PhD Macro Core Part I:
Lecture 1 – A Simple Dynamic Exchange Economy I

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Today

- Setup an Exchange Economic Model
- Define a Competitive Equilibrium
- Solve an Equilibrium
- Pareto Optimality & First Welfare Theorem

General Principles for Specifying a Model

- Households: Preferences and endowments of commodities
- Households: Optimize preferences over a constraint set w/ initial endowments and market prices
- Firms: Production technology that transforms commodities (inputs) into other commodities (outputs)
- Firms: maximize (expected) profits, subject to their production plans being technologically feasible
- Government: Policy instruments (taxes, money supply, etc.)

An Example Economy with 2 Agents

- Time is discrete and indexed by $t = 0, 1, 2, \ldots$

- An allocation is a sequence $\left(c^1, c^2\right) = \left\{\left(c_t^1, c_t^2\right)\right\}_{t=0}^{\infty}$:

$$u\left(c^i\right) = \sum_{t=0}^{\infty} \beta^t \ln\left(c_t^i\right) \text{ with } \beta \in (0, 1)$$

- Agents have deterministic endowment streams $e^i = \left\{e_t^i\right\}_{t=0}^{\infty}$ of the consumption goods:

$$e_t^1 = \begin{cases} 2 & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases}$$

$$e_t^2 = \begin{cases} 0 & \text{if } t \text{ is even} \\ 2 & \text{if } t \text{ is odd} \end{cases}$$

An Example Economy with 2 Agents

- Given a sequence of prices $\{p_t\}_{t=0}^{\infty}$ households solve the following optimization problem

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln\left(c_t^i\right)$$

$$\sum_{t=0}^{\infty} p_t c_t^i \leqslant \sum_{t=0}^{\infty} p_t e_t^i$$

$$c_t^i \geqslant 0 \text{ for all } t$$

- Note that the budget constraint can be rewritten as

$$\sum_{t=0}^{\infty} p_t \left(e_t^i - c_t^i\right) \geqslant 0$$

- The quantity $e_t^i - c_t^i$ is the net trade of consumption which may be positive or negative

Definition of Competitive Equilibrium

- **Definition:** Competitive Arrow-Debreu equilibrium are

$$\textbf{prices } \{\hat{p}_t\}_{t=0}^{\infty} \text{ and } \textbf{allocations } \left(\left\{\hat{c}_t^i\right\}_{t=0}^{\infty}\right)_{i=1,2}, \text{ such that}$$

1. **[Agents Optimization]** Given $\{\hat{p}_t\}_{t=0}^{\infty}$, for $i = 1, 2$, $\left\{\hat{c}_t^i\right\}_{t=0}^{\infty}$ solves

$$\max_{\left\{c_t^i\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln\left(c_t^i\right)$$

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i \leqslant \sum_{t=0}^{\infty} \hat{p}_t e_t^i, \quad c_t^i \geqslant 0 \text{ for all } t$$

2. **[Market clearing]** $\left\{\hat{c}_t^i\right\}_{t=0}^{\infty}$ satisfy

$$\hat{c}_t^1 + \hat{c}_t^2 = e_t^1 + e_t^2 \text{ for all } t$$

- The elements of an equilibrium are **allocations** and **prices**

Solving for the Equilibrium: FOCs

- Attach the Lagrange multiplier λ_i to the budget constraint. FOCs are:

$$\frac{\beta^t}{c_t^i} = \lambda_i p_t, \qquad \frac{\beta^{t+1}}{c_{t+1}^i} = \lambda_i p_{t+1}$$

  and hence

$$p_{t+1} c_{t+1}^i = \beta p_t c_t^i \text{ for all } t$$

- The solutions are functions of prices as $c_t^i = c_t^i\left(\{p_t\}_{t=0}^{\infty}\right)$, then marketing clearing becomes

$$c_t^1\left(\{p_t\}_{t=0}^{\infty}\right) + c_t^2\left(\{p_t\}_{t=0}^{\infty}\right) = e_t^1 + e_t^2 \text{ for all } t$$

