Solving Non-Linear Real Arithmetic Formulas with Virtual Substitution

Satisfiability Seminar

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Abstract

We present the virtual substitution method as published in [1] which is an incomplete quantifier elimination procedure for non-linear real arithmetic formulas.The solution approach relies on some substitution rules of virtual substitution. The rules can be applied only on real-arithmetic formulas which has a degree not higher than 2. If the degree is more than 2, the quantifier elimination procedure becomes complex.

1 Introduction

Quantifier/ variable elimination is a fundamental problem in elementary real arithmetic Virtual substitution method provides an incomplete solution to the problem. In 1993, the concept of virtual substitution was first introduced [1]. Virtual substitution was already used earlier to quantifier elimination for linear real arithmetic formulas. Now, virtual substitution has extended to quantifier elimination for non-linear real arithmetic formulas. But, this method is restricted to formulas that are linear or quadratic in the quantified variable. It means virtual substitution method is applicable if quantified variables has a degree of at most two. In addition, applying the method iteratively to eliminate quantified variables may increase the degrees of remaining variables, thus violating the degree restrictions.
Section 2 consists preliminaries with some definitions. How to construct the real zeros and virtual substitution rules by which we can eliminate quantified variables from non-linear real arithmetic formulas are explained in Section 3. In the Section 4 we conclude the paper.

2 Preliminaries

Real arithmetic (RA) is a first-order theory $(\mathbb{R}, +, ., 0, 1, <)$ over the reals with addition and multiplication. A RA could be in the form of terms, constraints or formulas which can be built upon constants $0, 1$ and real-valued variables x according to the following syntax:
Terms: $t \quad := \quad 0 \quad | \quad 1 \quad | \quad x \quad | \quad t+t \quad | \quad t \cdot t$
Constrains: $c \quad := \quad t < t$
Formulas: $\varphi \quad := \quad c \quad | \quad \neg\varphi \quad | \quad \varphi \wedge \varphi \quad | \quad \exists x \cdot \varphi$
A term is a product of an integer coefficient and a monomial. A monomial is the product

of variables and the empty product represents the constant 1. Constraint is a condition of a RA problem that the solution must satisfy. Syntactic sugar (more clear syntax) are: $\leq, =, \neq, \forall, \vee, \rightarrow, \ldots$. A variable x is said to be a bound variable if x occurs in a formula $\exists x \varphi$, otherwise x is called free variable means not bounded in φ. A formula with no free variables are called a sentence.
There are two types of RA: Linear real arithmetic (LRA) and Non-linear real arithmetic (NRA). LRA is a first-order theory over the reals with addition only, whereas the NRA first-order theory over the reals with addition and multiplication.
A polynomial $P \in R[x]$ is a sum of terms, each term being a variable raised to a power and multiplied by a coefficient from some coefficient ring R. If a polynomial has only one variable (i.e. its coefficient are variable-free), it is called univariate, else it is called multivariate:

$$p(x) = a_d x^d + a_{d-1} x^{d-1} + \ldots + a_0 x^0$$

Where $a_0, a_1, \ldots a_d \in R$.
The degree of $P(x)$ is the maximal $0 \leq k \leq d$ such that $a_k \neq 0$.
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Example 1 *The following expression is a non-linear quantifier-free real arithmetic formula where the set of all polynomials in φ is $P = \{p_1, p_2, p_3\}$:*

$$\varphi = (\underbrace{(x^2 + 2x + 4z \leq 0)}_{p_1} \vee \underbrace{(yx^2 + 6y^3x + 4z = 0)}_{p_2}) \wedge \underbrace{(6y^3x + 4z < 0)}_{p_3}$$

Where, x^2, x, yx^2, y^3x and z are the monomials, $2x, 6y^3x$ and $4z$ are the terms. Degree of $p_1(x), p_2(x)$ and $p_3(x)$ are $2, 2$ and 1, respectively

We are interested in solving non-linear real arithmetic formula where variables $x_1, x_1, \ldots, x_n$ are bounded in $\exists x \ldots \exists x_n \varphi$ with φ quantifier-free. To find solutions we will apply virtual substitution (introduced in Section 3.2) by eliminating all bound variables x in φ recursively.

