Solving Non-Linear Real Arithmetic Formulas with Virtual Substitution

Author: Aklima Zaman

Supervision: Erika Ábrahám

Theory of Hybrid Systems - Informatik 2 - RWTH-Aachen

Satisfiability Checking Seminar, Winter-16/17

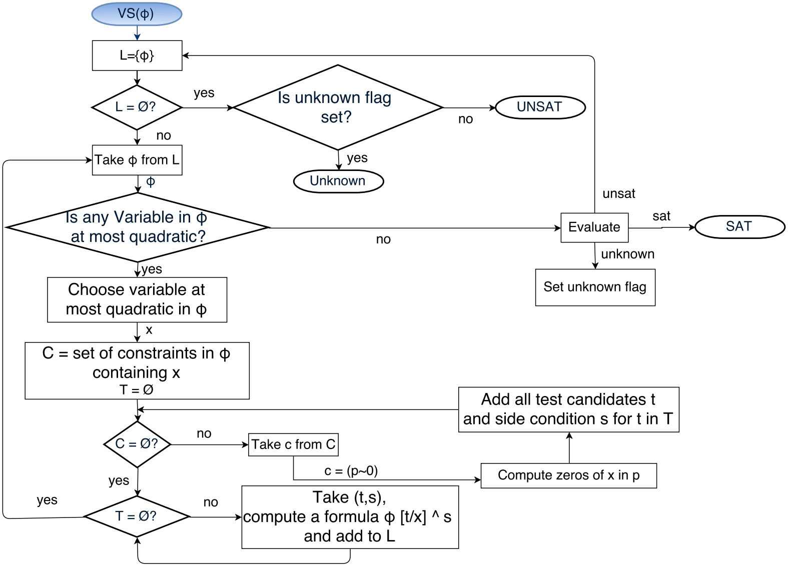
Outline

- Motivation
- Real Arithmetic Formula
- Virtual Substitution
  - Sign Invariant Regions
  - Compute Zeros
  - Compute Test Candidates
  - Virtual Substitution Rules

Motivation

- Other related methods
  - interval constraint propagation
  - cylindrical algebraic decomposition
- Virtual substitution
  - applicable only to sub-language
  - eliminates quantified variables up to degree 4

Flow Chart of Virtual Substitution



Real Arithmetic Formula

- Real arithmetic (RA) formula has the following syntax:
  polynomials: $t := \quad 0 \mid 1 \mid x \mid t + t \mid t - t \mid t \cdot t$
  constraints: $c := \quad t < t$
  formulas: $\varphi := \quad c \mid \neg\varphi \mid \varphi \wedge \varphi \mid \exists x \cdot \varphi$

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- Polynomial $p(x) \in Z[x_1 \ldots, x_n][x]$ normal form:

$$p(x) = a_d x^d + a_{d-1} x^{d-1} + \ldots + a_0 x^0$$

Example:

$$\varphi = ((\underbrace{(x^2+2x+4z)}_{p_1} \leq 0 \vee \underbrace{(yx^2+6y^3x+4z)}_{p_2} = 0)$$

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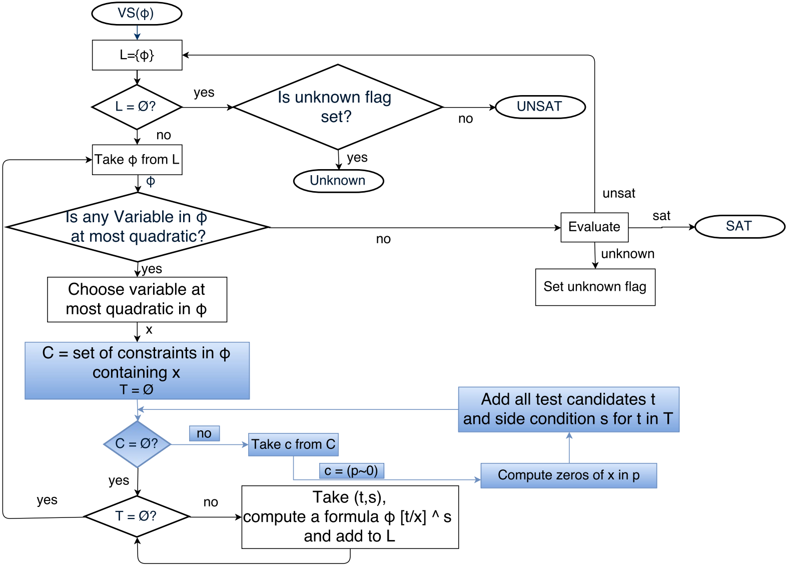


