PhD Macro Core Part I:
Lecture 12 – Overlapping Generations Model

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Why OLG Model?

- The Neoclassical growth model has a representative agent
- No way to discuss implications of heterogeneity
- Life-cycle/generations are important examples of heterogeneity
- OLG model allows discussion of these issues
- More generally, it allows discussion of issues that arise with
  - Heterogeneity
  - Infinite number of agents
- Seminal papers: Allais (1947), Samuelson (1958), Diamond (1965)

Content

- Today: OLG Model Setup
- Today: Dynamic Efficiency (i.e., over-accumulation of capital)
- Next: Social Security (i.e., old age pension systems)
- Next: Public Debt
- Next: Money/Bubbles

Setup

- Two generations: Young and Old
- Each lives for two periods (discrete time)
- Young work, consume, save
- Old consume and dissave (do not work)
- Common extensions: Many generations or Perpetual youth model (Blanchard, 1985)
- Two generation version is particularly simple
  because it precludes intertemporal trade (no one meets twice!)

Setup

- L_t individuals are born at time t
- Exogenous population growth at rate n:

$$L_{t+1} = (1+n)L_t$$

- Each young agent supplies 1 unit of labor
- "Youth" need not be due to birth. It could be immigration or the binding of a borrowing constraint.

Production

- Production function:

$$Y_t = F\left(K_t, A_t L_t\right)$$

- Exogenous productivity growth:

$$A_{t+1} = (1+g)A_t$$

- Perfect competition in factor markets yields:

$$r_t = f'\left(k_t\right), \quad w_t = f\left(k_t\right) - r_t k_t$$

  - r_t is the return on savings held from period $t-1$ to t
  - w_t is the wage per effective unit of labor

Households

- Preferences of households born at t:

$$U_t = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2t+1}^{1-\theta}}{1-\theta}$$

- Budget constraints:

$$C_{1t} + s_t = w_t A_t$$
$$C_{2t+1} = (1 + r_{t+1})\, s_t$$

- S_t is savings of young at time t

- Old consumes both interest and principle

- We are assuming no depreciation of capital (for simplicity)

Household Optimization

- We can plug budget constraints into U_t to get

$$U_t = \frac{(w_t A_t - s_t)^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{((1+r_{t+1})\, s_t)^{1-\theta}}{1-\theta}$$

- Differentiating with respect to s_t yields:

$$-(w_t A_t - s_t)^{-\theta} + \frac{1+r_{t+1}}{1+\rho} ((1+r_{t+1})\, s_t)^{-\theta} = 0$$

- Rearranging and using budget constraints again:

$$C_{1t}^{-\theta} = \frac{1+r_{t+1}}{1+\rho} C_{2t+1}^{-\theta}$$

- This is the consumption Euler equation

Household Consumption Function

- Combining the budget constraints:

$$C_{1t} + \frac{1}{1 + r_{t+1}} C_{2t+1} = A_t w_t$$

- Rearranging Euler equation:

$$C_{2t+1} = \left(\frac{1 + r_{t+1}}{1 + \rho} \right)^{1/\theta} C_{1t}$$

- Combining these two:

$$C_{1t} + \frac{(1 + r_{t+1})^{(1-\theta)/\theta}}{(1 + \rho)^{1/\theta}} C_{1t} = A_t w_t$$

Consumption and Saving

- Solving for C_{1t} yields:

$$C_{1t} = \frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{(1-\theta)\theta}} A_t w_t$$

- Young spend some fraction of labor income on time 1 consumption

- Savings:

$$s_t = A_t w_t - C_{1t} = \frac{(1+r_{t+1})^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{(1-\theta)\theta}} A_t w_t$$

- Young save a complementary fraction of their labor income

Savings: Comparative Statics

$$s_t = \frac{(1+r_{t+1})^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{(1-\theta)/\theta}} A_t w_t$$

- Savings unambiguously increase in wage income (Both C_{1t} and C_{2t+1} are normal goods)
- Effect of a change in r_{t+1} is ambiguous
- Change in r_{t+1} has both an income effect and a substitution effect
  - Increase in r_{t+1} decreases price of C_{2t+1} (which increases savings)
  - Increase in r_{t+1} increases feasible consumption set (which decreases savings)

Savings: Comparative Statics

$$s_t = \frac{(1+r_{t+1})^{(1-\theta)/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{(1-\theta)/\theta}} A_t w_t$$