- We could solve

$$p_{t+1}\left(c_{t+1}^1 + c_{t+1}^2\right) = \beta p_t\left(c_t^1 + c_t^2\right)$$
$$p_{t+1}\left(e_{t+1}^1 + e_{t+1}^2\right) = \beta p_t\left(e_t^1 + e_t^2\right)$$

  and hence

$$p_{t+1} = \beta p_t$$

Solving for the Equilibrium: Prices

- The **equilibrium prices** are (we can set $\hat{p}_0 = 1$)

$$\hat{p}_t = \beta^t \hat{p}_0 = \beta^t$$

- The left-hand side of the budget constraint becomes

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i = c_0^i \sum_{t=0}^{\infty} \beta^t = \frac{c_0^i}{1-\beta}, \qquad \text{for } i = 1, 2$$

- But Agent 1 is richer. Specifically, the right-hand side of the two agents are

$$\sum_{t=0}^{\infty} \hat{p}_t e_t^1 = 2 \sum_{t=0}^{\infty} \beta^{2t} = \frac{2}{1-\beta^2}$$

$$\sum_{t=0}^{\infty} \hat{p}_t e_t^2 = 2\beta \sum_{t=0}^{\infty} \beta^{2t} = \frac{2\beta}{1-\beta^2}$$

Solving for the Equilibrium: Allocations

- The **equilibrium allocation** is then given by

$$\hat{c}_t^1 = \hat{c}_0^1 = (1-\beta)\frac{2}{1-\beta^2} = \frac{2}{1+\beta} > 1$$
$$\hat{c}_t^2 = \hat{c}_0^2 = (1-\beta)\frac{2\beta}{1-\beta^2} = \frac{2\beta}{1+\beta} < 1$$

- Market clearing condition obviously satisfied

$$\hat{c}_t^1 + \hat{c}_t^2 = 2 = \hat{e}_t^1 + \hat{e}_t^2 \text{ for all } t$$

- Does trade make them better? Certainly!

- Without trade: $u\left(e_t^i\right) = -\infty$ when $e^i = 0$

- with trade: $u\left(\hat{c}^2\right) = \sum_{t=0}^{\infty} \beta^t \ln\left(\frac{2\beta}{1+\beta}\right) = \frac{\ln\left(\frac{2\beta}{1+\beta}\right)}{1-\beta} > -\infty$ (only show agent 2)

Pareto Optimality

- Demonstrates that for this economy, a competitive equilibrium is socially optimal

- Pareto efficiency (also referred to as Pareto optimality)
  - An allocation is Pareto efficient if it is feasible and if there is no other feasible allocation that makes no household worse off and at least one household strictly better off.

- **Definition:** An allocation $\left\{\left(c_t^1, c_t^2\right)\right\}_{t=0}^{\infty}$ is Pareto efficient if it is feasible and if there is no other feasible allocation $\left\{\left(\tilde{c}_t^1, \tilde{c}_t^2\right)\right\}_{t=0}^{\infty}$ such that

$$u\left(\tilde{c}^i\right) \geqslant u\left(c^i\right) \text{ for both } i = 1, 2$$
$$u\left(\tilde{c}^i\right) > u\left(c^i\right) \text{ for at least one } i = 1, 2$$

- Pareto efficiency has **nothing to do with fairness** in any sense

- Every competitive equilibrium allocation for the economy described above is Pareto efficient

The First Welfare Theorem

- Conditional on below conditions,
  - The preference relation is locally non-satiated for each consumer (from MWG)
  - Agents (consumers and firms in a production economy) take prices as given
  - Markets are complete (negligible transaction costs & perfect information)
  - Agents behave rationally
  - No externalities
- Every competitive equilibrium allocation that satisfies FWT is Pareto optimal

Pareto Optimality of Competitive Equilibrium

- **Proposition:** Let $\left(\{\hat{c}_t^i\}_{t=0}^{\infty}\right)_{i=1,2}$ be a competitive equilibrium allocation. Then $\left(\{\hat{c}_t^i\}_{t=0}^{\infty}\right)_{i=1,2}$ is Pareto efficient.

