Topic

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Abstract

We present a procedure as published in [1] which computes all the solutions of non-linear equalities with virtual substitution.The solution approach relies on some substitution rules of virtual substitution. The rules can be applied only on the real-arithmetic formulas which is linear or quadratic.

1 Introduction

Quantifier/ variable elimination for elementary real algebra is a fundamental problem. This problem can be solved easily by using virtual solution. In 1993, the concept of virtual substitution was first introduced. Initially it was a procedure to eliminate quantifier/variable elimination for linear real arithmetic formulas.Further, virtual substitution became a procedure of quantifier elimination for non-linear arithmetic formulas.Virtual substitution cannot eliminate quantified variables whose degree is higher than 2.

Section 2 consists some preliminaries with some definitions. How to construct the real zeros is explained in section 3. In the next section we will know some substitution rules by which we can eliminate variables from a formula with an example. In the section 4 an idea to eliminate quantifiers is explained by an example and conclude in the last section.

2 Preliminaries

2.1 Virtual Substitution

Virtual substitution is a procedure to eliminate a quantified variable. Let $\varphi^{\mathbb{R}}$ is a quantifier-free real-arithmetic formula where $x \in p(x)$ and $p(x) \sim 0, \sim \in \{=, <, >, \leq, \geq, \neq\}$ is a constraint of $\varphi^{\mathbb{R}}$. Degree of x in $p(x)$ must be ≤ 2. Then, after quantifier elimination by virtual substitution we get the following equivalence,

$$\exists x.\varphi^{\mathbb{R}} \iff \bigvee_{t \in T(x,\varphi^{\mathbb{R}})} (\varphi^{\mathbb{R}}[t\backslash\backslash x] \wedge C_t)$$

where T is a finite set of all possible test candidates for x and C_t is a side condition of $t \in T$.

2.2 Test Candidates and Side Condition

To solve non-linear equalities with virtual substitution first we have to choose a variable, $x \in p(x)$ to eliminate and then compute all possible test candidates. $\varphi^{\mathbb{R}}$ is satisfied if there is a test candidate $t \in T$ such that $\varphi^{\mathbb{R}}[t\backslash\backslash x] = p_1[t\backslash\backslash x] \wedge \cdots p_n[t\backslash\backslash x] \wedge C_t$ is satisfiable.

So, the indices of the substitutions are the side conditions of the test candidate it considers and the labels on the edges to a substitutions are the constraints which provide test candidate. A detailed explanation of how to construct test candidates with side condition is provided in the section 3.1.

2.3 Square Root Expression

A square root expression has the form,

$$\frac{p + q\sqrt{r}}{s} \qquad \text{, where } p, q, r, s \in P$$

and the set of all square root can be expressed by,

$$SqrtEx := \{\frac{p + q\sqrt{r}}{s} |\ p, q, r, s \in P\}$$

Definition 2.1 (Polynomial) *A polynomial is a mathematical expression consisting of a sum of terms, each term including a variable or variables raised to a power and multiplied by a coefficient. If a polynomial has only one variable, it is called univariate. An univariate of degree d has the following form where $a_d \neq 0$,*

$$p(x) = a_d x^d + a_{d-1} x^{d-1} + \ldots + a_0 x^0$$

If a polynomial has two or more variables, it is called multivariate. A multivariate (two variables) of degree d has the following form where $a_{dd} \neq 0$,

$$p(x, y) = a_{dd} x^d y^d + a_{d(d-1)} x^d y^{d-1} + a_{(d-1)d} x^{d-1} y^d + \ldots + a_{10} x^1 y^0 + a_{10} x^0 y^1 + a_{00} x^0 y^0$$

