PhD Macro Core Part I:
Lecture 3 – Neoclassical Growth I

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Today

- Setup the Neoclassical Growth Model
- Social Planner's Problem (From Sequential to Recursive)
- Simplified Solution with Guess and Verify
- General Solutions: VFI & PFI

Setup the Neoclassical Growth Model

- Discrete time $t = 0, 1, 2, \ldots$

- Aggregate output Y_t is produced with physical capital K_t and labor L_t

$$Y_t = F\left(K_t, A_t L_t\right)$$

  with labor-augmenting productivity A_t

- Physical capital depreciates at rate δ

$$K_{t+1} = (1-\delta)K_t + I_t, \quad 0 < \delta < 1, \quad K_0 > 0$$

- Goods may be either consumed or invested

$$C_t + I_t = Y_t$$

- Gives the sequence of resource constraints, one for each date

$$C_t + K_{t+1} = F\left(K_t, L_t\right) + (1-\delta)K_t, \quad K_0 > 0$$

Aggregate Production Function

- Each input has a positive marginal product

$$F_K(K,L) > 0, \quad F_L(K,L) > 0$$

- Each input suffers from diminishing returns

$$F_{KK}(K,L) < 0, \quad F_{LL}(K,L) < 0$$

- Constant returns to scale, i.e., if both inputs scaled by a common factor $c > 0$, then

$$F(cK, cL) = cF(K,L)$$

- Some analysis is simplified by assuming the Inada conditions

$$F_K(0,L) = F_L(K,0) = \infty$$
$$F_K(\infty,L) = F_L(K,\infty) = 0$$

and that both inputs are essential, i.e., $F(0,L) = F(K,0) = 0$

Efficiency Labor Unit Format

- In efficiency units

$$y \equiv \frac{Y}{AL}, \quad k \equiv \frac{K}{AL}, \ldots \qquad \text{etc}$$

- Using constant returns to scale

$$y = \frac{Y}{AL} = \frac{F(K, AL)}{AL} = F\left(\frac{K}{AL}, 1\right) = F(k, 1)$$

- Suppose constant $L_t = L$ and $A_t = A$, the resource constraint is simply

$$c_t + k_{t+1} = F(k, 1) + (1 - \delta)k_t \equiv f(k_t), \quad k_0 > 0$$

- Since all households are representative, it is the same to solve per household problem

Optimal Growth: Pareto Optimal Allocations

- The optimal growth satisfies the FWT, so it is Pareto optimal

- **Definition:** An allocation $\{c_t, k_t, l_t\}_{t=0}^{\infty}$ is feasible if for all $t \geqslant 0$

$$\begin{aligned} F(k_t, l_t) &= c_t + k_{t+1} - (1-\delta)k_t \\ c_t &\geqslant 0, k_t \geqslant 0, 0 \leqslant l_t \leqslant 1 \\ k_0 &\leqslant \bar{k}_0 \end{aligned}$$

- **Definition:** An allocation $\{c_t, k_t, l_t\}_{t=0}^{\infty}$ is Pareto efficient if it is feasible and there is no other feasible allocation $\left\{\hat{c}_t, \hat{k}_t, \hat{l}_t\right\}_{t=0}^{\infty}$ such that

$$\sum_{t=0}^{\infty} \beta^t U(\hat{c}_t) > \sum_{t=0}^{\infty} \beta^t U(c_t)$$

- Therefore, we can just focus on the social planner's problem; also, we know the optimal $l_t = 1$

Social Planner's Problem: Sequential Formulation

- Social planner chooses stream $c_t \geqslant 0$ to maximize

$$\sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

  subject to a sequence of resource constraints

$$c_t + k_{t+1} = f\left(k_t\right), \quad k_0 > 0$$

- Infinite horizon keeps model 'stationary', no life-cycle effects
- Can be decentralized, focus on planner's problem for simplicity

Social Planner's Problem: FOCs

- Lagrangian with multiplier $\lambda_t \geqslant 0$ for each resource constraint

