PhD Macro Core Part I:
Lecture 7 – Dynamic Programming III

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Fall 2024

Today

- Model with risk: Stochastic optimal growth model
- Sequential approach carrying histories and using contingent plans
- Recursive approach using dynamic programming
- Some background on Markov chains

Our Goal

- Solve a stochastic optimal growth model

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \mathbb{E}\left\{\sum_{t=0}^{\infty} \beta^t u(c_t)\right\}, \quad 0 < \beta < 1$$

  subject to the sequence of constraints, with productivity shock z_t,

$$c_t, k_{t+1} \geqslant 0, \quad \text{and} \quad c_t + k_{t+1} \leqslant z_t f(k_t)$$

  with the given initial conditions

$$k_0, z_0 > 0$$

- Problem takes as an input an exogenous stochastic process for $\{z_t\}$

- Delivers endogenous stochastic processes $\{c_t\}$ and $\{k_t\}$

Contingent Plans

- In the deterministic problem, choose deterministic sequences $\{c_t\}$ and $\{k_t\}$

- In the stochastic problem, choose stochastic processes

- The stochastic processes can be interpreted as contingent plans

    - $\{c_t\} = \{c_t(z_0, z_1, z_2, \ldots, z_t, \mathbb{E}z_{t+1}, \mathbb{E}z_{t+2}, \ldots)\}$

    - $\{k_t\} = \{k_t(z_0, z_1, z_2, \ldots, z_t, \mathbb{E}z_{t+1}, \mathbb{E}z_{t+2}, \ldots)\}$

- Every realization of the past $z_\tau, \tau \leqslant t$ matters!

- Every expectation of the future $\mathbb{E}z_\tau, \tau \geqslant t$ matters!

The Past: Histories

- Let z^t denote a history of realizations of the shock up to and including date t

$$z^t \equiv (z_0, z_1, \ldots, z_t) = \left(z^{t-1}, z_t\right)$$

- Let $c_t\left(z^t\right)$ and $k_{t+1}\left(z^t\right)$ denote contingent plans for consumption and capital accumulation only conditional on the history z^t

- History of realizations z^t known at t but unknown as of $t = 0$

- So $c_t\left(z^t\right)$ and $k_{t+1}\left(z^t\right)$ unknown as of $t = 0$

The Future: Expected Utility

- Outcomes are ranked according to the expected utility criterion

$$\mathbb{E}\left\{\sum_{t=0}^{\infty} \beta^t u\left(c_t\right)\right\}$$

- Involves taking expectations with respect to the probability distribution of the random variable $\{z^t\}_{t=0}^{\infty}$ if standing at time $t = 0$, consequently, $\{c_t\left(z^t\right)\}_{t=0}^{\infty}$

- For simplicity, let z_t be a discrete random variable and let $\pi_t\left(z^t\right)$ denote the probability of z^t as of date $t = 0$. Then

$$\mathbb{E}\left\{\sum_{t=0}^{\infty} \beta^t u\left(c_t\right)\right\} = \sum_{t=0}^{\infty} \sum_{z^t} \beta^t u\left(c_t\left(z^t\right)\right) \pi_t\left(z^t\right)$$

Restate the Goal: Sequence Problem

- Stochastic optimal growth model restated

$$\max_{\{c_t(z^t),k_{t+1}(z^t)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \sum_{z^t} \beta^t u\left(c_t\left(z^t\right)\right) \pi_t\left(z^t\right)$$

subject to the sequence of resource constraints

$$c_t\left(z^t\right) + k_{t+1}\left(z^t\right) \leqslant z_t\left(z^t\right) f\left(k_t\left(z^{t-1}\right)\right), \qquad \text{for all } z^t$$

and the non-negativity conditions

$$c_t\left(z^t\right), k_{t+1}\left(z^t\right) \geqslant 0, \qquad \text{for all } z^t$$

- Takes as given the sequence of probabilities $\pi_t\left(z^t\right)$, the initial conditions $k_0, z_0 > 0$, etc

