Solving Non-Linear Real Arithmetic Formulas with Virtual Substitution

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Outline

- Motivation
- Preliminaries
- Sign Invariant Regions
- Compute Zeros
- Compute Test Candidates
- Virtual Substitution
- Virtual Substitution Rules

Motivation

- Other related methods
  - Interval Constraint Propagation
  - Cylindrical Algebraic Decomposition
- Virtual substitution
  - Complete for a sub-language
  - Eliminates quantified variables up to degree 4

Preliminaries

- Real arithmetic (RA) formula has the following syntax:
  terms: $t := \; 0 \mid 1 \mid x \mid t + t \mid t - t \mid t \cdot t$
  constraints: $c := \; t < t$
  formulas: $\varphi := \; c \mid \neg\varphi \mid \varphi \wedge \varphi \mid \exists x \cdot \varphi$

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- A polynomial $P(x) \in Z[x_1 \ldots, x_n][x]$ has following form:

$$p(x) = a_d x^d + a_{d-1} x^{d-1} + \ldots + a_0 x^0$$

Example:

$$\varphi = (\underbrace{(x^2 + 2x + 4z)}_{p_1} \leq 0 \vee \underbrace{(yx^2 + 6y^3 x + 4z)}_{p_2} = 0)$$

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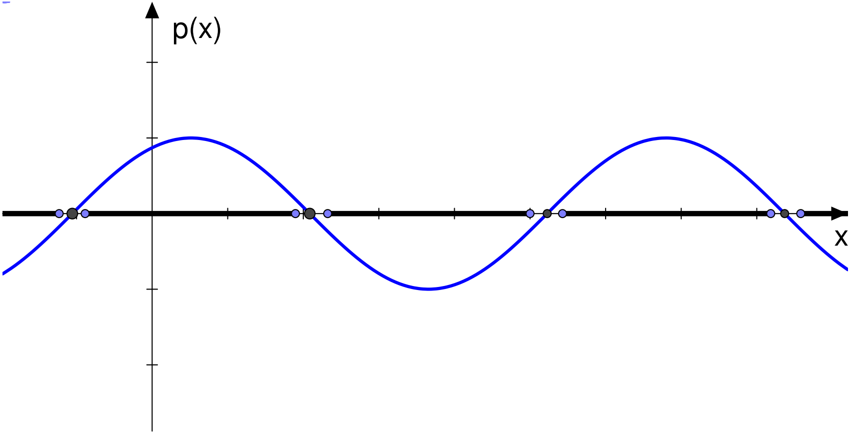
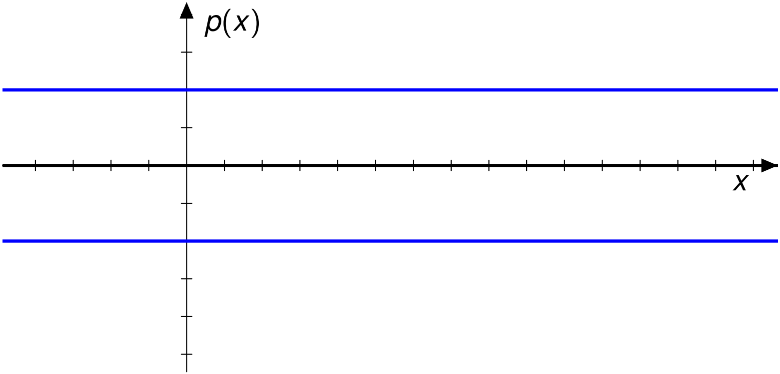
Sign Invariant Regions



