Computing Minimal Unsatisfiable Subsets of Clause Sets

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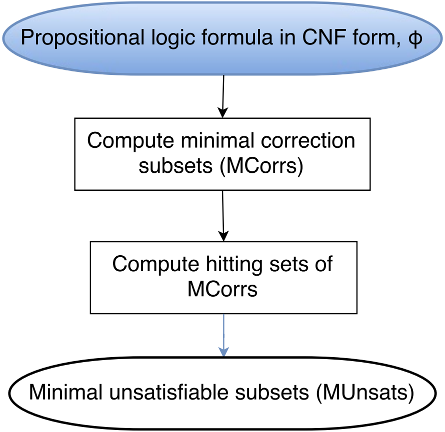
Theory of Hybrid Systems - Informatik 2 - RWTH-Aachen

Satisfiability Checking Seminar, Winter-16/17

Outline

- Propositional Logic Formula
- Minimal Unsatisfiable Subsets and Minimal Correction Subsets
- Duality of Minimal Unsatisfiable and Correction Subset
- Algorithms for Computing all Minimal Unsatisfiable Subsets

FlowChart


Propositional logic formula in CNF form, ϕ
Compute minimal correction subsets (MCorrs)
Compute hitting sets of MCorrs
Minimal unsatisfiable subsets (MUnsats)

Propositional Logic Formula

x

variable

Propositional Logic Formula

x $\qquad\qquad$ $\neg x$

variable

Propositional Logic Formula

x $\qquad\qquad$ $\neg x$

literals

Propositional Logic Formula

$$\overbrace{x \quad \vee \quad \neg x}^{\textit{clause}}$$

literals

Propositional Logic Formula

$$\varphi := \underbrace{\overbrace{(x \vee \neg x)}^{\textit{clause}} \wedge (x \vee \neg y)}_{\text{conjunctive normal form}}$$

literals

Problem Statement

$$\begin{aligned}\varphi &:= (x \vee \neg x) \wedge (\neg x \vee y) \\ &:= (T \vee F) \wedge (F \vee T) \\ &:= T \longrightarrow \text{ SAT}\end{aligned}$$

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\varphi &:= (x \vee \neg x) \wedge (\neg x \vee y) \\
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\end{aligned}
$$

$$
\begin{aligned}
\varphi &:= (x) \wedge (\neg x) \wedge (\neg x \vee y) \wedge (\neg x \vee \neg y) \\
&:= (\textcolor{red}{T}) \wedge (\textcolor{red}{F}) \wedge (\textcolor{red}{F} \vee \textcolor{red}{T}) \wedge (\textcolor{red}{F} \vee \textcolor{red}{F}) \\
&:= \textcolor{red}{F} \longrightarrow \ \textcolor{red}{\text{UNSAT}}
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unsatisfiable subset = $\{ \{(x), (\neg x), (\neg x \vee \neg y)\}$

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$$\text{unsatisfiable subset} = \{\, \{(x), (\neg x), (\neg x \vee \neg y)\}\\ , \{(x), (\neg x \vee y), (\neg x \vee \neg y)\}\}$$

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& ,\{(x), (\neg x \vee y), (\neg x \vee \neg y)\} \\
& ,\{(x), (\neg x)\}\ \}
\end{aligned}
$$

Minimal Unsatisfiable Subset (MUnsat):

(x)	$(\neg x)$	$(\neg x \vee y)$	$(\neg x \vee \neg y)$	MUnsat	MinimumUnsat
$\lightning$	$\lightning$			$\lightning$	$\lightning$
$\lightning$	$\lightning$		$\lightning$		
$\lightning$		$\lightning$	$\lightning$	$\lightning$	

$$\varphi = \underbrace{(x)}_{C_1} \wedge \underbrace{(\neg x)}_{C_2} \wedge \underbrace{(\neg x \vee y)}_{C_3} \wedge \underbrace{(\neg x \vee \neg y)}_{C_4}$$

Minimal Unsatisfiable Subset (MUnsat):

