Computational Economics Lecture 9: Heterogeneous Firm Models without Aggregate Uncertainty

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Outline

1. **Motivation**
2. **The Hopenhayn Model**
3. **Distribution Dynamics**
4. **Computation**
5. **Applications**

Motivation

- We have learned how to solve the heterogeneous household models
- How about firms? Many applications that span over
    - Business cycles: investment, employment, adjustment costs, financial shocks;
    - Development and growth: misallocation, financial development;
    - International trade, labor, etc.
    - Finance, marketing, management, etc.
- We would like to start with a canonical model: Hopenhayn, 1992
- "Entry, exit, and firm dynamics in long run equilibrium," Econometrica (1992)

The Hopenhayn Model: Overview

- Workhorse model of industry dynamics
- Steady-state model without aggregate uncertainty: firms enter, grow/decline, and exit, but the overall distribution of firms is unchanging
- Endogenous stationary distribution of firm-size, etc, straightforward comparative statics
- Self-interested, competitive firms with no strategic interactions

The Hopenhayn Model: Setup

- Continuum of firms, each measure zero, produce with decreasing-return-to-scale
- No aggregate risk: deterministic paths for output and input prices (taken as given)
- But idiosyncratic risk: individual firm productivities follow a first-order Markov process
- Fixed cost to enter, fixed cost to operate each period

The Hopenhayn Model: Setup

- Time $t = 0, 1, 2, \ldots$
- Output and input prices p and w taken as given
- Output y produced with labor n given productivity a

$$y = af(n)$$

- Static profits

$$\pi(a, p, w) := \max_n [paf(n) - wn - k]$$

  where $k > 0$ is the per-period fixed cost of operating
- Let $n(a, p, w)$ denote optimal employment and let $y(a, p, w)$ denote associated output

The Hopenhayn Model: Setup

- Assumptions:
  - $n(\cdot)$, $y(\cdot)$, $\pi(\cdot)$ are all strictly increasing in productivity a
  - Productivity draws follow a first-order Markov process with distribution $F\left(a' \mid a\right)$
  - $F(\cdot \mid a)$ is strictly decreasing in a
  - Entrants pay sunk cost $k_e > 0$ to draw initial productivity a_0 from separate distribution $G(a)$
- Timing within a period:
  - Incumbents decide to stay or exit, entrants decide to enter or not
  - Incumbents that stay pay k, entrants pay k_e
  - After paying k or k_e, operating firms learn their productivity draws

Optimization: Incumbent's Problem

- Let $z = \{p_t, w_t\}_{t=0}^{\infty}$ denote sequence of prices a firm takes as given
- Let $v_t(a, z)$ denote the value of incumbency to a firm with current productivity draw a
- Bellman equation for an incumbent firm

$$v_t(a, z) = \pi\left(a, p_t, w_t\right) + \beta \max\left[0, \int v_{t+1}\left(a', z\right) dF\left(a' \mid a\right)\right]$$

- An exit threshold $a_t^*(z)$ such that firm exits if $a_t < a_t^*(z)$, solves

$$\int v_{t+1}\left(a', z\right) dF\left(a' \mid a^*\right) = 0$$

(for interior cases)

Optimization: Entrant's Problem

- Potential entrants are ex-ante identical
- Pay $k_e > 0$ to enter, initial draw from $G(a)$ if they do
- Start producing next period
- Let $m_t \geq 0$ denote the mass of entrants, free entry condition

$$\beta \int v_{t+1}(a, z) dG(a) \leq k_e$$

with strict equality whenever $m_t > 0$

Distribution Dynamics: Aggregate State $\mu_t(\mathcal{A})$

- Let $\mu_t(\mathcal{A})$ be the measure of incumbents with productivity $a \in \mathcal{A}$
- $\mu_t(\mathcal{A})$ is the state variable for the aggregate economy
- $\mu_t(\mathcal{A})$ is endogenous and, in general, evolves over time

Law of Motion for the State

- The measure of incumbents with productivity $a \in [0, a')$ at $t+1$ is

$$\mu_{t+1}\left([0, a')\right) = \int F\left(a' \mid a\right) \mathbf{1}\left[a \geq a_t^*\right] \mu_t(da) + m_{t+1} G\left(a'\right), \quad \text{all } a'$$

  (suppressing the dependence on z)

- Suppose we discretize to a grid with N elements. Then, this is a linear system of the form

$$\boldsymbol{\mu}_{t+1} = \boldsymbol{\Psi}_t \boldsymbol{\mu}_t + m_{t+1}\mathbf{g}$$

  where $\boldsymbol{\Psi}$ is a $N \times N$ matrix that depends on the productivity process and exit threshold a_t^*, where $\boldsymbol{\mu}$ and $\mathbf{g}$ are $N \times 1$ vectors, and where m is a scalar

Industry Demand and Supply

- Industry demand curve $D(p)$, exogenous
- Industry supply curve, endogenous

$$Y = \int y\left(a, p_t, w_t\right) \mu_t(da)$$

- Market clears when

$$Y = D\left(p_t\right)$$

- Choose either p_t or w_t as numeraire. We will choose $w_t = 1$

Equilibrium

- Given an initial distribution μ_0, a perfect foresight equilibrium consists of sequences

$$\{p_t, m_t, a_t^*, \mu_t\}_{t=0}^{\infty}$$

  such that

  - (i) the goods market clears,
  - (ii) incumbents make optimal exit decisions,
  - (iii) no further incentives to enter,
  - (iv) distribution μ_t defined recursively by the law of motion above

