(d) stationary distribution $\Rightarrow$ $X_{k+1} = X_k \quad k \to \infty$

$$T = \begin{bmatrix} 0.8 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.6 \\ 0 & 0.2 & 0.2 \end{bmatrix}$$

$$X_s = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.6 \\ 0 & 0.2 & 0.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = T X_s$$

$$\Rightarrow \begin{cases} 0.8x_1 + 0.4x_2 + 0.2x_3 = x_1 \\ 0.2x_1 + 0.4x_2 + 0.6x_3 = x_2 \\ 0 + 0.2x_2 + 0.2x_3 = x_3 \end{cases}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{9}{14} \\ \frac{4}{14} \\ \frac{1}{14} \end{bmatrix} \#$$

(e)

$$H_p(X) = -\sum_x p(x) \log_2 p(x)$$

$$= -\left( \frac{9}{14} \times \log_2\left(\frac{9}{14}\right) + \frac{4}{14} \times \log_2\left(\frac{4}{14}\right) + \frac{1}{14} \times \log_2\left(\frac{1}{14}\right) \right)$$

$$\cong 1.1981$$

(f)

$$P(\text{yesterday} \mid \text{today}) = \frac{P(\text{today} \mid \text{yesterday})\, P(\text{yesterday})}{P(\text{today})}$$

use "y" as yesterday, "t" as today

assume $P(y)$ and $P(t)$ are stationary distribution of question (d)

① $$P(y=\text{sunny} \mid t=\text{sunny}) = \frac{P(t=\text{sunny} \mid y=\text{sunny})\, P(y=\text{sunny})}{P(t=\text{sunny})} = \frac{0.8 \times \frac{9}{14}}{\frac{9}{14}} = 0.8$$

② $$P(y=\text{cloudy} \mid t=\text{sunny}) = \frac{P(t=\text{sunny} \mid y=\text{cloudy})\, P(y=\text{cloudy})}{P(t=\text{sunny})} = \frac{0.4 \times \frac{4}{14}}{\frac{9}{14}} = 0.178$$

③ $$P(y=\text{rainy} \mid t=\text{sunny}) = \frac{P(t=\text{sunny} \mid y=\text{rainy})\, P(y=\text{rainy})}{P(t=\text{sunny})} = \frac{0.2 \times \frac{1}{14}}{\frac{9}{14}} = 0.022$$

④ $$P(y=\text{sunny} \mid t=\text{cloudy}) = \frac{P(t=\text{cloudy} \mid y=\text{sunny})\, P(y=\text{sunny})}{P(t=\text{cloudy})} = \frac{0.2 \times \frac{9}{14}}{\frac{4}{14}} = 0.45$$

⑤ $$P(y=\text{cloudy} \mid t=\text{cloudy}) = \frac{P(t=\text{cloudy} \mid y=\text{cloudy})\, P(y=\text{cloudy})}{P(t=\text{cloudy})} = \frac{0.4 \times \frac{4}{14}}{\frac{4}{14}} = 0.4$$

⑥ $$P(y=\text{rainy} \mid t=\text{cloudy}) = \frac{P(t=\text{cloudy} \mid y=\text{rainy})\,P(y=\text{rainy})}{P(t=\text{cloudy})}$$

$$= \frac{0.6 \times \frac{1}{14}}{\frac{4}{14}} = 0.15$$

⑦ $$P(y=\text{sunny} \mid t=\text{rainy}) = \frac{P(t=\text{rainy} \mid y=\text{sunny})\,P(y=\text{sunny})}{P(t=\text{rainy})}$$

$$= \frac{0 \times \frac{9}{14}}{\frac{1}{14}} = 0$$

⑧ $$P(y=\text{cloudy} \mid t=\text{rainy}) = \frac{P(t=\text{rainy} \mid y=\text{cloudy})\,P(y=\text{cloudy})}{P(t=\text{rainy})}$$

$$= \frac{0.2 \times \frac{4}{14}}{\frac{1}{14}} = 0.8$$

⑨. $$P(y=\text{rainy} \mid t=\text{rainy}) = \frac{P(t=\text{rainy} \mid y=\text{rainy})\,P(y=\text{rainy})}{P(t=\text{rainy})}$$

$$= \frac{0.2 \times \frac{1}{14}}{\frac{1}{14}} = 0.2$$

probability table of $P(y \mid t)$:

| | | yesterday | | |
|---|---|---|---|---|
| | | sunny | cloudy | rainy |
| today | sunny | 0.8 | 0.178 | 0.022 |
| | cloudy | 0.45 | 0.4 | 0.15 |
| | rainy | 0 | 0.8 | 0.2 |

