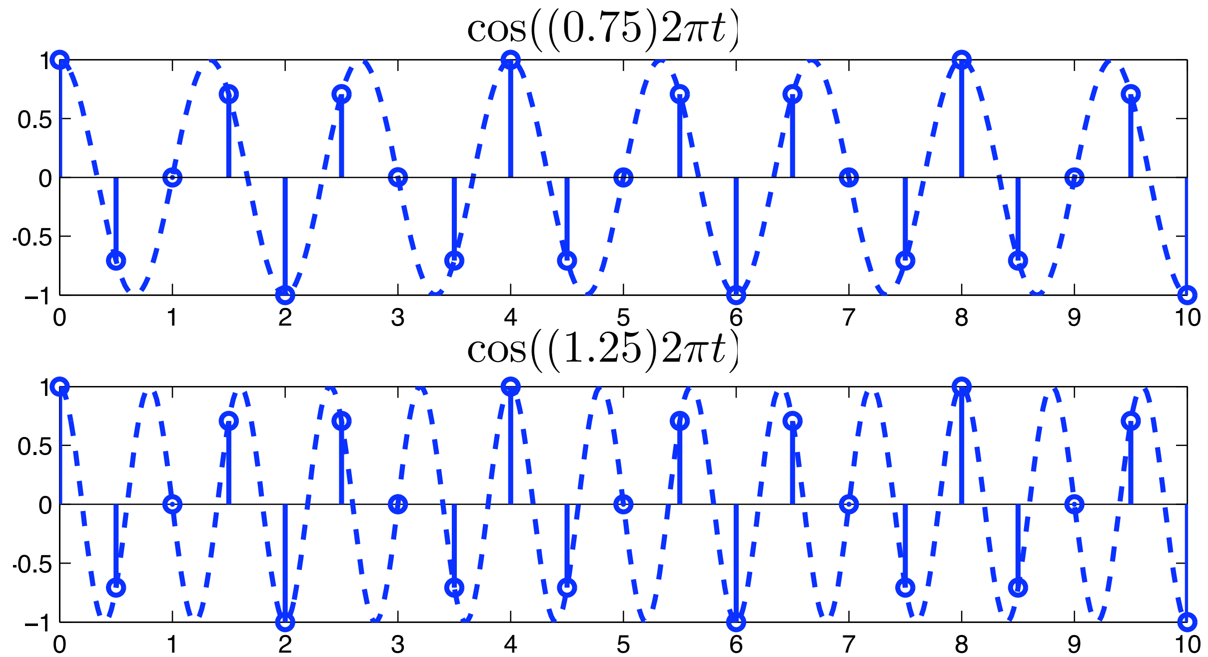
$$\omega_0 = (2\pi)(3000 \text{ Hz})$$
$$\omega_s = (2\pi)(8192 \text{ Hz})$$



The fact that we know the signal is bandlimited is a *very* powerful contraint on the reconstruction.

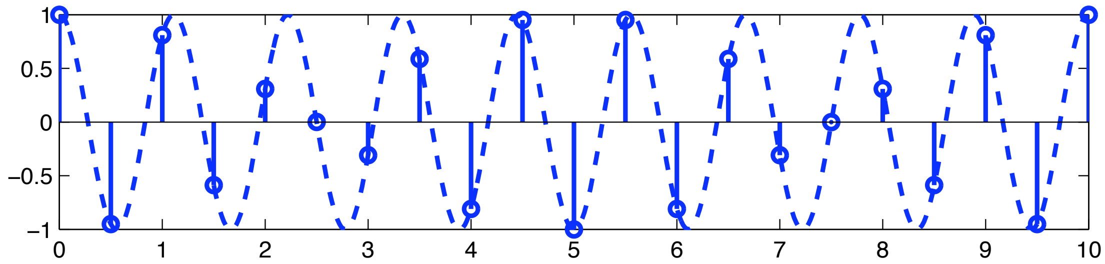
Aliased Frequencies

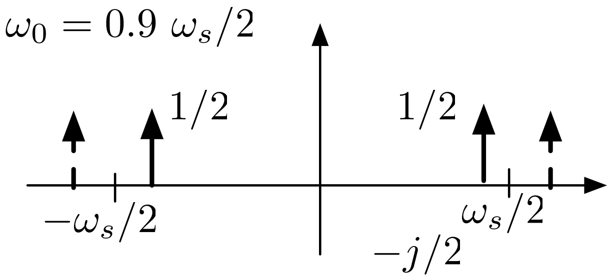
If the sampling rate is insufficient, there will be frequencies that are indistinguishable from each other,

