Self-Driving Car Assignment 2: Probability Bayesian

312512005 黃名諄

Exercises 2

Suppose we live at a place where days are either sunny, cloudy, or rainy. The weather transition function is a Markov chain with the following transition table:

		tomorrow will be...		
		sunny	cloudy	rainy
today it's...	sunny	.8	.2	0
	cloudy	.4	.4	.2
	rainy	.2	.6	.2

(a) Suppose Day 1 is a sunny day. What is the probability of the following sequence of days: Day2 = cloudy,Day3 = cloudy, Day4 = rainy?

Ans:

day1(sunny) to day2(cloudy) : 0.2
day2(cloudy) to day3(cloudy) : 0.4
day3(cloudy) to day4(rainy) : 0.2
1* 0.2* 0.4* 0.2 = 0.016

(b)Write a simulator that can randomly generate sequences of “weathers” from this state transition function.

Ans:

HW2 > 312512005_HW2.py > main

```python
import numpy as np

def generate_weather_sequence(tran_matrix, initial, days):

    weather_sequence = [initial]
    for i in range(days-1):
        current = weather_sequence[-1]
        next = np.random.choice(len(tran_matrix), p=tran_matrix[current])
        weather_sequence.append(next)

    #print weather_sequence
    weather = ['sunny', 'cloudy', 'rainy']
    print("randomly generate weather sequences: ")
    for i in range(len(weather_sequence)):
        if weather_sequence[i]==0:
            print("Day {:d} : {:s} ,".format(i+1, weather[0]),end='')
            print(" the probability of tomorrow\'s weather will be: ", tran_matrix[0])
        elif weather_sequence[i]==1:
            print("Day {:d} : {:s} ,".format(i+1, weather[1]),end='')
            print(" the probability of tomorrow\'s weather will be: ", tran_matrix[1])
        elif weather_sequence[i]==2:
            print("Day {:d} : {:s} ,".format(i+1, weather[2]),end='')
            print(" the probability of tomorrow\'s weather will be: ", tran_matrix[2])
```

```python

def stationary_distribution(tran_matrix, initial, days):
    if initial==0:
        weather_prob=np.array([1,0,0])
    elif initial==1:
        weather_prob=np.array([0,1,0])
    else:
        weather_prob=np.array([0,0,1])

    #count the probability of weather by Markov chain
    for i in range(days):
        weather_prob = np.dot(weather_prob, tran_matrix)
        print("The probability of the weather: ", weather_prob)

def main():
    weather_transition_matrix = np.array([
        [0.8, 0.2, 0],
        [0.4, 0.4, 0.2],
        [0.2, 0.6, 0.2]
    ])

    num_days = int(input('Number of days you want to generate the weather sequence \n'))
    first_day_weather = int(input('Today\'s weather is? 1.sunny, 2.cloudy, 3.rainy\n'))



    #generate weather sequence based on given probabilities table
    generate_weather_sequence(weather_transition_matrix,first_day_weather-1,num_days)
    print("-" * 40)
    #count stationary weather distributions
    print('The probability of the stationary distributions of the weather:')
    stationary_distribution(weather_transition_matrix, first_day_weather-1 ,num_days)

if __name__ == '__main__':
    main()
```

demo result:

1. number of days = 10, today(day1) is sunny

```
Number of days you want to generate the weather sequence
10
Today's weather is? 1.sunny, 2.cloudy, 3.rainy
1
randomly generate weather sequences:
Day 1 : sunny , the probability of tomorrow's weather will be:  [0.8 0.2 0. ]
Day 2 : sunny , the probability of tomorrow's weather will be:  [0.8 0.2 0. ]
Day 3 : sunny , the probability of tomorrow's weather will be:  [0.8 0.2 0. ]
Day 4 : cloudy , the probability of tomorrow's weather will be:  [0.4 0.4 0.2]
Day 5 : cloudy , the probability of tomorrow's weather will be:  [0.4 0.4 0.2]
Day 6 : cloudy , the probability of tomorrow's weather will be:  [0.4 0.4 0.2]
Day 7 : cloudy , the probability of tomorrow's weather will be:  [0.4 0.4 0.2]
Day 8 : cloudy , the probability of tomorrow's weather will be:  [0.4 0.4 0.2]
Day 9 : rainy , the probability of tomorrow's weather will be:  [0.2 0.6 0.2]
Day 10 : cloudy , the probability of tomorrow's weather will be:  [0.4 0.4 0.2]
```