3 Solving Non-linear Real Arithmetic formulas with Virtual Substitution

3.1 Test Candidates and Side Conditions

Let $p(x) = ax^2 + bx + c$ is a quadratic of variable x where $p \in P$ and a, b, c do not contain x. We assume $p(x) \sim 0, \sim \in \{=, <, >, \leq, \geq, \neq\}$. Now, the solution equations for x in $p(x)$ considers the following four cases:

$$x_0 = -\frac{c}{b} \qquad , \text{if } a = 0 \wedge b \neq 0 \tag{3.1}$$

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad , \text{if } a \neq 0 \wedge b^2 - 4ac \geq 0 \tag{3.2}$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad , \text{if } a \neq 0 \wedge b^2 - 4ac \geq 0 \tag{3.3}$$

$$x_3 = -\infty \qquad , \text{if } a = 0 \wedge b = 0 \tag{3.4}$$

Here x_0, x_1, x_2 and x_3 are the real zeros of p in x and the conditions on the right are called side conditions.

The constraint $p(x) \sim 0$ has solution intervals $(-\infty, x_0)$ or (x_i, x_{i+1})or $[x_i, x_i]$or $[x_2, x_2]$ or (x_2, ∞) where $0 \leq i \leq 1$. So, the endpoints of these intervals are $-\infty, x_i$ and x_2 which are the real zeros. As the real zeros are symbolic and we cannot order them to pick the solutions. Hence, we choose a left endpoint of each left closed interval, a left endpoint of each left opened interval plus an infinitesimal ϵ as sample solutions shown in the Figure 1. These sample solutions are called test candidates.

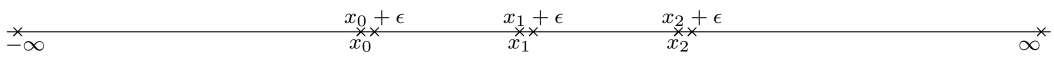

Figure 1: Test Candidates

To solve a non-linear real arithmetic formula φ, first we have to choose a variable $x, x \in p(x)$ to eliminate and then compute all possible test candidates of x. φ is satisfiable if there is a test candidate $t \in T$ such that $(\varphi[t\backslash\backslash x] = p_1[t\backslash\backslash x] \wedge \cdots p_n[t\backslash\backslash x]) \wedge$(side Condition) is satisfiable, but direct substitution is not possible as test candidates are not algebraic $(\sqrt{}, \epsilon, -\infty, \%, \ldots)$. Thus, virtual substitution rules need to be introduced.

3.2 Virtual Substitution

Virtual substitution is a procedure to eliminate a quantified variable. Let φ be a quantifier-free real arithmetic formula where $x \in p(x)$ and $p(x) \sim 0, \sim \in \{=, <, >, \leq, \geq, \neq\}$ is a constraint of $\varphi^{\mathbb{R}}$. Degree of x in $p(x)$ must be at most two. Then, quantifier elimination by virtual substitution is based on the following equivalence:

$$\exists x \varphi \iff \bigvee_{t \in T(x, \varphi)} (\varphi[t\backslash\backslash x] \wedge C_t)$$

Where T is a finite set of all possible test candidates for x and C_t is a side condition of test candidate $t \in T$.

Virtual Substitution is a restricted but very efficient procedure to solve non-linear equalities. In the paper [1] author explored an extension of the ideas in [2] from the linear to the quadratic case. For linear case the idea was to eliminate a quantifier from $\exists x \varphi$ by replacing x in φ with t that may involve improper expressions such as $\pm\infty$ or ϵ. However, $\varphi[t\backslash\backslash x]$ is defined in such a way that these improper expressions do not occur in the resulting formula.