(a) Assume $x_t$ is the weather of day $t$

$z_t$ is the measurement of the sensor on day $t$

Let $S$ = sunny , $C$ = cloudy , $R$ = rainy

from measurement model ⇒ if observation = rainy, then the actual weather 100% is rainy

⇒ $x_4$ must be rainy

所求 ⇒ $P(x_5=S \mid z_5=S, x_4=R)$

由 Bayes rule 和 Markov Assumption

① $P(x_5=S \mid z_5=S, x_4=R)$

$$= \frac{P(z_5=S \mid x_5=S, x_4=R)\,P(x_5=S \mid x_4=R)}{P(z_5=S \mid x_4=R)}$$

$$= \eta\, P(z_5=S \mid x_5=S)\,P(x_5=S \mid x_4=R)$$

$$= \eta\ 0.6 \times 0.2 = 0.12\eta$$

② $P(x_5=C \mid z_5=S, x_4=R)$

$$= \frac{P(z_5=S \mid x_5=C, x_4=R)\,P(x_5=C \mid x_4=R)}{P(z_5=S \mid x_4=R)}$$

$$= \eta\, P(z_5=S \mid x_5=C)\,P(x_5=C \mid x_4=R)$$

$$= \eta \times 0.3 \times 0.6 = 0.18\eta$$

③ $P(x_5=R \mid z_5=S, x_4=R)$

$$= \frac{P(z_5=S \mid x_5=R, x_4=R)\,P(x_5=R \mid x_4=R)}{P(z_5=S, x_4=R)}$$

$$= \eta \times P(z_5=S \mid x_5=R)\,P(x_5=R \mid x_4=R)$$

$= \eta \times 0 \times 0.2 = 0$

$0.12\eta + 0.18\eta + 0 = 1 \Rightarrow \eta = \frac{1}{0.12+0.18} = \frac{10}{3}$

$\Rightarrow$ 所求 $P(x_5 = S \mid z_5 = S, x_4 = R) = 0.12\eta = 0.12 \times \frac{10}{3}$

$= 0.4$ #

(b)

Assume $x_t$ is the weather of day $t$

$z_t$ is the measurement of the sensor on day $t$

Let $S$ = sunny , $C$ = cloudy , $R$ = rainy

由 Bayes rule 和 Markov Assumption

Known: $x_1 = S$ , $z_2 = S$, $z_3 = S$ , $z_4 = R$

Way 1 : only the data available to the day in question is used

① day 2 :

$$P(x_2 \mid z_2 = S, x_1 = S)$$

$$= \frac{P(z_2 = S \mid x_2, x_1 = S)\, P(x_2 \mid x_1 = S)}{P(z_2 = S \mid x_1 = S)}$$

$$= \eta\, P(z_2 = S \mid x_2)\, P(x_2 \mid x_1 = S)$$

$$\Rightarrow P(x_2 = S \mid z_2 = S, x_1 = S) = \eta \cdot P(z_2 = S \mid x_2 = S)\, P(x_2 = S \mid x_1 = S)$$

$$= \eta \cdot 0.6 \cdot 0.8 = 0.48\eta$$

$$P(x_2 = C \mid z_2 = S, x_1 = S) = \eta \cdot P(z_2 = S \mid x_2 = C)\, P(x_2 = C \mid x_1 = S)$$

$$= \eta \cdot 0.3 \cdot 0.2 = 0.06\eta$$

$$P(x_2 = R \mid z_2 = S, x_1 = S) = \eta \cdot P(z_2 = S \mid x_2 = R)\, P(x_2 = R \mid x_1 = S)$$

$$= \eta \cdot 0 \cdot 0 = 0$$

$$\eta = \frac{1}{0.48+0.06} = \frac{50}{27} \cong 1.85$$

$\eta$代回 $\Rightarrow$ $P(x_2=S \mid z_2, x_1) = \frac{8}{9} \cong 0.89$

$P(x_2=C \mid z_2, x_1) = \frac{1}{9} \cong 0.11$

$P(x_2=R \mid z_2, x_1) = 0$ #

② Day 3:

$$P(x_3 \mid z_{2:3}, x_1)$$

$$= \frac{P(z_3 \mid x_3, z_2, x_1)\, P(x_3 \mid z_2, x_1)}{P(z_3 \mid z_2, x_1)}$$

$$= \eta\, P(z_3=S \mid x_3, z_2=S, x_1=S)\, P(x_3 \mid z_2=S, x_1=S)$$

$$= \eta\, P(z_3=S \mid x_3) \sum_{x_2} P(x_3 \mid x_2, z_2=S, x_1=S)\, P(x_2 \mid z_2=S, x_1=S)$$

$$= \eta\, P(z_3=S \mid x_3) \sum_{x_2} P(x_3 \mid x_2)\, P(x_2 \mid z_2=S, x_1=S)$$

$$\Rightarrow P(x_3=S \mid z_{2:3}, x_1) = \eta\, P(z_3=S \mid x_3=S) \sum_{x_2} P(x_3=S \mid x_2)\, P(x_2 \mid z_2=S, x_1=S)$$

$$= \eta \cdot 0.6 \cdot \left(0.8 \times \frac{8}{9} + 0.4 \times \frac{1}{9} + 0.2 \times 0\right)$$

$$= \frac{34}{75}\eta$$

$$P(x_3=C \mid z_{2:3}, x_1) = \eta\, P(z_3=S \mid x_3=C) \sum_{x_2} P(x_3=C \mid x_2)\, P(x_2 \mid z_2=S, x_1=S)$$

$$= \eta \cdot 0.3 \cdot \left(0.2 \times \frac{8}{9} + 0.4 \times \frac{1}{9} + 0.6 \times 0\right)$$

$$= \frac{1}{15}\eta$$

$$P(x_3=R \mid z_{2:3}, x_1) = \eta\, P(z_3=S \mid x_3=R) \sum_{x_2} P(x_3=R \mid x_2)\, P(x_2 \mid z_2=S, x_1=S)$$

$$= 0$$

$$\eta = \frac{1}{\frac{34}{75}+\frac{1}{15}} = \frac{25}{13} \cong 1.923$$

$$\Rightarrow \begin{aligned} &P(x_3=S \mid z_{2:3}, x_1) = \frac{34}{75} \times \frac{25}{13} = \frac{34}{39} \cong 0.87 \\ &P(x_3=C \mid z_{2:3}, x_1) = \frac{1}{15} \times \frac{25}{13} = \frac{5}{39} \cong 0.13 \\ &P(x_3=R \mid z_{2:3}, x_1) = 0 \end{aligned}$$

#

③ Day 4:

由表知, observation = rainy, 則 weather 100% is rainy

$$z_4 = R \Rightarrow x_4 = R$$

用一樣的方式推導也能得到相同結果

$$P(x_4 \mid z_{2:4}, x_1)$$

$$= \frac{P(z_4 \mid x_4, z_{2:3}, x_1)\, P(x_4 \mid z_{2:3}, x_1)}{P(z_4 \mid z_{2:3}, x_1)}$$

$$= \eta\, P(z_4=R \mid x_4, z_{2:3}, x_1=S)\, P(x_4 \mid z_{2:3}, x_1=S)$$

$$= \eta\, P(z_4=R \mid x_4) \sum_{x_3} P(x_4 \mid x_3, z_{2:3}, x_1=S)\, P(x_3 \mid z_{2:3}, x_1=S)$$

$$= \eta\, P(z_4=R \mid x_4) \sum_{x_3} P(x_4 \mid x_3)\, P(x_3 \mid z_{2:3}, x_1=S)$$

$$\Rightarrow P(x_4=S \mid z_{2:4}, x_1) = \eta \times 0 \times \sum_{x_3} P(x_4=S \mid x_3) P(x_3 \mid z_{2:3}, x_1=S) = 0$$

$$P(x_4=C \mid z_{2:4}, x_1) = \eta \times 0 \times \sum_{x_3} P(x_4=C \mid x_3) P(x_3 \mid z_{2:3}, x_1=S) = 0$$

$$P(x_4=R \mid z_{2:4}, x_1) = \eta \times 1 \times (0 + 0.2 \times 0.13 + 0.2 \times 0) = 0.026\eta$$

$$\eta = \frac{1}{0.026}$$

$$\Rightarrow \begin{aligned} &P(x_4=S \mid z_{2:4}, x_1) = 0 \\ &P(x_4=C \mid z_{2:4}, x_1) = 0 \\ &P(x_4=R \mid z_{2:4}, x_1) = 1 \end{aligned}$$