Virtual Substitution

- Quantifier elimination procedure:

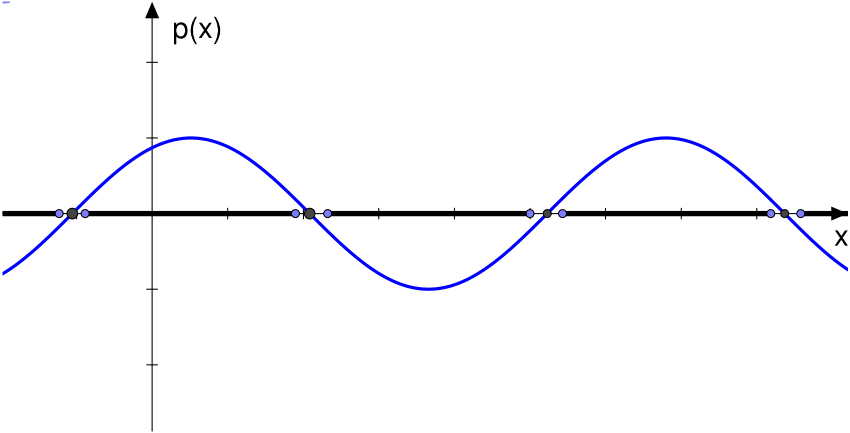
$$\exists x_1 \ldots \exists x_n \cdot \varphi \equiv \exists x_1 \ldots \exists x_{n-1} \cdot \psi$$

  where φ, ψ quantifier free.

- Quantifier elimination by virtual substitution:

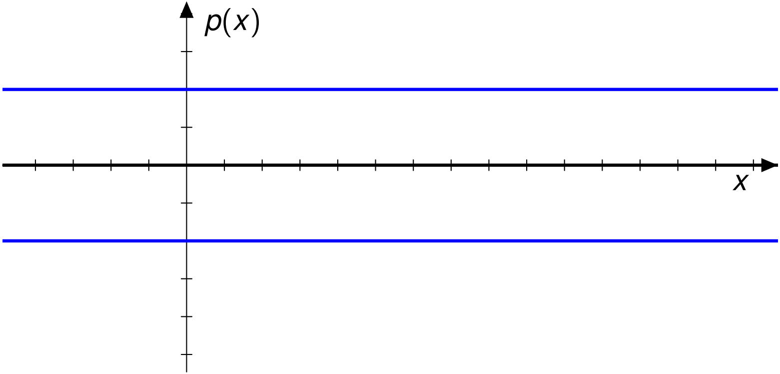
$$\exists x_1 \ldots \exists x_n \cdot \varphi \equiv \exists x_1 \ldots \exists x_{n-1} \cdot \bigvee_{t \in T} (\varphi[t//x] \wedge S_t)$$

Sign Invariant Regions



Compute Zeros

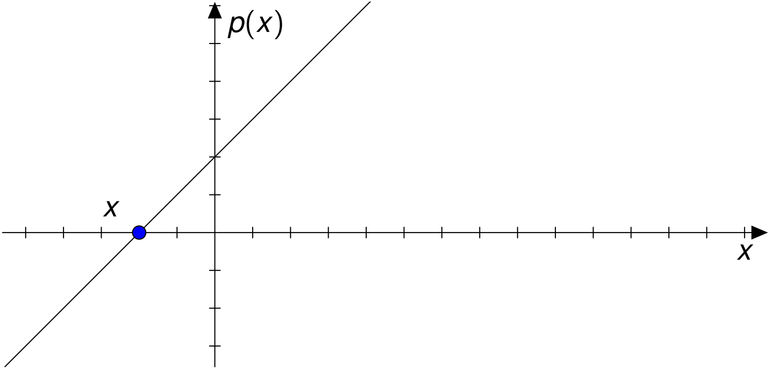
$p(x) = ax^2 + bx + c$ side condition: $a = 0 \wedge b = 0$



constant polynomial $\Rightarrow$ constant zero or non zeros

Compute Zeros

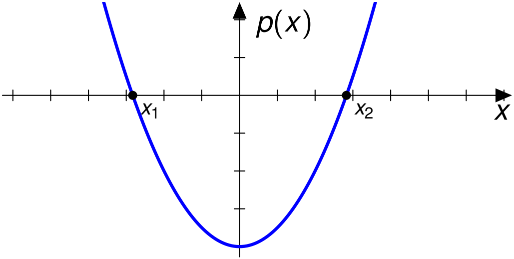
$p(x) = ax^2 + bx + c$ side condition: $a = 0 \wedge b \neq 0$



