Formulas and Tables

(these will be included in your final exam)

Regression

- Simple regression line: $a_1 = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$, $a_0 = \bar{y} - a_1\bar{x}$
- Standard error of the estimate (simple): $s_{y/x} = \sqrt{\frac{S_r}{n-2}}$
- Quadratic regression matrix:

$$\begin{bmatrix} n & \sum x & \sum x^2 \\ \sum x & \sum x^2 & \sum x^3 \\ \sum x^2 & \sum x^3 & \sum x^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum xy \\ \sum x^2 y \end{bmatrix}$$

- Multiple regression matrix (two independent variables):

$$\begin{bmatrix} n & \sum x_1 & \sum x_2 \\ \sum x_1 & \sum x_1^2 & \sum x_1 x_2 \\ \sum x_2 & \sum x_1 x_2 & \sum x_2^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum y \\ \sum x_1 y \\ \sum x_2 y \end{bmatrix}$$

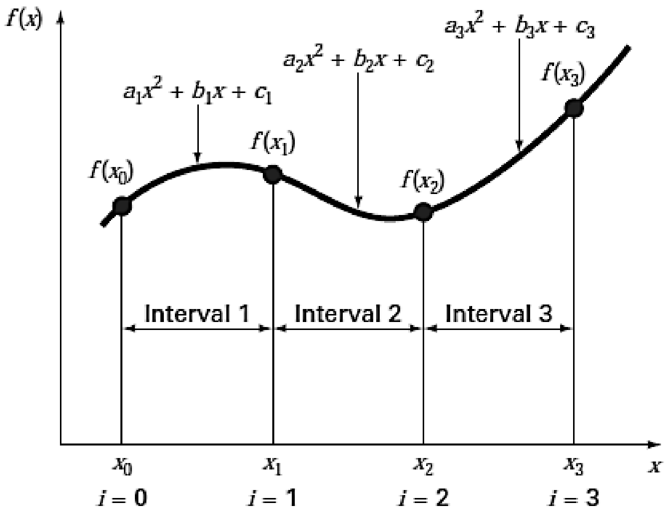
- Standard error of the estimate (polynomial/multiple): $s_{y/x} = \sqrt{\frac{S_r}{n-(m+1)}}$
- Correlation coefficient: $r = \sqrt{\frac{S_t - S_r}{S_t}}$

Polynomial Interpolation

- Newton's form (n+1 data points): $f_n(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)(x-x_1) + ... + b_n(x-x_0)...(x-x_{n-1})$
  $b_0 = y_0$
  $b_1 = \frac{y_1 - y_0}{x_1 - x_0}$
  $b_2 = \frac{\frac{y_2-y_1}{x_2-x_1} - \frac{y_1-y_0}{x_1-x_0}}{x_2 - x_0}$
  $b_n = [y_0, y_1, ..., y_n]$
- Lagrange's form (n data points): $f_{n-1}(x) = L_1 y_1 + L_2 y_2 + ... + L_n y_n$
  $L_i = \prod_{j \neq i}^{n} \frac{(x - x_j)}{(x_i - x_j)}$

Splines

Quadratic splines diagram:



Quadratic splines conditions:

1. : Each quadratic spline must pass through the data points at each endpoint of the interval where it is defined.
2. : The values of the first derivatives of adjacent quadratic splines must be equal at the knot where they meet.

Numerical Integration

Integral Formulas

- Trapezoid rule (single application): $I = \frac{(b-a)(f(a)+f(b))}{2}$
- Trapezoid rule (composite): $I = (b-a)\frac{f(x_0)+2\sum_{i=1}^{n-1} f(x_i)+f(x_n)}{2n}$
- Simpson's 1/3 rule (single): $I = \frac{(b-a)}{6}(f(a)+4f(m)+f(b))$
- Simpson's 1/3 rule (composite): $I = \frac{(b-a)}{3n}(f(x_0)+4\sum_{i=1,3,5,\ldots}^{n-1} f(x_i)+2\sum_{j=2,4,6,\ldots}^{n-2} f(x_j)+f(x_n))$
- Simpson's 3/8 rule (single): $I = \frac{(b-a)}{8}(f(x_0)+3f(x_1)+3f(x_2)+f(x_3))$

Error Estimation

- Trapezoid rule (single application): $E_a = -\frac{1}{12}(f'(b)-f'(a))(b-a)^2$
- Trapezoid rule (composite): $E_a = -\frac{(b-a)^2}{12n^2}(f'(b)-f'(a))$
- Simpson's 1/3 rule (single): $E_a = -\frac{(b-a)^4}{2880}(f^{(3)}(b)-f^{(3)}(a))$
- Simpson's 1/3 rule (composite): $E_a = -\frac{(b-a)^4}{180n^4}(f^{(3)}(b)-f^{(3)}(a))$
- Simpson's 3/8 rule (single): $E_a = -\frac{(b-a)^4}{6480}(f^{(3)}(b)-f^{(3)}(a))$