2. number of days = 10, today(day1) is cloudy

```
Number of days you want to generate the weather sequence
10
Today's weather is? 1.sunny, 2.cloudy, 3.rainy
2
randomly generate weather sequences:
Day 1 : cloudy , the probability of tomorrow's weather will be:  [0.4 0.4 0.2]
Day 2 : cloudy , the probability of tomorrow's weather will be:  [0.4 0.4 0.2]
Day 3 : sunny , the probability of tomorrow's weather will be:  [0.8 0.2 0. ]
Day 4 : sunny , the probability of tomorrow's weather will be:  [0.8 0.2 0. ]
Day 5 : sunny , the probability of tomorrow's weather will be:  [0.8 0.2 0. ]
Day 6 : sunny , the probability of tomorrow's weather will be:  [0.8 0.2 0. ]
Day 7 : sunny , the probability of tomorrow's weather will be:  [0.8 0.2 0. ]
Day 8 : sunny , the probability of tomorrow's weather will be:  [0.8 0.2 0. ]
Day 9 : sunny , the probability of tomorrow's weather will be:  [0.8 0.2 0. ]
Day 10 : cloudy , the probability of tomorrow's weather will be:  [0.4 0.4 0.2]
```

3. number of days = 10, today(day1) is rainy

```
Number of days you want to generate the weather sequence
10
Today's weather is? 1.sunny, 2.cloudy, 3.rainy
3
randomly generate weather sequences:
Day 1 : rainy , the probability of tomorrow's weather will be:  [0.2 0.6 0.2]
Day 2 : cloudy , the probability of tomorrow's weather will be:  [0.4 0.4 0.2]
Day 3 : sunny , the probability of tomorrow's weather will be:  [0.8 0.2 0. ]
Day 4 : sunny , the probability of tomorrow's weather will be:  [0.8 0.2 0. ]
Day 5 : sunny , the probability of tomorrow's weather will be:  [0.8 0.2 0. ]
Day 6 : cloudy , the probability of tomorrow's weather will be:  [0.4 0.4 0.2]
Day 7 : rainy , the probability of tomorrow's weather will be:  [0.2 0.6 0.2]
Day 8 : rainy , the probability of tomorrow's weather will be:  [0.2 0.6 0.2]
Day 9 : sunny , the probability of tomorrow's weather will be:  [0.8 0.2 0. ]
Day 10 : sunny , the probability of tomorrow's weather will be:  [0.8 0.2 0. ]
```

(c)Use your simulator to determine the stationary distribution of this Markov chain. The stationary distribution measures the probability that a random day will be sunny, cloudy, or rainy.

Ans:

1. number of days = 10, today(day1) is sunny

```
----------------------------------------
The probability of the stationary distributions of the weather:
The probability of the weather:  [0.8 0.2 0. ]
The probability of the weather:  [0.72 0.24 0.04]
The probability of the weather:  [0.68  0.264 0.056]
The probability of the weather:  [0.6608 0.2752 0.064 ]
The probability of the weather:  [0.65152 0.28064 0.06784]
The probability of the weather:  [0.64704  0.283264 0.069696]
The probability of the weather:  [0.6448768 0.2845312 0.070592 ]
The probability of the weather:  [0.64383232 0.28514304 0.07102464]
The probability of the weather:  [0.643328   0.28543846 0.07123354]
The probability of the weather:  [0.64308449 0.28558111 0.0713344 ]
```

2. number of days = 10, today(day1) is cloudy

```
The probability of the stationary distributions of the weather:
The probability of the weather:  [0.4 0.4 0.2]
The probability of the weather:  [0.52 0.36 0.12]
The probability of the weather:  [0.584 0.32  0.096]
The probability of the weather:  [0.6144 0.3024 0.0832]
The probability of the weather:  [0.62912 0.29376 0.07712]
The probability of the weather:  [0.636224 0.2896   0.074176]
The probability of the weather:  [0.6396544 0.2875904 0.0727552]
The probability of the weather:  [0.64131072 0.28662016 0.07206912]
The probability of the weather:  [0.64211046 0.28615168 0.07173786]
The probability of the weather:  [0.64249661 0.28592548 0.07157791]
```