$x = -c/b$

Compute Zeros

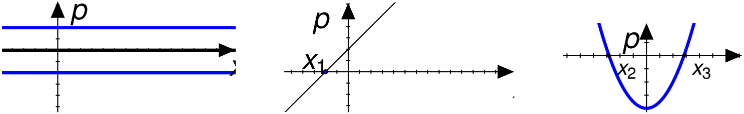
$p(x) = ax^2 + bx + c$ side condition: $a \neq 0 \wedge b^2 - 4ac \geq 0$



$x_1 = \frac{-b-\sqrt{b^2-4ac}}{2a}$, $x_2 = \frac{-b+\sqrt{b^2-4ac}}{2a}$

Compute Test Candidates

Possible solution intervals for x on $p \sim 0$:



Constraints	$-\infty$	x_1	$x_1 + \epsilon$	x_2	$x_2 + \epsilon$	x_3	$x_3 + \epsilon$
$p = 0$	-	✓	-	✓	-	✓	-
$p > 0$, $p < 0$, $p \neq 0$	✓	-	✓	-	✓	-	✓
$p \geqslant 0, p \leqslant 0$	✓	✓	-	✓	-	✓	-

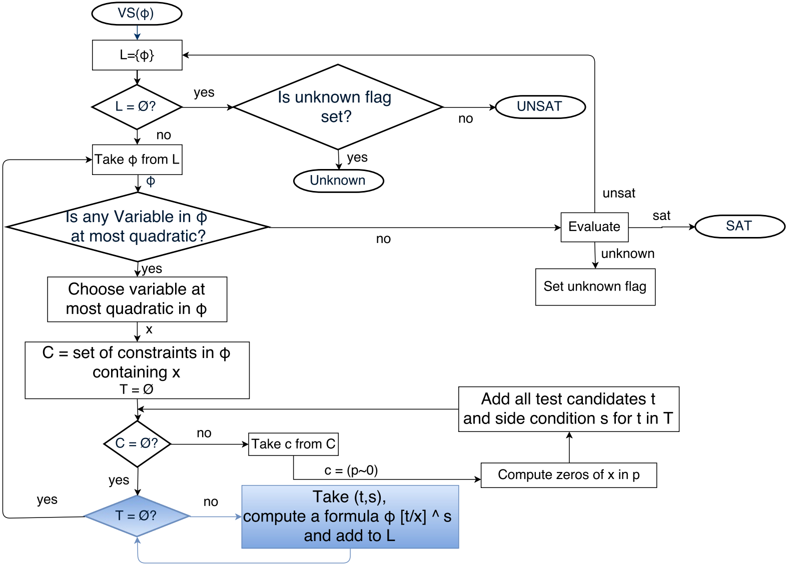
Example

$$\varphi := ((\underbrace{xy - 1}_{p_1} = 0) \wedge \underbrace{y^2 - 1}_{p_2} < 0)$$

Elimination of y:

	constraints	test candidates
1.	from all constraints	$-\infty$
2.	$p_1 = 0$	$1/x \quad \text{if} \quad x \neq 0$
3.	$p_2 < 0$	$1 + \epsilon$
4.	$p_2 < 0$	$-1 + \epsilon$

Example



VS(ϕ)
L={ϕ}
L = Ø?
yes
Is unknown flag set?
no
UNSAT
yes
Unknown
no
Take ϕ from L
ϕ
Is any Variable in ϕ at most quadratic?
no
unsat
Evaluate
sat
SAT
unknown
Set unknown flag
yes
Choose variable at most quadratic in ϕ
x
C = set of constraints in ϕ containing x
T = Ø
Add all test candidates t and side condition s for t in T
C = Ø?
no
Take c from C
c = (p~0)
Compute zeros of x in p
yes
yes
T = Ø?
no
Take (t,s), compute a formula ϕ [t/x] ^ s and add to L

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$$\begin{array}{lllll} \exists x \cdot \exists y \cdot \varphi & \leftrightarrow & \exists x \cdot & (\varphi[-\infty//y]) & & \vee \\ & & & (\varphi[\frac{1}{x}//y] & \wedge x \neq 0) & \vee \\ & & & (\varphi[1 + \epsilon//y]) & & \vee \\ & & & (\varphi[-1 + \epsilon//y]) & & \end{array}$$

