Computational Economics Lecture 7: Heterogeneous Agent Models without Aggregate Uncertainty

Min Fang

University of Florida

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Outline

1. **Motivation**
2. **The Aiyagari Model**
3. **Stationary Distribution**
4. **Transitional Dynamics**

Motivation

- Many models involve heterogeneous agents instead of one agent
- The reason is nowadays very obvious:
  - The reality features agent heterogeneity and has fewer assumptions
  - Heterogeneous agent models could match the microdata much better
  - Heterogeneous agent models connect macro to other fields much better
- We will start with a canonical heterogeneous agent model: (Aiyagari, 1994)
- It is a general equilibrium heterogeneous household model with incomplete markets of household consumption-savings problem without aggregate uncertainty in the economy
- It is usually the first heterogeneous agent (household) model everyone learned
- Later, we will learn how to do heterogeneous firms (just as I showed)

Outline of the Model

- Mainly follow lectures from Prof. Jesus Fernandez-Villaverde
- Continuum of households (vs. models with finite number/types of agents)
- One firm renting aggregate capital
- No aggregate uncertainty
- Individuals are subject to idiosyncratic shocks to their labor income
- Incomplete markets (can only trade finite type of assets)

Households

- Continuum of measure 1 of households.
- Preferences for household i :

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

- Budget constraint:

$$c_t + a_{t+1} = w_t y_t + (1 + r_t) a_t$$

- We could consider a hand-to-mouth (i.e., autarky) variation: $c_t = w_t y_t$.
- Initial conditions $y_0, a_0 \geq 0$.
- Borrowing constraint $a_{t+1} \geq 0$.

Labor Endowment

- Stochastic labor endowment process $\{y_t\}_{t=0}^{\infty}$:

$$y_t \in Y = \{y_1, y_2, \ldots y_N\}$$

- Markov process with transitions $\pi\left(y' \mid y\right) > 0$
- Common for all households, but realizations are specific for each individual
- Law of large numbers: $\pi\left(y' \mid y\right)$ is also the deterministic fraction of the population that has this particular transition
- Unique stationary distribution associated with π, denoted by Π
- Total labor endowment in the economy at each point of time:

$$L = \sum_{y} \Pi(y) y$$

One Firm

- Perfectly competitive firm with neoclassical technology:

$$Y_t = F\left(K_t, L_t\right)$$

- Depreciation rate: $0 < \delta < 1$.
- Aggregate resource constraint:

$$C_t + K_{t+1} - (1-\delta)K_t = F\left(K_t, L_t\right)$$

- The only net asset in the economy is physical capital
- No state-contingent claims (i.e. incomplete markets)
- Households have all the ownership of the firm

Recursive Formulation

- (a, y): household state
- $\Phi(a, y)$: aggregate state variable
- $A = [0, \infty)$: set of possible asset holdings
- $B(A)$: Borel σ-algebra of A
- Y: set of possible labor endowment realizations
- $P(Y)$: power set of Y
- $Z = A \times Y$ and $B(Z) = P(Y) \times B(A)$
- $\mathcal{M}$ the set of all probability measures on the measurable space $(Z, B(Z))$

Recursive Formulation

- Household problem in recursive formulation:

$$v(a, y; \Phi) = \max_{c \geq 0, a' \geq 0} u(c) + \beta \sum_{y' \in Y} \pi\left(y' \mid y\right) v\left(a', y'; \Phi'\right)$$

$$\text{s.t. } c + a' = w(\Phi) y + (1 + r(\Phi)) a$$

$$\Phi' = H(\Phi)$$

- Function $H : \mathcal{M} \rightarrow \mathcal{M}$ is called the aggregate "law of motion"

- Note the complexity of the operator

Recursive Competitive Equilibrium

A RCE is value function $v : Z \times \mathcal{M} \to R$, household policy functions $a', c : Z \times \mathcal{M} \to R$, firm policy functions $K, L : \mathcal{M} \to R$, pricing functions $r, w : \mathcal{M} \to R$ and law of motion $H : \mathcal{M} \to \mathcal{M}$ s.t.