3. number of days = 10, today(day1) is rainy

```
The probability of the stationary distributions of the weather:
The probability of the weather:  [0.2 0.6 0.2]
The probability of the weather:  [0.44 0.4  0.16]
The probability of the weather:  [0.544 0.344 0.112]
The probability of the weather:  [0.5952 0.3136 0.0912]
The probability of the weather:  [0.61984 0.2992  0.08096]
The probability of the weather:  [0.631744 0.292224 0.076032]
The probability of the weather:  [0.6374912 0.2888576 0.0736512]
The probability of the weather:  [0.64026624 0.287232   0.07250176]
The probability of the weather:  [0.64160614 0.2864471  0.07194675]
The probability of the weather:  [0.64225311 0.28606812 0.07167877]
```

(d)Can you devise a closed-form solution to calculating the stationary distribution based on the state transition matrix above?

Ans:

(d) stationary distribution $\Rightarrow X_{k+1} = X_k \quad k \to \infty$

$$T = \begin{bmatrix} 0.8 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.6 \\ 0 & 0.2 & 0.2 \end{bmatrix}$$

$$X_s = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.6 \\ 0 & 0.2 & 0.2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = T X_s$$

$$\Rightarrow \begin{cases} 0.8x_1 + 0.4x_2 + 0.2x_3 = x_1 \\ 0.2x_1 + 0.4x_2 + 0.6x_3 = x_2 \\ 0 + 0.2x_2 + 0.2x_3 = x_3 \end{cases}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{9}{14} \\ \frac{4}{14} \\ \frac{1}{14} \end{bmatrix} \#$$

(e) What is the entropy of the stationary distribution?

Ans:

(e)

$$H_p(X) = -\sum_x p(x)\log_2 p(x)$$

$$= -\left(\frac{9}{14}\times\log_2\left(\frac{9}{14}\right) + \frac{4}{14}\times\log_2\left(\frac{4}{14}\right) + \frac{1}{14}\times\log_2\left(\frac{1}{14}\right)\right)$$

$$\cong 1.1981$$

(f)Using Bayes rule, compute the probability table of yesterday's weather given today's weather. (It is okay to provide the probabilities numerically, and it is also okay to rely on results from previous questions in this exercise.)

Ans:

(f)

$$P(\text{yesterday} \mid \text{today}) = \frac{P(\text{today} \mid \text{yesterday})\, P(\text{yesterday})}{P(\text{today})}$$

use "y" as yesterday, "t" as today

assume $P(y)$ and $P(t)$ are stationary distribution of question (d)

① $$P(y=\text{sunny} \mid t=\text{sunny}) = \frac{P(t=\text{sunny} \mid y=\text{sunny})\, P(y=\text{sunny})}{P(t=\text{sunny})}$$

$$= \frac{0.8\times\frac{9}{14}}{\frac{9}{14}} = 0.8$$

② $$P(y=\text{cloudy} \mid t=\text{sunny}) = \frac{P(t=\text{sunny} \mid y=\text{cloudy})\, P(y=\text{cloudy})}{P(t=\text{sunny})}$$

$$= \frac{0.4\times\frac{4}{14}}{\frac{9}{14}} = 0.178$$

③ $$P(y=\text{rainy} \mid t=\text{sunny}) = \frac{P(t=\text{sunny} \mid y=\text{rainy})\, P(y=\text{rainy})}{P(t=\text{sunny})}$$

$$= \frac{0.2\times\frac{1}{14}}{\frac{9}{14}} = 0.022$$

④ $$P(y=sunny \mid t=cloudy) = \frac{P(t=cloudy \mid y=sunny)\, P(y=sunny)}{P(t=cloudy)}$$

$$= \frac{0.2 \times \frac{9}{14}}{\frac{4}{14}} = 0.45$$

⑤ $$P(y=cloudy \mid t=cloudy) = \frac{P(t=cloudy \mid y=cloudy)\, P(y=cloudy)}{P(t=cloudy)}$$

