PhD Macro Core Part I: Lecture 13 – Real Business Cycles

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Real Business Cycles

- One of the earliest questions in macroeconomics
- Why do economic cycles exist?
- How do they work (what are the propagation mechanisms)?
- Can we do anything about it?
- We will think about this from the perspective of the Real Business Cycles (RBC) Model.

A Bit of History

- 1950s-1970s:
  - Keynesian models estimated via simultaneous equations with aggregate data were the alternative for studying economic cycles.
  - Neoclassical models were used for long-term growth.
- Early 1970s:
  - Keynesian models failed to deal with supply shocks.
  - Methodologically did not survive Lucas's Critique and the rational expectations revolution.
- Early 1980s:
  - Kydland and Prescott's (1982): The Great Successor of the Rational Expectations Revolution
  - The RBC was responsible for the birth of DSGE (Dynamic Stochastic General Equilibrium)

Lucas Critique (1976)

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Carnegie-Rochester Conference Series on Public Policy
Volume 1, 1976, Pages 19-46

Econometric policy evaluation: A critique

Robert E. Lucas Jr

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Fun Fact: Lucas (1976) is the first paper published in the Carnegie-Rochester Conference

The Real Business Cycles (1982/1983)

Fun Fact: Kydland-Prescott (The Carnegie Team) vs Long-Plosser (The Rochester Team)

Today

- How to solve the standard RBC model.
- The trade-off between work and leisure.
- How to log-linearize and approximate the model solution around the steady state.

Part I: Stylized Facts of Business Cycles

Business Cycle Facts: How to Measure?

- Hodrick and Prescott (1980) established rigorously the facts about business cycles.
- The first challenge is to separate the economic cycle from the long-term trend.
- The most common method is to filter the data using the HP filter (Hodrick and Prescott).

HP Filter

- Let y_t be a time series (in logs). We want to decompose the series into a trend, y_t^g, and a cyclical component (residue), y_t^c, $y_t = y_t^g + y_t^c$:

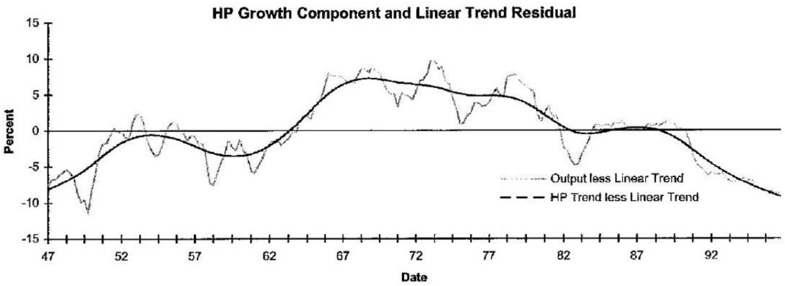
$$\min_{\{y_t^g\}_{t=1}^T} \sum_{t=2}^{T-1} \left\{ (y_t - y_t^g)^2 + \lambda \left[\left(y_{t+1}^g - y_t^g \right) - \left(y_t^g - y_{t-1}^g \right) \right]^2 \right\} + (y_T - y_T^g)^2 + (y_1 - y_1^g)^2$$

- The higher λ is, the more weight is given to variations in the growth rate of the trend component.

- If $\lambda = 0$, y_t^g equals y_t. If $\lambda = \infty$, y_t^g is a linear trend.

- Hodrick and Prescott's rule is to choose $\lambda = 1600$ for quarterly series and $\lambda = 400$ for annual ones.

HP Filter



- Fig1 above: Trend and business cycle in US real output. Sample period is 1947:1-1996:4. Source: King and Rebelo (1999).
- HP filter extracts much more low-frequency information than a simple linear trend.
- Eliminates series components with periodicities greater than about eight years.

Business Cycle Facts: Volatility

1. Consumption is less volatile than output. Durable goods consumption is more volatile.
2. Investment is three times more volatile than output.
3. Government spending is less volatile than output.
4. Hours worked are equally volatile as output.
5. Labor productivity is less volatile than output.

Business Cycle Facts: Correlations

- Comovement: (correlation between two series)
    1. Most variables are procyclical, i.e., exhibit positive contemporaneous correlation with output.
    2. Real wages, government spending, and capital stock are basically acyclical.
- Persistence: (autocorrelation)
    1. Most variables are highly persistent: $\rho = 0.8 \sim 0.9$.

Business Cycle Facts

Table 1
Business cycle statistics for the US Economy

	Standard deviation	Relative standard deviation	First-order autocorrelation	Contemporaneous correlation with output
Y	1.81	1.00	0.84	1.00
C	1.35	0.74	0.80	0.88
I	5.30	2.93	0.87	0.80
N	1.79	0.99	0.88	0.88
Y/N	1.02	0.56	0.74	0.55
w	0.68	0.38	0.66	0.12
r	0.30	0.16	0.60	–0.35
A	0.98	0.54	0.74	0.78

[a] All variables are in logarithms (with the exception of the real interest rate) and have been detrended with the HP filter. Data sources are described in Stock and Watson (1999), who created the real rate using VAR inflation expectations. Our notation in this table corresponds to that in the text, so that Y is per capita output, C is per capita consumption, I is per capita investment, N is per capita hours, w is the real wage (compensation per hour), r is the real interest rate, and A is total factor productivity.

Source: King and Rebelo (1999).