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$\lightning$	$\lightning$		$\lightning$		
$\lightning$		$\lightning$	$\lightning$	$\lightning$	

Minimal Correction Subset (MCorr):

(x)	$(\neg x)$	$(\neg x \vee y)$	$(\neg x \vee \neg y)$
$\checkmark$			
	$\checkmark$	$\checkmark$	
	$\checkmark$		$\checkmark$

$$\varphi = \underbrace{(x)}_{C_1} \wedge \underbrace{(\neg x)}_{C_2} \wedge \underbrace{(\neg x \vee y)}_{C_3} \wedge \underbrace{(\neg x \vee \neg y)}_{C_4}$$

Duality of Minimal Unsatisfiable and Correction Subset

- **Hitting Sets:**

Variable Set:	$\mathrm{D} = \{w, x, y, z\}$
Collection Set:	$\Omega = \{\{w, x\}, \{x, y, z\}\}$
Hitting Set:	$H = \{\{w, z\}, \{x, y\}, \{x\}, \ldots\}$

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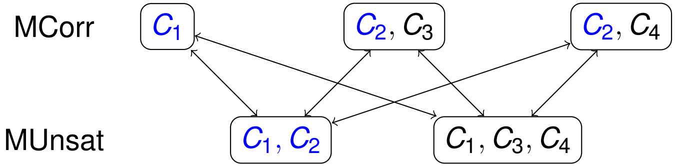
- Minimal hitting sets of the set of MCorr:



$$\varphi = \underbrace{(x)}_{C_1} \wedge \underbrace{(\neg x)}_{C_2} \wedge \underbrace{(\neg x \vee y)}_{C_3} \wedge \underbrace{(\neg x \vee \neg y)}_{C_4}$$

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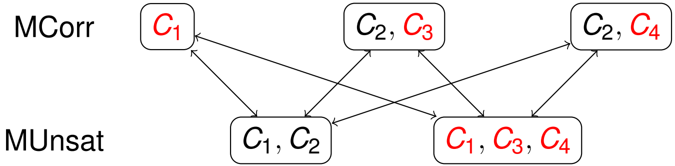

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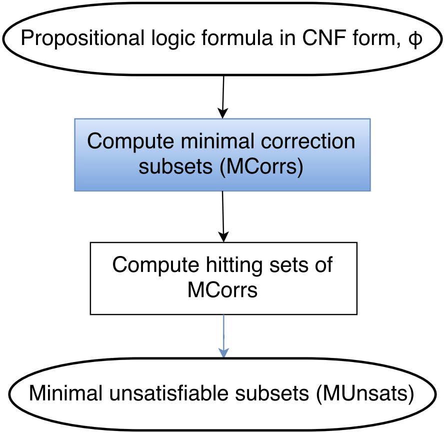
FlowChart


Propositional logic formula in CNF form, ϕ
Compute minimal correction subsets (MCorrs)
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Minimal unsatisfiable subsets (MUnsats)

Algorithm: Computing all MCorrs

1. Augment CNF with clause selector variables

$$
\begin{array}{lccccccc}
\varphi = & (x) & \wedge & (\neg x) & \wedge & (\neg x \vee y) & \wedge & (\neg x \vee \neg y) \\
\varphi' = & (\neg w_1 \vee x) & \wedge & (\neg w_2 \vee \neg x) & \wedge & (\neg w_3 \vee \neg x \vee y) & \wedge & (\neg w_4 \vee \neg x \vee \neg y)
\end{array}
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2. All MCorrs are found incrementally

$$\varphi' = (\neg \mathbf{false} \vee x) \wedge (\neg w_2 \vee \neg x) \wedge (\neg w_3 \vee \neg x \vee y) \wedge (\neg w_4 \vee \neg x \vee \neg y)$$

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3. Add blocking clauses to block old solutions

$$\varphi' = \varphi' \wedge w_1$$

Example: Computing all MCorrs

$$\varphi \quad = \quad (x) \wedge (\neg x) \wedge (\neg x \vee y) \wedge (\neg x \vee \neg y)$$

