Computational Economics Lecture 8:
Heterogeneous Agent Models with Aggregate Uncertainty

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Outline

Motivation

- We have learned how to solve the Aiyagari model (no aggregate uncertainty)
- However, in reality, aggregate uncertainty is everywhere: booms, recessions, monetary/fiscal policies, industrial policies, educational policies, etc
- We need to be able to solve models with aggregate uncertainty
- We would like to start with a basic model: Krusell and Smith, 1998

The Krusell-Smith Model

- A bare bone model with aggregate uncertainty (two states)
- Aggregate production function

$$Y_t = s_t F(K_t, L_t)$$

  where $\{s_t\}$ is a sequence of random variables

- Let

$$s_t \in \{s_b, s_g\} = S$$

  with $s_b < s_g$ and conditional probabilities $\pi(s' \mid s)$.

- s_b is an economic recession and s_g is an expansion
- Easy to extend to richer specifications

The Krusell-Smith Model

- Idiosyncratic labor productivity y_t

$$y_t \in Y = \{y_u, y_e\}, \quad \text{with} y_u < y_e$$

- y_u stands for the household being unemployed and y_e stands for being employed.
- The distribution of y_t is correlated with aggregate productivity s_t
- Probability of being unemployed is higher during recessions than during expansions
- Let π be a 4×4 matrix with entry

$$\pi\left(y', s' \mid y, s\right) > 0$$

  that gives the conditional probability of individual productivity y', aggregate state s' tomorrow, conditional on (y, s) today

Cross-sectional Distributions

- Consistency requires that:

$$\sum_{y' \in Y} \pi\left(y', s' \mid y, s\right) = \pi\left(s' \mid s\right) \text{ all } y \in Y, \text{ all } s, s' \in S$$

- Law of large numbers: idiosyncratic risk averages out, only aggregate risk determines the number of agents in states $y \in Y$.

- Assume that, cross-sectionally, the fraction of the population in idiosyncratic state $y = y_u$ is only a function of the aggregate state s. Denote the cross-sectional distribution by $\Pi_s(y)$.

- This assumption imposes additional restrictions on $\pi\left(y', s' \mid y, s\right)$:

$$\Pi_{s'}\left(y'\right) = \sum_{y \in Y} \frac{\pi\left(y', s' \mid y, s\right)}{\pi\left(s' \mid s\right)} \Pi_s(y) \text{ for all } s, s' \in S$$

Recursive Formulation

- Individual state variables (a, y).
- Aggregate state variables (s, Φ).
- Recursive formulation of the household problem:

$$v(a, y, s, \Phi) = \max_{c, a' \geq 0} \left\{ U(c) + \beta \sum_{y' \in Y} \sum_{s' \in S} \pi \left(y', s' \mid y, s \right) v \left(a', y', s', \Phi' \right) \right\}$$

$$\text{s.t. } c + a' = w(s, \Phi) y + (1 + r(s, \Phi)) a$$

$$\Phi' = H \left(s, \Phi, s' \right)$$

Recursive Competitive Equilibrium

A RCE is value function $v : Z \times S \times \mathcal{M} \rightarrow R$, household policy functions $c, a' : Z \times S \times \mathcal{M} \rightarrow R$, firm policy functions $K, L : S \times \mathcal{M} \rightarrow R$, pricing functions $r, w : S \times \mathcal{M} \rightarrow R$, aggregate law of motion $H : S \times \mathcal{M} \times S \rightarrow \mathcal{M}$ s.t.

1. v, a', c are measurable wrt $\mathcal{B}(S)$, v satisfies the household's Bellman equation and a', c are the associated policy functions, given $r()$ and $w()$
2. K, L satisfy, given $r()$ and $w()$
$$r(s,\Phi) = F_K(K(s,\Phi), L(s,\Phi)) - \delta$$
$$w(s,\Phi) = F_L(K(s,\Phi), L(s,\Phi))$$
3. For all $\Phi \in \mathcal{M}$ and all $s \in S$
$$K(H(s,\Phi)) = \int a'(a,y,s,\Phi) d\Phi$$
$$L(s,\Phi) = \int y d\Phi$$
$$\int c(a,y,s,\Phi) d\Phi + \int a'(a,y,s,\Phi) d\Phi =$$
$$F(K(s,\Phi), L(s,\Phi)) + (1-\delta) K(s,\Phi)$$
4. Aggregate law of motion H is generated by exogenous Markov chain π and policy a'.

Transition Functions

- Define $Q_{\Phi,s,s'} : Z \times \mathcal{B}(Z) \rightarrow [0,1]$ by:

$$Q_{\Phi,s,s'}((a,y),(\mathcal{A},\mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \begin{cases} \pi\left(y', s' \mid y, s\right) & \text{if } a'(a,y,s,\phi) \in \mathcal{A} \\ 0 \text{ else} & \end{cases}$$

- Aggregate law of motion:

$$\Phi'(\mathcal{A},\mathcal{Y}) = \left(H\left(s,\Phi,s'\right)\right)(\mathcal{A},\mathcal{Y}) = \int Q_{\Phi,s,s'}((a,y),(\mathcal{A},\mathcal{Y}))\Phi(da \times dy)$$

Keeping Track of Wealth Distribution

- Key challenge: wealth distribution Φ is an infinite-dimensional object.
- Why do agents need to keep track of Φ? In order to forecast future capital stock and, with it, future prices.
- But for K' need entire Φ since:

$$K' = \int a'(a, y, s, \Phi) d\Phi$$

- If a' were linear in a, with the same slope for all $y \in Y$, exact aggregation obtained and average capital stock today is a sufficient statistic for the average capital stock tomorrow.
- Krusell and Smith's proposal: approximate distribution Φ with a finite set of moments.