Business Cycle Facts

TABLE 1
EMERGING VS. DEVELOPED MARKETS (Averages)

	Emerging Markets	Developed Markets
$\sigma(Y)$	$\boxed{2.74}$ (.12)	1.34 (.05)
$\sigma(\Delta Y)$	1.87 (.09)	.95 (.04)
$\rho(Y)$	.76 (.02)	.75 (.03)
$\rho(\Delta Y)$	.23 (.04)	.09 (.03)
$\sigma(C)/\sigma(Y)$	$\boxed{1.45}$ (.02)	.94 (.04)
$\sigma(I)/\sigma(Y)$	3.91 (.01)	3.41 (.01)
$\sigma(TB/Y)$	$\boxed{3.22}$ (.17)	1.02 (.03)
$\rho(TB/Y, Y)$	−.51 (.04)	−.17 (.04)
$\rho(C, Y)$	.72 (.04)	.66 (.04)
$\rho(I, Y)$	.77 (.04)	.67 (.04)

NOTE.—This table lists average values of the moments for the group of emerging (13) and developed (13) economies. The values for each country separately are reported in table 2. Data are Hodrick-Prescott filtered using a smoothing parameter of 1,600. The standard deviations are in percentages. The standard errors for the averages were computed assuming independence across countries. The definition of an emerging market follows the classification in Standard & Poor's (2000).

Source: Aguiar and Gopinath (2007).

Business Cycle Facts

Table 1
Standard deviation of filtered series

Variable	USA	Brazil I	Brazil II	Brazil III	Standard	Working capital
Output	1.7	3.1	2.9	2.7	2.8	2.8
Consumption	1.3	2.2	2.2	2.1	2.0	2.1
Investment	5.3	7.2	7.2	7.0	7.3	7.6
Labor	1.6	–	–	–	2.7	2.9
Labor-PIM	–	3.7	3.7	3.8	–	–
Labor-PME	–	1.3	1.3	1.4	–	–
Interest rate	0.43	4.1	4.6	4.3	5.5	5.5

Table 2
Contemporaneous correlation with output of filtered series

Variable	USA	Brazil I	Brazil II	Brazil III	Standard	Working capital
Consumption	0.83	0.82	0.92	0.91	0.98	0.97
Investment	0.90	0.78	0.86	0.85	0.95	0.93
Labor	0.86	–	–	–	1.0	0.98
Labor-PIM	–	0.64	0.46	0.45	–	–
Labor-PME	–	0.40	0.45	0.46	–	–
Interest rate	−0.23	−0.24	−0.34	−0.32	0.05	−0.21

Source: Kanczuk (2004).

Part II: Standard RBC Models

Standard RBC Model

- The most basic version of the RBC model is a neoclassical growth model with stochastic technological shocks and elastic labor supply (leisure decision).

- Environment:
  - Discrete time, representative household living infinitely many periods.
  - Household owns capital (alternatively, the firm can own capital).
  - No population or technological growth (i.e., no long-term trend growth).
  - Competitive markets.
  - No government.

Preferences

- The representative household values consumption, C_t, and leisure, L_t, and has expected utility:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(C_t, L_t\right)$$

  where $\beta \in (0, 1)$, and u is increasing, concave, twice differentiable, and satisfies Inada conditions.

- Temporal endowment: one unit of time that can be divided into labor, N_t, and leisure, L_t:

$$L_t + N_t = 1 \quad \forall t$$

- Budget constraint (standard):

$$C_t + K_{t+1} \leqslant \left(1 + r_t - \delta\right) K_t + w_t N_t + \Pi_t \quad \forall t$$

  along with a no-Ponzi condition and $K_0 > 0$.

Technology

- Production function: $Y_t = Z_t F\left(K_t, A_t N_t\right)$.

- Usual assumptions: constant returns to scale $\left(\Pi_t = 0\right)$ and Inada conditions.

- Technological shocks:
  - A_t (Labor-augmenting Technological Change) $\Rightarrow$ Deterministic trend of long-term growth. Assume for simplicity that $A_t = 1$.
  - $Z_t \Rightarrow$ Stochastic productivity shocks around the trend.

- From the firm's problem, we derive the input demand equation (which implies that the price of capital and labor equals their marginal product)

$$r_t = Z_t F_K\left(K_t, N_t\right)$$
$$w_t = Z_t F_N\left(K_t, N_t\right)$$

Shocks

- The stochastic process of technological shock follows an $\mathrm{AR}(1)$ process:

$$\log\left(Z_t\right) = \rho \log\left(Z_{t-1}\right) + \sigma \varepsilon_t$$

- Where:
  - $-1 < \rho < 1$ represents the persistence of the $\mathrm{AR}(1)$;
  - $\sigma > 0$ captures the variance;
  - The stochastic innovation ε_t is an iid process with mean 0 and standard deviation 1;

- The unconditional mean of the process is $\mathbb{E}\left[\log\left(Z_t\right)\right] = 0$ (could be different than zero).

- Suppose $\log\left(Z_0\right)$ equals the unconditional mean.

Equilibrium Conditions

- Equilibrium requires that at every t:
  - Goods market is in equilibrium:

    $$Y_t = Z_t F(K_t, N_t) = C_t + I_t \quad \forall t$$

    where I_t is given by the capital law of motion: $K_{t+1} = I_t + (1-\delta)K_t$.

