Tutorial 8 the Mid-term Test

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COMP210
Discrete Structure

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Problem 1 (10pt)

A TVB television poll of 151 people found that 68 watched "Lives of Omission", 61 watched "Men with no shadows"; 52 watched "Be home for dinner"; 16 watched both "Lives of Omission" and "Men with no shadows"; 25 watched both "Men with no shadows" and "Be home for dinner"; 19 watched both "Lives of Omission" and "Be home for dinner"; and 26 watched none of these shows. How many persons watched all three shows? Justify your answer.

$$A = \{people\ who\ watched\ “Lives\ of\ Omission''\} \quad = 68$$
$$B = \{people\ who\ watched\ “Men\ with\ no\ shadows''\} \quad = 61$$
$$C = \{people\ who\ watched\ “Be\ home\ for\ dinner''\} \quad = 52$$

Problem 1 (10pt) cont.

$$A \cap B = 16 \qquad B \cap C = 25 \qquad A \cap C = 19$$

$$\overline{A \cup B \cup C} = 26 \qquad A \cup B \cup C = 151 - 26 = 125$$

To find

$$\begin{aligned} A \cap B \cap C &= (A \cup B \cup C) + (A \cap B) + (B \cap C) + (A \cap C) \\ &\quad - A - B - C \\ &= 125 + 16 + 25 + 19 - 68 - 61 - 52 = 4 \end{aligned}$$

$\blacksquare$

Problem 2 (20pt)

True or False:

- Let A = {1, 3, 5}, B = {3, 4}, A-B={5} False
- If $p \rightarrow q$, then $\neg(p \wedge \neg q)$ True
- $\exists x$, s.t., $x^2 + x + 1 < 0$ False
- Assume the musical will be scheduled if and only if both Jay and Marry show up on time. Since Jay is not here, the musical will be canceled. True
- Let the domain and co–domain be real numbers. Then, f(x)=3x is bijective. True

Problem 2 (20pt) cont.

True or False:

- If f(x) is onto, its inverse function exists — False
- Relation {(1,2), (2,1)} is reflexive. — False
- If $\lfloor x \rfloor = \lceil x \rceil$, then x is an integer number — True
- $\forall y \in Z^{+}$, x mod y is non-negative — True
- A relation can be both anti-symmetric and symmetric. — True

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Problem 3 (30pt)

Prove or disapprove the following statements:

a. Let $H_k = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$. *Then,*
$\sum_{i=1}^{n} H_i = (n+1)H_n - n$ *for all* $n \geq 1$.

True

prove by induction

Base case:
$\sum_{i=1}^{1} H_i = H_1 = 1$ $\qquad (1+1)\cdot H_1 - 1 = 1$

Induction:
For n=k, $\sum_{i=1}^{k} H_i = (k+1)H_k - k$ is True

Problem 3 (30pt) cont.

when n=k+1,

$$\begin{aligned}\sum_{i=1}^{k+1} H_i &= \sum_{i=1}^{k} H_i + H_{k+1} \\ &= (k+1)H_k - k + H_{k+1} \\ &= (k+1)(H_{k+1} - \frac{1}{k+1}) - k + H_{k+1} \\ &= (k+2)H_{k+1} - (k+1)\end{aligned}$$

$\blacksquare$

Problem 3 (30pt) cont.

Prove or disapprove the following statements:

b. $2^n - 1$ *is prime for all* $n \geq 1$

False disprove by example

When n=4, $2^n - 1 = 16 - 1 = 15$ is not a prime number.

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Problem 3 (30pt) cont.

Prove or disapprove the following statements:

c. There are no positive integer solutions to the equation $x^2 - y^2 = 1$

True

prove by contradiction

If there exist positive integer x, y, then x+y, x-y are both integers.

$\Rightarrow$

$$\begin{cases} x + y = 1 \\ x - y = 1 \end{cases} \qquad \begin{cases} x + y = -1 \\ x - y = -1 \end{cases}$$

Problem 3 (30pt) cont.

$$\begin{cases} x + y = 1 \\ x - y = 1 \end{cases} \quad\quad \begin{cases} x + y = -1 \\ x - y = -1 \end{cases}$$

$$\Downarrow \quad\quad \Downarrow$$

$$\begin{cases} x = 1 \\ y = 0 \end{cases} \quad\quad \begin{cases} x = -1 \\ y = 0 \end{cases}$$

Contradiction.