Substitution of a Minus Infinity

Assume $(p(x) = ax^2 + bx + c) < 0$ and test candidate is $-\infty$

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\begin{aligned}
p(x) < 0[-\infty//x] \quad = \quad & \underbrace{a < 0}_{\text{Case 1}} && \vee \\
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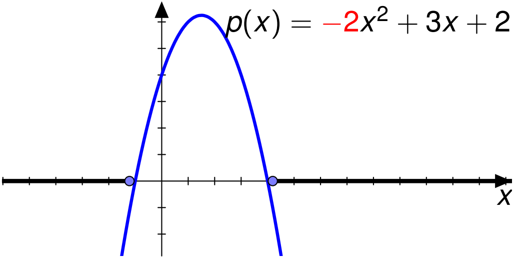

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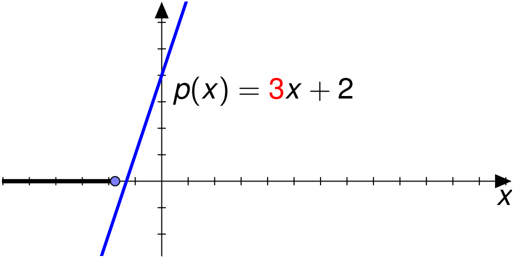

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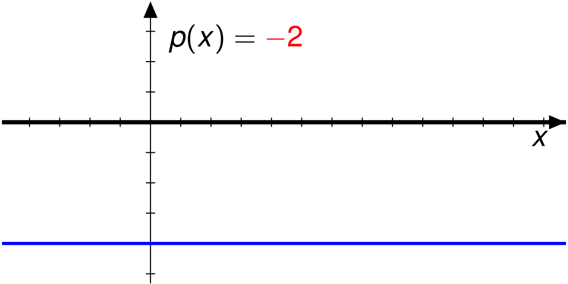

Example

$$\exists x \cdot \exists y \cdot ((xy - 1 = 0) \wedge y^2 - 1 < 0)$$

$$\begin{aligned} p(x) < 0[-\infty//x] \quad = \quad (a < 0) \quad & \vee \quad (a = 0 \wedge b > 0) \\ & \vee \quad (a = 0 \wedge b > 0 \wedge c < 0) \end{aligned}$$

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$$\begin{aligned} \exists x \cdot \quad (\quad & (xy - 1 = 0)[-\infty//y] \\ \wedge \quad & (y^2 - 1 < 0)[-\infty//y] \;) \end{aligned}$$

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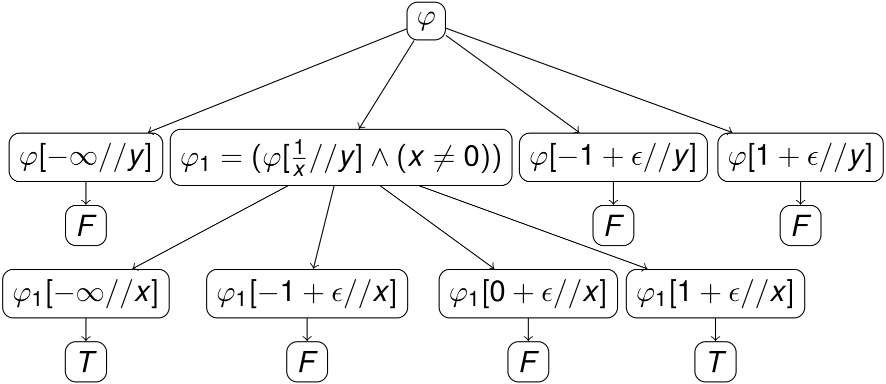
Example: Search Tree

$$\varphi := ((\underbrace{xy - 1}_{p_1} = 0) \wedge \underbrace{y^2 - 1}_{p_2} < 0)$$



Outlook

- Implemented in SMT-RAT and Redlog
- Strengths:
  - Better than Fourier Motzkin
  - Efficient than cylindrical algebraic decomposition
- Weaknesses:
  - Unable to handle the formulas of degree > 4
  - Incomplete
  - Exponentially Complex

Reference

- V. Weispfenning, *Quantifier elimination for real algebra - the quadratic case and beyond*. Appl. Algebra Eng. Commun. Comput, 1997.
- R. Loss, V. Weispfenning, *Applying linear quantifier elimination*. The computer Journal 36 (1993), pp. 450-462.