Example: Computing all MCorrs

$$\varphi \quad = \quad (\neg w_1 \vee x) \wedge (\neg w_2 \vee \neg x) \wedge (\neg w_3 \vee \neg x \vee y) \wedge (\neg w_4 \vee \neg x \vee \neg y)$$

1. Add clause-selector variables.

Example: Computing all MCorrs

$$\varphi \quad = \quad (\neg w_1 \vee x) \wedge (\neg w_2 \vee \neg x) \wedge (\neg w_3 \vee \neg x \vee y) \wedge (\neg w_4 \vee \neg x \vee \neg y)$$

1. Add clause-selector variables.
2. Add AtMost constraint.

$$AtMost(\{w_1, w_2, w_3, w_4\}, 1)$$

Example: Computing all MCorrs

$$\varphi = (\neg w_1 \vee x) \wedge (\neg w_2 \vee \neg x) \wedge (\neg w_3 \vee \neg x \vee y) \wedge (\neg w_4 \vee \neg x \vee \neg y) \wedge (w_1)$$

1. Add clause-selector variables.
2. Add AtMost constraint.
3. First solution : w_1 is **false**. Add blocking clause and a MCorr.

MCorrs

1. $\{(x)\}$

$$AtMost(\{w_1, w_2, w_3, w_4\}, 1)$$

Example: Computing all MCorrs

$$\varphi = (\neg w_1 \vee x) \wedge (\neg w_2 \vee \neg x) \wedge (\neg w_3 \vee \neg x \vee y) \wedge (\neg w_4 \vee \neg x \vee \neg y) \wedge (w_1)$$

1. Add clause-selector variables.
2. Add AtMost constraint.
3. First solution : w_1 is **false**. Add blocking clause and a MCorr.
4. No further solutions, increment AtMost.

MCorrs

1. $\{(x)\}$

$$AtMost(\{w_1, w_2, w_3, w_4\}, 2)$$

Example: Computing all MCorrs

$$\varphi = (\neg w_1 \vee x) \wedge (\neg w_2 \vee \neg x) \wedge (\neg w_3 \vee \neg x \vee y) \wedge (\neg w_4 \vee \neg x \vee \neg y) \wedge (w_1) \wedge (w_2 \vee w_3)$$

1. Add clause-selector variables.
2. Add AtMost constraint.
3. First solution : w_1 is **false**. Add blocking clause and a MCorr.
4. No further solutions, increment AtMost.
5. Second solution : w_2 and w_3 are **false**. Add blocking clause and another MCorrs.

MCorrs

1. $\{(x)\}$
2. $\{(\neg x), (\neg x \vee y)\}$

$$AtMost(\{w_1, w_2, w_3, w_4\}, 2)$$

Example: Computing all MCorrs

$$\varphi = (\neg w_1 \vee x) \wedge (\neg w_2 \vee \neg x) \wedge (\neg w_3 \vee \neg x \vee y) \wedge (\neg w_4 \vee \neg x \vee \neg y)$$
$$\wedge\, (w_1) \wedge (w_2 \vee w_3) \wedge (w_2 \vee w_4)$$

1. Add clause-selector variables.
2. Add AtMost constraint.
3. First solution : w_1 is **false**. Add blocking clause and a MCorr.
4. No further solutions, increment AtMost.
5. Second solution : w_2 and w_3 are **false**. Add blocking clause and another MCorrs.
6. Third solution : w_2 and w_4 are **false**. Add blocking clause and another MCorrs.

MCorrs

1. $\{(x)\}$
2. $\{(\neg x), (\neg x \vee y)\}$
3. $\{(\neg x), (\neg x \vee \neg y)\}$

