Topic

Event

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Abstract

Write a short abstract. Do not give to much details here, but arouse the readers interest. A nice opportunity to comment the text is given by the following: Especially if you write in german, you sometimes need to specify the hyphanation, as e.g. for the word thisisaverylongwordwhosehyphanationmustbedefined.

1 Introduction

Begin your paper with an introduction into your topic.

1.1 hallo

tschuess.

2 Preliminaries

2.1 Virtual Substitution

In 1993, the concept of Virtual Substitution (VS) was first introduced. Initially it was a procedure to eliminate quantifier/variable elimination for linear real arithmetic formulas. Further, VS became a procedure of quantifier elimination for non-linear arithmetic formulas. But one of the most significant limitation of VS is that it cannot eliminate quantified variables whose degree is higher than 2.

VS is a procedure to eliminate a quantified variable. Let $\varphi^{\mathbb{R}}$ is a quantifier-free real-arithmetic formula where $x \in p(x)$ and $p(x) \sim 0, \sim \in \{=, <, >, \leq, \geq, \neq\}$ is a constraint of $\varphi^{\mathbb{R}}$. Degree of x in $p(x)$ must be ≤ 2. Then, after quantifier elimination by VS we get the following equivalence,

$$\exists x.\varphi^{\mathbb{R}} \iff \bigvee_{t \in T(x,\varphi^{\mathbb{R}})} (\varphi^{\mathbb{R}}[t \backslash\backslash x] \wedge C_t)$$

where T is a finite set of all possible test candidates for x and C_t is a side condition of $t \in T$.

2.2 Test Candidates and Side Condition

To solve non-linear equalities with VS first we have to choose a variable, $x \in p(x)$ to eliminate and then compute all possible test candidates(TCs). $\varphi^{\mathbb{R}}$ is satisfied if there is a test candidate (TC) $t \in T$ such that $\varphi^{\mathbb{R}}[t\backslash\backslash x] = p_1[t\backslash\backslash x] \wedge \cdots p_n[t\backslash\backslash x] \wedge C_t$ is satisfiable.

So, the indices of the substitutions are the side conditions of the TC it considers and the labels on the edges to a substitutions are the constraints which provide TC. A detailed explanation of how to construct TCs with side condition is provided in the section 3.1.

2.3 Square Root Expression

A square root expression(SRE) has the form,

$$\frac{p + q\sqrt{r}}{s} \qquad \text{, where } p, q, r, s \in P$$

and the set of all square root can be expressed by,

$$SqrtEx := \{\frac{p + q\sqrt{r}}{s} |\ p, q, r, s \in P\}$$

Definition 2.1 (Polynomial) *A polynomial is a mathematical expression consisting of a sum of terms, each term including a variable or variables raised to a power and multiplied by a coefficient. If a polynomial has only one variable, it is called univariate. An univariate of degree d has the following form where $a_d \neq 0$,*

$$p(x) = a_d x^d + a_{d-1} x^{d-1} + \ldots + a_0 x^0$$

If a polynomial has two or more variables, it is called multivariate. A multivariate (two variables) of degree d has the following form where $a_{dd} \neq 0$,

$$p(x, y) = a_{dd} x^d y^d + a_{d(d-1)} x^d y^{d-1} + a_{(d-1)d} x^{d-1} y^d + \ldots + a_{10} x^1 y^0 + a_{10} x^0 y^1 + a_{00} x^0 y^0$$