$$= \frac{0.4 \times \frac{4}{14}}{\frac{4}{14}} = 0.4$$

⑥ $$P(y=rainy \mid t=cloudy) = \frac{P(t=cloudy \mid y=rainy)\, P(y=rainy)}{P(t=cloudy)}$$

$$= \frac{0.6 \times \frac{1}{14}}{\frac{4}{14}} = 0.15$$

⑦ $$P(y=sunny \mid t=rainy) = \frac{P(t=rainy \mid y=sunny)\, P(y=sunny)}{P(t=rainy)}$$

$$= \frac{0 \times \frac{9}{14}}{\frac{1}{14}} = 0$$

⑧ $$P(y=cloudy \mid t=rainy) = \frac{P(t=rainy \mid y=cloudy)\, P(y=cloudy)}{P(t=rainy)}$$

$$= \frac{0.2 \times \frac{4}{14}}{\frac{1}{14}} = 0.8$$

⑨ $$P(y=rainy \mid t=rainy) = \frac{P(t=rainy \mid y=rainy)\, P(y=rainy)}{P(t=rainy)}$$

$$= \frac{0.2 \times \frac{1}{14}}{\frac{1}{14}} = 0.2$$

probability table of $P(y|t)$:

		yesterday		
		Sunny	cloudy	rainy
today	Sunny	0.8	0.178	0.022
	cloudy	0.45	0.4	0.15
	rainy	0	0.8	0.2

(g)Suppose we added seasons to our model. The state transition function above would only apply to the Summer, whereas different ones would apply to Winter, Spring, and Fall. Would this violate the Markov property of this process? Explain your answer.

Ans:

The Markov property dictates that the future state of the system depends solely on its current state. We can view it as a Markov chain that is different depends on the season(time varying).Therefore, it do not violate Markov assumption because future state of the system still depends on its current state only just within each different season.

Furthermore, we can incorporate the season as a state variable $\boldsymbol{s_t}$ into the transition matrix for consideration. Use it to count the transition probability:

$$p(x_{t+1}, s_{t+1}|x_{1:t}, s_{1:t}) = p(x_{t+1}, s_{t+1}|x_t, s_t)$$

Exercises 3

Suppose that we cannot observe the weather directly, but instead rely on a sensor. The problem is that our sensor is noisy. Its measurements are governed by the following measurement model:

		our sensor tells us...		
		sunny	cloudy	rainy
the actual weather is...	sunny	.6	.4	0
	cloudy	.3	.7	0
	rainy	0	0	1

(a) Suppose Day 1 is sunny (this is known for a fact), and in the subsequent four days our sensor observes cloudy, cloudy, rainy, sunny. What is the probability that Day 5 is indeed sunny as predicted by our sensor?

Ans:

(a)

Assume x_t is the weather of day t

z_t is the measurement of the sensor on day t

Let S = sunny , C = cloudy , R = rainy

from measurement model ⇒ if observation = rainy, then the actual weather 100% is rainy

⇒ x_4 must be rainy

所求 ⇒ $P(x_5=S \mid z_5=S, x_4=R)$

由 Bayes rule 和 Markov Assumption

① $P(x_5=S \mid z_5=S, x_4=R)$

$$= \frac{P(z_5=S \mid x_5=S, x_4=R)\,P(x_5=S \mid x_4=R)}{P(z_5=S \mid x_4=R)}$$

$$= \eta\, P(z_5=S \mid x_5=S)\,P(x_5=S \mid x_4=R)$$

$$= \eta\ 0.6 \times 0.2 = 0.12\eta$$

② $P(x_5=C \mid z_5=S, x_4=R)$

$$= \frac{P(z_5=S \mid x_5=C, x_4=R)\,P(x_5=C \mid x_4=R)}{P(z_5=S \mid x_4=R)}$$

$$= \eta\, P(z_5=S \mid x_5=C)\,P(x_5=C \mid x_4=R)$$

$$= \eta \times 0.3 \times 0.6 = 0.18\eta$$

③ $P(x_5=R \mid z_5=S, x_4=R)$

$$= \frac{P(z_5=S \mid x_5=R, x_4=R)\,P(x_5=R \mid x_4=R)}{P(z_5=S, x_4=R)}$$

$$= \eta \times P(z_5=S \mid x_5=R)\,P(x_5=R \mid x_4=R)$$

$= \eta \times 0 \times 0.2 = 0$

$0.12\eta + 0.18\eta + 0 = 1 \Rightarrow \eta = \frac{1}{0.12+0.18} = \frac{10}{3}$

