PhD Macro Core Part I:
Lecture 1 – A Simple Dynamic Exchange Economy II

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Today

- Compute Equilibrium using Social Planner’s Problem
- Second Welfare Theorem
- Sequential Markets Equilibrium

Compute Equilibrium using Social Planner's Problem

- Compute for dynamic general equilibrium models would be much harder
- However, if the welfare theorem(s) hold, we have:
  - Competitive equilibrium allocations are Pareto optimal (last lecture)
  - Compute all Pareto-optimal allocations by solving an appropriate social planner problem
  - Isolate those who are, in fact, competitive equilibrium allocations
- This is Negishi's (1960) method to compute equilibria

Social Planner's Problem: Setup

- For a Pareto weights $\alpha^i \geqslant 0$, consider the following social planner problem

$$
\begin{aligned}
& \max_{\{(c_t^1, c_t^2)\}_{t=0}^{\infty}} \alpha^1 u\left(c^1\right) + \alpha^2 u\left(c^2\right) \\
= & \max_{\{(c_t^1, c_t^2)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\alpha^1 \ln\left(c_t^1\right) + \alpha^2 \ln\left(c_t^2\right)\right] \\
\text{s.t.} \quad & c_t^i \geqslant 0 \text{ for all } i, \text{ all } t \\
& c_t^1 + c_t^2 = e_t^1 + e_t^2 \equiv 2 \text{ for all } t
\end{aligned} \tag{2.11}
$$

- Optimal consumption choices are functions of $\alpha = \left(\alpha^1, \alpha^2\right)$

$$
\left\{\left(c_t^1, c_t^2\right)\right\}_{t=0}^{\infty} = \left\{\left(c_t^1(\alpha), c_t^2(\alpha)\right)\right\}_{t=0}^{\infty}
$$

Social Planner's Problem: Properties

- **Proposition:** Any allocation $\left\{\left(c_t^1, c_t^2\right)\right\}_{t=0}^{\infty}$ that solves the solves the social planners problem (2.11) for some vector of Pareto weights $\alpha > 0$ is Pareto efficient.
- **Proposition:** Conversely, any Pareto efficient allocation $\left\{\left(c_t^1, c_t^2\right)\right\}_{t=0}^{\infty}$ is the solution to the social planners problem (2.11) for some vector of Pareto weights $\alpha \geqslant 0$, $\alpha \neq 0$.
- Proofs omitted
- So, we can characterize the set of all Pareto efficient allocations by varying the α 's
- All that matters are the relative weights α^1/α^2

Solve Social Planner's Problem: FOCs

- Solve the planner's problem for arbitrary $\alpha \geqslant 0$
- Attach Lagrange multipliers $\frac{\mu_t}{2}$ to the resource constraints
- The first order of necessary conditions are

$$\frac{\alpha^1 \beta^t}{c_t^1} = \frac{\mu_t}{2}$$

$$\frac{\alpha^2 \beta^t}{c_t^2} = \frac{\mu_t}{2}$$

Combining yields

$$\frac{c_t^1}{c_t^2} = \frac{\alpha^1}{\alpha^2} \tag{2.12}$$

$$c_t^1 = \frac{\alpha^1}{\alpha^2} c_t^2 \tag{2.13}$$

Solve Social Planner's Problem: Lagrange Multipliers

- Using the resource constraint in conjunction with (2.13) yields

$$
\begin{aligned}
c_t^1 + c_t^2 &= 2 \\
\frac{\alpha^1}{\alpha^2} c_t^2 + c_t^2 &= 2 \\
c_t^2 &= 2/\left(1 + \alpha^1/\alpha^2\right) = c_t^2(\alpha) \\
c_t^1 &= 2/\left(1 + \alpha^2/\alpha^1\right) = c_t^1(\alpha)
\end{aligned}
$$

- The Lagrange multipliers are given by

$$\mu_t = \left(\alpha^1 + \alpha^2\right) \beta^t$$

- We can normalize $\alpha^1 + \alpha^2 = 1$, so the Lagrange multipliers are then

$$\mu_t = \beta^t$$

Solve Social Planner's Problem: Map to C.E.