- Savings increase in r_{t+1} if $(1+r_{t+1})^{(1-\theta)/\theta}$ is increasing in r_{t+1}

$$\frac{d}{dr}(1+r)^{(1-\theta)/\theta} = \frac{1-\theta}{\theta}(1+r)^{(1-\theta)/\theta}$$

- Savings increase in r_{t+1} if $\theta < 1$, i.e., if IES $\equiv 1/\theta > 1$
- If IES $\equiv 1/\theta < 1$, the substitution effect is strong and overwhelms the income effect
- If IES = 1 (log utility) saving is unaffected by r_{t+1}

Evolution of Capital Stock

- Savings of young at time t become capital stock at time $t+1$:

$$K_{t+1} = s_t L_t$$

- Using notation from Romer (2019): $s_t = s\left(r_{t+1}\right) A_t w_t$

$$K_{t+1} = s\left(r_{t+1}\right) A_t w_t L_t$$

- Dividing through by $A_{t+1} L_{t+1}$ yields:

$$k_{t+1} = \frac{s\left(r_{t+1}\right) w_t}{(1+n)(1+g)}$$

where $k_t = K_t / \left(A_t L_t\right)$

Evolution of Capital Stock

- Plugging in for w_t and r_{t+1}:

$$k_{t+1} = \frac{s\left(f'\left(k_{t+1}\right)\right)\left[f\left(k_t\right) - k_t f'\left(k_t\right)\right]}{(1+n)(1+g)}$$

- Implicitly defines k_{t+1} as a function of k_t
- Let's call this function the "savings locus."
- Steady state when $k_{t+1} = k_t$

Evolution of Capital Stock

$$k_{t+1} = \frac{s\left(f'\left(k_{t+1}\right)\right)\left[f\left(k_t\right) - k_t f'\left(k_t\right)\right]}{(1+n)(1+g)}$$

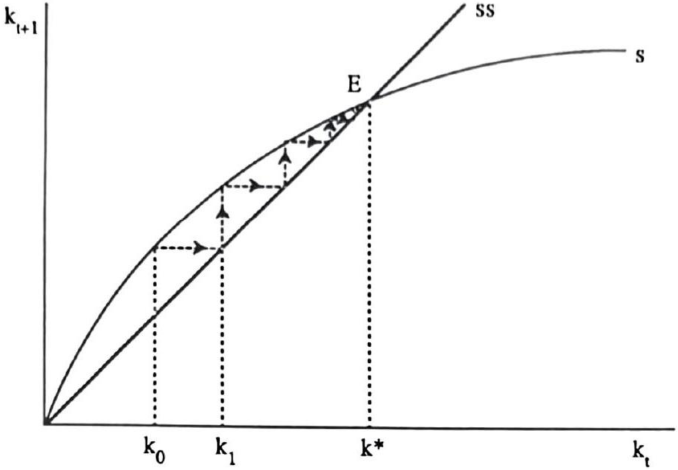
- Let's start by considering a special case:
  - Logarithmic utility (i.e., $\theta = 1$)
  - Cobb-Douglas production function $\left(y = k^{\alpha}\right)$
- In this case:

$$s\left(r_{t+1}\right) = \frac{1}{2+\rho} \quad \text{and} \quad f(k) - kf'(k) = k^{\alpha} - \alpha k^{\alpha} = (1-\alpha)k^{\alpha}$$

- So, we have:

$$k_{t+1} = \frac{(1-\alpha)}{(1+n)(1+g)(2+\rho)} k_t^{\alpha}$$

Evolution of Capital Stock: Special Case



Evolution of Capital Stock: Special Case

- In this special case:
  - There is a single steady state (with positive capital)
  - The steady state is locally stable
- What is it that makes the steady state locally stable?