$$AtMost(\{w_1, w_2, w_3, w_4\}, 2)$$

Example: Computing all MCorrs

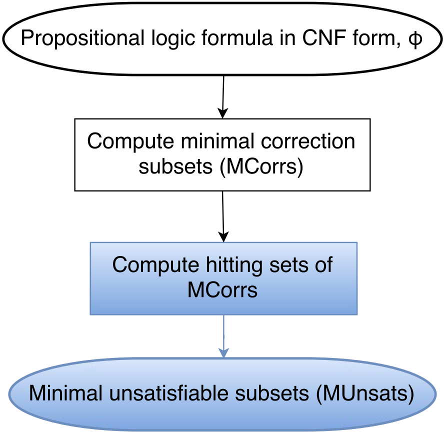
$$\varphi = (\neg w_1 \vee x) \wedge (\neg w_2 \vee \neg x) \wedge (\neg w_3 \vee \neg x \vee y) \wedge (\neg w_4 \vee \neg x \vee \neg y) \wedge (w_1) \wedge (w_2 \vee w_3) \wedge (w_2 \vee w_4)$$

1. Add clause-selector variables.
2. Add AtMost constraint.
3. First solution : w_1 is **false**. Add blocking clause and a MCorr.
4. No further solutions, increment AtMost.
5. Second solution : w_2 and w_3 are **false**. Add blocking clause and another MCorrs.
6. Third solution : w_2 and w_4 are **false**. Add blocking clause and another MCorrs.
7. No further solutions, even without AtMost constraint.

MCorrs

1. $\{(x)\}$
2. $\{(\neg x), (\neg x \vee y)\}$
3. $\{(\neg x), (\neg x \vee \neg y)\}$

FlowChart



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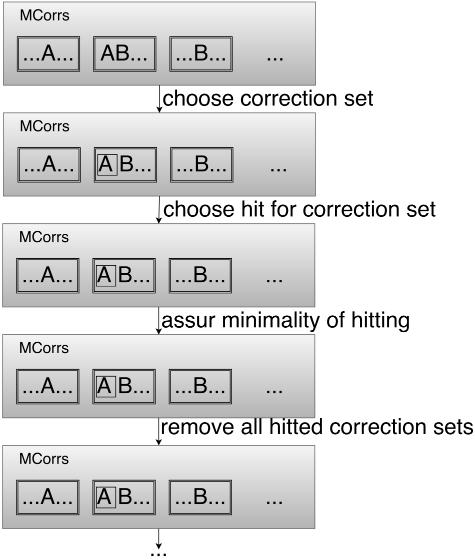
Compute minimal correction subsets (MCorrs)
Compute hitting sets of MCorrs
Minimal unsatisfiable subsets (MUnsats)

Example: Computing All Minimal Hitting Sets of MUnsats



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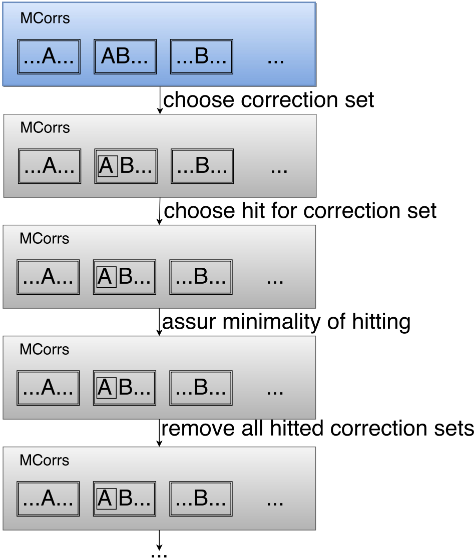

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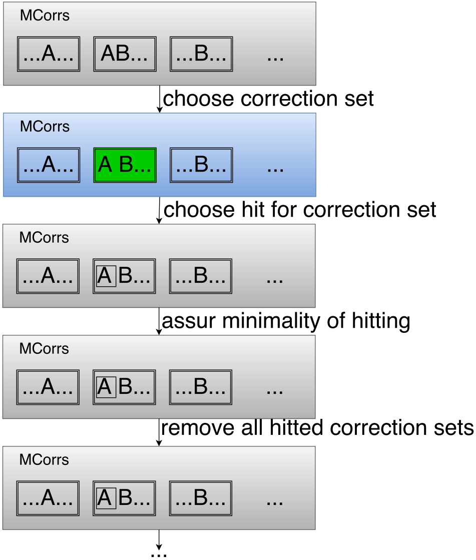