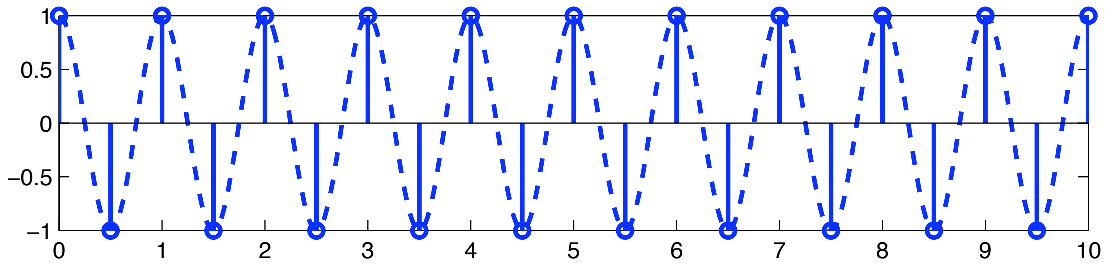


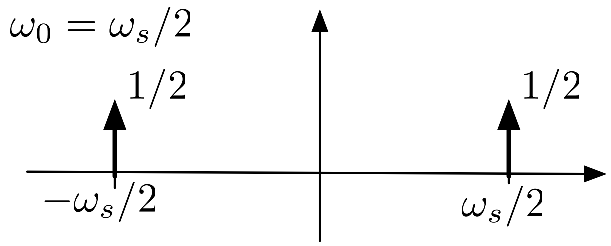
Both $\cos((0.75)2\pi t)$ and $\cos((1.25)2\pi t)$ and have exactly the same samples, when sampled at 2 Hz.

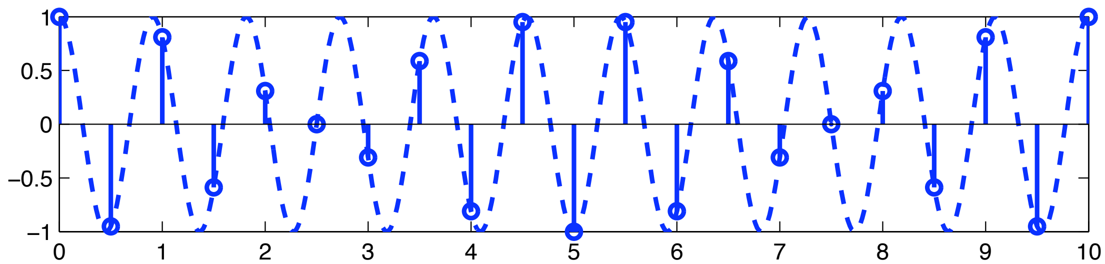
A cosine sampled just below its Nyquist rate looks the same as a cosine just above its Nyquist rate