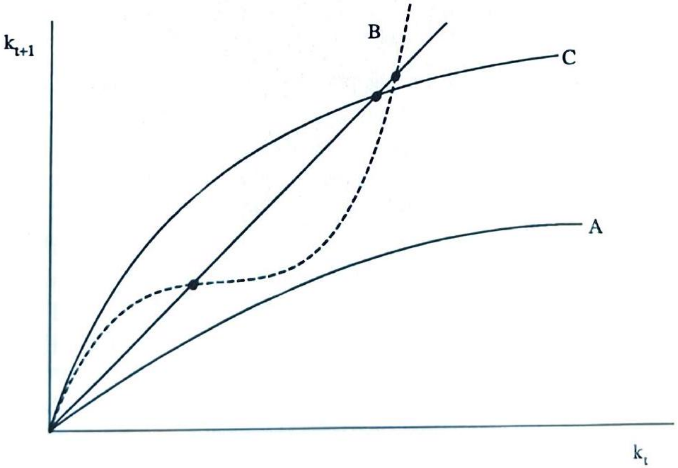
$$\left.\frac{dk_{t+1}}{dk_t}\right|_{ss} < 1$$

Evolution of Capital Stock: More Generally

$$k_{t+1} = \frac{s\left(f'\left(k_{t+1}\right)\right)\left[f\left(k_t\right) - k_t f'\left(k_t\right)\right]}{(1+n)(1+g)}$$

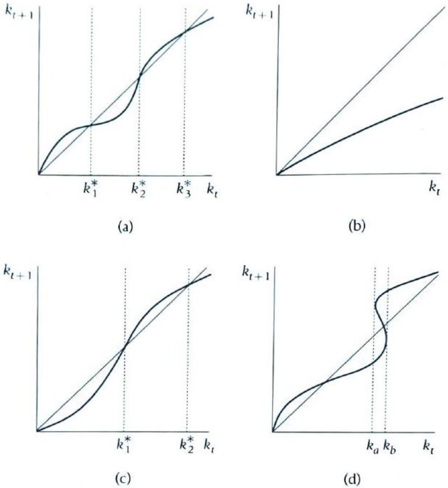
- More generally, the savings locus can take many different shapes
- This can lead to various types of pathologies
- No steady state with positive capital
- Multiple steady states with positive capital
- Multiple equilibria

Evolution of Capital Stock: More Generally



Evolution of Capital Stock: More Generally

- Various possibilities for the relationship between $\boldsymbol{k_t}$ and $\boldsymbol{k_{t+1}}$



Evolution of Capital Stock: More Generally

$$k_{t+1} = \frac{s\left(f'\left(k_{t+1}\right)\right)\left[f\left(k_t\right) - k_t f'\left(k_t\right)\right]}{(1+n)(1+g)}$$

- We can rewrite this as follows:

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \underbrace{s\left(r_{t+1}\right)}_{\text{savings rate}} \underbrace{\frac{f\left(k_t\right) - k_t f'\left(k_t\right)}{f\left(k_t\right)}}_{\text{labor share}} \underbrace{f\left(k_t\right)}_{\text{output per person}}$$

- $f(k)$ concave (diminishing returns)
- With log utility $s(r)$ constant, with Cobb-Douglas labor share constant
- Multiple steady states: need sharply rising savings rate or labor share

Welfare

- Common in macro to compare market outcome to outcome from "planner's problem"
- Conceptually simple in a model with a representative agent
  (planner will maximize that agent's welfare)
- Not as simple in model with heterogeneous agents such as OLG model
- How should planners weigh the welfare of different generations?
- However, Pareto optimality is still unambiguous

Welfare

- Is market outcome Pareto optimal in the OLG model?
- Turns out this is not necessarily the case
- Economy may accumulate "too much" capital
- If so, it is possible to make everyone better off

Golden Rule Capital

- Let’s consider log-utility, Cobb-Douglas production case
- Let’s also assume $g = 0$ for simplicity and focus on steady state
- Golden Rule capital stock:
- Capital stock that yields the highest steady-state consumption per effective unit of labor
- It Never makes sense to have more capital than Golden Rule capital
- In this case, less capital would give more consumption
- "the economy staggers under the weight of the need to maintain the per capita capital stock constant." (Blanchard and Fischer, 1989)

Resource Constraint

- Economy's resource constraint:

$$K_t + F\left(K_t, A_t L_t\right) = K_{t+1} + C_{1t}L_t + C_{2t}L_{t-1}$$

- Divide through by $A_t L_t$

$$k_t + f\left(k_t\right) = (1+n)k_{t+1} + A_t^{-1}c_t$$

where $c_t = C_{1t} + (1+n)^{-1}C_{2t}$ (weighted average of young and old consumption)

- In steady state with $g = 0$:

$$A^{-1}c = f(k) - nk$$

Golden Rule Capital

- In steady state with $g = 0$
$$A^{-1}c = f(k) - nk$$
- c is maximized when
$$f'(k_{GK}) = n$$
which implicitly gives the Golden Rule capital stock