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) + \sum_{t=0}^{\infty} \lambda_t \left[f\left(k_t\right) - c_t - k_{t+1}\right]$$

- Key first order conditions

$$\begin{aligned} c_t : \quad & \beta^t u'\left(c_t\right) - \lambda_t &= 0 \\ k_{t+1} : \quad & -\lambda_t + \lambda_{t+1} f'\left(k_{t+1}\right) &= 0 \\ \lambda_t : \quad & f\left(k_t\right) - c_t - k_{t+1} &= 0 \end{aligned}$$

These are held on every date

Consumption Euler Equation

- Eliminating the Lagrange multipliers

$$u'(c_t) = \beta u'(c_{t+1}) \left[f'(k_{t+1}) + 1 - \delta\right]$$

- Same as last class if we recognize that the 'return on capital' is

$$R_{t+1} = f'(k_{t+1})$$

- Planner equates marginal rate of substitution (MRS) between t and $t+1$ with marginal rate of transformation (MRT)

- MRS between t and $t+1$

$$\frac{u'(c_t)}{\beta u'(c_{t+1})}$$

- MRT between t and $t+1$

$$f'(k_{t+1})$$

From Sequential Formulation to Recursive Formulation

- We could rewrite the sequential formulation as

$$
\begin{aligned}
v\left(k_0\right) &= \max \sum_{t=0}^{\infty} \beta^t u\left(f\left(k_t\right)-k_{t+1}\right) \\
&= \max \left\{u\left(f\left(k_0\right)-k_1\right)+\beta \sum_{t=1}^{\infty} \beta^{t-1} u\left(f\left(k_t\right)-k_{t+1}\right)\right\} \\
&= \max \left\{u\left(f\left(k_0\right)-k_1\right)+\beta\left[\max \sum_{t=1}^{\infty} \beta^{t-1} u\left(f\left(k_t\right)-k_{t+1}\right)\right]\right\}
\end{aligned}
$$

- Eventually, we have

$$
v\left(k_0\right)=\max_{\substack{0 \leqslant k_1 \leqslant f\left(k_0\right) \\ k_0 \text{ given}}}\left\{u\left(f\left(k_0\right)-k_1\right)+\beta v\left(k_1\right)\right\}
$$

Solve Recursive Formulation: An Example

- The recursive problem is not straightforward to solve since we do not know $v(k')$

$$v(k) = \max_{0 \leqslant k' \leqslant f(k)} \{u(f(k) - k') + \beta v(k')\}$$

- We will show in the future that it derives the same Euler equations

- Today, we will guess and verify in a simplified example w/

$$u(c) = ln(c), \quad \delta = 1, \quad f(k) = k^{\alpha}$$

the functional equation becomes

$$v(k) = \max_{0 \leqslant k' \leqslant k^{\alpha}} \{\ln(k^{\alpha} - k') + \beta v(k')\}$$

Solve Recursive Formulation: Guess and Verify

- Let us guess

$$v(k) = A + B \ln(k)$$

where A and B are unknown coefficients to be determined.

- The method consists of three steps:

- Step 1: Solve the maximization problem on the right-hand side, given the guess for v, i.e., solve

$$\max_{0 \leqslant k' \leqslant k^\alpha} \{\ln\left(k^a - k'\right) + \beta\left(A + B\ln\left(k'\right)\right)\}$$

where the FOC yields

$$\frac{1}{k^\alpha - k'} = \frac{\beta B}{k'}$$

$$k' = \frac{\beta B k^\alpha}{1 + \beta B} \tag{3.3}$$

Solve Recursive Formulation: Guess and Verify

- Step 2: Evaluate the right-hand side at the optimal solution $k' = \frac{\beta B k^{\alpha}}{1+\beta B}$. This yields

$$\begin{aligned} \text{RHS} &= \ln\left(k^{a} - k'\right) + \beta\left(A + B\ln\left(k'\right)\right) \\ &= \ln\left(\frac{k^{\alpha}}{1+\beta B}\right) + \beta A + \beta B \ln\left(\frac{\beta B k^{\alpha}}{1+\beta B}\right) \\ &= -\ln(1+\beta B) + \alpha\ln(k) + \beta A + \beta B\ln\left(\frac{\beta B}{1+\beta B}\right) + \alpha\beta B\ln(k) \end{aligned}$$

- In order for our guess to solve the functional equation, the left-hand side of the functional equation, which we have guessed to equal $\text{LHS} = A + B\ln(k)$ must equal the right-hand side, which we just found, for all possible values of k.