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$$k = \frac{u + q\sqrt{r}}{s} \quad \text{with } u, q, r, s \text{ polynomials.}$$

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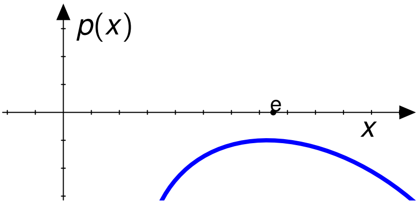

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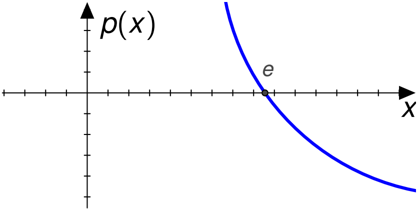

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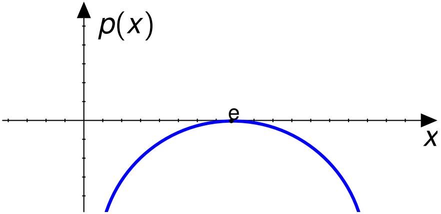

Example

$\exists x \cdot \exists y \cdot ((xy - 1 = 0) \wedge y^2 - 1 < 0)$

Elimination of y:

2.Test candidate: $\frac{1}{x}$ *if* $x \neq 0$

$$\begin{array}{lll} \exists x \cdot & (& (xy - 1 = 0)[\frac{1}{x}//y] \\ & \wedge & (y^2 - 1 < 0)[\frac{1}{x}//y] \\ & \wedge & x \neq 0 \;) \end{array}$$

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 & \wedge & x \neq 0 \ ) \\
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 & \wedge & ((1 > 0) \wedge 1 - x^2 < 0 \vee (1 < 0 \wedge 1 - x^2 < 0))
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$$\begin{array}{lll} \exists x \cdot & (& (xy - 1 = 0)[\frac{1}{x}//y] \\ & \wedge & (y^2 - 1 < 0)[\frac{1}{x}//y] \\ & \wedge & x \neq 0 \;) \\ \Leftrightarrow \exists x \cdot & (& (0 = 0) \\ & \wedge & ((1 > 0) \wedge 1 - x^2 < 0 \vee (1 < 0 \wedge 1 - x^2 < 0)) \\ & \wedge & x \neq 0 \;) \end{array}$$

Example

$$\exists x \cdot \exists y \cdot ((xy - 1 = 0) \wedge y^2 - 1 < 0)$$

Elimination of y:

2.Test candidate: $\frac{1}{x}$ *if* $x \neq 0$

$$\begin{array}{lll}
\exists x \cdot & (& (xy - 1 = 0)[\frac{1}{x}//y] \\
 & \wedge & (y^2 - 1 < 0)[\frac{1}{x}//y] \\
 & \wedge & x \neq 0\ ) \\
\Leftrightarrow \exists x \cdot & (& (0 = 0) \\
 & \wedge & ((1 > 0) \wedge 1 - x^2 < 0 \vee (1 < 0 \wedge 1 - x^2 < 0)) \\
 & \wedge & x \neq 0\ ) \\
\Leftrightarrow \exists x \cdot & (& (1 - x^2 < 0) \\
 & \wedge & x \neq 0\ )
\end{array}$$

Example

$\exists x \cdot (\ 1 - x^2 < 0 \ \wedge x \neq 0 \ )$

Elimination of x:

1. Test candidate: $-\infty$

$$
\begin{aligned}
& (1 - x_2 < 0)[-\infty//x] \\
= \quad & (-1 < 0 \vee (-1 = 0 \wedge 0 > 0) \vee (-1 = 0 \wedge 0 = 0 \wedge 1 < 0)) \\
= \quad & \textbf{true}
\end{aligned}
$$

Example

$\exists x \cdot (\ \textbf{true} \wedge x \neq 0 \ )$

Elimination of x:

1. Test candidate: $-\infty$

$$
\begin{aligned}
& (x \neq 0)[-\infty//x] \\
= \quad & (1 \neq 0 \vee 0 \neq 0) \\
= \quad & \textbf{true}
\end{aligned}
$$

Example

$\exists x \cdot ($ **true** $\wedge$ **true** $)$

Elimination of x:

1. Test candidate: $-\infty$

$$
\begin{aligned}
& (x \neq 0)[-\infty // x] \\
= \quad & (1 \neq 0 \vee 0 \neq 0) \\
= \quad & \textbf{true}
\end{aligned}
$$