1. v, a', c are measurable with respect to $\mathcal{B}(Z)$, v satisfies Bellman equation and a', c are the policy functions, given $r()$ and $w()$.
2. K, L satisfy, given $r()$ and $w()$

$$\begin{aligned} r(\Phi) &= F_K(K(\Phi), L(\Phi)) - \delta \\ w(\Phi) &= F_L(K(\Phi), L(\Phi)) \end{aligned}$$

3. For all $\Phi \in \mathcal{M}$, $L(\Phi) = \int y d\Phi$ and

$$K'\left(\Phi'\right) = K(H(\Phi)) = \int a'(a, y; \Phi) d\Phi$$

$$\int c(a, y; \Phi) d\Phi + \int a'(a, y; \Phi) d\Phi = F(K(\Phi), L(\Phi)) + (1 - \delta) K(\Phi)$$

4. Aggregate law of motion H is generated by π and a'.

Transition Functions

- Define transition function $Q_\Phi : Z \times \mathcal{B}(Z) \to [0,1]$ by

$$Q_\Phi((a,y),(\mathcal{A},\mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \begin{cases} \pi(y' \mid y) & \text{if } a'(a,y;\Phi) \in \mathcal{A} \\ 0 & \text{else} \end{cases}$$

for all $(a,y) \in Z$ and all $(\mathcal{A},\mathcal{Y}) \in \mathcal{B}(Z)$.

- $Q_\Phi((a,y),(\mathcal{A},\mathcal{Y}))$ is the probability that an agent with current assets a and current income y ends up with assets a' in $\mathcal{A}$ tomorrow and income y' in $\mathcal{Y}$ tomorrow.

- Hence

$$\begin{aligned} \Phi'(\mathcal{A},\mathcal{Y}) &= (H(\Phi))(\mathcal{A},\mathcal{Y}) \\ &= \int Q_\Phi((a,y),(\mathcal{A},\mathcal{Y}))\Phi(da \times dy) \end{aligned}$$

Stationary RCE

A stationary RCE is value function $v : Z \to R$, household policy functions $a', c : Z \to R$, firm policies K, L, prices r, w and a measure $\Phi \in \mathcal{M}$ such that

1. v, a', c are measurable with respect to $B(Z)$, v satisfies the household's Bellman equation and a', c are associated policy functions, given r, w.
2. K, L satisfy, given r, w :
$$r = F_k(K, L) - \delta$$
$$w = F_L(K, L)$$
3. $L = \int y d\Phi$ and $K = \int a'(a, y) d\Phi$ and
$$\int c(a, y) d\Phi + \int a'(a, y) d\Phi = F(K, L) + (1 - \delta)K$$
4. Let Q be transition function induced by π and a'. $\forall (\mathcal{A}, \mathcal{Y}) \in \mathcal{B}(Z)$
$$\Phi(\mathcal{A}, \mathcal{Y}) = \int Q((a, y), (\mathcal{A}, \mathcal{Y})) d\Phi$$

Characterizing the Stationary RCE

- Recall that L is exogenously given.
- Thus, from
$$r = F_k(K, L) - \delta, \quad w = F_L(K, L)$$
we can get w as a function of r (with $w'(r) < 0$).
- Example:
$$Y = K^{\alpha} L^{1-\alpha}$$
with
$$r = \alpha K^{\alpha-1} L^{1-\alpha} - \delta \Rightarrow K = \left(\frac{r+\delta}{\alpha} \right)^{\frac{1}{\alpha-1}} L$$
and
$$w = (1-\alpha) K^{\alpha} L^{1-\alpha} = (1-\alpha) \alpha^{\frac{\alpha}{1-\alpha}} (r+\delta)^{\frac{\alpha}{\alpha-1}} L$$

Existence and Uniqueness

- By Walras' law, we only need to check input market clearing
- Define asset market clearing condition:

$$K = K(r) = \int a'(a, y) d\Phi \equiv Ea(r)$$

- Then

$$r = F_k(K(r), L) - \delta$$

- Existence and uniqueness of stationary RCE boils down to one equation
- From assumptions on production function, $K(r)$ is continuous, strictly decreasing function on $r \in (-\delta, \infty)$ with

$$\lim_{r \to -\delta} K(r) = \infty$$

$$\lim_{r \to \infty} K(r) = 0$$

- We can ensure existence but cannot ensure uniqueness (leave for self-study)

Interest Rate in Equilibrium

- Complete markets model: $r^{CM} = \frac{1}{\beta} - 1$
- With incomplete markets: $r^{*} < r^{CM}$
- Why? Over-accumulation of capital and over-saving (because of precautionary reasons: liquidity constraints, prudence, or both)
- Policy implications

Computation of the Canonical Aiyagari Model

It involves three steps:

1. Fix an $r \in \left(-\delta, \frac{1}{\beta} - 1\right)$. For a fixed r, solve the household's recursive problem. This yields a value function v_r and decision rules a'_r, c_r.
2. The policy function a'_r and π induce Markov transition function Q_r. Compute the unique stationary measure Φ_r associated with this transition function.
3. Compute aggregation and calculate the excess demand for capital:

$$d(r) = K(r) - Ea(r)$$

   If small enough, stop; if not, adjust r (Which direction?)

