PhD Macro Core Part I:
Lecture 4 – Neoclassical Growth II

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Today

- Steady State and Modified Golden Rule
- Competitive Equilibrium Definitions

Social Planner's Problem

- Social planner chooses stream $c_t \geqslant 0$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

subject to a sequence of resource constraints

$$c_t + k_{t+1} = f\left(k_t\right) = k_t^{\alpha} + (1-\delta)k_t, \quad k_0 > 0$$

- Infinite horizon keeps model stationary, no life-cycle effects

- Can be decentralized, focus on planner's problem for simplicity

Dynamical System from the Solution

- Gives a system of two nonlinear difference equations in c_t, k_t

$$u'(c_t) = \beta u'(c_{t+1}) f'(k_{t+1})$$

and

$$c_t + k_{t+1} = f(k_t)$$

- Two boundary conditions: (i) initial $k_0 > 0$ given, and (ii) the transversality condition

$$\lim_{T\to\infty} \beta^T u'(c_T) k_{T+1} = 0$$

(analogous to $k_{T+1} = 0$ we would have in finite-horizon model)

Steady State

- Steady state where $\Delta c_t = 0$ and $\Delta k_t = 0$. Let c^*, k^* denote steady state values.

- These are determined by

$$1 = \beta f'(k^*)$$

  and

$$c^* + k^* = f(k^*)$$

- Steady state Euler equation pins down k^*, resource constraint then determines c^*, in particular

$$c^* = (k^*)^\alpha - \delta k^*$$

Modified Golden Rule

- Let $C(k)$ denote consumption sustained by holding k_t fixed at k

$$C(k) \equiv (k)^{\alpha} - \delta k$$

- $C(k)$ is maximized at the 'golden rule' level, where

$$f'(k) = 1, \quad \text{or} \quad \alpha k^{\alpha-1} - \delta = 0$$

- Steady state capital stock determined by

$$f'(k) = \frac{1}{\beta} > 1$$

- Hence, steady state capital is less than the golden rule level

Qualitative Dynamics

- Consumption dynamics

$$c_{t+1} > c_t \quad \Leftrightarrow \quad k_{t+1} < k^*$$

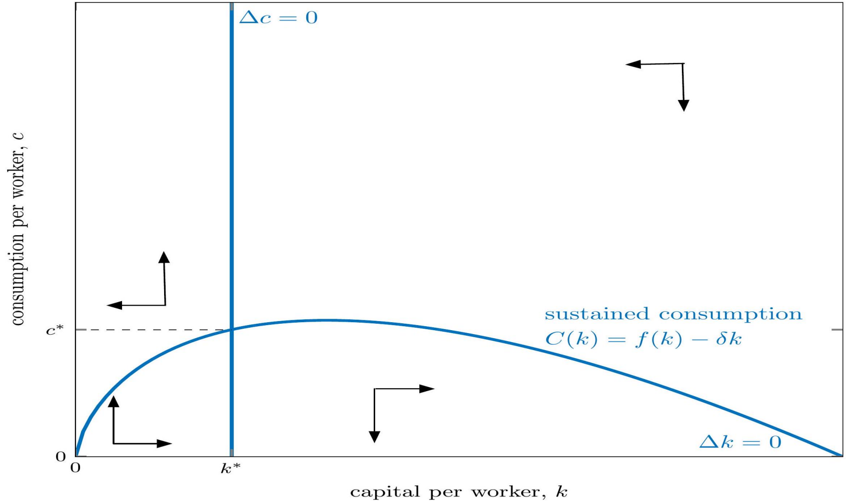
- Capital dynamics

$$k_{t+1} > k_t \quad \Leftrightarrow \quad c_t < C\left(k_t\right)$$

- Divides k_t, c_t space into four regions.

- Flows can be analyzed with a two-dimensional phase diagram

Phase Diagram in k_t, c_t Space



Determining c_0

- Capital k_0 is pre-determined (historically given) at date $t = 0$

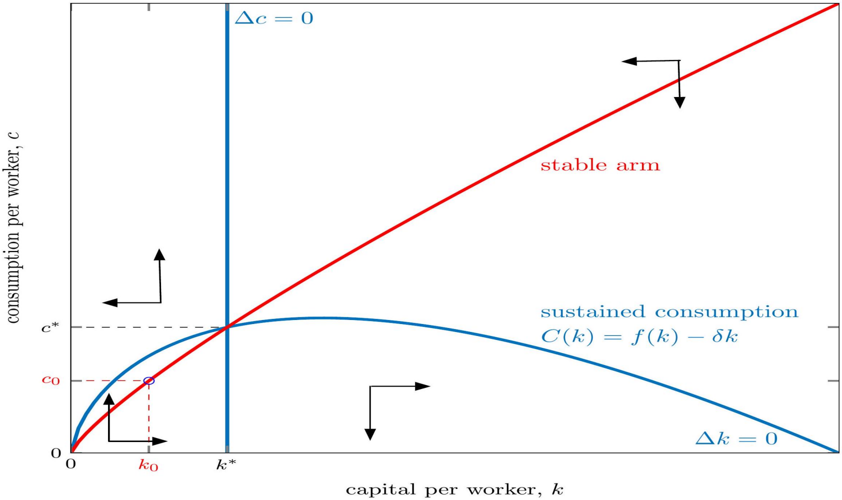
- Consumption c_0 not pre-determined, can 'jump' within feasible set