The following expression is a quantifier-free real-arithmetic formula where a, b, c are the polynomials and the set of all polynomials in $\varphi^{\mathbb{R}}$ is $P = \{a, b, c\}$,

$$\varphi^{\mathbb{R}} = (a \leq 0 \vee b = 0) \wedge (b < 0 \vee c \neq 0)$$

Let, $p(x) = ax^2 + bx + c = 0$ is a quadratic equation of variable x where $a, b, c \in P$ and $x \notin a \cup b \cup c$. Now, the solution formula for x in $p(x) = a_d x^d + a_{d-1} x^{d-1} + \ldots + a_0 x^0$ considers the following four cases,

$$x_0 = -\frac{c}{b} \qquad \text{, if } a = 0 \wedge b \neq 0 \tag{2.1}$$

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{, if } a \neq 0 \wedge b^2 - 4ac \geq 0 \tag{2.2}$$

$$x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad \text{, if } a \neq 0 \wedge b^2 - 4ac \geq 0 \tag{2.3}$$

$$x_3 = -\infty \qquad \text{, if } a = 0 \wedge b = 0 \tag{2.4}$$

Note that, x_0 is a real zero of $p(x)$ for linear equation, for quadratic equation x_1 and x_2 are two real zeros of $p(x)$. x_4 is any real number which is also a solution for x.

Now, we can express the symbolic zero of x in a polynomial, which is quadratic in x by a square root expression $\frac{p+q\sqrt{r}}{s}$ as given in table 2.1.

Remark We can construct test candidates by the comparison with square root expression (table 2.1) and also considering that test candidates can be supplemented by an infinitesimal ε.

Table 2.1 Comparison with square root expression $\frac{p+q\sqrt{r}}{s}$

Equation No.	p	q	r	s
2.1	$-c$	0	1	b
2.2	$-b$	1	b^2-4ac	$2a$
2.3	$-b$	-1	b^2-4ac	$2a$
2.4	0	1	0	0

3 Solving Non-linear Equalities with Virtual Substitution

Virtual Substitution is a restricted but very efficient procedure to solve non-linear equalities. In the paper [1] author explored an extension of the ideas in [2] from the linear to the quadratic case. For linear case the idea was to eliminate a quantifier from $\exists x\varphi$ by replacing x in φ with t that may involve improper expressions such as $\pm\infty$ or ϵ. However, $\varphi[t\backslash\backslash x]$ is defined in such a way that these improper expressions do not occur in the resulting formula.

Author extended these idea to various quadratic cases. The cases are the substitution of square root expressions and the substitution of infinitesimal expressions in formulas.

In virtual substitution, first a variable is replaced by test candidate and to perform the replacement we need to construct test candidates. An univariate real-arithmetic formula is satisfiable if and only if there is one test candidate for which satisfies formula and the side conditions of t holds. For multivariate real-arithmetic formula the virtual substitution method continues with the elimination of the next variable.

In this section we will see how we can apply virtual substitution. Let us consider a multivariate real-arithmetic formula which we will use in this section,

$$\varphi = \underbrace{(x^2y + x + y = 0)}_{p_1} \wedge \underbrace{(y^2 - 2 < 0)}_{p_2}$$

3.1 Constructing test candidates with side condition

First we will eliminate x from φ. To construct the test candidates for x, we have to compute SqrtEx for x. Also we need to consider an infinitesimal ϵ.

Definition 3.1 (Construction of Test Candidates) *The set of all test candidates is defined by,*

$$TCS := SqrtEx \cup \{t + \epsilon \mid t \in SqrtEx\}$$

The set of test candidates for x in $p(x) = ax^2 + bx + c \sim 0$ is defined by,

$$(x, p(x) \sim 0) \quad \mapsto \quad \begin{cases} \{-\infty, \dfrac{-c}{b}, \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}\} & , \textit{ if } \sim \textit{ is weak} \\ \\ \{-\infty, \dfrac{-c}{b} + \epsilon, \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} + \epsilon\} & , \textit{ otherwise} \end{cases}$$

where $a, b, c \in P$, $x \notin a \cup b \cup c$ and weak means $\{=, \leq, \geq\}$

The side condition of a test candidate is defined by,

$$C_t : t \quad \mapsto \quad \begin{cases} C_{t'} & , \textit{ if } t = t' + \epsilon \\ \\ \{s \neq 0 \wedge r \geq 0\} & , \textit{ if } t = \frac{p + q\sqrt{r}}{s} \\ \\ \textit{true} & , \textit{ Otherwise} \end{cases}$$

where p, q, r and s are polynomials and t' is a test candidate where $\epsilon \notin t'$.

