Computational Economics Lecture 10: Heterogeneous Firm Models with Aggregate Uncertainty

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Outline

1. **Motivation**
2. **The Khan-Thomas Model**
3. **Distribution Dynamics**
4. **Computation**
5. **Applications**

Motivation

- We have learned how to solve the heterogeneous firm models w/o aggregate uncertainty
- How about aggregate uncertainty?
  - Just like (Krusell and Smith, 1998), aggregate uncertainty makes the solution really hard
  - But it brings benefits: We can now study GE effects of shocks and policies
- We would like to start with a canonical model: (Khan and Thomas, 2008)
- "Idiosyncratic shocks and the role of nonconvexities in plant and aggregate investment dynamics," Econometrica (2008)
- This model is often used as a benchmark for new solution methods
- I follow largely Hanbaek Lee's notation

Model Overview

- IT studies heterogeneous establishments under aggregate productivity fluctuations
- Establishment-level nonlinear investment dynamics: (S, s) cycle
- Macro-level log-linear investment dynamics: strong general equilibrium effect
- Basic ingredients:
    - Heterogeneous idiosyncratic productivity process under the incomplete market
    - Aggregate TFP fluctuations
    - The fixed cost, $\xi \sim_{iid}$ Unif $[0, \bar{\xi}]$: smoothing the kink of the value function
    - A small-scale investment is allowed, which is not subject to a fixed cost
    - Value function normalization steps
    - Non-trivial market clearing condition
    - Representative household and competitive factor market

Production

- At the beginning of period t, a firm i is given with $(k_{it}, z_{it}; S_t)$:
- k_{it} : Pre-determined establishment-level capital stock.
  $-z_{it}$: Establishment-level idiosyncratic productivity (AR(1) process).
- $S_t = \{A_t, \Phi_t\}$: A_t is aggregate productivity (AR(1) process);
  Φ_t is the distribution of individual establishments.
- Cobb-Douglas production function with DRS ($\alpha + \gamma < 1$):

$$f\left(k_{it}, z_{it}; S_t\right) = A_t z_{it} \left(k_{it}\right)^{\alpha} n_{it}^{\gamma}$$

- Operating profit due to DRS: $\pi\left(k_{it}, z_{it}; S_t\right) = \max_{n_{it}} f\left(k_{it}, z_{it}; S_t\right) - w_t n_{it}$
- Operating profit $=$ Dividends $(D_{it}) +$ Investment (I_{it})
- The objective function is maximizing the firm value:

$$J_{it} = \max_{\{D_{it}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \frac{1}{R_t} D_{it}$$

Investment

- A firm needs to decide k_{it+1} by choosing I_{it}:

$$k_{it+1} = (1-\delta)k_{it} + I_{it}$$

- Two options in the investment scale: large/small
- If $I_{it} \in \Omega\left(k_{it}\right) := [\nu k_{it}, \nu k_{it}]$, then there is no fixed cost. ($\nu < \delta$)
- If $I_{it} \notin \Omega\left(k_{it}\right) := [\nu k_{it}, \nu k_{it}]$, then a fixed cost $\xi_{it} \sim_{iid}$ Unif$([0, \bar{\xi}])$: Why?
- Role of fixed adjustment cost:
- (1) Inaction period: no lumpy investment - inside the (S, s) cycle.
- (2) Large adjustment: Jumping from s to S

Household

- A representative household consumes, supplies labor and saves.

$$V(a;S) = \max_{c,a',I_H} \log(c) - \eta I_H + \beta \mathbb{E} V\left(a'; S'\right)$$
$$\text{s.t. } c + \int \Gamma_{S,S'} q\left(S,S'\right) a\left(S'\right) dS' = w(S) I_H + \int a(S) dS$$
$$G_\phi(S) = \Phi', \quad G_A(A) = A', \quad S = \{\Phi, A\}$$

- a: an equity portfolio, Φ : distribution of firms
  A: aggregate productivity, c : consumption
  a' : a state-contingent future saving portfolio, I_H : labor supply (indivisible)
  q : state-contingent bond price, w : wage

- Households hold the equity of firms as their wealth

- Stochastic discount factor:

$$q\left(S,S'\right) = \beta \frac{C(S)}{C\left(S'\right)}$$

  defines $p(S) := \frac{1}{C(S)}$, which will be extensively used after the normalization.