  - Prices, (r_t, w_t), are those that equate supply (household) and demand (firms) in the capital and labor markets:

    $$K_t^s = K_t^d \quad \forall t$$
    $$N_t^s = 1 - L_t^* = N_t^d \quad \forall t$$

Household's Problem

- Solve the following problem for FOCs:

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(C_t, 1 - N_t\right) + \lambda_t \left(\left(1 + r_t - \delta\right) K_t + w_t N_t - C_t - K_{t+1}\right)$$

  - $\beta^t u_C\left(C_t, L_t\right) = \lambda_t \quad \forall t;$
  - $\beta^t u_L\left(C_t, L_t\right) = \lambda_t w_t \quad \forall t;$
  - $\lambda_t = \mathbb{E}_t\left(1 + r_{t+1} - \delta\right)\lambda_{t+1} \quad \forall t.$

- Implies the following conditions (at every t):

$$\begin{aligned} u_C\left(C_t, L_t\right) &= \beta \mathbb{E}_t\left[\left(1 + r_{t+1} - \delta\right) u_C\left(C_{t+1}, L_{t+1}\right)\right] \\ u_L\left(C_t, L_t\right) &= u_C\left(C_t, L_t\right) w_t \end{aligned} \tag{LS}$$

- The traditional Euler Equation plus an intratemporal equation determining labor supply.

Decentralized Equilibrium

- The system of equations gives item Decentralized equilibrium for every t:

$$
\begin{aligned}
u_C\left(C_t, L_t\right) &= \beta \mathbb{E}_t\left[\left(1 + r_{t+1} - \delta\right) u_C\left(C_{t+1}, L_{t+1}\right)\right] \\
u_L\left(C_t, L_t\right) &= u_C\left(C_t, L_t\right) w_t \\
L_t + N_t &= 1 \\
r_t &= Z_t F_K\left(K_t, N_t\right) \\
w_t &= Z_t F_N\left(K_t, N_t\right) \\
Y_t &= I_t + C_t \\
K_{t+1} &= I_t + (1-\delta) K_t \\
Y_t &= Z_t F\left(K_t, N_t\right) \\
\log\left(Z_t\right) &= \rho \log\left(Z_{t-1}\right) + \sigma \varepsilon_t
\end{aligned}
$$

along with the TVC and K_0 given.

Decentralized Equilibrium

- Difference with respect to the deterministic neoclassical growth model:
  - Labor-leisure decision: Labor supply equation + time constraint.
  - Stochastic process of productivity.
- Depending on the functional forms, the system can be reduced to 3 equations + TVC and K_0.
- Euler Eq. + resource constraint + stochastic process of Z_t.
- First + Second Welfare Theorems hold, and the decentralized eqm equals the planner's solution.

Functional Forms

- Cobb-Douglas production function: $F(K,N) = K^{\alpha}N^{1-\alpha}$ with $\alpha \in (0,1)$.

- Utility: If the model has positive exogenous growth (i.e., $A_{t+1}/A_t > 1$), the utility ensuring constant hours worked on the Balanced-Growth Path (King-Plosser-Rebelo preferences) is:

$$u(C,L) = \begin{cases} \frac{(Cv(L))^{1-\sigma}-1}{1-\sigma}, & \sigma > 0, \sigma \neq 1 \\ \log(C) + \log(v(L)), & \sigma = 1 \end{cases}$$

- We will use:

$$u(C,L) = \log(C) + \theta\frac{L^{1-\phi}-1}{1-\phi}$$

where ϕ governs the elasticity of labor supply.

Functional Forms

- Applying the functional forms and reducing the system:

$$\frac{1}{C_t} = \beta \mathbb{E}_t \left[\left(1 + Z_{t+1} \alpha \left(K_{t+1}/N_{t+1} \right)^{\alpha-1} - \delta \right) \frac{1}{C_{t+1}} \right] \tag{1}$$

$$\theta \left(1 - N_t\right)^{-\phi} = \frac{Z_t (1-\alpha) \left(K_t/N_t\right)^{\alpha}}{C_t} \tag{2}$$

$$Z_t K_t^{\alpha} N_t^{1-\alpha} = K_{t+1} - (1-\delta) K_t + C_t \tag{3}$$

$$\log \left(Z_{t+1}\right) = \rho \log \left(Z_t\right) + \sigma \varepsilon_{t+1} \tag{4}$$

- Given a sequence of shocks $\{\varepsilon_t\}_{t=0}^{\infty}$, this system (+TVC, K_0 and Z_0) characterizes the optimal allocations.

Steady State

- Define the steady state in the unconditional mean non-stochastic $(\sigma = 0) : Z^* = 1$, $K_{t+1} = K_t = K^*$, $C_{t+1} = C_t = C^*$, and $N_t = N^*$.

- We can solve the system:

- Use (1) and write the capital-labor ratio K/N in terms of the parameters.

- Use (3), K/N, and find C/N in terms of the parameters.

- Use (2), K/N, C/N, and find N in terms of the parameters (note that there exists a 1-1 map between N^* and θ).

- Given the chosen functional forms, the system doesn't have an analytical solution.

Part III: Model Properties

Labor-Leisure Decision

- Before assessing the impact of technological shocks, it's important to understand the impact of an elastic labor supply.

- How do agents respond to a wage increase?

$$\theta\left(1-N_t\right)^{-\phi}=\frac{w_t}{C_t}=w_t\lambda_t$$

- Suppose for a moment that C_t is constant. An increase in w_t increases N_t: this is the substitution effect.

- A wage increase also makes families wealthier: they want to consume more goods and leisure (less work): income effect.

- For realistic calibrations, the substitution effect dominates the income effect.

Intratemporal Labor Substitution: Frisch Elasticity

- The elasticity of labor supply with respect to the wage while keeping the marginal utility of wealth constant (λ) is known as the Frisch Elasticity.