- If we can find coefficients A, B for which this is true, we have found a solution to the functional equation.

Solve Recursive Formulation: Guess and Verify

- Step 3: Equating LHS and RHS yields

$$A + B\ln(k) = -\ln(1+\beta B) + \alpha\ln(k) + \beta A + \beta B\ln\left(\frac{\beta B}{1+\beta B}\right) + \alpha\beta B\ln(k)$$

$$(B - \alpha(1+\beta B))\ln(k) = -A - \ln(1+\beta B) + \beta A + \beta B\ln\left(\frac{\beta B}{1+\beta B}\right) \tag{3.4}$$

- But this equation has to hold for every capital stock k. The right-hand side of (3.4) does not depend on k, but the left-hand side does. Hence, the right-hand side is a constant.

$$B = \frac{\alpha}{1-\alpha\beta}$$

$$\begin{aligned} 0 &= -A - \ln(1+\beta B) + \beta A + \beta B\ln\left(\frac{\beta B}{1+\beta B}\right) \\ &= -A - \ln\left(\frac{1}{1-\alpha\beta}\right) + \beta A + \frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) \end{aligned}$$

Solve Recursive Formulation: Guess and Verify

- Solving this mess for A yields

$$A = \frac{1}{1-\beta}\left[\frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) + \ln(1-\alpha\beta)\right]$$

- We can also determine the optimal policy function $k' = g(k)$ by plugging in $B = \frac{\alpha}{1-\alpha\beta}$ into (3.3):

$$\begin{aligned} g(k) &= \frac{\beta B k^{\alpha}}{1+\beta B} \\ &= \alpha\beta k^{\alpha} \end{aligned}$$

- Hence, our guess was correct: the function $v^*(k) = A + B\ln(k)$, with A, B as determined above, solves the functional equation, with associated policy function $g(k) = \alpha\beta k^{\alpha}$.

Value Function Iteration: Analytical Approach

- Guess and verify will not work in most cases; we need more general methods
- It is still "guess", but we do not just "verify"; we improve on our naive initial guess
- Consider instead the following iterative procedure for our previous example
  1. Guess an arbitrary function $v_0(k)$. For concreteness let's take $v_0(k) = 0$ for all
  2. Proceed by solving
  $$v_1(k) = \max_{0 \leqslant k' \leqslant k^\alpha} \{\ln(k^\alpha - k') + \beta v_0(k')\}$$
  Note that we can solve the maximization problem on the right-hand side since we know v_0 (since we have guessed it). In particular, since $v_0(k') = 0$ for all k' we have as optimal solution
  $$k' = g_1(k) = 0 \text{ for all } k$$
  Plugging this back in, we get
  $$v_1(k) = \ln(k^\alpha - 0) + \beta v_0(0) = \ln k^\alpha = \alpha \ln k$$

Value Function Iteration: Analytical Approach

- Consider instead the following iterative procedure for our previous example

3. Now we can solve

$$v_2(k) = \max_{0 \leqslant k' \leqslant k^\alpha} \left\{ \ln\left(k^\alpha - k'\right) + \beta v_1\left(k'\right) \right\}$$

4. Since we know v_1 and so forth. By iterating on the recursion

$$v_{n+1}(k) = \max_{0 \leqslant k' \leqslant k^\alpha} \left\{ \ln\left(k^\alpha - k'\right) + \beta v_n\left(k'\right) \right\}$$

- We obtain a sequence of value functions $\{v_n\}_{n=0}^{\infty}$ and policy functions $\{g_n\}_{n=1}^{\infty}$.

- Hopefully, these sequences will converge to the solution v^* and associated policy g^*.