Firm's Recursive Formulation

$$J(k,z;S) = \pi(k,z;S) + (1-\delta)k + \int_0^{\bar{\xi}} \max\left\{R^*(k,z;S) - w(S)\xi, R^c(k,z;S)\right\} dG_\xi(\xi)$$

$$R^*(k,z;S) = \max_{k'} -k' - c\left(k,k'\right) + \mathbb{E} m\left(S,S'\right) J\left(k',z';S'\right)$$

$$R^c(k,z;S) = \max_{k^c-(1-\delta)k \in \Omega(k)} -k^c - c\left(k,k^c\right) + \mathbb{E} m\left(S,S'\right) J\left(k^c,z';S'\right)$$

(Operating profit) $\pi(z,k;S) := \max_{n_d} zAk^\alpha n_d^\gamma - w(S)n_d$ (nd : labor demand)

(Constrained investment) $I^c \in \Omega(k) := [-k\nu, k\nu] \quad (\nu < \delta)$

(Idiosyncratic productivity) $z' = G_z(z)$(AR(1) process)

(Stochastic discount factor) $m\left(S,S'\right) = \beta\left(C(S)/C\left(S'\right)\right)$

(Aggregate states) $S = \{A, \Phi\}$

(Aggregate law of motion) $\Phi' := H(S), A' = G_A(A)(AR(1)$ process),

Aggregate Economy

- National account tracking is important for efficient GE computation.

$$
\begin{aligned}
Y &= C + I = C + (\tilde{I} + \text{ Adj.Cost }) \\
C &= Y - I \\
&= (\Pi + W * L) - I \\
&= (\Pi - I) + W * L \\
&= \underbrace{D}_{\text{Dividend income}} + \underbrace{W * L}_{\text{Labor income}}
\end{aligned}
$$

- Therefore, consumption is total dividends plus total labor expenses.
- After obtaining the distribution of firms, we compute the total dividend and labor expenses. Then, we obtain the consumption.
- Why does consumption matter? It determines $w(S)$ and $q\left(S, S'\right)$

Market Clearing

- From the intra-temporal labor supply optimality condition:

$$\begin{aligned}\eta &= \lambda(\mu(S);S)w(S)\\ &= p(S)w(S)\end{aligned}$$

- Therefore, if $p(S)$ is known, then $w(S)$ is determined
- We still need to know SDF, $q\left(S, S'\right)$, to solve the problem
- However, some normalization eases the problem: $p(S)$ is the only price!
- Multiply $p(S) = 1/C(S)$ on the both sides of the value function identity
    - $\widetilde{J}(k, z; S) = p(S)J(k, z; S)$, $\widetilde{R}^*(k, z; S) = p(S)R^*(k, z; S)$, $\widetilde{R}^c(k, z; S) = p(S)R^c(k, z; S)$
- There is no closed-form to determine $p(S)$
- The notorious internal loop:
    1. Guess $p(S)$
    2. Using distribution $\Phi(S)$ to compute aggregate consumption $C(S) = D(S) + w(S)L(S)$
    3. Compute $p^{update}(S) = 1/c(S)$, and repeat the steps until $\left\|p(S) - p^{update}(S)\right\| <$ tol

The Extensive Margin

$$\int_0^{\bar{\xi}} \max\left\{R^*(k,z;S) - w(S)\xi, R^c(k,z;S)\right\} dG_\xi(\xi) \tag{1}$$

