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# **Computational Economics Lecture 7: Heterogeneous Agent Models without Aggregate Uncertainty**

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# Outline

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1. **Motivation**
2. **The Aiyagari Model**
3. **Stationary Distribution**
4. **Transitional Dynamics**

## Motivation

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- Many models involve heterogeneous agents instead of one agent
- The reason is nowadays very obvious:
  - The reality features agent heterogeneity and has fewer assumptions
  - Heterogeneous agent models could match the microdata much better
  - Heterogeneous agent models connect macro to other fields much better
- We will start with a canonical heterogeneous agent model: (Aiyagari, 1994)
- It is a general equilibrium heterogeneous household model with incomplete markets of household consumption-savings problem without aggregate uncertainty in the economy
- It is usually the first heterogeneous agent (household) model everyone learned
- Later, we will learn how to do heterogeneous firms (just as I showed)

## Outline of the Model

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- Mainly follow lectures from Prof. Jesus Fernandez-Villaverde
- Continuum of households (vs. models with finite number/types of agents)
- One firm renting aggregate capital
- No aggregate uncertainty
- Individuals are subject to idiosyncratic shocks to their labor income
- Incomplete markets (can only trade finite type of assets)

# Households

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- Continuum of measure 1 of households.
- Preferences for household  $i$  :

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

- Budget constraint:

$$c_t + a_{t+1} = w_t y_t + (1 + r_t) a_t$$

- We could consider a hand-to-mouth (i.e., autarky) variation:  $c_t = w_t y_t$ .
- Initial conditions  $y_0, a_0 \geq 0$ .
- Borrowing constraint  $a_{t+1} \geq 0$ .

## Labor Endowment

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- Stochastic labor endowment process  $\{y_t\}_{t=0}^{\infty}$  :

$$y_t \in Y = \{y_1, y_2, \dots, y_N\}$$

- Markov process with transitions  $\pi(y' | y) > 0$
- Common for all households, but realizations are specific for each individual
- Law of large numbers:  $\pi(y' | y)$  is also the deterministic fraction of the population that has this particular transition
- Unique stationary distribution associated with  $\pi$ , denoted by  $\Pi$
- Total labor endowment in the economy at each point of time:

$$L = \sum_y \Pi(y)y$$

## One Firm

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- Perfectly competitive firm with neoclassical technology:

$$Y_t = F(K_t, L_t)$$

- Depreciation rate:  $0 < \delta < 1$ .
- Aggregate resource constraint:

$$C_t + K_{t+1} - (1 - \delta)K_t = F(K_t, L_t)$$

- The only net asset in the economy is physical capital
- No state-contingent claims (i.e. incomplete markets)
- Households have all the ownership of the firm

## Recursive Formulation

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- $(a, y)$ : household state
- $\Phi(a, y)$ : aggregate state variable
- $A = [0, \infty)$ : set of possible asset holdings
- $B(A)$ : Borel  $\sigma$ -algebra of  $A$
- $Y$ : set of possible labor endowment realizations
- $P(Y)$ : power set of  $Y$
- $Z = A \times Y$  and  $B(Z) = P(Y) \times B(A)$
- $\mathcal{M}$  the set of all probability measures on the measurable space  $(Z, B(Z))$



## Recursive Formulation

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- Household problem in recursive formulation:

$$\begin{aligned} v(a, y; \Phi) &= \max_{c \geq 0, a' \geq 0} u(c) + \beta \sum_{y' \in Y} \pi(y' | y) v(a', y'; \Phi') \\ \text{s.t. } c + a' &= w(\Phi)y + (1 + r(\Phi))a \\ \Phi' &= H(\Phi) \end{aligned}$$

- Function  $H : \mathcal{M} \rightarrow \mathcal{M}$  is called the aggregate "law of motion"
- Note the complexity of the operator

## Recursive Competitive Equilibrium

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A RCE is value function  $v : Z \times \mathcal{M} \rightarrow R$ , household policy functions  $a', c : Z \times \mathcal{M} \rightarrow R$ , firm policy functions  $K, L : \mathcal{M} \rightarrow R$ , pricing functions  $r, w : \mathcal{M} \rightarrow R$  and law of motion  $H : \mathcal{M} \rightarrow \mathcal{M}$  s.t.

