PhD Macro Core Part I:
Lecture 9 – Consumption-based Asset Pricing

Min Fang
University of Florida

Fall 2024

Today

- First Application of Stochastic Dynamic Programming
- Asset Prices in an Endowment Economy
- Contingent Claims
- Implications for Asset Returns

Overview

- Endowment economy
- Single type of durable asset: a fruit tree
- Asset delivers a flow of non-durable consumption goods (dividends)
  - each tree produces a random harvest of fruit
- We want to see how this durable asset is priced in equilibrium

Setup

- Time $t = 0, 1, 2, \ldots$

- Representative consumer, endowed with initial asset holdings k_0

- Stochastic flow of consumption goods y_t per unit of the asset with conditional distribution

$$F(y' \mid y) = \text{Prob}\left[y_{t+1} \leqslant y' \mid y_t = y\right]$$

- Price p_t of buying the asset at date t, taken as given

- Price p_t is 'ex-dividend' (asset bought at t, first dividend at $t+1$)

Recursive Problem

- Representative consumer takes as given a pricing function $p(y)$

- Bellman equation can be written

$$v(k, y) = \max_{k' \geqslant 0} \left[u(c) + \beta \int v\left(k', y'\right) dF\left(y' \mid y\right) \right]$$

  subject to

$$c + p(y)k' \leqslant (p(y) + y)k$$

- The RHS of the budget constraint is the consumer's wealth

$$w \equiv (p(y) + y)k$$

- Let $k' = g(k, y)$ denote the policy function implied by the maximization on the RHS of the Bellman equation

Recursive Problem

- Alternative Bellman equation using wealth as the state variable

$$v(w, y) = \max_{c \geqslant 0} \left[u(c) + \beta \int v\left(w', y'\right) dF\left(y' \mid y\right) \right]$$

subject to

$$w' = R\left(y', y\right)(w - c)$$

- The term $R\left(y', y\right)$ is the gross return on the asset

$$R\left(y', y\right) \equiv \frac{p\left(y'\right) + y'}{p(y)}$$

(also the return on wealth since here there is only one asset)

Recursive Competitive Equilibrium

- A recursive competitive equilibrium is a value function $v(k, y)$, policy function $g(k, y)$ and pricing function $p(y)$ such that:

  (i) taking $p(y)$ as given, $v(k, y)$ and $g(k, y)$ solve the consumer's recursive problem, and

  (ii) the asset market clears

$$g(k, y) = k_0, \qquad \text{for all } k, y$$

- If the asset market clears, the budget constraint implies

$$c + p(y)k_0 = (p(y) + y)k_0$$

  so that we have the goods market clearing condition

$$c = yk_0$$

Recursive Competitive Equilibrium

- Normalize initial asset holdings to $k_0 = 1$
- Then equilibrium consumption allocation is

$$c = y$$

- What does this consumption allocation imply for the asset prices?

Characterizing Asset Prices

- The first order condition for the consumer can be written

$$u_1(c)p(y) = \beta \int v_1\left(k', y'\right) dF\left(y' \mid y\right)$$

where it is understood that

$$c = (p(y) + y)k - p(y)g(k, y)$$

and where $u_1(c)$ and $v_1(k, y)$ denote first derivatives

- Writing

$$v(k, y) = u((p(y) + y)k - p(y)g(k, y)) + \beta \int v\left(g(k, y), y'\right) dF\left(y' \mid y\right)$$

the envelope condition gives

$$v_1(k, y) = u_1(c)(p(y) + y)$$

Characterizing Asset Prices

- Combining the first order and envelope conditions gives

$$u_1(c) = \beta \int u_1\left(c'\right) \frac{p\left(y'\right) + y'}{p(y)} dF\left(y' \mid y\right)$$

  where it is understood that c, c' are evaluated at the optimum

- Notice that this is the same as

$$u_1(c) = \beta \int u_1\left(c'\right) R\left(y', y\right) dF\left(y' \mid y\right)$$

  which, in our usual time-series notation, is

$$u_1\left(c_t\right) = \beta \mathbb{E}_t \left\{u_1\left(c_{t+1}\right) R_{t+1}\right\}, \quad R_{t+1} = \frac{p_{t+1} + y_{t+1}}{p_t}$$