Each side condition of the test candidates confirms that each test candidate exists. The side condition of the test candidate $-\infty$ is valid because, it does not relate to a zero.

To eliminate x first we get the following test candidates,

$$x_0 = -y \qquad , \text{if } y = 0 \wedge 1 \neq 0 \tag{3.1}$$

$$x_1 = \frac{-1 + \sqrt{1^2 - 4y^2}}{2y} \qquad , \text{if } y \neq 0 \wedge 1^2 - 4y^2 \geq 0 \tag{3.2}$$

$$x_2 = \frac{-1 - \sqrt{1^2 - 4y^2}}{2y} \qquad , \text{if } y \neq 0 \wedge 1^2 - 4y^2 \geq 0 \tag{3.3}$$

$$x_3 = -\infty \qquad , \text{if } y = 0 \wedge 1 = 0 \tag{3.4}$$

Here, C_{x_3} is invalid. So, x_3 does not exist for p_1.

Now, we will put all the test candidates of x in φ and we will get following three quantifier free and x free real-arithmetic formulas,

$$\varphi_1 = \varphi[x_0 \backslash\backslash x] = (y^3 + 2y = 0) \wedge (y^2 - 2 < 0) \wedge (y = 0) \wedge (1 \neq 0) \tag{3.5}$$

$$\varphi_2 = \varphi[x_1 \backslash\backslash x] = (p_3 = 0) \wedge (y^2 - 2 < 0) \wedge (y \neq 0) \wedge (1 - 4y^2 \geq 0) \tag{3.6}$$

$$\varphi_3 = \varphi[x_2 \backslash\backslash x] = (p_4 = 0) \wedge (y^2 - 2 < 0) \wedge (y \neq 0) \wedge (1 - 4y^2 \geq 0) \tag{3.7}$$

where,

$$p_3 = (\underbrace{(\frac{-1 + \sqrt{1 - 4y^2}}{2y})^2 y + (\frac{-1 + \sqrt{1 - 4y^2}}{2y}) + y}_{\text{it will be always 0 as it is a real zero}} = 0)$$

and

$$p_4 = (\underbrace{(\frac{-1 - \sqrt{1 - 4y^2}}{2y})^2 y + (\frac{-1 - \sqrt{1 - 4y^2}}{2y}) + y}_{\text{it will be always 0 as it is a real zero}} = 0)$$

To eliminate y from φ_1,φ_2,φ_3 we construct the test candidates for y given in Table 3.1 where we marked the valid and invalid side conditions.

Table 3.1 Test Candidates for y with side conditions

	Test Candidate	Side Condition	Validation of Side Condition
φ_1,φ_2,φ_3	$\sqrt{2}+\epsilon$	$1 \neq 0 \wedge 8 \geq 0$	✓
	$-\sqrt{2}+\epsilon$	$1 \neq 0 \wedge 8 \geq 0$	✓
	$-\infty$	true	✓
φ_1	0	$0 = 0 \wedge 1 \neq 0$	✓
φ_2,φ_3	$2+\epsilon$	$1 = 0 \wedge 0 \neq 0$	x
	$0+\epsilon$	$0 \neq 0 \wedge -2 \neq 0$	x
	$\frac{1}{2}$	$-4 \neq 0 \wedge 16 \geq 0$	✓
	$-\frac{1}{2}$	$-4 \neq 0 \wedge 16 \geq 0$	✓
	$\frac{-1}{0}$}invalid value	-	-

3.2 Substitute Variables by Test Candidates Virtually

We have already seen the construction of test candidates for a given multivariate real-arithmetic formula φ. φ is satisfiable if and only if one of the test candidates is a solution of φ. If all test candidates satisfy φ, it is said to be valid.