Setup the Lagrangian Problem

- Lagrangian with stochastic multiplier $\lambda_t\left(z^t\right) \geqslant 0$ for each constraint

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{z^t} \beta^t u\left(c_t\left(z^t\right)\right) \pi_t\left(z^t\right) + \sum_{t=0}^{\infty} \sum_{z^t} \lambda_t\left(z^t\right) \left[z_t\left(z^t\right) f\left(k_t\left(z^{t-1}\right)\right) - c_t\left(z^t\right) - k_{t+1}\left(z^t\right)\right]$$

- First order condition for $c_t\left(z^t\right)$ can be written

$$\beta^t u'\left(c_t\left(z^t\right)\right) \pi_t\left(z^t\right) = \lambda_t\left(z^t\right)$$

- First order condition for $k_{t+1}\left(z^t\right)$ can be written

$$\lambda_t\left(z^t\right) = \sum_{z'|z^t} \lambda_{t+1}\left(z^t, z'\right) \left[z_{t+1}\left(z^t, z'\right) f'\left(k_{t+1}\left(z^t\right)\right)\right]$$

where the sum is taken over all states z' that immediately follow z^t

Solve the Lagrangian Problem

- Eliminating the Lagrange multipliers gives

$$u'\left(c_t\left(s^t\right)\right) = \beta \sum_{z' \mid z^t} u'\left(c_{t+1}\left(z^t, z'\right)\right)\left[z_{t+1}\left(z^t, z'\right) f'\left(k_{t+1}\left(z^t\right)\right)\right] \frac{\pi_{t+1}\left(z^t, z'\right)}{\pi_t\left(z^t\right)}$$

- To interpret this condition, notice that

$$\frac{\pi_{t+1}\left(z^t, z'\right)}{\pi_t\left(z^t\right)} = \text{Prob}\left[z' \mid z^t\right]$$

- This is the conditional probability of $z_{t+1} = z'$ given the history z^t

- Thus, RHS involves a conditional expectation

Solution: Consumption Euler Equation

- In more familiar time-series notation, this is just

$$u'(c_t) = \beta \mathbb{E}_t \{u'(c_{t+1}) z_{t+1} f'(k_{t+1})\}$$

- A stochastic version of the consumption Euler equation
- Are we done? Yes!
- Are the solutions easy to keep tracking? No!
- $\mathbb{E}_t$ is time-dependent!

Introduce the Markov Processes

- Goal: To get rid of the time-dependent properties
- A (first-order) Markov process has the property that, conditional on the current z_t, future realizations are independent of z^{t-1}.
- In this sense, the current z_t is a sufficient statistic for the past
- Markov processes are recursive, and so are a natural setting for dynamic programming approaches
- To begin with, let's consider z_t with discrete support, usually referred to as a Markov chain

Markov Chains

- A finite Markov chain is a triple $(z, \boldsymbol{P}, \psi_0)$ where
  - z is an n-vector listing the possible states (outcomes) of the chain
  - $\boldsymbol{P}$ is an $n \times n$ probability transition matrix
  - ψ_0 is an n-vector recording the initial distribution over the states
- Restrictions

$$0 \leqslant p_{ij} \leqslant 1, \quad \text{and} \quad \sum_{j=1}^{n} p_{ij} = 1 \quad \text{for all } i = 1, \ldots, n$$

$$0 \leqslant \psi_{0,i} \leqslant 1, \quad \text{and} \quad \sum_{i=1}^{n} \psi_{0,i} = 1$$