- Taking the log in the intratemporal decision (and using the fact that C_t is constant):

$$\log(1-N_t) = -\frac{1}{\phi}\log w_t + \frac{1}{\phi}\log C_t + \frac{1}{\phi}\log\theta$$

$$d\log(1-N_t) = -\frac{1}{\phi}d\log w_t$$

- Using $d\log(1-N_t) = -\frac{N_t}{1-N_t}d\log N_t$:

$$\frac{d\log N_t}{d\log w_t} = \frac{1}{\phi}\left(\frac{1-N_t}{N_t}\right)$$

- ϕ governs the labor force response to a wage increase (considering the income effect constant).

Intertemporal Labor Substitution: C, S, L

- To understand the labor force response to a change in wages, we can think of the HH choosing three variables: Consumption today, consumption tomorrow (savings), and leisure (negative work).

- Using the Euler Equation and substituting the labor supply equation (ignore uncertainty):

$$\left(\frac{1-N_{t+1}}{1-N_t}\right)^{\phi} = \beta\left(1+r_{t+1}-\delta\right)\left(\frac{w_t}{w_{t+1}}\right)$$

- If the wage is higher today than tomorrow, agents prefer to work more today than tomorrow.

- If the interest rate is higher, agents prefer to work today (to save more) and relax tomorrow.

Technological Shocks

- What happens when there's a positive technological shock ($\uparrow Z_t$)?

- Capital is predetermined and doesn't respond immediately. Labor and consumption jump to the new optimal trajectory (recall they are jump variables).
  - $\uparrow w_t = MPN_t = (1-\alpha) Z_t \left(K_t / N_t\right)^{\alpha}$
  - $\uparrow r_t = MPK_t = \alpha Z_t \left(K_t / N_t\right)^{\alpha - 1}$

- Important: how persistent is Z_t ? Let's consider the two extreme cases:
  - Transitory shock: $\rho = 0$, and Z returns to its steady-state value in $t+1$.
  - Permanent shock: $\rho = 1$, and Z permanently alters its value in the steady state.

- Realistically, the shock will be between these two extremes.

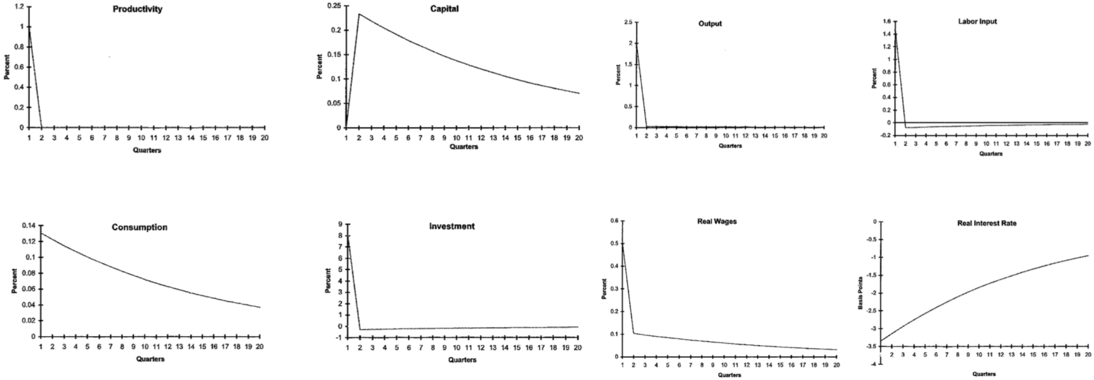
Transitory Technological Shock

- Suppose a completely transitory positive shock $(\rho = 0)$.
- High intertemporal substitution of labor: the agent will work a lot today since tomorrow Z_t returns "to normal".
- The increase in labor amplifies production at t.
- However, the effect on the agent's permanent income is very small: the productivity shock lasts only one period! This makes:
- The income effect on consumption is low: consumption increases but very little.
- The income effect on leisure is also low: substitution effect on work clearly dominates the income effect.

Transitory Technological Shock

- Since the shock is transitory, production returns almost to the steady-state value in $t+1$, $t+2, \ldots$, etc.
- The income difference between the present (t) and the future $(t+1, \ldots)$ causes investment to be very high in t but disappears in $t+1$.
- In the transitory shock:
- some amplification in production via labor supply...
- ...but very little persistence!
- The model is unable to generate internal propagation.

Transitory Technological Shock



Source: King and Rebelo (2002).

Permanent Technological Shock

- Suppose a completely permanent positive shock $(\rho = 1)$.
- New steady state with a higher capital-labor ratio.
- The household's permanent income is much higher than in the transitory case: income effect increases consumption at t (and in all periods).
- Capital is still very low at t: there are incentives to reduce consumption and invest more at t.
- For reasonable parameters, the first effect dominates, and consumption jumps upwards.
- With capital accumulation, the second effect diminishes over time, and consumption keeps increasing: $C_{t+1}/C_t = R_{t+1}\beta$

Permanent Technological Shock

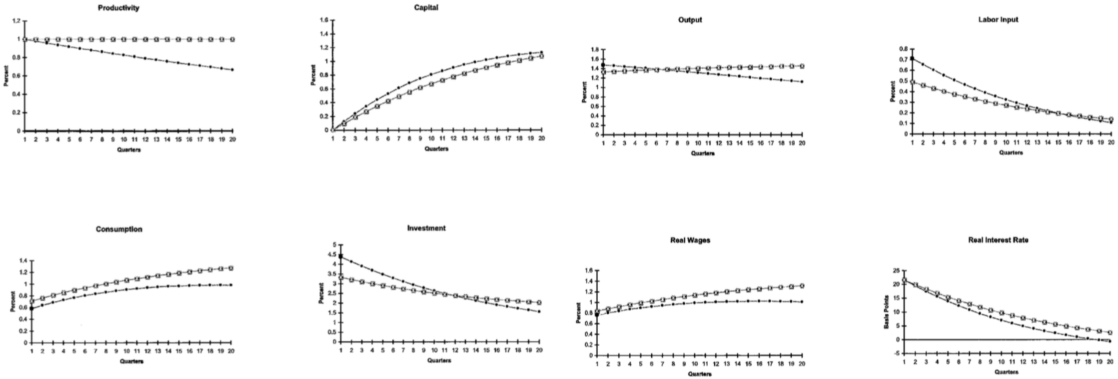
- Amplification of labor is much smaller than in the transitory case.
  - As in the transitory case, an increase in $Z \Rightarrow$ increases the demand for labor.
  - Unlike the transitory case, the income effect is much higher! Increases consumption and decreases labor supply!