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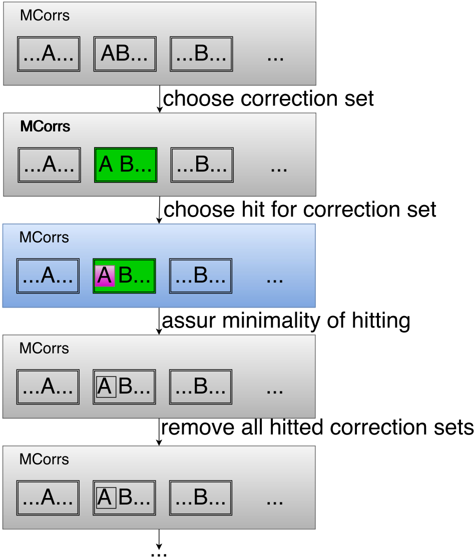

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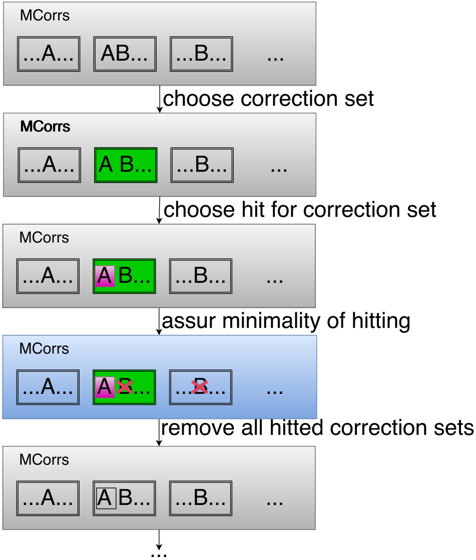
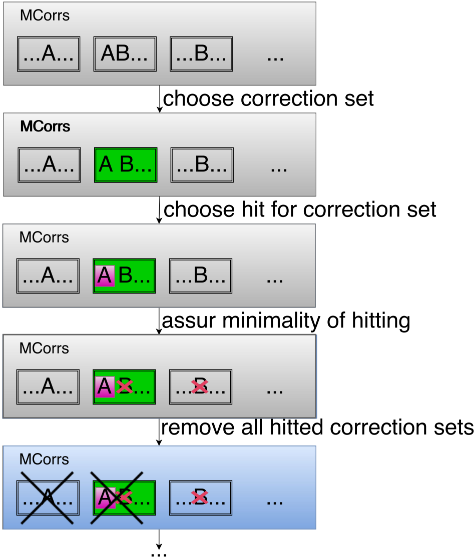


Example: Computing All Minimal Hitting Sets of MUnsats



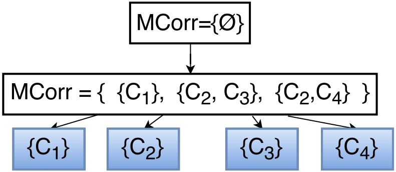
Example: Computing All Hitting Sets of MUnsats

MCorr={Ø}

$\downarrow$

MCorr = { $\{C_1\}$, $\{C_2, C_3\}$, $\{C_2, C_4\}$ }

Example: Computing All Hitting Sets of MUnsats



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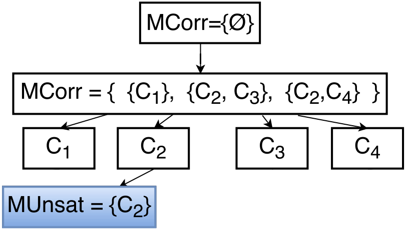

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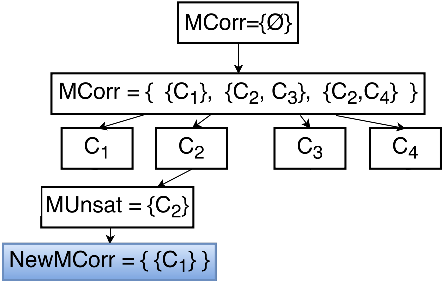