Step 1: Solve the Household's Recursive Problem

It is very standard as what you have learned so far

- Any acceptable solution method for recursive problems is valid: i.e., VFI
- However, speed is at a premium
- Thus, value function iteration might not be fast enough
- Standard "tricks": monotonicity and concavity
- Be smart about initial guesses in the updates
- Fix variable values in steady state, not parameters!
- Also, explore multi-grid schemes

Step 2: Compute the Stationary Distribution

- Grid. Suppose $A = \{a_1, \ldots, a_M\}$.
- Then Φ is $M * N \times 1$ column vector and $Q = \left(q_{ij,kl}\right)$ is $M * N \times M * N$ matrix with

$$q_{ij,kl} = \Pr\left(\left(a', y'\right) = (a_k, y_l) \mid (a, y) = (a_i, y_l)\right)$$

- Stationary measure Φ satisfies matrix equation

$$\Phi = Q^T \phi$$

- Φ is (rescaled) eigenvector associated with eigenvalue $\lambda = 1$ of Q^T
- Q^T is a stochastic matrix and thus has at least one unit eigenvalue. If it has more than one unit eigenvalue, a continuum of stationary measures

Step 2: Compute the Stationary Distribution

- Variation of grid method I: allocate mass between two grid points according to relative distance (It is time to remember the Young method)
- Variation of grid method II: uniform mass between two grid points
- Both cases: sufficiently small grid; otherwise, no convergence
- Simulation (Stochastic or Non-Stochastic)
- Convergence

Transitional Dynamics

- Often, we are interested in the effects of the change in a parameter of the model (transitory or permanent).
- We want to compute both the new steady state and the transitional dynamics.
- Example: permanent introduction of a capital income tax at rate τ. Receipts are rebated lump-sum to households as the government transfers T.

Model with a Capital Income Tax

- State space: $Z = Y \times \mathbf{R}_+$, the set of all possible (y, a)
- Let $\mathcal{B}(Z) = \mathcal{P}(Y) \times \mathcal{B}(\mathbf{R}_+)$and $\mathbf{M}$ be the set of all finite measures on the measurable space $(Z, \mathcal{B}(Z))$
- Household problem:

$$v_t(a, y) = \max_{c \geq 0, a' \geq 0} u(c) + \beta \sum_{y' \in Y} \pi\left(y' \mid y\right) v_{t+1}\left(a', y'\right)$$

$$\text{s.t. } c + a' = w_t y + \left(1 + \left(1 - \tau_t\right) r_t\right) a + T_t$$

A Competitive Equilibrium with Taxes

Given initial distribution Φ_0 and fiscal legislation $\{\tau_t\}_{t=0}^{\infty}$, a competitive equilibrium is sequence of functions for the household $\{v_t, c_t, a_{t+1} : Z \to \mathbf{R}\}_{t=0}^{\infty}$, sequence of firm production plans $\{L_t, K_t\}_{t=0}^{\infty}$, factor prices $\{w_t, r_t\}_{t=0}^{\infty}$, government transfers $\{T_t\}_{t=0}^{\infty}$, and sequence of measures $\{\Phi\}_{t=1}^{\infty}$ s.t. $\forall t$,

- Given $\{w_t, r_t\}$ and $\{T_t, \tau_t\}$ the functions $\{v_t\}$ solve Bellman equation in t and $\{c_t, a_{t+1}\}$ are associated policy functions.
- Prices w_t and r_t satisfy

$$w_t = F_L(K_t, L_t)$$
$$r_t = F_K(K_t, L_t) - \delta$$

- Government Budget Constraint: for all $t \geq 0$

$$T_t = \tau_t r_t K_t$$

A Competitive Equilibrium with Taxes

- Market Clearing:

$$\int c_t\left(a_t, y_t\right) d\Phi_t + K_{t+1} = F\left(K_t, L_t\right) + (1-\delta) K_t$$