$$0 \leqslant c_0 \leqslant C(k_0) + k_0$$

- Consumption c_0 jumps to the 'stable arm' of the dynamical system

- Initial consumption is the one degree of freedom that can be used to avoid undesirable trajectories

The Stable Arm



Local Dynamics

- Let $\hat{x}_t$ denote the log-deviation of x_t from its steady state value

$$\hat{x}_t \equiv \log\left(\frac{x_t}{x^*}\right) \approx \frac{x_t - x^*}{x^*}$$

- Can show that local to steady state c^*, k^* dynamics given by

$$\begin{pmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 - \frac{\beta f''(k^*)c^*}{\sigma(c^*)} & \frac{f''(k^*)k^*}{\sigma(c^*)} \\ -\frac{c^*}{k^*} & \frac{1}{\beta} \end{pmatrix} \begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix}$$

  where $\frac{1}{\sigma(c)}$ is the intertemporal elasticity of substitution

$$\frac{1}{\sigma(c)} = -\frac{u'(c)}{u''(c)c} > 0$$

- This coefficient matrix has one stable and one unstable root

Local Dynamics

- The Solution to this system has the form

$$\hat{k}_{t+1} = \psi_{kk}\hat{k}_t, \quad \text{and} \quad \hat{c}_t = \psi_{ck}\hat{k}_t$$

where ψ_{kk} is the stable root of the coefficient matrix and where ψ_{ck} is the slope of the stable arm

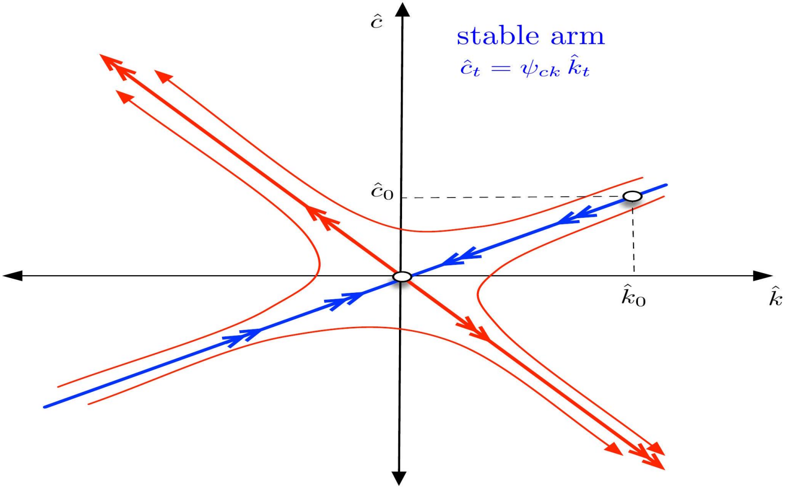
- In particular, $\psi_{kk} \in (0, 1)$ is the stable root of the quadratic

$$\psi_{kk}^2 - \left(1 - \frac{\beta f''(k^*)\, c^*}{\sigma(c^*)} + \frac{1}{\beta}\right)\psi_{kk} + \frac{1}{\beta} = 0$$