- **Proof:** The proof will be by contradiction; Suppose that $\left(\{\hat{c}_t^i\}_{t=0}^{\infty}\right)_{i=1,2}$ is not Pareto efficient.

- Then, by the definition of Pareto efficiency, there exists another feasible allocation $\left(\{\tilde{c}_t^i\}_{t=0}^{\infty}\right)_{i=1,2}$:

$$
\begin{aligned}
u\left(\tilde{c}^i\right) &\geqslant u\left(\hat{c}^i\right) \text{ for both } i = 1,2 \\
u\left(\tilde{c}^i\right) &> u\left(\hat{c}^i\right) \text{ for at least one } i = 1,2
\end{aligned}
$$

  Without loss of generality, assume that the strict inequality holds for $i = 1$.

Proof: Step 1

- Show that
$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^1 > \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^1$$
where $\{\hat{p}_t\}_{t=0}^{\infty}$ are the equilibrium prices associated with $\left(\left\{\hat{c}_t^i\right\}_{t=0}^{\infty}\right)_{i=1,2}$. If not, i.e., if
$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^1 \leqslant \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^1$$
then for agent 1 the $\tilde{}$-allocation is better (remember $u\left(\tilde{c}^1\right) > u\left(\hat{c}^1\right)$ is assumed) and not more expensive, which cannot be the case since $\left\{\hat{c}_t^1\right\}_{t=0}^{\infty}$ is part of a competitive equilibrium, i.e. maximizes agent 1's utility given equilibrium prices.

- Hence
$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^1 > \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^1 \tag{2.9}$$

Proof: Step 2

- Show that
$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 \geqslant \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2$$
  If not, then
$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 < \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2$$
  But then there exists a $\delta > 0$ such that
$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 + \delta \leqslant \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2$$

- Remember that we normalized $\hat{p}_0 = 1$. Now define a new allocation for agent 2, by
$$\check{c}_t^2 = \tilde{c}_t^2 \text{ for all } t \geqslant 1$$
$$\check{c}_0^2 = \tilde{c}_0^2 + \delta \text{ for } t = 0$$

Proof: Step 2

- Obviously

$$\sum_{t=0}^{\infty} \hat{p}_t \check{c}_t^2 = \sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 + \delta \leqslant \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2$$

  and

$$u\left(\check{c}^2\right) > u\left(\tilde{c}^2\right) \geqslant u\left(\hat{c}^2\right)$$

  which can't be the case since $\left\{\hat{c}_t^2\right\}_{t=0}^{\infty}$ is part of a competitive equilibrium, i.e. maximizes agent 2's utility given equilibrium prices.

- Hence

$$\sum_{t=0}^{\infty} \hat{p}_t \tilde{c}_t^2 \geqslant \sum_{t=0}^{\infty} \hat{p}_t \hat{c}_t^2 \tag{2.10}$$

Proof: Step 3

- Now sum equations (2.9) and (2.10) to obtain

$$\sum_{t=0}^{\infty} \hat{p}_t \left(\tilde{c}_t^1 + \tilde{c}_t^2\right) > \sum_{t=0}^{\infty} \hat{p}_t \left(\hat{c}_t^1 + \hat{c}_t^2\right)$$

  But since both allocations are feasible (the allocation $\left(\{\hat{c}_t^i\}_{t=0}^{\infty}\right)_{i=1,2}$ because it is an equilibrium allocation, the allocation $\left(\{\tilde{c}_t^i\}_{t=0}^{\infty}\right)_{i=1,2}$ by assumption) we have that

$$\tilde{c}_t^1 + \tilde{c}_t^2 = e_t^1 + e_t^2 = \hat{c}_t^1 + \hat{c}_t^2 \text{ for all } t$$

  and thus

$$\sum_{t=0}^{\infty} \hat{p}_t \left(e_t^1 + e_t^2\right) > \sum_{t=0}^{\infty} \hat{p}_t \left(e_t^1 + e_t^2\right)$$

- Our desired contradiction.

References

- [DK] Dirk Krueger, Macroeconomic Theory (2015)
  - Chapter 2: A Simple Dynamic Economy

- Please refer to the book for all other proofs