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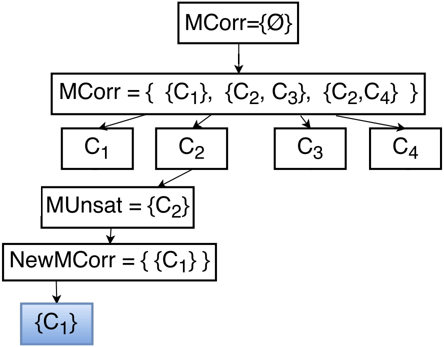

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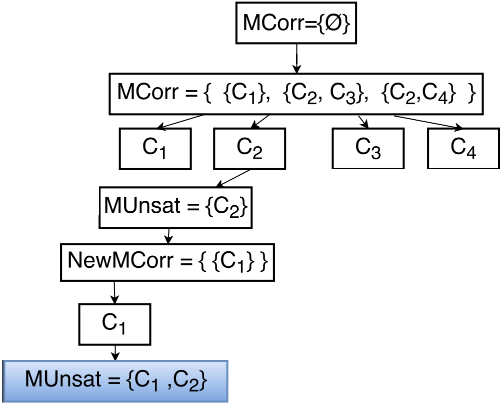

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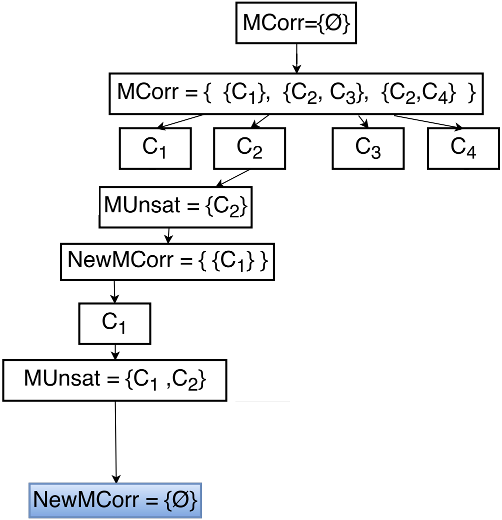

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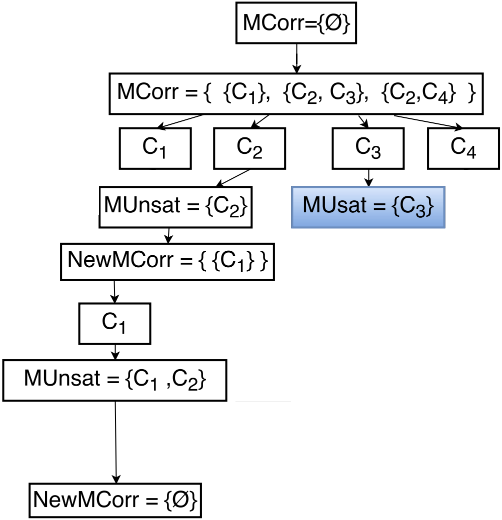

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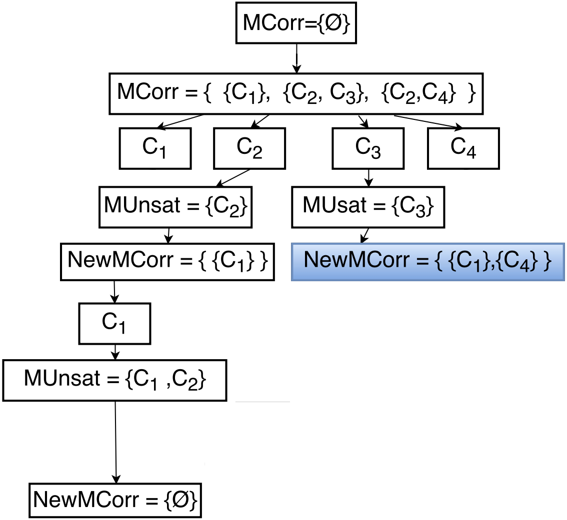