- The set of Pareto efficient allocations is given by

$$PO = \left\{ \left\{ \left(c_t^1, c_t^2 \right) \right\}_{t=0}^{\infty} : c_t^1 = 2/\left(1 + \alpha^2/\alpha^1\right) \text{ and } c_t^2 = 2/\left(1 + \alpha^1/\alpha^2\right) \right\}$$

- Compare FOCs of the competitive market and the social planner's problem:

$$\frac{\beta^t}{c_t^i} = \lambda_i p_t, \quad \text{and} \quad \frac{\alpha^i \beta^t}{c_t^i} = \frac{\mu_t}{2}$$

- This suggests that if we were to construct an equilibrium with allocations from the set PO:

$$p_t = \mu_t = \beta^t, \quad \text{and} \quad \lambda_i = \frac{1}{2\alpha^i}$$

Solve Social Planner's Problem: Final Steps

- Define the transfer functions $t^i(\alpha), i = 1, 2$ by

$$t^i(\alpha) = \sum_t \mu_t \left[c_t^i(\alpha) - e_t^i\right], \qquad \text{with "right" at C.E. } t^i(\alpha) = 0$$

- Using the Lagrange multipliers before normalization, we note that

$$t^i(\alpha) = \sum_t \mu_t \left[c_t^i(\alpha) - e_t^i\right] = \sum_{t=0}^{\infty} \left(\alpha^1 + \alpha^2\right) \beta^t \left[c_t^i(\alpha) - e_t^i\right]$$

- Fortunately, the transfer functions sum to zero for all α, since

$$\sum_{i=1}^{2} t^i(\alpha) = \sum_{i=1}^{2} \sum_t \mu_t \left[c_t^i(\alpha) - e_t^i\right] = \sum_t \mu_t \sum_{i=1}^{2} \left[c_t^i(\alpha) - e_t^i\right] = 0$$

Solve Social Planner's Problem: Final Steps

- Doing so yields

$$
\begin{aligned}
t^1(\alpha) &= \sum_{t=0}^{\infty} \left(\alpha^1 + \alpha^2\right) \beta^t \left[c_t^1(\alpha) - e_t^1\right] = \sum_{t=0}^{\infty} \beta^t \left[2/\left(1 + \alpha^2/\alpha^1\right) - e_t^1\right] \\
&= \frac{2}{(1-\beta)\left(1 + \alpha^2/\alpha^1\right)} - \frac{2}{1-\beta^2} = 0
\end{aligned}
$$

therefore,

$$
\alpha^2/\alpha^1 = \beta
$$

- Finally, the corresponding consumption allocations are

$$
\begin{aligned}
c_t^1 &= \frac{2}{1 + \alpha^2/\alpha^1} = \frac{2}{1+\beta} \\
c_t^2 &= \frac{2}{1 + \alpha^1/\alpha^2} = \frac{2\beta}{1+\beta}
\end{aligned}
$$

Solve Social Planner's Problem: Remarks

1. Solve the social planner's problem for Pareto efficient allocations indexed by the Pareto weights
2. Compute transfers, indexed by α, necessary to make the efficient allocation affordable. Prices use Lagrange multipliers on the resource constraints in the planners' problem
3. Find the normalized Pareto weight(s) $\hat{\alpha}$ that makes the transfer functions 0
4. The Pareto efficient allocations corresponding to $\hat{\alpha}$ are equilibrium allocations; the supporting equilibrium prices are (multiples of) the Lagrange multipliers from the planning problem

Second Welfare Theorem

- The second welfare theorem provides a converse to the first welfare theorem.

- **Core:** Under suitable assumptions, it states that for any Pareto-optimal allocation, a price system exists such that the allocation and price system form a competitive equilibrium.

- **Assumption:**
  - All assumptions of the first theorem; in addition:
  - The preference relation is locally non-satiated and convex for each consumer
  - The production set is convex for each firm
  - For the step from price quasi-equilibrium to price equilibrium with transfers: The initial endowment of each agent is strictly positive.

- Proof omitted (from MWG)

Sequential Markets Equilibrium: Setup

- Let r_{t+1} denote the interest rate on one period bonds from period t to period $t+1$.

- A one-period bond is a promise (contract) to pay 1 unit of the consumption good in period $t+1$ in exchange for $\frac{1}{1+r_{t+1}}$ units of the consumption good in period t.

- We can interpret $q_t \equiv \frac{1}{1+r_{t+1}}$ as the relative price of one unit of the consumption good in period $t+1$ in terms of the period t consumption good.

- Let a_{t+1}^i denote the amount of such bonds purchased by agent i in period t and carried over to period $t+1$. (If $a_{t+1}^i < 0$, we can interpret this as the agent taking out a one-period loan.)