Author extended these idea to various quadratic cases. The cases are the substitution of square root expressions and the substitution of infinitesimal expressions in formulas.

In virtual substitution, first a variable is replaced by test candidate and to perform the replacement we need to construct test candidates. An univariate real-arithmetic formula is satisfiable if and only if there is one test candidate for which satisfies formula and the side conditions of t holds. For multivariate real-arithmetic formula the virtual substitution method continues with the elimination of the next variable.

3.3 Constructing test candidates with side condition

Let a quadratic equation $p(x) = p_1x^2 + p_2x + p_3 = 0$ in the variable x where $p_1, p_2, p_3 \in P$ and x does not contain in p_1, p_2 and p_3. Now, we get the solution formula for x in $p(x)$ as follows:

$$x_0 = -\frac{p_3}{p_2} \qquad , \text{if } p_1 = 0 \wedge p_2 \neq 0 \tag{3.5}$$

$$x_1 = \frac{-p_2 + \sqrt{p_2^2 - 4p_1p_3}}{2p_1} \qquad , \text{if } p_1 \neq 0 \wedge p_2^2 - 4p_1p_3 \geq 0 \tag{3.6}$$

$$x_2 = \frac{-p_2 - \sqrt{p_2^2 - 4p_1p_3}}{2a} \qquad , \text{if } p_1 \neq 0 \wedge p_2^2 - 4p_1p_3 \geq 0 \tag{3.7}$$

$$x_3 = -\infty \qquad , \text{if } p_1 = 0 \wedge p_2 = 0 \wedge p_3 = 0 \tag{3.8}$$

Here x_0, x_1 and x_2 are the symbolic zeros of x in $p(x)$.

Definition 3.1 (Square Root Expression) *A square root expression has the following form:*

$$\frac{p + q\sqrt{r}}{s} \qquad , \textit{ where } p, q, r, s \in P$$

and the set of all square root expression is denoted as follows:

$$SqrtEx = \{\frac{p + q\sqrt{r}}{s} |\ p, q, r, s \in P\}$$

Table 3.1 Comparison with square root expression $\frac{p+q\sqrt{r}}{s}$

Equation No.	p	q	r	s
3.5	$-p_3$	0	1	p_2
3.6	$-p_2$	1	$p_2^2 - 4p_1p_3$	$2p_1$
3.7	$-p_2$	-1	$p_2^2 - 4p_1p_3$	$2p_1$

Now, we can express the symbolic zero of x in $p(x)$, which is quadratic in x by a square root expression $\frac{p+q\sqrt{r}}{s}$ as given in Table 3.1.
So, for x in $p(x)$, $SqrtEx = \{-\frac{p_3}{p_2}, \frac{-p_2 \pm \sqrt{p_2^2 - 4p_1p_3}}{2p_1}\}$. Now, We can construct the set of all test candidates for x in $p(x) \sim 0$ in terms of $SE \in SqrtEx$ which is given as follows:

$$t = \begin{cases} \{-\infty, SE & , \text{if } \sim \text{ is weak} \\ \\ \{-\infty, SE + \epsilon\} & , \text{otherwise} \end{cases}$$

Here weak means $\{=, \leq, \geq\}$.
And the side condition of each test candidate t is defined by:

$$C_t = \begin{cases} C_{t'} & , \text{if } t = t' + \epsilon \\ \\ \{s \neq 0 \wedge r \geq 0\} & , \text{if } t = \frac{p+q\sqrt{r}}{s} \\ \\ true & , \text{Otherwise} \end{cases}$$

Here t' is a test candidate which does not contain ϵ.
Each side condition of the test candidates confirms that each test candidate indeed exists.
The side condition of the test candidate $-\infty$ is valid because, it does not relate to a zero.