Then, there exists $\xi^*(k,z;S)$ such that

$$\begin{aligned} R^*(k,z;S) - w(S)\xi &> R^c(k,z;S) \quad \text{if } \xi < \xi^*(k,z;S) \\ R^*(k,z;S) - w(S)\xi &\leq R^c(k,z;S) \quad \text{if } \xi \geq \xi^*(k,z;S) \end{aligned}$$

Especially, $\xi^*(k,z;S) = \frac{R^*(k,z;S) - R^c(k,z;S)}{w(S)}$ is the closed-form characterization:

- In the support of ξ, $[0, \xi^*)$ corresponds to large-scale investment and $\left[\xi^*, \bar{\xi}\right]$ corresponds to small-scale investment: define $\psi(k,z;S) := \frac{\min\left\{\xi^*(k,z;S), \bar{\xi}\right\}}{\bar{\xi}}$
- With probability $\psi(k,z;S)$, a firm makes a large-scale investment.
- Eq (1) becomes a linear combination form: No Kink! (c.f., Discrete choice model)

$$\psi(k,z;S)\left(R^*(k,z;S) - w(S)\frac{\xi^*(k,z;S)}{2}\right) + (1 - \psi(k,z;S))\left(R^c(k,z;S)\right)$$

**Skip the RCE definition here*

Computation

1. Set the parametric law of motion: assumption
2. Guess the parameters: $\#(S) \times 2 \times 2$
3. Solution (optimization)
   - VFI/PFI/EGM/Projection method
   - Interpolation
4. Simulation and internal loop for price p
   - Simulation
   - Aggregation
   - Update p until convergence
5. Update the parameters with (Krusell and Smith, 1998)'s method!
6. After convergence, verify the assumption

Parametric Law of Motion

There are two layers of choices:

- First, we need to set what are sufficient statistics to characterize the dynamics of the individual state distribution
    - A good candidate is the first moment of the endogenous individual state
    - As in Krusell and Smith (1998), by tracking only K_t, the aggregate prices are also characterized
    - Then, we need to decide the parametric form of the law of motion
    - So start from the following parameter guesses $\left(\alpha_S^K, \beta_S^K; \alpha_S^p, \beta_S^p\right)$

$$\log\left(K_{t+1}\right) = \alpha_S^K + \beta_S^K \log\left(K_t\right) \text{ when } S_t = S$$
$$\log\left(p_t\right) = \alpha_S^p + \beta_S^p \log\left(K_t\right) \text{ when } S_t = S$$

- K_t does not immediately give p_t. But it should give some inference on p_t!

Value Function Iteration

Now we know $p(S)$ if we are given with K. The pseudo-code is as follows:

1. Guess $J^{(n)} : \mathcal{K} \times \mathcal{Z} \times \mathcal{A} \times \mathcal{K}^{Agg} \rightarrow \mathbb{R}$
2. Solve for the policy function, $g_k^{(n)}$ (using monotonicity: $g_k^{(n)} \geq g_{\widetilde{k}}^{(n)}$ for $k \geq \widetilde{k}$).
    - We have K, so we know (p, K') from the law of motion.
    - Interpolate the value functions over K' to have $J^{(n)}\left(\cdot, z'; A', K'\right)$
    - Then, the problem becomes a typical VFI.
3. Update $J^{(n+1)}$ using the policy function, $g_k^{(n)}$.
4. Update $J^{(n+2)}$ using the policy function, $g_k^{(n)}$.
   ...
5. Update $J^{(n+m)}$ using the policy function, $g_k^{(n)}$.
6. Check if $\left\| J^{(n+m)} - J^{(n+m-1)} \right\|_p < Tol$
    - If yes, the solution converged.
    - If no, go back to step 1.

