Computational Economics Lecture 12: Advanced Methods for Heterogeneous Agent Models with Aggregate Uncertainty

(just touching base here; for self-study if interested)

Min Fang

University of Florida

Spring 2025

Outline

1. **Motivation**
2. **State-space Approaches**
3. **Sequence-space Approaches**
4. **Deep Learning Approaches**
5. **Conclusions**

Motivation

- Why are people so obsessed with solving the HAM models?
- Because it is on the intersection of micro and macro
    - Macro: having more reliable micro-foundations and rich heterogeneities
    - Micro: could see its effects on the aggregate economy and vice versa
- Again, the task is still very difficult!
- And (Krusell and Smith, 1998)-ish methods are not very scalable and reliable
- The field developed different approaches, generally in three directions:
    - **State-space Approaches**
    - **Sequence-space Approaches**
    - **Deep Learning Approaches**
- We will briefly introduce them today, since most of you won't need it

State-space Approaches: Overview

- What are state-space models?
    - State variables: that summarize all relevant past information
    - Law of motion: for how the state evolves
    - Decision rules: mapping states to optimal choices
- For instance, (Krusell and Smith, 1998) is one example of state-space models. But it is a very costly-to-compute state-space model by carrying aggregate states.
- You solve for all possible (individual and aggregate) states in the KS model
- This is harder when the individual states S is large
- But it is quite efficient under many aggregate states N
- A key concept is "**perturbation**" (Dynare-ish approach) of the aggregate dynamics

State-space Approaches: Perturbation

- What is perturbation?
    - Given that the steady state has been calculated
    - Take a Taylor expansion of the policy rules
    - Find the coefficients of this Taylor expansion
    - It is as if the policy rules are perturbed around the steady state
- This is not new! It is the basic method for Dynare! (Schmitt-Grohé and Uribe, 2004)
- We used it last semester on *log-linearized Neoclassical Growth Models*
- What is the difference?
- The difference is that we only perturb the aggregate states
- But we solve the individual policy rules fully

State-space Approaches: Contributions

- The following papers contributed a LOT to this direction:
  - Michael Reiter. "Solving heterogeneous-agent models by projection and perturbation.", Journal of Economic Dynamics and Control, 2009 (Reiter, 2009)
  - Thomas Winberry. "A method for solving and estimating heterogeneous agent macro models.", Quantitative Economics, 2018 (Winberry, 2018)
  - Ahn, SeHyoun, Greg Kaplan, Benjamin Moll, Thomas Winberry, and Christian Wolf. "When inequality matters for macro and macro matters for inequality." NBER Macroeconomics Annual 32, 2018 (Ahn et al., 2018)
  - Bayer, Christian, and Ralph Luetticke. "Solving discrete time heterogeneous agent models with aggregate risk and many idiosyncratic states by perturbation." Quantitative Economics, 2020 (Bayer and Luetticke, 2020)

State-space Approaches: Computation

- Step 1: Approximate the model's equilibrium objects - the distribution, law of motion, firm value, Bellman equation (often infinite-dimensional)- using finite-dimensional global approximations concerning individual state variables.
- Step 2: Compute the stationary equilibrium of the approximated model without aggregate shocks but still with idiosyncratic shocks.
- Step 3: Compute the aggregate dynamic of the approximated model by perturbing it around the stationary equilibrium (just use Dynare!).

Sequence-space Approaches: Overview

- What are sequence-space models?
  - Sequence variables: that summarize all relevant responses to MIT aggregate shocks
  - MIT aggregate shocks: for how the sequence variables evolve
  - Decision rules: mapping shocks to optimal impulse responses
- This is harder when the aggregate states N is large
- But it is quite efficient under many individual states S
- A key concept is "**Sequence-Space Jacobian**"

State-space Approaches: Sequence-Space Jacobian

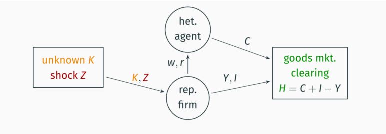
- What is the Sequence-Space Jacobian?
  - Math: it is just a fancy name for a sequence of Jacobian matrices of a vector-valued function of several variables, of all their first-order partial derivatives
  - Economics: the derivatives of perfect foresight equilibrium mappings between aggregate sequences around the steady state
- This is not new! It is the basic idea in MIT shocks! (Boppart, Krusell, and Mitman, 2018)
- We did not explicitly use it, but we have done IRFs of MIT shocks
- What is the difference?
- The difference is that we want to calculate SSJs directly
- But we solve the individual policy rules fully