Example 2 *Let us consider a non-linear real arithmetic formula as follows:*

$$\varphi = \underbrace{(x^2y + x + y = 0)}_{p_1} \wedge \underbrace{(y^2 - 2 < 0)}_{p_2}$$

First we will eliminate x and then y from φ. To eliminate x first we get the following test candidates of x in p_1 with side conditions:

$$x_0 = -y \qquad , \textit{ if } y = 0 \wedge 1 \neq 0 \tag{3.9}$$

$$x_1 = \frac{-1 + \sqrt{1^2 - 4y^2}}{2y} \qquad , \textit{ if } y \neq 0 \wedge 1^2 - 4y^2 \geq 0 \tag{3.10}$$

$$x_2 = \frac{-1 - \sqrt{1^2 - 4y^2}}{2y} \qquad , \textit{ if } y \neq 0 \wedge 1^2 - 4y^2 \geq 0 \tag{3.11}$$

$$x_3 = -\infty \qquad , \textit{ if } y = 0 \wedge 1 = 0 \wedge y = 0 \tag{3.12}$$

Here x_3 does not exist for p_1 as C_{x_3} is invalid. Now, we will put all the test candidates of x in φ and get following three real-arithmetic formulas without x:

$$\varphi_1 = \varphi[x_0 \backslash\backslash x] = (y^3 + 2y = 0) \wedge \underbrace{(y^2 - 2 < 0)}_{p_2} \wedge \underbrace{(y = 0)}_{p_6} \wedge (1 \neq 0) \tag{3.13}$$

$$\varphi_2 = \varphi[x_1 \backslash\backslash x] = p_3 \wedge \underbrace{(y^2 - 2 < 0)}_{p_2} \wedge \underbrace{(y \neq 0)}_{p_7} \wedge \underbrace{(1 - 4y^2 \geq 0)}_{p_8} \tag{3.14}$$

$$\varphi_3 = \varphi[x_2 \backslash\backslash x] = p_4 \wedge \underbrace{(y^2 - 2 < 0)}_{p_2} \wedge \underbrace{(y \neq 0)}_{p_7} \wedge \underbrace{(1 - 4y^2 \geq 0)}_{p_8} \tag{3.15}$$

Where

$$p_3 = \underbrace{((\frac{-1 + \sqrt{1 - 4y^2}}{2y})^2 y + (\frac{-1 + \sqrt{1 - 4y^2}}{2y}) + y = 0)}_{\textit{it will be always 0 as it is a real zero of } x \textit{ in } p_1}$$

and

$$p_4 = \underbrace{((\frac{-1 - \sqrt{1 - 4y^2}}{2y})^2 y + (\frac{-1 - \sqrt{1 - 4y^2}}{2y}) + y = 0)}_{\textit{it will be always 0 as it is a real zero of } x \textit{ in } p_1}$$

We alreday eliminated x and now, we will eliminate y from φ_1,φ_2,φ_3. We have shown all the test candidates of y in p_2, p_6, p_7 and p_8 in the Table 3.2 where we marked the valid and invalid side conditions.

Note that there is a polynomial $y^3 + 2y = 0$ in φ_1 where y has a degree of 3. We did not consider this polynomial to construct test candidate as the degree is higher than 2. If the constructed test candidates for y in other polynomials of φ_1 satisfy $y^3 + 2y = 0$, φ is satisfiable. φ_2 and φ_3 have become same as p_3 and p_4 are already satisfied. Further, we will consider φ_2 and φ_3 as φ_4:

$$\varphi_4 = \underbrace{(y^2 - 2 < 0)}_{p_2} \wedge \underbrace{(y \neq 0)}_{p_7} \wedge \underbrace{(1 - 4y^2 \geq 0)}_{p_8}$$