1.  $v, a', c$  are measurable with respect to  $\mathcal{B}(Z)$ ,  $v$  satisfies Bellman equation and  $a', c$  are the policy functions, given  $r()$  and  $w()$ .
2.  $K, L$  satisfy, given  $r()$  and  $w()$

$$r(\Phi) = F_K(K(\Phi), L(\Phi)) - \delta$$

$$w(\Phi) = F_L(K(\Phi), L(\Phi))$$

3. For all  $\Phi \in \mathcal{M}$ ,  $L(\Phi) = \int y d\Phi$  and

$$K'(\Phi') = K(H(\Phi)) = \int a'(a, y; \Phi) d\Phi$$

$$\int c(a, y; \Phi) d\Phi + \int a'(a, y; \Phi) d\Phi = F(K(\Phi), L(\Phi)) + (1 - \delta)K(\Phi)$$

4. Aggregate law of motion  $H$  is generated by  $\pi$  and  $a'$ .

## Transition Functions

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- Define transition function  $Q_\Phi : Z \times \mathcal{B}(Z) \rightarrow [0, 1]$  by

$$Q_\Phi((a, y), (\mathcal{A}, \mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \begin{cases} \pi(y' | y) & \text{if } a'(a, y; \Phi) \in \mathcal{A} \\ 0 & \text{else} \end{cases}$$

for all  $(a, y) \in Z$  and all  $(\mathcal{A}, \mathcal{Y}) \in \mathcal{B}(Z)$ .

- $Q_\Phi((a, y), (\mathcal{A}, \mathcal{Y}))$  is the probability that an agent with current assets  $a$  and current income  $y$  ends up with assets  $a'$  in  $\mathcal{A}$  tomorrow and income  $y'$  in  $\mathcal{Y}$  tomorrow.
- Hence

$$\begin{aligned} \Phi'(\mathcal{A}, \mathcal{Y}) &= (H(\Phi))(\mathcal{A}, \mathcal{Y}) \\ &= \int Q_\Phi((a, y), (\mathcal{A}, \mathcal{Y})) \Phi(da \times dy) \end{aligned}$$

## Stationary RCE

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A stationary RCE is value function  $v : Z \rightarrow R$ , household policy functions  $a', c : Z \rightarrow R$ , firm policies  $K, L$ , prices  $r, w$  and a measure  $\Phi \in \mathcal{M}$  such that

1.  $v, a', c$  are measurable with respect to  $B(Z)$ ,  $v$  satisfies the household's Bellman equation and  $a', c$  are associated policy functions, given  $r, w$ .
2.  $K, L$  satisfy, given  $r, w$  :

$$r = F_K(K, L) - \delta$$

$$w = F_L(K, L)$$

3.  $L = \int y d\Phi$  and  $K = \int a'(a, y) d\Phi$  and

$$\int c(a, y) d\Phi + \int a'(a, y) d\Phi = F(K, L) + (1 - \delta)K$$

4. Let  $Q$  be transition function induced by  $\pi$  and  $a'. \forall (\mathcal{A}, \mathcal{Y}) \in \mathcal{B}(Z)$

$$\Phi(\mathcal{A}, \mathcal{Y}) = \int Q((a, y), (\mathcal{A}, \mathcal{Y})) d\Phi$$

## Characterizing the Stationary RCE

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- Recall that  $L$  is exogenously given.
- Thus, from

$$r = F_K(K, L) - \delta, \quad w = F_L(K, L)$$

we can get  $w$  as a function of  $r$  ( with  $w'(r) < 0$  ).

- Example:

$$Y = K^\alpha L^{1-\alpha}$$

with

$$r = \alpha K^{\alpha-1} L^{1-\alpha} - \delta \Rightarrow K = \left( \frac{r + \delta}{\alpha} \right)^{\frac{1}{\alpha-1}} L$$

and

$$w = (1 - \alpha) K^\alpha L^{1-\alpha} = (1 - \alpha) \alpha^{\frac{\alpha}{1-\alpha}} (r + \delta)^{\frac{\alpha}{\alpha-1}} L$$

## Existence and Uniqueness

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- By Walras' law, we only need to check input market clearing
- Define asset market clearing condition:

$$K = K(r) = \int a'(a, y) d\Phi \equiv Ea(r)$$

- Then

$$r = F_k(K(r), L) - \delta$$

- Existence and uniqueness of stationary RCE boils down to one equation
- From assumptions on production function,  $K(r)$  is continuous, strictly decreasing function on  $r \in (-\delta, \infty)$  with

$$\lim_{r \rightarrow -\delta} K(r) = \infty$$

$$\lim_{r \rightarrow \infty} K(r) = 0$$

- We can ensure existence but cannot ensure uniqueness (leave for self-study)

## Interest Rate in Equilibrium

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- Complete markets model:  $r^{CM} = \frac{1}{\beta} - 1$
- With incomplete markets:  $r^* < r^{CM}$
- Why? Over-accumulation of capital and over-saving (because of precautionary reasons: liquidity constraints, prudence, or both)
- Policy implications

## Computation of the Canonical Aiyagari Model

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It involves three steps:

1. Fix an  $r \in \left(-\delta, \frac{1}{\beta} - 1\right)$ . For a fixed  $r$ , solve the household's recursive problem. This yields a value function  $v_r$  and decision rules  $a'_r, c_r$ .
2. The policy function  $a'_r$  and  $\pi$  induce Markov transition function  $Q_r$ . Compute the unique stationary measure  $\Phi_r$  associated with this transition function.
3. Compute aggregation and calculate the excess demand for capital:

$$d(r) = K(r) - Ea(r)$$

If small enough, stop; if not, adjust  $r$  (Which direction?)