Equilibrium asset prices

- In equilibrium $c = y$, so equilibrium asset prices $p(y)$ solve

$$u_1(y) = \beta \int u_1\left(y'\right) \frac{p\left(y'\right) + y'}{p(y)} dF\left(y' \mid y\right)$$

or

$$p(y) = \beta \int \frac{u_1\left(y'\right)}{u_1(y)} \left(p\left(y'\right) + y'\right) dF\left(y' \mid y\right)$$

- The equilibrium pricing function $p(y)$ is a fixed point of this functional equation

- This functional equation is linear, so this basically reduces to solving a linear algebra problem (we'll see some examples)

Iterated Euler Equations

- In time-series notation, we have

$$p_t = \mathbb{E}_t \left\{ \beta \frac{u_1\left(y_{t+1}\right)}{u_1\left(y_t\right)} \left(p_{t+1} + y_{t+1}\right) \right\}$$

- But

$$p_{t+1} = \mathbb{E}_{t+1} \left\{ \beta \frac{u_1\left(y_{t+2}\right)}{u_1\left(y_{t+1}\right)} \left(p_{t+2} + y_{t+2}\right) \right\}$$

Iterated Euler Equations

- Substituting for p_{t+1}

$$
\begin{aligned}
p_t &= \mathbb{E}_t \left\{ \beta \frac{u_1(y_{t+1})}{u_1(y_t)} \left(\mathbb{E}_{t+1} \left\{ \beta \frac{u_1(y_{t+2})}{u_1(y_{t+1})} (p_{t+2} + y_{t+2}) + y_{t+1} \right) \right\} \right\} \\
&= \mathbb{E}_t \left\{ \beta \frac{u_1(y_{t+1})}{u_1(y_t)} y_{t+1} + \beta^2 \frac{u_1(y_{t+2})}{u_1(y_t)} (y_{t+2} + p_{t+2}) \right\}
\end{aligned}
$$

  where the second line uses the law of iterated expectations

- More generally, we have, iterating forward T times,

$$
p_t = \mathbb{E}_t \left\{ \sum_{j=1}^{T} \beta^j \frac{u_1(y_{t+j})}{u_1(y_t)} y_{t+j} \right\} + \mathbb{E}_t \left\{ \beta^T \frac{u_1(y_{t+T})}{u_1(y_t)} p_{t+T} \right\}, \text{ in the limit, we have:}
$$

$$
p_t = \mathbb{E}_t \left\{ \sum_{j=1}^{\infty} \beta^j \frac{u_1(y_{t+j})}{u_1(y_t)} y_{t+j} \right\} + \mathbb{E}_t \left\{ \lim_{T \to \infty} \beta^T \frac{u_1(y_{t+T})}{u_1(y_t)} p_{t+T} \right\}
$$

Iterated Euler Equations

- Think of this as
$$p_t = \text{ fundamental component } + \text{ speculative component}$$

- In equilibrium, the speculative component is zero. To see why, suppose not. For example,
$$\mathbb{E}_t \left\{ \beta^T u_1 \left(y_{t+T} \right) p_{t+T} \right\} > 0$$
Then, the marginal value of selling the asset exceeds the value of consuming its dividends forever
$$u_1 \left(y_t \right) p_t > \mathbb{E}_t \left\{ \sum_{j=1}^{\infty} \beta^j u_1 \left(y_{t+j} \right) y_{t+j} \right\}$$
So everyone would want to sell the asset, driving its price down

- Likewise, if the speculative component is < 0, then everyone would prefer to buy it, driving its price up

Iterated Euler Equations

- Thus, in this model, equilibrium asset prices are given by the fundamental component

$$p_t = \mathbb{E}_t \left\{ \sum_{j=1}^{\infty} \beta^j \frac{u_1\left(y_{t+j}\right)}{u_1\left(y_t\right)} y_{t+j} \right\}$$

