PhD Macro Core Part I:
Lecture 10 – Consumption-savings Problems

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Today

- Second Application of Stochastic Dynamic Programming
- Consumption-savings Problems
- Review of Linear-quadratic Permanent Income Theory
- Effects of Income Uncertainty in More General Settings

Review of Permanent Income Theory

- Time $t = 0, 1, 2, \ldots$
- Single agent with risk-averse preferences

$$\mathbb{E}\left\{\sum_{t=0}^{\infty} \beta^t u\left(c_t\right)\right\}, \quad 0 < \beta < 1$$

- Flow budget constraint

$$a_{t+1} = R\left(a_t + y_t - c_t\right)$$

given some stochastic process for income y_t

- Consumption Euler equation

$$u'\left(c_t\right) = \beta R \mathbb{E}_t\left\{u'\left(c_{t+1}\right)\right\}$$

Hall (1978)

- Strict version of the permanent income hypothesis (PIH)

- Quadratic utility

$$u(c) = c - \frac{b}{2}c^2, \quad b > 0$$

- interest rate equals the rate of time preference

$$\beta R = 1$$

- Then consumption Euler equation simply implies

$$c_t = \mathbb{E}_t\{c_{t+1}\}$$

- Implies consumption is a martingale (e.g., a random walk)

- More generally, marginal utility is a martingale

Iterating Forward

- At $t = 0$ we have

$$a_1 = R\left(a_0 + y_0 - c_0\right)$$

- At $t = 1$ we have

$$a_2 = R\left(a_1 + y_1 - c_1\right) = R^2\left(a_0 + y_0 - c_0\right) + R\left(y_1 - c_1\right)$$

- At $t = 2$ we have

$$a_3 = R^3\left(a_0 + y_0 - c_0\right) + R^2\left(y_1 - c_1\right) + R\left(y_2 - c_2\right)$$

Iterating Forward

- Iterating this out to some arbitrary date T

$$a_{T+1} = R^{T+1} a_0 + \sum_{t=0}^{T} R^{T+1-t} \left(y_t - c_t \right)$$

- Dividing both sides by R^{T+1} and rearranging

$$\sum_{t=0}^{T} R^{-t} c_t + R^{-(T+1)} a_{T+1} = a_0 + \sum_{t=0}^{T} R^{-t} y_t$$

- Taking $T \to \infty$ and imposing the no-Ponzi-scheme constraint

$$\sum_{t=0}^{\infty} R^{-t} c_t = a_0 + \sum_{t=0}^{\infty} R^{-t} y_t$$

Intertemporal Budget Constraint

- Nothing special about period $t = 0$ so write this as

$$\sum_{j=0}^{\infty} R^{-j} c_{t+j} = a_t + \sum_{j=0}^{\infty} R^{-j} y_{t+j}$$

- Also holds in expectation

$$\mathbb{E}_t \left\{ \sum_{j=0}^{\infty} R^{-j} c_{t+j} \right\} = a_t + \mathbb{E}_t \left\{ \sum_{j=0}^{\infty} R^{-j} y_{t+j} \right\}$$

Solving for Consumption

- Interchanging the sum and expectations

$$\sum_{j=0}^{\infty} R^{-j}\mathbb{E}_t\{c_{t+j}\} = a_t + \sum_{j=0}^{\infty} R^{-j}\mathbb{E}_t\{y_{t+j}\}$$

- But from the consumption Euler equation and the law of iterated expectations

$$\mathbb{E}_t\{c_{t+j}\} = c_t \quad \text{for all } j$$

Solving for Consumption

- This gives the solution

$$c_t = \frac{r}{1+r}\left(a_t + \sum_{j=0}^{\infty} R^{-j}\mathbb{E}_t\left\{y_{t+j}\right\}\right), \quad R = 1+r$$

- It is customary to refer to a_t as 'financial wealth; and to define 'human wealth' h_t by

$$h_t \equiv \sum_{j=0}^{\infty} R^{-j}\mathbb{E}_t\left\{y_{t+j}\right\}$$

- Consumption out of total wealth (i.e., 'permanent income') is

$$c_t = \frac{r}{1+r}w_t = (1-\beta)w_t \quad w_t \equiv a_t + h_t$$

Certainty Equivalence

- Solution exhibits certainty equivalence. Optimal c_t policy depends only on expected y_{t+j}
- Higher moments do not matter. In particular, income risk (volatility of y_{t+j}) does not matter for optimal c_t
- This is because of the linear-quadratic specification
- Volatility of y_{t+j} matters for payoffs - the agent is risk averse - but with quadratic utility, volatility doesn't matter for optimal policy