$\Rightarrow$ 所求 $P(X_5 = S \mid Z_5 = S, X_4 = R) = 0.12\eta = 0.12 \times \frac{10}{3}$

$= 0.4$ #

(b) Once again, suppose Day 1 is known to be sunny. At Days 2 through 4, the sensor measures sunny, sunny, rainy. For each of the Days 2 through 4, what is the most likely weather on that day? Answer the question in two ways: one in which only the data available to the day in question is used, and one in hindsight, where data from future days is also available.

(b)

Assume X_t is the weather of day t

Z_t is the measurement of the sensor on day t

Let S = sunny , C = cloudy , R = rainy

由 Bayes rule 和 Markov Assumption

Known: $X_1 = S$, $Z_2 = S$, $Z_3 = S$, $Z_4 = R$

Way 1 : only the data available to the day in question is used

① day 2:

$$P(X_2 \mid Z_2 = S, X_1 = S)$$

$$= \frac{P(Z_2 = S \mid X_2, X_1 = S)\, P(X_2 \mid X_1 = S)}{P(Z_2 = S \mid X_1 = S)}$$

$$= \eta\, P(Z_2 = S \mid X_2)\, P(X_2 \mid X_1 = S)$$

$$\Rightarrow P(X_2 = S \mid Z_2 = S, X_1 = S) = \eta \cdot P(Z_2 = S \mid X_2 = S)\, P(X_2 = S \mid X_1 = S)$$

$$= \eta \cdot 0.6 \cdot 0.8 = 0.48\eta$$

$$P(X_2 = C \mid Z_2 = S, X_1 = S) = \eta \cdot P(Z_2 = S \mid X_2 = C)\, P(X_2 = C \mid X_1 = S)$$

$$= \eta \cdot 0.3 \cdot 0.2 = 0.06\eta$$

$$P(X_2 = R \mid Z_2 = S, X_1 = S) = \eta \cdot P(Z_2 = S \mid X_2 = R)\, P(X_2 = R \mid X_1 = S)$$

$$= \eta \cdot 0 \cdot 0 = 0$$

$$\eta = \frac{1}{0.48+0.06} = \frac{50}{27} \cong 1.85$$

η代回 $\Rightarrow$

$$P(x_2=S \mid z_2, x_1) = \frac{8}{9} \cong 0.89$$

$$P(x_2=C \mid z_2, x_1) = \frac{1}{9} \cong 0.11$$

$$P(x_2=R \mid z_2, x_1) = 0$$

#

② Day 3:

$$P(x_3 \mid z_{2:3}, x_1)$$

$$= \frac{P(z_3 \mid x_3, z_2, x_1)\, P(x_3 \mid z_2, x_1)}{P(z_3 \mid z_2, x_1)}$$

$$= \eta\, P(z_3=S \mid x_3, z_2=S, x_1=S)\, P(x_3 \mid z_2=S, x_1=S)$$

$$= \eta\, P(z_3=S \mid x_3) \sum_{x_2} P(x_3 \mid x_2, z_2=S, x_1=S)\, P(x_2 \mid z_2=S, x_1=S)$$

$$= \eta\, P(z_3=S \mid x_3) \sum_{x_2} P(x_3 \mid x_2)\, P(x_2 \mid z_2=S, x_1=S)$$

$$\Rightarrow P(x_3=S \mid z_{2:3}, x_1) = \eta\, P(z_3=S \mid x_3=S) \sum_{x_2} P(x_3=S \mid x_2)\, P(x_2 \mid z_2=S, x_1=S)$$

$$= \eta \cdot 0.6 \cdot \left(0.8 \times \frac{8}{9} + 0.4 \times \frac{1}{9} + 0.2 \times 0\right)$$