(i.e., the expected discounted value of the dividend stream)

- Dividends are discounted from $t + j$ back to t using the **stochastic discount factor**

$$M_{t,t+j} = \beta^j \frac{u_1\left(y_{t+j}\right)}{u_1\left(y_t\right)}$$

Example: Log Utility

- Suppose $u(c) = \log c$ so that $u_1(c) = 1/c$. Then

$$p_t = \mathbb{E}_t \left\{ \sum_{j=1}^{\infty} \beta^j \frac{1/y_{t+j}}{1/y_t} y_{t+j} \right\}$$

$$= \mathbb{E}_t \left\{ \sum_{j=1}^{\infty} \beta^j y_t \right\} = \frac{\beta}{1-\beta} y_t$$

- So, for log utility, the equilibrium pricing function is

$$p(y) = \frac{\beta}{1-\beta} y$$

When y is high, consumers seek smooth consumption by buying assets, and asset prices rise to ensure $k = 1$. When y is low, consumers seek to smooth consumption by selling assets, and asset prices fall to again ensure $k = 1$

Example: Log Utility

- Constant price/dividend ratio

$$\frac{p_t}{y_t} = \frac{\beta}{1-\beta} = \frac{1}{\rho}$$

where $\rho = \frac{1}{\beta} - 1$ is the pure rate of time preference

- Capital gains

$$\frac{p_{t+1} - p_t}{p_t} = \frac{y_{t+1} - y_t}{y_t}$$

- Gross return

$$R_{t+1} = \frac{p_{t+1} + y_{t+1}}{p_t} = \frac{1}{\beta}\frac{y_{t+1}}{y_t} = (1+\rho)\frac{y_{t+1}}{y_t}$$

Example: Log Utility

- In this example, equilibrium price p_t does not depend on properties of expected future y_{t+j}.
- Why not?
- Suppose y_{t+j} is expected to be high. This will tend to drive up demand for the asset
- But high y_{t+j} means $u_1\left(y_{t+j}\right)$ is low. This will tend to drive down demand for the asset
- With log utility, these two effects exactly cancel (c.f., income and substitution effects)

Example: CRRA with IID Dividend Growth

- Suppose $u_1(c) = c^{-\sigma}$ and $g_{t+1} \equiv y_{t+1}/y_t$ is IID over time. Equilibrium prices are given by

$$p_t = \mathbb{E}_t \left\{ \sum_{j=1}^{\infty} \beta^j \left(\frac{y_{t+j}}{y_t} \right)^{-\sigma} y_{t+j} \right\}$$

  Dividing both sides by y_t, price/dividend ratio given by

$$\frac{p_t}{y_t} = \mathbb{E}_t \left\{ \sum_{j=1}^{\infty} \beta^j \left(\frac{y_{t+j}}{y_t} \right)^{1-\sigma} \right\}$$

- Notice that

$$\frac{y_{t+j}}{y_t} = \frac{y_{t+j}}{y_{t+j-1}} \times \cdots \times \frac{y_{t+1}}{y_t} = \prod_{i=1}^{j} g_{t+i}$$

Example: CRRA with IID Dividend Growth

- Since dividend growth is IID

$$\mathbb{E}_t\left\{\prod_{i=1}^{j} g_{t+i}^{1-\sigma}\right\} = \left(\mathbb{E}\left[g^{1-\sigma}\right]\right)^j = \delta^j, \quad \delta \equiv \mathbb{E}\left[g^{1-\sigma}\right]$$

- So, the equilibrium price/dividend ratio is

$$\frac{p_t}{y_t} = \sum_{j=1}^{\infty}(\beta\delta)^j = \frac{\beta\delta}{1-\beta\delta}$$

  and equilibrium pricing function is

$$p(y) = \frac{\beta\delta}{1-\beta\delta}y$$

- Price/dividend ratio again constant, etc, but now coefficient depends on g_{t+1} distribution and risk aversion