Consumption Dynamics

- Change in consumption

$$\Delta c_t \equiv c_t - c_{t-1} = c_t - \mathbb{E}_{t-1}\{c_t\} = \frac{r}{1+r}\left(w_t - \mathbb{E}_{t-1}\{w_t\}\right)$$

  driven purely by innovations to permanent income

- Since $a_t = \mathbb{E}_{t-1}\{a_t\}$, these innovations are given by

$$w_t - \mathbb{E}_{t-1}\{w_t\} = \sum_{j=0}^{\infty} R^{-j}\left(\mathbb{E}_t - \mathbb{E}_{t-1}\right)\{y_{t+j}\}$$

  so that

$$\Delta c_t = \frac{r}{1+r}\sum_{j=0}^{\infty} R^{-j}\left(\mathbb{E}_t - \mathbb{E}_{t-1}\right)\{y_{t+j}\}$$

- In short, changes in consumption are proportional to revisions in expected income due to the arrival of new information

Permanent and Transitory Shocks

- Example: suppose income has a permanent component z_t and a transitory component u_t as in

$$
\begin{aligned}
y_t &= z_t + u_t \\
z_t &= z_{t-1} + \varepsilon_t
\end{aligned}
$$

  where the shocks u_t and ε_t are IID over time, independent of each other, and have mean zero

- What are the revisions $(\mathbb{E}_t - \mathbb{E}_{t-1})\{y_{t+j}\}$ for this process?

Revisions to Expected Income

- For $j = 0$ we have

$$\begin{aligned}(\mathbb{E}_t - \mathbb{E}_{t-1})\{y_t\} &= (\mathbb{E}_t - \mathbb{E}_{t-1})(y_{t-1} + u_t - u_{t-1} + \varepsilon_t) \\ &= u_t + \varepsilon_t\end{aligned}$$

- For $j = 1$ we have

$$\begin{aligned}(\mathbb{E}_t - \mathbb{E}_{t-1})\{y_{t+1}\} &= (\mathbb{E}_t - \mathbb{E}_{t-1})(y_t + u_{t+1} - u_t + \varepsilon_{t+1}) \\ &= u_t + \varepsilon_t + (\mathbb{E}_t - \mathbb{E}_{t-1})(u_{t+1} - u_t + \varepsilon_{t+1}) \\ &= \varepsilon_t\end{aligned}$$

Revisions to Expected Income

- Continuing in the same way

$$(\mathbb{E}_t - \mathbb{E}_{t-1})\{y_{t+j}\} = \varepsilon_t \qquad \text{for any } j \geqslant 1$$

- Hence, for this example

$$\begin{aligned}
\Delta c_t &= \frac{r}{1+r} \sum_{j=0}^{\infty} R^{-j} (\mathbb{E}_t - \mathbb{E}_{t-1})\{y_{t+j}\} \\
&= \frac{r}{1+r} \left(u_t + \varepsilon_t + \sum_{j=1}^{\infty} R^{-j} \varepsilon_t \right) \\
&= \frac{r}{1+r} \left(u_t + \sum_{j=0}^{\infty} R^{-j} \varepsilon_t \right)
\end{aligned}$$

Response to Permanent and Transitory Shocks

- This simplifies to

$$\Delta c_t = \varepsilon_t + \frac{r}{1+r} u_t$$

- In this example, consumption responds 1-for-1 to permanent shocks ε_t but is much less responsive to transitory shocks u_t

Saving Motives

- Three basic motives for saving/dissaving
  - (i) intertemporal substitution $-\beta$ vs R, operates even if y_t is deterministic
  - (ii) consumption smoothing - the desire to smooth consumption over different income shocks, operates even if utility is quadratic
  - (iii) precautionary saving - insurance against future income risk, need to go beyond certainty equivalence

Precautionary Saving

- Two period example

- Single agent with risk-averse preferences

$$u(c_0) + \beta \mathbb{E}\{u(c_1)\}$$

- Budget constraints

$$c_0 + a_1 = y_0, \quad \text{and} \quad c_1 = Ra_1 + y_1$$

- Stochastic income $y_1 \sim F(y_1)$

- Choose a_1 to maximize

$$u(y_0 - a_1) = \beta \int u(Ra_1 + y_1)\, dF(y_1)$$

Precautionary Saving Example

- Suppose $\beta R = 1$, no intertemporal substitution motive
- Consumption Euler equation

$$u'(y_0 - a_1) = \int u'(Ra_1 + y_1)\, dF(y_1)$$

- Since $u''(c) < 0$, LHS strictly increasing in a_1 and RHS strictly decreasing in a_1
- Pins down a_1 and hence $c_0 = y_0 - a_1$