## Step 1: Solve the Household's Recursive Problem

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It is very standard as what you have learned so far

- Any acceptable solution method for recursive problems is valid: i.e., VFI
- However, speed is at a premium
- Thus, value function iteration might not be fast enough
- Standard "tricks": monotonicity and concavity
- Be smart about initial guesses in the updates
- Fix variable values in steady state, not parameters!
- Also, explore multi-grid schemes

## Step 2: Compute the Stationary Distribution

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- Grid. Suppose  $A = \{a_1, \dots, a_M\}$ .
- Then  $\Phi$  is  $M * N \times 1$  column vector and  $Q = (q_{ij,kl})$  is  $M * N \times M * N$  matrix with

$$q_{ij,kl} = \Pr((a', y') = (a_k, y_l) \mid (a, y) = (a_i, y_l))$$

- Stationary measure  $\Phi$  satisfies matrix equation

$$\Phi = Q^T \phi$$

- $\Phi$  is (rescaled) eigenvector associated with eigenvalue  $\lambda = 1$  of  $Q^T$
- $Q^T$  is a stochastic matrix and thus has at least one unit eigenvalue. If it has more than one unit eigenvalue, a continuum of stationary measures

## Step 2: Compute the Stationary Distribution

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- Variation of grid method I: allocate mass between two grid points according to relative distance (It is time to remember the Young method)
- Variation of grid method II: uniform mass between two grid points
- Both cases: sufficiently small grid; otherwise, no convergence
- Simulation (Stochastic or Non-Stochastic)
- Convergence

## Transitional Dynamics

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- Often, we are interested in the effects of the change in a parameter of the model (transitory or permanent).
- We want to compute both the new steady state and the transitional dynamics.
- Example: permanent introduction of a capital income tax at rate  $\tau$ . Receipts are rebated lump-sum to households as the government transfers  $T$ .

## Model with a Capital Income Tax

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- State space:  $Z = Y \times \mathbf{R}_+$ , the set of all possible  $(y, a)$
- Let  $\mathcal{B}(Z) = \mathcal{P}(Y) \times \mathcal{B}(\mathbf{R}_+)$  and  $\mathbf{M}$  be the set of all finite measures on the measurable space  $(Z, \mathcal{B}(Z))$
- Household problem:

$$v_t(a, y) = \max_{c \geq 0, a' \geq 0} u(c) + \beta \sum_{y' \in Y} \pi(y' | y) v_{t+1}(a', y')$$
$$\text{s.t. } c + a' = w_t y + (1 + (1 - \tau_t) r_t) a + T_t$$

## A Competitive Equilibrium with Taxes

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Given initial distribution  $\Phi_0$  and fiscal legislation  $\{\tau_t\}_{t=0}^{\infty}$ , a competitive equilibrium is sequence of functions for the household  $\{v_t, c_t, a_{t+1} : Z \rightarrow \mathbf{R}\}_{t=0}^{\infty}$ , sequence of firm production plans  $\{L_t, K_t\}_{t=0}^{\infty}$ , factor prices  $\{w_t, r_t\}_{t=0}^{\infty}$ , government transfers  $\{T_t\}_{t=0}^{\infty}$ , and sequence of measures  $\{\Phi\}_{t=1}^{\infty}$  s.t.  $\forall t$ ,

- Given  $\{w_t, r_t\}$  and  $\{T_t, \tau_t\}$  the functions  $\{v_t\}$  solve Bellman equation in  $t$  and  $\{c_t, a_{t+1}\}$  are associated policy functions.
- Prices  $w_t$  and  $r_t$  satisfy

$$\begin{aligned}w_t &= F_L(K_t, L_t) \\ r_t &= F_K(K_t, L_t) - \delta\end{aligned}$$