Table 3.2 Test Candidates for y with side conditions

	Test Candidate	Side Condition	Validation of Side Condition
p_2	$\sqrt{2}+\epsilon$	$1 \neq 0 \wedge 8 \geq 0$	✓
	$-\sqrt{2}+\epsilon$	$1 \neq 0 \wedge 8 \geq 0$	✓
	$2+\epsilon$	$1 = 0 \wedge 0 \neq 0$	**x**
	$-\infty$	true	✓
p_6	0	$0 = 0 \wedge 1 \neq 0$	✓
	$-\infty$	true	✓
p_7	$0+\epsilon$	$0 \neq 0 \wedge -2 \neq 0$	**x**
	$-\infty$	true	✓
p_8	$\frac{1}{2}$	$-4 \neq 0 \wedge 16 \geq 0$	✓
	$-\frac{1}{2}$	$-4 \neq 0 \wedge 16 \geq 0$	✓
	$\frac{-1}{0}$}invalid value	-	✓
	$-\infty$	true	✓

3.4 Virtual Substitution Rules

We have already seen the construction of test candidates of a non-linear real arithmetic formula φ. φ is satisfiable if and only if at least one test candidate for x and y are the solution of φ. If all test candidates for x and y satisfy φ, it is said to be valid.
We eliminated x already and currently we have φ_1 and φ_4 from which y is needed to eliminate. Substituting directly the test candidates of y in φ_1 and φ_4 are not possible. Thus, we will introduce virtual substitution rules.

3.4.1 Substitution of Square Root Expressions

Let, t is a test candidate for x where $t = \frac{p_1+q_1\sqrt{r}}{s_1}$ is a square root expression. Let consider a constraint $p = 0$ and we want to substitute t for x in it. If we substitute t for all occurrences of x in p, we can transform the result into $\frac{p_2+q_2\sqrt{r}}{s_2}$ which is also a square root expression where p_2, q_2 and $s_2 \in P$. It has to be mentioned that, the radicand (r) still remains the same. This transformation is possible as the summation and multiplication result of two square root expression with the same radicand r is a square root expression with radican r.

- Summation,

$$\frac{p_3+q_3\sqrt{r}}{s_3}+\frac{p_4+q_4\sqrt{r}}{s_4} = \frac{s_4(p_3+q_3\sqrt{r})+s_3(p_4+q_4\sqrt{r})}{s_3s_4} = \frac{s_4p_3+s_4q_3\sqrt{r}+s_3p_4+s_3q_4\sqrt{r}}{s_3s_4}$$

$$= \frac{\overbrace{s_4p_3+s_3p_4}^{p_2}+(\overbrace{s_4q_3+s_3q_4}^{q_2})\sqrt{r}}{\underbrace{s_3s_4}_{s_2}}$$

- Multiplication,

$$\frac{p_3+q_3\sqrt{r}}{s_3}*\frac{p_4+q_4\sqrt{r}}{s_4} = \frac{(p_3+q_3\sqrt{r})(p_4+q_4\sqrt{r})}{s_3s_4} = \frac{p_3p_4+p_3q_4\sqrt{r}+p_4q_3\sqrt{r}+q_3\sqrt{r}q_4\sqrt{r}}{s_3s_4}$$

$$= \frac{\overbrace{p_3p_4}^{p_2} + (\overbrace{p_3q_4 + p_4q_3 + q_3q_4}^{q_2})\sqrt{r}}{\underbrace{s_3s_4}_{s_2}}$$

The equation $\frac{p_2+q_2\sqrt{r}}{s_2} = 0$ holds if and only if $p_2 + q_2\sqrt{r} = 0$. It holds if and only if either $(p_2 = 0 \wedge q_2 = 0)$ or p_2 and q_2 have different signs with same absolute value, i.e., $|p_2| = |q_2\sqrt{r}|$. So, after substitution, we get the following real arithmetic formula without x:

$$(p = 0)[t\backslash\backslash x] = (p_2q_2 \leq 0) \wedge (p_2^2 - q_2^2 r = 0)$$