Income Risk

- How does saving a_1 respond to greater income risk?
- Consider mean-preserving spread. Write $y_1 = \bar{y}_1 + \varepsilon$ with mean $\bar{y}_1$ and mean zero risk $\varepsilon \sim G(\varepsilon)$
- Now write the consumption Euler equation

$$u'\left(y_0 - a_1\right) = \int u'\left(Ra_1 + \bar{y}_1 + \varepsilon\right) dG(\varepsilon)$$

- If marginal utility is convex, i.e., if $u'''(c) > 0$, then by Jensen's inequality we have

$$\int u'\left(Ra_1 + \bar{y}_1 + \varepsilon\right) dG(\varepsilon) > u'\left(Ra_1 + \bar{y}_1\right)$$

- So if marginal utility is convex, income risk leads to more saving

Prudence

- Risk aversion refers to the curvature of utility function $u(c)$. 'Prudence' refers to the curvature of marginal utility function $u'(c)$

- CRRA utility function
$$u(c) = \frac{c^{1-\alpha} - 1}{1-\alpha}, \quad \alpha > 0$$
Risk aversion $u''(c) < 0$ and prudence $u'''(c) > 0$

- Quadratic utility function
$$u(c) = c - \frac{b}{2}c^2, \quad b > 0$$
Risk aversion $u''(c) < 0$ but no prudence $u'''(c) = 0$

Dynamic Version

- Finite periods $t = 0, 1, \ldots, T$
- Budget constraints
$$c_t + a_{t+1} = Ra_t + y_t$$
- IID income shocks $y_t \sim F\left(y_t\right)$
- Bellman equation
$$V_t(a,y) = \max_{a'} \left[u\left(Ra + y - a'\right) + \beta \int V_{t+1}\left(a', y'\right) dF\left(y'\right) \right]$$
- Finite horizon will let us do backwards induction from $t = T$ given that $a_{T+1} = 0$ so that
$$V_T(a,y) = u(Ra + y)$$

Cash-on-hand

- Define 'cash-on-hand' from RHS of the budget constraint

$$x_t \equiv Ra_t + y_t$$

- Evolves according to

$$x_{t+1} = Ra_{t+1} + y_{t+1} = R\left(x_t - c_t\right) + y_{t+1}$$

- Terminal condition can be written

$$V_T(x) = u(x)$$

- So $V_T(x)$ inherits all properties of $u(x)$ and hence exhibits prudence if $u(x)$ does

Backwards Induction

- Then, for one period earlier

$$V_{T-1}(x) = \max_c \left[u(c) + \beta \int u\left(R(x-c) + y'\right) dF\left(y'\right) \right]$$

since $V_T\left(x'\right) = u\left(x'\right)$ and $x' = R(x-c) + y'$

- If $u'''(c) > 0$ mean-preserving spread will decrease optimal c, saving increases as insurance against more income risk

- Note $V_{T-1}(x)$ is the sum of concave functions, hence concave and by envelope theorem

$$V'''_{T-1}(x) = \beta R^3 \int u'''\left(R(x-c) + y'\right) dF\left(y'\right) > 0$$

Again, the value function inherits key properties of the utility function

Backwards Induction

- One period even earlier

$$V_{T-2}(x) = \max_c \left[u(c) + \beta \int V_{T-1}\left(R(x-c) + y'\right) dF\left(y'\right) \right]$$

- Note $V_{T-2}(x)$ is sum of concave functions hence concave and again

$$V_{T-2}'''(x) = \beta R^3 \int V_{T-1}'''\left(R(x-c) + y'\right) dF\left(y'\right) > 0$$

- Iterate all the way back to

$$V_0(x) = \max_c \left[u(c) + \beta \int V_1\left(R(x-c) + y'\right) dF\left(y'\right) \right]$$

- At each step of iteration $V_t(x)$ is concave and exhibits prudence

References

- [SLR] Sargent and Ljungqvist, Recursive Macroeconomic Theory (4th)
  - Chapter 12: Recursive Competitive Equilibrium: II
- [AKMM] Azzimonti, Krusell, McKay, and Mukoyama, Macroeconomics
  - Chapter 10: Consumption