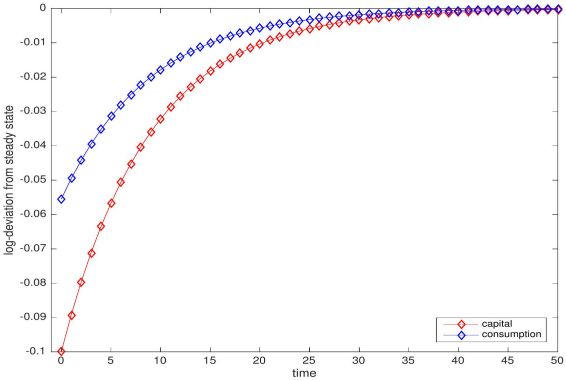
and then

$$\psi_{ck} = \left(\frac{1}{\beta} - \psi_{kk}\right)\frac{k^*}{c^*} > 0$$

Local Dynamics



Example: Transition to Steady State

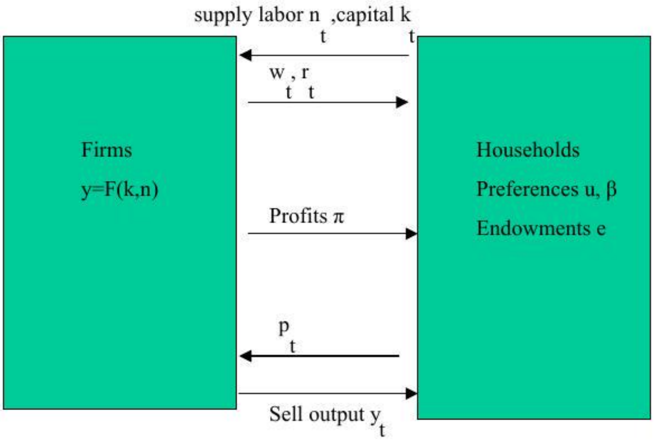


Initial capital $\hat{k}_0 = -0.1$ (i.e., 10% below steady state).

Capital $\hat{k}_{t+1} = \psi_{kk}\hat{k}_t$ and consumption $\hat{c}_t = \psi_{ck}\hat{k}_t$ with $\psi_{kk} = 0.89$ and $\psi_{ck} = 0.56$.

Back to Equilibrium

- Eventually, we want to formally define a production economy equilibrium



Arrow-Debreu Equilibrium

- **Definition:** A Competitive Equilibrium (Arrow-Debreu Equilibrium) consists of prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ and allocations for the firm $\left\{k_t^d, n_t^d, y_t\right\}_{t=0}^{\infty}$ and the household $\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}_{t=0}^{\infty}$ such that
  1. Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of the representative firm $\left\{k_t^d, n_t^d, y_t\right\}_{t=0}^{\infty}$ solves firm's maximization problem
  2. Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the allocation of the representative household $\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}_{t=0}^{\infty}$ solves household's maximization problem
  3. All markets clear

$$y_t = c_t + i_t \text{ (Goods Market)}$$
$$n_t^d = n_t^s \text{ (Labor Market)}$$
$$k_t^d = k_t^s \text{ (Capital Services Market)}$$

Sequential Markets Equilibrium

- **Definition:** A sequential markets equilibrium is a sequence of prices $\{w_t, r_t\}_{t=0}^{\infty}$, allocations for the representative household $\left\{c_t, k_{t+1}^s\right\}_{t=0}^{\infty}$ and allocations for the representative firm $\left\{n_t^d, k_t^d\right\}_{t=0}^{\infty}$ s.t.
  1. Given k_0 and $\{w_t, r_t\}_{t=0}^{\infty}$, allocations for the representative household $\left\{c_t, k_{t+1}^s\right\}_{t=0}^{\infty}$ solve the household maximization problem (3.18)
  2. For each $t \geqslant 0$, given (w_t, r_t) the firm allocation $\left(n_t^d, k_t^d\right)$ solves the firms' maximization problem (3.19).
  3. Markets clear: for all $t \geqslant 0$

$$
\begin{aligned}
n_t^d &= 1 \\
k_t^d &= k_t^s \\
F\left(k_t^d, n_t^d\right) &= c_t + k_{t+1}^s - (1-\delta)k_t^s
\end{aligned}
$$

Recursive Competitive Equilibrium

- **Definition:** A recursive competitive equilibrium is a value function $v : \mathbf{R}_+^2 \to \mathbf{R}$ and policy functions $C, G : \mathbf{R}_+^2 \to \mathbf{R}_+$ for the representative household, pricing functions $w, r : \mathbf{R}_+ \to \mathbf{R}_+$ and an aggregate law of motion $H : \mathbf{R}_+ \to \mathbf{R}_+$ such that
  1. Given the functions w, r and H, the value function v solves the Bellman equation (3.20) and C, G are the associated policy functions.
  2. The pricing functions satisfy (3.21)-(3.22).
  3. Consistency
  $$H(K) = G(K, K)$$
  4. For all $K \in \mathbf{R}_+$
  $$C(K, K) + G(K, K) = F(K, 1) + (1 - \delta)K$$

Why Different Equilibrium Definitions?

- They are eventually the same thing since our economy satisfies FWT & SWT
- But they are defined under different market conditions
  - AD Eqm. is the most ideal; agents could trade anything
  - SM Eqm. is assuming only trading one-period bond
  - RC Eqm. assumes we transferred their problem in a recursive format (but essentially, agents are trading one-period bonds as in the SM Eqm.)
- These definitions fix ideas of an aggregate economy

References

- [DK] Dirk Krueger, Macroeconomic Theory (2015)
  - Chapter 2: A Simple Dynamic Economy
- Please refer to the book for all other proofs
- Please refer to the book for the characterization of the competitive equilibrium