Market Steady State

- OLG savings locus:

$$k_{t+1} = \frac{(1-\alpha)}{(1+n)(1+g)(2+\rho)} k_{t+1}^{\alpha}$$

- With $g = 0$ and in steady state:

$$k^* = \frac{(1-\alpha)}{(1+n)(2+\rho)} k^{*\alpha}$$

which simplifies to

$$k^* = \left[\frac{(1-\alpha)}{(1+n)(2+\rho)}\right]^{1/(1-\alpha)}$$

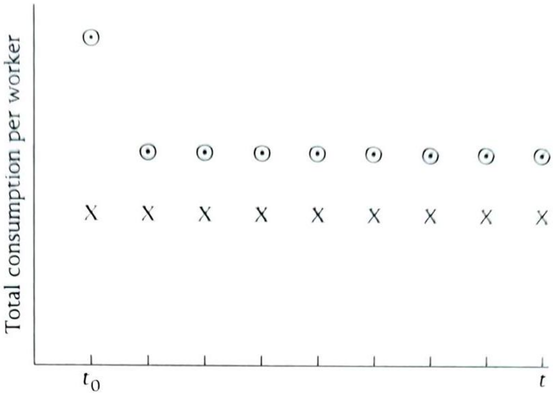
Market Steady State

- If
$$k^* = \left[\frac{(1-\alpha)}{(1+n)(2+\rho)}\right]^{1/(1-\alpha)}$$
then
$$f'(k^*) = \alpha k^{*\alpha-1} = \frac{\alpha}{1-\alpha}(1+n)(2+\rho)$$
- We have ignored depreciation. If $f(k) = k^\alpha - \delta k$:
$$f'(k^*) = \frac{\alpha}{1-\alpha}(1+n)(2+\rho) - \delta$$
- Recall that $r = f'(k)$. So, we have
$$r^* = \frac{\alpha}{1-\alpha}(1+n)(2+\rho) - \delta$$

Dynamic Inefficiency

- If $r^* < n$, the economy has more capital than the Golden Rule capital
- This outcome is Pareto inefficient
- Economy is said to be dynamically inefficient
- Suppose in some period t_0, social planner cuts capital to k_{GK}
- In period t_0: More resources available for consumption due to cut
- In periods $t > t_0$: More resources available for consumption because nk falls more than $f(k)$
- This policy change can thus make everyone better off

Dynamic Inefficiency



- X maintaining k at $k^* > k_{GR}$ & $\odot$ reducing k to k_{GR} in period t_0

Dynamic Inefficiency

- Only technology available to households to transfer resources from when they are young to when they are old is capital accumulation
- At the margin, the return on this technology is

$$r = f'(k)$$

- If households are patient enough, they will accumulate capital to the point where $r < n$
- They have no private reason to pay any attention to n

Dynamic Inefficiency

- Society has another technology for transferring resources from the young to the old
- The government can simply:
  - Take d units from each young
  - Give $(1+n)d$ units to each old
- Notice that the "return" on this technology is n
  (because the old generation is less populous than the young)
- Must be repeated forever to be a Pareto improvement
- If $r < n$, this "government technology" is better than what is available to people "in the market"
  (i.e., through saving or bilateral trade)

Dynamic Inefficiency

- With growth in output per person $(g \neq 0)$ we get
- Economy is dynamically efficient if $r^* > g + n$
- Economy is dynamically inefficient if $r^* < g + n$
- This suggests a way to test dynamic efficiency
- Complication: Which interest rate to use? (More on this later.)

Why Inefficiency?

- It may seem puzzling that the market equilibrium is inefficient
- What is the failure of the First Welfare Theorem?
- All markets are competitive
- All agents are rational
- Property rights are well-defined and costlessly enforced
- Isn't this enough?