We eliminated x already and currently we have φ_1, φ_2 and φ_3 from which y is needed to be eliminate. In φ_1, there is a constraint $y^3 + 2y = 0$ which has a degree of 3. We will not consider this constraint for substitution as the degree is higher than 2. If the constructed test candidates of y from the other constrains in φ_1 satisfy this constraint, φ is satisfiable. φ_2 and φ_3 have become same as p_3 and p_4 are already satisfied. Further, we will represent φ_2 and φ_3 as φ_4.

$$\varphi_4 = (y^2 - 2 < 0) \wedge (y \neq 0) \wedge (1 - 4y^2 \geq 0)$$

In this section, we will eliminate y by substituting the test candidates and check if the formula is satisfiable. To perform substitution the major substitution rules are described in the following.

3.2.1 Substitution of Square Root Expressions

Let, t is a test candidate for x where $t = \frac{p_1+q_1\sqrt{r}}{s_1}$ is a square root expression. Let consider a constraint $p = 0$ and we want to substitute t for x in it. If we substitute t for all occurrences of x in p, we can transform the result into $\frac{p_2+q_2\sqrt{r}}{s_2}$ which is also a square root expression where p_2, q_2 and $s_2 \in P$. It has to be mentioned that, the radicand still remains the same which is r. This transformation is possible as the summation and multiplication result of two square root expression with the same radicand is a square root expression of that radicand.

- Summation,

$$\frac{p_3 + q_3\sqrt{r}}{s_3} + \frac{p_4 + q_4\sqrt{r}}{s_4} = \frac{s_4(p_3 + q_3\sqrt{r}) + s_3(p_4 + q_4\sqrt{r})}{s_3 s_4} = \frac{s_4p_3 + s_4q_3\sqrt{r} + s_3p_4 + s_3q_4\sqrt{r}}{s_3s_4}$$

$$= \frac{\overbrace{s_4p_3 + s_3p_4}^{p_2} + (\overbrace{s_4q_3 + s_3q_4}^{q_2})\sqrt{r}}{\underbrace{s_3s_4}_{s_2}}$$

- Multiplication,

$$\frac{p_3 + q_3\sqrt{r}}{s_3} * \frac{p_4 + q_4\sqrt{r}}{s_4} = \frac{(p_3 + q_3\sqrt{r})(p_4 + q_4\sqrt{r})}{s_3 s_4} = \frac{p_3p_4 + p_3q_4\sqrt{r} + p_4q_3\sqrt{r} + q_3\sqrt{r}q_4\sqrt{r}}{s_3s_4}$$

$$= \frac{\overbrace{p_3p_4}^{p_2} + (\overbrace{p_3q_4 + p_4q_3 + q_3q_4}^{q_2})\sqrt{r}}{\underbrace{s_3s_4}_{s_2}}$$

The equation $\frac{p_2+q_2\sqrt{r}}{s_2} = 0$ holds if and only if $p_2 + q_2\sqrt{r} = 0$. It holds if and only if either $(p_2 = 0 \wedge q_2 = 0)$ or p_2 and q_2 have different signs with same absolute value, i.e., $|p_2| = |q_2\sqrt{r}|$. So, after substitution, we get the following quantifier free real arithmetic formula,

$$(p = 0)[t\backslash\backslash x] = (p_2q_2 \le 0) \wedge (p_2^2 - q_2^2 r = 0)$$

3.2.2 Substitution of Infinitesimal Expressions

Let, $t + \epsilon$ is a test candidate for x where $t \neq -\infty$ and $\epsilon \notin t$. Let consider a constraint $p < 0$ and we want to substitute t for x in it. Note that x should be occurred at most quadratic in p. After substitution we will get the following quantifier-free real-arithmetic formula,

$$(p < 0)[t + \epsilon\backslash\backslash x] = \underbrace{((p < 0)[t\backslash\backslash x])}_{\text{Case 1}} \vee \underbrace{((p = 0)[t\backslash\backslash x] \wedge (p' < 0)[t\backslash\backslash x])}_{\text{Case 2}} \vee$$

$$\underbrace{((p = 0)[t\backslash\backslash x] \wedge (p' = 0)[t\backslash\backslash x] \wedge (p'' < 0[t\backslash\backslash x])}_{\text{Case 3}}$$

where p' and p'' are the fist and second derivative of p for x, respectively. $(p < 0)[t+\epsilon\backslash\backslash x]$ holds if and only if any of the three cases hold for $x = t$

- Case 1 states that if we substitute t for all x in $p < 0$ and it holds, there must be a value in the right of t. It means if x has this value, after substitution in p, it will still evaluate to a negative value.