Discussion

- Individually, a consumer perceives net return on asset to be r_{t+1}
- But the social net return on the asset is 0 (resources spent on assets do not deliver more resources in the future)
- The general equilibrium consequence of every individual trying to save at rate r_{t+1} is a social return of 0

Contingent Claims

- An Arrow security is an asset that delivers one unit of consumption if and only if a particular state is realized

- Let $q(y', y)$ denote the price in state y of an Arrow security that delivers one unit of consumption if and only if y' is realized next period

- Suppose the representative consumer can trade in a complete set of Arrow securities

- Let $\boldsymbol{a}'$ denote the representative consumer's portfolio of Arrow securities with typical element $a(y')$

Dynamic Programming Problem

- Pricing functions $q\left(y', y\right)$ and $p(y)$ taken as given
- Bellman equation can be written

$$v(\boldsymbol{a}, k, y) = \max_{\boldsymbol{a}', k'} \left[u(c) + \beta \int v\left(\boldsymbol{a}', k', y'\right) dF\left(y' \mid y\right) \right]$$

subject to

$$c + p(y)k' + \int q\left(y', y\right) a\left(y'\right) dy' \leqslant (p(y) + y)k + a(y)$$

- As before, can also use wealth as a state variable

Characterizing Asset Prices

- First order conditions for each $a(y')$ are

$$u_1(c)q(y',y) = \beta v_1(\boldsymbol{a}',k',y') f(y' \mid y)$$

where $f(y' \mid y)$ is the density associated with $F(y' \mid y)$

- First order condition for k' is

$$u_1(c)p(y) = \beta \int v_2(\boldsymbol{a}',k',y')\, dF(y' \mid y)$$

- Envelope conditions

$$v_1(\boldsymbol{a},k,y) = u_1(c)$$

and

$$v_2(\boldsymbol{a},k,y) = u_1(c)(p(y)+y)$$

Equilibrium Asset Prices

- In equilibrium again have $c = y$
- Equilibrium prices of Arrow securities are therefore

$$q\left(y', y\right) = \beta \frac{u_1\left(y'\right)}{u_1(y)} f\left(y' \mid y\right)$$

- Equilibrium price of the durable asset again solves

$$p(y) = \beta \int \frac{u_1\left(y'\right)}{u_1(y)} \left(p\left(y'\right) + y'\right) dF\left(y' \mid y\right)$$

Pricing Other Assets

- Consider an asset i that pays $x_i\left(y'\right)$ in state y'

- By no-arbitrage, this asset will have a price equal to

$$q_i(y) = \int q\left(y', y\right) x_i\left(y'\right) dy'$$

- And so in equilibrium

$$q_i(y) = \beta \int \frac{u_1\left(y'\right)}{u_1(y)} x_i\left(y'\right) dF\left(y' \mid y\right)$$

(i.e., of the form $q_i = \mathbb{E}\left[M x_i\right]$ where M is the one-period SDF)

- Payoff $x_i\left(y'\right)$ could be anything, e.g., payoffs of some exotic option

Asset Returns

- Let $R^{i}\left(y', y'\right)$ denote the gross return on such an asset

$$R^{i}\left(y', y\right) = \frac{x_{i}\left(y'\right)}{q_{i}(y)}$$

- Can then restate these conditions in terms of asset returns

$$1 = \beta \int \frac{u_{1}\left(y'\right)}{u_{1}(y)} R^{i}\left(y', y\right) dF\left(y' \mid y\right)$$

- Or in more standard time-series notation

$$1 = \mathbb{E}_{t}\left\{\beta \frac{u_{1}\left(y_{t+1}\right)}{u_{1}\left(y_{t}\right)} R_{t+1}^{i}\right\}$$