3.4.2 Substitution of Infinitesimal Expressions

Let $t + \epsilon$ is a test candidate for x where $t \neq -\infty$ and $\epsilon \notin t$. Let us consider a constraint $p < 0$ and we want to substitute t for x in it. Note that x should be occurred at most quadratic in p. After substitution we will get the following real arithmetic formula which does not have x:

$$(p < 0)[t + \epsilon\backslash\backslash x] = \underbrace{((p < 0)[t\backslash\backslash x])}_{\text{Case 1}} \vee \underbrace{((p = 0)[t\backslash\backslash x] \wedge (p' < 0)[t\backslash\backslash x])}_{\text{Case 2}} \vee$$
$$\underbrace{((p = 0)[t\backslash\backslash x] \wedge (p' = 0)[t\backslash\backslash x] \wedge (p'' < 0[t\backslash\backslash x])}_{\text{Case 3}}$$

Where p' and p'' are the first and second derivative of p for x, respectively. $(p < 0)[t+\epsilon\backslash\backslash x]$ holds if and only if any of the three cases hold for $x = t$.

- Case 1 states that if we substitute t for all x in p and it holds, there must be a satisfying assignment for $p < 0$ in the right of t. It means substituting t for x in p will still evaluate to a negative value.
- Case 2 states that if we substitute t for all x in p, it will evaluate to a zero. Also if we move to the right of t, p decreases only when $(p' < 0)[t\backslash\backslash x]$. So, there must be a value in the right of t such that p evaluates to a negative value.
- In case 3 $p < 0[t\backslash\backslash x]$ holds if and only if for $x = t$ in p, p' are equal to 0, but p'' evaluates to 0. It means, there must be a value from t to positive x-direction for which $p < 0$.

Example 3 *Let us consider our example formula $\varphi = (x^2y + x + y = 0) \wedge (y^2 - 2 < 0)$ for which now we have φ_1 and φ_4. Both of this formula contain $p_2 = (y^2 - 2 < 0)$ and we have the test candidates $\pm\sqrt{2} + \epsilon$ for y in p_2.*

- *For $t = \sqrt{2} + \epsilon$,*

  $$(y^2 - 2 < 0)[\sqrt{2} + \epsilon\backslash\backslash y] = \underbrace{((y^2 - 2 < 0)[\sqrt{2}\backslash\backslash y])}_{\textit{Case 1}} \vee \underbrace{((y^2 - 2 = 0)[\sqrt{2}\backslash\backslash y] \wedge (2y < 0)[\sqrt{2}\backslash\backslash x])}_{\textit{Case 2}} \vee$$
  $$\underbrace{((y^2 - 2 = 0)[\sqrt{2}\backslash\backslash y] \wedge (2y = 0)[\sqrt{2}\backslash\backslash y] \wedge (2 < 0[\sqrt{2}\backslash\backslash y])}_{\textit{Case 3}} \wedge \underbrace{1 \neq 0 \wedge 8 \geq 0}_{\textit{side condition}}$$

  Here $y^2 - 2 \not< 0$ for $(y^2 - 2 < 0)[\sqrt{2} + \epsilon\backslash\backslash y]$. Because, from the Figure 2 we can see that case 1 does not hold for $\sqrt{2}$. If we move to the right of $\sqrt{2}$, p_2 is increasing instead of decreasing. So, case 2 and case 3 also do not hold.

- *For $t = -\sqrt{2} + \epsilon$,*

$$(y^2 - 2 < 0)[-\sqrt{2} + \epsilon \backslash\backslash y] = \underbrace{((y^2 - 2 < 0)[-\sqrt{2}\backslash\backslash y])}_{\textit{Case 1}} \ \vee$$

$$\underbrace{((y^2 - 2 = 0)[-\sqrt{2}\backslash\backslash y] \wedge (2y < 0)[-\sqrt{2}\backslash\backslash x])}_{\textit{Case 2}} \ \vee$$

$$\underbrace{((y^2 - 2 = 0)[-\sqrt{2}\backslash\backslash y] \wedge (2y = 0)[-\sqrt{2}\backslash\backslash y] \wedge (2 < 0[-\sqrt{2}\backslash\backslash y])}_{\textit{Case 3}} \ \wedge \ \underbrace{1 \neq 0 \ \wedge \ 8 \geq 0}_{\textit{side}}$$