Simulation of the Aggregate Economy

- Simulate a long enough aggregate shock path using Γ_S. (Not idiosyncratic shocks)
    - Start from an initial guess $\mathcal{H}_0$. Compute the corresponding K_0. And then the internal loop.
    - Guess p_0, and solve the problem to get g_{d0} and g_{l0}.
    - Compute c_0 using $(\mathcal{H}_0, g_{d0}, g_{l0}, w_0)$.
    - Compute $p_0^{\text{update}} = 1/c_0$, and repeat the steps until $\left\| p_0 - p_0^{\text{update}} \right\| < \text{tol}$.
    - So, we have $\left(K_0, p_0^{\text{converged}}\right)$.
- Let $\mathcal{H}_0$ evolve to $\mathcal{H}_1$ using $g_{a0}^{\text{converged}}$, and compute the corresponding K_1.
    - Guess p_1, and solve the problem to get g_{d1} and $g_{/1}$.
    - Compute c_1 using $(\mathcal{H}_1, g_{d1}, g_{11}, w_1)$.
    - Compute $p_1^{\text{update}} = 1/c_1$, and repeat the steps until $\left\| p_1 - p_1^{\text{update}} \right\| < \text{tol}$.
    - So, we have $\left(K_1, p_1^{\text{converged}}\right)$.
- By repeating this process, we obtain $\left\{K_t, p_t^{\text{converged}}\right\}_{t=0}^{T}$.

Updating the Parameters

- We have $\left\{ K_t, p_t^{\text{converged}} \right\}_{t=\text{ burnIn}}^{T}$ and $\{S_t\}_{t=\text{ burnIn}}^{T}$.
- Fit the time series into the parametric form of the law of motion to estimate the parameters: $\left(\alpha_S^K, \beta_S^K; \alpha_S^p, \beta_S^p\right)$

$$\log\left(K_{t+1}\right) = \alpha_S^K + \beta_S^K \log\left(K_t\right) \text{ when } S_t = S$$
$$\log\left(p_t^{\text{converged}}\right) = \alpha_S^p + \beta_S^p \log\left(K_t\right) \text{ when } S_t = S$$

- If the parameter estimates are not close to the guess, return to the initial step.
- Otherwise, the solution is converged.
- Check R^2 as the first check for the validity of the parametric form.

Model Results: PE Investment Rates

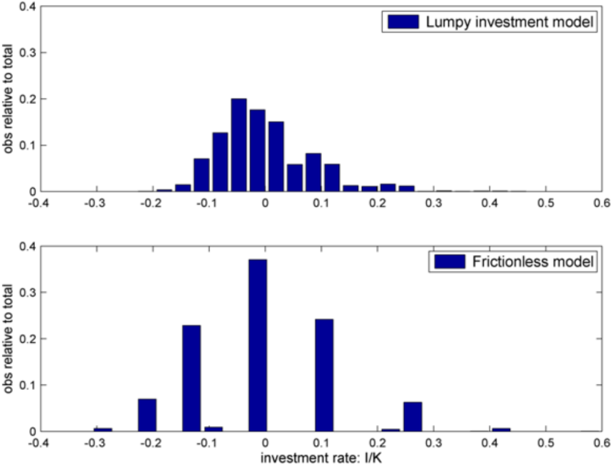

FIGURE 1.—Distribution of aggregate investment rates in partial equilibrium.

Model Results: PE Asymmetry

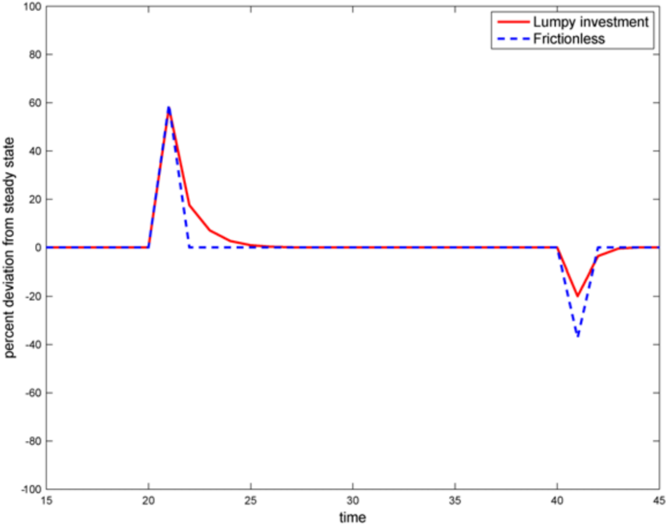

FIGURE 2.—Partial equilibrium aggregate capital responses.