$$\theta\left(1-N_t\right)^{-\phi}=\frac{\uparrow w_t}{\uparrow C_t}$$

- Generally, the substitution effect dominates at t, but the dynamics are much more complex since both C_t and w_t increase in the future due to capital accumulation.

- In the permanent shock, the model has even less amplification: The effect of permanent income silences the labor response.

Permanent Technological Shock



Source: King and Rebelo (2002).

Quantify Real Business Cycles

- A suggestion by Kydland and Prescott (1982) was to assess the usefulness of the theory by judging whether the data simulated by the model can replicate the data from the economy.

- In RBC, the focus is on the second moments (standard deviation, correlation, and autocorrelation) of the (filtered) variables of interest.

- To simulate the model, we need to:
  - Choose functional forms and the model period (annual, quarterly, etc.).
  - Choose parameters consistent with long-term facts and/or microeconomic studies.
  - Find the policy functions of the model.
  - Simulate a sequence of random variables (ε_t) and compute the endogenous variables $(y_t, c_t, k_{t+1}, \ldots)$ using the policy functions.

Calibration and Model Fit

Calibration

- We already have the functional forms. The model period will be quarterly (same as the data).

- A portion of the parameters are chosen to be consistent with the Kaldor Facts:

  - $1-\alpha \Rightarrow$ fraction of labor income in national income (usually 2/3).

  - $\beta = \frac{1}{(1+\bar{r}-\delta)} \Rightarrow$ annual interest rate 6.5% (quarterly $0.065/4$). The formula needs adjustment if the model has population/technological growth.

  - $\delta \Rightarrow 10\%$ per year (or 2.5% quarterly).

  - $n, g \Rightarrow$ if the model has exogenous population and technological growth.

  - Intertemporal elasticity of substitution / risk aversion $= 1 \Rightarrow$ log utility. Consistent with micro estimates (between 1 and 3).

Calibration

- Conditional on the parameters and ϕ, there exists a one-to-one mapping between $\bar{L}$ and θ.

$$\theta(1-\bar{N})^{-\phi} = \frac{w}{\bar{C}} = \frac{(1-\alpha)\bar{Y}}{\bar{N}\bar{C}} \Leftrightarrow \bar{N}(1-\bar{N})^{-\phi} = \theta \underbrace{\frac{(1-\alpha)\bar{Y}}{\bar{C}}}_{\text{parameters}}$$

- Typical calibrations assume the agent works from 20% to 33% of the available time: $\bar{N} = 1/3 \Rightarrow$ choose θ consistent with this value.
  - Frisch Elasticity: if $\phi = 1$ (log), $\bar{N} = 1/3$ implies a Frisch elasticity of 2. Inconsistent value with micro studies!
  - Microeconomic estimates vary between 0 and 0.5.
  - Problem: Lower elasticity implies an even weaker amplification of the RBC.
  - How to interpret this difference? Intensive/extensive margin? Unemployment?

Calibration

- Technological shock parameters: ρ and σ.

- Estimate the Solow Residual (SR) from the production function: $\log Y_t = \log SR_t + \alpha \log K_t + (1-\alpha) \log L_t$.

- Detrend (SR) from the long-term trend and recover only the short-term fluctuations Z_t (using HP-filter or a linear trend).

- Estimate an AR(1):

$$\log Z_t = \rho Z_{t-1} + \sigma \varepsilon_t \tag{5}$$

- Shock is quite persistent: $\rho = 0.979$ and $\sigma = 0.0072$.

Quantitative Performance

Table 3
Business cycle statistics for basic RBC model[a,b]

	Standard deviation	Relative standard deviation	First-order autocorrelation	Contemporaneous correlation with output
Y	1.39	1.00	0.72	1.00
C	0.61	0.44	0.79	0.94
I	4.09	2.95	0.71	0.99
N	0.67	0.48	0.71	0.97
Y/N	0.75	0.54	0.76	0.98
w	0.75	0.54	0.76	0.98
r	0.05	0.04	0.71	0.95
A	0.94	0.68	0.72	1.00

Table 1
Business cycle statistics for the US Economy

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Y/N	1.02	0.56	0.74	0.55
w	0.68	0.38	0.66	0.12
r	0.30	0.16	0.60	−0.35
A	0.98	0.54	0.74	0.78

- Strengths: (i) The shock generates good output fluctuation; (ii) Consumption is less volatile than output; (iii) Investment is more volatile than output; (iv) Variables have good autocorrelation; (v) Most variables are procyclical;

- Weaknesses: (i) Little volatility in labor; (ii) Does not generate volatility in the interest rate; (iii) Wages and interest rates are too procyclical; (iv) Basically, all autocorrelation comes from the shock.

Solution Method: Log-linearization

Local Methods

- The solution to the RBC model consists of a system of nonlinear difference equations $\Rightarrow$ no closed-form analytical solution!
- One way to solve the problem is to use Dynamic Programming.
- Another way is to use local methods:
- (Log)-Linearize the equations of the problem (Euler Equations, feasibility, etc.) around a point, usually at the deterministic steady state.
- Write the problem in a system of linear difference equations.
- Check the stability of the system and solve for the (linear) policy functions.

