Computational Economics Lecture 4: Intro to Numerical Optimization

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Outline

1. **Motivation**
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3. **Brent's Method**
4. **Newton's Method**
5. **Gradient Descent**
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Motivation

- Numerical optimization is almost needed in every model to be solved
- Today, we will cover some of the most commonly used methods
- Please read the handout by Prof. Paul Klein: Notes on Numerical Optimization
- Think about that we just want to find a minimizer of a function
- We will first talk about the non-gradient-based method: Slow but reliable
- We will then cover the gradient-based method: faster but riskier
- In reality, we always use a mixture of some of these methods

Golden Section Search (Gradient Free)

- To find a minimizer in one dimension of a single-troughed known function $f(x)$
- The golden section search indicates the search location between points

$$\frac{a+b}{a} = \frac{a}{b} = \varphi.$$
$$\frac{1+\varphi}{\varphi} = \varphi.$$
$$\varphi^2 - \varphi - 1 = 0.$$
$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61803$$

- It starts with three points x_1, x_2, x_3 such that $f(x_1) > f(x_2)$ and $f(x_3) > f(x_2)$:

$$\frac{x_3 - x_2}{x_2 - x_1} = \varphi, \quad \text{initial construction}$$

- If f is single-troughed, we can be sure that the minimizer lies between x_1 and x_3
- The question is how to find the minimizer by updating the new point of search

Golden Section Search: Case 1

- It starts with three points x_1, x_2, x_3 such that $f(x_1) > f(x_2)$ and $f(x_3) > f(x_2)$:

$$\frac{x_3 - x_2}{x_2 - x_1} = \varphi, \quad \text{initial construction}$$

- We do not know if x_2 is the minimizer or not, so we keep searching
- The next point x_4 where we evaluate f should be chosen in this way too, i.e.

$$\frac{x_3 - x_4}{x_4 - x_2} = \varphi$$

- This is desirable: The next point we evaluate f is in the larger sub-interval (x_2, x_3).
- The idea is to minimize the significance of bad luck

Updating: Case 1

- Follow the golden search rule: $a + c = b \to c/a = a/b \to b/a = \varphi$
- If $f(x_4) = f_{4a}$, search interval $a + c$; if $f(x_4) = f_{4b}$, search interval c

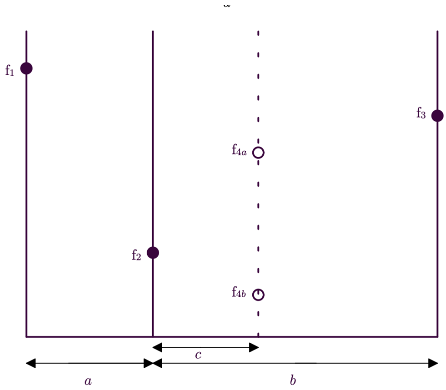

Figure 1: Golden section search: case 1

Updating: Case 2

- The Same rule applied if the opposite side of intervals
- If $f(x_4) = f_{4a}$, search interval $b + c$; if $f(x_4) = f_{4b}$, search interval $a - c$

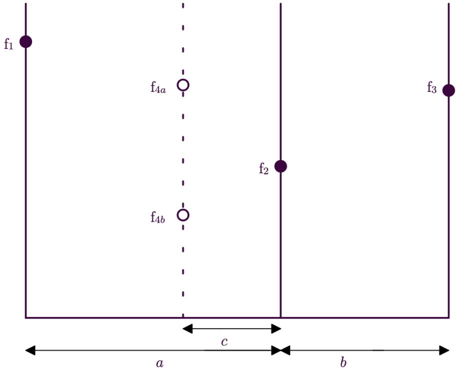

Figure 2: Golden section search: case 2

When to Stop?

- It is tempting to think that you can bracket the solution x^* in a range as small as $(1-\epsilon)x^* < x^* < (1+\epsilon)x^*$ where ϵ is machine precision. However, that is not so!

$$f(x) \approx f\left(x^*\right) + f'\left(x^*\right)\left(x - x^*\right) + \frac{1}{2} f''\left(x^*\right)\left(x - x^*\right)^2$$

- The second term is zero, and the third term will be negligible compared to the first (that is, will be a factor ϵ smaller and so will be an additive zero in finite precision) whenever

$$\frac{1}{2} f''\left(x^*\right)\left(x - x^*\right)^2 < \epsilon f\left(x^*\right)$$

or

$$\frac{x - x^*}{x^*} < \sqrt{\epsilon}\sqrt{\frac{2\left|f\left(x^*\right)\right|}{\left(x^*\right)^2 f''\left(x^*\right)}}$$

- Therefore, it is hopeless to ask for bracketing with a width of less than $\sqrt{\epsilon}$!

Brent's (Parabola) Method (Gradient-based)

- If $f(\cdot)$ is smooth (is continuously differentiable), then approximating it by a parabola and taking as the new approximation of the minimizer the minimizer of the parabola.
- Then the following number minimizes the parabola that goes through the points $(a, f(a))$, $(b, f(b))$ and $(c, f(c))$ for $a < b < c$:

$$x = b - \frac{1}{2} \frac{(b-a)^2[f(b) - f(c)] - (b-c)^2[f(b) - f(a)]}{(b-a)[f(b) - f(c)] - (b-c)[f(b) - f(a)]}$$

- If $a < x < b$, then the new three points are a, x and b
- If $b < x < c$, then the new three points are b, x and c
- The above parabola method is fast but could easily become numerically unstable.
- Brent's method switches between inverse parabolic interpolation (when it is "acceptable") as above and golden section search (safe choice).

Newton's Method (Gradient-based)

- Newton's method of root finding to the equation $f'(x) = 0$
- The first-order approximation around the n^{th} approximation x_n of the true solution x^* is

$$f'(x^*) \approx f'(x_n) + f''(x_n) * (x^* - x_n)$$

  where $f'(x_n)$ is the gradient of f at x_n and $f''(x_n)$ is the Hessian at x_n.

- The gradient $\nabla f(x)$ of a function is that it points in the direction of steepest ascent at x
- Evidently $f'(x^*) = 0$, so we can solve for $\Delta x_n = x^* - x_n$ by solving

$$f''(x_n)\,\Delta x_n = -f'(x_n)$$

  and then defining

$$x_{n+1} = x_n + \gamma \Delta x_n$$

  where $0 < \gamma \leq 1$ is the choice of how greedy we are (too greedy is not good)

Gradient Descent

- To descend on the gradient $\nabla f(x)$, we of course want to move in the opposite direction
- The only question is how far
- Denoting the distance traveled in each iteration by α_k:

$$x_{k+1} = x_k - \alpha_k \nabla f\left(x_k\right)$$

- But how to choose α_k ?
- A natural option would seem to be to minimize f along the line $x = x_k - \alpha \nabla f\left(x_k\right)$ and that is precisely what is typically done, i.e. α_k solves

$$\min_{\alpha} f\left(x_k - \alpha \nabla f\left(x_k\right)\right)$$

- This problem can be solved using the method of the golden section or Brent's method
- It is also quite good for two-dimensional problems
- For higher dimensions, please read the Nelder-Mead Method (self-read)

In Practice

- What do we need to know about the optimization methods?
- (1) In old times (two decades before), everybody tended to write their own optimization
- (2) We still want you to understand what happens when you call packages
- Optimization in practice:
  - (1) Stay in lower dimensions as possible (Cut into steps/dimension reduction)
  - (2) Use gradient-based methods as possible (Limit discontinuity in your problem)
  - (3) Well-define the support for your solution (Constrain your solution)
- We will figure out details in the following lectures