Problem with Infinity

- Things can get complicated when there are an infinite number of agents

- Consider "government technology" discussed above:

  - Take one from each young and give $1 + n$ to each old
    (Recall that the young generation is more populous)

  - Do this again next period, and so on

  - If return to saving is less than n, this makes everyone better off

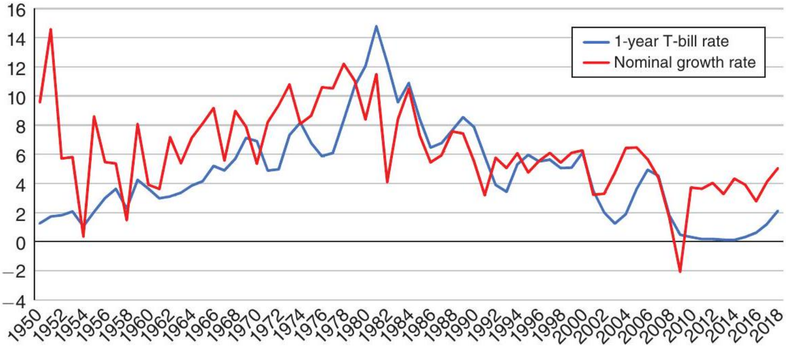
- This scheme only works if there are an infinite number of generations

- FWT holds with infinite agents if the present value of endowments is finite
  (which does not hold if the economy is dynamically inefficient)

Public Debt

- When $r < n$, government can issue debt at no cost
- Suppose government borrows B from each young person
- Next period it owes $(1 + r)B$ to each old.
- Suppose it again borrows B from each young
- Since there are $(1 + n)$ young for each old, it borrows $(1 + n)B$ for each $(1 + r)B$ that it owes
- System is self-financing as long as $r < n$!!
- With growth, the relevant issue is perhaps the debt-to-GDP ratio.
  The relevant condition is then $r < g$

More Public Debt?



Should We Issue More Public Debt?

- Looks like $r < g$ much of the time
- So, it looks like public debt is a "free lunch."
- Does this mean we should issue more?
- Well, public debt "crowds out" private capital
- But with $r < g$, isn't there overaccumulation of capital?
- Not so fast! Relevant r for dynamic efficiency is not necessarily the same as for debt sustainability

Should We Issue More Public Debt?

Blanchard (2019):

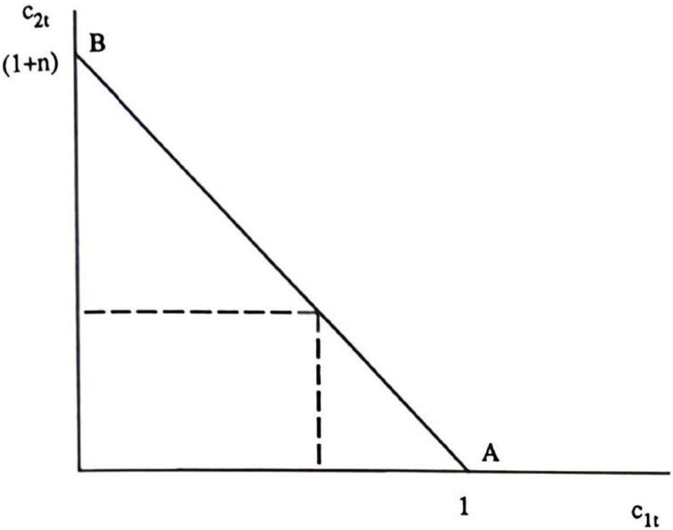
- Two types of welfare effects of more debt:
  - Lower capital accumulation
  - Induced changes in returns to labor and capital
- Relevant interest rate for first of these:
  - Safe rate because the safe rate is the "risk-adjusted" rate of return on capital
- Relevant interest rate for the second of these:
- Average (risky) marginal return on capital
- Welfare effects of more debt ambiguous

OLG Without Capital & Production

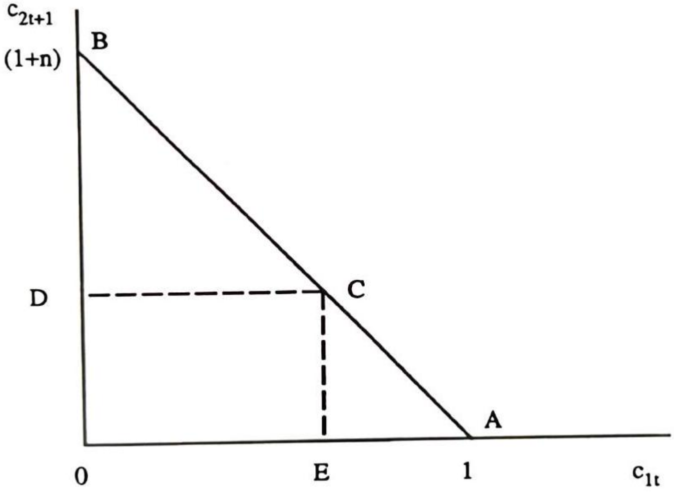
Consider the following simpler setting:

- Two generation OLG model: Young and Old
- Population growth: $L_t = (1+n)^t$
- No production / No capital
- Each young individual endowed with 1 unit of consumption good
- Old receive no endowment
- Consumption good is perishable
- Individuals have standard utility function $U\left(C_{1t}, C_{2t+1}\right)$

Society's Consumption Possibility



Individual's Consumption Possibility



Barter Equilibrium

- Given this set of possibilities, the individual would choose an "interior" point (e.g., C on the last slide)
- However, this is not attainable through bilateral trade
- Initial old have nothing to offer
- Initial young would like to exchange goods today for goods next period, but next period's young not yet born
- No trade possible!!
- "Market outcome" is A on the last slide, which is highly Pareto-inefficient

Shadow Interest Rate

- Intertemporal trade not possible. So, there is no actual interest rate
- But we can define a "shadow interest rate."
- I.e., interest rate that would make young happy not to trade
- For "normal preferences", this interest rate would be -100% (i.e., if $U'(C) \to \infty$ as $C \to 0$)
- So, this simple case is clearly a case of $r < n + g$

Pay-As-You-Go Pension System

- Suppose the government transferred an amount $d < 1$ from young to old from period t onward
- Initial old obviously much better off | Young and all future generations are also better off
- No longer destitute in old age.
- For moderate d, an increase in d is a Pareto improvement
- Marginal cost: $U'(1-d)$ | Marginal benefit: $(1+n)U'((1+n)d)(1+\rho)^{-1}$
- Increase in d is a Pareto improvement as long as

$$(1+n)\frac{U'((1+n)d)}{(1+\rho)} > U'(1-d) \quad = \quad 1+n > 1+r$$

(Recall that $(1+r)^{-1} = U'\left(C_{t+1}\right) / \left(U'\left(C_t\right)(1+\rho)\right)$)

Two Kinds of Pension System

1. Fully Funded
   - Government forces young to save (buy capital)
   - No effect on capital accumulation if people are fully rational (and forced saving is not too large)
   - Increases capital accumulation if people are myopic
2. Pay-as-You-Go
   - Government taxes young and gives proceeds to current old
   - Reduces capital accumulation if people are fully rational
   - Welfare improving even with rational agents if economy is dynamically inefficient $(r < n + g)$ (See Blanchard and Fischer (1989, ch. 3.2))

Inter-grenational Risk Sharing

- We have ignored risk up until now
- Risk introduces another source of inefficiency in OLG models
- Efficient intergenerational risk sharing is not possible
- Suppose there is a shock at time t :
- Efficient to smooth the shock over infinite future
- This will not happen in an OLG model
- Gov. pension system can help bring about efficient risk-sharing
- Ball and Mankiw (2007) take a "first stab" at this

Pure Fiat Money

- Consider again the simple barter economy
- Suppose at $t = 0$ the government gives old H units of (completely divisible) inherently useless green pieces of paper
- Let's call these pieces of paper money
- Suppose the old and every future generation believe they will be able to exchange goods for money at a price P_t in period t
- P_t is the price level in this economy
- If this is an equilibrium, individuals can trade:
- Buy money for goods when young

Household Problem

- Maximize

$$U_t = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2t+1}^{1-\theta}}{1-\theta}$$

  subject to

$$P_t\left(1 - C_{1t}\right) = M_t^d$$

$$P_{t+1}C_{2t+1} = M_t^d$$

- Plugging constraints into objective, differentiating, setting the result to zero, and rearranging yields:

$$\frac{M_t^d}{P_t} = \frac{1}{1 + (1+\rho)^{1/\theta}\Pi_{t+1}^{(\theta-1)/\theta}} \quad \text{where} \quad \Pi_{t+1} = \frac{P_{t+1}}{P_t}$$

- This is the money demand function, as the savings function

Money Demand

$$\frac{M_t^d}{P_t} = \frac{1}{1 + (1+\rho)^{1/\theta}\Pi_{t+1}^{(\theta-1)/\theta}}$$