Local Methods

- Linearization is part of a general class of local solutions called Perturbation Methods.
- In practice, linearization (as well as its distant cousin, the Linear Quadratic Approximation) are equivalent to first-order Perturbation.
- Most packages that solve DSGE models on the computer- Matlab (Dynare), Python (PyMacLab), Julia (SolveDSGE.jl), etc - use Perturbation.
- For more details on Perturbation, see Fernández-Villaverde, Rubio-Ramírez, & Schorfheide (2016, Handbook of Macro) and Schmitt-Grohé & Uribe (2004, JEDC).

Local Methods

- Dynamic Programming
- Global Method (solution is a nonlinear policy function).
- Slow (curse of dimensionality!).
- Captures nonlinearities, asymmetries, etc.
- Can be applied to non-convexities, discrete choice.
- Perturbation Methods.
- Local Method (what happens when the shock is very large? Covid?)
- Fast.

A Road Map of Solving by Log-linearization

1. Find the equations determining the equilibrium (Euler Equations, resource constraint, etc.) [Done]

2. Compute the deterministic steady state. [Done]

3. Linearize the necessary conditions in the neighborhood of the SS and write in a system of linear difference equations like (or similar to):

$$B \begin{bmatrix} k_{t+1} \\ \mathbb{E}_t c_{t+1} \end{bmatrix} = A \begin{bmatrix} k_t \\ c_t \end{bmatrix} + C z_t$$

4. Find the solution of the system using the method of undetermined coefficients (Uhlig (1998)) or methods for solving linear models with rational expectations (Blanchard & Kahn (1980), Sims (2002), and others).

5. Use the (linear) decision rules to simulate the model, find impulse-response functions, etc.

Log-linearization

- Log-linearize or linearize?

- Suppose an aggregate variable X_t, where $\bar{X}$ is its value in steady state.

$$\tilde{x}_t = \log\left(\frac{X_t}{\bar{X}}\right) = \underbrace{\log\left(X_t\right) - \log(\bar{X})}_{\%\text{ deviations from steady state}} \approx \frac{X_t - \bar{X}}{\bar{X}}$$

- Note that we can rewrite $X_t = \bar{X}e^{\tilde{x}_t}$ and $e^{\tilde{x}_t} \approx 1 + \tilde{x}_t$.

- Our goal is to write the model variables in % deviations from steady state.

- Linearizing the model (without the log) would make interpretation difficult. Deviations would be in absolute level from SS instead of %.

General Loglinearization Rule

- With a multiplicative/exponential function, just apply the log directly. For example, production function $Y_t = Z_t K_t^{\alpha} N_t^{1-\alpha}$:

$$\underbrace{\log(Y_t) - \log(\bar{Y})}_{\tilde{y}_t} = \underbrace{\log(Z_t) - \log(\bar{Z})}_{\tilde{z}_t} + \alpha \underbrace{\left(\log(K_t) - \log(\bar{K})\right)}_{\tilde{k}_t} + (1-\alpha) \underbrace{\left(\log(N_t) - \log(\bar{N})\right)}_{\tilde{n}_t}$$

- With more complex functions, a general rule is needed.

- Remember the first-order Taylor expansion around a point $(\bar{X}, \bar{Y})$:

$$f(X,Y) = f(\bar{X},\bar{Y}) + f_x(\bar{X},\bar{Y})(X-\bar{X}) + f_y(\bar{X},\bar{Y})(Y-\bar{Y})$$

General Loglinearization Rule

- Suppose you want to loglinearize the function $Z_t = f(X_t, Y_t)$ around the steady state $\bar{Z} = f(\bar{X}, \bar{Y})$:

$$Z_t = \underbrace{f(\bar{X}, \bar{Y})}_{\bar{Z}} + f_x(\bar{X}, \bar{Y}) \left(X_t - \bar{X}\right) + f_y(\bar{X}, \bar{Y}) \left(Y_t - \bar{Y}\right)$$

$$\left(\frac{Z_t - \bar{Z}}{\bar{Z}}\right) = f_x(\bar{X}, \bar{Y}) \frac{\bar{X}}{\bar{Z}} \left(\frac{X_t - \bar{X}}{\bar{X}}\right) + f_y(\bar{X}, \bar{Y}) \frac{\bar{Y}}{\bar{Z}} \left(\frac{Y_t - \bar{Y}}{\bar{Y}}\right)$$

$$\tilde{z}_t = \bar{X} \frac{f_x(\bar{X}, \bar{Y})}{f(\bar{X}, \bar{Y})} \tilde{x}_t + \bar{Y} \frac{f_y(\bar{X}, \bar{Y})}{f(\bar{X}, \bar{Y})} \tilde{y}_t$$

- Since $\bar{X}$, $\bar{Y}$, $f(\bar{X}, \bar{Y})$ are parameter functions, $\tilde{z}_t$ is a linear function of $\tilde{x}_t$ and $\tilde{y}_t$.