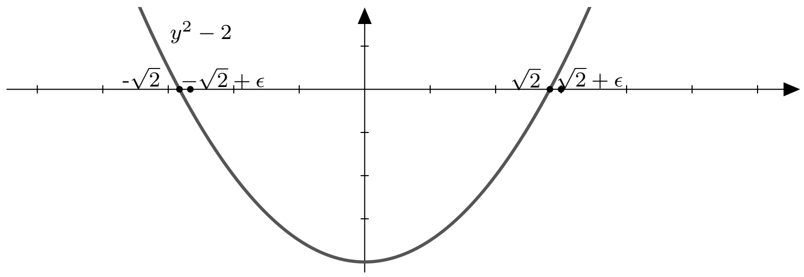

Figure 2: Substitution of Infinitesimal Expressions in $y^2 - 2 < 0$

From the Figure 2, we can see that case 1 does not hold for $-\sqrt{2}$. But, case 2 holds as the constraint is decreasing for at least one point in the right of $-\sqrt{2}$. So, $y^2 - 2 < 0$ for $y = -\sqrt{2} + \epsilon$

3.4.3 Substitution of a Minus Infinity

Let $t = -\infty$ be a test candidate for x in $p < 0$. t cannot have any other values rather than $-\infty$. If we substitute $x = t$ in $p < 0$, we get the following formula:

$$p < 0[-\infty\backslash\backslash x] = (a < 0) \ \vee \ (a = 0 \wedge b > 0) \ \vee \ (a = 0 \wedge b = 0 \wedge c < 0)$$

Where $p = ax^2 + bx + c$ and a, b, c do not contain x.

Example 4 *Assume $\varphi_4 = (y^2 - 2 < 0) \wedge (y \neq 0) \wedge (1 - 4y^2 \geq 0)$ which we get from our example formula φ. By substituting $y = -\infty$ in $p_2 = (y^2 - 2 < 0)$ of φ_4 we get the following:*

$$y^2 - 2 < 0[-\infty\backslash\backslash y] = (1 < 0) \ \vee \ (1 = 0 \wedge 0 > 0) \ \vee \ (1 = 0 \wedge 0 = 0 \wedge -2 < 0)$$

Here $a = 1, b = 0$ and $c = -2$.

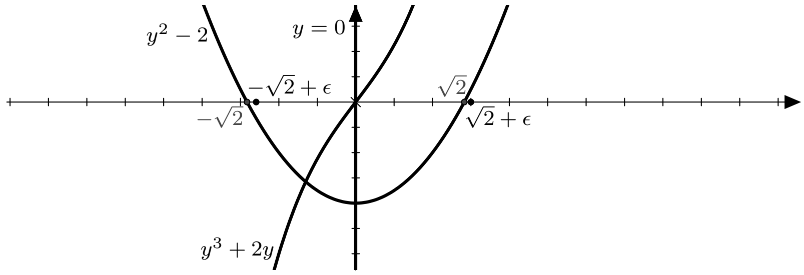

Figure 3: Solutions of y in φ_1 where $x = x_0$

3.5 Quantifier Elimination with the Virtual Substitution

Let φ is a quantifier-free real arithmetic formula where $x \in \varphi$ and x has at most quadratic in φ. After eliminating existential and universal qualifier with virtual substitution we will get the following equivalences, respectively:

$$\exists x \varphi^{\mathbb{R}} \iff \bigvee_{t \in T(x,\varphi^{\mathbb{R}})} (\varphi^{\mathbb{R}}[t\backslash\backslash x] \wedge C_t) \tag{3.16}$$

$$\forall x \varphi^{\mathbb{R}} \iff \bigwedge_{t \in T(x,\varphi^{\mathbb{R}})} (C_t \rightarrow \varphi^{\mathbb{R}}[t\backslash\backslash x]) \tag{3.17}$$