#

Way 2 : data from future days is also available

① Day 2 :

$$P(x_2=S \mid z_{2:4}, x_1)$$

$$= \frac{P(z_{2:4} \mid x_2=S, x_1=S)\, P(x_2=S \mid x_1=S)}{P(z_{2:4} \mid x_1=S)}$$

$$= \eta P(z_{2:4} \mid x_2=S, x_1=S)\, P(x_2=S \mid x_1=S)$$

$$= \eta P(z_2 \mid x_2=S, x_1=S)\, P(z_{3:4} \mid x_2=S, x_1=S)\, P(x_2=S \mid x_1=S)$$

$$= \eta P(z_2 \mid x_2=S)\, P(x_2=S \mid x_1=S) \sum_{x_3} P(z_{3:4} \mid x_3, x_2=S)\, P(x_3 \mid x_2=S)$$

$$= \eta P(z_2 \mid x_2=S)\, P(x_2=S \mid x_1=S) \sum_{x_3} P(z_3 \mid x_3, x_2=S)\, P(z_4 \mid x_3, x_2=S)\, P(x_3 \mid x_2=S)$$

$$= \eta P(z_2 \mid x_2=S)\, P(x_2=S \mid x_1=S) \sum_{x_3} P(z_3 \mid x_3)\, P(x_3 \mid x_2=S)\, P(z_4 \mid x_3)$$

$$= \eta P(z_2 \mid x_2=S)\, P(x_2=S \mid x_1=S) \sum_{x_3} \left[ P(z_3 \mid x_3)\, P(x_3 \mid x_2=S) \sum_{x_4} P(z_4 \mid x_4, x_3)\, P(x_4 \mid x_3) \right]$$

$$= \eta P(z_2 \mid x_2=S)\, P(x_2=S \mid x_1=S) \sum_{x_3} \left[ P(z_3 \mid x_3)\, P(x_3 \mid x_2=S) \sum_{x_4} P(z_4 \mid x_4)\, P(x_4 \mid x_3) \right]$$

$\because z_4=R \Rightarrow x_4=R$

$$= \eta P(z_2 \mid x_2=S)\, P(x_2=S \mid x_1=S) \sum_{x_3} \left[ P(z_3 \mid x_3)\, P(x_3 \mid x_2=S)\, P(z_4=R \mid x_4=R)\, P(x_4=R \mid x_3) \right]$$

$$\Rightarrow P(x_2=S \mid z_{2:4}, x_1) = \eta P(z_2 \mid x_2=S)\, P(x_2=S \mid x_1=S) \sum_{x_3} \left[ P(z_3 \mid x_3)\, P(x_3 \mid x_2=S)\, P(z_4=R \mid x_4=R)\, P(x_4=R \mid x_3) \right]$$

$$= \eta \cdot 0.6 \cdot 0.8 \cdot (0.6 \times 0.8 \times 1 \times 0 + 0.3 \times 0.2 \times 1 \times 0.2 + 0 \times 0 \times 1 \times 0.2)$$

$$= 0.00576\, \eta$$

同理可得：

$$\Rightarrow P(x_2=C \mid z_{2:4}, x_1)$$

$$= \eta P(z_2 \mid x_2=C)\, P(x_2=C \mid x_1=S) \sum_{x_3} \left[ P(z_3 \mid x_3)\, P(x_3 \mid x_2=C)\, P(z_4=R \mid x_4=R)\, P(x_4=R \mid x_3) \right]$$

$$= \eta \cdot 0.3 \cdot 0.2 \cdot (0.6 \times 0.4 \times 1 \times 0 + 0.3 \times 0.4 \times 1 \times 0.2 + 0 \times 0.2 \times 1 \times 0.2)$$

$$= 0.00144\, \eta$$

$$\Rightarrow P(x_2=R \mid z_{2:4}, x_1)$$
$$= \eta \underbrace{P(z_2 \mid x_2=R)}_{=0} P(x_2=R \mid x_1=S) \sum_{x_3} [P(z_3 \mid x_3) P(x_3 \mid x_2=R) P(z_4=R \mid x_4=R) P(x_4=R \mid x_3)]$$
$$= 0$$

$$\eta = \frac{1}{0.00576+0.00144} = \frac{1250}{9} \cong 138.89$$

$$\Rightarrow P(x_2=S \mid z_{2:4}, x_1) = 0.00576\,\eta = 0.8$$
$$P(x_2=C \mid z_{2:4}, x_1) = 0.00144\,\eta = 0.2$$
$$P(x_2=R \mid z_{2:4}, x_1) = 0$$
#