Sequence-space Approaches: Contributions

- The following papers contributed a LOT to this direction:
  - Boppart, Timo, Per Krusell, and Kurt Mitman. "Exploiting MIT shocks in heterogeneous-agent economies: the impulse response as a numerical derivative." Journal of Economic Dynamics and Control, 2018 (Boppart, Krusell, and Mitman, 2018)
  - Auclert, Adrien, Bence Bardóczy, Matthew Rognlie, and Ludwig Straub. "Using the sequence-space Jacobian to solve and estimate heterogeneous-agent models." Econometrica, 2021 (Auclert et al., 2021)
  - Auclert, Adrien, Evan Majic, Matt Rognlie, Ludwig Straub. "Thinking Big: Determinacy and Large-Scale Solutions in the Sequence Space." Working Paper

Sequence-space Approaches: Computation

- Step 1: Write the model as a collection of blocks along a DAG (directed acyclic graph).



- Step 1: Solve the problems of each block given the inputs and calculate the aggregate outputs of the block, which only depend on the inputs. For example, the het. agent block only takes in $\{w, r\}$ and outputs $\{C\}$.
- Step 2: Compute all the necessary Jacobians
- Step 3: Use Jacobians to do what ever you like
  (three slides below, credit to Marlon Azinovic-Yang)

Sequence-space Approaches: Computation Step 2

How does the model react to a sequence of shocks d$\mathbf{Z}$?
$d\mathbf{K}$ needs to be consistent with $\mathbf{H}(\mathbf{K}, \mathbf{Z}) = 0$.
By the implicit function theorem

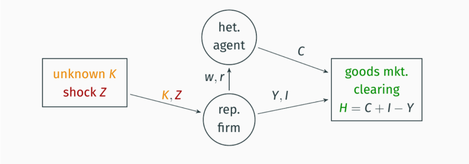
$$d\mathbf{K} = -\left(\underbrace{\frac{\partial \mathbf{H}}{\partial \mathbf{K}}}_{(n_{targets}T)\times(n_{targets}T)}\right)^{-1} \left(\frac{\partial \mathbf{H}}{\partial \mathbf{Z}}\right) d\mathbf{Z} \tag{2}$$

So what's a good way to get $\frac{\partial \mathbf{H}}{\partial \mathbf{K}}$ and $\frac{\partial \mathbf{H}}{\partial \mathbf{Z}}$?

Compute the sequence space Jacobians for each block and chain them with the chain rule!

Sequence-space Approaches: Computation Step 2

A block Jacobian is the derivatives of its outputs wrt its inputs.



- Heterogeneous agent block: two Jacobians: $\mathcal{J}^{C,w}_{t,s} := \frac{\partial C_t}{\partial w_s}$, $\mathcal{J}^{C,r}_{t,s} := \frac{\partial C_t}{\partial r_s}$
- Firm block: eight Jacobians: $\underbrace{\mathcal{J}^{w,\mathbf{K}}}_{T\times T}, \mathcal{J}^{w,\mathbf{Z}}, \mathcal{J}^{r,\mathbf{K}}, \mathcal{J}^{r,\mathbf{Z}}, \mathcal{J}^{Y,\mathbf{K}}, \mathcal{J}^{Y,\mathbf{Z}}, \mathcal{J}^{I,\mathbf{K}}, \mathcal{J}^{I,\mathbf{Z}}$
- Market clearing block: three Jacobians: $\mathcal{J}^{\mathbf{H},C}$, $\mathcal{J}^{\mathbf{H},I}$, $\mathcal{J}^{\mathbf{H},Y}$
- Now we use the chain rule to get