Computation

- Let n-dimensional vector m denote first n moments of asset distribution.
- Agents use an approximate law of motion:

$$m' = H_n(s, m)$$

- Agents are **boundedly rational** in the sense that moments of higher order than n of the current wealth distribution may help to more accurately forecast the first n moments tomorrow.
- Choose the number of moments and the functional form of the function H_n.

Computation

- Krusell and Smith pick $n = 1$ and pose

$$\log\left(K'\right) = a_s + b_s \log(K)$$

  for $s \in \{s_b, s_g\}$. Here (a_s, b_s) are parameters that need to be determined.

- Household problem

$$v(a, y, s, K) = \max_{c, a' \geq 0} \left\{ U(c) + \beta \sum_{y' \in Y} \sum_{s' \in S} \pi\left(y', s' \mid y, s\right) v\left(a', y', s', K'\right) \right\}$$

$$\text{s.t. } c + a' = w(s, K)y + (1 + r(s, K))a$$

$$\log\left(K'\right) = a_s + b_s \log(K)$$

- Reduction of the state space to a four-dimensional space $(a, y, s, K) \in \mathbf{R} \times Y \times S \times \mathbf{R}$.

Algorithm

- 1. Guess (a_s, b_s).
- 2. Solve households problem to obtain $a'(a, y, s, K)$.
- 3. Simulate for large number of T periods, large number N of households:
  - Initial conditions for economy (s_0, K_0), for each household (a_0^i, y_0^i).
  - Draw random sequences $\{s_t\}_{t=1}^{T}$ and $\left\{y_t^i\right\}_{t=1,i=1}^{T,N}$, use decision rule $a'(a, y, s, K)$, perceived law of motion for K to generate $\left\{a_t^i\right\}_{t=1,i=1}^{T,N}$.
  - Aggregate:

$$K_t = \frac{1}{N}\sum_{i=1}^{N} a_t^i$$

Algorithm

- 4. Run the regressions
$$\log\left(K'\right) = \alpha_s + \beta_s \log(K)$$
to estimate (α_s, β_s) for $s \in S$.
- 5. If the R^2 for this regression is high and $(\alpha_s, \beta_s) \approx (a_s, b_s)$ stop. An approximate equilibrium is found. Otherwise update guess for (a_s, b_s).
- 6. If guesses for (a_s, b_s) converge, but R^2 remains low, add higher moments to the aggregate law of motion and/or use different functional forms.

Assignment 2

- Individual assignment, but you can discuss with classmates
- Make presentable slides of your results (maybe some codes if you wish)
- Push codes and slides of the results to your GitHub (shared with me)
- Everyone will have 8 minutes to present the results on **Apr.2nd (DDL)**
- You will be graded depending on:
    - If you could deliver the correct results (40)
    - How clean and organized are your codes written (30)
    - How well you could deliver your results in a presentation (30)
- **The whole assignment is based on Handouts 05 Krusell and Smith (1998)**
- The calibration is on Page 6, Section 3.5 Calibration

Task 1: Solve the Aiyagari Model Stationary Distribution

- Shut down the aggregate shocks ($z^g = z^b = 0.99$, reduce to Aiyagari model)
- Task 1-1: Plot the value function of both employed and unemployed over wealth
- Task 1-2: Plot the policy function of both employed and unemployed over wealth
- Task 1-3: Plot the wealth distribution of both employed and unemployed over wealth
- Task 1-4: [Bonus] Try to parallel your VFI and show improvements in solution time

Task 2: Solve the Krusell-Smith Model

- Solve the Krusell-Smith Model as parameterized in the handouts with VFI
- Task 2-1: Compare your forecasting rule to the handouts: Coefficients and R^2 for both good and bad aggregate states. Explain why it could be different.
- Task 2-2: Plot the value function of all four types (BU,BE,GU,GE) over wealth
- Task 2-3: Plot the plot function of all four types (BU,BE,GU,GE) over wealth
- Task 2-4: Plot the wealth distribution of all four types (BU,BE,GU,GE) over wealth
- Task 2-5: [Bonus] Try to parallel your VFI and show improvements in solution time

Task 3: Shock Transmission Comparison

- Solve the IRFs of aggregate wealth dynamics following a shock to the aggregate state in the Aiyagari Model (Task 1) and Krusell-Smith Model (Task 2)
- In the Aiyagari Model (Task 1), the economy is initially at $z_0 = 0.99$, but receives a one-time unexpected MIT shock $z_1 = 1.01$, then $z_t = 0.99$ for any $t > 1$. We assume the economy goes back to its steady state after $T = 20$.
- In the Krusell-Smith Model (Task 2), the economy is initially at $z_0 = z^b = 0.99$, but receives a one-time shock $z_1 = z^g = 1.01$, then $z_t = 0.99$ for any $t > 1$.
- Task 3-1: Plot the aggregate wealth dynamics up to $T = 20$
- Task 3-2: Compare the whole computing time of the two methods

Appendix

References

Krusell, Per and Anthony A Smith Jr (1998). "Income and wealth heterogeneity in the macroeconomy". In: *Journal of political Economy* 106.5, pp. 867–896.