$$L_t = \int y_t d\Phi_t$$

$$K_{t+1} = \int a_{t+1}\left(a_t, y_t\right) d\Phi_t$$

- Aggregate Law of Motion: Define Markov transition functions $Q_t : Z \times \mathcal{B}(Z) \rightarrow [0,1]$ induced by the transition probabilities π and optimal policy $a_{t+1}(y, a)$ as

$$Q_t((a, y),(\mathcal{A}, \mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \begin{cases} \pi\left(y' \mid y\right) & \text{if } a_{t+1}(a, y) \in \mathcal{A} \\ 0 & \text{else} \end{cases}$$

for all $(a, y) \in Z$ and all $(\mathcal{A}, \mathcal{Y}) \in \mathcal{B}(Z)$. Then for all $(\mathcal{A}, \mathcal{Y}) \in \mathcal{B}(Z)$

$$\Phi_{t+1}(\mathcal{A}, \mathcal{Y}) = \left[\Gamma_t\left(\Phi_t\right)\right](\mathcal{A}, \mathcal{Y}) = \int Q_t((a, y),(\mathcal{A}, \mathcal{Y})) d\Phi_t$$

Stationary Equilibrium and Transitions

- A stationary equilibrium is an equilibrium such that all elements of the equilibrium that are indexed by t are constant over time.
- Transitions are likely to be asymptotic.
- However, assume that after T periods, the transition from old to new stationary equilibrium is completed.
- Under the assumption $v_T = v_\infty$, for a given sequence of prices $\{r_t, w_t\}_{t=1}^{T}$ household problem can be solved backwards.

Transition Dynamics Computation: Step 1

- Fix T.
- Compute stationary equilibrium $\Phi_0, v_0, r_0, w_0, K_0$ associated with $\tau = \tau_0 = 0$.
- Compute stationary equilibrium $\Phi_\infty, v_\infty, r_\infty, w_\infty, K_\infty$ associated with $\tau_\infty = \tau$. Assume that

$$\Phi_T, v_T, r_T, w_T, K_T = \Phi_\infty, v_\infty, r_\infty, w_\infty, K_\infty$$

- Guess sequence of capital stocks $\left\{\hat{K}_t\right\}_{t=1}^{T-1}$ The capital stock at time $t = 1$ is determined by decisions at time 0, $\hat{K}_1 = K_0$. Note that $L_t = L_0 = L$ is fixed. We also obtain

$$\hat{w}_t = F_L\left(\hat{K}_t, L\right)$$
$$\hat{r}_t = F_K\left(\hat{K}_t, L\right) - \delta$$
$$\hat{T}_t = \tau_t \hat{r}_t \hat{K}_t$$

Transition Dynamics Computation: Step 2

- Since we know $v_T(a, y)$ and $\left\{\hat{r}_t, \hat{w}_t, \hat{T}_t\right\}_{t=1}^{T-1}$ we can solve for $\{\hat{v}_t, \hat{c}_t, \hat{a}_{t+1}\}_{t=1}^{T-1}$ backwards.
- With policy functions $\{\hat{a}_{t+1}\}$ define transition laws $\left\{\hat{\Gamma}_t\right\}_{t=1}^{T-1}$. We know $\Phi_0 = \Phi_1$ from the initial stationary equilibrium. Iterate the distributions forward

$$\hat{\Phi}_{t+1} = \hat{\Gamma}_t\left(\hat{\Phi}_t\right)$$

  for $t = 1, \ldots, T-1$.
- With $\left\{\hat{\Phi}_t\right\}_{t=1}^{T}$ we can compute, for $t = 1, \ldots, T$.

$$\hat{A}_t = \int a d\hat{\Phi}_t$$

Transition Dynamics Computation: Step 3

- Test
$$\max_{1\leq t<T}\left|\hat{A}_t - \hat{K}_t\right| < \varepsilon$$
If yes, go to nthe ext step. If not, adjust your guesses for $\left\{\hat{K}_t\right\}_{t=1}^{T-1}$.

- Test
$$\left\|\hat{\Phi}_T - \Phi_T\right\| < \varepsilon$$
If yes, the transition converges smoothly into the new steady state and we are done and should save $\left\{\hat{v}_t, \hat{a}_{t+1}, \hat{c}_t, \hat{\Phi}_t, \hat{r}_t, \hat{w}_t, \hat{K}_t\right\}$. If not, increase T.

- We can be smart with the initial guess: compute associated RA transition.

Appendix

References

Aiyagari, S Rao (1994). “Uninsured idiosyncratic risk and aggregate saving”. In: *The Quarterly Journal of Economics* 109.3, pp. 659–684.