Interpretation

- Consider stochastic process $\{z_t\}$ induced by a Markov chain
- A realization of z_t takes on the value of one of the states in $\boldsymbol{z}$
- Elements p_{ij} of the transition matrix $\boldsymbol{P}$ have interpretation

$$p_{ij} = \text{Prob}\left[z_{t+1} = z_j \mid z_t = z_i\right]$$

- Elements $\psi_{0,i}$ of the initial distribution $\boldsymbol{\psi}_0$ have interpretation

$$\psi_{0,i} = \text{Prob}\left[z_0 = z_i\right]$$

Transitions: Probability Notation

- Let the vector ψ_t be the distribution over z at t, with elements

$$\psi_{t,i} = \text{Prob}\left[z_t = z_i\right]$$

- Using the transition probabilities gives

$$\psi_{1,i} = \sum_{j=1}^{n} \text{Prob}\left[z_1 = z_i \mid z_0 = z_j\right] \text{Prob}\left[z_0 = z_j\right]$$

$$\vdots$$

$$\psi_{t+1,i} = \sum_{j=1}^{n} \text{Prob}\left[z_{t+1} = z_i \mid z_t = z_j\right] \text{Prob}\left[z_t = z_j\right]$$

Transitions: Matrix Notation

- Collecting these together in matrix notation, we see that

$$\psi_1 = \boldsymbol{P}^\top \psi_0$$
$$\vdots$$
$$\psi_{t+1} = \boldsymbol{P}^\top \psi_t, \quad t = 0, 1, \ldots$$

  where $\boldsymbol{P}^\top$ denotes the transpose of $\boldsymbol{P}$

- Evolves according to a deterministic difference equation

- Iterating forward from date $t = 0$, we have

$$\psi_t = \left(\boldsymbol{P}^\top\right)^t \psi_0$$

Stationary Distributions

- Stationary distribution ψ^* of Markov chain satisfies

$$\psi^* = \boldsymbol{P}^\top \psi^*$$

(i.e., a fixed point of the difference equation $\psi_{t+1} = \boldsymbol{P}^\top \psi_t$)

- Writing this as

$$\left(\boldsymbol{I} - \boldsymbol{P}^\top\right) \psi^* = \mathbf{0}$$

we see ψ^* is an eigenvector of $\boldsymbol{P}^\top$ associated with a unit-eigenvalue

- Requirement that $\sum_i \psi_i^* = 1$ is a normalization of the eigenvector

A 2×2 Example

- Consider a two-state Markov chain with a transition matrix

$$\boldsymbol{P} = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

- Stationary distribution solves (note the transpose)

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1-p & q \\ p & 1-q \end{pmatrix} \right] \begin{pmatrix} \psi_1^* \\ \psi_2^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Gives

$$\begin{pmatrix} \psi_1^* \\ \psi_2^* \end{pmatrix} = \begin{pmatrix} \frac{q}{p+q} \\ \frac{p}{p+q} \end{pmatrix}$$

(e.g., $q \to 0$ makes state 2 absorbing and state 1 transient, etc)

Markov chains with Continuous Support

- We can also consider Markov chains with continuous support
- Suppose z_t has continuous support with density $\psi_t(z)$
- Intuitively

$$\psi_{t+1}(z') = \int p(z' \mid z)\, \psi_t(z) dz$$

where $p(z' \mid z)$ is density for $z_{t+1} = z'$ conditional on $z_t = z$

- A stationary density $\psi^*(z)$ satisfies the fixed point condition

$$\psi^*(z') = \int p(z' \mid z)\, \psi^*(z) dz$$

Most Used: AR(1) Example

- Suppose $\{z_t\}$ is a linear Gaussian AR(1) process

$$z_{t+1} = (1-\rho)\mu + \rho z_t + \sigma \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \text{IID}\, N(0,1)$$

- Then

$$p(z' \mid z) = \frac{1}{\sigma}\phi\left(\frac{z' - (1-\rho)\mu - \rho z}{\sigma}\right)$$

where $\phi(\varepsilon)$ is the PDF of the standard normal distribution

$$\phi(\varepsilon) \equiv \frac{1}{\sqrt{2\pi}} e^{-\varepsilon^2/2}$$

Most Used: AR(1) Example

- If $|\rho| < 1$, then a unique, stable stationary density

$$\psi^*(z) = \frac{1}{\sigma^*}\phi\left(\frac{z-\mu}{\sigma^*}\right), \text{where} \sigma^* = \frac{\sigma}{\sqrt{1-\rho^2}}$$

- Discretization of continuous AR(1) in computer
  - Step 1: Choose grids $\{z_1, z_2, ..., z_n\}$ (wisely)
  - Step 2: Choose a probability matrix $\boldsymbol{P}$
  - Now a continuous AR(1) works just like a discrete one!