General Loglinearization Rule

- Production function $Y_t = f(Z_t, K_t, N_t) = Z_t K_t^{\alpha} N_t^{1-\alpha}$ in the general case:

$$\tilde{y}_t = \bar{Z}\frac{f_z(\bar{Z},\bar{K},\bar{N})}{f(\bar{Z},\bar{K},\bar{N})}\tilde{z}_t + \bar{K}\frac{f_k(\bar{Z},\bar{K},\bar{N})}{f(\bar{Z},\bar{K},\bar{N})}\tilde{k}_t + \bar{N}\frac{f_n(\bar{Z},\bar{K},\bar{N})}{f(\bar{Z},\bar{K},\bar{N})}\tilde{n}_t$$

- Parameters:

$$\bar{Z}\frac{f_z(\bar{Z},\bar{K},\bar{N})}{f(\bar{Z},\bar{K},\bar{N})} = \bar{Z}\frac{\bar{K}^{\alpha}\bar{N}^{1-\alpha}}{\bar{Z}\bar{K}^{\alpha}\bar{N}^{1-\alpha}} = 1$$

$$\bar{K}\frac{f_k(\bar{Z},\bar{K},\bar{N})}{f(\bar{Z},\bar{K},\bar{N})} = \bar{K}\frac{\alpha\bar{Z}\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}}{\bar{Z}\bar{K}^{\alpha}\bar{N}^{1-\alpha}} = \alpha$$

$$\bar{N}\frac{f_n(\bar{Z},\bar{K},\bar{N})}{f(\bar{Z},\bar{K},\bar{N})} = \bar{N}\frac{(1-\alpha)\bar{Z}\bar{K}^{\alpha}\bar{N}^{-\alpha}}{\bar{Z}\bar{K}^{\alpha}\bar{N}^{1-\alpha}} = 1-\alpha$$

Linearized Euler Equation

- Euler Equation: $\mathbb{E}_t\left[C_{t+1}^{-\gamma}R_{t+1}\beta\right] = C_t^{-\gamma}$
  - where $R_{t+1} \equiv 1 + r_{t+1} - \delta$ and $1/\gamma$ is the intertemporal substitution elasticity.
  - note that $\left(\overline{1 + r_{t+1} - \delta}\right) = \log\left(R_{t+1}\right) - \log(\bar{R}) \approx r_{t+1} - \bar{r} = \tilde{r}_{t+1}$.

- Rearranging and using the fact: $C_t = \bar{C}e^{\tilde{c}_t}$:

$$\mathbb{E}_t\left[\frac{C_{t+1}^{\gamma}}{C_t^{\gamma}}\right] = \mathbb{E}_t\left[R_{t+1}\beta\right], \quad \mathbb{E}_t\left[\frac{\bar{C}^{\gamma}e^{\gamma\tilde{c}_{t+1}}}{\bar{C}^{\gamma}e^{\gamma\tilde{c}_t}}\right] = \mathbb{E}_t\left[\bar{R}e^{\tilde{r}_{t+1}}\beta\right]$$

- Using $\bar{R}\beta = 1, e^{\gamma\left(\tilde{c}_{t+1} - \tilde{c}_t\right)} \approx 1 + \gamma\left(\tilde{c}_{t+1} - \tilde{c}_t\right)$ and $e^{\tilde{r}_{t+1}} \approx 1 + \tilde{r}_{t+1}$:

$$\mathbb{E}_t\left[1 + \gamma\left(\tilde{c}_{t+1} - \tilde{c}_t\right)\right] = \mathbb{E}_t\left[1 + \tilde{r}_{t+1}\right]$$

$$\mathbb{E}_t\left[\tilde{c}_{t+1}\right] - \tilde{c}_t = \frac{1}{\gamma}\mathbb{E}_t\left[\tilde{r}_{t+1}\right]$$

Log-linearized Euler Equation

- Risk aversion depends on the concavity of the utility function (second derivative of u).
- Remember that $\mathbb{E}_t\left[f\left(x_{t+1}\right)\right] = f\left(\mathbb{E}_t\left[x_{t+1}\right]\right)$ only if f is linear (Jensen's inequality).
- By linearizing the equation, we are assuming certainty equivalence, meaning that an increase in uncertainty about $\mathbb{E}_t\left[c_{t+1}\right]$ has no effect on the model's equilibrium.
- Only the shock distribution's first moment (mean) matters. Which parameter represents the shock variance?

$$\log Z_{t+1} = \rho \log Z_t + \sigma \varepsilon_{t+1}$$

Log-linearization

- After log-linearizing all equations, we have a system of the type:

$$
\begin{aligned}
&\text{(Production Function)} && \tilde{y}_t = \psi_1 \tilde{z}_t + \psi_2 \tilde{k}_t + \psi_3 \tilde{n}_t && (6)\\
&\text{(Mkt. Clearing)} && \tilde{k}_{t+1} = \psi_4 \tilde{k}_t + \psi_5 \tilde{y}_t + \psi_6 \tilde{c}_t && (7)\\
&\text{(Demand for K)} && \tilde{r}_t = \psi_7 \tilde{z}_t + \psi_8 \tilde{k}_t + \psi_9 \tilde{n}_t && (8)\\
&\text{(Demand for N)} && \tilde{w}_t = \psi_{10} \tilde{z}_t + \psi_{11} \tilde{k}_t + \psi_{12} \tilde{n}_t && (9)\\
&\text{(Supply of N)} && \tilde{w}_t = \psi_{13} \tilde{n}_t + \psi_{14} \tilde{c}_t && (10)\\
&\text{(Euler Equation)} && \mathbb{E}_t \left[\tilde{c}_{t+1}\right] = \psi_{15} \tilde{c}_t + \psi_{16} \mathbb{E}_t \left[\tilde{r}_{t+1}\right] && (11)\\
&\text{(Shock)} && \tilde{z}_{t+1} = \rho \tilde{z}_t + \sigma \varepsilon_{t+1} && (12)
\end{aligned}
$$

where the ψ 's are functions of parameters and variables in a steady state.