- Household i 's budget constraint in period t reads as

$$c_t^i + \frac{a_{t+1}^i}{(1+r_{t+1})} \leqslant e_t^i + a_t^i \tag{2.17}$$

Sequential Markets Equilibrium: Definition

- **Definition:** A Sequential Markets equilibrium is allocations $\left\{\left(\hat{c}_t^i, \hat{a}_{t+1}^i\right)_{i=1,2}\right\}_{t=0}^{\infty}$, interest rates (which is prices in SM equilibrium) $\{\hat{r}_{t+1}\}_{t=0}^{\infty}$ such that
  1. For $i = 1, 2$, given interest rates $\{\hat{r}_{t+1}\}_{t=0}^{\infty}$, $\left\{\hat{c}_t^i, \hat{a}_{t+1}^i\right\}_{t=0}^{\infty}$ solves

$$\max_{\left\{c_t^i, a_{t+1}^i\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln\left(c_t^i\right) \tag{2.18}$$

$$c_t^i + \frac{a_{t+1}^i}{\left(1+\hat{r}_{t+1}\right)} \leqslant e_t^i + a_t^i$$

$$c_t^i \geqslant 0 \text{ for all } t \tag{2.19}$$

$$a_{t+1}^i \geqslant -\bar{A}^i \tag{2.20}$$

  2. For all $t \geqslant 0$

$$\sum_{i=1}^{2} \hat{c}_t^i = \sum_{i=1}^{2} e_t^i, \quad \text{and} \quad \sum_{i=1}^{2} \hat{a}_{t+1}^i = 0$$

Sequential Markets Equilibrium: Properties

- **Proposition:** Let allocations $\left\{\left(\hat{c}_t^i\right)_{i=1,2}\right\}_{t=0}^{\infty}$ and prices $\{\hat{p}_t\}_{t=0}^{\infty}$ form an Arrow-Debreu equilibrium w/

$$\frac{\hat{p}_{t+1}}{\hat{p}_t} \leqslant \xi < 1 \text{ for all } t \tag{2.22}$$

Then: (1) there exist an S.M.E. w/ $\left(\bar{A}^i\right)_{i=1,2}$, $\left\{\left(\tilde{c}_t^i, \tilde{a}_{t+1}^i\right)_{i=1,2}\right\}_{t=0}^{\infty}$, and $\{\tilde{r}_{t+1}\}_{t=0}^{\infty}$ such that

$$\tilde{c}_t^i = \hat{c}_t^i \text{ for all } i \text{, all } t$$

(2) Reversely, let $\left\{\left(\hat{c}_t^i, \hat{a}_{t+1}^i\right)_{i=1,2}\right\}_{t=0}^{\infty}$ and $\{\hat{r}_{t+1}\}_{t=0}^{\infty}$ form an S.M.E. that satisfies

$$\begin{aligned} \hat{a}_{t+1}^i &> -\bar{A}^i \text{ for all } i, \text{ all } t \\ \hat{r}_{t+1} &\geqslant \varepsilon > 0 \text{ for all } t \end{aligned} \tag{2.23}$$

Then there exists a corresponding Arrow-Debreu equilibrium $\left\{\left(\tilde{c}_t^i\right)_{i=1,2}\right\}_{t=0}^{\infty}$, $\{\tilde{p}_t\}_{t=0}^{\infty}$ such that

$$\hat{c}_t^i = \tilde{c}_t^i \text{ for all } i, \text{ all } t$$

Sequential Markets Equilibrium: Proof Steps

- Step 1: Show that any consumption allocation that satisfies the sequence of SM budget constraints is also in the AD budget set.

  From this, it fairly directly follows that **an AD equilibria can be made into SM equilibria**.

- Step 2: Make sure that we can find a large enough borrowing limit $\bar{A}^i$ such that the asset holdings required to implement the AD consumption allocation as an S.M.E. do not violate any Ponzi constraint.

- Step 3: Argue that **an SM equilibrium can be made into an AD equilibrium**.

Dynamic Exchange Economy: Takeaways

- Introduced basic concepts of model setting
- Introduced basic concepts of markets (incomplete/complete)
- Introduced basic concepts of welfare theorems
- Introduced basic concepts of equilibrium definition & solutions
- A bit of proofs in macroeconomics

References

- [DK] Dirk Krueger, Macroeconomic Theory (2015)
  - Chapter 2: A Simple Dynamic Economy

- Please refer to the book for all other proofs