$$\frac{\partial \mathbf{H}}{\partial \mathbf{K}} = \mathcal{J}^{\mathbf{H},C}\mathcal{J}^{C,r}\mathcal{J}^{r,\mathbf{K}} + \mathcal{J}^{\mathbf{H},C}\mathcal{J}^{C,w}\mathcal{J}^{w,\mathbf{K}} + \mathcal{J}^{\mathbf{H},I}\mathcal{J}^{I,\mathbf{K}} + \mathcal{J}^{\mathbf{H},Y}\mathcal{J}^{Y,\mathbf{K}} \tag{3}$$

$$\frac{\partial \mathbf{H}}{\partial \mathbf{Z}} = \mathcal{J}^{\mathbf{H},C}\mathcal{J}^{C,r}\mathcal{J}^{r,\mathbf{Z}} + \mathcal{J}^{\mathbf{H},C}\mathcal{J}^{C,w}\mathcal{J}^{w,\mathbf{Z}} + \mathcal{J}^{\mathbf{H},I}\mathcal{J}^{I,\mathbf{Z}} + \mathcal{J}^{\mathbf{H},Y}\mathcal{J}^{Y,\mathbf{Z}} \tag{4}$$

Sequence-space Approaches: Computation Step 3

Suppose we want GE response to shock $d\mathbf{Z}$ (e.g. AR(1)).
$\mathbf{K}$ needs to be consistent with $\mathbf{H}(\mathbf{K}, \mathbf{Z}) = 0$.
Equipped with all the Jacobians, we can first get

$$d\mathbf{K} = -\left(\underbrace{\frac{\partial \mathbf{H}}{\partial \mathbf{K}}}_{(n_{targets}T)\times(n_{targets}T)} \right)^{-1} \left(\frac{\partial \mathbf{H}}{\partial \mathbf{Z}} \right) d\mathbf{Z} \tag{5}$$

and can than get any other GE impulse response we want from plain matrix multiplications. E.g. IRF of output:

$$d\mathbf{Y} = \mathcal{J}^{Y,\mathbf{K}} d\mathbf{K} + \mathcal{J}^{Y,\mathbf{Z}} d\mathbf{Z}. \tag{6}$$

Deep-learning Approaches: Overview

- Deep learning folks want to solve the problems more generally (and more brutally)
- Goal: directly approximate the policy (or value) functions using deep learning
- The basic setup IS as follows:
- 1. Natural sampling: simulated paths of the economy or grid points.
- 2. Natural loss function: the implied error in the equilibrium/optimality conditions.
- 3. Natural minimization algorithm: Stochastic gradient descent or variations.

Conclusions

- Many advanced methods to solve much harder questions
- Quite exciting emergence of these new methods in last 10 years!
- See the handout folder for additional slides

Appendix

References

Ahn, SeHyoun et al. (2018). “When inequality matters for macro and macro matters for inequality”. In: *NBER macroeconomics annual* 32.1, pp. 1–75.

Auclert, Adrien et al. (2021). “Using the sequence-space Jacobian to solve and estimate heterogeneous-agent models”. In: *Econometrica* 89.5, pp. 2375–2408.

Bayer, Christian and Ralph Luetticke (2020). “Solving discrete time heterogeneous agent models with aggregate risk and many idiosyncratic states by perturbation”. In: *Quantitative Economics* 11.4, pp. 1253–1288.

Boppart, Timo, Per Krusell, and Kurt Mitman (2018). “Exploiting MIT shocks in heterogeneous-agent economies: the impulse response as a numerical derivative”. In: *Journal of Economic Dynamics and Control* 89, pp. 68–92.

Krusell, Per and Anthony A Smith Jr (1998). “Income and wealth heterogeneity in the macroeconomy”. In: *Journal of political Economy* 106.5, pp. 867–896.

Reiter, Michael (2009). “Solving heterogeneous-agent models by projection and perturbation”. In: *Journal of Economic Dynamics and Control* 33.3, pp. 649–665.

Schmitt-Grohé, Stephanie and Martın Uribe (2004). "Solving dynamic general equilibrium models using a second-order approximation to the policy function". In: *Journal of economic dynamics and control* 28.4, pp. 755–775.

Winberry, Thomas (2018). "A method for solving and estimating heterogeneous agent macro models". In: *Quantitative Economics* 9.3, pp. 1123–1151.