Log-linearization

- From the equations, we can reduce the system to:

$$\tilde{k}_{t+1} = \lambda_1 \tilde{k}_t + \lambda_2 \tilde{z}_t + \lambda_3 \tilde{c}_t \tag{13}$$

$$\mathbb{E}_t \left[\tilde{c}_{t+1}\right] = \lambda_4 \mathbb{E}_t \tilde{z}_{t+1} + \lambda_5 \mathbb{E}_t \tilde{k}_{t+1} + \lambda_6 \tilde{c}_t \tag{14}$$

$$\tilde{z}_{t+1} = \rho \tilde{z}_t + \sigma \varepsilon_{t+1} \tag{15}$$

  where the λ 's are functions of the ψ 's.

- This step is not strictly necessary, and we can include intratemporal variables ($\tilde{r}_t, \tilde{w}_t$, etc).

- From now on, we can solve the system in two ways:
  - Method of undetermined coefficients (Uhlig (1998), Campbell (1994)). Depending on the model, solving it by pen and paper is possible.
  - Use a rational expectations linear model solver (Blanchard & Kahn (1980), Klein (1999), Sims (2002), Rendahl (2017)).

Undetermined Coefficients

- The idea is to guess that the policy functions are linear functions of the states $\left(\tilde{k}_t, \tilde{z}_t\right)$:

$$\tilde{k}_{t+1} = \eta_{kk}\tilde{k}_t + \eta_{kz}\tilde{z}_t \tag{16}$$

$$\tilde{c}_t = \eta_{ck}\tilde{k}_t + \eta_{cz}\tilde{z}_t \tag{17}$$

- Using (17) into (13):

$$\tilde{k}_{t+1} = \left(\lambda_1 + \lambda_3\eta_{ck}\right)\tilde{k}_t + \left(\lambda_2 + \lambda_3\eta_{cz}\right)\tilde{z}_t$$

- that is, the undetermined coefficients need to satisfy the equations:

$$\lambda_1 + \lambda_3\eta_{ck} = \eta_{kk} \tag{18}$$

$$\lambda_2 + \lambda_3\eta_{cz} = \eta_{kz} \tag{19}$$

Undetermined Coefficients

- Iterating (17) one period forward, and using (15) and (16):

$$\mathbb{E}_t \tilde{c}_{t+1} = \eta_{ck}\eta_{kk}\tilde{k}_t + (\eta_{ck}\eta_{kz} + \eta_{cz}\rho)\,\tilde{z}_t$$

- Using (17), (16) and (15) in (14):

$$\mathbb{E}_t \tilde{c}_{t+1} = (\lambda_5\eta_{kk} + \lambda_6\eta_{ck})\,\tilde{k}_t + (\rho\lambda_4 + \lambda_5\eta_{kz} + \lambda_6\eta_{cz})\,\tilde{z}_t$$

- that is, the undetermined coefficients need to satisfy the equations:

$$\eta_{ck}\eta_{kk} = \lambda_5\eta_{kk} + \lambda_6\eta_{ck} \tag{20}$$

$$\eta_{ck}\eta_{kz} + \eta_{cz}\rho = \rho\lambda_4 + \lambda_5\eta_{kz} + \lambda_6\eta_{cz} \tag{21}$$

- Finally, we have a system of 4 equations, (18), (19), (20), (21), and 4 unknowns $(\eta_{kk}, \eta_{kz}, \eta_{ck}, \eta_{cz})$.

Undetermined Coefficients

- The system of 4 equations will yield a quadratic equation in η_{kk}.
- Two possible solutions for η_{kk} :
- We are interested in the stable solution $\eta_{kk} < 1$.
- The solution $\eta_{kk} > 1$ is explosive ($\tilde{k}_{t+1}$ tends to infinity).
- The existence of a unique stable solution depends on the parameter values of the model.
- If both solutions $\eta_{kk} < 1 \Rightarrow$ multiple solutions.
- If both solutions $\eta_{kk} \geqslant 1 \Rightarrow$ no solution.
- The RBC model is quite robust to parameters, other models require more care.

Undetermined Coefficients

- The method of undetermined coefficients can be generalized (in matrix form).

- Again the system will collapse to a quadratic (matrix) equation and the system will be stable if the number of generalized eigenvalues inside the unit circle $(|\lambda| \leqslant 1)$ equals the number of predetermined states (endogenous).

- Other methods can be used to solve the system: Blanchard-Kahn (1980), Sims (2002), Klein (2000), Rendahl (2017).

- They all involve tedious manipulations of the system in matrix form.

- For more information: McCandless (2008), Canova (2007), Fernandez-Villaverde's notes.

Blanchard-Kahn Conditions

- Stability conditions can be checked directly in the linear difference equations.

- Suppose x is a $(n \times 1)$ vector of predetermined variables ($\tilde{k}_t$ in RBC), y is a ($m \times 1$) vector of non-predetermined variables (jump variables, $\tilde{c}_t$ in RBC), and z a ($k \times 1$) vector of exogenous states ($\tilde{z}_t$ in RBC).

$$\begin{bmatrix} x_{t+1} \\ \mathbb{E}_t y_{t+1} \end{bmatrix} = F \begin{bmatrix} x_t \\ y_t \end{bmatrix} + G z_t$$

  where F is a $(n+m) \times (n+m)$ matrix and G a $(n+m) \times k$ matrix.

- (Proposition) Blanchard-Kahn Conditions (1980): let h be the number of eigenvalues of F outside the unit circle $(|\lambda| > 1)$.
  - If $h = m$, the system has a unique stable solution.
  - If $h > m$, the system has no solution.
  - If $h < m$, the system is indeterminate (infinitely many solutions).