- Π_{t+1} is the (inverse of the) rate of return on money
- Effect of an increase in Π_{t+1} on money demand ambiguous
- If $\theta > 1$, higher Π_{t+1} leads to lower money demand (substitution effect dominates)
- If $\theta < 1$, higher Π_{t+1} leads to higher money demand (income effect dominates)
- Let's denote the money demand function:

$$\frac{M_t^d}{P_t} = L\left(\Pi_{t+1}\right)$$

Equilibrium with Money

- Money demand equal to money supply:

$$(1+n)^t M_t^d = H$$

- Also true in period $t+1$

$$(1+n)^t M_t^d = (1+n)^{t+1} M_{t+1}^d$$

- Dividing by P_t on both sides:

$$\frac{M_t^d}{P_t} = (1+n)\frac{P_{t+1}}{P_t}\frac{M_{t+1}^d}{P_{t+1}}$$

- Plugging in for money demand:

$$L\left(\Pi_{t+1}\right) = (1+n)\Pi_{t+1} L\left(\Pi_{t+2}\right)$$

Equilibrium with Money

$$L\left(\Pi_t\right) = (1+n)\Pi_t L\left(\Pi_{t+1}\right)$$

- Consider a steady state where

$$\Pi_t = \Pi_{t+1} = \bar{\Pi}$$

- Then we have that

$$L(\bar{\Pi}) = (1+n)\bar{\Pi}L(\bar{\Pi})$$

- This simplifies to

$$\bar{\Pi} = (1+n)^{-1}$$

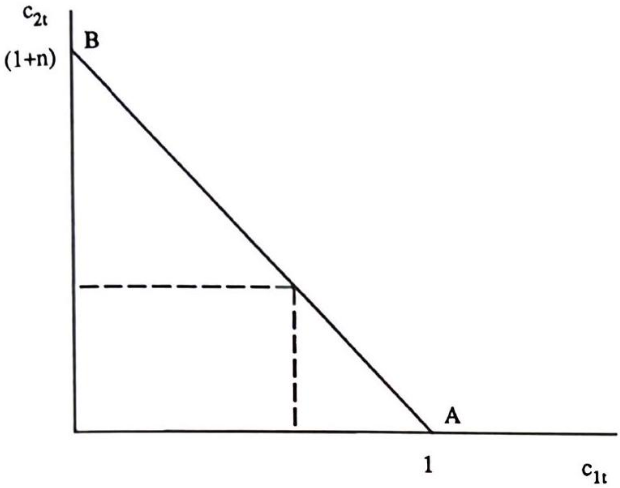
Equilibrium with Money

- This means that there is an equilibrium of the model with a constant inflation rate equal to $(1+n)^{-1}$
- Return on holding money is Π^{-1}
- In equilibrium with a constant inflation rate, the return on holding money is

$$\bar{\Pi}^{-1} = (1+n)$$

- This is the "golden rule" return on assets in this economy
- Money allows the economy to reach an efficient equilibrium

Consumption Possibility with Money



Fiat Money in OLG Model

- Money is intrinsically worthless in this model
- Yet, it is valued in equilibrium
- Valued because everyone believes it will continue to be valued
- Not just valued, it allows the economy to reach Pareto efficient outcome!

Fiat Money and Time Horizon

- For money to be valued, the economy must go on forever
- If world ends at time T, money will not be valued in period T
- If money not valued in period T, also not valued in period T - 1
- Many other equilibria, including one where money is not valued
- If people don't believe money will be valued tomorrow, it will not be valued today
- Lots of equilibria in between

Fragility of Monetary Equilibrium

- In simple economy $r < n$
- In an economy with assets with $r > n$, there is no monetary equilibrium (Blanchard and Fischer, 1989, ch. 4.1)
- Monetary equilibrium only exists when the economy is dynamically inefficient
- Money plays the same role as the government pension system

Money and OLG Model

- In the OLG model, money is only valued if it is not dominated in the rate of return
- In reality, money is dominated in rate of return
- In the OLG model, money is a store of value
- In reality, money is a unit of account (and medium of exchange)
- OLG model doesn't capture some crucial features of money

Rational Bubbles

- In the OLG model, money can be valued even though it pays no dividends
- Example of a "rational bubble"
- Bubble: Asset that has a higher price than the discounted value of future dividends
- Bubbles can arise in OLG model (Tirole, 1985; Blanchard and Fischer, 1989, ch. 5)
- Bubbles can arise in some other settings as well (Santos and Woodford, 1997)