Compute Zeros

$p(x) = ax^2 + bx + c$



$x_0 = -\infty$
side condition: $a = 0 \wedge b = 0$

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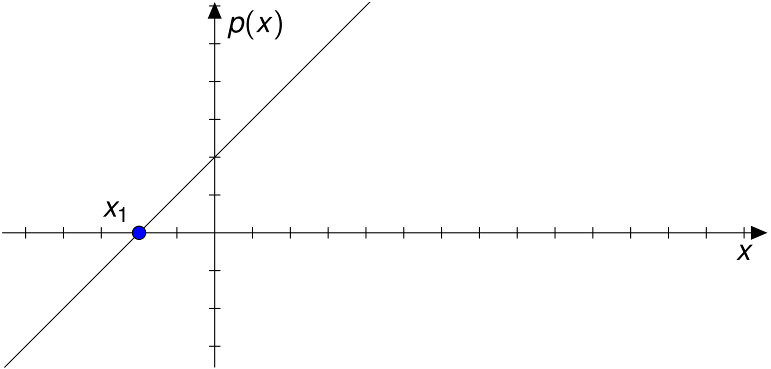
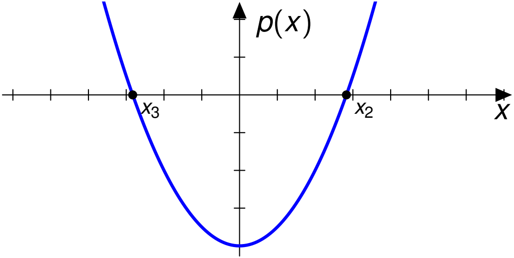


$x_1 = -b/c$
side condition: $a = 0 \wedge b \neq 0$

Compute Zeros

$$p(x) = ax^2 + bx + c$$

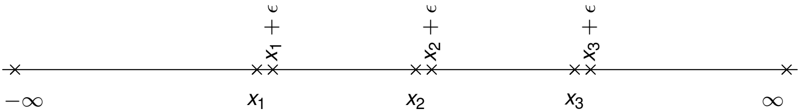


$x_2 = \frac{-b+\sqrt{b^2-4ac}}{2a}$, $x_3 = \frac{-b-\sqrt{b^2-4ac}}{2a}$
side condition: $a \neq 0 \wedge b^2 - 4ac \geq 0$

Compute Test Candidates

Possible solution intervals for x om $p \sim 0$:

Constraints	$-\infty$	x_1	$x_1 + \epsilon$	x_2	$x_2 + \epsilon$	x_3	$x_3 + \epsilon$
$p = 0$	-	$\checkmark$	-	$\checkmark$	-	$\checkmark$	-
$p > 0$, $p < 0$, $p \neq 0$	$\checkmark$	-	$\checkmark$	-	$\checkmark$	-	$\checkmark$
$p \geqslant 0, p \leqslant 0$	$\checkmark$	$\checkmark$	-	$\checkmark$	-	$\checkmark$	-



Virtual Substitution

- Virtual Substitution is an existential quantifier elimination procedure:

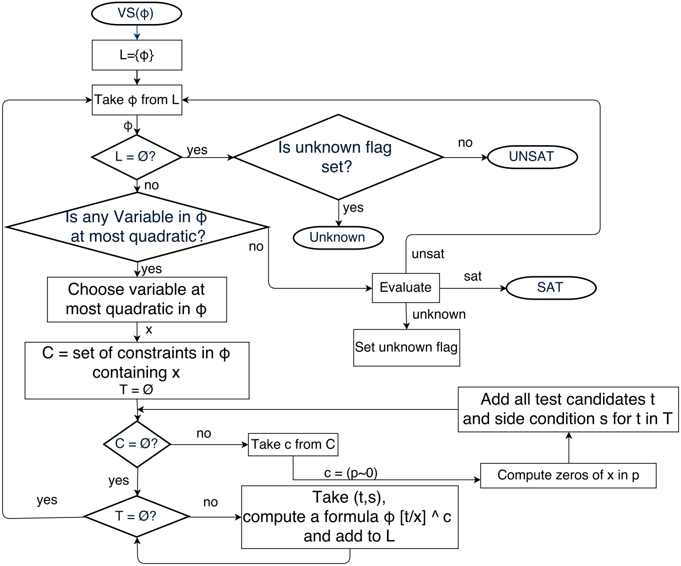
$$\exists x_1 \dots \exists x_n \cdot \varphi' \rightarrow \exists x_1 \dots \exists x_{n-1} \cdot \psi'$$

  where φ', ψ' quantifier free and $\exists x_1 \dots \exists x_n \cdot \varphi' \equiv \exists x_1 \dots \exists x_{n-1} \cdot \psi'$