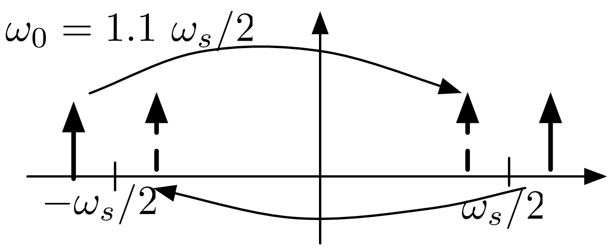




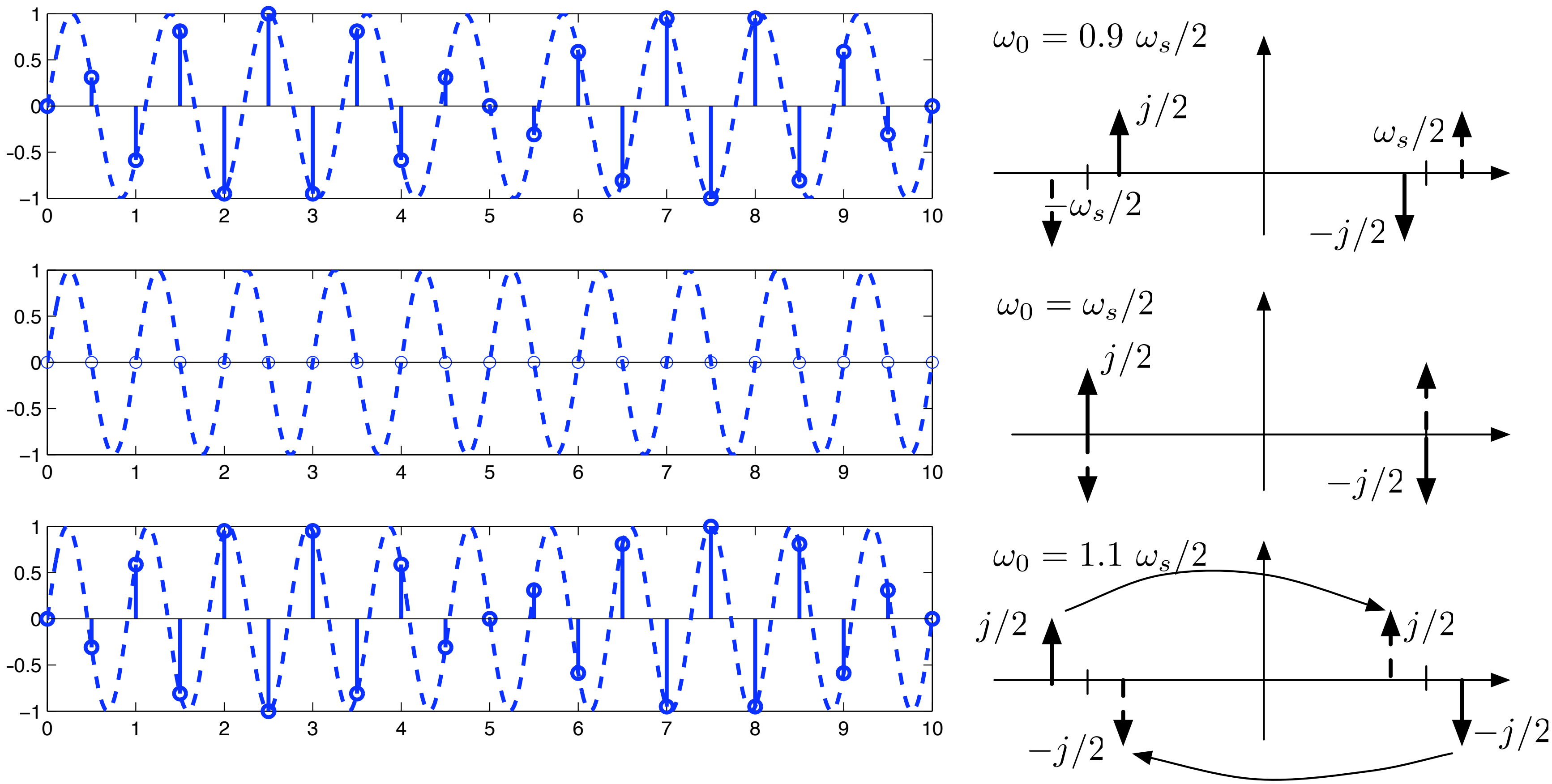








A sine sampled just below its Nyquist rate looks the same as the negative of sine sampled just above its Nyquist rate.

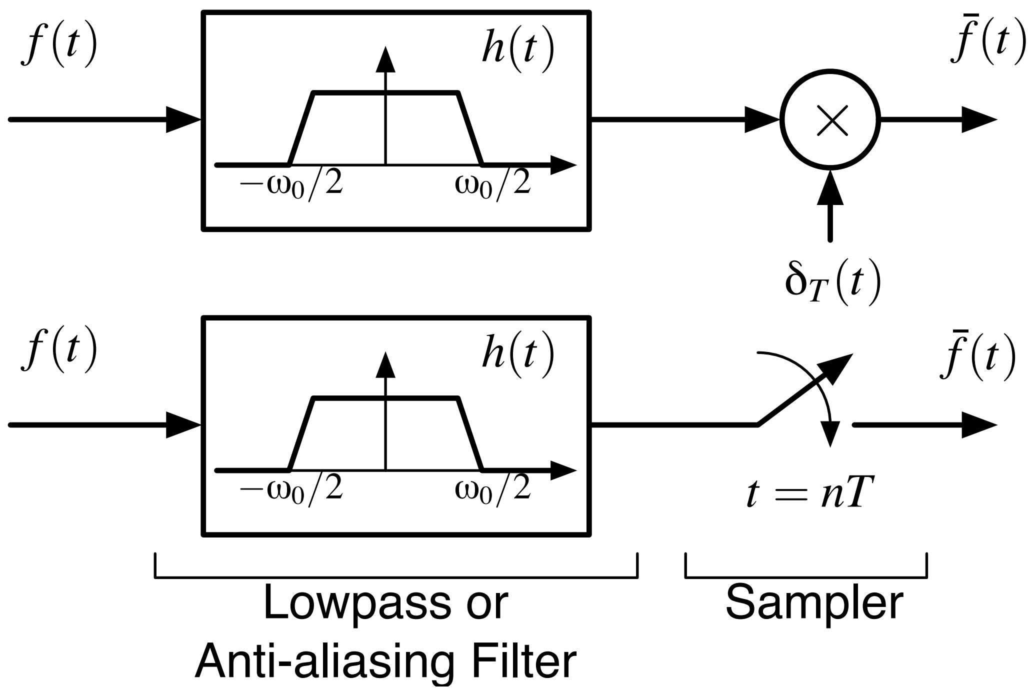


This is the same as negating the frequency. As the frequency of a sine increases past its Nyquist rate, it wraps around to negative frequency.

Minimizing Aliasing

Once a signal has been sampled, there is no way to eliminate aliasing (unless we have some other information about the signal).

Aliasing is minimized by first lowpass filtering the signal, then sampling:



Any practical anti-aliasing filter will not be identically zero outside of its passband. Some aliasing will always occur.

By bandlimiting with the anti-aliasing filter, we are choosing to distort the signal in a known way.

This is usually preferable to sampling the non-bandlimited signal, and having unknown artifacts from aliasing.

Bandlimiting also suppresses noise.

Discrete Time Signal Processing

- Frequently, we are only concerned with sampled signals
- Continuous time signals plus sampling are sufficient to understand these system
- However, we can simplify the analysis
  $\Rightarrow$ Discrete Time Fourier Transform (DTFT)
- Very closely related to the continuous time Fourier transform, but a much different character
- In practice, you really need to understand both. Most interesting systems are hybrid at some level!