Restate the Goal: Stochastic Dynamic Programming

- Suppose z_t is first-order Markov with conditional density

$$\pi\left(z' \mid z\right)$$

- Bellman equation for this problem

$$v(k, z) = \max_{k'} \left[u\left(zf(k) - k'\right) + \beta \int v\left(k', z'\right) \pi\left(z' \mid z\right) dz' \right]$$

- First order condition for k'

$$u'\left(zf(k) - k'\right) = \beta \int v_k\left(k', z'\right) \pi\left(z' \mid z\right) dz'$$

- Envelope condition

$$v_k(k, z) = u'\left(zf(k) - k'\right) zf'(k)$$

Solve the Stochastic Dynamic Programming

- Eliminating $v_k(k', z')$ using the envelope condition then gives

$$u'(zf(k) - k') = \beta \int u'(z'f(k') - k'') z'f'(k') \pi(z' \mid z) dz'$$

which, in our usual time-series notation, is just

$$u'(c_t) = \beta \mathbb{E}_t \{u'(c_{t+1}) z_{t+1} f'(k_{t+1})\}$$

where it is understood that $c_t = z_t f(k_t) - k_{t+1}$, etc.

Solution: Stochastic Dynamic Programming

- Let $k' = g(k, z)$ be the optimal policy that solves this dynamic programming problem
- This is a stochastic difference equation of the form

$$k_{t+1} = g\left(k_t, z_t\right)$$

- We cannot expect $\{k_t\}$ to converge to some steady state k^* (does not exist!)
- The Past: All summarized in the state (k_t, z_t)
- The Future: Depends on expectation $\mathbb{E}_t z_t$ and choice k_{t+1}

IID Example

- Suppose policy function has the multiplicative form $k_{t+1} = z_t g(k_t)$

- And that z_t is IID (independent and identically distributed) over time with cumulative distribution

$$H(z) \equiv \text{Prob}[z_t \leqslant z]$$

- Now consider the cumulative distribution of k at time t

$$\Psi_t(k) \equiv \text{Prob}[k_t \leqslant k]$$

- For example, for $t = 1$ we have

$$\Psi_1(k) = \text{Prob}[k_1 \leqslant k] = \text{Prob}[z_0 g(k_0) \leqslant k] = \text{Prob}\left[z_0 \leqslant \frac{k}{g(k_0)}\right]$$
$$= H\left(\frac{k}{g(k_0)}\right)$$

IID Example

- Let $P(k' \mid k)$ denote the conditional distribution

$$P(k' \mid k) \equiv \text{Prob}\left[k_{t+1} \leqslant k' \mid k_t = k\right]$$

- For this IID example, we have

$$P(k' \mid k) = H\left(\frac{k'}{g(k)}\right)$$

- Then cumulative distribution of k satisfies the law of motion

$$\Psi_{t+1}(k') = \int P(k' \mid k)\, d\Psi_t(k)$$

- So that if there is a density representation

$$\psi_{t+1}(k') = \int p(k' \mid k)\, \psi_t(k) dk$$

IID Example

- A stationary density $\psi^*(k)$ is a fixed point of this law of motion

$$\psi^*\left(k'\right) = \int p\left(k' \mid k\right)\psi^*(k)dk$$

- More generally, we would have a joint distribution over the state variables (k, z) induced by (i) the policy function $k' = g(k, z)$ and (ii) the exogenous conditional density $\pi\left(z' \mid z\right)$

- We would then look for a fixed point for that joint distribution

- We'll see lots of examples of this

References

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