Model Results: Distribution and Adjustment Prob.

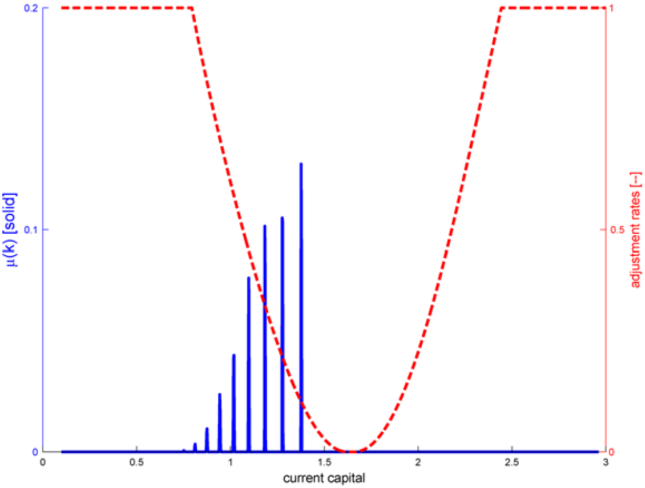

FIGURE 3.—Steady-state adjustment in the common productivity lumpy model.

Model Results: Distribution and Adjustment Prob.

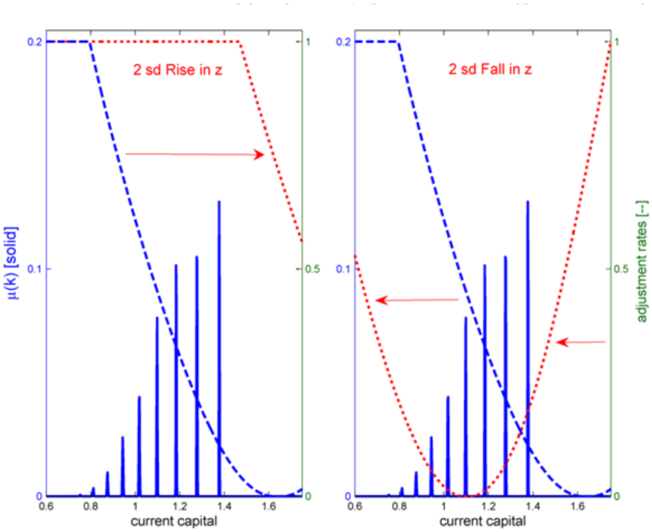

FIGURE 4.—Adjustment responses in the common productivity lumpy model.

Model Results: GE Distribution

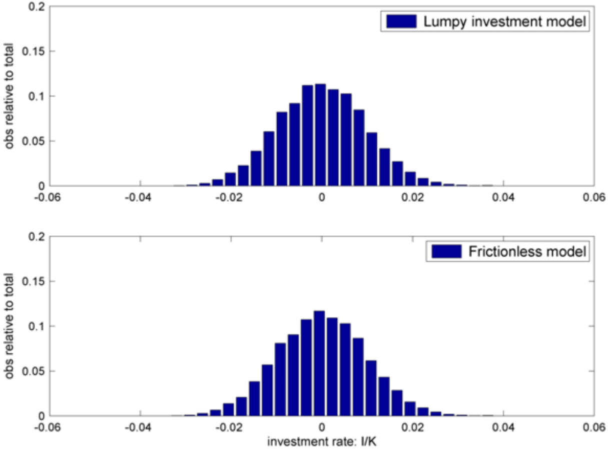

FIGURE 5.—Distribution of aggregate investment rates in general equilibrium.