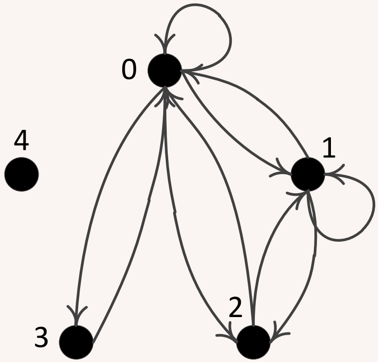
$\Rightarrow$ So the hypothesis is false, the original statement is true. $\blacksquare$

Problem 4 (30pt)

Draw the graph of the following relations on $\{0, 1, 2, 3, 4\}$. Determin if they are reflexive, transitive, symmetric, antisymmetric, partial order?

a. xRy *if* $x + y < 4$

$$\begin{aligned} xRy &= \{(x, y) \mid x + y < 4\} \\ &= \{\{0,0\}, \{0,1\}, \{0,2\}, \{0,3\}, \\ &\{1,0\}, \{1,1\}, \{1,2\}, \{2,0\}, \\ &\{2,1\}, \{3,0\}\} \end{aligned}$$



Symmetric

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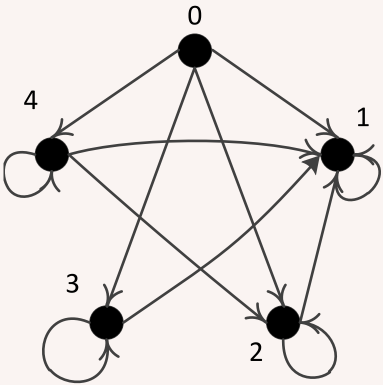
Problem 4 (30pt) cont.

Draw the graph of the following relations on $\{0, 1, 2, 3, 4\}$. Determin if they are reflexive, transitive, symmetric, antisymmetric, partial order?
b. xRy if x is dividable by y

$$xRy = \{(x, y) \mid x \equiv 0 \; mod(y)\}$$
$$= \{\{0,1\},\{0,2\},\{0,3\},\{0,4\},\{1,1\}, \{2,2\},\{3,3\},\{4,4\},\{2,1\},\{3,1\}, \{4,1\},\{4,2\}\}$$



Antisymmetric, Transitive

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Problem 4 (30pt) cont.

Draw the graphs of the following relations on {0, 1, 2, 3, 4}. Determin if they are reflexive, transitive, symmetric, antisymmetric, partial order?

c. $\{\{1,1\},\{2,2\},\{3,3\},\{4,4\},\{2,3\},\{3,4\},\{4,2\}\}$

Antisymmetric

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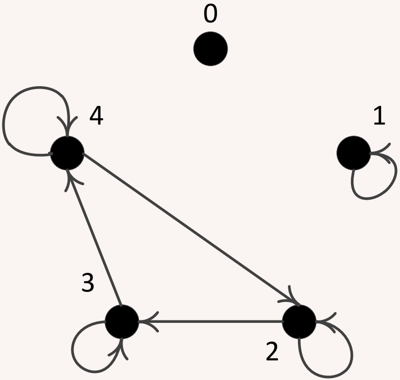

Problem 5 (10pt and 10pt bonus)

A plane convex set, subsequently abbreviated to "convex set", is a non-empty set X in the plane having the property that if x and y are any two points in X, the straight-line segment from x to y is also in X. The following figures illustrate:

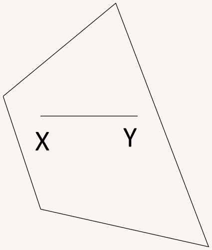

Figure: Convex set

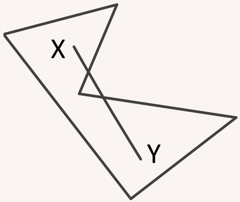

Figure: Non convex set

Problem 5 (10pt and 10pt bonus) cont.

a. Prove that if X and Y are convex sets and $X \cap Y$ is nonempty, then $X \cap Y$ is convex.

$\forall x, y \in X \cap Y$
Let the point z be on the straight line segment from x to y.

Since $x, y \in X$, X is convex $\Rightarrow z \in X$
Since $x, y \in Y$, Y is convex $\Rightarrow z \in Y$

$\Rightarrow z \in X \cap Y$

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Problem 5 (10pt and 10pt bonus) cont.

b. *Assume the following statement is correct:*

> *Lemma: Four convex sets are given in the plane. If every three of them have a nonempty intersection, then the intersection of all four sets is also nonempty.*

Prove Helly's theorem: Suppose that $X_1, X_2, \cdots, X_n, n \geq 4$ *are convex sets, each three of which have a common point. All* n *sets have a common point.*

Prove by indcution.

Base step: n=4 true by lemma

Problem 5 (10pt and 10pt bonus) cont.

Induction step: true when n=k

now consider n=k+1, for $X_1, X_2, \cdots, X_{k+1}$,
let $Y_1 = X_1 \cap X_{k+1}, Y_2 = X_2 \cap X_{k+1}, \cdots, Y_k = X_k \cap X_{k+1}$
a total of k sets.
By (a), all of them are convex.
consider any 3 sets:
$X_i \cap X_{k+1}$, $X_j \cap X_{k+1}$ and $X_l \cap X_{k+1}$ $i, j, l \in \{1, \cdots, k\}$
$= X_i \cap X_j \cap X_l \cap X_{k+1} \neq \phi$ by the base of the induction where we pick four sets X_i, X_j, X_l, X_{k+1}.
This implies that any three set among $Y_1, Y_2, \cdots, Y_k$ share a common point.
By induction hypothesis, $\bigcap_{i=1}^{k+1} X_i = \bigcap_{i=1}^{k} Y_i \neq \phi$