② Day 3 :

$$P(x_3=S \mid z_{2:4}, x_1)$$
$$= \frac{P(z_{2:4} \mid x_3=S, x_1=S)\, P(x_3=S \mid x_1=S)}{P(z_{2:4} \mid x_1=S)}$$
$$= \eta\, P(z_{2:4} \mid x_3=S, x_1=S) P(x_3=S \mid x_1=S)$$
$$= \eta\, P(z_4 \mid x_3=S, x_1=S) P(z_3 \mid x_3=S, x_1=S) P(z_2 \mid x_3=S, x_1=S) P(x_3=S \mid x_1=S)$$
$$= \eta\, P(z_3 \mid x_3=S) \sum_{x_4} [P(z_4 \mid x_4, x_3=S) P(x_4 \mid x_3=S)] \sum_{x_2} [P(z_2 \mid x_2, x_1=S) P(x_2 \mid x_1=S)]$$
$$\cdot \sum_{x_2} [P(x_3=S \mid x_2, x_1=S) P(x_2 \mid x_1=S)]$$
$$= \eta\, P(z_3 \mid x_3=S) \sum_{x_4} [P(z_4 \mid x_4) P(x_4 \mid x_3=S)] \sum_{x_2} [P(z_2 \mid x_2) P(x_2 \mid x_1=S)]$$
$$\cdot \sum_{x_2} [P(x_3=S \mid x_2) P(x_2 \mid x_1=S)]$$

$\because z_4=R \Rightarrow x_4=R$

$$= \eta\, P(z_3 \mid x_3=S)\, P(z_4=R \mid x_4=R) \underbrace{P(x_4=R \mid x_3=S)}_{=0} \sum_{x_2} [P(z_2 \mid x_2) P(x_2 \mid x_1=S)]$$
$$\cdot \sum_{x_2} [P(x_3=S \mid x_2) P(x_2 \mid x_1=S)]$$

$$\Rightarrow P(x_3=S \mid z_{2:4}, x_1) = 0$$

同理可得：

$\Rightarrow P(x_3=C \mid z_{2:4}, x_1) =$

$\eta\, P(z_3 \mid x_3=C)\, P(z_4=R \mid x_4=R)\, P(x_4=R \mid x_3=C) \sum_{x_2} \left[ P(z_2 \mid x_2) P(x_2 \mid x_1=S) \right]$

$\cdot \sum_{x_2} \left[ P(x_3=C \mid x_2)\, P(x_2 \mid x_1=S) \right]$

$= \eta \cdot 0.3 \cdot 1 \cdot 0.2 \cdot (0.6\times0.8 + 0.3\times0.2 + 0)(0.2\times0.8 + 0.4\times0.2 + 0)$

$= 0.007776\, \eta$

$\Rightarrow P(x_3=R \mid z_{2:4}, x_1)$

$= \eta\, \underbrace{P(z_3 \mid x_3=R)}_{0}\, P(z_4=R \mid x_4=R)\, P(x_4=R \mid x_3=R) \sum_{x_2} \left[ P(z_2 \mid x_2) P(x_2 \mid x_1=S) \right]$

$\cdot \sum_{x_2} \left[ P(x_3=R \mid x_2)\, P(x_2 \mid x_1=S) \right]$

$= 0$

$\eta = \dfrac{1}{0.007776}$

$\Rightarrow P(x_3=S \mid z_{2:4}, x_1) = 0$

$P(x_3=C \mid z_{2:4}, x_1) = 1$

$P(x_3=R \mid z_{2:4}, x_1) = 0$ #

④ Day 4：

前面也有提到

在 $z_4=R$ 條件下

$x_4$ 必為 rainy

$\Rightarrow P(x_4=S \mid z_{2:4}, x_1) = 0$

$P(x_4=C \mid z_{2:4}, x_1) = 0$

$P(x_4=R \mid z_{2:4}, x_1) = 1$ #