Model Results: GE Aggregate Effects

TABLE IV

AGGREGATE BUSINESS CYCLE MOMENTS

	Output	TFP[a]	Hours	Consump.	Invest.	Capital
A. Standard deviations relative to output[b]						
GE frictionless	(2.277)	0.602	0.645	0.429	3.562	0.494
GE lumpy	(2.264)	0.605	0.639	0.433	3.539	0.492
B. Contemporaneous correlations with output						
GE frictionless		1.000	0.955	0.895	0.976	0.034
GE lumpy		1.000	0.956	0.900	0.976	0.034

[a]Total factor productivity.

[b]The logarithm of each series is Hodrick–Prescott-filtered using a weight of 100. The output column of panel A reports percent standard deviations of output in parentheses.

Model Results: Perfect Aggregation!

TABLE A.II

FORECASTING RULES IN FULL LUMPY MODEL

Productivity[a]	β_0	β_1	S.E.	Adj. R^2
A. Forecasting m_1'				
z_1 (119 obs)	0.009	0.800	0.15e−3	1.0000
z_2 (298 obs)	0.016	0.798	0.22e−3	0.9999
z_3 (734 obs)	0.023	0.796	0.23e−3	0.9999
z_4 (1,208 obs)	0.030	0.795	0.26e−3	0.9999
z_5 (1,682 obs)	0.037	0.794	0.27e−3	0.9999
z_6 (1,871 obs)	0.044	0.079	0.28e−3	0.9999
z_7 (1,706 obs)	0.051	0.793	0.26e−3	0.9999
z_8 (1,237 obs)	0.058	0.792	0.24e−3	0.9999
z_9 (751 obs)	0.065	0.792	0.23e−3	0.9999
z_{10} (295 obs)	0.072	0.791	0.25e−3	0.9999
z_{11} (99 obs)	0.079	0.791	0.19e−3	0.9999
B. Forecasting p				
z_1 (119 obs)	0.994	−0.397	0.03e−3	1.0000
z_2 (298 obs)	0.986	−0.395	0.04e−3	1.0000
z_3 (734 obs)	0.977	−0.394	0.04e−3	1.0000
z_4 (1,208 obs)	0.968	−0.393	0.05e−3	1.0000
z_5 (1,682 obs)	0.958	−0.392	0.05e−3	1.0000
z_6 (1,871 obs)	0.949	−0.391	0.05e−3	1.0000
z_7 (1,706 obs)	0.940	−0.389	0.05e−3	1.0000
z_8 (1,237 obs)	0.931	−0.388	0.05e−3	1.0000
z_9 (751 obs)	0.921	−0.386	0.04e−3	1.0000
z_{10} (295 obs)	0.912	−0.384	0.05e−3	1.0000
z_{11} (99 obs)	0.903	−0.382	0.04e−3	1.0000

[a] Forecasting rules are conditional on current aggregate total factor productivity z_i. Each regression takes the form $\log(y) = \beta_0 + \beta_1 \log(m_1)$, where $y = m_1'$ or p.

Conclusion

- The KS solution method works really well because the model fits perfect aggregation!
- Is this result empirically supported?
- Recent literature shows probably not...
- So, if not using the KS method, how can we solve the model?

Appendix

References

Khan, Aubhik and Julia K Thomas (2008). “Idiosyncratic shocks and the role of nonconvexities in plant and aggregate investment dynamics”. In: *Econometrica* 76.2, pp. 395–436.

Krusell, Per and Anthony A Smith Jr (1998). “Income and wealth heterogeneity in the macroeconomy”. In: *Journal of political Economy* 106.5, pp. 867–896.