- Case 2 states that if we substitute t for all x in p, it will evaluate to a zero. Also if we move to the right of t, p will decrease only when $(p' < 0)[t\backslash\backslash x]$. So, there must be a value in the right of t so that p will evaluate to a negative value

- In case 3 $p < 0[t\backslash\backslash x]$ holds if and only if for $x = t$ p, p' are equal to 0, but p'' evaluates to 0. It means, there must be a value from t to positive x-direction for which $p < 0$.

Let us consider our example φ. We already eliminated x from φ and currently, we have φ_1 and φ_4. In both of this formula, there is a constraint $y^2 - 2 < 0$ from which we already constructed the test candidates $\pm\sqrt{2} + \epsilon$.

- For $t = \sqrt{2}$,

$$(y^2-2 < 0)[\sqrt{2}+\epsilon\backslash\backslash y] = \underbrace{((y^2 - 2 < 0)[\sqrt{2}\backslash\backslash y])}_{\text{Case 1}} \vee \underbrace{((y^2 - 2 = 0)[\sqrt{2}\backslash\backslash y] \wedge (2y < 0)[\sqrt{2}\backslash\backslash x])}_{\text{Case 2}} \vee$$

$$\underbrace{((y^2-2=0)[\sqrt{2}\backslash\backslash y] \wedge (2y=0)[\sqrt{2}\backslash\backslash y] \wedge (2<0[\sqrt{2}\backslash\backslash y])}_{\text{Case 3}} \ \wedge \ 1 \neq 0 \ \wedge \ 8 \geq 0$$

Here, $y^2 - 2 \nless 0$ for $y = \sqrt{2} + \epsilon$. Because, from the figure 1 we can see that for $\sqrt{2}$ case 1 does not hold. If we move to the right of $\sqrt{2}$, the constraint is increasing instead of decreasing. So case 2 and case 3 also do not hold.

- For $t = -\sqrt{2}$,

$$(y^2-2<0)[-\sqrt{2}+\epsilon\backslash\backslash y] = \underbrace{((y^2-2<0)[-\sqrt{2}\backslash\backslash y])}_{\text{Case 1}} \ \vee$$

$$\underbrace{((y^2-2=0)[-\sqrt{2}\backslash\backslash y] \wedge (2y<0)[-\sqrt{2}\backslash\backslash x])}_{\text{Case 2}} \ \vee$$

$$\underbrace{((y^2-2=0)[-\sqrt{2}\backslash\backslash y] \wedge (2y=0)[-\sqrt{2}\backslash\backslash y] \wedge (2<0[-\sqrt{2}\backslash\backslash y])}_{\text{Case 3}} \ \wedge \ 1 \neq 0 \ \wedge \ 8 \geq 0$$

From the figure 1, we can see that case 1 does not hold for $-\sqrt{2}$. But, case 2 holds as the constraint is decreasing for at least one point to right of $-\sqrt{2}$. So, $y^2 - 2 < 0$ for $y = -\sqrt{2} + \epsilon$