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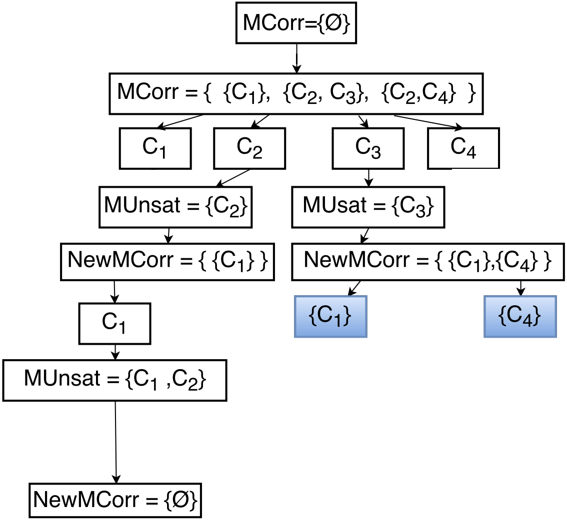

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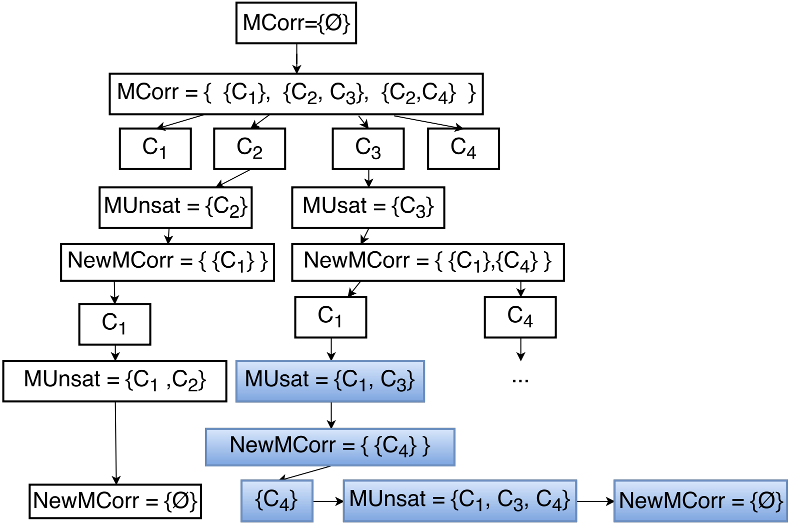

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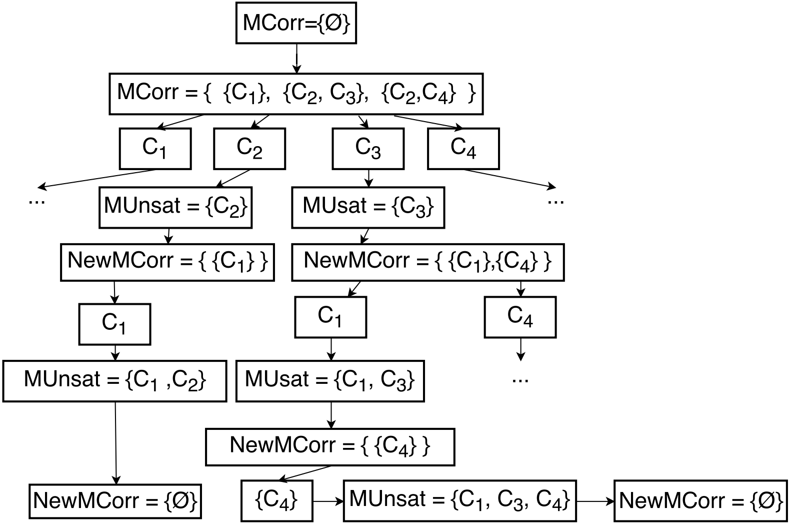

Reference

- Mark H. Liffiton, Karem A. Sakallah, *Algorithms for Computing Minimal Unsatisfiable Subsets of Constraints*. Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor 48109-2121.
- Alexander Nadel, Vadim Ryvchin, Ofer Strichman *Ultimately Incremental SAT*. https://ie.technion.ac.il/~ofers/publications/sat14.pdf