- Quantifier elimination by virtual substitution is based on the following equivalence:

$$\exists x_1 \dots \exists x_n \cdot \varphi' \equiv \exists x_1 \dots \exists x_{n-1} \cdot \bigvee_{t \in T} \varphi'[t \backslash\backslash x] \wedge S_t$$

Flow Chart



VS(ϕ)
L={ϕ}
Take ϕ from L
ϕ
L = Ø?
yes
no
Is unknown flag set?
no
UNSAT
yes
Unknown
Is any Variable in ϕ at most quadratic?
no
yes
Choose variable at most quadratic in ϕ
x
C = set of constraints in ϕ containing x
T = Ø
unsat
Evaluate
sat
SAT
unknown
Set unknown flag
Add all test candidates t and side condition s for t in T
C = Ø?
no
Take c from C
c = (p~0)
Compute zeros of x in p
yes
yes
T = Ø?
no
Take (t,s), compute a formula ϕ [t/x] ^ c and add to L

Example

$$\varphi := ((\underbrace{xy - 1}_{p_1} = 0) \wedge \underbrace{y^2 - 1}_{p_2} < 0)$$

Elimination of y:

	constraints	test candidates
1.	from all constraints	$-\infty$
2.	$p_1 = 0$	$1/x$ if $x \neq 0$
3.	$p_2 < 0$	$1 + \epsilon$
4.	$p_2 < 0$	$-1 + \epsilon$

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$$\begin{array}{lllll} \exists x \cdot \exists y \cdot \varphi & \leftrightarrow & \exists x \cdot & (\varphi[-\infty/y]) & \vee \\ & & & (\varphi[\frac{1}{x}/y] \quad \wedge x \neq 0) & \vee \\ & & & (\varphi[1 + \epsilon/y]) & \vee \\ & & & (\varphi[-1 + \epsilon/y]) & \end{array}$$

Virtual Substitution Rules

1. Substitution of Square Root Expressions
2. Substitution of Infinitesimal Expressions
3. Substitution of a Minus Infinity

Substitution of Square Root Expression

- A square root expression has following form:

$$\color{red}{k = \frac{u + q\sqrt{r}}{s}} \quad \text{with } u, q, r, s \text{ polynomials.}$$

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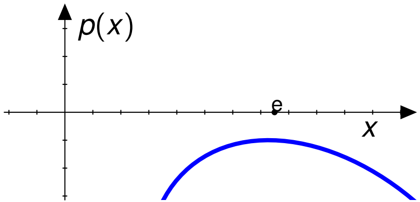

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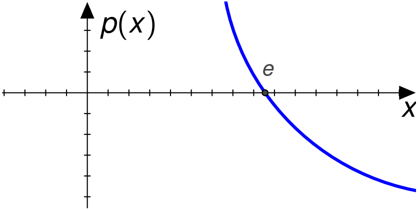

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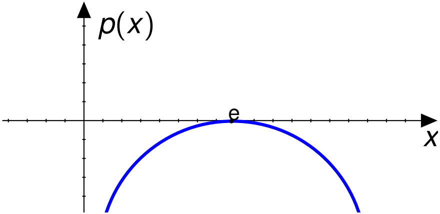

Substitution of a Minus Infinity

- Assume $p(x) = ax^2 + bx + c < 0$ and test candidate is $-\infty$

$$p(x) < 0[-\infty/x] \quad = \quad \underbrace{a < 0}_{Case1} \quad \wedge$$
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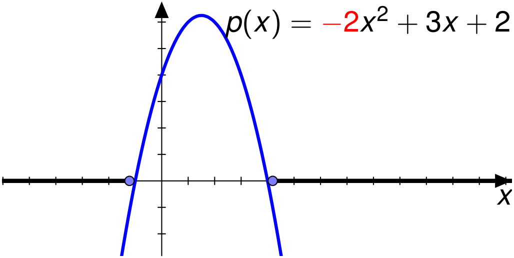
$$\underbrace{a = 0 \wedge b = 0 \wedge c < 0}_{Case3}$$