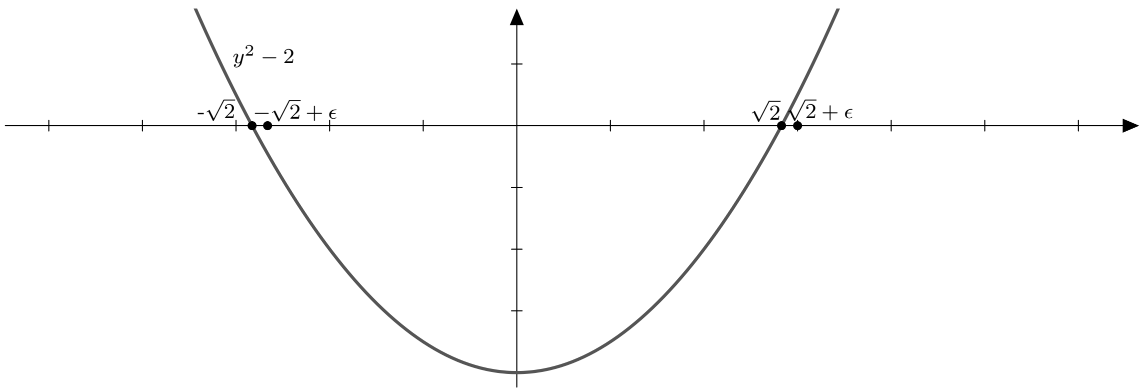

Figure 1: Substitution of Infinitesimal Expressions in $y^2 - 2 < 0$

3.2.3 Substitution of a Minus Infinity

Let $t = -\infty$ is a test candidate of x for $p < 0$. t cannot have any other values rather than $-\infty$. If we substitute $x = t$ in $p < 0$, then we will get the following formula,

$$p < 0[-\infty\backslash\backslash x] = (a < 0) \ \vee \ (a = 0 \wedge b > 0) \ \vee \ (a = 0 \wedge b = 0 \wedge c < 0)$$

where, $p = ax^2 + bx + c$.

In our example, for the constraint $y^2 - 2 < 0$ of φ_4,

$$y^2 - 2 < 0[-\infty\backslash\backslash y] = (1 < 0) \ \vee \ (1 = 0 \wedge 0 > 0) \ \vee \ (1 = 0 \ wedge 0 = 0 \wedge -2 < 0)$$

Here, $a = 1, b = 0$ and $c = -2$.

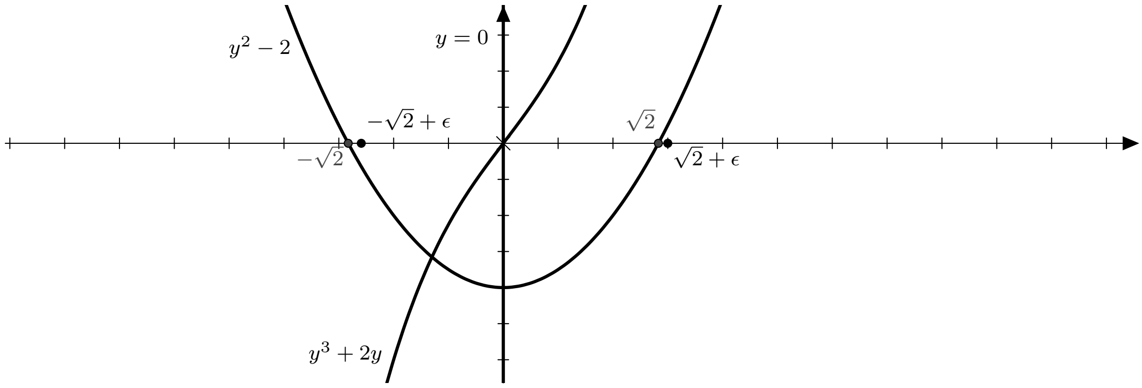

Figure 2: Solutions of y in φ_1 where $x = x_0$

4 Quantifier Elimination with the Virtual Substitution

Let $\varphi^{\mathbb{R}}$ is a quantifier-free real-arithmetic formula where $x \in \varphi^{\mathbb{R}}$ and x has at most degree 2 in $\varphi^{\mathbb{R}}$. After eliminating existential qualifier with virtual substitution we will get the followings,

$$\exists x.\varphi^{\mathbb{R}} \Longleftrightarrow \bigvee_{t \in T(x,\varphi^{\mathbb{R}})} (\varphi^{\mathbb{R}}[t \backslash\backslash x] \wedge C_t)$$

Again, let us consider $\exists x.\varphi$ where, $\varphi = (x^2y + x + y = 0) \wedge (y^2 - 2 < 0)$. We already constructed all the test candidates for x and y. Also we get to know about substitution rules with virtual substitution. By using the substitution rules, we eliminate all occurrences of x and y from φ. Then we get the following equivalence holds.

$$\exists x \exists y.\varphi \Longleftrightarrow \bigvee_{t_1 \in T(x,\varphi)} (\bigvee_{t \in T(y,\varphi')} (\varphi'[t_2 \backslash\backslash y] \wedge C_{t_2}))$$

where, $\varphi' = (\varphi[t_1 \backslash\backslash x] \wedge C_{t_1})$.