$$= \frac{34}{75}\eta$$

$$P(x_3=C \mid z_{2:3}, x_1) = \eta\, P(z_3=S \mid x_3=C) \sum_{x_2} P(x_3=C \mid x_2)\, P(x_2 \mid z_2=S, x_1=S)$$

$$= \eta \cdot 0.3 \cdot \left(0.2 \times \frac{8}{9} + 0.4 \times \frac{1}{9} + 0.6 \times 0\right)$$

$$= \frac{1}{15}\eta$$

$$P(x_3=R \mid z_{2:3}, x_1) = \eta\, P(z_3=S \mid x_3=R) \sum_{x_2} P(x_3=R \mid x_2)\, P(x_2 \mid z_2=S, x_1=S)$$

$$= 0$$

$$\eta = \frac{1}{\frac{34}{75}+\frac{1}{15}} = \frac{25}{13} \cong 1.923$$

$$\Rightarrow \quad P(x_3=S \mid z_{2:3}, x_1) = \frac{34}{75} \times \frac{25}{13} = \frac{34}{39} \cong 0.87$$

$$P(x_3=C \mid z_{2:3}, x_1) = \frac{1}{15} \times \frac{25}{13} = \frac{5}{39} \cong 0.13$$

$$P(x_3=R \mid z_{2:3}, x_1) = 0 \qquad \#$$

③ Day 4:

由表知, observation = rainy, 則 weather 100% is rainy

$$z_4 = R \Rightarrow x_4 = R$$

用一樣的方式推導也能得到相同結果

$$P(x_4 \mid z_{2:4}, x_1)$$

$$= \frac{P(z_4 \mid x_4, z_{2:3}, x_1)\, P(x_4 \mid z_{2:3}, x_1)}{P(z_4 \mid z_{2:3}, x_1)}$$

$$= \eta\, P(z_4=R \mid x_4, z_{2:3}, x_1=S)\, P(x_4 \mid z_{2:3}, x_1=S)$$

$$= \eta\, P(z_4=R \mid x_4) \sum_{x_3} P(x_4 \mid x_3, z_{2:3}, x_1=S)\, P(x_3 \mid z_{2:3}, x_1=S)$$

$$= \eta\, P(z_4=R \mid x_4) \sum_{x_3} P(x_4 \mid x_3)\, P(x_3 \mid z_{2:3}, x_1=S)$$

$$\Rightarrow P(x_4=S \mid z_{2:4}, x_1) = \eta \times 0 \times \sum_{x_3} P(x_4=S \mid x_3) P(x_3 \mid z_{2:3}, x_1=S) = 0$$

$$P(x_4=C \mid z_{2:4}, x_1) = \eta \times 0 \times \sum_{x_3} P(x_4=C \mid x_3) P(x_3 \mid z_{2:3}, x_1=S) = 0$$

$$P(x_4=R \mid z_{2:4}, x_1) = \eta \times 1 \times (0 + 0.2 \times 0.13 + 0.2 \times 0) = 0.026\eta$$

$$\eta = \frac{1}{0.026}$$

$$\Rightarrow \quad P(x_4=S \mid z_{2:4}, x_1) = 0$$

$$P(x_4=C \mid z_{2:4}, x_1) = 0$$

$$P(x_4=R \mid z_{2:4}, x_1) = 1 \qquad \#$$

Way 2 : data from future days is also available

① Day 2 :

$$P(x_2=S \mid z_{2:4}, x_1)$$

$$= \frac{P(z_{2:4} \mid x_2=S, x_1=S)\, P(x_2=S \mid x_1=S)}{P(z_{2:4} \mid x_1=S)}$$

$$= \eta P(z_{2:4} \mid x_2=S, x_1=S)\, P(x_2=S \mid x_1=S)$$

$$= \eta P(z_2 \mid x_2=S, x_1=S)\, P(z_{3:4} \mid x_2=S, x_1=S)\, P(x_2=S \mid x_1=S)$$

$$= \eta P(z_2 \mid x_2=S)\, P(x_2=S \mid x_1=S) \sum_{x_3} P(z_{3:4} \mid x_3, x_2=S)\, P(x_3 \mid x_2=S)$$