The following expression is a quantifier-free real-arithmetic formula where a, b, c are the polynomials and the set of all polynomials in $\varphi^{\mathbb{R}}$ is $P = \{a, b, c\}$,

$$\varphi^{\mathbb{R}} = (a \leq 0 \vee b = 0) \wedge (b < 0 \vee c \neq 0)$$

Let, $p(x) = ax^2 + bx + c = 0$ is a quadratic equation of variable x where $a, b, c \in P$ and $x \notin a \cup b \cup c$. Now, the solution formula for x in $p(x) = a_d x^d + a_{d-1} x^{d-1} + \ldots + a_0 x^0$ considers the following four cases,

$$x_0 = -\frac{c}{b} \qquad \text{, if } a = 0 \wedge b \neq 0 \tag{2.1}$$

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{, if } a \neq 0 \wedge b^2 - 4ac \geq 0 \tag{2.2}$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad \text{, if } a \neq 0 \wedge b^2 - 4ac \geq 0 \tag{2.3}$$

$$x_3 = -\infty \qquad \text{, if } a = 0 \wedge b = 0 \tag{2.4}$$

Note that, x_0 is a real zero of $p(x)$ for linear equation, for quadratic equation x_1 and x_2 are two real zeros of $p(x)$. x_4 is any real number which is also a solution for x.

Now, we can express the symbolic zero of x in a polynomial, which is quadratic in x by a SRE $\frac{p+q\sqrt{r}}{s}$ as given in table 2.1.
Remark We can construct TCs by the comparison with SRE (table 2.1) and also considering that TCs can be supplemented by an infinitesimal ε.

Table 2.1 Comparison with SRE $\frac{p+q\sqrt{r}}{s}$

Equation No.	p	q	r	s
2.1	$-c$	0	1	b
2.2	$-b$	1	b^2-4ac	$2a$
2.3	$-b$	-1	b^2-4ac	$2a$
2.4	0	1	0	0

3 Solving Non-linear Equalities with Virtual Substitution

VS is a restricted but very efficient procedure to solve non-linear equalities. In the paper [1] author explored an extension of the ideas in [2] from the linear to the quadratic case. For linear case the idea was to eliminate a quantifier from $\exists x\varphi$ by replacing x in φ with t that may involve improper expressions such as $\pm\infty$ or ϵ. However, $\varphi[t\backslash\backslash x]$ is defined in such a way that these improper expressions do not occur in the resulting formula.

Author extended these idea to various quadratic cases. The cases are the substitution of SREs and the substitution of infinitesimal expressions in formulas.

In virtual substitution, first a variable is replaced by test candidate and to perform the replacement we need to construct test candidates. An univariate real-arithmetic formula is satisfiable if and only if there is one test candidate for which satisfies formula and the side conditions of t holds. For multivariate real-arithmetic formula the virtual substitution method continues with the elimination of the next variable.

In this section we will see how we can apply virtual substitution. Let us consider a multivariate real-arithmetic formula which we will use in this section,

$$\varphi^{\mathbb{R}} = \underbrace{(x^2y + x + y = 0)}_{p_1} \wedge \underbrace{(y^2 - 2 < 0)}_{p_2}$$

3.1 Constructing test candidates with side condition

First we will eliminate x from $\varphi^{\mathbb{R}}$. To construct the test candidates for x, we have to compute SqrtEx for x. Also we need to consider an infinitesimal ϵ.

Definition 3.1 (Construction of Test Candidates) *The set of all test candidates is defined by,*

$$TCS := SqrtEx \cup \{t + \epsilon \mid t \in SqrtEx\}$$

The set of test candidates for x in $p(x) = ax^2 + bx + c \sim 0$ is defined by,

$$(x, p(x) \sim 0) \quad \mapsto \quad \begin{cases} \{-\infty, \dfrac{-c}{b}, \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}\} & , \textit{if } \sim \textit{ is weak} \\ \\ \{-\infty, \dfrac{-c}{b} + \epsilon, \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} + \epsilon\} & , \textit{otherwise} \end{cases}$$

where $a, b, c \in P$, $x \notin a \cup b \cup c$ and weak means $\{=, \leq, \geq\}$

The side condition of a test candidate is defined by,

$$C_t : t \quad \mapsto \quad \begin{cases} C_{t'} & , \textit{if } t = t' + \epsilon \\ \\ \{s \neq 0 \wedge r \geq 0\} & , \textit{if } t = \frac{p + q\sqrt{r}}{s} \\ \\ \textit{true} & , \textit{Otherwise} \end{cases}$$

where p, q, r and s are polynomials and t' is a test candidate where $\epsilon \notin t'$.