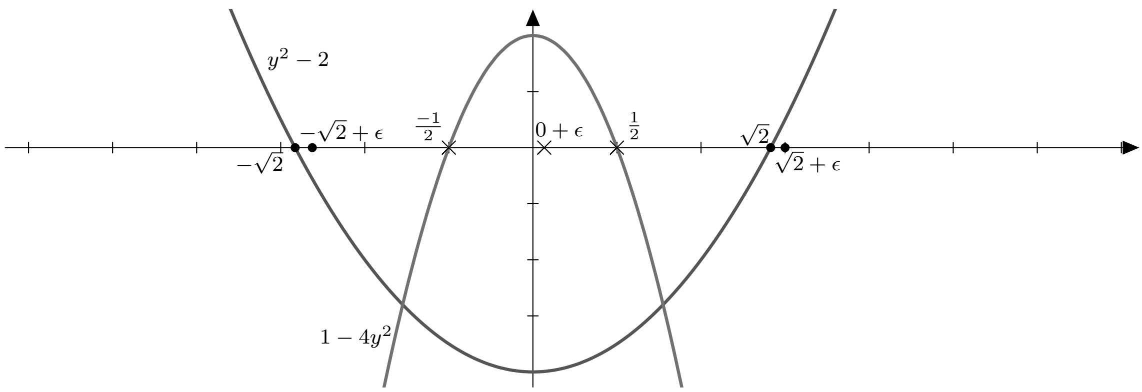

Figure 3: Solution of y in φ_4 where $x = x_0$ and x_2

The solutions of x and y for φ is shown in the figure 2 and 3. Also we can have an overview of virtual substitution though figure 4.

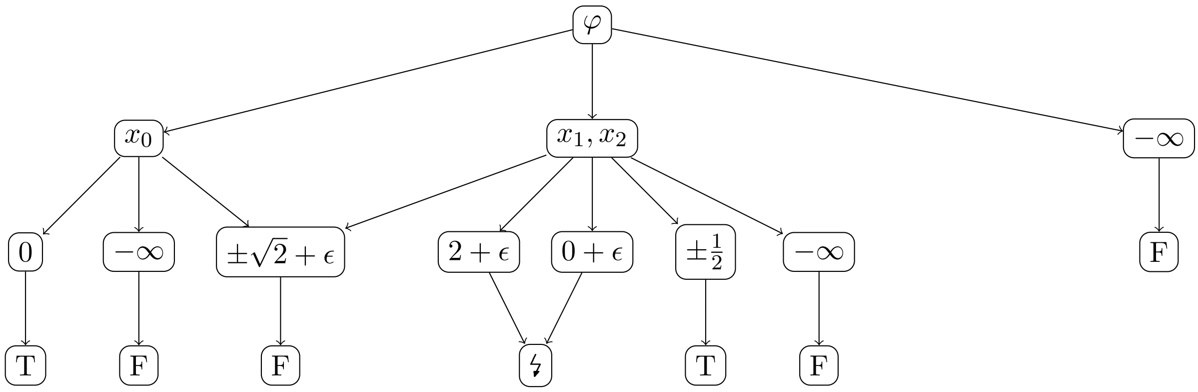

Figure 4: Example of Virtual Substitution.

5 Conclusion

We have described the solving technique of non-linear equalities with virtual substitution step by step. It has two steps. One is to construct test candidates with side conditions and another one is to replace a variable by a test candidate in a formula. It has already said that the replacement is based on some substitution rules.Finally, we can have the solutions for which the formula is satisfied.

References

[1] V. Weispfenning, *Quantifier elimination for real algebra - the quadratic case and beyond*. Appl. Algebra Eng. Commun. Comput, 1997.

[2] R. Loss, V. Weispfenning, *Applying linear quantifier elimination.* The computer Journal 36 (1993), pp. 450-462.