$$= \eta P(z_2 \mid x_2=S)\, P(x_2=S \mid x_1=S) \sum_{x_3} P(z_3 \mid x_3, x_2=S)\, P(z_4 \mid x_3, x_2=S)\, P(x_3 \mid x_2=S)$$

$$= \eta P(z_2 \mid x_2=S)\, P(x_2=S \mid x_1=S) \sum_{x_3} P(z_3 \mid x_3)\, P(x_3 \mid x_2=S)\, P(z_4 \mid x_3)$$

$$= \eta P(z_2 \mid x_2=S)\, P(x_2=S \mid x_1=S) \sum_{x_3} \Big[P(z_3 \mid x_3)\, P(x_3 \mid x_2=S) \sum_{x_4} P(z_4 \mid x_4, x_3)\, P(x_4 \mid x_3)\Big]$$

$$= \eta P(z_2 \mid x_2=S)\, P(x_2=S \mid x_1=S) \sum_{x_3} \Big[P(z_3 \mid x_3)\, P(x_3 \mid x_2=S) \sum_{x_4} P(z_4 \mid x_4)\, P(x_4 \mid x_3)\Big]$$

$\because z_4=R \Rightarrow x_4=R$

$$= \eta P(z_2 \mid x_2=S)\, P(x_2=S \mid x_1=S) \sum_{x_3} \Big[P(z_3 \mid x_3)\, P(x_3 \mid x_2=S)\, P(z_4=R \mid x_4=R)\, P(x_4=R \mid x_3)\Big]$$

$$\Rightarrow P(x_2=S \mid z_{2:4}, x_1) = \eta P(z_2 \mid x_2=S)\, P(x_2=S \mid x_1=S) \sum_{x_3} \Big[P(z_3 \mid x_3)\, P(x_3 \mid x_2=S)\, P(z_4=R \mid x_4=R)\, P(x_4=R \mid x_3)\Big]$$

$$= \eta \cdot 0.6 \cdot 0.8 \cdot (0.6 \times 0.8 \times 1 \times 0 + 0.3 \times 0.2 \times 1 \times 0.2 + 0 \times 0 \times 1 \times 0.2)$$

$$= 0.00576\,\eta$$

同理可得：

$$\Rightarrow P(x_2=C \mid z_{2:4}, x_1)$$

$$= \eta P(z_2 \mid x_2=C)\, P(x_2=C \mid x_1=S) \sum_{x_3} \Big[P(z_3 \mid x_3)\, P(x_3 \mid x_2=C)\, P(z_4=R \mid x_4=R)\, P(x_4=R \mid x_3)\Big]$$

$$= \eta \cdot 0.3 \cdot 0.2 \cdot (0.6 \times 0.4 \times 1 \times 0 + 0.3 \times 0.4 \times 1 \times 0.2 + 0 \times 0.2 \times 1 \times 0.2)$$

$$= 0.00144\,\eta$$

$$\Rightarrow P(x_2=R \mid z_{2:4}, x_1)$$

$$= \eta \underbrace{P(z_2 \mid x_2=R)}_{=0} P(x_2=R \mid x_1=S) \sum_{x_3} \left[P(z_3 \mid x_3) P(x_3 \mid x_2=R) P(z_4=R \mid x_4=R) P(x_4=R \mid x_3) \right]$$

$$= 0$$

$$\eta = \frac{1}{0.00576+0.00144} = \frac{1250}{9} \cong 138.89$$

$$\Rightarrow P(x_2=S \mid z_{2:4}, x_1) = 0.00576\,\eta = 0.8$$

$$P(x_2=C \mid z_{2:4}, x_1) = 0.00144\,\eta = 0.2$$

$$P(x_2=R \mid z_{2:4}, x_1) = 0$$

\#

② Day 3 :

$$P(x_3=S \mid z_{2:4}, x_1)$$

$$= \frac{P(z_{2:4} \mid x_3=S, x_1=S) P(x_3=S \mid x_1=S)}{P(z_{2:4} \mid x_1=S)}$$

$$= \eta P(z_{2:4} \mid x_3=S, x_1=S) P(x_3=S \mid x_1=S)$$

$$= \eta P(z_4 \mid x_3=S, x_1=S) P(z_3 \mid x_3=S, x_1=S) P(z_2 \mid x_3=S, x_1=S) P(x_3=S \mid x_1=S)$$