- Government Budget Constraint: for all  $t \geq 0$

$$T_t = \tau_t r_t K_t$$

## A Competitive Equilibrium with Taxes

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- Market Clearing:

$$\int c_t(a_t, y_t) d\Phi_t + K_{t+1} = F(K_t, L_t) + (1 - \delta)K_t$$

$$L_t = \int y_t d\Phi_t$$

$$K_{t+1} = \int a_{t+1}(a_t, y_t) d\Phi_t$$

- Aggregate Law of Motion: Define Markov transition functions  $Q_t : Z \times \mathcal{B}(Z) \rightarrow [0, 1]$  induced by the transition probabilities  $\pi$  and optimal policy  $a_{t+1}(y, a)$  as

$$Q_t((a, y), (\mathcal{A}, \mathcal{Y})) = \sum_{y' \in \mathcal{Y}} \begin{cases} \pi(y' | y) & \text{if } a_{t+1}(a, y) \in \mathcal{A} \\ 0 & \text{else} \end{cases}$$

for all  $(a, y) \in Z$  and all  $(\mathcal{A}, \mathcal{Y}) \in \mathcal{B}(Z)$ . Then for all  $(\mathcal{A}, \mathcal{Y}) \in \mathcal{B}(Z)$

$$\Phi_{t+1}(\mathcal{A}, \mathcal{Y}) = [\Gamma_t(\Phi_t)](\mathcal{A}, \mathcal{Y}) = \int Q_t((a, y), (\mathcal{A}, \mathcal{Y})) d\Phi_t$$

## Stationary Equilibrium and Transitions

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- A stationary equilibrium is an equilibrium such that all elements of the equilibrium that are indexed by  $t$  are constant over time.
- Transitions are likely to be asymptotic.
- However, assume that after  $T$  periods, the transition from old to new stationary equilibrium is completed.
- Under the assumption  $v_T = v_\infty$ , for a given sequence of prices  $\{r_t, w_t\}_{t=1}^T$  household problem can be solved backwards.



## Transition Dynamics Computation: Step 1

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- Fix  $T$ .
- Compute stationary equilibrium  $\Phi_0, v_0, r_0, w_0, K_0$  associated with  $\tau = \tau_0 = 0$ .
- Compute stationary equilibrium  $\Phi_\infty, v_\infty, r_\infty, w_\infty, K_\infty$  associated with  $\tau_\infty = \tau$ . Assume that

$$\Phi_T, v_T, r_T, w_T, K_T = \Phi_\infty, v_\infty, r_\infty, w_\infty, K_\infty$$

- Guess sequence of capital stocks  $\{\hat{K}_t\}_{t=1}^{T-1}$ . The capital stock at time  $t = 1$  is determined by decisions at time 0,  $\hat{K}_1 = K_0$ . Note that  $L_t = L_0 = L$  is fixed. We also obtain

$$\hat{w}_t = F_L(\hat{K}_t, L)$$

$$\hat{r}_t = F_K(\hat{K}_t, L) - \delta$$

$$\hat{T}_t = \tau_t \hat{r}_t \hat{K}_t$$

## Transition Dynamics Computation: Step 2

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- Since we know  $v_T(a, y)$  and  $\{\hat{r}_t, \hat{w}_t, \hat{T}_t\}_{t=1}^{T-1}$  we can solve for  $\{\hat{v}_t, \hat{c}_t, \hat{a}_{t+1}\}_{t=1}^{T-1}$  backwards.
- With policy functions  $\{\hat{a}_{t+1}\}$  define transition laws  $\{\hat{\Gamma}_t\}_{t=1}^{T-1}$ . We know  $\Phi_0 = \Phi_1$  from the initial stationary equilibrium. Iterate the distributions forward

$$\hat{\Phi}_{t+1} = \hat{\Gamma}_t(\hat{\Phi}_t)$$

for  $t = 1, \dots, T - 1$ .

- With  $\{\hat{\Phi}_t\}_{t=1}^T$  we can compute, for  $t = 1, \dots, T$ .

$$\hat{A}_t = \int a d\hat{\Phi}_t$$

## Transition Dynamics Computation: Step 3

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- Test

$$\max_{1 \leq t < T} |\hat{A}_t - \hat{K}_t| < \varepsilon$$

If yes, go to the next step. If not, adjust your guesses for  $\{\hat{K}_t\}_{t=1}^{T-1}$ .

- Test

$$\|\hat{\Phi}_T - \Phi_T\| < \varepsilon$$

If yes, the transition converges smoothly into the new steady state and we are done and should save  $\{\hat{v}_t, \hat{a}_{t+1}, \hat{c}_t, \hat{\Phi}_t, \hat{r}_t, \hat{w}_t, \hat{K}_t\}$ . If not, increase  $T$ .

- We can be smart with the initial guess: compute associated RA transition.

# Appendix

## REFERENCES

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Aiyagari, S Rao (1994). “Uninsured idiosyncratic risk and aggregate saving”. In: *The Quarterly Journal of Economics* 109.3, pp. 659–684.