Romberg Integration

1st column

$R(n,0)$ is the composite trapezoid rule estimate of the integral using width $h = \frac{b-a}{2^n}$

$$
\begin{array}{lllll}
R(0,0) & & & & \\
R(1,0) & R(1,1) & & & \\
R(2,0) & R(2,1) & R(2,2) & & \\
R(3,0) & R(3,1) & R(3,2) & R(3,3) & \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
$$

$R(n,0) = I(h_n) =$ composite trapezoid rule with $h_n = \frac{b-a}{2^n}$

$R(n,k) = \frac{4^k R(n,k-1) - R(n-1,k-1)}{4^k - 1}$ when $n \geq k > 0$

E.g. $R(1,1) = \frac{4R(1,0) - R(0,0)}{3}$

Numerical Differentiation

Forward Estimates

	Error
First Derivative	
$f'(x_i) = \dfrac{f(x_{i+1}) - f(x_i)}{h}$	$O(h)$
$f'(x_i) = \dfrac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h}$	$O(h^2)$
Second Derivative	
$f''(x_i) = \dfrac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2}$	$O(h)$
$f''(x_i) = \dfrac{-f(x_{i+3}) + 4f(x_{i+2}) - 5f(x_{i+1}) + 2f(x_i)}{h^2}$	$O(h^2)$
Third Derivative	
$f'''(x_i) = \dfrac{f(x_{i+3}) - 3f(x_{i+2}) + 3f(x_{i+1}) - f(x_i)}{h^3}$	$O(h)$
$f'''(x_i) = \dfrac{-3f(x_{i+4}) + 14f(x_{i+3}) - 24f(x_{i+2}) + 18f(x_{i+1}) - 5f(x_i)}{2h^3}$	$O(h^2)$
Fourth Derivative	
$f''''(x_i) = \dfrac{f(x_{i+4}) - 4f(x_{i+3}) + 6f(x_{i+2}) - 4f(x_{i+1}) + f(x_i)}{h^4}$	$O(h)$
$f''''(x_i) = \dfrac{-2f(x_{i+5}) + 11f(x_{i+4}) - 24f(x_{i+3}) + 26f(x_{i+2}) - 14f(x_{i+1}) + 3f(x_i)}{h^4}$	$O(h^2)$

Backward Estimates

First Derivative | Error

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h} \qquad O(h)$$

$$f'(x_i) = \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2h} \qquad O(h^2)$$

Second Derivative

$$f''(x_i) = \frac{f(x_i) - 2f(x_{i-1}) + f(x_{i-2})}{h^2} \qquad O(h)$$

$$f''(x_i) = \frac{2f(x_i) - 5f(x_{i-1}) + 4f(x_{i-2}) - f(x_{i-3})}{h^2} \qquad O(h^2)$$

Third Derivative

$$f'''(x_i) = \frac{f(x_i) - 3f(x_{i-1}) + 3f(x_{i-2}) - f(x_{i-3})}{h^3} \qquad O(h)$$

$$f'''(x_i) = \frac{5f(x_i) - 18f(x_{i-1}) + 24f(x_{i-2}) - 14f(x_{i-3}) + 3f(x_{i-4})}{2h^3} \qquad O(h^2)$$

Fourth Derivative

$$f''''(x_i) = \frac{f(x_i) - 4f(x_{i-1}) + 6f(x_{i-2}) - 4f(x_{i-3}) + f(x_{i-4})}{h^4} \qquad O(h)$$

$$f''''(x_i) = \frac{3f(x_i) - 14f(x_{i-1}) + 26f(x_{i-2}) - 24f(x_{i-3}) + 11f(x_{i-4}) - 2f(x_{i-5})}{h^4} \qquad O(h^2)$$

Centered Estimates

First Derivative | Error

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} \qquad O(h^2)$$

$$f'(x_i) = \frac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2})}{12h} \qquad O(h^4)$$

Second Derivative

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} \qquad O(h^2)$$

$$f''(x_i) = \frac{-f(x_{i+2}) + 16f(x_{i+1}) - 30f(x_i) + 16f(x_{i-1}) - f(x_{i-2})}{12h^2} \qquad O(h^4)$$

Third Derivative

$$f'''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + 2f(x_{i-1}) - f(x_{i-2})}{2h^3} \qquad O(h^2)$$

$$f'''(x_i) = \frac{-f(x_{i+3}) + 8f(x_{i+2}) - 13f(x_{i+1}) + 13f(x_{i-1}) - 8f(x_{i-2}) + f(x_{i-3})}{8h^3} \qquad O(h^4)$$

Fourth Derivative

$$f''''(x_i) = \frac{f(x_{i+2}) - 4f(x_{i+1}) + 6f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{h^4} \qquad O(h^2)$$

$$f''''(x_i) = \frac{-f(x_{i+3}) + 12f(x_{i+2}) - 39f(x_{i+1}) + 56f(x_i) - 39f(x_{i-1}) + 12f(x_{i-2}) - f(x_{i-3})}{6h^4} \qquad O(h^4)$$