$$= \eta P(z_3 \mid x_3=S) \sum_{x_4} \left[P(z_4 \mid x_4, x_3=S) P(x_4 \mid x_3=S) \right] \sum_{x_2} \left[P(z_2 \mid x_2, x_1=S) P(x_2 \mid x_1=S) \right]$$

$$\cdot \sum_{x_2} \left[P(x_3=S \mid x_2, x_1=S) P(x_2 \mid x_1=S) \right]$$

$$= \eta P(z_3 \mid x_3=S) \sum_{x_4} \left[P(z_4 \mid x_4) P(x_4 \mid x_3=S) \right] \sum_{x_2} \left[P(z_2 \mid x_2) P(x_2 \mid x_1=S) \right]$$

$$\cdot \sum_{x_2} \left[P(x_3=S \mid x_2) P(x_2 \mid x_1=S) \right]$$

$\because z_4=R \Rightarrow x_4=R$

$$= \eta P(z_3 \mid x_3=S) P(z_4=R \mid x_4=R) \underbrace{P(x_4=R \mid x_3=S)}_{=0} \sum_{x_2} \left[P(z_2 \mid x_2) P(x_2 \mid x_1=S) \right]$$

$$\cdot \sum_{x_2} \left[P(x_3=S \mid x_2) P(x_2 \mid x_1=S) \right]$$

$$\Rightarrow P(x_3=S \mid z_{2:4}, x_1) = 0$$

同理可得：

$\Rightarrow P(x_3=C \mid z_{2:4}, x_1) =$

$\eta\, P(z_3 \mid x_3=C)\, P(z_4=R \mid x_4=R)\, P(x_4=R \mid x_3=C) \sum_{x_2} [P(z_2 \mid x_2) P(x_2 \mid x_1=S)]$

$\cdot \sum_{x_2} [P(x_3=C \mid x_2)\, P(x_2 \mid x_1=S)]$

$= \eta \cdot 0.3 \cdot 1 \cdot 0.2 \cdot (0.6\times 0.8 + 0.3\times 0.2 + 0)(0.2\times 0.8 + 0.4\times 0.2 + 0)$

$= 0.007776\ \eta$

$\Rightarrow P(x_3=R \mid z_{2:4}, x_1)$

$= \eta\, \underbrace{P(z_3 \mid x_3=R)}_{0}\, P(z_4=R \mid x_4=R)\, P(x_4=R \mid x_3=R) \sum_{x_2} [P(z_2 \mid x_2) P(x_2 \mid x_1=S)]$

$\cdot \sum_{x_2} [P(x_3=R \mid x_2)\, P(x_2 \mid x_1=S)]$

$= 0$

$\eta = \frac{1}{0.007776}$

$\Rightarrow P(x_3=S \mid z_{2:4}, x_1) = 0$

$P(x_3=C \mid z_{2:4}, x_1) = 1$

$P(x_3=R \mid z_{2:4}, x_1) = 0$ #

④ Day 4：

前面也有提到

在 $z_4=R$ 條件下

x_4 必為 rainy

$\Rightarrow P(x_4=S \mid z_{2:4}, x_1) = 0$

$P(x_4=C \mid z_{2:4}, x_1) = 0$

$P(x_4=R \mid z_{2:4}, x_1) = 1$ #

(c) Consider the same situation (Day 1 is sunny, the measurements for Days 2, 3, and 4 are sunny, sunny, rainy). What is the most likely sequence of weather for Days 2 through 4? What is the probability of this most likely sequence?

Ans:

i. Use way1 method:

most likely sequence: Day 1 is sunny, Day 2 is sunny, Day 3 is sunny, Day 4 is rainy

probability of this sequence:

$$P \approx 1 \times 0.89 \times 0.87 \times 1 = 0.7743$$

ii. Use way2 method:

most likely sequence: Day 1 is sunny, Day 2 is sunny, Day 3 is cloudy, Day 4 is rainy

probability of this sequence:

$$P = 1 \times 0.8 \times 1 \times 1 = 0.8$$