Now, let us consider $\exists y \exists x \varphi$ where, $\varphi = (x^2y + x + y = 0) \wedge (y^2 - 2 < 0)$. We already constructed all the test candidates for x and y in all the polynomials of φ. Also we get to know about virtual substitution rules. Based on virtual substitution rules, we can eliminate all occurrences of x and y from φ by successively eliminating all bound variables (x, y) starting with the inner most one.

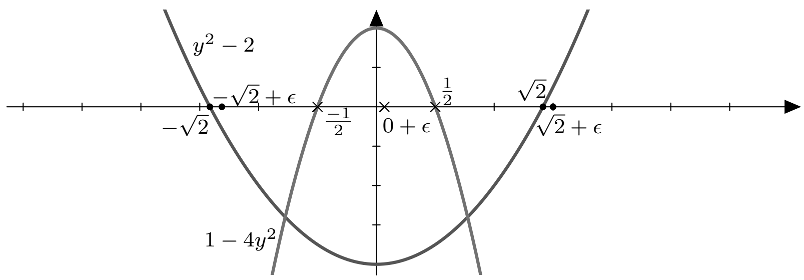

Figure 4: Solution of y in φ_4 where $x = x_0$ and x_2

The solution values for x and y in φ is shown in the figure 3 and 4. Also Figure5 illustrates an overview of solving non-linear real arithmetic formula with virtual substitution.
We get the proof of equation (4.1) which we have explained throughout this paper by an example of non-linear real arithmetic formula. Equation (4.2) can be implied by equation

(4.1) as follows:

$$
\begin{aligned}
\forall x.\varphi &\iff \neg\exists x.\neg\varphi && (3.18)\\
&\iff \neg\Big(\bigvee_{t\in T(x,\neg\varphi)} (\neg\varphi[t\backslash\backslash x]\wedge C_t)\Big) && (3.19)\\
&\iff \bigwedge_{t\in T(x,\neg\varphi)} (\varphi[t\backslash\backslash x]\vee\neg C_t) && (3.20)\\
&\iff \bigwedge_{t\in T(x,\varphi)} (C_t\rightarrow\varphi[t\backslash\backslash x]) && (3.21)
\end{aligned}
$$

Where we get

- Equation 4.4 by Equation 4.1.
- Equation 4.5 and Equation 4.6 after applying $\neg((\neg\varphi_1)\wedge\varphi_2) = (\varphi_1\vee\neg\varphi_2)$ and $(\varphi_1\vee\neg\varphi_2) = (\varphi_2\rightarrow\varphi_1)$, respectively.

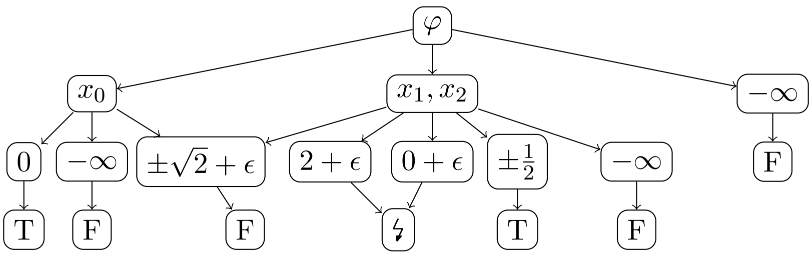

Figure 5: Example of Virtual Substitution.

4 Conclusion

We have described the solving technique of non-linear real arithmetic with virtual substitution step by step. The solving technique has two steps. One is to construct test candidates with side conditions and another one is to replace all occurrences of a variable in a non-linear real arithmetic formula by a test candidate. This replacement is based on virtual substitution rules. Finally, we find solution values for which the formula is satisfied.

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