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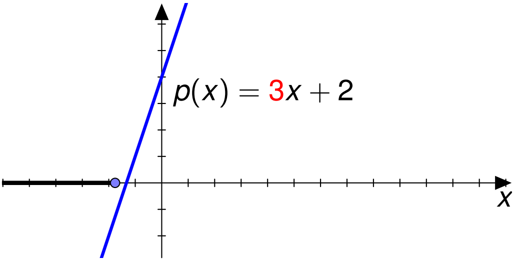

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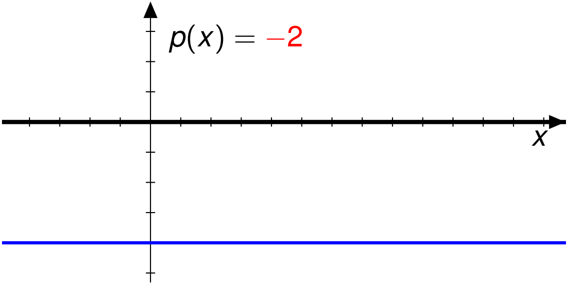

Example

$$\exists x \cdot \exists y \cdot ((xy - 1 = 0) \wedge y^2 - 1 < 0)$$

Elimination of y:

1.Test candidate: $-\infty$

$$\begin{aligned} \exists x \cdot \quad (\quad & (xy - 1 = 0)[-\infty/y] \\ \wedge \quad & (y^2 - 1 < 0)[-\infty/y] \;) \end{aligned}$$

Example

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Elimination of y:

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$$\begin{array}{lll} \exists x \cdot & (& (xy - 1 = 0)[-\infty/y] \\ & \wedge & (y^2 - 1 < 0)[-\infty/y] \;) \\ \Leftrightarrow \exists x \cdot & (& (x = 0 \wedge -1 = 0) \end{array}$$

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Example

$$\exists x \cdot \exists y \cdot ((xy - 1 = 0) \wedge y^2 - 1 < 0)$$

Elimination of y:

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Example

$\exists x \cdot \exists y \cdot ((xy - 1 = 0) \land y^2 - 1 < 0)$
Elimination of y:
2.Test candidate: $\frac{1}{x}$ *if* $x \neq 0$

$$\begin{array}{lll}
\exists x \cdot & (& (xy - 1 = 0)[\frac{1}{x}/y] \\
 & \land & (y^2 - 1 < 0)[\frac{1}{x}/y] \\
 & \land & x \neq 0 \ ) \\
\Leftrightarrow \exists x \cdot & (& (0 = 0) \\
 & \land & ((1 > 0) \land 1 - x^2 < 0 \lor (1 < 0 \land 1 - x^2 < 0)) \\
 & \land & x \neq 0 \ ) \\
\Leftrightarrow \exists x \cdot & (& (1 - x^2 < 0) \\
 & \land & x \neq 0 \ )
\end{array}$$

Example

$\exists x \cdot (1 - x^2 < 0 \ \wedge x \neq 0 \ )$

Elimination of x:

1. Test candidate: $-\infty$

$$
\begin{aligned}
& (1 - x_2 < 0)[-\infty/x] \\
= \quad & (-1 < 0 \vee (-1 = 0 \wedge 0 > 0) \vee (-1 = 0 \wedge 0 = 0 \wedge 1 < 0)) \\
= \quad & \textbf{true}
\end{aligned}
$$

Example

$\exists x \cdot \left(\textbf{true} \wedge x \neq 0 \right)$

Elimination of x:

1. Test candidate: $-\infty$

$$
\begin{aligned}
& (x \neq 0)[-\infty/x] \\
= \quad & (1 \neq 0 \vee 0 \neq 0) \\
= \quad & \textbf{true}
\end{aligned}
$$

Example

$\exists x \cdot ($ **true** $\wedge$ **true** $)$

Elimination of x:

1. Test candidate: $-\infty$

$$
\begin{aligned}
& (x \neq 0)[-\infty / x] \\
= \quad & (1 \neq 0 \vee 0 \neq 0) \\
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Example: Flow Chart

$$\varphi := ((\underbrace{xy-1}_{p_1} = 0) \wedge \underbrace{y^2-1}_{p_2} < 0)$$