Risk Free Asset

- Consider a sure claim to a unit of consumption at the next period

- This has $x(y') = 1$ for all y' and has price

$$q_f(y) = \int q(y', y)\, 1 dy' = \beta \int \frac{u_1(y')}{u_1(y)} dF(y' \mid y)$$

  and return $R^f(y)$ independent of y' (in this sense it is risk-free)

$$R^f(y) = \frac{1}{q_f(y)}$$

- Hence, in time-series notation, we can write

$$1 = \mathbb{E}_t \left\{ \beta \frac{u_1(y_{t+1})}{u_1(y_t)} R_t^f \right\}, \text{ so, risk-free return is } R_t^f = 1/\mathbb{E}_t \left\{ \beta \frac{u_1(y_{t+1})}{u_1(y_t)} \right\}$$

Consumption-based Asset Pricing

- Return on any asset i

$$1 = \mathbb{E}_t \left\{ M_{t+1} R^i_{t+1} \right\}, \quad M_{t+1} = \beta \frac{u_1\left(c_{t+1}\right)}{u_1\left(c_t\right)}$$

where M_{t+1} is the one-period stochastic discount factor (SDF)

- Return on a risk-free asset

$$1 = \mathbb{E}_t \left\{ M_{t+1} R^f_t \right\}$$

- Expanding the expectation of the product gives

$$1 = \mathbb{E}_t \{M_{t+1}\} \mathbb{E}_t \left\{ R^i_{t+1} \right\} + \mathrm{Cov}_t \left\{ M_{t+1}, R^i_{t+1} \right\}$$

Expected Excess Returns

- Since $R_t^f = 1/\mathbb{E}_t\{M_{t+1}\}$ we can write this as

$$1 = \frac{1}{R_t^f}\mathbb{E}_t\left\{R_{t+1}^i\right\} + \mathrm{Cov}_t\left\{M_{t+1}, R_{t+1}^i\right\}$$

  or

$$\mathbb{E}_t\left\{R_{t+1}^i\right\} - R_t^f = -R_t^f\,\mathrm{Cov}_t\left\{M_{t+1}, R_{t+1}^i\right\}$$

- All assets have an expected return equal to the risk-free return plus a risk premium (which may be positive or negative)

Risk Premia

- The risk premia are given by

$$-R_t^f \operatorname{Cov}_t \left\{M_{t+1}, R_{t+1}^i\right\} = -\frac{\operatorname{Cov}_t \left\{u_1\left(c_{t+1}\right), R_{t+1}^i\right\}}{\mathbb{E}_t \left\{u_1\left(c_{t+1}\right)\right\}}$$

- In general, these risk premia are time-varying via the conditioning information (but we will see examples where they are constant)

- What determines risk premia is not the variance of returns, but rather how those returns covary with consumption

- Investors do not care about the volatility of their portfolio per se; it depends on how that translates to volatility in consumption

Risk Premia

- Asset returns that covary negatively with M_{t+1} deliver high payoffs when marginal utility is low - these are a bad hedge, will be in low demand, and carry a high-risk premium

- (assets that covary positively with c_{t+1} make consumption more volatile)

- Asset returns that covary positively with M_{t+1} deliver high payoffs when marginal utility is high - these are a good hedge, will be in high demand, and carry a low (or negative) risk premium

- (assets that covary negatively with c_{t+1} make consumption less volatile)

Idiosyncratic Risk is Not Priced

- Only that part of an asset return that is correlated with the aggregate M_{t+1} leads to a risk adjustment (positive or negative)

- The idiosyncratic component in returns offers no better (or worse) hedging opportunities and so is not priced

Consumption CAPM

- This is a version of the capital asset pricing model (CAPM) except that covariance with the 'market return' is replaced with covariance with the SDF

- This setting where
$$M_{t+1} = \beta \frac{u_1\left(c_{t+1}\right)}{u_1\left(c_t\right)}$$
is often referred to as the consumption-CAPM

- Although elegant and intuitive, it is difficult to reconcile this model with data on stock and bond returns (huge literature on the 'equity premium puzzle', the 'risk-free rate puzzle' etc, etc)

References

- [SLR] Sargent and Ljungqvist, Recursive Macroeconomic Theory (4th)
  - Chapter 13: Asset Pricing Theory
- [AKMM] Azzimonti, Krusell, McKay, and Mukoyama, Macroeconomics
  - Chapter 15: Asset Prices