- We will focus on a stationary equilibrium, constants

$$(p^*, m^*, a^*, \mu^*)$$

  that corresponds to a steady state of the dynamical system

Computation

- Step 1. Guess output price p_0. For this price, solve the incumbent's dynamic programming problem

$$v(a, p_0) = \pi(a, p_0) + \beta \max\left[0, \int v\left(a', p_0\right) dF\left(a' \mid a\right)\right]$$

The solution to this problem also implies the optimal exit rule, i.e., the $a^*(p_0)$ that solves

$$\int v\left(a', p_0\right) dF\left(a' \mid a^*\right) = 0$$

- Step 2. Check that this price p_0 satisfies the free-entry condition

$$\beta \int v\left(a', p_0\right) dG\left(a'\right) = k_e$$

For example, if the LHS is too high, then go back to Step 1 and guess a new price $p_1 < p_0$. Continue until a price p^* is found that solves the free-entry condition

Computation

- Step 3. Guess a measure of entrants, m_0. Given this, calculate the stationary distribution μ_0. This solves the linear system

$$\mu_0\left(\left[0, a'\right)\right) = \int_{a \geq a^*(p^*)} F\left(a' \mid a\right) \mu_0(da) + m_0 G\left(a'\right), \quad \text{for all } a'$$

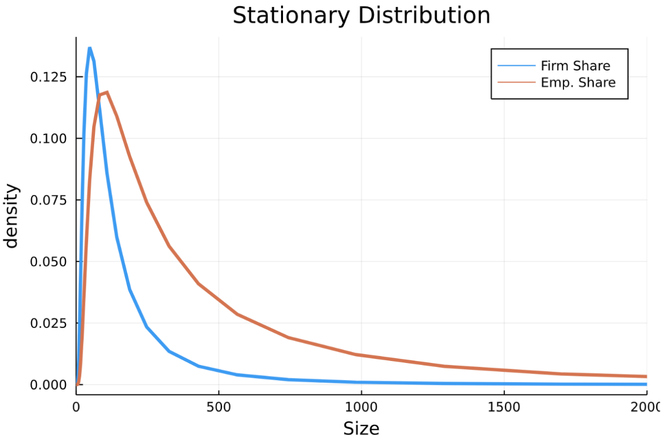
  Observe that the RHS depends on the price found at Step 2 via the exit threshold $a^*\left(p^*\right)$

- Step 4. Given this μ_0, calculate the total industry supply and check the market clearing condition

$$Y = \int y\left(a, p^*\right) \mu_0(da) = D\left(p^*\right)$$

  For example, if the LHS is too low, then go back to Step 3 and guess new entrants $m_1 > m_0$. Continue until a m^* is found that solves the market-clearing condition

Example



Stationary Distribution
Firm Share
Emp. Share
density
0.000
0.025
0.050
0.075
0.100
0.125
0
500
1000
1500
2000
Size

Implications: Increase in entry cost k_e

- Increases expected discounted profits
- Decreases exit threshold a^*
  - Less selection, incumbents make more profits, more continue
  - Increases the average age of firms
- Decreases entrants m^*
- Decreases entry/exit rate $m^*/\mu^*(\mathbb{R})$
- Increases price p^*

Implications: Increase in entry cost k_e

- Ambiguous implications for firm-size distribution:
- (1) Price effect, higher k_e increases price p^*
  hence incumbents increase output $y\left(a, p^*\right)$ and employment $n\left(a, p^*\right)$
- (2) Selection effect, higher k_e reduces productivity threshold a^*
  hence, more incumbent firms are relatively low-productivity firms

Conclusion

- Firm Dynamics Model: Open the aggregate production function black box
- At this point, we abstract from aggregate uncertainty
- At this point, we also abstract from capital. Introducing capital without some sort of friction does not change the analysis (but it introduces more interesting questions)
- Many extensions in various topics:
    - Capital Frictions: Cooley and Quadrini, 2001, Gomes, 2001, Cooper and Haltiwanger, 2006
    - Innovation and Development: Klette and Kortum, 2004, Akcigit and Kerr, 2018
    - International Trade: Melitz, 2003, Edmond, Midrigan, and Xu, 2015

Appendix

References

Akcigit, Ufuk and William R Kerr (2018). “Growth through heterogeneous innovations”. In: *Journal of Political Economy* 126.4, pp. 1374–1443.

Cooley, Thomas F and Vincenzo Quadrini (2001). “Financial markets and firm dynamics”. In: *American economic review* 91.5, pp. 1286–1310.

Cooper, Russell W and John C Haltiwanger (2006). “On the nature of capital adjustment costs”. In: *The Review of Economic Studies* 73.3, pp. 611–633.

Edmond, Chris, Virgiliu Midrigan, and Daniel Yi Xu (2015). “Competition, markups, and the gains from international trade”. In: *American Economic Review* 105.10, pp. 3183–3221.

Gomes, Joao F (2001). “Financing investment”. In: *American Economic Review* 91.5, pp. 1263–1285.

Hopenhayn, Hugo A (1992). “Entry, exit, and firm dynamics in long run equilibrium”. In: *Econometrica: Journal of the Econometric Society*, pp. 1127–1150.

Klette, Tor Jakob and Samuel Kortum (2004). “Innovating firms and aggregate innovation”. In: *Journal of political economy* 112.5, pp. 986–1018.

Melitz, Marc J (2003). “The impact of trade on intra-industry reallocations and aggregate industry productivity”. In: *econometrica* 71.6, pp. 1695–1725.