Euler Equation Approach (Policy Function Iteration)

- The infinite horizon case: The problem was to solve

$$v\left(\bar{k}_0\right) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u\left(f\left(k_t\right) - k_{t+1}\right)$$
$$0 \leqslant k_{t+1} \leqslant f\left(k_t\right)$$
$$k_0 = \bar{k}_0 > 0 \text{ given}$$

  The first-order conditions constitute the necessary conditions for an optimal

$$\beta u'\left(f\left(k_{t+1}\right) - k_{t+2}\right) f'\left(k_{t+1}\right) = u'\left(f\left(k_t\right) - k_{t+1}\right) \quad \text{for all } t = 0, \ldots, t, \ldots \tag{3.8}$$

- However, we have no terminal condition since there is no terminal time period

- The transversality condition (TVC) substitutes for the missing terminal condition

Euler Equation Approach: The Transversality Condition

- Let us first state and then interpret the TVC

$$\lim_{t\to\infty} \underbrace{\beta^t u'\left(f\left(k_t\right)-k_{t+1}\right) f'\left(k_t\right)}\ \underbrace{k_t}\ = 0$$

value in discounted total

utility terms of one more $= 0$

unit of capital stock

- Often one can find an alternative statement of the TVC in the literature:

$$\lim_{t\to\infty} \lambda_t k_{t+1} = 0$$

where λ_t is the Lagrange multiplier on the constraint

$$c_t + k_{t+1} = f\left(k_t\right)$$

hence, the TVC becomes

$$\lim_{t\to\infty} \beta^t u'\left(f\left(k_t\right)-k_{t+1}\right) k_{t+1} = 0$$

Euler Equation Approach: Analytical Approach

- Take TVC as given, solve the simple example $u(c) = ln(c), \quad \delta = 1, \quad f(k) = k^{\alpha}$

- Define $z_t = k_{t+1}/f(k_t) = k'/k^{\alpha}$ as the saving rate, the TVC becomes

$$
\begin{aligned}
&\lim_{t\to\infty} \beta^t u'\left(f\left(k_t\right) - k_{t+1}\right) f'\left(k_t\right) k_t \\
&= \lim_{t\to\infty} \frac{\alpha\beta^t k_t^{\alpha}}{k_t^{\alpha} - k_{t+1}} = \lim_{t\to\infty} \frac{\alpha\beta^t}{1 - \frac{k_{t+1}}{k_t^{\alpha}}} \\
&= \lim_{t\to\infty} \frac{\alpha\beta^t}{1 - z_t}
\end{aligned}
$$

- Repeat the first-order difference equation derived from the Euler equations

$$
z_{t+1} = 1 + \alpha\beta - \frac{\alpha\beta}{z_t}
$$

Euler Equation Approach: Analytical Approach

- **Guess and verify:** Only one guess for z_0 yields a sequence that does not violate the TVC or the non-negativity constraint on capital or consumption.
    1. $z_0 < \alpha\beta$. In finite time $z_t < 0$, violating the nonnegativity constraint on capital
    2. $z_0 > \alpha\beta$. Then from Figure 3 we see that $\lim_{t\to\infty} z_t = 1$. (Note that, in fact, every $z_0 > 1$ violates the nonnegativity of consumption and hence is not admissible as a starting value). We will argue that all these paths violate the TVC.
    3. $z_0 = \alpha\beta$. Then $z_t = \alpha\beta$ for all $t > 0$. For this path (which obviously satisfies the Euler equations) we have that
    $$\lim_{t\to\infty} \frac{\alpha\beta^t}{1 - z_t} = \lim_{t\to\infty} \frac{\alpha\beta^t}{1 - \alpha\beta} = 0$$
- The Euler approach directly solves the policy function $z_t = z_0$ or $k' = g(k) = \alpha\beta k^\alpha$

References

- [DK] Dirk Krueger, Macroeconomic Theory (2015)
  - Chapter 2: A Simple Dynamic Economy

- Please refer to the book for all other proofs

- Please refer to the book for the finite case of the Euler approach