Each side condition of the test candidates confirms that each test candidate exists. The side condition of the test candidate $-\infty$ is valid because, it does not relate to a zero.

To eliminate x first we get the following test candidates,

$$x_0 = -y \quad , \text{if } y = 0 \wedge 1 \neq 0 \tag{3.1}$$

$$x_1 = \frac{-1 + \sqrt{1^2 - 4y^2}}{2y} \quad , \text{if } y \neq 0 \wedge 1^2 - 4y^2 \geq 0 \tag{3.2}$$

$$x_2 = \frac{-1 - \sqrt{1^2 - 4y^2}}{2y} \quad , \text{if } y \neq 0 \wedge 1^2 - 4y^2 \geq 0 \tag{3.3}$$

$$x_3 = -\infty \quad , \text{if } y = 0 \wedge 1 = 0 \tag{3.4}$$

Here, C_{x_3} is invalid. So, x_3 does not exist for p_1.

Now, we will put all the test candidates of x in $\varphi^{\mathbb{R}}$ and we will get following three quantifier free and x free real-arithmetic formulas,

$$\varphi_1 = \varphi^{\mathbb{R}}(x_0 \backslash\backslash x) = (y^3 + 2y = 0) \wedge (y^2 - 2 < 0) \wedge (y = 0) \wedge (1 \neq 0) \tag{3.5}$$

$$\varphi_2 = \varphi^{\mathbb{R}}(x_1 \backslash\backslash x) = (p_3 = 0) \wedge (y^2 - 2 < 0) \wedge (y \neq 0) \wedge (1 - 4y^2 \geq 0) \tag{3.6}$$

$$\varphi_3 = \varphi^{\mathbb{R}}(x_2 \backslash\backslash x) = (p_4 = 0) \wedge (y^2 - 2 < 0) \wedge (y \neq 0) \wedge (1 - 4y^2 \geq 0) \tag{3.7}$$

where,

$$p_3 = (\underbrace{(\frac{-1 + \sqrt{1 - 4y^2}}{2y})^2 y + (\frac{-1 + \sqrt{1 - 4y^2}}{2y}) + y = 0}_{\text{it will be always 0 as it is a real zero}})$$

and

$$p_4 = (\underbrace{(\frac{-1 - \sqrt{1 - 4y^2}}{2y})^2 y + (\frac{-1 - \sqrt{1 - 4y^2}}{2y}) + y = 0}_{\text{it will be always 0 as it is a real zero}})$$

To eliminate y from φ_1,φ_2,φ_3 we construct the test candidates for y given in Table 3.1 where we marked the valid and invalid side conditions.

Table 3.1 Test Candidates for y with side conditions

	Test Candidate	Side Condition	Validation of Side Condition
φ_1,φ_2,φ_3	$\sqrt{2}+\epsilon$	$1 \neq 0 \wedge 8 \geq 0$	✓
	$-\sqrt{2}+\epsilon$	$1 \neq 0 \wedge 8 \geq 0$	✓
	$-\infty$	true	✓
φ_2,φ_3	$2+\epsilon$	$1 = 0 \wedge 0 \neq 0$	**x**
	$0+\epsilon$	$0 \neq 0 \wedge -2 \neq 0$	**x**
	$\frac{1}{2}$	$-4 \neq 0 \wedge 16 \geq 0$	✓
	$-\frac{1}{2}$	$-4 \neq 0 \wedge 16 \geq 0$	✓
	$\frac{-1}{0}$}invalid value	-	-

4 Conclusion

Give a conclusion on your topic. Give a few sentences to summarize the topic. If possible, point out the quality of the result and give ∃ a small prospect of subsequent works.